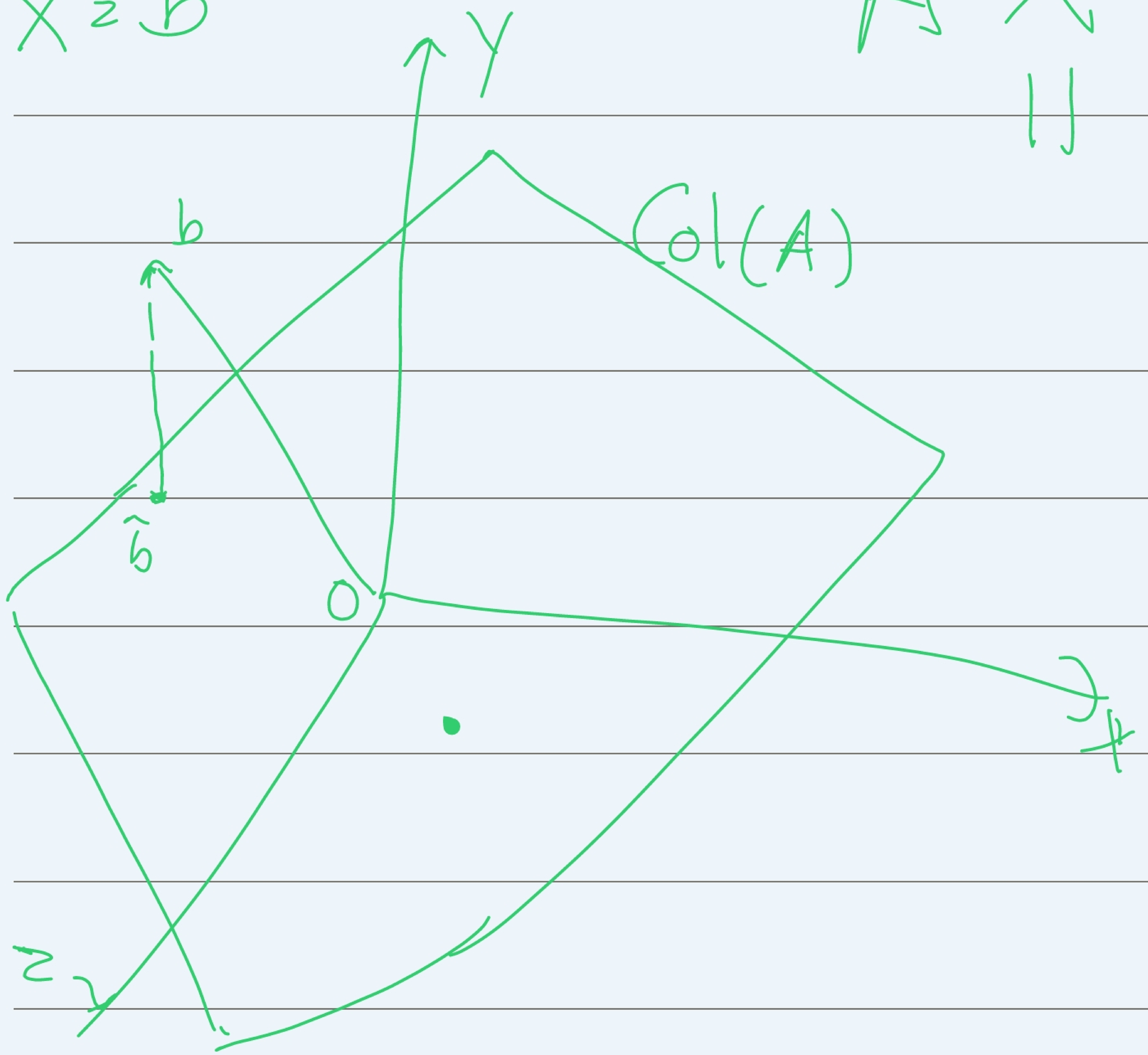
Inconsistency

$A x = b$ does not have sol^n.

$\Updownarrow$

$b \notin Col(A)$

$A x = \hat{b}$



Col(A) = Set of all linear combinations of columns of A.

