1. 1) 3×3 matrix, & 3 basic variables, so, it only has a trivial sol^n so it's linearly independent

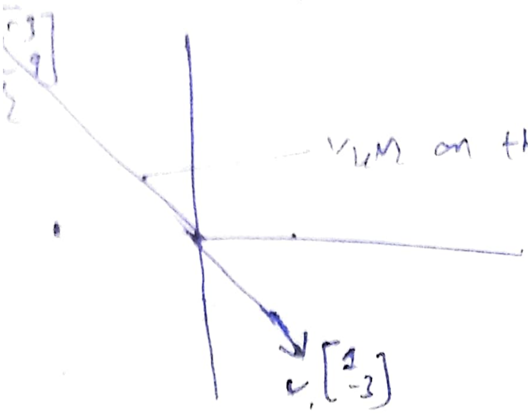
we have,

2) $$\begin{bmatrix} 2 & 8 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

again, no free variables, so linearly independent.

3) $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 9 \end{bmatrix}$, $v_2 = -3v_1$
v_1 $\quad$ v_2

so, they are linearly dependent



4) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $\begin{bmatrix} -2 \\ -8 \end{bmatrix}$, v_1 & v_2 not a multiple of each other
v_1 $\quad$ v_2

so, they are linearly independent

5. $$\begin{bmatrix} 0 & -8 & 5 & 0 \\ 3 & -7 & 4 & 0 \\ -1 & 5 & -4 & 0 \\ 1 & -3 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 2 & 0 \\ 3 & -7 & 4 & 0 \\ -1 & 5 & -4 & 0 \\ 0 & -8 & 5 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -8 & 5 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -8 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

no, free variable, so, they are linearly independent

6) $\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & -3 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ -4 & -3 & 0 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 4 & -9 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 12 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 12 & 0 \end{bmatrix}$ no free variables, only trivial soln. exists.

7) $\begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ -2 & -7 & 5 & 1 & 0 \\ -4 & -5 & 7 & 5 & 0 \end{bmatrix}$, There are only 3 rows & 4 vectors,

so, atmost only 3 pivot positions or basic variable possible, so, one variable will <u>always be free</u>.

so, linearly dependent.

8) $\begin{bmatrix} 1 & -3 & 3 & -2 \\ -3 & 7 & -1 & 2 \\ 0 & 1 & -4 & 3 \end{bmatrix}$, same as 8, only 3 rows & 4 coln, so

Atmost 3 basic variables & 1 variable will always be free, so, linearly dependent.

4) a) for v_3 to be in span of $\{v_1, v_2\}$;

$x_1 v_1 + x_2 v_2 = v_3$, has a soln.

$v_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$

$$\begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & 6 & h \end{bmatrix} = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 12 & h-10 \end{bmatrix}$$

we have $0 = 8$, so, the eqn has no nontrivial soln.

So, v_3 is not in $\{v_1, v_2\}$ for any value of h

b) for, $\{v_1, v_2, v_3\}$ to be linearly dependent,

$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$, must have non-trivial soln.

$$\begin{bmatrix} 1 & -3 & 5 & 0 \\ -3 & 9 & -7 & 0 \\ 2 & -6 & h & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 12 & h-10 & 0 \end{bmatrix}$$

so, we have, $x_3 = 0$,

$x_1 = -5x_3 + 3x_2$,

x_2 = free. so 1 free ~~vario~~ variable,

So, the eqn has a non trivial soln.

Thus, $\{v_1, v_2, v_3\}$ is linearly dependent set for all h

10) a) $v_1 = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}$ $v_2 = \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ -9 \\ h \end{bmatrix}$

for v_3 to be in span $\{v_1, v_2\}$

$x_1 v_1 + x_2 v_2 = v_3$ must have non-trivial sol^n

$$\begin{bmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & h+6 \end{bmatrix}$$

we have $0 = 1$, so no sol^n. exist,

So, v_3 not in span$\{v_1, v_3\}$

b) $\{v_1, v_2, v_3\}$ to be linearly dependent

$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$ must have non-trivial sol^n

$$\begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h+6 & 0 \end{bmatrix},$$

$x_3 = 0$

$x_1 = -2x_3 + 2x_2$

$x_2 =$ free, for any value of h.

So, sol^n. exist & $\{v_1, v_2, v_3\}$ linearly dependent

11) for the vector's to be linearly dependent,

$x_1 v_1 + x_2 v_2 + \dots + x_p v_p = 0$, must have non-trivial sol^n.

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ -1 & -5 & 5 & 0 \\ 4 & 7 & h & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -5 & h+4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & h-6 & 0 \end{bmatrix}$$

$x_2 = 2x_3$

$x_1 = x_3 - 3x_2$,

So, for ~~any~~ value of $h - 6 = 0$

$h = 6$,

So, the vectors are linearly dependent if & only if $h = 6$

$$\begin{bmatrix} 2 & -6 & 8 & 0 \\ 4 & 7 & h & 0 \\ 1 & 3 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & 0 \\ 0 & 5 & h+16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, we have 2 ~~low~~ pivot position.

$$x_2 = \left(\frac{h+16}{5}\right) x_3$$

for any value of h, x_3 is free.

13> $$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -9 & h & 0 \\ -3 & 6 & -9 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & h-15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is free, so for any value of h,

~~so~~ so,

14> $$\begin{bmatrix} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ 3 & 8 & h & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 23 & h-3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & h-26 & 0 \end{bmatrix}$$

~~x_3 is free, for any value of h.~~

The eqn. $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$ has a nontrivial soln. only if $h-26=0$, $h=26$.

Thus, the vectors are linearly dependent if and only if $h=26$.

