1, 5, 2, 5, 7, 5, 9, 5, 4, 3, 2, 1,

6, 8, 7, 9, 1, 5

1 ⟶ 3

2 ⟶ 2

3 ⟶ 1

4 ⟶ 1

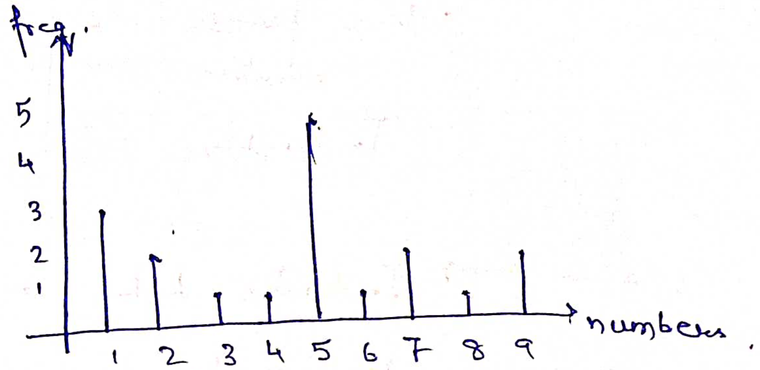
5 ⟶ 5

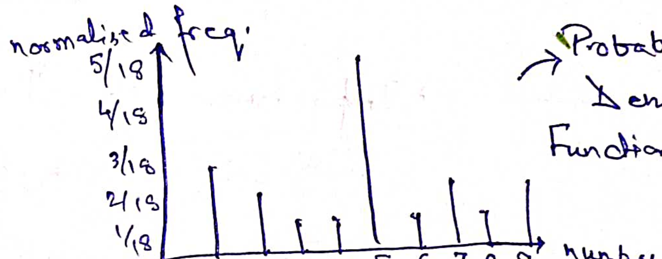
6 ⟶ 1

7 ⟶ 2

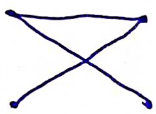
8 ⟶ 1

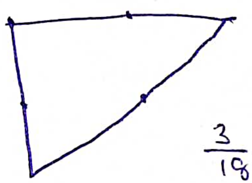
9 ⟶ 2





→ Probability Density Function (PDF)





$$\frac{3}{18} + \frac{2}{18} + \frac{1}{18} + \frac{1}{18} + \frac{5}{18} + \frac{1}{18} + \frac{2}{18} + \frac{1}{18} + \frac{2}{18} = 1$$

$$\left[\begin{array}{l} \text{Mid} \longrightarrow 35\% \\ \text{End} \longrightarrow 45\% \\ \text{Quiz} \longrightarrow 20\% \end{array}\right]$$

Foundation

27/08/2024

Real world	Engineering World
• Events occurs (Temp change, rain etc) $\xrightarrow{\text{Measurement}}$	Numbers.
• We 'perceive' events →	Number System
• Inductive	Deductive

→ Include ALL, once

No redundancy, no exclusion } → Probability

→ Importance of Probability

- Stock Market Fluctuations
- Temp. variations over a period of time
- Ages of children in a class
- Random Analysis.

} Random (non-deterministic)

$$p(A) = \frac{N_A}{N} \Rightarrow \text{Catching average characteristic.}$$

→ Axiomatic defination of Probability:

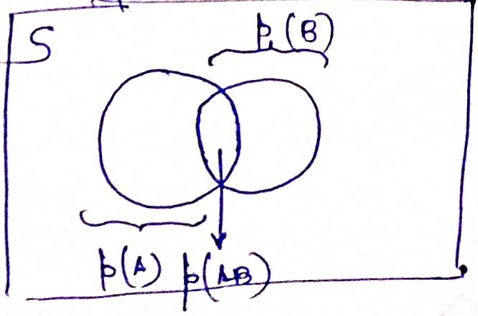
(i) $p(A) \geq 0$

(ii) $p(S) = 1$

} → $P(A) \in [0, 1]$

(iii) $p(A+B) = p(A) + p(B) - \cancel{p(A \cap B)}\ p(AB)$

~~$p(A+B) = p(B) + p(\bar{A})$~~



27/08/2024

Foundation

Real world	Engineering World
• Events occurs (Temp change, rain etc) $\xrightarrow{\text{Measurement}}$	Numbers.
• We perceive events $\longrightarrow$	Number System
• Inductive	Deductive

→ Include ALL, once
No redundancy, no exclusion } → Probability

→ Importance of Probability
- Stock Market Fluctuations
- Temp. variations over a period of time
- Ages of children in a class
- Random Analysis.

} Random (non-deterministic)

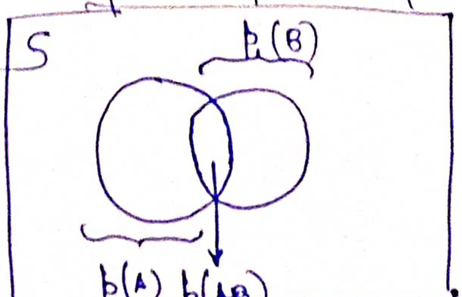
$$p(A) = \frac{N_A}{N} \Rightarrow \text{Catching average characteristics.}$$

→ Axiomatic defination of Probability:

(i) $p(A) \geq 0$
(ii) $p(S) = 1$

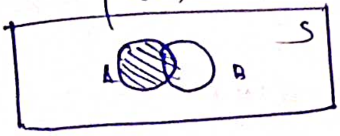
} → $P(A) \in [0, 1]$

(iii) $p(A+B) = p(A) + p(B) - \cancel{p(A \cap B)}\ p(AB)$

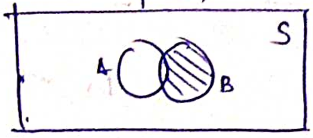


~~$p(A+B) = p(B) + p(A)$~~

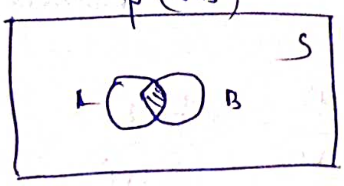
$p(A)$



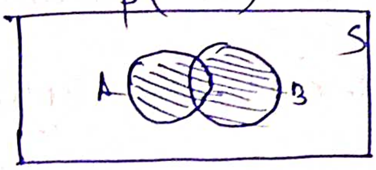
$p(B)$



$p(AB)$



$p(A+B)$



$\longrightarrow$ Mutually exclusive event :

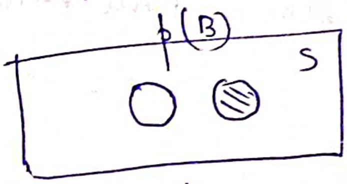
$$p(AB) = \{\phi\}$$

$$\Updownarrow$$

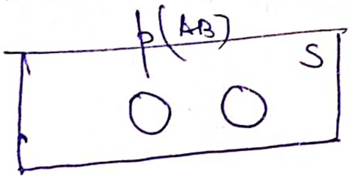
$$p(A+B) = p(A) + p(B)$$

$p(A)$

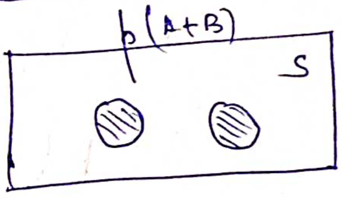
$p(B)$



$p(AB)$



$p(A+B)$



Q) Consider two events, A and B, such that $B \subset A$, then $p(A) \geq p(B)$

Prove by Venn diagram $\longrightarrow$ $A = B + \bar{B} A$

Conditional Probability -

$$\boxed{p(A/M) = \frac{p(AM)}{p(M)}}$$

given → M; M → occurred; $p(AM)$ → Joint form (intersection)

(A & B are dependent)

If $A \subset M \longrightarrow p(A/M) = \frac{p(AM)}{p(M)} = \frac{p(A)}{p(M)}$

If $M \subset A \longrightarrow p(A/M) = \frac{p(AM)}{p(M)} = \frac{p(M)}{p(M)} = 1$

exclusive

$p((A+B)/M) = \cancel{\frac{p((A+B)M)}{p(M)}}\ p(A/M) + p(B/M)$

exclusive event

Example : ~~Fair~~ Fair die equation $\quad S = \{1,2,3,4,5,6\}$

$A = \{f_2\} \longrightarrow p(A) = 1/6$

$M = \{f_2, f_4, f_6\} \longrightarrow p(M) = 1/2$

$p(A/M) = \frac{1/6}{1/2} = \frac{1}{3}$

$$\boxed{\text{Conditioning} \longrightarrow \text{Reduces Variance}}$$

Total Probability

$S = \underbrace{\{A_1, A_2, A_3 \dots A_n\}}_{}$ such that $A_iA_j = \{\phi\}$

mutually exclusive

Let B is a event which intersects all elements in S

$B = B \cdot S = B \cdot (A_1 + A_2 + \dots + A_n)$

$= BA_1 + BA_2 + \dots + BA_n$

$p(B) = p(BA_1) + p(BA_2) + \dots + p(BA_n)$

$$\boxed{p(B) = p(B/A_1)\,p(A_1) + p(B/A_2)\,p(A_2) \cdots + p(B/A_n)\,p(A_n)}$$

→ Total probability

Bayes Theorem

$AB = BA$, $p(AB) = p(BA)$

$p(A/B)\,p(B) = p(B/A)\,p(A)$

$$\boxed{p(A/B) = \frac{p(B/A)\,p(A)}{p(B)}}$$

→ likelihood.

→ prior

→ comes from total probability

Example: There are 3 machines ①, ②, ③

	defective	production
m – ①	1%	40%
m – ②	3%	20%
m – ③	2%	40%

If we pick up a productive & find it defective, how to find out that it is from which machine

→ $p(D_1)$ = Productive but defective from ①

$= 0.01$

$p(D_2) = 0.03$, $p(D_3) = 0.02$

$$\cancel{p(M_i/B)} = \frac{p(D/M_i)\,p(M_i)}{\cancel{p(D)}}$$

$$p(M_i/D) = \frac{p(D/M_i)\,p(M_i)}{\Sigma\, p(D/M_i)\cdot p(M_i)}$$

$$p(M_1/D) = \frac{0.01 \times 0.4}{(0.01 \times 0.4) + (0.03 \times 0.2) + (0.02 \times 0.4)}$$

$$= \frac{0.004}{0.004 + 0.006 + 0.008} = \frac{0.004}{0.018} = 2/9$$

$$p(M_2/D) = \frac{0.03 \times 0.2}{0.018} = \frac{0.006}{0.018} = \frac{1}{3}$$

$$p(M_3/D) = \frac{0.02 \times 0.4}{0.018} = \frac{0.008}{0.018} = 4/9$$

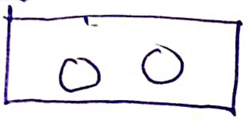
Independent event

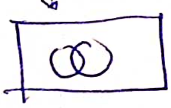
$$\boxed{p(AB) = p(A) \cdot p(B)} \longrightarrow p(A/B) = p(A)$$

