MA101: Linear Algebra and Matrices: Course Content

Matrices and Linear systems: Matrix operations(addition, multiplication), Block-Partitioned Matrices and Block Operations, Elementary Row and Column Operations, Determinant and its Properties, Cofactor Expansion, Rank of a Matrix, The System of Linear Equations: Ax=b, Row Reduction and Echelon forms (Gaussian Elimination), Gauss Jordan Method for marix inversion

Canonical Factorizations: Eigenvalues and Eigenvectors, Companion Matrices and Characteristic Polynomial, Method of Danilevsky for Characteristic Polynomial, diagonalization-Matrices with a Full-Set of Eigenvectors, The Cayley-Hamilton Theorem, Triangulization and Unitary Diagonalization of a Matrix, Schur's Lemma and the Spectral Theorem, QR-Decomposition, QR-Algorithm, Singular Value Decomposition.

Vector Spaces: Vector Space over the set real numbers (Field), Linear Independence of Vectors, Bases in a Vector Space, Dimension of a Vector Space, Direct Sum Decomposition of a Vector Space, Linear Transformation (LT), Change of Bases, Canonical forms, Rank of a LT.

Numerical methods: Iterative methods (Jacobi, Gauss-Seidel, Relaxation) for linear systems, computing eigenvalues and eigenvectors.

Evaluation and Grading policy:

Assignments: 15%

Attendance: 5%

Mini-Project: 5%

Mid-semester Exam.: 30% (Premid: 10% Midsem:10% online, 10% remote)

End-semester Exam.: 45% (Pre-End: 15%, Endsem:15% online, 15% remote)

Evaluation and Grading policy:

Assignments: 15%

Attendance: 5%

Mini-Project: 5%

Mid-semester Exam.: 30% (Premid: 10% Midsem:10% online, 10% remote)

End-semester Exam.: 45% (Pre-End: 15%, Endsem:15% online, 15% remote)

Text & Reference books:

Linear Algebra and its Applications, David C. Lay, 4th Ed, Pearson, 2016.

Introduction to Linear Algebra, Gilbert Strang, 5th Ed, SIAM, 2016.

Linear Algebra, Kunze Ray, Hoffman Kenneth, 2nd Ed, Phi Learning, 2014.

Fundamentals of Matrix Computations, David S. Watkins, 3rd ed, Wiley.

Video lectures:

https://www.youtube.com/watch?v=7UJ4CFRGd-U&list=PLE7DDD91010BC51F8

https://www.youtube.com/watch?v=LJ-LoJhbBA4&list=PLbMVogVj5nJQ2vsW$_hmyvVfO4GYWaaPp$7

Question: What is a matrix?

Video lectures:

https://www.youtube.com/watch?v=7UJ4CFRGd-U&list=PLE7DDD91010BC51F8

https://www.youtube.com/watch?v=LJ-LoJhbBA4&list=PLbMVogVj5nJQ2vsW$_{h}myvVfO4GYWaaPp$7

Question: What is a matrix?

An arrangement of objects in rows and columns.

Video lectures:

https://www.youtube.com/watch?v=7UJ4CFRGd-U&list=PLE7DDD91010BC51F8

https://www.youtube.com/watch?v=LJ-LoJhbBA4&list=PLbMVogVj5nJQ2vsW$_{h}myvVfO4GYWaaPp$7

Question: What is a matrix?

An arrangement of objects in rows and columns.

Is that it?

Video lectures:

https://www.youtube.com/watch?v=7UJ4CFRGd-U&list=PLE7DDD91010BC51F8

https://www.youtube.com/watch?v=LJ-LoJhbBA4&list=PLbMVogVj5nJQ2vsW$_{h}myvVfO4GYWaaPp$7

Question: What is a matrix?

An arrangement of objects in rows and columns.

Is that it?

It can be viewed as a function (linear map).

Video lectures:

https://www.youtube.com/watch?v=7UJ4CFRGd-U&list=PLE7DDD91010BC51F8
https://www.youtube.com/watch?v=LJ-LoJhbBA4&list=PLbMVogVj5nJQ2vsW$_h myvVfO4GYWaaPp$7

Question: What is a matrix?

An arrangement of objects in rows and columns.

Is that it?

It can be viewed as a function (linear map).

Question: Why to study matrices?

Video lectures:

https://www.youtube.com/watch?v=7UJ4CFRGd-U&list=PLE7DDD91010BC51F8

https://www.youtube.com/watch?v=LJ-LoJhbBA4&list=PLbMVogVj5nJQ2vsW$_{h}myvVfO4GYWaaPp7$

Question: What is a matrix?

