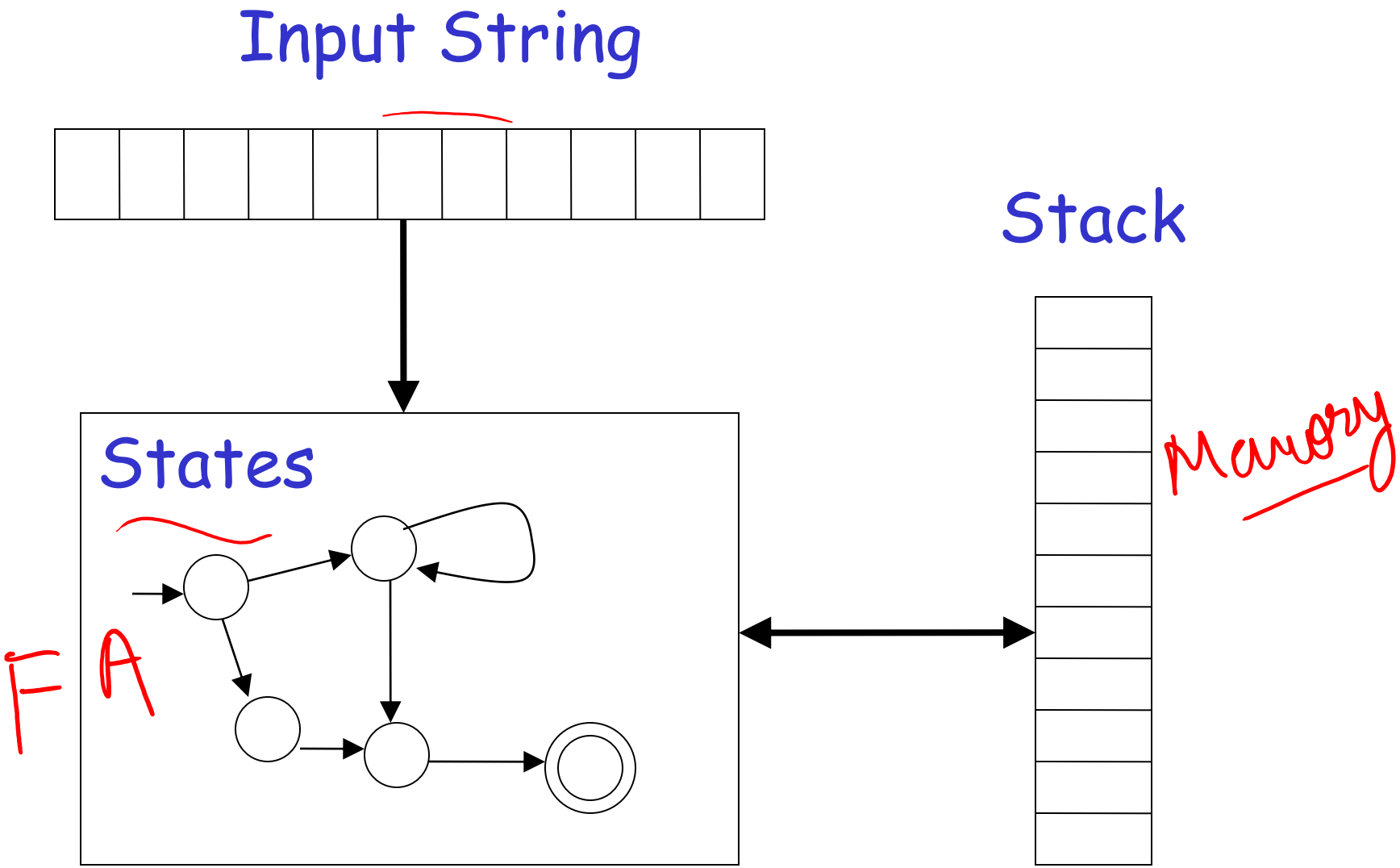
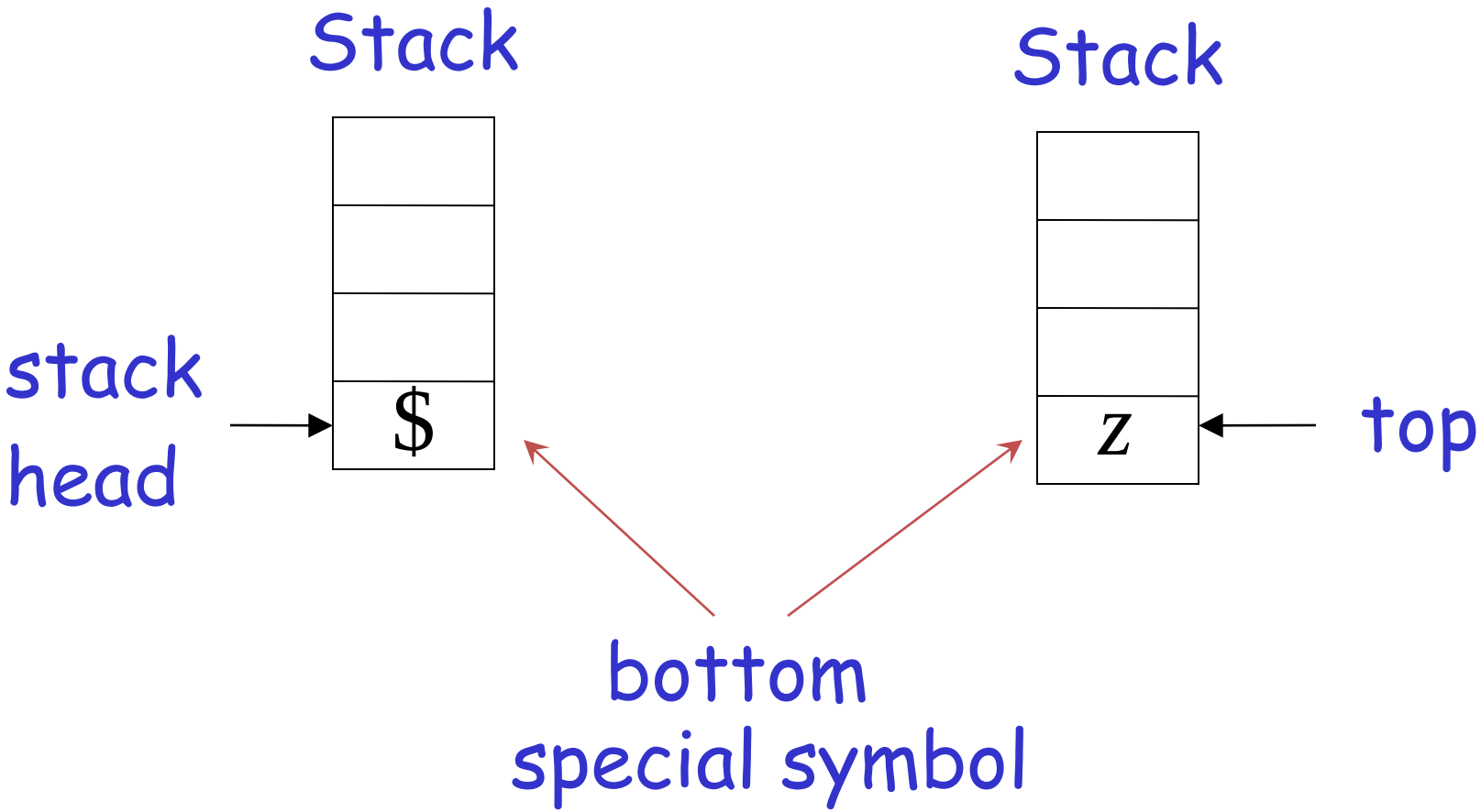
Pushdown Automata
PDAs

