Complex Eigenvalues

$P = [v_1 v_2 \cdots v_n]$ $AP = PD$

$D = \text{diag}(\lambda_1 \ldots \lambda_n) \Rightarrow A\underline{v_i = \lambda_i v_i}$

$\Leftarrow A = PDP^{-1}$ $D, P \in M_n(\underline{\mathbb{R}})$

A is diagonalizable over $\mathbb{R}$ $\Leftrightarrow$ A has n lin. indep. eigenvectors

Consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Its characteristic equation is $\lambda^2 + 1 = 0 \Rightarrow \lambda = \underline{\pm i}$

$D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ or $D = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$

$\not\exists$ $M_n(\underline{\mathbb{R}})$

Given

By defn

$A = PDP^{-1}$

columns of P are eigenvectors corresponding to eigvalues = diag. entries of D

$\overline{(a+ib)} = a-ib$

$\bar{u} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_n \end{bmatrix}$

$u = \begin{bmatrix} i \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

Observe. $v = \bar{u}$ or $u = \bar{v}$

Consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Its characteristic equation is $\lambda^2 + 1 = 0$. Hence eigenvalues of A are complex numbers $i, -i$.

Qustion: How to find its eigenvectors? Work in $\mathbb{C}^n = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} / a_i \in \mathbb{C} \right\}$

$\mathrm{Null}(\underbrace{A - iI}_{\in M_n(\mathbb{C})})$

$A - iI : \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \xrightarrow{R_2 \to R_2 - iR_1} \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix}$

$\mathrm{Null}(A + iI) =$

$A = \begin{array}{c} \begin{array}{cc} u & v \end{array} \\ \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}$

$c \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} ic \\ c \end{bmatrix}$

$A = \underset{P}{\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}} \underset{P^{-1}}{\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}^{-1}}$

$-ix = y = c$

$x = \frac{-1}{i} c$

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Qustion: How to find its eigenvectors?

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Real and Imaginary Parts of Vectors

For each $x \in \mathbb{C}^n$, there exists $u, v \in \mathbb{R}^n$ such that $x = u + iv$.

$$\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+i \\ 2-i \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + i \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Real and Imaginary Parts of Vectors

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If $A \in M_n(\mathbb{R})$ and $Ax = \lambda.x$ then

$$A\overline{x} = \overline{\lambda}.\overline{x}$$ ✓ $\Rightarrow \overline{x}$ is an e.vector for e.v. $\overline{\lambda}$.

(obtained by taking complex conjugates on both sides).

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Question: What do you observe here?

$\begin{bmatrix} a+ib & 0 & 0 \\ 0 & a-ib & 0 \\ 0 & 0 & \gamma \end{bmatrix}$

Ex $A \in M_3(\mathbb{R})$ with e.v. $a-ib,\ a+ib,\ \gamma$
Find P & $C \in M_n(\mathbb{R})$, $A = PCP^{-1}$.

$\mathrm{Re}(x)$ $\quad$ $\mathrm{Im}(x)$

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$M_n(\mathbb{C}) \ni A : PDP^{-1}$; $P \to$ e.v. ; $D \to$ e.values

(obtained by taking complex conjugates on both sides).
Question: What do you observe here?
If λ, x are eigenvalue and eigenvector of a matrix with real entries than $\overline{\lambda},\ \overline{x}$ are eigenvalue and eigenvector of that matrix.

$P, C \in M_n(\mathbb{R})$ over $\mathbb{R}$

Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then

$a+ib \to \mathbf{v}$

$$\boxed{A = PCP^{-1}}, \quad \text{where} \quad P = [\,\mathrm{Re}\,\mathbf{v} \quad \mathrm{Im}\,\mathbf{v}\,] \quad \text{and} \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$\lambda = i$ $\quad a = 0,\ b = -1$

$v = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P^{-1}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Calay Hamilton theorem

(Recall) Given a square matrix A, the characteristic equation of A is the polynomial equation

$$det(A - \lambda.I) = 0$$

Its real or complex roots are called eigenvalues.
Question: Is it possible to find its "matrix root"?

Calay Hamilton theorem

(Recall) Given a square matrix A, the characteristic equation of A is the polynomial equation

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Theorem (Calay-Hamilton theorem)

A square matrix A satisfies its own characteristic equation, i.e. $det(A - \lambda.I)|_{\lambda=A} = 0$.

If $det(A - \lambda.I) = \sum_{i=0}^{n} a_i \lambda^i$ then $\sum_{i=0}^{n} a_i A^i = 0$, zero matrix.

Example

poly of eign $= \det(A-\lambda I) = 0 \to$ get λ

$\lambda = \pm i \Leftarrow \lambda^2 + 1 = 0$

$Ax = \lambda x$

$A - \lambda I \longrightarrow$ REF

$\underline{\text{Null}}\,(A - \lambda I)$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \longrightarrow \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix}$

$c \begin{bmatrix} +i \\ 1 \end{bmatrix} = \begin{bmatrix} +ic \\ c \end{bmatrix}$

A

$-ix - c = 0$

$\det(A - \lambda I)$

$x = \frac{c}{-i} = +ic$

$A - \lambda I = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$

$(-\lambda)^2 - (-1) = 0$ $\quad$ ch. eq^n?

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Theorem

Given a square matrix A, if its minimal polynomial is a product of distinct linear factors over $\mathbb{R}$ then A is diagonalizable over $\mathbb{R}$.