Recall Span$\{v_1, \ldots, v_m\}$: $A = \{ c_1 v_1 + c_2 v_2 + \cdots + c_m v_m \mid c_i \in \mathbb{R} \}$

A is a vector space.

$A \ni u = d_1 v_1 + d_2 v_2 + \cdots + d_m v_m$ for some real no. d_i

$\Leftarrow$ $cu \in A$

$cu = c d_1 v_1 + \cdots + c d_m v_m$

$\Leftarrow A \ni v = e_1 v_1 + e_2 v_2 + \cdots + e_m v_m$ for some real no. e_i

$A \ni u + v = (d_1 + e_1) v_1 + (d_2 + e_2) v_2 + \cdots + (d_m + e_m) v_m$

Subspaces of $\mathbb{R}^n$ (Vector space)

Recall Span$\{\mathrm{v}_1, ..., v_m\}$:

A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

a. The zero vector is in H.

b. For each u and v in H, the sum $u + v$ is in H.

c. For each u in H and each scalar c, the vector $c.u \in H$.

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In short, Linear Combination of vectors of H lies in H.

Subspaces of $\mathbb{R}^n$ (Vector space)

If $v_1, v_2 \in V$ then $a+b+c=0=d+e+f$

$v_1+v_2 = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix} \Rightarrow v_1+v_2 \in V.$

$\alpha(a+b+c)=0 \Rightarrow \alpha . v_1 \in V$ for any $\alpha \in \mathbb{R}$

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Give examples of subspaces.

$\mathbb{R}^n, \{0\}$ are called trivial subspaces of $\mathbb{R}^n$.

$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x+y+z=0 \right\}$

$v_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad v_2 = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$

$\text{Span}\{v_1\} = \left\{ \begin{bmatrix} c \\ 2c \\ 3c \end{bmatrix} \middle| c \in \mathbb{R} \right\}$

$y = 2x$

$z = 3x$

$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\text{Span}\{v_1, v_2\}$ is a subspace of $\mathbb{R}^3$

$= \left\{ \begin{bmatrix} c_1+c_2 \\ 2(c_1+c_2) \\ 3c_1+c_2 \end{bmatrix} \middle| c_1, c_2 \in \mathbb{R} \right\}$

Subspaces of $\mathbb{R}^n$ (Vector space)

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Give examples of subspaces.

$\mathbb{R}^n, \{0\}$ are called trivial subspaces of $\mathbb{R}^n$.

Is the set of all solutions of $AX = b$ a subspace?

$Av_1 = 0 = Av_2$

$\Rightarrow A(v_1 + v_2) = 0$

$A(c.v_1) = c.0 = 0$

Given A & b

Yes if $b = 0$

$\{X \mid Ax = b\} = X_p + \{X_h \mid AX_h = 0\}$ — Subspace

$AX_p = b$

Subspaces of $\mathbb{R}^n$ (Vector space)

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Give examples of subspaces.

$\mathbb{R}^n, \{0\}$ are called trivial subspaces of $\mathbb{R}^n$.

Is the set of all solutions of $AX = b$ a subspace? W

Is the plane of vectors (b_1, b_2, b_3) with first component $b_1 = 1$ a subspace of $\mathbb{R}^3$? No.

$x = 1$

$W = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \;\middle|\; \begin{matrix} b_1 = 1 \\ b_2, b_3 \in \mathbb{R} \end{matrix} \right\}$ $0 \notin$

$\underset{\in W}{\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}} + \underset{\in W}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} = \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix} \notin W$

Column Space

Recall, The column space of a matrix A is the set $Col(A)$ of all linear combinations of the columns of A. $m \times n$

$$A0 = 0 \Rightarrow 0 \in Col(A) = \{b \mid \underline{AX = b} \text{ for some vector } X\}$$

consistant

is a subspace of $\mathbb{R}^m$.

