Linear Independence

Let A be a matrix of order $m \times n$. We say columns of A to be linearly independent if $AX = 0$ has only trivial (0) solution.

Linear Independence

Let A be a matrix of order $m \times n$. We say columns of A to be linearly independent if $AX = 0$ has only trivial (0) solution.

Definition

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ is said to be linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$$

has only trivial solution.

Linear Independence

Let A be a matrix of order $m \times n$. We say columns of A to be linearly independent if $AX = 0$ has only trivial (0) solution.

Definition

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ is said to be linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$$

has only trivial solution.
Else $\{v_1, v_2, \ldots, v_n\}$ is called linearly dependent.

Linear Independence

Let A be a matrix of order $m \times n$. We say columns of A to be linearly independent if $AX = 0$ has only trivial (0) solution.

Definition

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ is said to be linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$$

has only trivial solution.
Else $\{v_1, v_2, \ldots, v_n\}$ is called linearly dependent.

$\{v_1, v_2, \dots, v_n\}$ is called linearly dependent iff one of v_i is a linear combination of others.

$\{v_1, v_2, \ldots, v_n\}$ is called linearly dependent iff one of v_i is a linear combination of others.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ is always linearly dependent if $n > m$.

$\{v_1, v_2, \ldots, v_n\}$ is called linearly dependent iff one of v_i is a linear combination of others.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ is always linearly dependent if $n > m$.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ is always linearly dependent if $v_i = 0$ for some i.

Linear Transformation

Recall that a matrix A of order $m \times n$ can be thought as a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ in following way:

$$u \to Au$$

Linear Transformation

Recall that a matrix A of order $m \times n$ can be thought as a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ in following way:

$$u \to Au$$

We can observe that
$A(u+v) = Au + Av$ for all $u, v \in \mathbb{R}^n$

Recall that a matrix A of order $m \times n$ can be thought as a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ in following way:

$$u \to Au$$

We can observe that
$A(u+v) = Au + Av$ for all $u, v \in \mathbb{R}^n$
$A(cu) = c.Au$ for any real number c.
This motivates to define

Definition (Linear Transformation)

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be Linear Transformation if
$T(u+v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$

Linear Transformation

Recall that a matrix A of order $m \times n$ can be thought as a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ in following way:

$$u \to Au$$

We can observe that
$A(u+v) = Au + Av$ for all $u, v \in \mathbb{R}^n$
$A(cu) = c.Au$ for any real number c.
This motivates to define

Definition (Linear Transformation)

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be Linear Transformation if
$T(u+v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$
$T(cu) = c.T(u)$ for any real number c.

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ as $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x \\ 0 \end{bmatrix}$

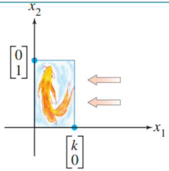
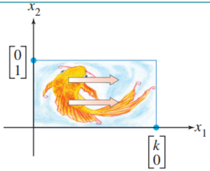
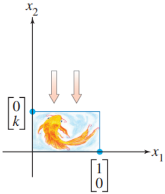
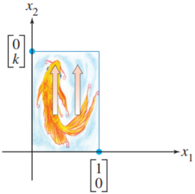
Projection onto X axis.

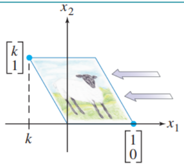
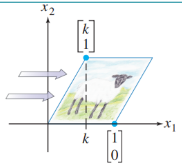
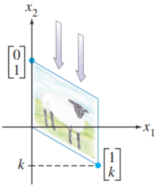
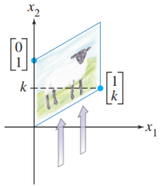
Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ as $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ scaling by 2.

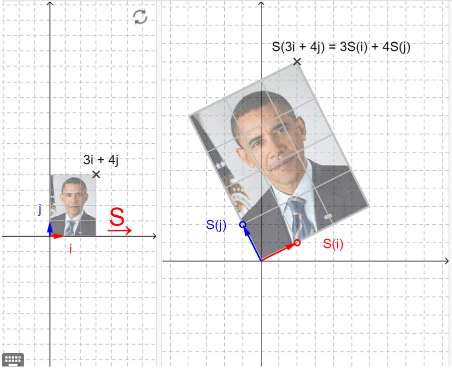
Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ as $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$

rotation by 45 degree.

