Vector Space over $\mathbb{R}$

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A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

A Subspace Spanned by a Set

For $v_1, v_2, \ldots, v_p \in V$
$\text{Span}\{v_1, v_2, \ldots, v_p\} =$ set of all linear combinations of $v_1, v_2, \ldots, v_p$.

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Linear transformation

Let V_1, V_2 be two vector spaces over $\mathbb{R}$. A function $T : V_1 \to V_2$ is called linear transformation if

$$T(u+v) = T(u) + T(v)$$
$$\text{T(c.u)=c.T(u)}$$

for all $u, v \in V_1$ and $c \in \mathbb{R}$.

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V_1, V_2-two vector spaces over $\mathbb{R}$ are said to be isomorphic if there exists a linear transformation $T : V_1 \to V_2$ such that T is bijective.

Basis

A set of vectors $B = \{v_1, v_2, \ldots, v_p\}$ is said to be basis of a vector space V if B is linearly independent and $\text{Span}(B) = V$.

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The Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V, and let $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

a. If one of the vectors in S—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

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Dimension of a vector space V is the number of elements in any basis of V.