$b_1, b_2 \in Col(A)$

$AX_1 = b_1$
$AX_2 = b_2 \Rightarrow A(X_1 + X_2) = \underline{b_1 + b_2}$ is satisfied

$c \cdot b = A(c \cdot X_1) \Rightarrow$ prop c

$$A = \begin{bmatrix} V_1 | V_2 | \cdots | V_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

X

$\underline{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n V_n}$ for $\alpha_i \in \mathbb{R}$

Column Space

Recall, The column space of a matrix A is the set $Col(A)$ of all linear combinations of the columns of A.

$$Col(A) = \{b | AX = b \text{ for some vector } X\}$$

is a subspace of $\mathbb{R}^m$.

Find $Col(A)$, where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \end{bmatrix}$

$$W = \left\{ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \middle| \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ for some } x_1, x_2, x_3 \right\}$$

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in W \qquad A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \in W \qquad A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \in W$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a - c \\ a + 2b \end{bmatrix} \in W \text{ for all } a, b, c \in \mathbb{R}$$

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is a subspace of $\mathbb{R}^m$.

Find $Col(A)$, where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \end{bmatrix}$

Row space of A is the set $Row(A)$ of all linear combinations of rows of A.

$Row(A) = Col(A^T)$

Column Space

Recall, The column space of a matrix A is the set $Col(A)$ of all linear combinations of the columns of A.

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Find $Col(A)$, where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \end{bmatrix}$

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Col(A)=Row(A^T)

Null Space of a Matrix

The null space of a matrix $A_{m \times n}$ is the set $Nul(A)$ of all solutions of the homogeneous equation $AX = 0$.

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$Nul(A)$ is a subspace of $\mathbb{R}^n$.

Find $Nul(A)$ where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{2} & 1 \end{bmatrix}$

$$\begin{cases} 2y + z = 0 \\ x - z = 0 \end{cases}$$

$z = c$

$y = -\frac{1}{2}c$

$x = c$

$$Nul(A) = \left\{ c\begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

4 Fundamental subspaces

For a matrix $A_{m\times n}$, we get Col(A)
Col(A^T) $= Row(A)$
Null(A) : $\{x \mid Ax=0\}$
Null(A^T) : $\{y \mid A^Ty=0\}$

$y^TA=0$

$[\quad '' \quad]_{1\times m} A_{m\times n}=0$

$\cancel{AB}$

$A_{m\times n}$

$A^T_{n\times m}\, y_{m\times 1}$

$y^T_{1\times m}$

L.I.

$\begin{bmatrix} v_1 & v_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $c_1 v_1 + c_2 v_2 = 0 \Rightarrow c_1 = c_2 = 0$ only!

A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H

$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x+y+z=0 \right\}$

$x = -y - z$, $y = d$, $z = c$

$\begin{bmatrix} -c-d \\ d \\ c \end{bmatrix}$

Que: Is $\{v_1, v_2\}$ L.I.?

$\begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Basis for H is $\{v_1, v_2\}$

$\operatorname{Span}\{v_1, v_2\} = H = \left\{ c \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{v_1} + d \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{v_2} \middle| \; c, d \in \mathbb{R} \right\}$

A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

Find basis for $Col(A), Nul(A)$, where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \end{bmatrix}$

$Nul(A) = \left\{ c \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix} \middle| c \in \mathbb{R} \right\} = \left\{ \frac{c}{2} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \middle| c \in \mathbb{R} \right\}$

$= \left\{ d \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \middle| d \in \mathbb{R} \right\}$

$\left\{ \frac{d}{10} \begin{bmatrix} 20 \\ -10 \\ 20 \end{bmatrix} \middle| d \in \mathbb{R} \right\}$

$\left\{ e \begin{bmatrix} 20 \\ -10 \\ 20 \end{bmatrix} \middle| e \in \mathbb{R} \right\}$

Basis for $Nul(A) = \left\{ \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 20 \\ -10 \\ 20 \end{bmatrix} \right\}$

$\begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{2} & 1 \end{bmatrix}$ Basis for $Col(A)$

$= \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$

$Row(A)$:

Basis for a Subspace

$Col(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} \middle| c_i \in \mathbb{R} \right\}$

$= \left\{ d \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e \begin{bmatrix} 0 \\ 2 \end{bmatrix} \middle| d, e \in \mathbb{R} \right\}$

$(-1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

Find basis for $Col(A), Nul(A)$, where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \end{bmatrix}$

Theorem

The pivot columns of a matrix A form a basis for the column space of A.

