Gaussian Elimination: Matrix form AX=b

Example: $2x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + 2z = 9.$

Note that the last column is the RHS column vector b.

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

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The last matrix corresponds to
$z = 2$,
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Gaussian Algorithm

For Row Echelon Form

1. Find the leftmost nonzero column.
2. Select a nonzero entry in that column (pivot). Bring it on top by interchanging rows
3. Use Elementary row operations to make all entries below that nonzero entry 0.
4. Ignore that row and columns before (including of leading entry) above found column. Apply steps 1-3 for remaining submatrix. Go on till all rows are exhausted.

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   For Reduced Row Echelon Form

5. Make all leading entries 1 by elementary row operation ($R_i \to cR_i$)
6. Make all entries in a column above leading 1 zero by elementary row operation ($R_i \to R_i - cR_j$)

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 \\ 3 & -7 & 8 & -5 \\ 3 & -9 & 12 & -9 \end{bmatrix}$$

Reduced Row Echelon Form (RREF) of A is

$$A' = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Theorem

1. A linear system is inconsistent if and only if REF or RREF of its augmented matrix has a row of the form $0\ 0\ \cdots\ 0\ b$*, for some nonzero* b*.*
2. A linear system is has unique solution if and only if no. of nonzero rows and no. of columns of REF or RREF of its augmented matrix are same.

Triangular factorization: $A = LU$ Given a square matrix A, we can find L a lower Triangular matrix with 1s on the diagonal and U upper Triangular matrix such that $A = LU$.

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

Calculation of A^{-1}: Gauss-Jordan Method

Let $AA^{-1} = I$
If X_1 is a first column of A^{-1} then $AX_1 = e_1$, where
$e_1 = [1\ 0\ \cdots\ 0]^T$.
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Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$.
Consider

$$[A|I] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{array}\right]$$

Perform elementary row operations on $[A|I]$ to convert A into I.
Then I will change to A^{-1}.

Vectors in $\mathbb{R}^n$:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ is a vector in } \mathbb{R}^2.$$

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \text{ is a vector in } \mathbb{R}^3.$$

Vector addition and scalar multiplication are done like matrices.

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

Linear Combinations

Consider a linear system

$$
\begin{aligned}
x_2 - 4x_3 &= 8 \\
2x_1 - 3x_2 + 2x_3 &= 1 \\
5x_1 - 8x_2 + 7x_3 &= 1
\end{aligned}
$$

It can be written as

$$x_1 \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -3 \\ -8 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}$$

LHS of $=$ is called linear combination of

$$u = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -3 \\ -8 \end{bmatrix} \;\&\; w = \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix}$$

Above equation is called vector equation.

Linear system and Vector Equation

A linear system $AX = b$ is same as $x_1 C_1 + x_2 C_2 + \cdots + x_n C_n = b$ where C_i is i^{th} column of A.

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Hence we need to study linear combination of columns of A.
Column space of A= set of all linear combinations of columns of A.

$$Col(A) = \{x_1C_1 + x_2C_2 + \cdots + x_nC_n | x_i \in \mathbb{R} \text{ for all } i\}$$

$\therefore$ $AX = b$ has a solution iff $b \in Col(A)$.

If $v_1, v_2, ..., v_p \in \mathbb{R}^n$, then
Span $\{v_1, v_2, ..., v_p\} = \{c_1 v_1 + c_2 v_2 + ... + c_p v_p | c_1, c_2, ..., c_p \in \mathbb{R}\}$,
the set of all linear combinations of $v_1, v_2, ..., v_p$.

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the set of all linear combinations of $v_1, v_2, ..., v_p$.
Let

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Span $\{v\} = \{c.v | c \in \mathbb{R}\} = \{\begin{bmatrix} c \\ 0 \end{bmatrix} | c \in \mathbb{R}\}$

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Span $\{v\} = \{c.v | c \in \mathbb{R}\} = \{\begin{bmatrix} c \\ 0 \end{bmatrix} | c \in \mathbb{R}\}$ Let $u \in \mathbb{R}^n$. Then $span\{u\} = \{x.u | x \in \mathbb{R}\}$.
Geometrically it is a line passing through origin and point u.
Let $u, v \in \mathbb{R}^2$. Then $span\{u, v\} = \{x.u + y.v | x, y \in \mathbb{R}\} = \mathbb{R}^2$
Geometrically, what will it be?

Let $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$ $\quad Span\{v\} \subseteq Span\{u,v\}$

$xu + yv = b$ $\quad \left[\begin{array}{cc|c} 2 & 1 & b_1 \\ 3 & 0 & b_2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 1 & b_1 \\ 0 & -\frac{3}{2} & b_2 - \frac{3}{2} b_1 \end{array}\right]$

If $v_1, v_2, ..., v_p \in \mathbb{R}^n$, then
Span $\{v_1, v_2, ..., v_p\} = \{c_1v_1 + c_2v_2 + ... + c_pv_p | c_1, c_2, ..., c_p \in \mathbb{R}\}$, the set of all linear combinations of $v_1, v_2, ..., v_p$.
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Geometrically, what will it be?
For a matrix A,
$Col(A) := span\{C_1, C_2, \ldots, C_n\} \subseteq \mathbb{R}^m$

The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A.

