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Ghulam Mustafa
The Islamia University of Bahawalpur

Faheem Khan
University of Sargodha

Abdul Ghaffar
Balochistan University of Information Technology, Engineering and Management Sci…

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The m-Point Approximating Subdivision Scheme

G. Mustafa*, **F. Khan****, and **A. Ghaffar*****

(Submitted by A.V. Lapin)

Department of Mathematics, The Islamia University of Bahawalpur, Pakistan

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Abstract—In this paper, the m-point approximating subdivision scheme with one parameter is proposed and analyzed where $m > 1$. Smoothness of schemes is higher in comparison with the existing binary and ternary subdivision schemes. The proposed scheme also generalizes several subdivision schemes.

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1. INTRODUCTION

Computer Aided Geometric Design (CAGD) is a branch of applied Mathematics concerned with algorithms for the design of smooth curves/surfaces. One common approach to the design of curves/surfaces which related to CAGD is the subdivision schemes. It is an algorithm technique to generate smooth curves and surfaces as a sequence of successively refined control polygons. At each refinement level, new points are added into the existing polygon and the original points remain existed or discarded in all subsequent sequences of control polygons. The number of points inserted at level $k+1$ between two consecutive points from level k is called arity of the scheme. In the case when number of points inserted are $2, 3, \ldots, n$ the subdivision schemes are called binary, ternary,..., n-ary respectively. For more details on n-ary subdivision schemes, we may refer to the thesis of N. Aspert [1] and Kwan [9].

Now a days wide variety of approximating and interpolating schemes have been proposed in the literature which posses shape parameters. In this paper we offer family of approximating schemes and give comparison with other existing schemes. The crucial issue that is smoothness of schemes has been discussed by Laurent polynomial method for a certain range of parameter.

In the following section we present brief introduction about preliminary concepts used in this work. In Section 3 we present and analyze our schemes. We give comparison of our schemes with other schemes in Section 4.

2. PRELIMINARIES

A general form of univariate subdivision scheme S which maps a polygon $f^k = \{f_i^k\}_{i\in\mathbb{Z}}$ to a refined polygon $f^{k+1} = \{f_i^{k+1}\}_{i\in\mathbb{Z}}$ is defined by

$$\begin{cases} f_{2i}^{k+1} = \sum\limits_{j\in\mathbb{Z}} \alpha_{2j} f_{i-j}^k, \\ f_{2i+1}^{k+1} = \sum\limits_{j\in\mathbb{Z}} \alpha_{2j+1} f_{i-j}^k, \end{cases} \tag{2.1}$$

*E-mail: mustafa_rakib@yahoo.com
**E-mail: fahimscholar@gmail.com
***E-mail: gulzarkhan143@yahoo.com

where the set $\alpha = \{\alpha_i : i \in \mathbb{Z}\}$ of coefficients is called mask of the subdivision scheme. A necessary condition for the uniform convergence of the subdivision scheme (2.1) is that

$$\sum_{j\in\mathbb{Z}} \alpha_{2j} = \sum_{j\in\mathbb{Z}} \alpha_{2j+1} = 1. \tag{2.2}$$

For the analysis of subdivision scheme with mask α, it is very practical to consider the z-transform of the mask,

$$\alpha(z) = \sum_{i\in\mathbb{Z}} \alpha_i z^i, \tag{2.3}$$

which is usually called the *symbol /Laurent polynomial* of the scheme.

From (2.2) and (2.3) the Laurent polynomial of a convergent subdivision scheme satisfies

$$\alpha(-1) = 0 \quad \text{and} \quad \alpha(1) = 2. \tag{2.4}$$

This condition guarantees existence of a related subdivision scheme for the divided differences of the original control points and the existence of associated Laurent polynomial $\alpha^{(1)}(z)$.

$$\alpha^{(1)}(z) = \frac{2z}{1+z}\alpha(z).$$

The subdivision scheme S_1 with *symbol* $\alpha^{(1)}(z)$ is related to scheme S with *symbol* $\alpha(z)$ by the following Theorem.

