# Determinant & It's Properties

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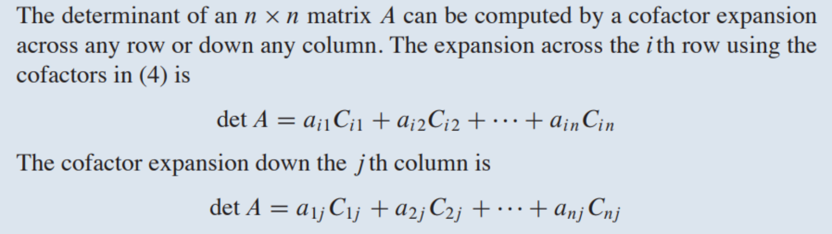
where $A_{ij}$ is the matrix obtained from $A$ by deleting $i^{th}$ row and $j^{th}$ column.

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a(d) - b(c)$$

Let $A = [a_{ij}]$ be a $n \times n$ matrix. Define cofactors as, for each $i, j$,

$$C_{ij} := (-1)^{i+j} det(A_{ij})$$

### Theorem



The determinant of an $n \times n$ matrix $A$ can be computed by a cofactor expansion across any row or down any column. The expansion across the $i$th row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the $j$th column is

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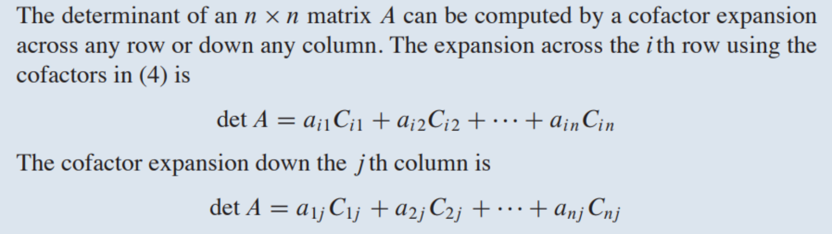

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## Corollary

$$det(A) = det(A^T)$$

**Theorem**

*If $A$ is a triangular matrix then $\det(A)$ is the product of diagonal entries.*

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Note: If a computer performs one trillion multiplications per second, it would have to run for more than 500,000 years to compute determinant of a $25 \times 25$ matrix.
We need better "definition" or properties of determinant to calculate determinant easily.

Let's start with new definition of determinant:

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### Definition (Determinant)

Let $\mathbb{M}_n(\mathbb{R})$ denote the set of all $n \times n$ matrix with entries from $\mathbb{R}$. Then det: $\mathbb{M}_n(\mathbb{R}) \to \mathbb{R}$ is a function which satisfies
(1) For each $i$, det is a linear function of the $i^{th}$ row, when other $n-1$ rows are fixed.

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Verify your definition of determinant satisfies all 3 properties.

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Another definition of determinant: Let $A = [a_{ij}]$ be a $n \times n$ matrix.

$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \underline{a_{1\sigma(1)}}\, \underline{a_{2\sigma(2)} \cdots} a_{n\sigma(n)}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \det(A) = \underline{a_{11}a_{22}} - \underline{a_{12}a_{21}}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$\sigma : \{1,2\} \to \{1,2\}$

$\sigma(1)=1 \quad \sigma(2)=2$

$\sigma' : \{1,2\} \to \{1,2\}$

$\sigma'(1)=2 \quad \sigma'(2)=$

# Applications of determinant

$(1,2)$ ✓ $(1,3)$ X $(2,3)$ X $\quad \mathrm{sign}(\sigma) = (-1)^1 = -1$

~~Area/Volume~~:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$\sigma : \{1,2,3\} \longrightarrow \{1,2,3\}$ bijective fun$^n$

$$\underset{id.}{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}} \quad \underset{\sigma_1}{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}} \quad \underset{\sigma_2}{\begin{pmatrix} x & & y \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}} \quad \underset{\sigma_3}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}} \quad \underset{\sigma_4}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}} \quad \underset{\sigma_5}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}} \begin{matrix} \rightarrow \text{domain} \\ \rightarrow \text{codom} \end{matrix}$$

$$\mathrm{sign}(\sigma) := (-1)^r \quad \text{wh } r = \#\left\{ (x,y) \,\middle|\, \begin{matrix} x < y \\ \sigma(x) > \sigma(y) \end{matrix} \right\}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

