1 Kinetic Energy and Kinetic part of the Action

NOTE: The period of the path is $T = \pi$.

Define the action as

$$A = \int_0^\pi L dt = \int_0^\pi [K - V]\, dt = \int_0^\pi K dt - \int_0^\pi V dt = A_K + A_V$$

The velocity of a particle with respect to the rest frame is $v' = v - \Omega x$, where Ω is the the generator of the rotation.

Then, using non-symmetric bra-ket notation[1], the Kinetic Energy of a single particle is

$$K = \frac{1}{2} m \langle v|v\rangle = \frac{1}{2} m \left[\underbrace{\langle v|v\rangle}_{\text{Center of mass}} + \underbrace{\langle \Omega r|\Omega r\rangle}_{\text{Centrifugal}} - \Big(\underbrace{\langle \Omega r|v\rangle + \langle v|\Omega r\rangle}_{\text{Coriolis}} \Big) \right]$$

Define, then,

$$\begin{aligned} K_{\text{CM}} &= \frac{1}{2} m \langle v|v\rangle \\ K_{\text{Cf}} &= \frac{1}{2} m \langle \Omega r|\Omega r\rangle \\ K_{\text{Cor}} &= -\frac{1}{2} m \left(\langle \Omega r|v\rangle + \langle v|\Omega r\rangle \right) \end{aligned}$$

Consider nolw the representation of the path in terms of the Fourier coefficients A_k, and the velocity

$$\begin{aligned} |x(t)\rangle &= \left(1 - \frac{t}{\pi}\right) |x_0\rangle + \frac{t}{\pi} |x_1\rangle + \sum_{k=1}^{F} \sin(kt) |A_k\rangle \\ |v(t)\rangle = |\dot{x}(t)\rangle &= -\frac{1}{\pi} |x_0\rangle + \frac{1}{\pi} |x_1\rangle + \sum_{k=1}^{F} k \cos(kt) |A_k\rangle \end{aligned}$$

To compute the action, we need to integrate K and obtain the kinetic part of the action, $\mathcal{K}$. In particular, there are three different contributions:

Center of mass contribution

$$\mathcal{K}_{\text{CM}} = \frac{1}{2} m \int_0^\pi \langle v|v\rangle\, dt = \frac{1}{2} m \left[\frac{1}{\pi} (\langle x_0|x_0\rangle + \langle x_1|x_1\rangle - \langle x_1|x_0\rangle - \langle x_0|x_1\rangle) + \frac{\pi}{2} \sum_{k=1}^{F} k^2 \langle A_k|A_k\rangle \right]$$

[1] $\langle u|v\rangle \neq \langle v|u\rangle$

Centrifugal contribution

$$\mathcal{K}_{\mathrm{Cf}} =\frac{1}{2}m\int_0^\pi \langle \Omega x|\Omega x\rangle \, dt = \frac{1}{2}m\int_0^\pi \ \langle x|\Omega^\dagger\Omega|x\rangle \,\mathrm{d}t = -\frac{1}{2}m\int_0^\pi \ \langle x|\Omega^2|x\rangle =$$

$$= -\frac{1}{2}m\Bigg\{\frac{\pi}{3}\big[\,\langle x_0|\Omega^2|x_0\rangle +\ \langle x_1|\Omega^2|x_1\rangle\big] + \frac{\pi}{6}\big[\,\langle x_0|\Omega^2|x_1\rangle +\ \langle x_1|\Omega^2|x_0\rangle\big]+$$

$$+\sum_{k=1}^{F}\left[\frac{1}{k}\big(\,\langle x_0|\Omega^2|A_k\rangle +\ \langle A_k|\Omega^2|x_0\rangle\big) + \frac{(-1)^k}{k}\big(\,\langle x_1|\Omega^2|A_k\rangle +\ \langle A_k|\Omega^2|x_1\rangle\big) + \frac{\pi}{2}\,\langle A_k|\Omega^2|A_k\rangle\right]\Bigg\}$$

Coriolis contribution

$$\mathcal{K}_{\mathrm{Cor}} =\frac{1}{2}m\int_0^\pi (\langle\Omega x|v\rangle + \langle v|\Omega x\rangle)dt = \frac{1}{2}m\int_0^\pi (\,\langle v|\Omega|x\rangle -\ \langle x|\Omega|v\rangle)dt =$$

$$=\frac{1}{2}m\Bigg\{\,\langle x_0|\Omega|x_1\rangle -\ \langle x_1|\Omega|x_0\rangle + \frac{4}{\pi}\sum_{\substack{k=1\ldots F\\ k\ \text{even}}}\frac{1}{k}\Big[\,\langle x_0|\Omega|A_k\rangle -\ \langle A_k|\Omega|x_0\rangle -\ \langle x_1|\Omega|A_k\rangle +\ \langle A_k|\Omega|x_1\rangle\,\Big]+$$

$$+\sum_{\substack{k,j=1\ldots F\\ k+j\ \text{even}}}\frac{k^2+j^2}{k^2-j^2}\,\langle A_k|\Omega|A_j\rangle\Bigg\}$$

2 Potential and Potential part of the Action