Theorem 2.1 [4]. *Let S denote a subdivision scheme with symbol $\alpha(z)$ satisfying (2.2). Then there exist a subdivision scheme S_1 with the property*

$$\Delta f^k = S_1 \Delta f^{k-1},$$

where $f^k = S^k f^0$ and $\Delta f^k = \{(\Delta f^k)_i = 2^k(f_{i+1}^k - f_i^k) : i \in \mathbb{Z}\}$. Furthermore, S is a uniformly convergent if and only if $\frac{1}{2}S_1$ converges uniformly to the zero function for all initial data f^0 in the sense that

$$\lim_{k\to\infty} \left(\frac{1}{2}S_1\right)^k f^0 = 0. \tag{2.5}$$

A scheme S_1 satisfying (2.5) for all initial data $f^0 = \{f_i^0 : i \in \mathbb{Z}\}$ is termed contractive. By Theorem 2.1, the convergence of S is equivalent to checking whether S_1 is contractive, which is then equivalent to checking whether $\left|\left|\left(\frac{1}{2}S_1\right)^L\right|\right|_\infty < 1$ for some integer $L > 0$.

Since there are two rules for computing the values at next refinement level, one with even coefficients of the mask and one with odd coefficient of the mask, we define the norm

$$||S||_\infty = \max\left\{\sum_{j\in\mathbb{Z}} |\alpha_{2j}|, \sum_{j\in\mathbb{Z}} |\alpha_{2j+1}|\right\},$$

and

$$\left|\left|\left(\frac{1}{2}S_n\right)^L\right|\right|_\infty = \max\left\{\sum_{j\in\mathbb{Z}} \left|b_{i+2^L j}^{[n,L]}\right| : i = 0, 1, \ldots, 2^L - 1\right\},$$

where

$$b^{[n,L]}(z) = \frac{1}{2^L}\prod_{j=0}^{L-1} \alpha^{(n)}(z^{2^j}), \quad \alpha^{(n)}(z) = \frac{2z}{1+z}\alpha^{(n-1)}(z). \tag{2.6}$$

Theorem 2.2 [4]. *Let $\alpha(z) = \left(\frac{1+z}{2}\right)^n q(z)$. If S_q is convergent, then $S_\alpha^\infty \in C^n(\mathbb{R})$ for any initial data f^0.*

3. THE m-POINT APPROXIMATING SCHEME

In this section we present the $(\lambda+2)$-point subdivision scheme with one parameter. The Laurent polynomial of scheme is defined by

$$\alpha_{[\lambda+2]}(z)=\frac{1}{2^{2\lambda}}(1+z)^{2\lambda}\left(\left(\frac{1}{6}+\omega\right)+\left(\frac{5}{6}-\omega\right)z+\left(\frac{5}{6}-\omega\right)z^2+\left(\frac{1}{6}+\omega\right)z^3\right), \tag{3.1}$$

where $\lambda=0,1,\ldots,t$.

3.1. Analysis of the Schemes

Here we analyze smoothness of proposed schemes by using Laurent polynomial method.

2-point scheme. The mask of scheme for $\lambda=0$ from (3.1) is

$$\alpha_{[2]}=\left[\ldots,0,0,\frac{1}{6}+\omega,\frac{5}{6}-\omega,\frac{5}{6}-\omega,\frac{1}{6}+\omega,0,0,\ldots\right].$$

For C^0 continuity we require that $\alpha_{[2]}$ satisfy (2.2), which it does and $\left\|\left(\frac{1}{2}S_1^{[2]}\right)^L\right\|_\infty<1$, we have

$$\alpha_{[2]}^{(1)}=\left[..,0,0,\frac{1}{3}+2\omega,\frac{4}{3}-4\omega,\frac{1}{3}+2\omega,0,0,\ldots\right]$$

for $-\frac{1}{6}<\omega<\frac{1}{3}$ and $L=1$,

$$\Rightarrow\quad\left\|\frac{1}{2}S_1^{[2]}\right\|_\infty=\max\left\{\left|\frac{2}{3}-2\omega\right|,2\left|\frac{1}{6}+\omega\right|\right\}<1. \tag{3.2}$$

Hence, by Theorem 2.1, scheme is C^0.

