Vectors in $\mathbb{R}^n$:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ is a vector in } \mathbb{R}^2.$$

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \text{ is a vector in } \mathbb{R}^3.$$

Vector addition and scalar multiplication are done like matrices.

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

Linear Combinations

Consider a linear system

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

It can be written as

$$x_1 \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -3 \\ -8 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}$$

LHS of $=$ is called linear combination of

$$u = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -3 \\ -8 \end{bmatrix} \;\&\; w = \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix}$$

Above equation is called vector equation.

Linear system and Vector Equation

A linear system $AX = b$ is same as $x_1C_1 + x_2C_2 + \cdots + x_nC_n = b$ where C_i is i^{th} column of A.

A linear system $AX = b$ is same as $x_1 C_1 + x_2 C_2 + \cdots + x_n C_n = b$ where C_i is i^{th} column of A.

Theorem

$AX = b$ has a solution if and only if b is a linear combination of columns of A.

Linear system and Vector Equation

A linear system $AX = b$ is same as $x_1 C_1 + x_2 C_2 + \cdots + x_n C_n = b$ where C_i is i^{th} column of A.

Theorem

$AX = b$ has a solution if and only if b is a linear combination of columns of A.

Hence we need to study linear combination of columns of A.

Linear system and Vector Equation

A linear system $AX = b$ is same as $x_1 C_1 + x_2 C_2 + \cdots + x_n C_n = b$ where C_i is i^{th} column of A.

Theorem

$AX = b$ has a solution if and only if b is a linear combination of columns of A.

Hence we need to study linear combination of columns of A.
Column space of A= set of all linear combinations of columns of A.

$$Col(A) = \{x_1 C_1 + x_2 C_2 + \cdots + x_n C_n | x_i \in \mathbb{R} \text{ for all } i\}$$

$\therefore\ AX = b$ has a solution iff $b \in Col(A)$.

If $v_1, v_2, ..., v_p \in \mathbb{R}^n$, then
Span $\{v_1, v_2, ..., v_p\} = \{c_1 v_1 + c_2 v_2 + ... + c_p v_p | c_1, c_2, ..., c_p \in \mathbb{R}\}$,
the set of all linear combinations of $v_1, v_2, ..., v_p$.

If $v_1, v_2, ..., v_p \in \mathbb{R}^n$, then
Span $\{v_1, v_2, ..., v_p\} = \{c_1 v_1 + c_2 v_2 + ... + c_p v_p | c_1, c_2, ..., c_p \in \mathbb{R}\}$,
the set of all linear combinations of $v_1, v_2, ..., v_p$.
Let

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\text{Span } \{v\} = \{c.v | c \in \mathbb{R}\} = \{ \begin{bmatrix} c \\ 0 \end{bmatrix} | c \in \mathbb{R}\}$$

If $v_1, v_2, ..., v_p \in \mathbb{R}^n$, then
Span $\{v_1, v_2, ..., v_p\} = \{c_1 v_1 + c_2 v_2 + ... + c_p v_p | c_1, c_2, ..., c_p \in \mathbb{R}\}$, the set of all linear combinations of $v_1, v_2, ..., v_p$.
Let

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Span $\{v\} = \{c.v | c \in \mathbb{R}\} = \{ \begin{bmatrix} c \\ 0 \end{bmatrix} | c \in \mathbb{R}\}$ Let $u \in \mathbb{R}^n$. Then $span\{u\} = \{x.u | x \in \mathbb{R}\}$.
Geometrically it is a line passing through origin and point u.
Let $u, v \in \mathbb{R}^2$. Then $span\{u, v\} = \{x.u + y.v | x, y \in \mathbb{R}\}$.
Geometrically, what will it be?

