MA 201: Probability and Statistics
Tutorial 03

Important Instructions:

- Submit a separate notebook dedicated solely to tutorials to the faculty. Loose sheets will not be accepted.
- Ensure all numerical answers are rounded to three decimal places, for example, 0.1385 becomes 0.139.
- Submit the coding assignment on the same day in MATLAB, uploading only .m files. Other file formats will not be accepted.

Coding Question (At last)

Question 1

a) Let X ~ Bernoulli(p). Compute Var(X).
b) Let Y ~ Bin(n, p). Show Var(Y) = $n\,p(1 - p)$.
c) Suppose $X1$, $X2$, … , Xn are independent and all have the same standard deviation σ = 2. Let X be the average of $X1$, … , Xn. What is the standard deviation of X?

Question 2

a) Suppose X ~ Bernoulli(p). Find $E[X]$. (This is important! Remember it!)
b) Suppose $Y = X1 + X2 + … + X12$, where each Xi ~ Bernoulli(0.25). Find $E[Y]$.

Question 3

In a parallel universe, two powerful creatures, Xorak and Yvandor, control two worlds. The values of X and Y represent the energy levels of their worlds. These energy levels are given by the following joint probability mass function (PMF):

$$p(x,y) = \begin{cases} \frac{2}{15} & \text{for } (x,y) = (1,1) \\ \frac{3}{15} & \text{for } (x,y) = (1,2) \\ \frac{4}{15} & \text{for } (x,y) = (2,1) \\ \frac{6}{15} & \text{for } (x,y) = (2,2) \\ 0 & \text{otherwise} \end{cases}$$

The creatures have discovered that understanding their world's energy will give them an edge in future battles, but the calculations are tricky.

a) Compute the expected energy level E(X) and E(Y).
b) Find the variance Var(X) and Var(Y).
c) Compute the covariance COV(X,Y) and the correlation coefficient between X and Y.
d) Calculate the joint expectation E(XY).
e) Suppose Xorak uses a Bernoulli random variable Z with probability p=0.7 to decide whether to increase his world's energy. If X is the energy level of Xorak's world, calculate the expected value and variance of X given that Z=1.

Question 4

A quality control engineer tests the quality of produced computers. Suppose that 5% of computers have defects, and defects occur independently of each other.

a) Find the probability of exactly 3 defective computers in a shipment of twenty.
b) Find the probability that the engineer has to test at least 5 computers in order to find 2 defective ones.

Question 5

A group of adventurers is searching for a treasure buried on a mysterious island. The coordinates of the treasure, X and Y, are continuous random variables that follow the joint probability density function (PDF):

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{8}e^{-\frac{x}{2}}e^{-y} & \text{for } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

The map they found uses transformed coordinates U and V, where:

- U = the square of the distance from the origin
- V = the ratio of the two coordinates (Y/X)

The adventurers need to interpret the transformed coordinates to find the treasure. Find the Jacobian matrix for the transformation (X,Y)→(U,V).

Question 6

Let X and Y be the number of hardware failures in two computer labs in a given month. The joint distribution of X and Y is given in the table below.

			x	
$\mathbf{P_{X,Y}(x,y)}$		**0**	**1**	**2**
	0	0.52	0.20	0.04
y	**1**	0.14	0.02	0.01
	2	0.06	0.01	0

a) Compute the probability of at least one hardware failure.
b) From the given distribution, are X and Y independent? Why or why not?

Question 7

In a mystical realm, a trickster spirit offers a hero two challenges to prove their worth. The first challenge involves flipping a trick coin where the probability of heads is p=0.7. The outcome X is either 1 (heads) or 0 (tails). In the second challenge, the hero performs a ritual consisting of 6 coin flips, where the number of heads Y follows a Binomial distribution with parameters n=6 and p=0.7. However, the number of heads influences the outcome of a new flip, which follows a conditional Bernoulli distribution.

The trickster requires the hero to prove their calculation skills:

a) Calculate the expected value and variance of a single coin flip X.
b) Compute the expected value and variance of the number of heads Y from the 6-flip ritual.
c) The trickster introduces a dependency such that

$$P(X = 1|Y = k) = \frac{k+1}{7}$$

where k is the number of heads from the ritual. Calculate E(X∣Y), E(XY), COV(X,Y) and the correlation coefficient between X and Y.

d) Determine the probability that the final coin flip results in heads given that the hero obtained exactly 4 heads in the ritual.

e) The hero needs to understand the total effect of the ritual and the final flip on their power. Compute the expected value and variance of the power level, where the power level is defined as P=X+Y.

Coding Question

In this final challenge, the Trickster presents the hero with a complex problem. The stages of the challenge are:

Stage 1: Two Dependent Binomial Trials -

➔ The hero flips a biased coin n times in the first trial, with a probability p of getting heads. The number of heads follows a **Binomial distribution**.

➔ Depending on the number of heads obtained in the first trial, the hero's chances change for the second trial. If the number of heads in the first trial is more than n/2, the probability of heads increases by Δp for the second trial. Otherwise, the probability stays the same as in the first trial.

➔ The hero performs another n flips in the second trial using this updated probability.

Stage 2: Conditional Bernoulli Test -

➔ After completing both trials, the hero must do a final flip. This flip follows a **Bernoulli distribution**, where the probability depends on the total number of heads obtained in both the first and second trials.

➔ The probability of heads for this final flip is given by:

$$P(X = 1|Y_1 = k_1, Y_2 = k_2) = \frac{k_1 + k_2 + 2}{2n + 2}$$

➔ where k1 is the number of heads in the first trial, and k2 is the number of heads in the second trial.

Stage 3: Visualization of Results -

The hero must create the following visualizations to understand the patterns:

➔ **Histogram**: Show the number of heads in both the first and second trials.

➔ **Conditional Probability Plot**: Compare the theoretical and empirical probabilities of heads in the final test.

➔ **CDF Plot**: Plot the **Cumulative Distribution Function (CDF)** for the total heads in both trials combined.

Your Task -

Write a Matlab program (.m) that simulates this process and generates the following plots:

- A **histogram** of the number of heads from both the first and second trials.
- A **plot** comparing the **theoretical** & **empirical conditional probabilities** for the final flip.
- A **CDF plot** showing the cumulative distribution of the total heads from both trials.

For this question, you can refer to the *tut3Hint.m* file attached, which contains an incomplete code. Your task is to fill in the blanks to complete it. This will help you code faster and in the right direction

Inputs/Parameters -

- Probability p of heads in the first trial.
- Increase in probability Δp if more than n/2 heads are obtained in the first trial.
- Number of flips n per trial.
- Number of trials N to simulate.

Pseudo Code -

```
% Define Inputs and Initialize result arrays

% Loop through N simulations

    % First Trial: Binomial simulation

    % Adjust probability for second trial based on k1

    % Second Trial: Binomial simulation

    % Final Bernoulli Trial based on k1 and k2

% Plot results: Histograms, CDFs, etc.
```

END