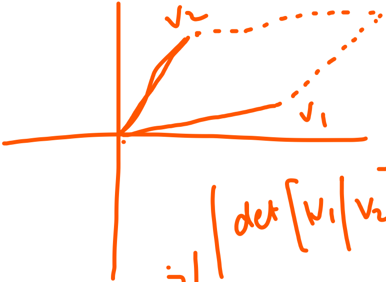
$$\begin{pmatrix} x & y \\ 1, & 2 \\ 3 & 1 \end{pmatrix} ✓ \quad \begin{pmatrix} x & y \\ 1, & 3 \\ 3 & 2 \end{pmatrix} ✓ \quad \begin{pmatrix} x & y \\ 2, & 3 \\ 1 & 2 \end{pmatrix} X$$

$$\underbrace{(-1)^2 = 1}_{\mathrm{sign}(\sigma)}$$

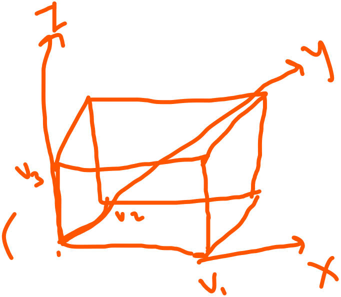
# Applications of determinant

Area/Volume:

(1) If $A$ is a $n \times n$ matrix with columns denoted as $C_1, \ldots, C_n$. Then $|det(A)|$ gives the volume of a box formed by $C_1, \ldots, C_n$.



$$\left|\det\left[v_1 \middle| v_2\right]\right|$$



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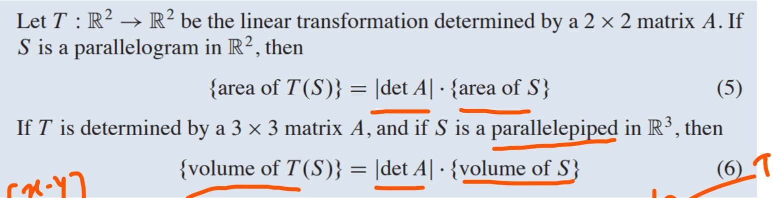


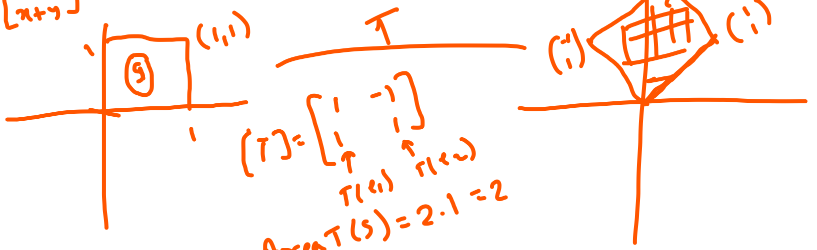
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a $2 \times 2$ matrix $A$. If $S$ is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = \underline{|\det A|} \cdot \{\underline{\text{area of } S}\} \tag{5}$$

If $T$ is determined by a $3 \times 3$ matrix $A$, and if $S$ is a parallelepiped in $\mathbb{R}^3$, then

$$\{\underline{\text{volume of } T(S)}\} = \underline{|\det A|} \cdot \{\underline{\text{volume of } S}\} \tag{6}$$