If $v_1, v_2, ..., v_p \in \mathbb{R}^n$, then
Span $\{v_1, v_2, ..., v_p\} = \{c_1 v_1 + c_2 v_2 + ... + c_p v_p | c_1, c_2, ..., c_p \in \mathbb{R}\}$,
the set of all linear combinations of $v_1, v_2, ..., v_p$.
Let

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Span $\{v\} = \{c.v | c \in \mathbb{R}\} = \{\begin{bmatrix} c \\ 0 \end{bmatrix} | c \in \mathbb{R}\}$ Let $u \in \mathbb{R}^n$. Then
$span\{u\} = \{x.u | x \in \mathbb{R}\}$.
Geometrically it is a line passing through origin and point u.
Let $u, v \in \mathbb{R}^2$. Then $span\{u, v\} = \{x.u + y.v | x, y \in \mathbb{R}\}$.
Geometrically, what will it be?
For a matrix A,
$Col(A) := span\{C_1, C_2, \ldots, C_n\} \subseteq \mathbb{R}^m$

The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A.

The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A.

Theorem

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent

1. *For each $b \in \mathbb{R}^m$, the equation $AX = b$ has a solution.*
2. *Each $b \in \mathbb{R}^m$, is a linear combination of the columns of A.*
3. *$Col(A) = \mathbb{R}^m$.*
4. *A has a pivot position in every row.*

The homogeneous equation $AX = 0$ has a nontrivial solution if and only if the REF(A) has at least one free variable.
Suppose augmented form $[A|0]$ is

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}$$

Homogeneous Linear System

The homogeneous equation $AX = 0$ has a nontrivial solution if and only if the REF(A) has at least one free variable.
Suppose augmented form $[A|0]$ is

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

Non homogeneous linear system

Theorem

Suppose the equation $Ax = b$ is consistent for some given b, and let p be a solution (i.e., $Ap = b$). Then the solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$, where v_h is any solution of the homogeneous equation $Ax = 0$.

Non homogeneous linear system

Theorem

Suppose the equation $Ax = b$ is consistent for some given b, and let p be a solution (i.e., $Ap = b$). Then the solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$, where v_h is any solution of the homogeneous equation $Ax = 0$.

WRITING A SOLUTION SET (OF A CONSISTENT SYSTEM) IN PARAMETRIC VECTOR FORM

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

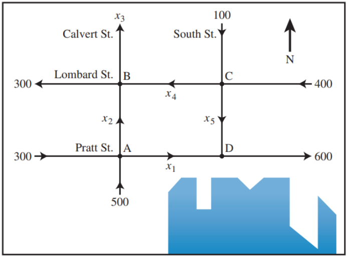
Write down the solutions of $AX = 0$ in parametric form where $A =$

$$\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chemical Equations

Consider a chemical reaction of propane gas with oxygen to form carbon dioxide and water.
$x_1 C_3H_8 + x_2 O_2 = x_3 CO_2 + x_4 H_2O$,
where x_i are the no. of molecules

Network flow



Linear Independence

Let A be a matrix of order $m \times n$. We say columns of A to be linearly independent if $AX = 0$ has only trivial (0) solution.

Linear Independence

Let A be a matrix of order $m \times n$. We say columns of A to be linearly independent if $AX = 0$ has only trivial (0) solution.

Definition

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ is said to be linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0 \equiv AX = 0$$

has only trivial solution.

$$A = \left[v_1 \mid v_2 \mid \cdots \mid v_n \right] \xrightarrow{REF}$$

If # of pivot entries = no. of columns then L.I.

Linear Independence

Let A be a matrix of order $m \times n$. We say columns of A to be linearly independent if $AX = 0$ has only trivial (0) solution.

Definition

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ is said to be linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$$

has only trivial solution.
Else $\{v_1, v_2, \ldots, v_n\}$ is called linearly dependent.

Linear Independence

Let A be a matrix of order $m \times n$. We say columns of A to be linearly independent if $AX = 0$ has only trivial (0) solution.

Definition

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ is said to be linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$$

has only trivial solution.
Else $\{v_1, v_2, \ldots, v_n\}$ is called linearly dependent.

$\{v_1, v_2, \ldots, v_n\}$ is called linearly dependent iff one of v_i is a linear combination of others.

