08 February 2024 11:42

Set of all lin. combinations of col. of A

$$\text{Col}(A) = \{ b \mid b = Ax \text{ for some } x \}$$

$$\{ x \mid Ax = 0 \}$$

Suppose $Ax_1 = b_1$ & $Ax_2 = b_2$

$\Rightarrow b_1, b_2 \in \text{Col}(A)$

Check $b_1 + b_2 = A(x_1 + x_2)$

$\Rightarrow b_1 + b_2 \in \text{Col}(A)$.

$\alpha b_1 = A(\alpha x_1) \quad \forall \alpha \in \mathbb{R}$

$\Rightarrow \alpha b_1 \in \text{Col}(A)$.

Null(A)

$\{ x \mid Ax = 0 \}$

$Ax_1 = 0 = Ax_2$

$A(x_1 + x_2) = 0$

$A(\alpha x_1) = 0$

Col(A) & Null(A) are closed under addition & sclar multiplication

Subspace of $\mathbb{R}^n$

$V \subseteq \mathbb{R}^n$ is called subspace of $\mathbb{R}^n$

if 1) $0 \in V$

2) $v_1, v_2 \in V \Rightarrow v_1 + v_2 \in V$

3) $\alpha v_1 \in V \quad \forall \alpha \in \mathbb{R}$

eg. Col(A), Null(A), $\mathbb{R}^n$, $\left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$

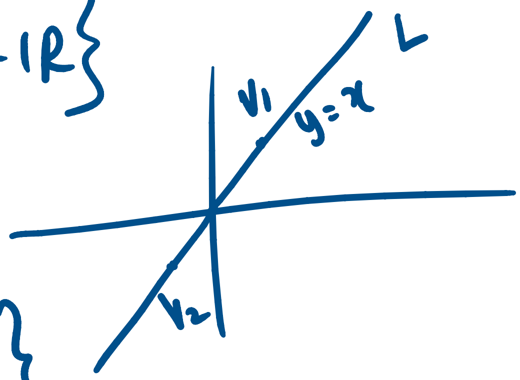
eg. Col(A), Null(A), $\mathbb{R}^n$, $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$

trivial subspaces of $\mathbb{R}^n$.

Basis of a subspace

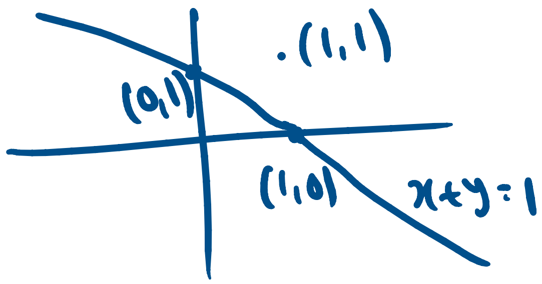
$L = \left\{\alpha\begin{bmatrix}1\\1\end{bmatrix} \middle| \alpha \in \mathbb{R}\right\}$

basis of $L = \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$



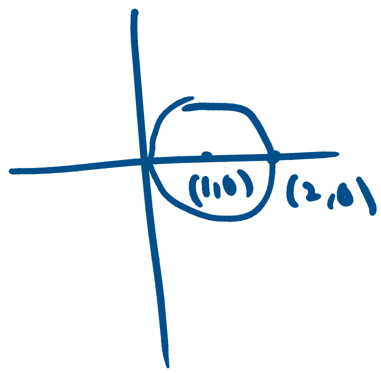
$L_2 = \left\{\begin{bmatrix}x\\y\end{bmatrix} \middle| x+y=1\right\}$

is not a subspace



$C = \left\{\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2 \middle| (x-1)^2+y^2=1\right\}$

$\begin{bmatrix}2\\0\end{bmatrix}+\begin{bmatrix}2\\0\end{bmatrix} \notin C$



Basis of subspace H

$\{v_1,..v_k\}$ = Collection of elements in H which is linearly independent & span H

$\alpha_1 v_1 + \alpha_2 v_2 \dots + \alpha_k v_k = 0$ has only trivial sol^n.

[...] $Ax=0$ has only ... sol^n.

For $A = [v_1\ v_2\ \dots\ v_k]$ $Ax = 0$ has only trivial sol^n.

Span H : for any $b \in H$
$Ax = b$ is consistent.

e.g. $H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, \underline{x+y+z=0} \right\}$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$v_5 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$H = \{ \underset{\mathbb{R}^3}{x} \mid \underline{[1\ 1\ 1]_{1\times 3}}\, x = 0 \}$

$= \underline{\text{Null}(A)}$ wh $A = [\boxed{1}\ 1\ 1] = \text{REF}(A)$

$\cancel{x_1}$
$x_2 = c$
$x_3 = d$
$x_4 = -c-d$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -c-d \\ c \\ d \end{bmatrix} = c\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \underline{\text{Span}\{v_1, v_2\} = H}$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Bx = 0 \Rightarrow x = 0$$

$$\text{REF}(B) = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ col^m of B are lin. indep.

$\{v_1, v_2\}$ is lin. indep. set

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$\Rightarrow$ Basis of $H = \{v_1, v_2\}$

dimension of $H = 2$

$$H_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \underset{\begin{bmatrix} 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0}{\underset{\shortparallel}{4x+3y+2z=0}} \right\}$$

$$= \text{Null}\left(\begin{bmatrix} 4 & 3 & 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} \frac{1}{4}(-3c-2d) \\ c \\ d \end{bmatrix} = c \underbrace{\begin{bmatrix} -3/4 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + d \underbrace{\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$$

$H_1 = \text{Span}\{v_1, v_2\}$

$$B_1 = \begin{bmatrix} -3/4 & -1/2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} -3/4 & -1/2 \\ 0 & -2/3 \\ 0 & 0 \end{bmatrix}$$

$-\frac{4}{3 \cdot 2}$

No free variable

$\Rightarrow$ col^{ns} of B_1 are lin. indep.

Basis of $H_1 = \left\{ \begin{bmatrix} -3/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$

ex

$$H_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{matrix} x_1 + x_2 + x_3 = 0 \\ 2x_1 + x_4 = 0 \end{matrix} \right\}$$

$H_2 = \text{Null}(A_2)$, ~~wh~~ $A_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

$$\text{REF}(A_2) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -d/2 \\ -\frac{1}{2}(2c-d) \\ c \\ d \end{bmatrix} = c\underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + d\underbrace{\begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$$

$$v_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & -1/2 \\ -1 & 1/2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{Null}(B_2) = \{0\}$$

$\Rightarrow \{v_1, v_2\}$ is L.I.

$$\text{Basis of } H_2 : \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\dim(H_2) = 2$$

$A_{m\times n}$ $\quad$ $\text{Col}(A)$

$\text{Rank}(A) = \dim(\text{Col}(A))$

$\text{Nullity}(A) = \dim(\text{Null}(A))$

$\text{Rank}(A) + \text{Nullity}(A) = n =$ no. of columns of A

Rank-Nullity thm

Find rank & Nullity of A

wh $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$

wh $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$