An arrangement of objects in rows and columns.

Is that it?

It can be viewed as a function (linear map).

Question: Why to study matrices?

Linear transformations, Hessian, quadratic forms, graphs, Image Processing, Deep/Machine learning, Artificial Intelligence,...............

Imagine an image of 10 MP (mega pixel)

Each pixel contain 3 value denoting saturation value of three colors i.e R G B, which results in some color. Now we need to manage 10,000,000 such values. Matrix makes it easy to store and handle.

Suppose you have sufficient no. of currency notes of denominations Rs. 10, Rs. 20, Rs. 50 and Rs. 100. Your canteen bill is Rs. 260. How many notes of each denomination will you need to pay?

Suppose you have sufficient no. of currency notes of denominations Rs. 10, Rs. 20, Rs. 50 and Rs. 100. Your canteen bill is Rs. 260. How many notes of each denomination will you need to pay?
Mathematical model: Let t, u, v and w be number of notes of denominations Rs. 10, Rs. 20, Rs. 50 and Rs. 100 respectively that you will pay.

Suppose you have sufficient no. of currency notes of denominations Rs. 10, Rs. 20, Rs. 50 and Rs. 100. Your canteen bill is Rs. 260. How many notes of each denomination will you need to pay?

Mathematical model: Let t, u, v and w be number of notes of denominations Rs. 10, Rs. 20, Rs. 50 and Rs. 100 respectively that you will pay.

$$10t + 20u + 50v + 100w = 260 \ldots (*)$$
$$t, u, v \geq 0$$

Suppose you have sufficient no. of currency notes of denominations Rs. 10, Rs. 20, Rs. 50 and Rs. 100. Your canteen bill is Rs. 260. How many notes of each denomination will you need to pay?
Mathematical model: Let t, u, v and w be number of notes of denominations Rs. 10, Rs. 20, Rs. 50 and Rs. 100 respectively that you will pay.

$$10t + 20u + 50v + 100w = 260 \ldots (*)$$
$$t, u, v \geq 0$$

Find all solutions to equation $(*)$ without non-negative constraints/conditions.

Covid Pandemic

Suppose 1 cr cases of Covid 19 are found in city X which has a population of 301 cr. Hence city was locked and nobody is allowed to go in or out. Every day 20% are getting infected and 10% were getting cured. How many will be infected (but not cured) after 10 days?

Covid Pandemic

Suppose 1 cr cases of Covid 19 are found in city X which has a population of 301 cr. Hence city was locked and nobody is allowed to go in or out. Every day 20% are getting infected and 10% were getting cured. How many will be infected (but not cured) after 10 days?

$$\begin{aligned}\text{Infected: } & 300 \times (0.20) + 1 \times (0.90) = 60.9 \\ \text{Healthy: } & 300 \times (0.80) + 1 \times (0.10) = 240.1\end{aligned}$$

Covid Pandemic

Suppose 1 cr cases of Covid 19 are found in city X which has a population of 301 cr. Hence city was locked and nobody is allowed to go in or out. Every day 20% are getting infected and 10% were getting cured. How many will be infected (but not cured) after 10 days?

$$\text{Infected: } 300 \times (0.20) + 1 \times (0.90) = 60.9$$
$$\text{Healthy: } 300 \times (0.80) + 1 \times (0.10) = 240.1$$
$$\begin{bmatrix} 0.2 & 0.9 \\ 0.8 & 0.1 \end{bmatrix} . \begin{bmatrix} 300 \\ 1 \end{bmatrix} = \begin{bmatrix} 60.9 \\ 240.1 \end{bmatrix}$$

$AX_0 = X_1$
$X_{10} = A^{10} X_0$

Let M be a packet which contains 5kg of wheat & 2 kg of rice and N be a packet which contains 2 kg of wheat & 5 kg of rice. How many packets of M and N should you buy to get 19kg of wheat and 15kgs of rice?

System of linear equations

Let M be a packet which contains 5kg of wheat & 2 kg of rice and N be a packet which contains 2 kg of wheat & 5 kg of rice. How many packets of M and N should you buy to get 19kg of wheat and 15kgs of rice?

Question: Is it possible to buy 39 kg of wheat and 39kg of rice?

System of linear equations

Let M be a packet which contains 5kg of wheat & 2 kg of rice and N be a packet which contains 2 kg of wheat & 5 kg of rice. How many packets of M and N should you buy to get 19kg of wheat and 15kgs of rice?