Mutually exclusive $\longleftrightarrow\!\!\!\!\!\!\times$ Independent event

Q) Can two event be mutually exclusive & statistically indepent at same time? ~~Draw venn dia~~

→ M.E. → $p(AB) = 0$, Independent → $p(AB) = p(A) \cdot p(B)$



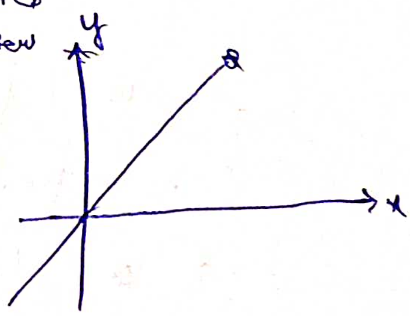


Random variable

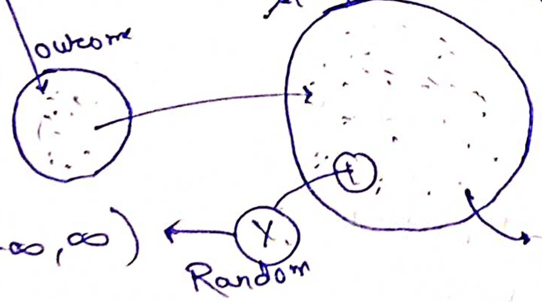
Random variable → always represented in capital letter

Function → $f: X \xrightarrow{\text{maps}} Y$

Domain (X), Codomain (Y)



Random exp



A = power set → consist of union, intersection, complement, null (no subtraction)

Probability Space = Sample space + Power Set + Probability

We have flexibility to define random variable

Random variable is a real-value function → $\boxed{X: S \longrightarrow R}$ ← condition

R → Real life

(i) $\{X \le x\}$ is an event $-\infty < x < \infty$

(ii) $p(x = \pm\infty) = 0$

Ex: Throwing of fair dice

$i = 1, 2, 3 \cdots 6$ $X(f_i) = 10i$ — gives full view

$X(f_1) = 10,\ X(f_2) = 20 \cdots$

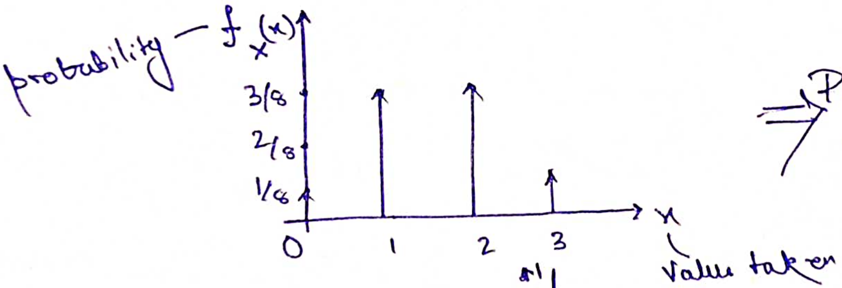
Example: A fair coin is tossed three times and faces shown up are observed

(a) Write sample space

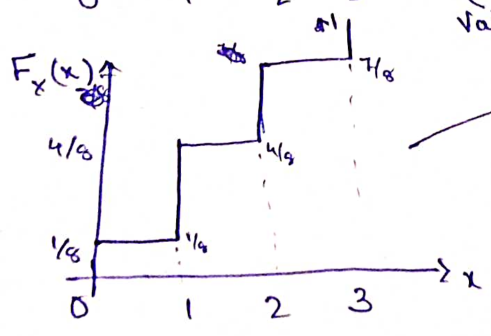
(b) If X is a random variable representing number of heads in each outcome. Draw PDF/CDF

(a) $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

(b) $X = \{3, 2, 2, 2, 1, 1, 1, 0\}$ — Random variable



⇒ Probability Density Function (PDF)



→ monotonically increasing graph

Cumulative distribution ⇒ function (CDF)

$$PDF \underset{\text{differentiate}}{\overset{\text{integrate}}{\rightleftharpoons}} CDF$$

Cumulative Distribution Function (CDF):

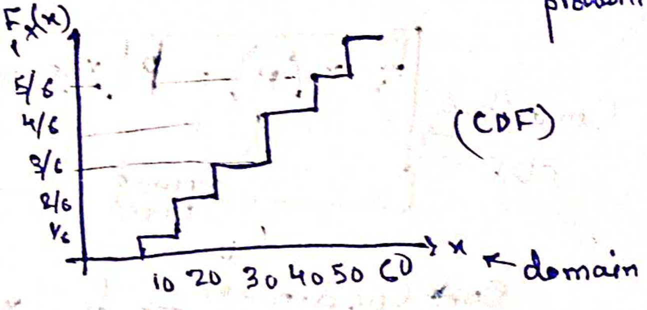
CDF $\longrightarrow$ $F_X(x) = P(X \le x)$ where $-\infty < x < +\infty$

X ↑ random variable; x ↖ values of X

Ex: Throw a fair dice

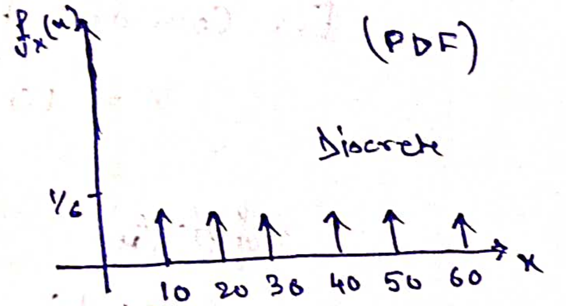
$S = \{f_1, f_2, \ldots, f_6\}$ → Flexibility to define random variable.