TABLE 2 Contractions and Expansions

Transformation	Image of the Unit Square	Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$  $k > 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$  $k > 1$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Transformation	Image of the Unit Square		Standard Matrix
Horizontal shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$



$$S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

$$S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$$

$$\begin{aligned} S\begin{pmatrix}3\\4\end{pmatrix} &= 3\,S\begin{pmatrix}1\\0\end{pmatrix} + 4\,S\begin{pmatrix}0\\1\end{pmatrix} \\ &= 3\begin{pmatrix}2\\1\end{pmatrix} + 4\begin{pmatrix}-1\\2\end{pmatrix} \\ &= \begin{pmatrix}2\\11\end{pmatrix} \end{aligned}$$

Represent S by $\begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}$

$$\begin{aligned} S\begin{pmatrix}3\\4\end{pmatrix} &= \begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} \\ &:= 3\begin{pmatrix}2\\1\end{pmatrix} + 4\begin{pmatrix}-1\\2\end{pmatrix} \end{aligned}$$

Is T([x])=[$|x|$] a linear transformation from $\mathbb{R} \to \mathbb{R}$?

Is T([x])=[$|x|$] a linear transformation from $\mathbb{R} \rightarrow \mathbb{R}$?

Is $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} sin(x_1) \\ 2x_2 \end{bmatrix})$ a linear transformation from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$?

Is $T([x])=[|x|]$ a linear transformation from $\mathbb{R} \to \mathbb{R}$?

Is $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} sin(x_1) \\ 2x_2 \end{bmatrix})$ a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$?

How does T look like?

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}_{m\times 1}$$

for some real numbers $a_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$.

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}_{m\times 1}$$

for some real numbers $a_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$.

T is represented by a matrix $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$

If $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}_{m\times 1}$$

for some real numbers $a_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$.

T is represented by a matrix $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$

This matrix is called standard matrix representation of T.

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$$

Question: Given T How to find its matrix representation-$[T]$?

Question: Given T How to find its matrix representation-$[T]$?

Answer: First column of $[T]$ is $T(e_1)$. where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Question: Given T How to find its matrix representation-$[T]$?

Answer: First column of $[T]$ is $T(e_1)$. where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Second column of $[T]$ is $T(e_2)$. where $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad T(e_2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$ – std matrix rep. of T.

Question: Given T How to find its matrix representation-$[T]$?

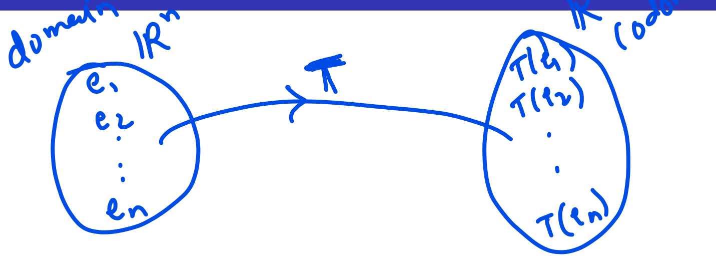
Answer: First column of $[T]$ is $T(e_1)$. where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Second column of $[T]$ is $T(e_2)$. where $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ = first column of A

So on and last column of $[T]$ is $T(e_n)$. where $e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Properties of L. T.



Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto if for each $b \in \mathbb{R}^m$ there exists $u \in \mathbb{R}^n$ such that $T(u) = b$

codomain

Onto : $\{$ $\boxed{\text{Range}(f) = \text{codomain}}$

$T(u) = b$

b

$\mathbb{R}^2 \leftarrow \mathbb{R}^2 : T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$

Onto

Let $\begin{bmatrix} 3 \\ 7 \end{bmatrix} \in$ codomain

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

For any $b \in$ codomain $\exists\, x \in$ domain s.t. $Ax = b$

$T: \mathbb{R}^2 \to \mathbb{R}^3 \quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+2y \\ x-y \end{bmatrix} \quad \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$

$[T] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix} \quad b, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ 1 & -1 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1}$

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto if for each $b \in \mathbb{R}^m$ there exists $u \in \mathbb{R}^n$ such that $T(u) = b$

$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & \boxed{1} & b_2 - b_1 \\ 0 & -2 & b_3 - b_1 \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + 2b_2 - 3b_1 \end{array}\right]$

If $b_3 + 2b_2 - 3b_1 \neq 0$ then system will be inconsistent

$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \notin \text{Range}(T)$

T is not onto

f is one to one if for each $y \in \text{Range}(f)$ $\exists$ unique $x \in$ domain of f s.t. $f(x) = y$

$f(x_1) = f(x_2) \implies x_1 = x_2$

$f(x_1) - f(x_2) = 0$

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one to one if $T(u) = 0$ then u must be 0.

If f is L.T, then f is 1-1 if $f(x_1 - x_2) = 0 \Rightarrow x_1 - x_2 = 0$

$$[T] = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -2 & 0 \\ 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = 0 = x_2$$

i.e. $U = 0$

$[T]X = 0$

All possible lin. combinations of col.

$T: \mathbb{R}^n \to \mathbb{R}^m$

T is onto $\quad$ $\underline{\text{Col}}([T]) = \mathbb{R}^m$ codomain

T is one-to-one $\quad$ Sol^n of $[T]X = 0$ is only $\{0\}$

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one to one if $T(u) = 0$ then u must be 0.

Properties of Linear Transformation

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one to one if $T(u) = 0$ then u must be 0.

Properties of Linear Transformation

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one to one if $T(u) = 0$ then u must be 0.

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto/surjective if for each $v \in \mathbb{R}^m$ there exists $u \in \mathbb{R}^n$ such that $T(u) = v$.

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto/surjective if for each $v \in \mathbb{R}^m$ there exists $u \in \mathbb{R}^n$ such that $T(u) = v$.

If T is invertible then $n = m$

$[T]^{-1}_{m\times n}$ exist.

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be bijective/invertible if T is one to one and onto.

$[T(e_1)|T(e_2)|\cdots T(e_n)]$ is invertible.

T

T

$\exists T^{-1}$ L.T. $T^{-1} : \mathbb{R}^m \to \mathbb{R}^n$

u

$u \to T(u) \longrightarrow T^{-1}(T(u)) = u$

Let T be the linear transformation whose standard matrix is

$$A \cdot \begin{bmatrix} \boxed{1} & -4 & 8 & 1 \\ 0 & \boxed{2} & -1 & 3 \\ 0 & 0 & 0 & \boxed{5} \end{bmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

$u = \begin{bmatrix} -6 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}$ $\quad Au = 0$

$$\left[\begin{array}{cccc|c} 1 & -4 & 8 & 1 & b_1 \\ 0 & 2 & -1 & 3 & b_2 \\ 0 & 0 & 0 & 5 & b_3 \end{array}\right]$$

For each $b \in \mathbb{R}^3$, does $\exists\, x \in \mathbb{R}^4$ $Ax = b$?

✓

$Col(A) = \mathbb{R}^3$? codomain

Range

$\Rightarrow T$ is onto.

Let T be the linear transformation whose standard matrix is $\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then: T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.

Let T be the linear transformation whose standard matrix is $\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then: T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
T is one-to-one if and only if the columns of A are linearly independent.

Let T be the linear transformation whose standard matrix is $\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then: T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
T is one-to-one if and only if the columns of A are linearly independent.

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. T is bijective iff $n = m$ and A is an invertible matrix.

Composite Transformation

If T_1, T_2 are two linear transformations then so is composite of T_1, T_2.

Composite Transformation

If T_1, T_2 are two linear transformations then so is composite of T_1, T_2.