Question: Is it possible to buy 39 kg of wheat and 39kg of rice?

Question: What if we have $M\&N$ containing 3 types of grains?

Entry of matrices.

System of linear equations

Let M be a packet which contains 5kg of wheat & 2 kg of rice and N be a packet which contains 2 kg of wheat & 5 kg of rice. How many packets of M and N should you buy to get 19kg of wheat and 15kgs of rice?

Question: Is it possible to buy 39 kg of wheat and 39kg of rice?

Question: What if we have $M\&N$ containing 3 types of grains?

Entry of matrices.

Matrix operations(addition, multiplication)

Size/Dimension of a matrix: A matrix with m rows and n columns is called an $m \times n$ matrix, where m & n are called its dimensions.

Matrix operations(addition, multiplication)

Size/Dimension of a matrix: A matrix with m rows and n columns is called an $m \times n$ matrix, where m & n are called its dimensions.

Addition of two matrices:

Question: When can we add two matrices?

Matrix operations(addition, multiplication)

Size/Dimension of a matrix: A matrix with m rows and n columns is called an $m \times n$ matrix, where m & n are called its dimensions.

Addition of two matrices:

Question: When can we add two matrices?

Ans: If they have same size/dimension.

Matrix operations(addition, multiplication)

Size/Dimension of a matrix: A matrix with m rows and n columns is called an $m \times n$ matrix, where m & n are called its dimensions.

Addition of two matrices:

Question: When can we add two matrices?

Ans: If they have same size/dimension.

If $A = [a_{ij}], B = [b_{ij}]$ then $A + B = [c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$

Matrix operations(addition, multiplication)

Size/Dimension of a matrix: A matrix with m rows and n columns is called an $m \times n$ matrix, where m & n are called its dimensions.

Addition of two matrices:

Question: When can we add two matrices?

Ans: If they have same size/dimension.

If $A = [a_{ij}], B = [b_{ij}]$ then $A + B = [c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$

Multiplition of two matrices:

Question: When can we multiply two matrices?

Matrix operations(addition, multiplication)

Size/Dimension of a matrix: A matrix with m rows and n columns is called an $m \times n$ matrix, where m & n are called its dimensions.

Addition of two matrices:

Question: When can we add two matrices?

Ans: If they have same size/dimension.

If $A = [a_{ij}], B = [b_{ij}]$ then $A + B = [c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$

Multiplition of two matrices:

Question: When can we multiply two matrices?

Ans: If no. of columns of first matrix = no. of rows of second matrix.

Matrix operations(addition, multiplication)

Size/Dimension of a matrix: A matrix with m rows and n columns is called an $m \times n$ matrix, where m & n are called its dimensions.

Addition of two matrices:

Question: When can we add two matrices?

Ans: If they have same size/dimension.

If $A = [a_{ij}], B = [b_{ij}]$ then $A + B = [c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$

Multiplition of two matrices:

Question: When can we multiply two matrices?

Ans: If no. of columns of first matrix = no. of rows of second matrix.

If $A = [a_{ij}]$ of order $m \times n$ & $B = [b_{ij}]$ of order $n \times p$ then AB is of size $m \times p$ with

$$AB = [c_{ij}], \text{ where } c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Matrix operations(addition, multiplication)

Size/Dimension of a matrix: A matrix with m rows and n columns is called an $m \times n$ matrix, where m & n are called its dimensions.

Addition of two matrices:

Question: When can we add two matrices?

Ans: If they have same size/dimension.

If $A = [a_{ij}], B = [b_{ij}]$ then $A + B = [c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$

Multiplition of two matrices:

Question: When can we multiply two matrices?

Ans: If no. of columns of first matrix = no. of rows of second matrix.

If $A = [a_{ij}]$ of order $m \times n$ & $B = [b_{ij}]$ of order $n \times p$ then AB is of size $m \times p$ with

$$AB = [c_{ij}], \text{ where } c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Question Is $AB = BA$ for any two matrices A, B?