3-point scheme. For this scheme we take $\lambda=1$ in (3.1)

$$\alpha_{[3]}=\left[\ldots,0,0,\frac{1}{24}+\frac{1}{4}\omega,\frac{7}{24}+\frac{1}{4}\omega,\frac{2}{3}-\frac{1}{2}\omega,\frac{2}{3}-\frac{1}{2}\omega,\frac{7}{24}+\frac{1}{4}\omega,\frac{1}{24}+\frac{1}{4}\omega,0,0,\ldots\right].$$

For C^0 continuity we require that $\alpha_{[3]}$ satisfy (2.2), which it does and $\left\|\left(\frac{1}{2}S_1^{[3]}\right)^L\right\|_\infty<1$, we have

$$\alpha_{[3]}^{(1)}=\left[..,0,0,\frac{1}{12}+\frac{1}{2}\omega,\frac{1}{2},\frac{5}{6}-\omega,\frac{1}{2},\frac{1}{12}+\frac{1}{2}\omega,0,0,\ldots\right]$$

for $-\frac{2}{3}<\omega<\frac{4}{3}$ and $L=1$,

$$\left\|\frac{1}{2}S_1^{[3]}\right\|_\infty=\max\left\{2\left|\frac{1}{24}+\frac{1}{4}\omega\right|+\left|\frac{5}{12}-\frac{1}{2}\omega\right|,\frac{1}{2}\right\}<1. \tag{3.3}$$

For C^1 continuity we require that $\alpha_{[3]}^{(1)}$ satisfy (2.2), which it does and $\left\|\left(\frac{1}{2}S_2^{[3]}\right)^L\right\|_\infty<1$, since $\alpha_{[3]}^{(2)}=\alpha_{[2]}$, so for $-\frac{2}{3}<\omega<\frac{4}{3}$ and $L=1$,

$$\left\|\frac{1}{2}S_2^{[3]}\right\|_\infty=\max\left\{\left|\frac{1}{12}+\frac{1}{2}\omega\right|+\left|\frac{5}{12}-\frac{1}{2}\omega\right|\right\}<1. \tag{3.4}$$

Next we have $\alpha_{[3]}^{(3)}=\alpha_{[2]}^{(1)}$. By (3.2) and Theorem 2.2, scheme is C^2 for $-\frac{1}{6}<\omega<\frac{1}{3}$.

4-point scheme. For $\lambda = 2$ in (3.1) we have the mask of 4-point scheme

$$\alpha_{[4]} = \Big[\ldots, 0, 0, \frac{1}{96} + \frac{1}{16}\omega, \frac{3}{32} + \frac{3}{16}\omega, \frac{31}{96} + \frac{1}{16}\omega, \frac{55}{96} - \frac{5}{16}\omega, \frac{55}{96} - \frac{5}{16}\omega,$$
$$\frac{31}{96} + \frac{1}{16}\omega, \frac{3}{32} + \frac{3}{16}\omega, \frac{1}{96} + \frac{1}{16}\omega, 0, 0, \ldots\Big].$$

For C^0 continuity we require that $\alpha_{[4]}$ satisfy (2.2), which it does and $\left\|\left(\frac{1}{2}S_1^{[4]}\right)^L\right\|_\infty < 1$, we have

$$\alpha_{[4]}^{(1)} = \Big[\ldots, 0, 0, \frac{1}{48} + \frac{1}{8}\omega, \frac{1}{6} + \frac{1}{4}\omega, \frac{23}{48} - \frac{1}{8}\omega, \frac{2}{3} - \frac{1}{2}\omega, \frac{23}{48} - \frac{1}{8}\omega, \frac{1}{6} + \frac{1}{4}\omega, \frac{1}{48} + \frac{1}{8}\omega, 0, 0, \ldots\Big]$$

for $-\frac{5}{3} < \omega < \frac{7}{3}$ and $L = 1$,

$$\left\|\frac{1}{2}S_1^{[4]}\right\|_\infty = \max\left\{2\left|\frac{1}{12} + \frac{1}{8}\omega\right| + \left|\frac{1}{3} - \frac{1}{4}\omega\right|, 2\left(\left|\frac{1}{96} + \frac{1}{16}\omega\right| + \left|\frac{23}{96} - \frac{1}{16}\omega\right|\right)\right\} < 1. \tag{3.5}$$

