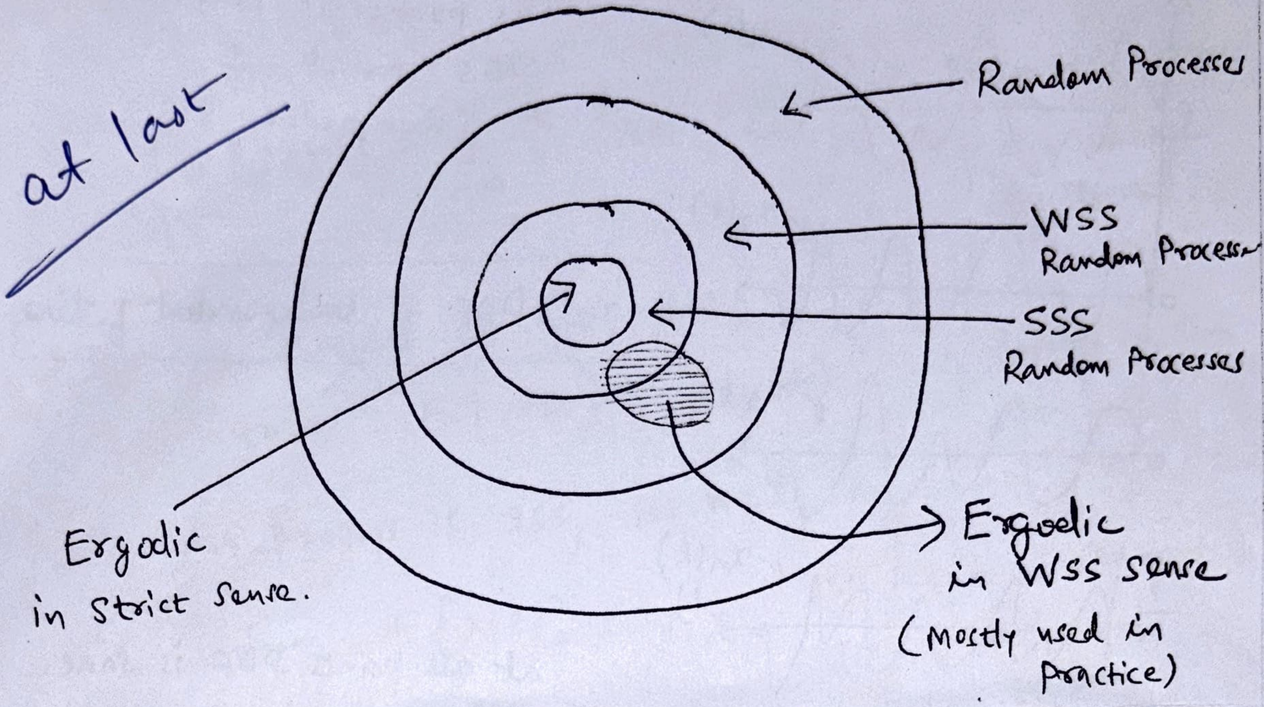
20.11.24
WED

* Types of Random Processes *



* Strict Sense Stationary (S.S.S) Random Process *

n^{th} order joint PDF

$$f_{X(t_1),X(t_2),\dots X(t_n)}(x_1, x_2, x_3, \dots x_n) = f_{X(t_1+T),\dots X(t_n+T)}(x_1, x_2, \dots, x_n)$$

$\therefore$ joint PDF is independent of time origin.

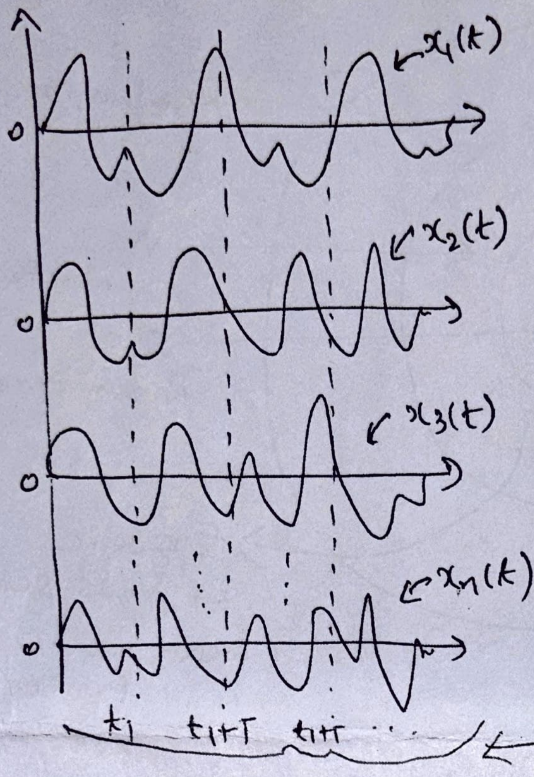
means everywhere, joint pdf is the same.