Example

Col(A) = X-Y plane

$$\underset{3\times 3}{A}\, X = b_{3\times 1}$$

$$b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{Take } \hat{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = (e_1^T b) \cdot e_1 + (e_2^T b) \cdot e_2$$

$$a = \begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ a_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

Innerproduct: $a \cdot b = a^T b = \sum_{i=1}^{n} a_i b_i$

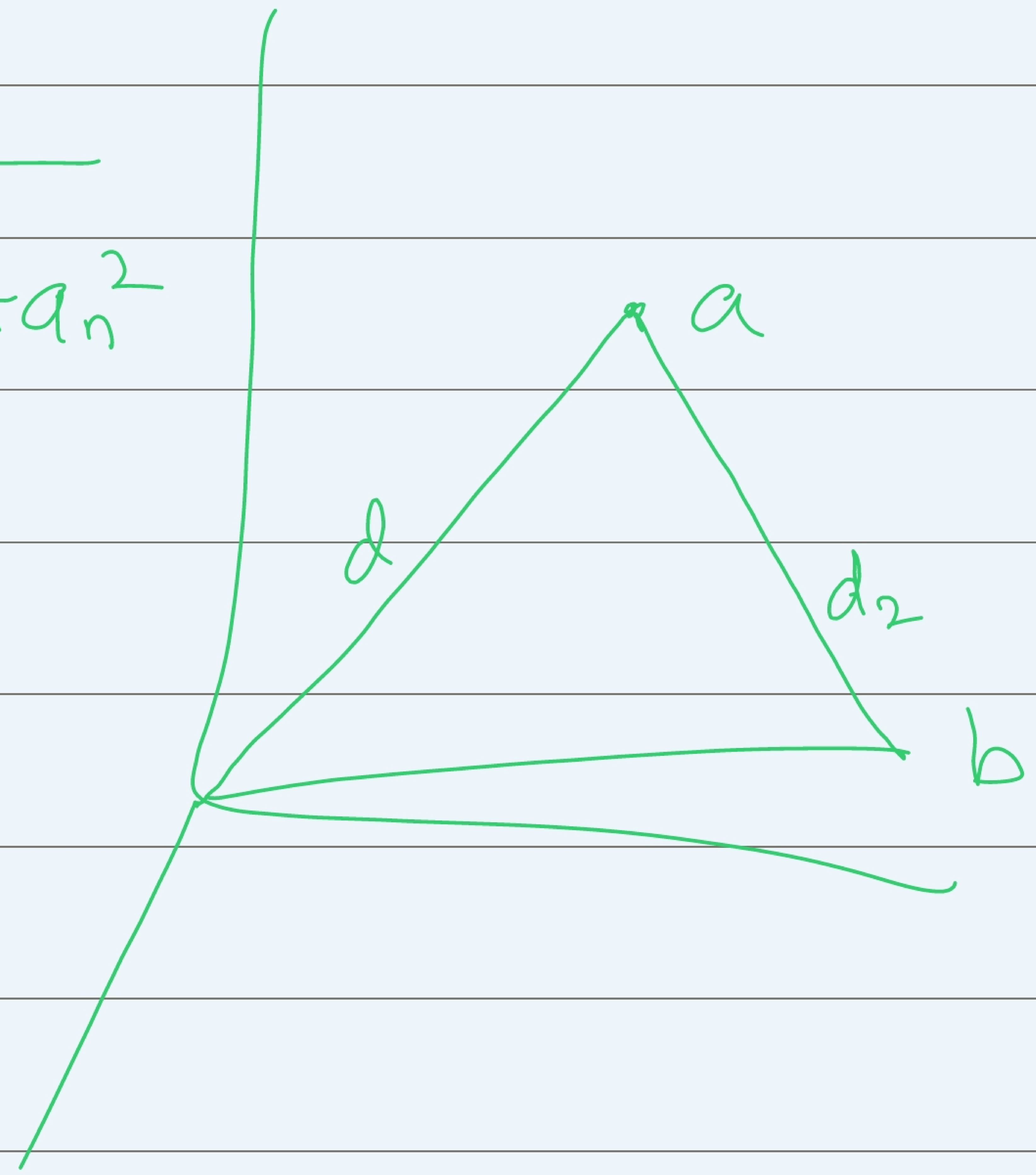
$$(\alpha a_1 + a_2) \cdot b = \alpha a_1 \cdot b + a_2 \cdot b$$

