4. Vector Space

Gen. of $\mathbb{R}^n$

Recall

m eq^{ns} (lin). & n variables.

Null(A) $\quad$ $AX = b$ $\quad$ matrix form.

$\{X \in \mathbb{R}^n \mid AX = 0\}$ $\quad$ $[A|b] \longrightarrow REF$

Subspace of $\mathbb{R}^m$

is ~~another~~ subspace of $\mathbb{R}^n$ $\quad$ $AX = b$ is consistent $\Longleftrightarrow$ $b \in Col(A)$.

If $AX = b$ is consistent then it has unique sol^n $\Longleftrightarrow$ $\det(A) \neq 0$ or $Col(A) = \mathbb{R}^n$ or $Null(A) = \{0\}$

$\cancel{9}$ $\quad c=d=1$

$\mathbb{R}^n$.

1) $u, v \in \mathbb{R}^n$, $u+v \in \mathbb{R}^n$ $\quad \begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

2) $c \in \mathbb{R}$, $u \in \mathbb{R}^n$, $c \cdot u \in \mathbb{R}^n$ $\quad (c+d) \cdot u \neq cu \quad d \cdot u$

3) $u+v = v+u \quad \forall u, v \in \mathbb{R}^n$ (commutativity) ~~Fourier transform~~

4) $u+(v+w) = (u+v)+w$

5) For each $u \in \mathbb{R}^n$, $\exists -u \in \mathbb{R}^n$ s.t. $u+(-u) = 0$ $\quad c=2$

6) $c \cdot (d \cdot u) = (cd) \cdot u \quad \forall u \in \mathbb{R}^n, c, d \in \mathbb{R}$. $\quad u = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, v = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

7) $1 \cdot u = u \quad \forall u \in \mathbb{R}^n$

8) $c \cdot (u+v) = c \cdot u + c \cdot v \quad \forall c \in \mathbb{R}, u, v \in \mathbb{R}^n$ $\quad LHS = 2 \cdot \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 6 & 8 \end{bmatrix}$

9) $(c+d) \cdot u = c \cdot u + d \cdot u$ $\quad RHS = \begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix}$

10. $u+0 = u \quad \forall u \in \mathbb{R}^n \quad 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

1) $\mathbb{R}_4[x] = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \mid a_i \in \mathbb{R} \}$ $\qquad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \in \mathbb{R}^5$

$$\sum a_i x^i + \sum b_i x^i = \sum (a_i + b_i) x^i$$

$$c \cdot \left(\sum a_i x^i\right) = \sum (c \cdot a_i) x^i$$

$$0 + 0 \cdot x + \cdots + 0 \cdot x^4 \in \mathbb{R}_4[x]$$

2) $\mathbb{R}[x]$ = set of all poly. ~~of~~ with coeff. from $\mathbb{R}$.

3) $M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a,b,c,d \in \mathbb{R} \right\}$ $\qquad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$$

$$\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}.$$

Define $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$

$$\alpha \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ c & d \end{bmatrix}$$

$$\alpha\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) = \alpha \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} = \begin{bmatrix} \alpha(a+a') & \alpha(b+b') \\ c+c' & d+d' \end{bmatrix}$$

(8) is satisfied $\quad = \begin{bmatrix} \alpha a & \alpha b \\ c & d \end{bmatrix} + \begin{bmatrix} \alpha a' & \alpha b' \\ c' & d' \end{bmatrix}$

$$= \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \alpha \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}.$$

$M_2(\mathbb{R})$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\alpha \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ 0 & 0 \end{bmatrix}$$

$1 \cdot A \neq A.$

$$A = \{ f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ is a function} \}$$

$\mathbb{R}^2 \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix}, \; x, y \in \mathbb{R}$

$$(f+g)(x) = f(x) + g(x) \quad \forall x \in \mathbb{R}$$

$$(c \cdot f)(x) = \underline{c \cdot f(x)} \quad \forall x \in \mathbb{R}$$

$\mathbb{Z}_2^2 \longrightarrow \begin{bmatrix} a \\ b \end{bmatrix}, \; a, b \in \mathbb{Z}_2 = \{0, 1\}$

makes A ~~makes~~ a vector space $\boxed{\text{over } \underline{\mathbb{R}}}$

$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in \mathbb{Z}_2 \right.$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \cancel{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$\boxed{11\;01\;00\;01}$

$m = \underline{01\;11\;00\;11}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\mathbb{Z}_2^{\cancel{2}2} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \cancel{\in} \middle| \; a, b \in \mathbb{Z}_2 \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a + c \pmod 2 \\ b + d \pmod 2 \end{bmatrix}$

$\alpha \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha a \pmod 2 \\ \alpha b \pmod 2 \end{bmatrix} \quad \alpha \in \mathbb{Z}_2$

is a vector space over $\mathbb{Z}_2$.

Fix prime p $\underline{\mathbb{Z}_p^n} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| \; a_i \in \mathbb{Z}_p \right\}$ is a v.s. over $\mathbb{Z}_p$

$\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$

$$M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a, b, c, d \in \mathbb{R} \right\}$$

Find $W \subseteq M_2(\mathbb{R})$ s.t. W is again a vector space over $\mathbb{R}$.

$W = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is a trivial subspace

$\boxed{\begin{array}{l} \text{For } u, v \in W \quad u + v \in W \\ \text{for } \alpha \in \mathbb{R},\ u \in W,\ \alpha \cdot u \in W \end{array}} \iff$ For any $u, v \in W$, $\alpha \in \mathbb{R}$. $u + \alpha v \in W$

$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a, b, c, d \in \mathbb{Z} \right\}$ X

$c = b = 0$

$$W = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \middle| \, a, \cancel{b}\, d \in \mathbb{R} \right\} \checkmark$$

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \alpha \begin{bmatrix} a' & 0 \\ 0 & d' \end{bmatrix} = \begin{bmatrix} a+\alpha a' & 0 \\ 0 & d+\alpha d' \end{bmatrix} \in W$$

$$W = \left\{ \begin{bmatrix} 0 & a \\ 0 & d \end{bmatrix} \middle| \, a, d \in \mathbb{R} \right\} \checkmark$$

$a_{11} = 1$

$$W = \left\{ \begin{bmatrix} 1 & a \\ 0 & d \end{bmatrix} \middle| \, a, d \in \mathbb{R} \right\} \times$$

$$W = \left\{ \overset{A}{\overset{\shortparallel}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}} \;\middle|\; \overset{ad-bc=0}{\det(A) \cancel{\neq} = 0} \right\}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \notin W$$

$$M_2(\mathbb{R}) \to \mathbb{R}^3$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

✓ $$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; b = c \right\}$$

$$\begin{bmatrix} a \\ b \\ b \\ d \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ d \end{bmatrix} \quad \text{bijective} \quad \phi$$

$$\phi(\alpha A + B) = \alpha\phi(A) + \phi(B).$$

wh A & B are symm.

$$W = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \middle/ a, b, d \in \mathbb{R} \right\}$$

$$= a\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A_1} + b\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A_2} + d\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_3}$$

$$= \underline{\text{Span}\{A_1, A_2, A_3\}}$$

$\{A_1, A_2, A_3\}$ is lin. indep. set. $\Longleftarrow$ $\underline{x_1 A_1 + x_2 A_2 + x_3 A_3} = 0$

$$\begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Longleftrightarrow x_1 = x_2 = x_3 = 0$$