Note:
1. Mostly random processes are NOT SSS (in practice).
2. Practically, some random processes found stationary upto some time interval.

Ex: Consider, $n=1$

$$f_{X(t_1)}(x_1) = f_{X(t_1+T)}(x)$$

The process is said to be SSS for $n=1$ (first order SSS).



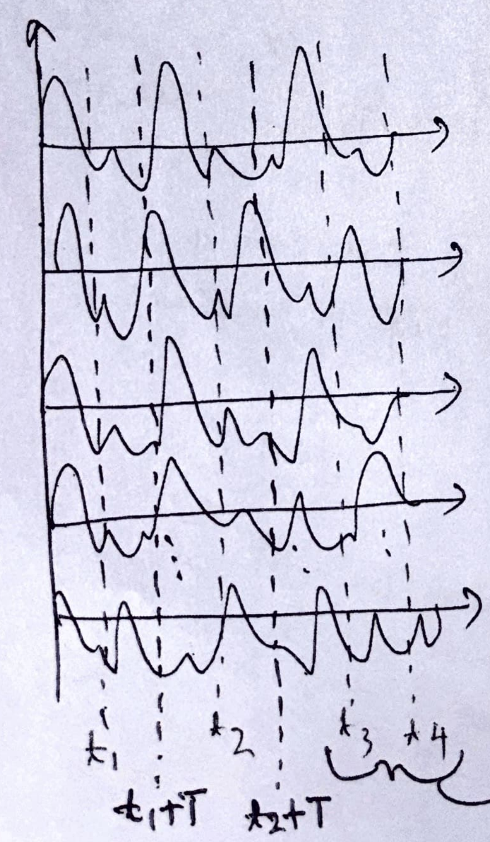
$\therefore$ PDF is independent of time

at all points PDF is same.

Ex: Consider $n=2$

$$f_{X(t_1)X(t_2)}(x_1, x_2; t_1, t_2) = f_{X(t_1+T), X(t_2+T)}(x_1, x_2; t_1+T, t_2+T)$$

Joint PDF



Joint PDF is dependent on time difference $(t_2 - t_1)$

If this time interval is same as $(t_1 \to t_2)$

Then the Joint PDF at t_3, t_4 will be same as t_1, t_2.

* Gross Characterization of Random Process *

⇒ Consider $n=1$

$$\boxed{E[X(t)] = \int_{-\infty}^{\infty} x \cdot f_{X(t)}(x) \cdot dx}$$

⇒ Ensemble average

∴ taking average at a time t.

- If the process is NOT SSS, then $E[X(t)] = \mu_X(t)$ ∴ depend on time.

- If the process is SSS, for $n=1$, then $E[X(t)] = \mu_X$ ∴ Mean is constant i.e., independent of time.

⇒ Consider $n=2$

$$\boxed{E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 \, f_{X(t_1)X(t_2)}(x_1, x_2) \cdot dx_1 \cdot dx_2}$$

This is called Autocorrelation function of RP denoted as, $R_X(t_1, t_2)$

- If the process is SSS then

$$E[X(t_1)X(t_2)] = E[X(t_1+T)X(t_2+T)]$$
$$= R_X(t_2 - t_1)$$

$\therefore R_X(t_1, t_2) = R_X(t_2 - t_1)$ ⇒ Autocorrelation funⁿ is funⁿ of time difference.

Note: For a random process, if expectation is independent of time (constant) and Autocorrelation fun$^{\underline{n}}$ is fun$^{\underline{n}}$ of time difference then such a random process is called

Wide-Sense Stationary (WSS)

This is applicable in practical situations.