Let Random variable $X(f_i) = 10i$ $[i = 1, 2, 3 \ldots, 6]$ → each having probability $\frac{1}{6}$



- Let $x > 60$, $F_X(60) = P(X \le 60) = 1$
- Let $50 \le x < 60$,

  $F_X(x) = 5/6$
- Let $40 \le x < 50$, $F_X(x) = 4/6$
- Let $30 \le x < 40$, $F_X(x) = 3/6$
- Let $20 \le x < 30$, $F_X(x) = 2/6$
- Let $10 \le x < 20$, $F_X(x) = 1/6$



Probability density Function (PDF):

Continuous random variable $\longrightarrow$ $f_X(x) = \frac{d}{dx} F_X(x)$ $\longleftrightarrow$ $f_X(x) = F_X(x) - F_X(x^-)$ ← Discrete random variable

CDF characteristics:-

(i) Known ~~de~~increasing function

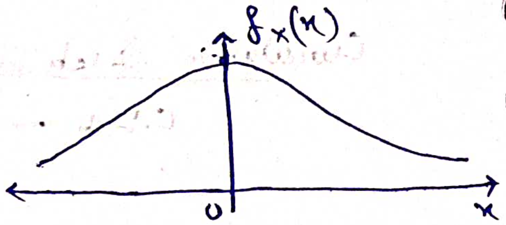
(ii) $P(X > x) = 1 - F_X(x)$

(iii) $F_X(x^-) = \lim_{\epsilon \to 0} F_X(x^-$

(iv) $P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$

Some well known distribution:

Discrete	Continuous
• Binomial distribution	• Normal or Gaussian distribution
• Poisson distribution	• Uniform distribution
	• Exponential distribution



↓
Normal
Gaussian

⊛ Conditional distribution:

$$p(A/B) = \frac{p(AB)}{p(B)}$$

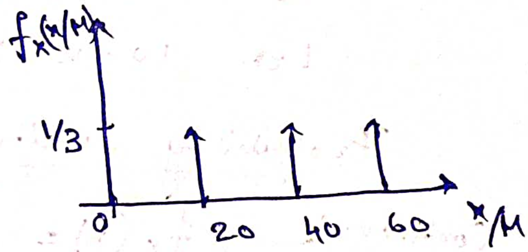
Conditional CDF →
$$\boxed{F_X(x/M) = P(X \le x \mid M) = \frac{p(X \le x, M)}{p(M)}}$$
occured event

Conditional PDF → $f_X(x/M) = \frac{d}{dx} F_X(x/M)$

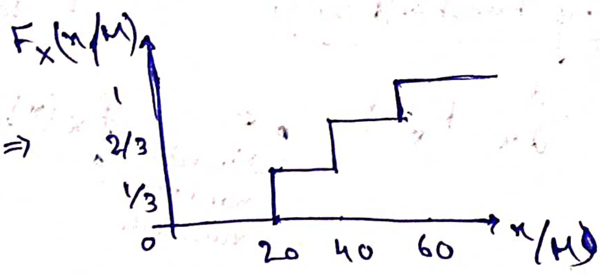
Ex: Consider fair dice exp where

$X = 10i, \quad M = \{f_2, f_4, f_6\}$

PDF →



Conditional CDF ⇒



• Let $x > 60$, $F_X(60) = P(x \le 60) = 1$

• Let $40 \le x < 60$, $F_X(x) = 2/3$

• Let $20 \le x < 40$, $F_X(x) = 1/3$

"Functions describe the world"

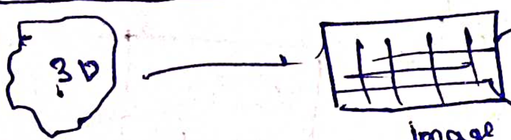


image (I)

Domain: [Independent Variable (Space, Time)]
↓
Space

Codomain: Pixel intensity

$\boxed{\begin{array}{l} \mathbb{R}\times\mathbb{R} \to 1D \\ \mathbb{R}^{x\times y} \to 2D \end{array}}$

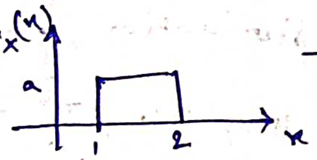
Image (I) : $(x, y) \longrightarrow \mathbb{R}^{x \times y}$

Q. Look around & write down 10 different event & define them as a function

Function of a Random Variable -

- Consider a random variable $\Rightarrow X : S \longrightarrow \mathbb{R}$
- Let us define another random variable $\Rightarrow Y = g(X)$