Let A, B, C be matrices of same dimension and $r, s \in \mathbb{R}$. Then

1. A+B=B+A
2. r.(A+B)=r.A+r.B
3. (A+B)+C=A+(B+C)
4. (r+s).A=r.A+s.A
5. r.(s.A)=(rs).A

Let A, B, C be matrices of same dimension and $r, s \in \mathbb{R}$. Then

1. A+B=B+A
2. r.(A+B)=r.A+r.B
3. (A+B)+C=A+(B+C)
4. (r+s).A=r.A+s.A
5. r.(s.A)=(rs).A

If $b_1, b_2, ..., b_p$ are columns of B then columns of AB are

Let A, B, C be matrices of same dimension and $r, s \in \mathbb{R}$. Then

1. A+B=B+A
2. r.(A+B)=r.A+r.B
3. (A+B)+C=A+(B+C)
4. (r+s).A=r.A+s.A
5. r.(s.A)=(rs).A

If $b_1, b_2, ..., b_p$ are columns of B then columns of AB are $Ab_1, Ab_2, ..., Ab_p$ respectively.

Let A, B, C be matrices of same dimension and $r, s \in \mathbb{R}$. Then

1. A+B=B+A
2. r.(A+B)=r.A+r.B
3. (A+B)+C=A+(B+C)
4. (r+s).A=r.A+s.A
5. r.(s.A)=(rs).A

If $b_1, b_2, ..., b_p$ are columns of B then columns of AB are $Ab_1, Ab_2, ..., Ab_p$ respectively.

What can you say about rows of AB?

Let A, B, C be matrices of same dimension and $r, s \in \mathbb{R}$. Then

1. A+B=B+A
2. r.(A+B)=r.A+r.B
3. (A+B)+C=A+(B+C)
4. (r+s).A=r.A+s.A
5. r.(s.A)=(rs).A

If $b_1, b_2, ..., b_p$ are columns of B then columns of AB are $Ab_1, Ab_2, ..., Ab_p$ respectively.

What can you say about rows of AB?

6. A.(B.C)=(A.B).C
7. A.(B+C)=A.B+A.C
8. (B+C).A=B.A+C.A
9. r.(A.B)=(r.A).B=A.(r.B)
10. I.A=A.I=A

Let A, B, C be matrices of same dimension and $r, s \in \mathbb{R}$. Then

1. A+B=B+A
2. r.(A+B)=r.A+r.B
3. (A+B)+C=A+(B+C)
4. (r+s).A=r.A+s.A
5. r.(s.A)=(rs).A

If $b_1, b_2, ..., b_p$ are columns of B then columns of AB are $Ab_1, Ab_2, ..., Ab_p$ respectively.
What can you say about rows of AB?

6. A.(B.C)=(A.B).C
7. A.(B+C)=A.B+A.C
8. (B+C).A=B.A+C.A
9. r.(A.B)=(r.A).B=A.(r.B)
10. I.A=A.I=A

Question: Does $AB = 0$ mean either $A = 0$ or $B = 0$?

Transpose of a matrix: If $A = [a_{ij}]$ is a matrix then $(i, j)^{th}$ entry of A^T is a_{ji}.

Transpose of a matrix: If $A = [a_{ij}]$ is a matrix then $(i,j)^{th}$ entry of A^T is a_{ji}.

1. $(A^T)^T = A$.
2. $(A+B)^T = A^T + B^T$
3. $(rA)^T = r.A^T$
4. $(AB)^T =$

Transpose of a matrix: If $A = [a_{ij}]$ is a matrix then $(i,j)^{th}$ entry of A^T is a_{ji}.

1. $(A^T)^T = A$.
2. $(A+B)^T = A^T + B^T$
3. $(rA)^T = r.A^T$
4. $(AB)^T = B^T A^T$.

Transpose of a matrix: If $A = [a_{ij}]$ is a matrix then $(i,j)^{th}$ entry of A^T is a_{ji}.

1. $(A^T)^T = A$.
2. $(A+B)^T = A^T + B^T$
3. $(rA)^T = r.A^T$
4. $(AB)^T = B^T A^T$.

Inverse of a Matrix: A matrix A is said to be invertible if there exists a matrix B such that

$$AB = BA = I.$$

B is called inverse of A.

Transpose of a matrix: If $A = [a_{ij}]$ is a matrix then $(i,j)^{th}$ entry of A^T is a_{ji}.
1. $(A^T)^T = A$.
2. $(A+B)^T = A^T + B^T$
3. $(rA)^T = r.A^T$
4. $(AB)^T = B^T A^T$.

Inverse of a Matrix: A matrix A is said to be invertible if there exists a matrix B such that

$$AB = BA = I.$$

B is called inverse of A.
If A is invertible than A must be square matrix.