WSS Random Process conditions

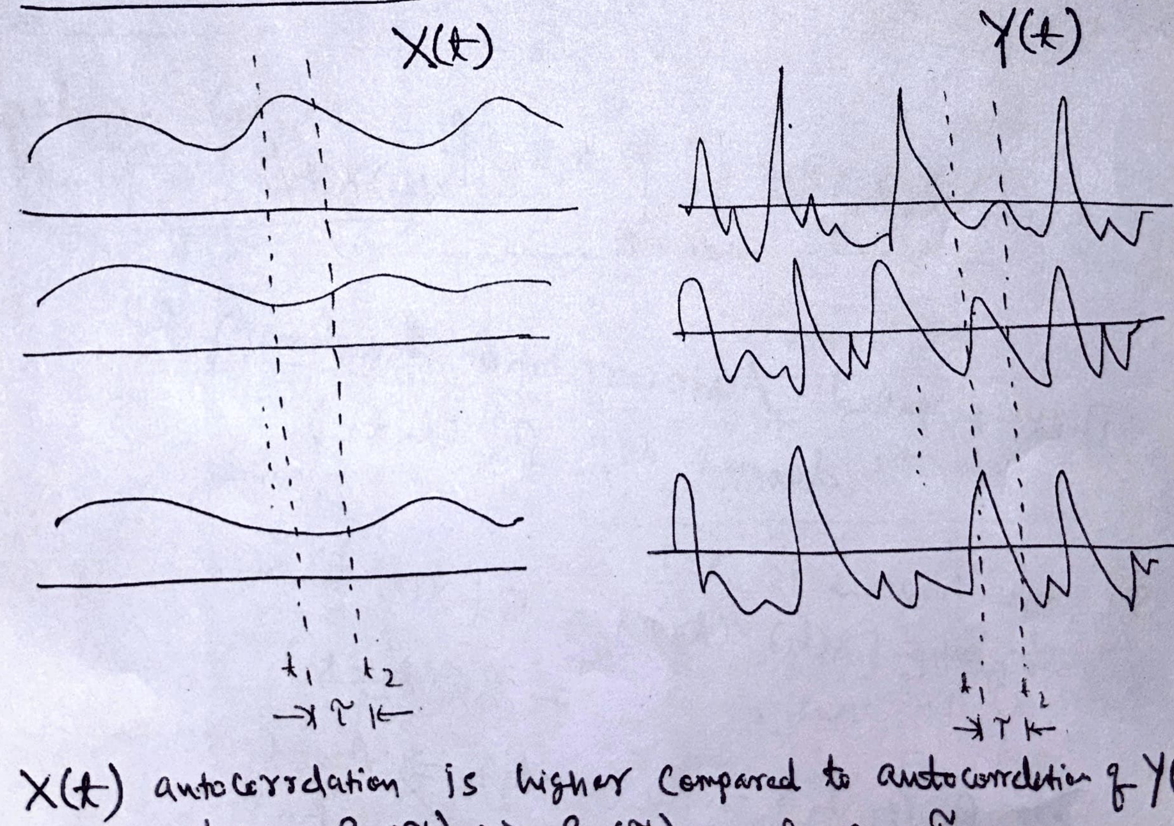
(1) $E[X(t)] = \mu_X$

(2) $E[X(t_1)X(t_2)] = R_X(t_1, t_2)$

$= R_X(t_2 - t_1)$

$= R_X(\tau)$

Intuitive Illustration:

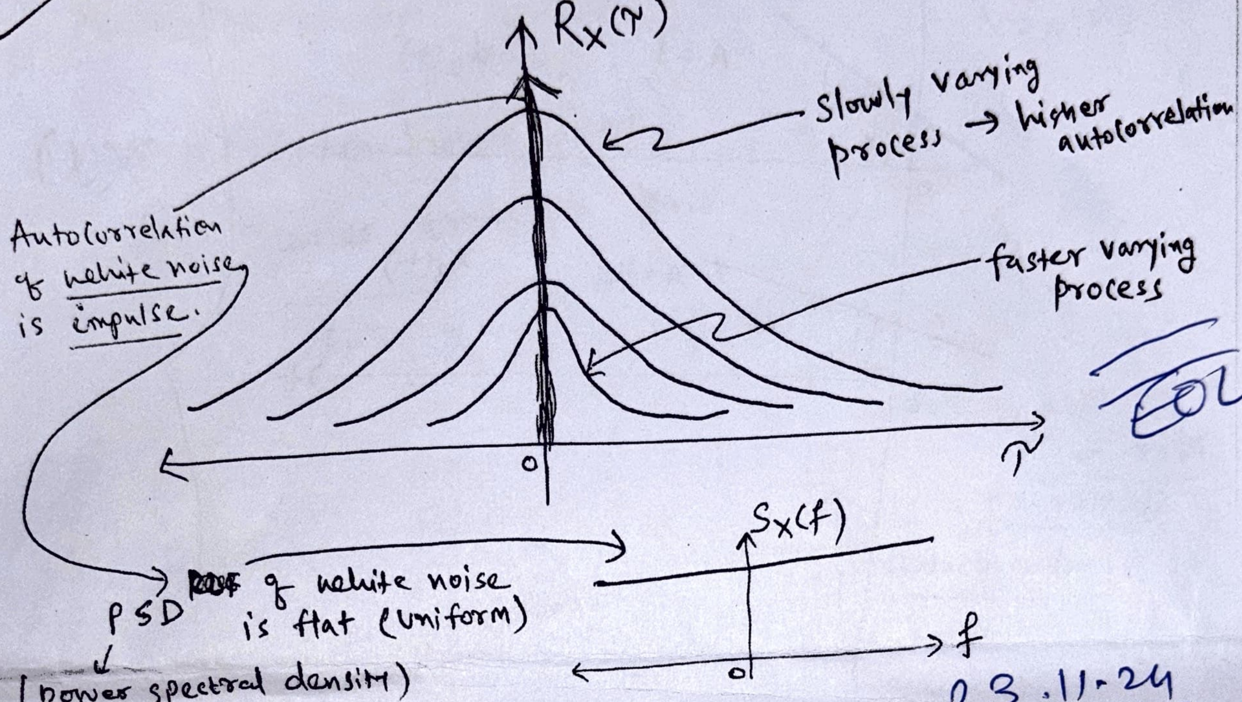


X(t) autocorrelation is higher compared to autocorrelation of Y(t)

i.e., $R_X(\tau) \gg R_Y(\tau)$, for same τ.

Practical point

Remark: Autocorrelation gives us how a random process overall behaves or looks like; without explicity observing the random process.



23.11.24
SAT

Example: Sketch ensemble of random process,

$$X(t) = At + b,$$

where, b is constant

A is a random variable uniformly distributed betⁿ $(-2, 2)$

(a) Just by looking at the ensemble, comment on the stationarity of the process.

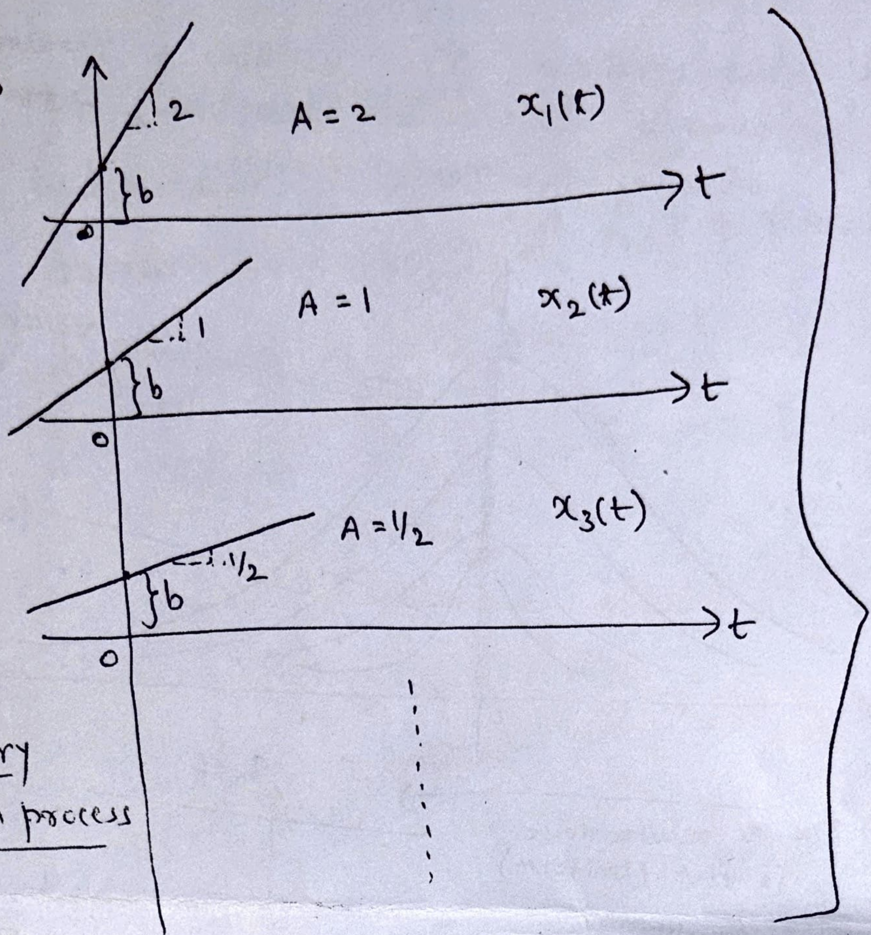
(b) Find autocorrelation funⁿ $R_{XX}(\tau)$.