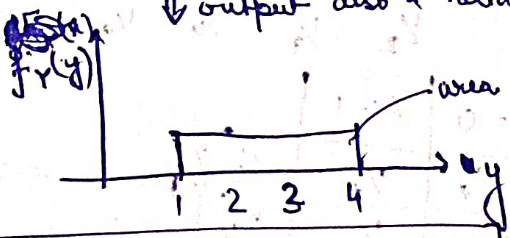
eg. $Y = X^2$



$\longrightarrow a = \frac{1}{b-a} = \frac{1}{2-1} = 1$

$\boxed{f_X(x) = \frac{1}{b-a}}$ (uniform PDF)

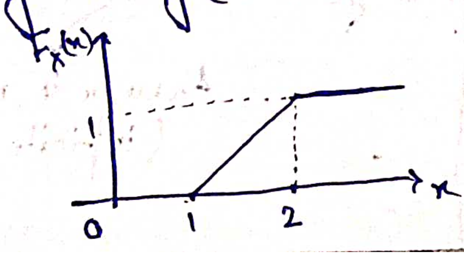
⇓ output also a valid PDF

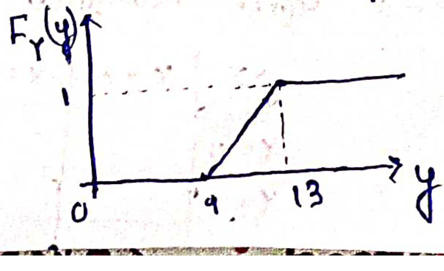


$\boxed{\text{A number map to another number}}$ → Function of a random variable

eg. $Y = g(X) = aX + b = 4X + 5$, $F_Y(y) = F_X\left(\frac{y-b}{a}\right)$

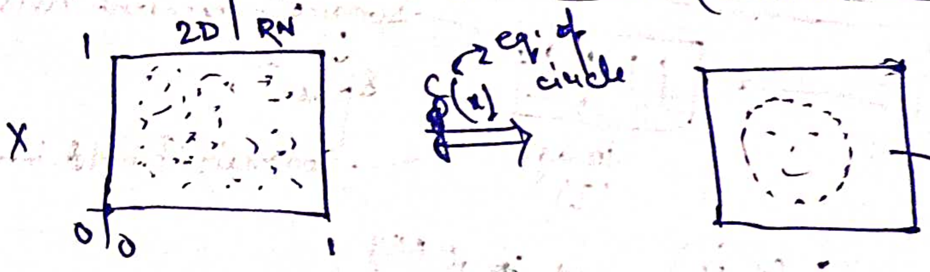
Linear Transformation of r.v.s





Probability starts when event starts

~~Linear transformation of rvs (random variable)~~



"learn"

Generative Adversarial Network (GAN)

Joint distribution

- Consider two random variables, X & Y
- Interested in ~~paired~~ joint behaviour of X & Y

Illustration : Speech waveform, images, web page, etc...

Characterisation : $F_{XY}(x,y) = P(X \le x, Y \le y)$

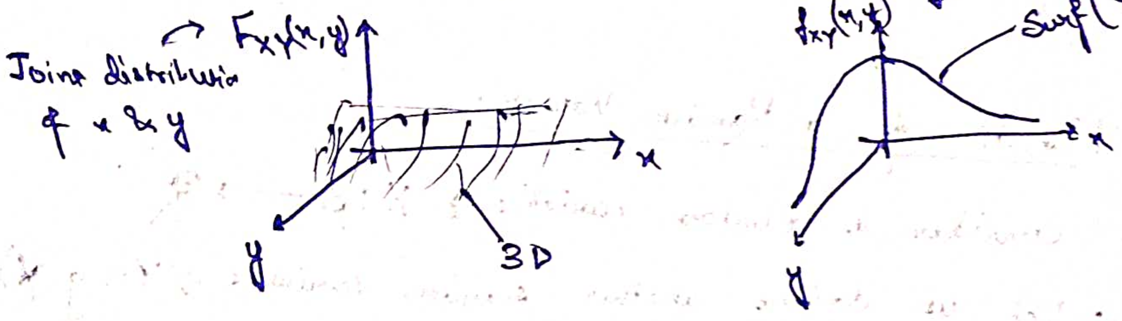


Illustration → Given the joint distribution, find marginal distribution

$S = \{HH, HT, TH, TT\}$ → tossing coins

$\downarrow$ 1/2, 1/8, 1/4, 1/8

Y \ X	X = 0	X = 1	
Y = 0	1/8	1/8	1/4
Y = 1	1/4	1/2	3/4
	3/8	5/8	

⟹ Marginal probability (Marginal distribution)

Illustration → Given marginal, find joint → Answer is not unique (infinite solutions)

Y \ X	X = 0	X = 1	
Y = 0			
Y = 1			

$$\text{Joint distribution} \underset{\substack{\text{not unique solution} \\ \text{(infinite solution)}}}{\overset{\text{unique solution}}{\rightleftarrows}} \text{Marginal distribution}$$

<u>Working Formulae</u> $\rightarrow$

Set of observations $\longrightarrow o_1, o_2, o_3, \dots, o_n$

$$\boxed{\text{Mean} = \bar{o} = u_o = \frac{1}{n}\sum_{i=1}^{n} o_i}$$

$$\boxed{\text{Variance}(o) = \frac{1}{n}\sum_{i=1}^{n} (o_i - \bar{o})^2}$$

Let consider other set of observation : $p_1, p_2, p_3, \dots, p_n$

For covariance $\longrightarrow$ consider pairs $(o_1, p_1), (o_2, p_2), \dots, (o_n, p_n)$

$$\boxed{\text{cov}(o, p) = \frac{1}{n}\sum_{i=1}^{n} o_i p_i - \bar{o}\cdot\bar{p}} \longrightarrow \text{can be '+ve', '-ve', or } 0$$