$\boxed{\text{Def}^n \quad a \cdot b = 0 \text{ then } a \,\&\, b \text{ are called orthogonal}}$

$$a \cdot a \geq 0, \quad a \cdot a = 0 \Longleftrightarrow a = 0$$

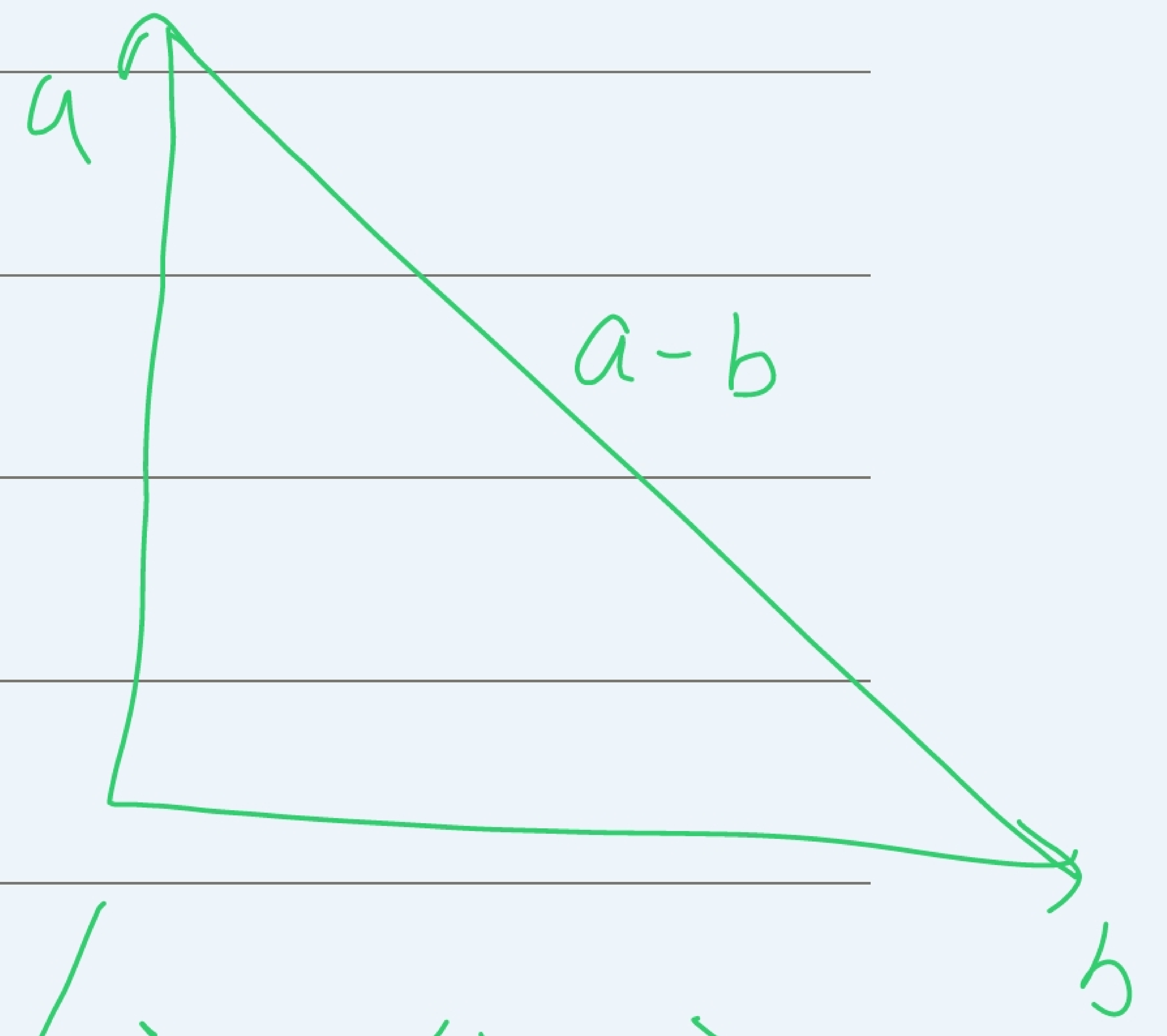
$$d = +\sqrt{a \cdot a} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