Note: The pivot columns of REF of A need not form a basis for the column space of A.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$Col(A) = \left\{ c \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| c \in \mathbb{R} \right\}$

Basis $= \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Line passing through $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Coordinate System

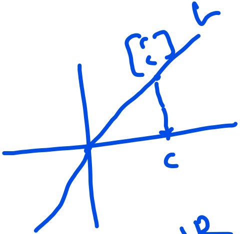
$100 \longrightarrow \begin{bmatrix} 100 \\ 100 \end{bmatrix} \qquad \begin{bmatrix} 50 \\ 50 \end{bmatrix} \longrightarrow 50$

Let $B = \{b_1, b_2, \ldots, b_p\}$ be a basis(fix the order) of subspace H.
For any vector $x \in H$, $\exists! c_1, c_2, \ldots, c_p \in \mathbb{R}$ such that

$$x = c_1.b_1 + c_2.b_2 + \cdots + c_p.b_p$$

$x \longleftrightarrow (c_1, c_2, \ldots, c_p)$

$H \longrightarrow \mathbb{R}^p$



$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$H = Col(A) = \left\{ c \begin{bmatrix} 1 \\ 1 \end{bmatrix} \,\middle|\, c \in \mathbb{R} \right\}$

Basis $= \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$f : L \longrightarrow \mathbb{R}$

$f((c,c)) = c$

$f\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = 2$

For any $x \in Col(A)$, $x = c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for some $c \in \mathbb{R}$

$\left\{ \begin{bmatrix} c \\ c \end{bmatrix} \middle| c \in \mathbb{R} \right\} = Col(A) \xrightarrow{f} \mathbb{R}$

$c \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow c$

1-1 correspondence.

Coordinate System

$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ Basis for $Col(A) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$

$= \left\{ c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\} \ni \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \leftarrow (1,1)$

Let $B = \{b_1, b_2, \ldots, b_p\}$ be a basis(fix the order) of subspace H.
For any vector $x \in H$, $\exists! c_1, c_2, \ldots, c_p \in \mathbb{R}$ such that

$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}$

$$x = c_1.b_1 + c_2.b_2 + \cdots + c_p.b_p$$

$9\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$Col(A) = span\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$ — B

Define a vector $[x]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$ in $\mathbb{R}^p$.

For any $v \in Col(A)$, $v = c\begin{bmatrix} 1 \\ -1 \end{bmatrix} + d\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$[x]_B$ is called the coordinate vector of x relative to basis B.

$v \to (c, d)$

$\begin{bmatrix} 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}_B$

$f : Col(A) \to \mathbb{R}^2$

$v = c\begin{bmatrix} 1 \\ -1 \end{bmatrix} + d\begin{bmatrix} 0 \\ 2 \end{bmatrix} \longrightarrow \begin{bmatrix} c \\ d \end{bmatrix}$

$Col(A) \ni \begin{bmatrix} 9 \\ 5 \end{bmatrix} = 9\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 2 \end{bmatrix} \leftarrow (9, 7)$

The Dimension of a Subspace

Definition

The dimension of a nonzero subspace H, denoted by $dim(H)$, is the number of vectors in any basis for H.
The dimension of the zero subspace $\{0\}$ is defined to be zero.

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Definition

The rank of a matrix A, denoted by rank(A), is the dimension of the column space of A.
Nullity of a matrix, denoted by nullity(A), is the dimension of the null space of A.

Theorem

For any matrix $A_{m\times n}$,

$$rank(A) + nullity(A) = n$$

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\operatorname{Col} A = \mathbb{R}^n$

o. $\dim \operatorname{Col} A = n$

p. $\operatorname{rank} A = n$

q. $\operatorname{Nul} A = \{\mathbf{0}\}$

r. $\dim \operatorname{Nul} A = 0$