<u>Corelation b/w sets of observation</u> $\Rightarrow$

$$\boxed{P_{op} = \frac{\text{cov}(o, p)}{\underbrace{\sqrt{\text{var}(o)}}_{\text{standard deviation}} \cdot \sqrt{\text{var}(p)}}} \Rightarrow \boxed{-1 \le P_{op} \le 1}$$

Ex :

$o \xrightarrow{\text{friends}} p$ $\qquad P_{op} = 1$

$p \xleftrightarrow{\text{enemy}} o$ $\qquad P_{op} = -1$

$p \perp o$ (strangers) $\qquad P_{op} = 0$

<u>Concept of Expectation</u> :

$\rightarrow o_1, o_2, \dots, o_n \longrightarrow$ observations

$\rightarrow f_1, f_2, \dots, f_n \longrightarrow$ frequency of appearing

$$\boxed{\bar{o} = \frac{1}{\sum_{i=1}^{n} f_i}\sum_{i=1}^{n} o_i \cdot f_i}$$

$$\bar{o} = \sum_{i=1}^{n} o_i \frac{f_i}{\sum_{i=1}^{n} f_i} \rightarrow \text{Relative frequency}$$

another name

$E(x) = \sum_i x_i p(x_i)$ Expectation = [Value of variable × its probability]

<u>Remarks</u> :- (i) Expectation is a linear operator

$E[ax+b] = a \cdot E(x) + b$ → Random Variable

(ii) $Var(X) = E[X-\mu]^2$

$$= \sum_x (x-\mu)^2 \cdot p(x) = \sum_x (x^2 - 2x\mu + \mu^2) \cdot p(x)$$

$$= \sum_x \left(x^2 \cdot p(x) - 2\mu x p(x) + \mu^2 \cdot p(x)\right)$$

$$\boxed{Var(x) = \sum_x x^2 \cdot p(x) - 2\mu \sum_x x \cdot p(x) + \mu^2 \sum_x p(x)}$$

$$\therefore Var(x) = E(X^2) - 2\mu E(x) + \mu^2 \underbrace{\sum_x p(x)}_{\to 1 \text{ why? } \to \text{PDF}}$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$\Rightarrow \boxed{Var(X) = E(X^2) - (E(X))^2}$$

* <u>Useful Identity</u> :

$$Var(aX+b) = E[(aX + b - a\mu - b)^2]$$

$$= E[a^2 (X-\mu)^2] = a^2 E[(x-\mu)^2]$$

$$\therefore \boxed{var(ax+b) = a^2 Var(X)}$$

Therefore, variance is not a linear operator

* Expectation & variance of function of two random variables : $g(x,y)$

from definition :- $$\boxed{E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) \underbrace{f_{xy}(x,y)}_{\text{Joint distribution}} \, dx \, dy}$$

For ex :- $g(x,y) = xy$

$$E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot f_{xy}(xy) \cdot dx \cdot dy$$

If $g(xy) = x + y$

$$\therefore E[g(xy)] = E[x+y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f_{xy}(x,y)\, dx \cdot dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \cdot f_{xy}(xy) \cdot dx\,dy$$

$$= \int_{-\infty}^{\infty} x\underbrace{\left(\int_{-\infty}^{\infty} f(x,y)\cdot dy\right)}_{\text{marginal distribution or pdf}} \cdot dx + \int_{-\infty}^{\infty} y\underbrace{\left(\int_{-\infty}^{\infty} f_{xy}(x,y)\cdot dx\right)}\cdot dy$$

marginal distribution or pdf

marginal density of x

marginal density of y

$$\int_{-\infty}^{\infty} x\overbrace{f(x)}\,dx + \int_{-\infty}^{\infty} y\overbrace{f(y)}\,dy$$

→ Now

$$Var(x+y) = E[(x+y)^2] - [E[x+y]]^2$$

$$= E[x^2 + y^2 + 2xy] - (m_x + m_y)^2$$

$$= E(x^2) + E(y^2) + 2E(xy) - m_x^2 - m_y^2 - 2M_xM_y$$

$$= [E(x^2) - m_x^2] + [E(y)^2 - m_y^2] + 2E(xy) - 2M_xM_y$$

$$Var(x+y) = Var(x) + Var(y) + 2\underbrace{[E(xy) - E(x)E(y)]}_{\text{covariance of }(x,y)}$$

In matrix form:-
$$Var(x,y) = \begin{array}{c|cc} & x & y \\ \hline x & Var(X) & cov(x,y) \\ y & cov(y,x) & var(y) \end{array}$$

→ always square matrix

→ Symmetric

→ Diagonals are non-negative

→ Positive semi-definite (means always invertible)

[Can be 3×3, if random variables are three]

$X \sim$ Bernoulli(p)

X	0	1
$p(x)$	$1-p$	p

$E(X) = \sum x\,p(x) = p$

$Var(X) = E(X^2) - (E(X))^2$

$Y = Y_1 + Y_2 + \dots + Y_n \longrightarrow$ Bernoulli (P)

$Bin(n, p) \sim y$

↓

iids of n Bernoulli(p)