$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$



$[T] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$T(e_1)$ $T(e_2)$

Area $T(S) = 2.1 = 2$

# Applications of determinant

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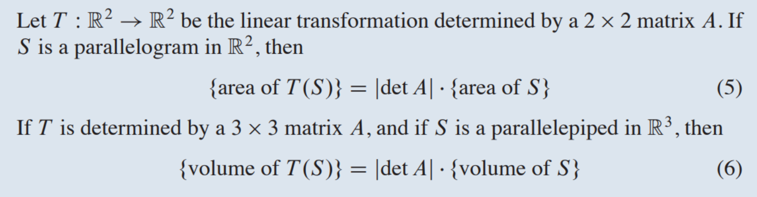

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$$A^{-1} = \frac{1}{\det(A)}[c_{ij}]^T,$$

where $c_{ij} = (-1)^{i+j} det(A_{ij})$ and $A_{ij}$ is a matrix obtained from $A$ by deleting $i^{th}$ row and $j^{th}$ column.

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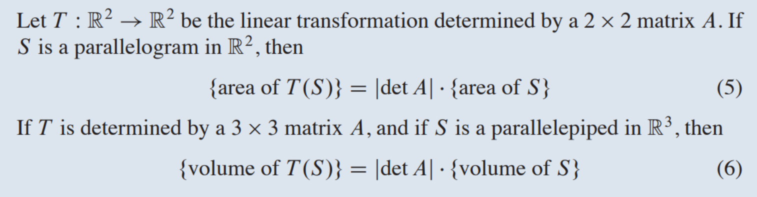

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(3) It also helps you to find solution to the linear system $AX = b$, when $A$ is invertible matrix.

# Determinant and REF

$\det(E_k) \cdots \det(E_1) \cdot \det(A) = \det(U)$

$E_k \cdots E_2 E_1 A = U$

RREF

Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges only.

$\det(A) = \det(E_1^{-1}) \cdots \det(E_k^{-1}) \det(U)$

$R_i \to cR_i$ X

$R_i \leftrightarrow R_j$

$R_i \to R_i + cR_j$ E

det $-1$

det $1$

$R_i \to R_i - cR_j$ $E^{-1}$

inverse of E corresp to $R_i \leftrightarrow R_j$ is E $(E^2 = Id.)$

# Determinant and REF

Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges only.
Then

$$\boxed{det(A) = (-1)^r det(U),}$$

efficient way to
cal. det(A).

where $r$ is the no. of row interchanges required to get $U$ =REF(A).

# Determinant of Block diagonal matrices

Let $A = \begin{bmatrix} A_1 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & A_r \end{bmatrix}$ be a matrix in block diagonal form, where $A_i$ are submatrices of $A$.

non-zero

$A_1$

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} A_2 \longrightarrow \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix}$$

$\det(A_1)$ $\quad \downarrow \det(A_2)$.

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Then

$$det(A) = det(A_1).det(A_2) \cdots det(A_r)$$

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Let $X = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$. Then

$det(X) = \underline{det(A).det(D)}$

# Determinant of Block diagonal matrices

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Then

$$det(A) = det(A_1).det(A_2)\cdots det(A_r)$$

Let $X = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$. Then

$det(X) = det(A).det(D)$

$X$ [illegible] $= \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} \cdot \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$

Therefore, $det(X) = det(A).det(D)$.

If $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $A$ is invertible then

$$det(X) = det(A).det\underline{(D - CA^{-1}B)}$$

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$$det(X) = \underline{det(A).det(D - CA^{-1}B)}$$

$$X = \underline{\begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \cdot \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}}$$

If $D$ is $\underline{\text{invertible}}$ then

$$det(X) = \underline{det(D).det(A - BD^{-1}C)}$$

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$C$

$$X = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}$$

$1 \qquad 1$

$$X = \begin{array}{c} A \\ \\ C \end{array} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 3 & 4 & 0 & 1 \\ \hline 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right] \begin{array}{c} B \\ \\ D \end{array}$$

$$det(X) = -2\, det\left( \nearrow \right)$$

$$D - CA^{-1}B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \left( -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$