$$d(a,b) = \|a-b\|_2 = d_2 = \sqrt{(a-b)\cdot(a-b)}$$



Orthogonal : $\overset{\langle a, b \rangle}{\cancel{a \cdot b}} = 0$

$$\|a\|^2 + \|b\|^2 = \underline{\|a-b\|^2}$$



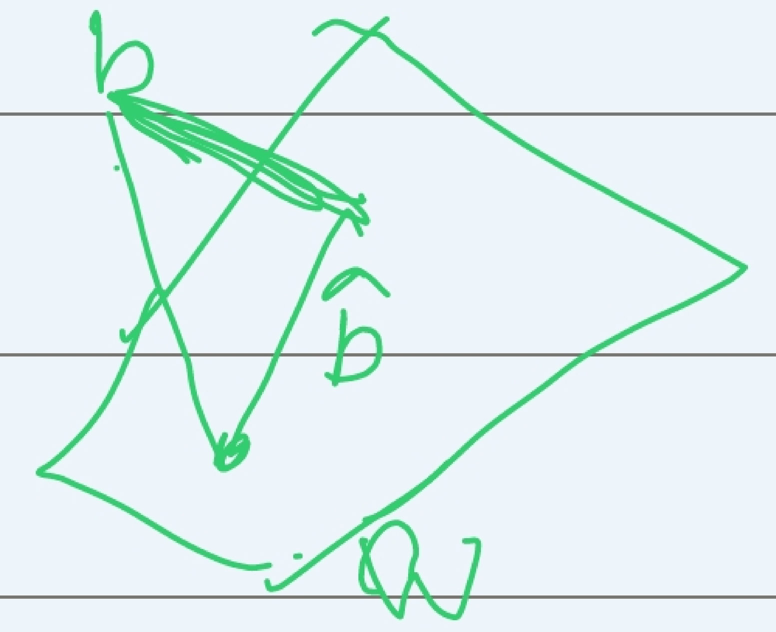
$$\text{RHS} = \langle a-b, a-b \rangle = \langle a, a-b \rangle - \langle b, a-b \rangle$$

$$= \underbrace{\langle a, a \rangle}_{\|a\|^2} - \underbrace{\cancel{\langle a, b \rangle}}_{0} - \underbrace{\cancel{\langle b, a \rangle}}_{0} + \underbrace{\langle b, b \rangle}_{\|b\|^2}$$

$W = \mathbb{R}^3 \subseteq \mathbb{R}^4$

$W^\perp = L$

$$W \oplus W^\perp = \mathbb{R}^n$$



$\mathbb{R}^n \supseteq W = \mathrm{Col}(A)$

$W = X$ axis in $\mathbb{R}^2$

$= \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \middle/ a \in \mathbb{R} \right\}$

$W^\perp = \left\{ \begin{bmatrix} 0 \\ b \end{bmatrix} \middle/ b \in \mathbb{R} \right\}$

$W^\perp = \{ u \in \mathbb{R}^n \;/\; \langle u, w \rangle = 0 \;\; \forall\, w \in W \}$

$0 \in W^\perp$

If $u_1, u_2 \in W^\perp$ $\quad u_1 + u_2 \in W^\perp$

$\langle u_1 + u_2, w \rangle = \langle u_1, w \rangle + \langle u_2, w \rangle$

$\mathrm{Col}(A)$

$W = \mathrm{Row}(A)$

$W^{\perp} = \mathrm{Null}(A) = \{u \mid Au = 0\}$

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}, \qquad Au = \begin{bmatrix} \langle R_1, u \rangle \\ \langle R_2, u \rangle \\ \vdots \\ \langle R_m, u \rangle \end{bmatrix}$$

$\mathrm{Col}(A)^{\perp} = \mathrm{Null}(A^T)$

Orthogonal basis

• $\underline{AA^T = I}$

$\{u_1, \dots, u_m\}$ is called orthogonal if $\langle u_i, u_j \rangle = 0$
$\forall\ i \neq j$

$\{\qquad\}$ orthonormal if ——— + $\langle u_i, u_i \rangle = 1$

$\{u_1, \dots, u_m\}$ is called orthogonal basis of ~~$\mathbb{R}$~~ W
if $\{\qquad\}$ is orthogonal & basis of W.