→ Independent & identical distance

$$cov(X, Y) = E[(X - E(X))(Y - E(Y))]$$
$$= E(XY) - E(X)E(Y)$$

3) ⓐ $E(XY) = \sum_{x,y} xy\, p(x, y)$

$$= 1 \cdot \frac{2}{15} + 2\left(\frac{7}{15}\right) + 4\left(\frac{6}{15}\right) =$$

Negative Binomial ⟶ For k no. of success in n trial,

$pr(\text{success}) = P$

$$PMF = {}^{n-1}C_{k-1}\, P^k (1-P)^{n-k}$$

Binomial $\Rightarrow {}^nC_k\, P^k (1-P)^{n-k}$

4) b) At least 5 computers to check ⇒

$k = 2$ ⟶ defective

$p(X = t) = {}^{t-1}C_{k-1}\, p^k (1-p)^{t-k}$ (negative binomial)

$p(X \geq 5) = 1 - P(X \leq 4)$

Jacobian Matrix $= J = \begin{pmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \end{pmatrix}$

$$f_{U,V}(u,v) = f_{xy}(x,y)\,|J|$$

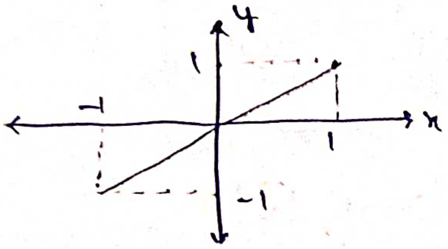
$X, Y \rightarrow$ uncorelated $\longrightarrow$ covariance $= 1$

$X, Y \rightarrow$ independent $\longrightarrow$ always uncorelated, but not vice-versa

$$E(X|Y) = E(E(XY/Y)) = E(Y\,E(X/Y))$$

Illustration: Covariance

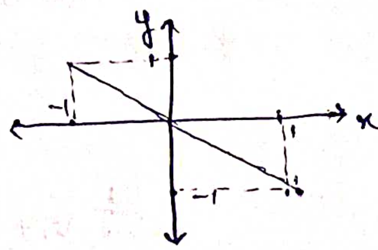
Fig: A



$p(1,1) = p(-1,-1) = \frac{1}{2}$

$x = y$ (relation)

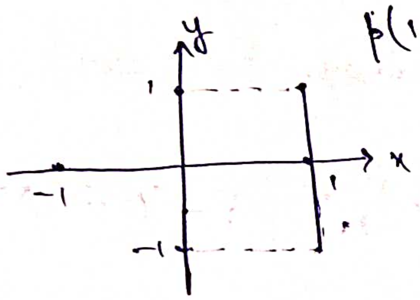
Fig B



$p(1,-1) = p(-1,1) = \frac{1}{2}$

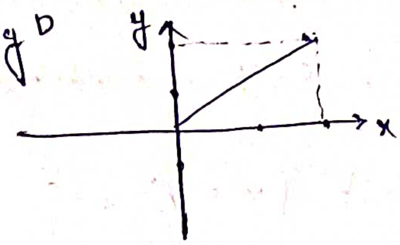
$x = -y$ (relation)

Fig C



$p(1,-1) = p(1,1) = \frac{1}{2}$

Fig D



$p(0,0) = p(2,2) = \frac{1}{2}$

Fig A $\quad E(X) = 0,\ E(Y) = 0$

$E(XY) = \sum_i \sum_j x_i y_j \, p(x_i, y_j)$

$= \left(1 \times 1 \times \frac{1}{2}\right) + \left(-1 \times -1 \times \frac{1}{2}\right)$

$= 1$

$cov(XY) = 1$

Fig B $\quad E(X) = 0,\ E(Y) = 0$

$E(XY) = \left(1 \times -1 \times \frac{1}{2}\right) + \left(1 \times -1 \times \frac{1}{2}\right)$

$= -\frac{1}{2} + -\frac{1}{2} = -1$

$cov(XY) = -1$

Fig C $\quad E(X) = 1,\ E(Y) = 0$

$E(XY) = 1 \times -1 \times \frac{1}{2} + 1 \times 1 \times \frac{1}{2}$

$= 0$

$cov(XY) = E(XY) - E(X)E(Y)$

$= 0$

Fig D $\quad E(X) = 1,\ E(Y) = 1$

$E(XY) = 0 \times 0 \times \frac{1}{2} + 2 \times 2 \times \frac{1}{2}$

$= 2$

$cov(XY) = 1$

The measure will be,

$$\boxed{E\left[(X - m_x)(Y - m_y)\right] \triangleq cov(X,Y)}$$

$cov(X,Y) = E\left[XY - X m_y - m_x Y + m_x m_y\right]$

$= E(XY) - m_y E(X) - m_x E(Y) + m_x m_y$

$$\boxed{cov(X,Y) = E(XY) - E(X)E(Y)}$$ when $m_y = E(Y)$

$$\mathrm{cov}(X,Y) = \begin{cases} +ve \longrightarrow X, Y \text{ in same direction} \\ 0 \longrightarrow \text{not related} \\ -ve \longrightarrow X, Y \text{ in opposite direction} \end{cases}$$

~~cov(X, Y)~~

If $\mathrm{cov}(X,Y) = 0 \longrightarrow E(XY) = E(X)\,E(Y)$

(such X & Y are called ~~uncorrelate~~ uncorelated random variable)