(c) Find Expectation $E[X(t)]$.

(d) Based on (b) and (c), comment whether the process is WSS?

Solution:

(a)



Not a stationary random process

(c) $E[X(t)] = E[At+b] = t\,E(A) + b = \boxed{b.}$ CONSTANT Ans. Expectation

$(\because A \sim U(-2,2)$

$\therefore E(A) = 0).$

(b) $R_X(\tau) = R_X(t_1, t_2) = E\{X(t_1)\, X(t_2)\}$

$= E[(At_1 + b)\,(At_2 + b)]$

$= E[A^2 t_1 t_2 + At_1 b + At_2 b + b^2]$

$= E[A^2]\, t_1 t_2 + \cancel{E[A]\, t_1 b}^{\,0} + \cancel{E[A]\, t_2 b}^{\,0} + b^2$

$= b^2 + E[A^2]\, t_1 t_2$ $\qquad \Big|\; t_1 t_2 \int_{-2}^{2} \frac{A^2}{4}\, da + b^2$

$= \boxed{\frac{4}{3} t_1 t_2 + b^2}$ $\qquad = \frac{4}{3} t_1 t_2 + b^2$

Autocorrelation fun^{n} is NOT dependent on time diff $(t_2 - t_1)$ but depend on t_1, t_2 itself.

$\therefore$ The X(t) is "NOT" a WSS. $\rightarrow$ Ans (d) $\left(\because \begin{smallmatrix}(b)\\ \text{and } (c)\end{smallmatrix}\right)$

* Wiener - Khinchine Theorem " (without proof)

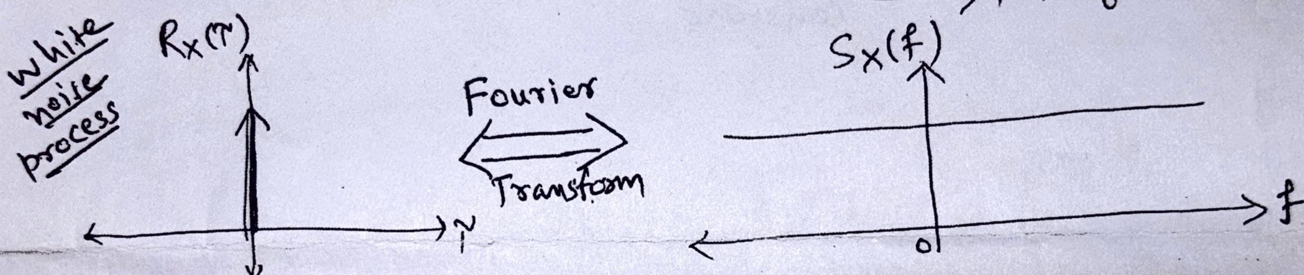
Autocorrelation fun^{n} $R_X(\tau)$ and Power spectral density (PSD) $S_X(f)$

(of a random process $X(t)$)

are Fourier transform pair.

$$\boxed{\mathcal{F}[R_X(\tau)] = S_X(f)}$$

→ PSD of $X(t)$.