$\{v_1, v_2, \ldots, v_n\}$ is called linearly dependent iff one of v_i is a linear combination of others.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ is always linearly dependent if $n > m$.

$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$

$\{v_1, v_2, \ldots, v_n\}$ is called linearly dependent iff one of v_i is a linear combination of others.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ is always linearly dependent if $n > m$.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ is always linearly dependent if $v_i = \underline{0}$ for some i.

$0 v_1 + 0 \cdot v_2 + \cdots + 1 \cdot 0 + \cdots + 0 \cdot v_n = 0$

$v_1 \quad v_2 \quad v_3$

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is L.D.

$0 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3 = 0$

$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ non trivial sol^n

Recall that a matrix A of order $m \times n$ can be thought as a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ in following way:

$$u \to Au$$

$\left\{ v_1 = \begin{bmatrix}1\\5\\7\end{bmatrix}, v_2 = \begin{bmatrix}3\\9\\8\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ is lin. indep.

$x_1 v_1 + x_2 v_2 = 0$

$AX = 0$ wh.

$[A|0] \begin{bmatrix}1 & 3 & 0\\5 & 9 & 0\\7 & 8 & 0\end{bmatrix} \to \begin{bmatrix}1 & 3 & 0\\0 & -6 & 0\\0 & -13 & 0\end{bmatrix}$

$R_3 \to R_3 - \frac{13}{6} R_2$

$\begin{bmatrix}1 & 3 & 0\\0 & -6 & 0\\0 & 0 & 0\end{bmatrix}$

$-6x_2 = 0 \Rightarrow x_2 = 0$

$x_1 + 3x_2 = 0 \Rightarrow x_1 = 0$

$A' = \begin{bmatrix}1 & 3 & 0\\5 & 9 & 0\\7 & 8 & 1\end{bmatrix}$

$A'X = 0$

$A'^{-1}A'X = 0$

$X = 0$

Linear Transformation

Recall that a matrix A of order $m \times n$ can be thought as a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ in following way:

$$u \rightarrow Au$$

We can observe that
$A(u+v) = Au + Av$ for all $u, v \in \mathbb{R}^n$

Linear Transformation

Recall that a matrix A of order $m \times n$ can be thought as a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ in following way:

$$u \to Au$$

We can observe that
$A(u+v) = Au + Av$ for all $u, v \in \mathbb{R}^n$
$A(cu) = c.Au$ for any real number c.
This motivates to define

Definition (Linear Transformation)

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be Linear Transformation if
$T(u+v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$

Linear Transformation

Recall that a matrix A of order $m \times n$ can be thought as a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ in following way:

$$T: \quad u \to Au$$

$V_{n\times1}$

$$\left(A_{m\times n}\, u_{n\times1}\right)_{m\times1}$$

We can observe that
$A(u+v) = Au + Av$ for all $u, v \in \mathbb{R}^n$
$A(cu) = c.Au$ for any real number c.
This motivates to define

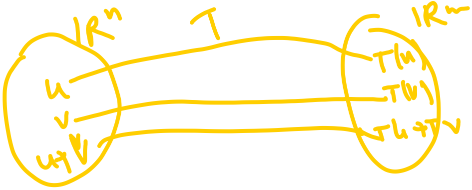
$A(u+v) \overset{?}{=} Au+Av$

Definition (Linear Transformation)

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be Linear Transformation if
linear $T(u+v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$
$T(cu) = c.T(u)$ for any real number c.

$A(B+C) = AB+AC$

$A(cu) = c(Au)$


$\mathbb{R}^n$
T
$\mathbb{R}^m$
u
v
$u+v$
$T(u)$
$T(v)$
$Tu+Tv$

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ as $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x \\ 0 \end{bmatrix}$ → L.T.

Projection onto X axis.