$W = \mathbb{R}^2$

$\|u_i\| = 1 \ \forall i$

$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}}$ is $\boxed{\text{o.n.b}}$ (also o.g.b) of $\mathbb{R}^2$

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is o.g.b of $\mathbb{R}^2$

$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^2$

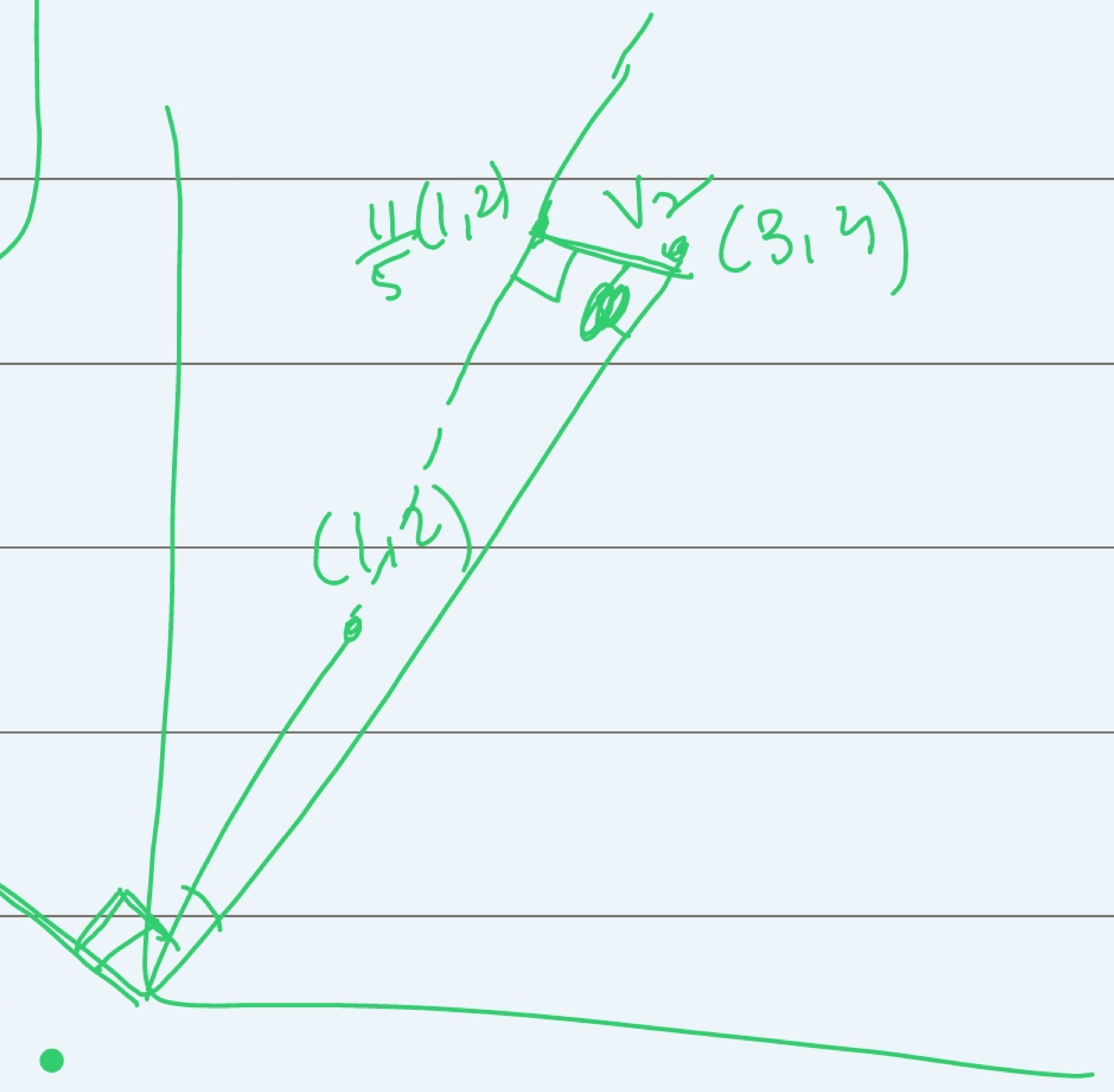
$$\underset{u_1}{\begin{bmatrix}1\\2\end{bmatrix}},\ \underset{u_2}{\begin{bmatrix}3\\4\end{bmatrix}} \longrightarrow \begin{bmatrix}1\\2\end{bmatrix},\ \begin{bmatrix}4/5\\-2/5\end{bmatrix}$$