For C^1 continuity we require that $\alpha_{[4]}^{(1)}$ satisfy (2.2), which it does and $\left\|\left(\frac{1}{2}S_2^{[4]}\right)^L\right\|_\infty < 1$, since $\alpha_{[4]}^{(2)} = \alpha_{[3]}$, so for $-\frac{17}{12} < \omega < \frac{13}{12}$ and $L = 1$,

$$\left\|\frac{1}{2}S_2^{[4]}\right\|_\infty = \max\left\{\left|\frac{1}{48} + \frac{1}{8}\omega\right| + \left|\frac{2}{3} - \frac{1}{2}\omega\right| + \left|\frac{7}{48} + \frac{1}{8}\omega\right|\right\} < 1. \tag{3.6}$$

Next we have $\alpha_{[4]}^{(3)} = \alpha_{[3]}^{(1)}$, $\alpha_{[4]}^{(4)} = \alpha_{[3]}^{(2)}$ and $\alpha_{[4]}^{(5)} = \alpha_{[2]}^{(1)}$, So by (3.3), (3.4), (3.2) and Theorem 2.2, scheme is C^4 for $-\frac{1}{6} < \omega < \frac{1}{3}$.

5-point scheme. From the Laurent polynomial (3.1) we have for $\lambda = 3$,

$$\alpha_{[5]} = \Big[\ldots, 0, 0, \frac{1}{384} + \frac{1}{64}\omega, \frac{11}{384} + \frac{5}{64}\omega, \frac{25}{192} + \frac{1}{8}\omega, \frac{63}{192}, \frac{49}{96} - \frac{7}{32}\omega, \frac{49}{96} - \frac{7}{32}\omega, \frac{63}{192}, \frac{25}{192} + \frac{1}{8}\omega,$$
$$\frac{11}{384} + \frac{5}{64}\omega, \frac{1}{384} + \frac{1}{64}\omega, 0, 0, \ldots\Big].$$

For C^0 continuity we require that $\alpha_{[5]}$ satisfy (2.2), which it does and $\left\|\left(\frac{1}{2}S_1^{[5]}\right)^L\right\|_\infty < 1$, we have

$$\alpha_{[5]}^{(1)} = \Big[\ldots, 0, 0, \frac{1}{192} + \frac{1}{32}\omega, \frac{5}{186} + \frac{1}{8}\omega, \frac{5}{24} + \frac{1}{8}\omega, \frac{43}{96} - \frac{1}{8}\omega, \frac{55}{96} - \frac{5}{16}\omega, \frac{43}{96} - \frac{1}{8}\omega, \frac{5}{24} + \frac{1}{8}\omega,$$
$$\frac{5}{186} + \frac{1}{8}\omega, \frac{1}{192} + \frac{1}{32}\omega, 0, 0, \ldots\Big],$$

for $-\frac{29}{12} < \omega < \frac{103}{30}$ and $L = 1$,

$$\left\|\frac{1}{2}S_1^{[5]}\right\|_\infty = \max\left\{2\left|\frac{1}{384} + \frac{1}{64}\omega\right| + 2\left|\frac{5}{48} + \frac{1}{16}\omega\right| + \left|\frac{55}{192} - \frac{5}{32}\omega\right|,\right. \tag{3.7}$$
$$\left. 2\left(\left|\frac{5}{192} + \frac{1}{16}\omega\right| + \left|\frac{43}{192} - \frac{1}{16}\omega\right|\right)\right\} < 1.$$

For C^1 continuity we require that $\alpha^{(1)}_{[5]}$ satisfy (2.2), which it does and $\left\|\left(\frac{1}{2}S_2^{[5]}\right)^L\right\|_\infty < 1$, since $\alpha^{(2)}_{[5]} = \alpha_{[4]}$, so for $-\frac{233}{96} < \omega < \frac{137}{40}$ and $L = 1$,

$$\left\|\frac{1}{2}S_2^{[5]}\right\|_\infty = \max\left\{\left|\frac{1}{192}+\frac{1}{32}\omega\right| + \left|\frac{3}{64}+\frac{3}{32}\omega\right| + \left|\frac{31}{192}+\frac{1}{32}\omega\right| + \left|\frac{55}{192}-\frac{5}{32}\omega\right|\right\} < 1. \tag{3.8}$$