Transpose of a matrix: If $A = [a_{ij}]$ is a matrix then $(i,j)^{th}$ entry of A^T is a_{ji}.
1. $(A^T)^T = A$.
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = r.A^T$
4. $(AB)^T = B^T A^T$.
Inverse of a Matrix: A matrix A is said to be invertible if there exists a matrix B such that

$$AB = BA = I.$$

B is called inverse of A.
If A is invertible than A must be square matrix.
5. $(A^{-1})^{-1} = A$.
6. $(AB)^{-1} = B^{-1}A^{-1}$.
7.$(A^T)^{-1} = (A^{-1})^T$.
Question: If A, B, C are square matrices of same dimension such that $AB = CA = I$ then is $B = C$?

Let A, B be matrices of same dimension.
Does $AB = I$ imply $BA = I$, i.e., A is invertible?

Let A, B be matrices of same dimension.
Does $AB = I$ imply $BA = I$, i.e., A is invertible?
We will see another characterization of invertible matrices using linear system, elementary matrices, linear map, etc.

Let A, B be matrices of same dimension.
Does $AB = I$ imply $BA = I$, i.e., A is invertible?
We will see another characterization of invertible matrices using linear system, elementary matrices, linear map, etc.

Block-Partitioned Matrices

We can partition a matrix into smaller matrices called blocks such as

$$M = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 1 \\ \hline 0 & 1 & 2 & 0 \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

$$\text{Here } A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}, D = (0).$$

Note: There are many ways to partition given matrix.

Block-Partitioned Matrices

We can partition a matrix into smaller matrices called blocks such as

$$M = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 1 \\ \hline 0 & 1 & 2 & 0 \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

$$\text{Here } A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ C = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix},\ D = (0).$$

Note: There are many ways to partition given matrix.
Question: What is the advantage of partitioning?

Block-Partitioned Matrices

We can partition a matrix into smaller matrices called blocks such as

$$M = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 1 \\ \hline 0 & 1 & 2 & 0 \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}$$

Here $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}$, $D = (0)$.

$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$

$B = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$D = \begin{bmatrix} 7 & 8 \end{bmatrix}$

$E = \begin{bmatrix} 9 \end{bmatrix}$

$F = \begin{bmatrix} 1 \end{bmatrix}$

$G = \begin{bmatrix} 0 & 1 \end{bmatrix}$

$H = \begin{bmatrix} 2 \end{bmatrix}$

$I = \begin{bmatrix} 0 \end{bmatrix}$

Note: There are many ways to partition given matrix.
Question: What is the advantage of partitioning?

$$M^2 = \begin{bmatrix} A^2 + BC & AB + BD \\ CA + DC & CB + D^2 \end{bmatrix}$$

Qsn Is det(M) = det(AD − BC) or det(AD − CB)

Block Matrix Inversion

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} \underbrace{A^{-1}(I + B(D - CA^{-1}B)^{-1}CA^{-1})}_{E} & \underbrace{-A^{-1}B(D - CA^{-1}B)^{-1}}_{F} \\ \underbrace{-(D - CA^{-1}B)^{-1}CA^{-1}}_{G} & \underbrace{(D - CA^{-1}B)^{-1}}_{H} \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$M \, M^{-1}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$CE + DG = 0$ $\quad AE + BG = I$

$CF + DH = I$ $\quad AF + BH = 0$

Block Matrix Inversion

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} A^{-1}(I + B(D - CA^{-1}B)^{-1}CA^{-1}) & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

A block diagonal matrix is a block matrix that is a square matrix such that the main diagonal blocks are square matrices and all off diagonal blocks are zero matrices. A block diagonal matrix and its inverse have the form:

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix} A^{-1} = \begin{bmatrix} A_1^{-1} & 0 & \cdots & 0 \\ 0 & A_2^{-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_n^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & A \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 & 0 \\ 0 & B^{-1} & 0 \\ 0 & 0 & A^{-1} \end{bmatrix}$$

$$\left[\begin{array}{cc|cc|cc} 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 7 & 8 & 6 & 0 \\ 0 & 0 & 9 & 10 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 3 & 4 \end{array}\right]$$

Block Matrix Inversion

$$M^{-1} = \begin{bmatrix} \frac{4}{-2} & \frac{-2}{-2} & 0 & 0 \\ \frac{-3}{-2} & \frac{1}{-2} & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & & \\ 0 & 0 & & \\ 0 & 0 & & \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} A^{-1}(I + B(D - CA^{-1}B)^{-1}CA^{-1}) & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

A block diagonal matrix is a block matrix that is a square matrix such that the main diagonal blocks are square matrices and all off diagonal blocks are zero matrices. A block diagonal matrix and its inverse have the form:

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix} \quad A^{-1} = \begin{bmatrix} A_1^{-1} & 0 & \cdots & 0 \\ 0 & A_2^{-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_n^{-1} \end{bmatrix}$$

Block partition is useful in many computer science applications, VLSI chip design, the Strassen algorithm for fast matrix multiplication, coding theory.