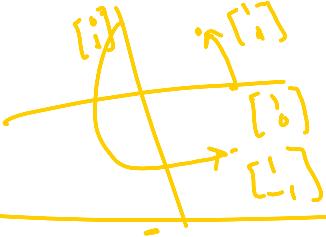
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x+z \\ y+w \end{bmatrix}\right) = \begin{bmatrix} x+z \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ 0 \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix} + T\begin{bmatrix} z \\ w \end{bmatrix}$$

$$T\left(c \cdot \begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} cx \\ 0 \end{bmatrix} = c \cdot \begin{bmatrix} x \\ 0 \end{bmatrix} = c \cdot T\begin{bmatrix} x \\ y \end{bmatrix}$$

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ as $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ scaling by 2.

$$T^{-1}: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+y \\ x-y \end{bmatrix}$$



Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ as $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$

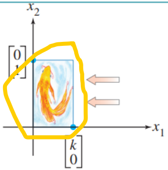
rotation by 45 degree.

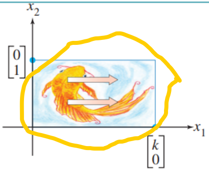
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x+z \\ y+w \end{bmatrix}\right) = \begin{bmatrix} x+z-y-w \\ x+y+z+w \end{bmatrix} = \begin{bmatrix} x-y \\ x+y \end{bmatrix} + \begin{bmatrix} z-w \\ z+w \end{bmatrix}$$

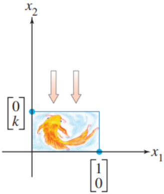
$$= T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + T\begin{bmatrix} z \\ w \end{bmatrix}$$

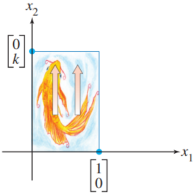
$$T\left(c \cdot \begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} cx - cy \\ cx + cy \end{bmatrix} = c\begin{bmatrix} x-y \\ x+y \end{bmatrix} = c \cdot T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

TABLE 2 Contractions and Expansions

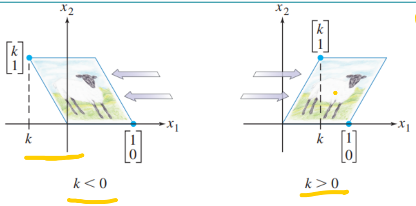
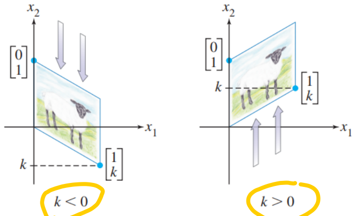
<table>
<tr><th>Transformation</th><th colspan="2">Image of the Unit Square</th><th>Standard Matrix</th></tr>
<tr><td>Horizontal contraction and expansion</td><td>

$0 < k < 1$</td><td>

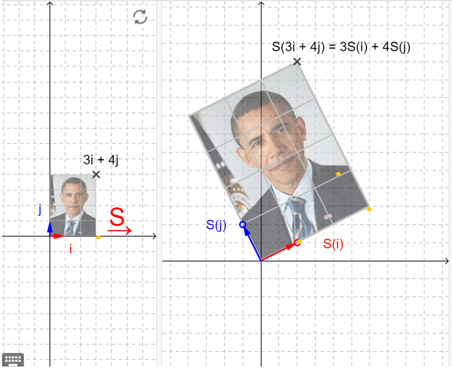
$k > 1$</td><td>$\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}$</td></tr>
<tr><td>Vertical contraction and expansion</td><td>

$0 < k < 1$</td><td>

$k > 1$</td><td>$\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$</td></tr>
</table>

Transformation	Image of the Unit Square	Standard Matrix
Horizontal shear		$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear		$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

$T(u) = Au$



$$S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

$$S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$$

$$\begin{aligned} S\begin{pmatrix}3\\4\end{pmatrix} &= 3\,S\begin{pmatrix}1\\0\end{pmatrix} + 4\,S\begin{pmatrix}0\\1\end{pmatrix} \\ &= 3\begin{pmatrix}2\\1\end{pmatrix} + 4\begin{pmatrix}-1\\2\end{pmatrix} \\ &= \begin{pmatrix}2\\11\end{pmatrix} \end{aligned}$$