Next we have $\alpha^{(3)}_{[5]} = \alpha^{(1)}_{[4]}$, $\alpha^{(4)}_{[5]} = \alpha^{(2)}_{[4]}$, $\alpha^{(5)}_{[5]} = \alpha^{(1)}_{[3]}$, $\alpha^{(6)}_{[5]} = \alpha^{(2)}_{[3]}$ and $\alpha^{(7)}_{[5]} = \alpha^{(1)}_{[2]}$. So by (3.5), (3.6), (3.3), (3.4), (3.2) and Theorem 2.2, scheme is C^6 for $-\frac{1}{6} < \omega < \frac{1}{3}$.

6-point scheme. The mask of this scheme is obtained for $\lambda = 4$

$$\alpha_{[6]} = \Big[\ldots,0,0,\frac{1}{1536}+\frac{1}{256}\omega,\frac{13}{1536}+\frac{7}{256}\omega,\frac{73}{1536}+\frac{19}{256}\omega,\frac{273}{1536}+\frac{21}{256}\omega,$$
$$\frac{249}{768}-\frac{6}{256}\omega,\frac{357}{768}-\frac{42}{256}\omega,\frac{357}{768}-\frac{42}{256}\omega,\frac{249}{768}-\frac{6}{256}\omega,\frac{273}{1536}+\frac{21}{256}\omega,\frac{73}{1536}+\frac{19}{256}\omega,$$
$$\frac{13}{1536}+\frac{7}{256}\omega,\frac{1}{1536}+\frac{1}{256}\omega,0,0,\ldots\Big].$$

For C^0 continuity we require that $\alpha_{[6]}$ satisfy (2.2), which it does and $\left\|\left(\frac{1}{2}S_1^{[6]}\right)^L\right\|_\infty < 1$, we have

$$\alpha^{(1)}_{[6]} = \Big[\ldots,0,0,\frac{1}{768}+\frac{1}{128}\omega,\frac{1}{64}+\frac{3}{64}\omega,\frac{61}{768}+\frac{13}{128}\omega,\frac{11}{48}+\frac{1}{16}\omega,\frac{161}{384}-\frac{7}{64}\omega,\frac{49}{96}-\frac{7}{32}\omega,$$
$$\frac{161}{384}-\frac{7}{64}\omega,\frac{11}{48}+\frac{1}{16}\omega,\frac{61}{768}+\frac{13}{128}\omega,\frac{1}{64}+\frac{3}{64}\omega,\frac{1}{768}+\frac{1}{128}\omega,0,0,\ldots\Big],$$

for $-\frac{95}{21} < \omega < \frac{97}{21}$ and $L = 1$,

$$\left\|\frac{1}{2}S_1^{[6]}\right\|_\infty = \max\left\{2\left|\frac{1}{128}+\frac{3}{128}\omega\right| + 2\left|\frac{11}{96}+\frac{1}{32}\omega\right| + \left|\frac{49}{192}-\frac{7}{64}\omega\right|,\right. \tag{3.9}$$
$$\left.2\left|\frac{1}{1536}+\frac{1}{256}\omega\right| + 2\left|\frac{61}{1536}+\frac{13}{256}\omega\right| + \left|\frac{161}{768}-\frac{7}{128}\omega\right|\right\} < 1.$$

For C^1 continuity we require that $\alpha^{(1)}_{[6]}$ satisfy (2.2), which it does and $\left\|\left(\frac{1}{2}S_2^{[6]}\right)^L\right\|_\infty < 1$, since $\alpha^{(2)}_{[6]} = \alpha_{[5]}$, so for $-\frac{317}{84} < \omega < \frac{97}{21}$ and $L = 1$

$$\left\|\frac{1}{2}S_2^{[6]}\right\|_\infty = \max\left\{\left|\frac{1}{768}+\frac{1}{128}\omega\right| + \left|\frac{25}{768}+\frac{1}{16}\omega\right| + \left|\frac{49}{384}-\frac{7}{64}\omega\right|\right. \tag{3.10}$$
$$\left.+\left|\frac{21}{128}\right| + \left|\frac{5}{128}+\frac{1}{768}\omega\right|\right\} < 1.$$