* Ergodic process * (just definition)

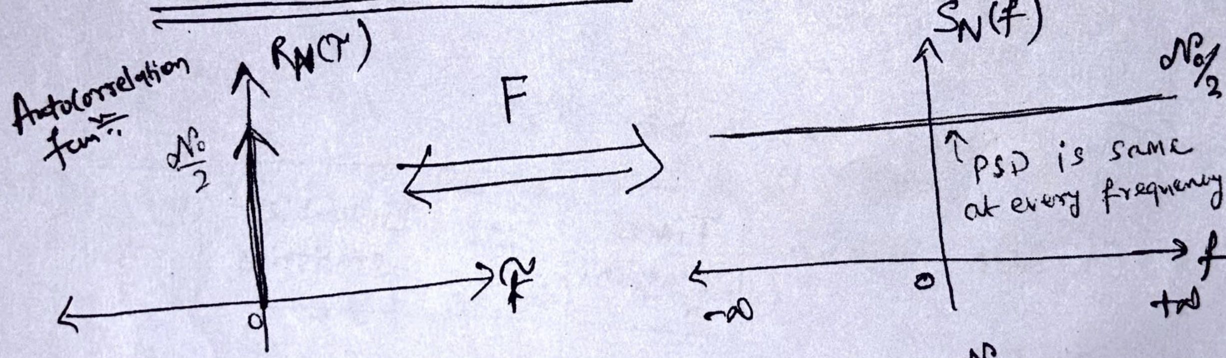
- The process has to be SSS.
- Ergodic when $\boxed{\text{Time statistics} = \text{Ensemble statistics}}$

Means: only one sample function is sufficient to determine characteristics of entire RP.

* AWGN *

- **additive** → **Data model**: signal + noise (data)
- **white** noise process → PSD of noise is constant
- **Gaussian** process → Joint PDF of noise is Gaussian
- **noise process**

White noise process: ~~X(t)~~ N(t)



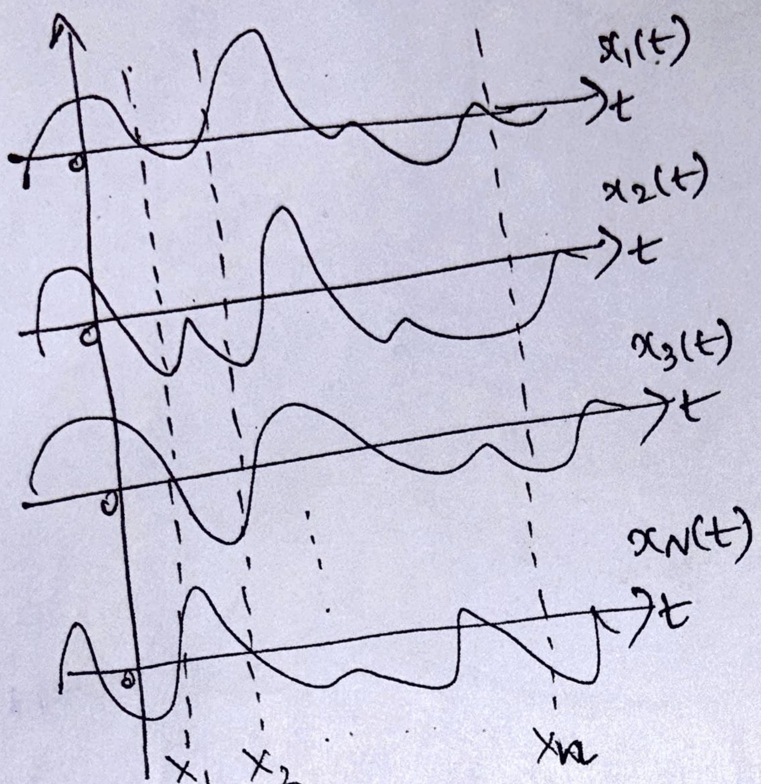
$$R_N(\tau) = \frac{N_0}{2}\,\delta(\tau)$$

$$S_N(f) = \frac{N_0}{2}$$

"**White**" colour contents all the colours, equally.

Similarly, "**white noise**" contents equal power (uniform PSD) at all frequencies.

Gaussian process: $X_1, X_2, \ldots, X_N$.



Remember,

$n=1$ first order PDF

$n=2$ second order (Two at a time) joint PDF

for any n

$n=1$

$n=2$

$\vdots$

the joint PDF will be Gaussian density fun. at every n.

$\therefore$ The process is called "Gaussian process".

White Gaussian Random process (WGN)

If PSD of $N(t)$ is comes out to be flat (Uniform) i.e., $R_{N}(\tau)$ an impulse.

Joint PDF at any n is Gaussian PDF.

$\Rightarrow$ Go back to first slide "Types of R.P."