Linear System

Definition

A linear System of m equations in n variables- $X_1, X_2, \dots, X_n$ *is*

$$\begin{gathered} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n = b_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n = b_2 \\ . \\ . \\ . \\ a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = b_m \end{gathered}$$

where $a_{ij}, b_j \in \mathbb{R}$

Linear System

Definition

A linear System of m equations in n variables- $X_1, X_2, \ldots, X_n$ *is*

$$
\begin{aligned}
a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n &= b_1 \\
a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n &= b_2 \\
&. \\
&. \\
&. \\
a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n &= b_m
\end{aligned}
$$

where $a_{ij}, b_j \in \mathbb{R}$

Let's look at linear system of 2 equations in 2 variables:

Solve the system: $(1)\ x + 2y = 3, \qquad (2)\ 3x + y = 4.$

Elimination of variables:

Eliminate x by $(2) - 3 \times (1)$ to get $y = 1$.

Cramer's Rule (determinant): $y = \dfrac{\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = \frac{4-9}{1-6} = 1$

$x = \dfrac{\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}}$

In either case, back substitution gives $\quad x = 1$

We could also solve for x first and use back substitution for y.

Comparison: For a large system, say 100 equations in 100 variables, elimination method is preferred, since computing the determinants of a 101 matrices of size 100×100 is time-consuming.

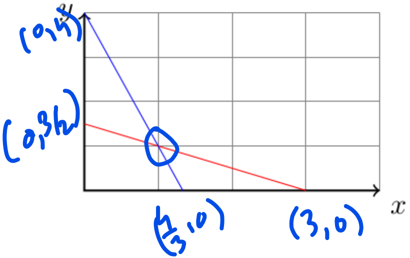
Geomtry of Linear Equations

$$\boxed{x+2y=3}$$

and

$$\boxed{3x+y=4}$$

represent lines in $\mathbb{R}^2$ passing through $\boxed{(0,3/2) \quad \text{and} \quad (3,0)}$ and through $\boxed{(0,4) \quad \text{and} \quad (4/3,0)}$ respectively.



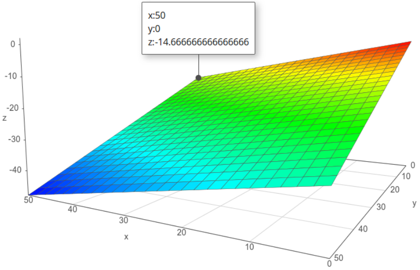
The intersection of the two lines is the unique point $(1,1)$. Hence $x=1$ and $y=1$ is the solution of above system of linear equations.

3 Equations in 3 Variables

A linear equation in 3 variables represents a plane in a 3-dimensional space $\mathbb{R}^3$.

$$x + 2y + 3z = 6$$

passes through $(0, 0, 2)$, $(0, 3, 0)$, $(6, 0, 0)$.



$x + 2y + 3z = 12$ passes through $(0, 0, 4)$, $(0, 6, 0)$, $(12, 0, 0)$. which is parallel to above plane.

Suppose we have 3 equations in 3 variables, means we have to find intersection of 3 planes, say P_1, P_2, P_3.

Suppose we have 3 equations in 3 variables, means we have to find intersection of 3 planes, say P_1, P_2, P_3.
This is same as finding intersection of line L with P_3 (Intersection of P_1, P_2 is L, if P_1, P_2 are not parallel).

Suppose we have 3 equations in 3 variables, means we have to find intersection of 3 planes, say P_1, P_2, P_3.
This is same as finding intersection of line L with P_3 (Intersection of P_1, P_2 is L, if P_1, P_2 are not parallel).

- If the line L does intersects with the plane P_3 , then the linear system has unique solution.

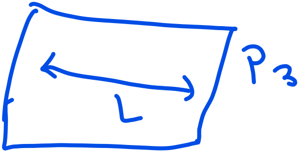
Suppose we have 3 equations in 3 variables, means we have to find intersection of 3 planes, say P_1, P_2, P_3.
This is same as finding intersection of line L with P_3 (Intersection of P_1, P_2 is L, if P_1, P_2 are not parallel).