Next we have $\alpha^{(3)}_{[6]} = \alpha^{(1)}_{[5]}$, $\alpha^{(4)}_{[6]} = \alpha^{(2)}_{[5]}$, $\alpha^{(5)}_{[6]} = \alpha^{(1)}_{[4]}$, $\alpha^{(6)}_{[6]} = \alpha^{(2)}_{[4]}$, $\alpha^{(7)}_{[6]} = \alpha^{(1)}_{[3]}$, $\alpha^{(8)}_{[6]} = \alpha^{(2)}_{[3]}$ and $\alpha^{(9)}_{[6]} = \alpha^{(1)}_{[2]}$. So by (3.7), (3.8), (3.5), (3.6), (3.3), (3.4), and (3.2), scheme is C^8 over $-\frac{1}{6} < \omega < \frac{1}{3}$.

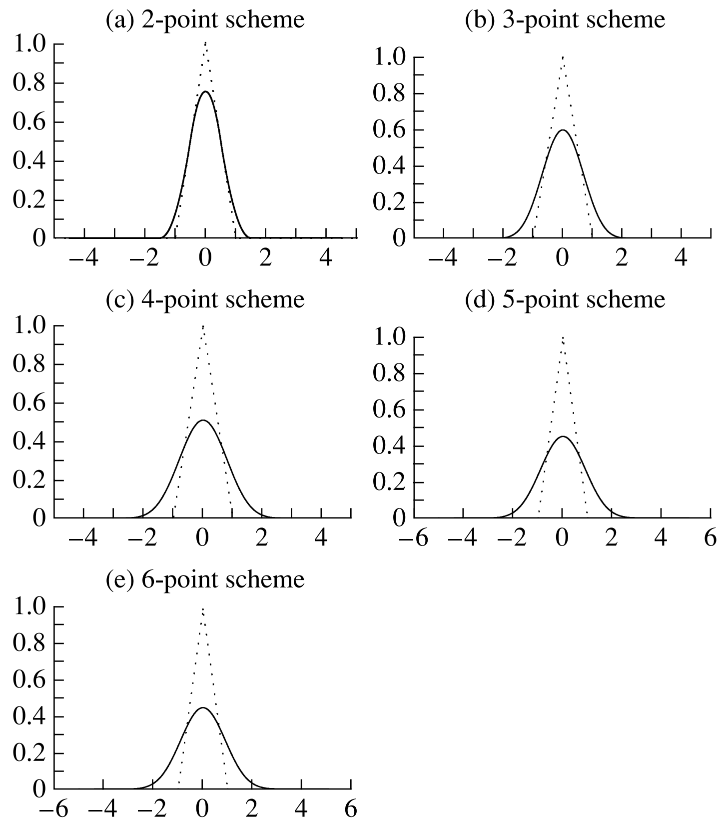


Fig. 1. The basic limit functions of proposed schemes at $\omega = 1/12$.

3.2. Support of Basic Limit Function

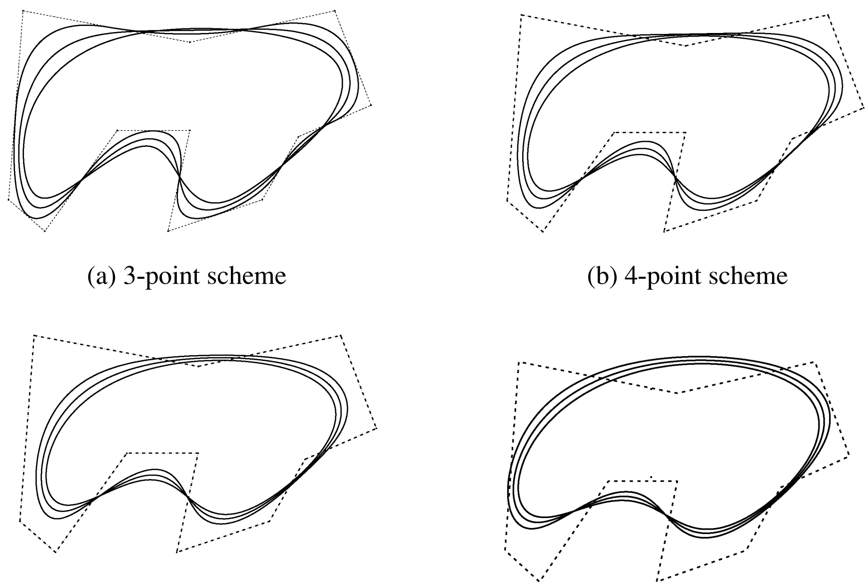
The basic function of a subdivision scheme is the limit function of proposed scheme for the following data