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Theorem

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent

1. For each $b \in \mathbb{R}^m$, the equation $AX = b$ has a solution.

2. Each $b \in \mathbb{R}^m$, is a linear combination of the columns of A.

3. $Col(A) = \mathbb{R}^m$.

4. A has a pivot position in every row.

Homogeneous Linear System

$\left\{ c' \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \middle| c' \in \mathbb{R} \right\}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left\{ c \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \middle| c \in \mathbb{R} \right\}$

The homogeneous equation $AX = 0$ has a nontrivial solution if and only if the REF(A) has at least one free variable.

Suppose augmented form $[A|0]$ is

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right]$$

$x_1 = \frac{4}{3}c$; $x_2 = 0$

$x_3 = c$

$X = 0$

$A(0) = 0$

$R_3 \to R_3 + 3R_2 \left[\begin{array}{ccc|c} \boxed{3} & 5 & -4 & 0 \\ 0 & \boxed{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow$ 0 is always a sol^n to $AX = 0$

x_1, x_2 are dependent variables

x_3 is free variable.

$3x_2 = 0$, $3x_1 + 5x_2 - 4x_3 = 0$

$x_3 = c$, $x_1 = \frac{4}{3}c$

Homogeneous Linear System

sol^n space for $AX=0$ is span$\{u, v\}$

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Suppose augmented form $[A|0]$ is

$$\left[\begin{array}{rrr|r} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

$AX = 0$

$$c\underbrace{\begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{u} + d\underbrace{\begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}}_{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9c+8d \\ 4c-5d \\ c \\ d \end{bmatrix}$$

x_3, x_4 free, $x_3 = c$, $x_4 = d$

$x_2 - 4x_3 + 5x_4 = 0 \Rightarrow x_2 = 4c - 5d$

$x_1 + 3x_2 - 3x_3 + 7x_4 = 0$

$x_1 = -3(4c-5d) + 3c - 7d$
$= -9c + 8d$

Non homogeneous linear system

Theorem

Suppose the equation $Ax = b$ is consistent for some given b, and let p be a solution (i.e., $Ap = b$). Then the solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$, where v_h is any solution of the homogeneous equation $Ax = 0$.

$Ap = b$

$Aq = b$

$A(p-q) = 0$

Let q be any soln to $Ax = b$

$A(q-p) = 0$, $\underbrace{q-p}_{v_h} \in$ Soln $(Ax=0)$

$q = p + v_h$

Non homogeneous linear system

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WRITING A SOLUTION SET (OF A CONSISTENT SYSTEM) IN PARAMETRIC VECTOR FORM

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation. dependent.
3. Write a typical solution **x** as a vector whose entries depend on the free variables, if any.
4. Decompose **x** into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Write down the solutions of $AX = 0$ in parametric form where $A =$

$$\left[\begin{array}{cccccc|c} 1 & -4 & -2 & 0 & 3 & -5 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_5 = 4x_6 + 3$, $x_3 = x_6 + 2$, $x_4 = 4x_2 + 2x_3 - 3x_5 + 5x_6 + 1$

$x_2 = c$, $x_4 = d$, $x_6 = e$

$Ax = b$

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + e\begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix} + c\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

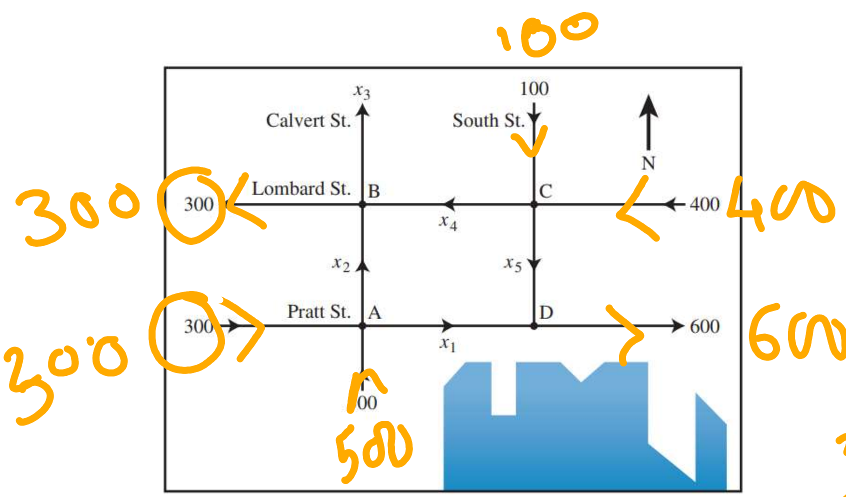
$$\begin{bmatrix} 4c - 5e \\ c \\ e \\ d \\ 4e \\ e \end{bmatrix} = e\begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix} + c\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Chemical Equations

Consider a chemical reaction of propane gas with oxygen to form carbon dioxide and water.
$x_1 C_3H_8 + x_2 O_2 = x_3 CO_2 + x_4 H_2O$,
where x_i are the no. of molecules

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Network flow



$x_1+x_2=800$

$x_2+x_4-x_3=300$

$x_4+x_5=500$

$x_4+x_5=600$

$x_3+900=1300$