- If the line L does intersects with the plane P_3 , then the linear system has unique solution.
- If the line L does not intersects with the plane P_3 , then the linear system has no solution.

Suppose we have 3 equations in 3 variables, means we have to find intersection of 3 planes, say P_1, P_2, P_3.
This is same as finding intersection of line L with P_3 (Intersection of P_1, P_2 is L, if P_1, P_2 are not parallel).

- If the line L does intersects with the plane P_3 , then the linear system has unique solution.
- If the line L does not intersects with the plane P_3 , then the linear system has no solution.
- If the line L is contained in the plane P_3, then the system has infinitely many solutions.
  In this case, every point of L is a solution.

$L \cap P_3$ No. point. L



$P_1 \cap P_2$ No $\cap P_3$ No L $L \cap P_3$ P_1

Suppose we have 3 equations in 3 variables, means we have to find intersection of 3 planes, say P_1, P_2, P_3.
This is same as finding intersection of line L with P_3 (Intersection of P_1, P_2 is L, if P_1, P_2 are not parallel).

- If the line L does intersects with the plane P_3 , then the linear system has unique solution.
- If the line L does not intersects with the plane P_3 , then the linear system has no solution.
- If the line L is contained in the plane P_3, then the system has infinitely many solutions.
  In this case, every point of L is a solution.

Question: Can we do the same when number of variables are > 3?

Gaussian Elimination: Unique solution

Example: $2x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + 2z = 9.$
Algorithm: Eliminate x from last 2 equations by
$(2) - 2(1)$, and $(3) + (1)$ to get the *equivalent system:*

$$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 8y + 3z = 14$$

The first *pivot* is 2, second pivot is -8. Eliminate y from the last equation to get an equivalent *triangular system*:

$$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad z = 2$$

Solve this triangular system by *back substitution*, we get

$$z = 2, \quad y = 1, \quad x = 1$$

Observe: This is the only possible solution!

Gaussian Elimination: No solution

Example: $\mathbf{2}x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + z = 9.$

$\boxed{\text{Step 1}}$ Eliminate x (using the 1st pivot 2) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 8y + 2z = 14$

$\boxed{\text{Step 2}}$: Eliminate y (using the 2nd pivot -8) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 0 = 2.$

The last equation shows that there is no solution,
i.e., the system is *inconsistent*.

Geometric reasoning: In Step 1, notice we get two distinct parallel planes $8y + 2z = 12$ and $8y + 2z = 14$.
They have no point in common.

Note: The planes in the original system were not parallel, but in an equivalent system, we get two distinct parallel planes!

Gaussian Elimination: Infinitely solution

$x = \frac{1}{2}(5-c-d)$ $\quad \boxed{2x+y+z=5}$ $\quad$ $y \to c$, $z \to d$

$2x+y+z=5$ $\qquad$ $4x+2y+2z=10$

$0=0$ $\qquad$ $0=0$

Example: $\mathbf{2}x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + z = 7.$

$\boxed{\text{Step 1}}$ Eliminate x (using the 1st pivot 2) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 8y + 2z = 12$

$\boxed{\text{Step 2}}$: Eliminate y (using the 2nd pivot -8) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 0 = 0.$

There are only two equations. For every value of z, values for x and y are obtained by back-substitution, e.g, $(1, 1, 2)$ or $\left(\frac{7}{4}, \frac{3}{2}, 0\right)$. Hence the system has infinitely many solutions.

Geometric reasoning: In Step 1, notice we get two parallel planes $-8y - 2z = 12$ and $8y + 2z = 12$.
They give the same plane. Hence we are looking at the intersection of the two planes, $2x + y + z = 5$ and $8x + 2z = 12$, which is a line.

A system of linear equations has exactly one of following

1. no solution; (inconsistent system)
2. exactly one solution; (consistent system)
3. infinitely many solutions. (consistent system)

A system of linear equations has exactly one of following

1. no solution; (inconsistent system)
2. exactly one solution; (consistent system)
3. infinitely many solutions. (consistent system)

Solve

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

Elementary Row Operations

1. $R_i \longleftrightarrow R_j$: Interchange of rows i and j.

Elementary Row Operations

1. $R_i \longleftrightarrow R_j$: Interchange of rows i and j.
2. $R_i \longrightarrow R_i + cR_j$: Adding a scalar multiple of row j to row i.

Elementary Row Operations

1. $R_i \longleftrightarrow R_j$: Interchange of rows i and j.
2. $R_i \longrightarrow R_i + cR_j$: Adding a scalar multiple of row j to row i.
3. $R_i \longrightarrow cR_i$:, $c \neq 0$: Multiply a non zero scalar to a row.