Remarks :- (i) $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{cov}(X,Y)$

if X & Y are uncorelated ⟶

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

(ii) Independent of two random variable ⟶ cov turns into diagonal matrix

$$f_{XY}(x,y) \triangleq f_X(x) \cdot f_Y(y)$$

(Joint PDF) — Marginal PDF of x & y

else ~~independent~~ n variable in n×1 forms a n×n matrix

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy$$

$$\Rightarrow E(XY) = \int x f(x)\, dx \cdot \int y f(y)\, dy$$

$$= E(X)\,E(Y)$$

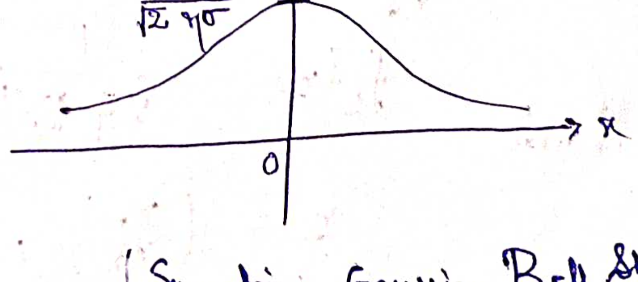
If X, Y are statistically independent ⟶ they are uncorrelated

~~If X, Y are uncorelated ⟶~~ (but vice-versa isn't true) ↓ Prove it

For Gaussian random variable ⟶ Independent ⟺ Uncorelated

Gaussian PDF :

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



σ^2 ⟶ Variance

μ ⟶ Mean

σ ⟶ Std. deviation

(Symmetric, Gaussian, Bell Shape)

n - Variance Gaussian PDF

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{|\Sigma|}}\, e^{-\frac{1}{2}\overbrace{(\underline{x}-\underline{\mu})^T\,\Sigma^{-1}\,(\underline{x}-\underline{\mu})}^{\text{Scalers (from 3 vectors)}}}$$

$(1\times n)$ $(n\times n)$ $(n\times 1) \rightarrow 1\times 1$

$|\Sigma| \rightarrow$ Determinant of Σ matrix

[$\underline{X} \rightarrow$ X bar , $\underline{x} \rightarrow$ x bar , $\Sigma \rightarrow$ Sigma].

Variance covariance matrix $= \Sigma =$

$$\begin{bmatrix} Var(X_1) & Cov(X_1 X_2) & \cdots & cov(X_1 X_m) \\ cov(X_2 X_1) & Var(X_2) & \cdots & \vdots \\ \vdots & & \ddots & \\ cov(X_n X_1) & \cdots & \cdots & Var(X_n) \end{bmatrix}$$

Vector Valued Random Variable:-

$$\underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\underline{X_1} = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{pmatrix} \xrightarrow{\text{Mean}} \mu_1$$

$$\underline{X_2} = \begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{pmatrix} \xrightarrow{\text{Mean}} \mu_2$$

$$\underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}$$

Reduced expression (Taking n=1) $\longrightarrow$ Univariate

$$\cancel{f_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{2\pi}}\, e^{-1/2}}\ (x-\mu)^T(\sigma^2)^{-1}(x-\mu)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} \rightarrow \text{Provides probability of Value of X at x}$$

Scalar

Taking $n=2 \Rightarrow$ (Bivariate)

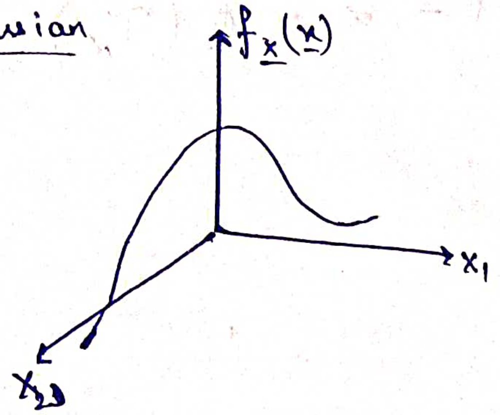
$$f_{\underline{X}}(\underline{x}) = \frac{1}{2\pi\,|\Sigma|}\, e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^T\Sigma^{-1}(\underline{x}-\underline{\mu})}$$

Scalar

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\quad \underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix},\quad \Sigma = \begin{pmatrix} Var(X_1) & Cov(X_2 X_1) \\ Cov(X_1 X_2) & Var(X_2) \end{pmatrix}$$

Bivariate Gaussian

$\underline{x} = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$



Properties of Gaussian distribution $X \sim \mathcal{N}(\mu, \sigma^2)$

(I) Normalisation of X & Standard Normal

$$\boxed{Y = \frac{X-\mu}{\sigma}} \qquad f_Y(y) \approx \mathcal{N}(0,1)$$

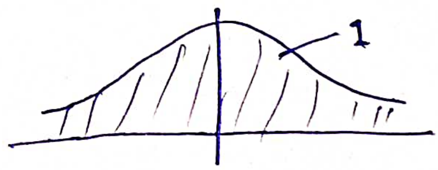
(II) Linear x^n of X (Transformation of X)

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

$$Y = \alpha X + \beta \implies f_Y(y) \sim \mathcal{N}(\alpha\mu+\beta, \alpha^2\sigma^2)$$

(III) PDF of Standard normal, $\phi_Z(z) \sim \mathcal{N}(0,1)$

$$\phi_z(z) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\left(-\frac{z^2}{2}\right)}$$



(IV) $\phi_Z(z) = \phi_Z(-z)$ ⟶ Symmetry of PDF

(V) CDF : $\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \phi_Z(z)\, dz$

(VI) $\Phi(-z) = 1 - \Phi(z)$