Assignment-1

MA 101

1. Suppose that $S = \{v_1, v_2, v_3\}$ is a linearly independent set of three vectors in $\mathbb{R}^5$. Prove that the set $T = \{2v_1 + 3v_2 + v_3,\ v_1 - v_2 + 2v_3,\ 2v_1 + v_2 - v_3\}$ is linearly independent. Suppose we will add another three vectors $v_1 + v_2,\ v_2 + v_3,\ v_1 + v_3$ into T. Then what can you say about the linearly independence of the new set of vectors T?

2. In a house there are a number of cats, dogs and peacocks as pets. There are a total of 108 legs and 30 tails. Also the house has twice as many as dogs as cats. How many of each pets are there in the house?

3. Let R be the triangle with vertices at $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3). Show that the area of the triangle R is $\dfrac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$.

4. Let A be a 4×4 matrix with $\det(A) = \frac{1}{2}$. Then Find (i) $\det(2A)$ (ii) $\det(-A)$ (iii) $\det(A^2)$ (iv) $\det(A^T)^{-1}$.

5. We know that a matrix Q is said to be orthogonal if $Q^TQ = I$. Show that for an orthogonal matrix Q, $\det(Q) = +1 \ or \ -1$.

6. Let the 4×4 matrix M is written as 2×2 block matrices in the form $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, where A, B, D are 2×2 matrices and 0 is the 2×2 zero matrix. Prove that $\det(M) = \det(A)\det(D)$.
   If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are 2×2 matrices, then can you say $\det(M) = \det(A)\det(D) - \det(B)\det(C)$? Justify your answer.

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation with $T(x_1, x_2) = (2x_1 - x_2,\ -3x_1 + x_2,\ 2x_1 - 3x_2)$. Can you find $X = (x_1, x_2)$ such that $T(x_1, x_2) = (1, 2, 3)$?

8. Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ are two linear transformations such that the compositions $T_1 o T_2$ and $T_2 o T_1$ are defined. Are $T_1 o T_2$ and $T_2 o T_1$ again linear transformations? Explain.
   If Yes, let A_1 be the standard matrix representation of T_1 and A_2 be the standard matrix representation of T_2, then what are the standard matrix representations of $T_1 o T_2$ and $T_2 o T_1$?

Note the following results:

Theorem 1: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix representation of T. Then

(a) T is onto if and only if the columns of A span $\mathbb{R}^n$.

(b) T is one-one if and only if the columns of A are linearly independent.

Theorem 2: If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is one-one and onto, then it is invertible.

9. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by $T(x_1, x_2, x_3, x_4) = (x_1 + x_2,\ x_2 + x_3,\ x_3 + x_4,\ 0)$. Is T one-one? Is T onto?

10. Is the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (3x_1 - x_2,\ x_1 + 4x_2,\ x_2)$ invertible? What can you say about the invertibility of the standard matrix A representing T?

11. Find LU decomposition of the famous Pascal matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}$. Can you generalize the resultant LU decomposition for any $n \times n$ Pascal matrix?

12. In the following matrix A, x means any value. Show that the determinant of A is 0, regardless of the x values.

$$A = \begin{pmatrix} x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{pmatrix}$$

13. By considering the determinant of the coefficient matrix, find a relationship between a and b for which the following system has exactly one solution.

$x - 2y + bz = 3$

$ax + 2z = 2$

$5x + 2y = 1.$

14. A large apartment building is to be built using modular construction techniques. The arrangement of apartments on any particular floor is to be chosen from one of the following three basic plans:

Plan A: has 18 apartments on one floor including 3 three-BHK units, 7 two-BHK units and 8 one-BHK units.

Plan B: has 16 apartments on one floor including 4 three-BHK units, 4 two-BHK units and 8 one-BHK units.

Plan C: has 17 apartments on one floor including 5 three-BHK units, 3 two-BHK units and 9 one-BHK units.

Suppose the building contains a total of x_1 floors of plan A, x_2 floors of plan B, and x_3 floors of plan C. Then

(a) What interpretation can be given to the vector $x_1 \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix}$?

(b) Write a formal linear combination of vectors that expresses the total numbers of three-BHK, two-BHK and one-BHK apartments contained in the building.

(c) Is it possible to design the building with exactly 66 three- BHK units, 74 two-BHK units, and 136 one-BHK units? If so, is there more than one way to do it? Explain your answer.

Solve following exercises using matlab/Scilab.

15. Compute $detA^TA$ and $det(AA^T)$ for several random 5×6 matrices and several random 6×7 matrices. What can you say about A^TA and AA^T when A has more columns than rows?

16. Solve:
$0.543(10)^{-3} * X_1 + 3.21 * X_2 = 3.87$
$4.32 * X_1 + 2.31 * X_2 = 4.92$
using only Gaussian elimination with back substitution to 3 significant digits.

17. Generate a random matrix A of size 6*6 and 5 vectors, b_i. Check whether $AX = b_i$ has solution for each i. Write down A, b_i, X_i.

    r1=rand(m,n): r1 is a m-by-n matrix containing real floating-point numbers drawn from a uniform distribution

18. Construct a linear systems ($AX = b$) which has infinitely many solutions. There should be 9 equations and constant vector (b) should be formed using your student id.

19. Write codes for finding inverse of a matrix a) using Gauss-Jordan method, and b) Cramer's rule. Choose a suitably large invertible matrix and find inverse using both methods and find time rquired for each method. You can use timeit command for measuring time.

20. Construct a 2×3 matrix A, not in echelon form, such that the solution of $Ax = 0$ is a line in $\mathbb{R}^3$. Find equation of that line.