$$f_i^0 = \begin{cases} 1, & i = 0 \\ 0, & i \neq 0. \end{cases} \tag{3.11}$$

Figure 1 shows the basic limit functions $\psi_{[\lambda+2]} = (S^{[\lambda+2]})^\infty f_i^0$ of proposed $(\lambda+2)$-point approximating schemes for $\lambda = 0, 1, 2, 3, 4,$ respectively. The following theorem is related to the support of the limit functions.

Theorem 3.1. *The basic limit functions $\psi_{[\lambda+2]}$ of proposed $(\lambda+2)$-point schemes have support width $s = 2\lambda + 3$, which implies that it vanishes outside the interval* $\left[-\frac{2\lambda+3}{2}, \frac{2\lambda+3}{2}\right]$.

Proof. Since the basic function is the limit function of the scheme for the data (3.11), its support width s can be determine by computing how for the effect of the non-zero vertex f_0^0 will propagate along by. As the mask of $(\lambda+2)$- scheme is $2(\lambda+2)$-long sequence by centering it on that vertex, the distances to the last of its left and right non-zero coefficients are equal to $\lambda+2$ and $\lambda+1$ respectively. At the first subdivision step we see that the vertices on the left and right sides of f_0^1 at $\frac{\lambda+2}{2}$ and $\frac{\lambda+1}{2}$ are the furthest non-zero new vertices. At each refinement, the distance on both sides is reduced by the factor $\frac{1}{2}$. At the next step of the scheme this will propagate along by $\frac{\lambda+2}{2}\frac{1}{2}$ on the left and $\frac{\lambda+1}{2}\frac{1}{2}$ on the right. Hence after k subdivision steps the furthest non-zero vertex on the left will be at



(a) 3-point scheme (b) 4-point scheme

(c) 5-point scheme (d) 6-point scheme

Fig. 2. Dotted polygons are subdivided 4 times by different schemes.

$(\lambda+2)\left(\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^k}\right)=\frac{\lambda+2}{2}\sum_{j=0}^{k-1}\frac{1}{2^j}$ and on the right will be at $(\lambda+1)\left(\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^k}\right)=\frac{\lambda+1}{2}\sum_{j=0}^{k-1}\frac{1}{2^j}$. So the total support width is $\frac{\lambda+2}{2}\sum_{j=0}^{\infty}\frac{1}{2^j}+\frac{\lambda+1}{2}\sum_{j=0}^{\infty}\frac{1}{2^j}=2\lambda+3$. □

Remark 3.1.

- If we replace 2λ by $n \geqslant 0$ in right hand side of (3.1) then for $w = 1/12$, $n \geqslant 0$ we get B-splines.
- For $n = 2$, $w = 1/12$ it becomes Hassan 3-point scheme [7].
- If we set $n = 2$, $w = -1/24$ we have mask of Siddiqi 3-point scheme [11].
- In case $n = 3$, $w = -2/3$ we get mask as 4-point interpolatory scheme [6].
- Also when we set $n = 4$, $w = -19/24$ this subdivision scheme becomes the 4-point approximating scheme [5].