Elementary Row Operations

1. $R_i \longleftrightarrow R_j$: Interchange of rows i and j.
2. $R_i \longrightarrow R_i + cR_j$: Adding a scalar multiple of row j to row i.
3. $R_i \longrightarrow cR_i$:, $c \neq 0$: Multiply a non zero scalar to a row.

Definition (Elementary matrices)

These are the matrices obtained from identity matrix by applying any one of 3 elementary row operations.

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to 3R_1 - R_2$

$R_1 \to 3R_1$

$R_1 \to R_1 - R_2$

1. $R_i \longleftrightarrow R_j$: Interchange of rows i and j.
2. $R_i \longrightarrow R_i + cR_j$: Adding a scalar multiple of row j to row i.
3. $R_i \longrightarrow cR_i$:, $c \neq 0$: Multiply a non zero scalar to a row.

Definition (Elementary matrices)

These are the matrices obtained from identity matrix by applying any one of 3 elementary row operations.

Find out examples of Elementary matrices and matrices which are not elementary.

(ae1) $R_i \longleftrightarrow R_j$ on $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

No possible

(ae2) $R_i \longrightarrow R_i + cR_j$.

N. possible

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Elementary Row Operations

1. $R_i \longleftrightarrow R_j$: Interchange of rows i and j.
2. $R_i \longrightarrow R_i + cR_j$: Adding a scalar multiple of row j to row i.
3. $R_i \longrightarrow cR_i$:, $c \neq 0$: Multiply a non zero scalar to a row.

Definition (Elementary matrices)

These are the matrices obtained from identity matrix by applying any one of 3 elementary row operations.

Find out examples of Elementary matrices and matrices which are not elementary.
Similary, Elementary Column Operations are defined.

Theorem

For each elementary row operation e, there exists another elementary row operation e_1 of same type such that $e_1(e(A)) = A$ for any matrix A.

(an 1) $e : R_i \longleftrightarrow R_j$

$e_1 = e$

2) $e : R_i \to R_i + cR_j$

$e_1 : R_i \to R_i - cR_j$

3) $e : R_i \to cR_i$

$e_1 : R_i \to \frac{1}{c} R_i$

$E_1 A x = E_1 b$

Theorem

For each elementary row operation e, there exists another elementary row operation e_1 of same type such that $e_1(e(A)) = A$ for any matrix A.

Definition

Let A, B be two $m \times n$ matrices over $\mathbb{R}$. We say A is row equivalent to B if B can be obtained from A by a finite sequence of elementary row operations.

Take A

E_1, E_2

$E_2 E_1 A = B$

$$\boxed{B = E_k E_{k-1} \cdots E_2 E_1 A}$$

$R_2 \to R_2 - R_1$

$$B = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A$$

$\underset{B}{\begin{bmatrix}2&4\\3&4\end{bmatrix}} \leftarrow A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ $R_1\to 2R_1$ $\underset{E_1}{\begin{bmatrix}2&0\\0&1\end{bmatrix}}$ | $E_1A=B$

Theorem

For each elementary row operation e, there exists another elementary row operation e_1 of same type such that $e_1(e(A)) = A$ for any matrix A.

Definition

Let A, B be two $m \times n$ matrices over $\mathbb{R}$. We say A is row equivalent to B if B can be obtained from A by a finite sequence of elementary row operations.

Theorem

If A, B are row equivalent then $AX = 0$ and $BX = 0$ have same set of solutions.

Note: $AX = b$ & $BX = b$ may not have same set of solutions.

Gaussian Elimination: Matrix form AX=b

Example: $2x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + 2z = 9.$

Note that the last column is the RHS column vector b.

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Gaussian Elimination: Matrix form AX=b

Example: $2x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + 2z = 9.$

Note that the last column is the RHS column vector b.

$$\begin{pmatrix} 2 & 1 & 1 & | & 5 \\ 4 & -6 & 0 & | & -2 \\ -2 & 7 & 2 & | & 9 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 & | & 5 \\ 0 & -8 & -2 & | & -12 \\ 0 & 8 & 3 & | & 14 \end{pmatrix} \to$$

$$\begin{pmatrix} 2 & 1 & 1 & | & 5 \\ 0 & -8 & -2 & | & -12 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

The last matrix corresponds to
$z = 2$,
$-8y - 2z = -12$,
$2x + y + z = 5$.