Represent S by $\begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}$

$$\begin{aligned} S\begin{pmatrix}3\\4\end{pmatrix} &= \begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} \\ &:= 3\begin{pmatrix}2\\1\end{pmatrix} + 4\begin{pmatrix}-1\\2\end{pmatrix} \end{aligned}$$

Is T([x])=[|x|] a linear transformation from $\mathbb{R} \to \mathbb{R}$? X

$T[x] \longrightarrow [|x|]$

$T([2]) = [2]$

$T([-2]) = [2]$

$T[-2+1] = T([-1]) = [1]$

$\neq$

$T([-2]) + T([1]) = [2] + [1]$

$=$

$[3]$

Is T([x])=[|x|] a linear transformation from $\mathbb{R} \to \mathbb{R}$?

Is $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} sin(x_1) \\ 2x_2 \end{bmatrix})$ a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$? NO.

$$T\left(\begin{bmatrix} \pi/2 \\ 0 \end{bmatrix} + \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix}\right) \neq T\begin{bmatrix} \pi/2 \\ 0 \end{bmatrix} + T\begin{bmatrix} \pi/2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Is T([x])=[$|x|$] a linear transformation from $\mathbb{R} \to \mathbb{R}$?

Is $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} sin(x_1) \\ 2x_2 \end{bmatrix})$ a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$?

How does T look like?

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}_{m\times 1}$$

homo. deg 1 poly in $x_1, x_2, \ldots x_n$

for some real numbers $a_{ij}, 1 \le i \le m, 1 \le j \le n$.

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$$

$1 \to a_{11}$, $a_{12} - -1$, a_{21} 1, a_{22} +1

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}_{m\times 1}$$

for some real numbers $a_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$.

T is represented by a matrix $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x-y \\ x+y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}_{m\times 1}$$

for some real numbers $a_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$.

T is represented by a matrix $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$

This matrix is called standard matrix representation of T.

Question: Given T How to find its matrix representation-$[T]$?

Question: Given T How to find its matrix representation-$[T]$?

Answer: First column of $[T]$ is $T(e_1)$. where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Question: Given T How to find its matrix representation-$[T]$?

Answer: First column of $[T]$ is $T(e_1)$. where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Second column of $[T]$ is $T(e_2)$. where $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Question: Given T How to find its matrix representation-$[T]$?

Answer: First column of $[T]$ is $T(e_1)$. where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Second column of $[T]$ is $T(e_2)$. where $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

So on and last column of $[T]$ is $T(e_n)$. where $e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Properties of L. T.

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto if for each $b \in \mathbb{R}^m$ there exists $u \in \mathbb{R}^n$ such that $T(u) = b$

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto if for each $b \in \mathbb{R}^m$ there exists $u \in \mathbb{R}^n$ such that $T(u) = b$

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one to one if $T(u) = 0$ then u must be 0.

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one to one if $T(u) = 0$ then u must be 0.

Properties of Linear Transformation

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one to one if $T(u) = 0$ then u must be 0.

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one to one if $T(u) = 0$ then u must be 0.

Properties of Linear Transformation

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto/surjective if for each $v \in \mathbb{R}^m$ there exists $u \in \mathbb{R}^n$ such that $T(u) = v$.

Properties of Linear Transformation

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto/surjective if for each $v \in \mathbb{R}^m$ there exists $u \in \mathbb{R}^n$ such that $T(u) = v$.

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be bijective/invertible if T is one to one and onto.

Let T be the linear transformation whose standard matrix is $\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

Let T be the linear transformation whose standard matrix is $\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then: T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.

Let T be the linear transformation whose standard matrix is $\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then: T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
T is one-to-one if and only if the columns of A are linearly independent.

Let T be the linear transformation whose standard matrix is $\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then: T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
T is one-to-one if and only if the columns of A are linearly independent.

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. T is bijective iff $n = m$ and A is an invertible matrix.

Composite Transformation

If T_1, T_2 are two linear transformations then so is composite of T_1, T_2.

Composite Transformation

If T_1, T_2 are two linear transformations then so is composite of T_1, T_2.