O.g.b

$$\langle u_1, v_2\rangle = 0$$

$$v_2 = u_2 - \frac{\langle u_2, u_1\rangle}{\langle u_1, u_1\rangle}\cdot u_1 \in \operatorname{Span}\{u_1, u_2\}$$

$$\begin{bmatrix}3\\4\end{bmatrix} - \frac{11}{5}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}4/5\\-2/5\end{bmatrix}$$



$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\mathbb{R}^3$$

$$\underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{V_1}, \underbrace{\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix}}_{V_2}, \underbrace{\begin{bmatrix}-2/3\\2/3\\2/3\end{bmatrix}}_{V_3} \leftarrow \underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{u_2}, \underbrace{\begin{bmatrix}0\\1\\1\end{bmatrix}}_{u_3} \text{ ~~is a~~ basis of } \mathbb{R}^3$$

$$\det\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix} = -2$$

$V \in \operatorname{Span}\{u_1, u_2, u_3\}$

$[V] = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$ s.t.

$V = c_1u_1 + c_2u_2 + c_3u_3$

$$V_2 = u_2 - \frac{\langle u_2, u_1\rangle}{\langle u_1, u_1\rangle}\cdot u_1 = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\1\end{bmatrix}$$

$$V_3 = u_3 - \frac{\langle u_3, u_1\rangle}{\langle u_1, u_1\rangle}\cdot u_1 - \frac{\langle u_3, V_2\rangle}{\langle V_2, V_2\rangle}\cdot V_2$$

$$= \begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1/2}{3/2}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix} = \begin{bmatrix}-1/2-1/6 = -2/3\\1-1/2+1/6 \quad 2/3\\2/3\end{bmatrix} = \begin{bmatrix}-2/3\\2/3\\2/3\end{bmatrix}$$

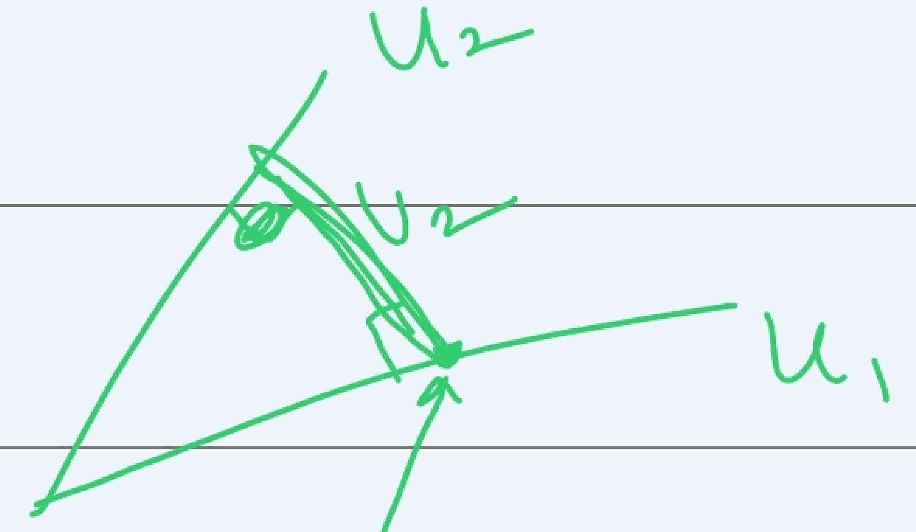
$$\underset{v}{\begin{bmatrix}3\\4\end{bmatrix}} = \underset{\langle v, e_1\rangle}{\underset{\parallel}{3}} \cdot \begin{bmatrix}1\\0\end{bmatrix} + \underset{\langle v, e_2\rangle}{\underset{\parallel}{4}} \cdot \begin{bmatrix}0\\1\end{bmatrix}$$

If $\underline{\{v_1, \ldots, v_m\}}$ is o.g.b. of W

& $w \in W$ then

$$\therefore w = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m \quad \text{where}$$

$$c_i = \frac{\langle w, v_i\rangle}{\langle v_i, v_i\rangle}$$



$$v_2 = \left\langle u_2 - \frac{\langle u_2, u_1\rangle}{\langle u_1, u_1\rangle} \cdot u_1 , u_1 \right\rangle$$

$$\langle u_1, v_2 \rangle = \langle u_2, u_1 \rangle - \langle u_1, u_1 \rangle$$