MA 201 Practice Problems

Question 1

Let $X \sim Bernoulli\ (n,p)$. Using Markov's inequality, find an upper bound on $P(X>=\alpha n)$ where $p<\alpha<1$. Evaluate the bound for $p = ½, \alpha=¾$

Question 2

Let $X \sim Bernoulli\ (n,p)$. Using Chebyshev's inequality, find an upper bound on $P(X>=\alpha n)$ where $p<\alpha<1$. Evaluate the bound for $p = ½, \alpha=¾$

Question 3

Suppose the expected runtime of QuickSort is $2n \log(n)$ operations/comparisons to sort an array of size n (we can show this using linearity of expectation with dependent indicator variables). Use Markov's inequality to bound the probability that QuickSort runs for longer than $20n \log(n)$ time.

Question 4

a) A coin is weighted so that its probability of landing on heads is 20%, independently of other flips. Suppose the coin is flipped 20 times. Use Markov's inequality to bound the probability it lands on heads at least 16 times.
b) Upper bound the probability it lands on heads at least 16 times out of 20 flips using Chebyshev's inequality

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Question 5

Alice sends a bit to Bob; this is some $X \in \{-1, 1\}$, and the probability of $X = -1$ or 1 is $1/2$ for each. However, the communication channel is noisy - in particular, it introduces some Gaussian noise $N \sim N(0, 1)$ (which is independent from the transmitted bit). Bob then receives $Y = X + N$, and wants to remove the noise and recover the original bit. Bob finds that $Y = y$, for some $y \in R$. Compute the probability $P[X = 1 \mid Y = y]$.

Question 6

Corrupted by their power, the judges running the popular game show America's Next Top Mathematician have been taking bribes from many of the contestants. Each episode, a given contestant is either allowed to stay on the show or is kicked off. If the contestant has been bribing the judges they will be allowed to stay with probability 1. If the contestant has not been bribing the judges, they will be allowed to stay with probability 1/3. Suppose that 1/4 of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds, i.e., if a contestant bribes them in the first round, they bribe them in the second round too (and vice versa).

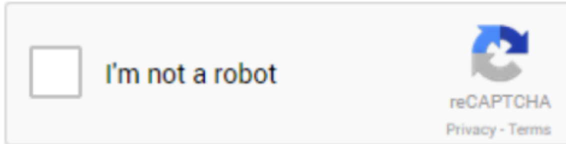
a) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that they were bribing the judges?
b) If you pick a random contestant, what is the probability that they are allowed to stay during both of the first two episodes?
c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that they get kicked off during the second episode?

Question 7

Let X and Y be two random variables and let r, s, t, and u be real numbers.
(a) Show that $\mathrm{Cov}(X + s, Y + u) = \mathrm{Cov}(X, Y)$.
(b) Show that $\mathrm{Cov}(rX, tY) = rt\mathrm{Cov}(X, Y)$.
(c) Show that $\mathrm{Cov}(rX + s, tY + u) = rt\mathrm{Cov}(X, Y)$

Question 8

Based on browser history, Google believes that there is a 0.2 probability that a particular visitor to a website is a robot. They decide to give the visitor a recaptcha (shown below).



Google presents the visitor with a box, 10mm by 10mm. The visitor must click inside the box to show that they are not a robot. You have observed that robots click uniformly in the box. However, the distance location of a human click has X location (mm from the left) and the Y location (mm from the right) distributed as independent normals both with mean $\mu = 5$ and $\sigma^2 = 4$:

a) What is the probability density function of a robot clicking $X = x$ mm from the left of the box and $Y = y$ mm from the top of the box?
b) What is the probability density function of a human clicking $X = x$ mm from the left of the box and $Y = y$ mm from the top of the box?
c) The visitor clicks in the box at (x = 6 mm, y = 6 mm). What is Google's new belief that the visitor is a robot?

Question 9

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} x+y & 0 \le x,y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E[XY^2]$.

Question 10

Using the Cauchy-Schwarz inequality, show that for any two random variables X and Y.

$$|\rho(X,Y)| \le 1.$$

Also, $|\rho(X,Y)| \le 1$, if and only if $Y=aX+b$ for some constants $a,b \in \mathrm{R}$.

Question 11 [Coding Question]

Write MATLAB programs to generate *Geometric(p)* and *Negative Binomial(i,p)* random Variables.

Hint: *Use inverse CDF for Geometric(p) and sum i Geometric(p) variables for Negative Binomial(i,p).*

END