Table 1. Comparison of proposed existing 4-point schemes

Scheme	Type	Support	C^n
Binary 4-point [2]	Interpolating	6	1
Binary 4-point [5]	Approximating	7	2
Binary 4-point [6]	Interpolating	6	1
Ternary 4-point [3]	Interpolating	5	2
Ternary 4-point [8]	Interpolating	5	2
Ternary 4-point [10]	Approximating	5.5	2
Proposed scheme	Approximating	7	4

Table 2. Comparison of proposed existing 5-point schemes

Scheme	Type	Support	C^n
Binary 5-point [12]	Approximating	9	4
Proposed scheme	Approximating	9	6

Table 3. Comparison of proposed existing 6-point schemes

Scheme	Type	Support	C^n
Binary 6-point [13]	Approximating	11	6
Binary 6-point [14]	Interpolating	10	2
Proposed scheme	Approximating	11	8

4. COMPARISON AND APPLICATION

Comparison of some proposed schemes with existing schemes are given in Tables 1, 2, and 3. In order to see the performance of the schemes, we have used different values of the parameter in the range $-\frac{1}{6} < \omega < \frac{1}{3}$. In Figure 2 results are shown at parametric values $\omega = -0.15$, $\omega = 1/12$ and $\omega = 0.30$. In this paper we offer m-point scheme but we have only discussed the continuity and comparisons upto 6-point schemes. Continuity of other schemes can be proved in the same way.

REFERENCES

1. N. Aspert, *Non-linear subdivision of univariate signals and discrete surfaces*, EPFL thesis, (2003).
2. C. Beccari, G. Casciola, and L. Romani, *A non-stationary uniform tension controlled interpolating 4-point scheme reproducing conics*, Computer Aided Geometric Design **24**, 1 (2007).
3. C. Beccari, G. Casciola, and L. Romani, *An interpolating 4-point C^2 ternary non-stationary subdivision scheme with tension control*, Computer Aided Geometric Design **24**, 210 (2007).
4. N. Dyn, *Interpolatory subdivision schemes and analysis of convergence & smoothness by the formalism of Laurent polynomials*, in: A. Iske, E. Quak, and M. S. Floater (eds.), *Tutorials on Multiresolution in Geometric Modelling*, (Springer, 2002), p. 51 (Chapter 2 and 3).
5. N. Dyn, M. S. Floater, and K. Horman, *A C^2 four-point subdivision scheme with fourth order accuracy and its extension, in mathematical methods for curves and surfaces:* Tromso, 2004, M. Daehlen, K. Morken, and L. L. Schumaker (eds.) (2005), p. 145.
6. N. Dyn, D. Levin, and J. A. Gregory, *A 4-point interpolatory subdivision scheme for curve design*, Computer Aided Geometric Design **4**, 257 (1987).
7. M. F. Hassan and N. A. Dodgson, *Ternary and 3-point univariate subdivision schemes*, Curve and surface Fitting: Saint-Malo, 2002, A. Cohen, J.-L. Merrien, and L. L. Schumaker (eds.) (Nash-boro Press, 2003), p. 199.
8. M. F. Hassan, I. P. Ivrissimitzis, N. A. Dodgson, and M. A. Sabin, *An interpolating 4-point C^2 ternary stationary subdivision scheme*, Computer Aided Geometric Design **19**, 1 (2002).
9. Kwan Pyo Ko, *A study on subdivision scheme-draft*, Dongseo University Busan South Korea, http://kowon.dongseo.ac.kr/∼kpko/publication/ 2004book.pdf (2007).
10. Kwan Pyo Ko, Byung-Gook Lee, and Gang Joon Yoon, *A ternary 4-point approximating subdivision scheme*, Applied Mathematics and Computation **190**, 1563 (2007).
11. S. S. Siddiqi and A. Nadeem, *A new three point approximating C^2 subdivision scheme*, Applied Mathematics Letters 707 (2007).
12. S. S. Siddiqi and A. Nadeem, *A new five-point approximating subdivision scheme*, International Journal of Computer Mathematics **85** (1), 65 (2008).
13. S. S. Siddiqi, A. Nadeem, *A C^6 approximating subdivision scheme*, Applied Mathematics Letters **21** (7), 722 (2008).
14. A. Weissman, *A 6-point interpolatory subdivision scheme for curve design* (M.Sc. Thesis, Tel Aviv University, 1990).