15> $$\begin{bmatrix} 5 & 2 & 1 & -1 \\ 1 & 8 & 3 & 7 \end{bmatrix}$$

At most we have 2 pivots,

Theorem, so we will be left with free variables, so the system is linearly dependent.

1.2) $\begin{bmatrix} [illegible] & & & 0 \\ 7 & 1 & 4 & 0 \\ 1 & [illegible] & 4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -? & 4 & 0 \\ 0 & (-5) & h+16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So we have 2 ~~basic~~ pivot position.

$x_2 = \dfrac{-(h+16)x_3}{-5}$

$= \left(\dfrac{h+16}{5}\right)x_3$

For any value of h, x_3 is free.

13) $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -9 & h & 0 \\ -3 & 6 & -9 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & h-15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

~~A~~ x_3 is free, ~~so~~ for any values of h,

~~exp~~ so,

14) $\begin{bmatrix} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ 3 & 8 & h & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 23 & h-3 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & h-26 & 0 \end{bmatrix}$

~~x_3 is free, for any value of h.~~

The eqn. $x_1v_1 + x_2v_2 + x_3v_3 = 0$ has a nontrivial soln. only if $h-26=0$, $h=26$.

Thus, the vectors are linearly dependent if and only if $h=26$.

1.5) $\begin{bmatrix} 5 & 2 & 1 & -1 \\ 1 & 8 & 3 & 7 \end{bmatrix}$

Theorem, ~~A~~, At most we have, 2 pivots, and will be left with free ~~p~~ variables, so the system is linearly dependent.

16) $\begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$,

so, $v_2 = \frac{3}{2} v_1$, so they are linearly dependent

17) $\begin{bmatrix} 3 & \boxed{0} & -6 \\ 5 & 0 & 5 \\ -1 & 0 & 4 \end{bmatrix}$

we have one zero vector, so the vectors are linearly dependent.

18) $\begin{bmatrix} 4 & -1 & 2 & 8 \\ 4 & 3 & 5 & 1 \end{bmatrix}$, Theorem 8, we have more vectors than entries in vectors

so, we will always have free variables

So, the vectors are linearly dependent.

19) both vectors are not multiple of each others

so they are linearly independent

20) we have 1 zero vector, so they are linearly dependent.

21) a) True, if the ~~eqn~~ $Ax = 0$, has no nontrivial soln.

then they are linearly independent

~~22~~ b) False, at least one vector needs to be a linear combination of other vectors.

c) True, ~~there are~~ Theorem 8, there are more vectors than entries in vector

21) d) True,

Since, x, y are independent &, {x, y, z} is linearly dependent, then z is a linear combination of x, y i.e. z is in Span{x, y}.

23) a) True, As Ex $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$,

False, we can't guarantee a free variable,

Ex $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ are linearly dependent

c) x & y are linearly independent, z in Span{x, y} i.e. z is a linear combination of x, y, therefore, {x, y, z} is a linearly dependent, as we have one vector (z), as a linear combination of (x, y).

d) False, soln. of b proves.

3) A is a 3x3 matrix with ~~lr~~ indepd coln.

$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, no free variables, so, no non-trivial soln.

24) 2x2 matrix linearly dependent.

$\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}$, or $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$, or

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

25) A is 4x2 matrix and linearly independent.

$\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, here both $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

26) 4x3 and all 3 vectors are linearly independent i.e. no free variables

$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$

27) All 5 colums of the 7x5 matrix A must be pivot coln.s, otherwise the eqn. Ax=0, would have a free variable, in which case the colums of A would be linearly dependent

28) ~~Same as~~ If the colum of 5x7 matrix span R^5, i.e. they are linearly independent, so, A has five pivot colums

29) A, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ B, $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

31) A non-trivial sol^n of $Ax = 0$,

$-(1)a_1 - (1)a_2 + a_3 = 0.$

so, $\begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

32) Similarly, sol^n for $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

30) a) If A is $m \times n$ matrix, then the columns of A are linearly independent if & only if A has $\underline{n}$ pivots

b) If it has less than $\underline{n}$ pivots, it means it has free variables so, it has non-trivial sol^n.

33) True, since, v_3 is a linear combination of v_1 & v_2. and $\{v_1, v_2, v_3, v_4\}$ so, it contains atleast one vector (v_3) which is a linear combination of (v_1, v_2).

34) v_3 is a zero vector, so then $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

True

35) True, ~~Yes~~, v_1 & v_2 are linearly independent

36) False, let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $v_4 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$, $v_3 = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$.

then, $v_4 = v_1 + v_2$, and so, $\{v_1, v_2, v_3, v_4\}$ is linearly dependent

37) True since, $\{v_1, v_2, v_3\}$ linearly dependent & part of $\{v_1, v_2, v_3, v_4\}$

38) True, prove by contradiction,

let v_1, v_2 & v_3 be ~~not~~ linearly dependent

i.e. $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$ (with atleast one of them $\neq 0$)

then, $x_1 v_1 + x_2 v_2 + x_3 v_3 + 0 v_4 = 0$, should be true,

but we know, its not, so, $\{v_1, v_2, v_3\}$ should also be linearly independent

$Ax=b$, has only one solution, we know it would the trivial solution as $Ax=b$ has solution $\vec{p}$ + solution of $Ax=0$

So, $Ax=0$ has no non trivial solution

i.e. columns of A are not linear combination of each other

and hence, linearly independent

A is a $m \times n$ matrix and has n pivot column

So, $m \geq n$, so, it has pivot in each column

So, the eqn $Ax=b$ has no free variable i.e. if there is a sol^n, it must be unique.