Pushdown Automaton -- PDA



Initial Stack Symbol

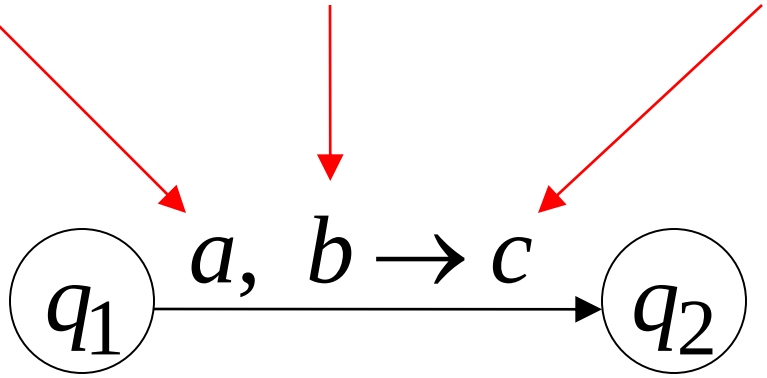


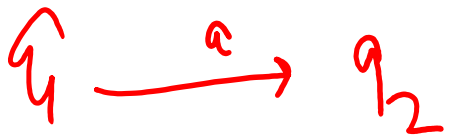
The States

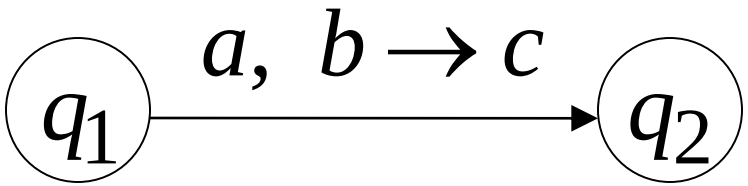
Input symbol

Pop symbol

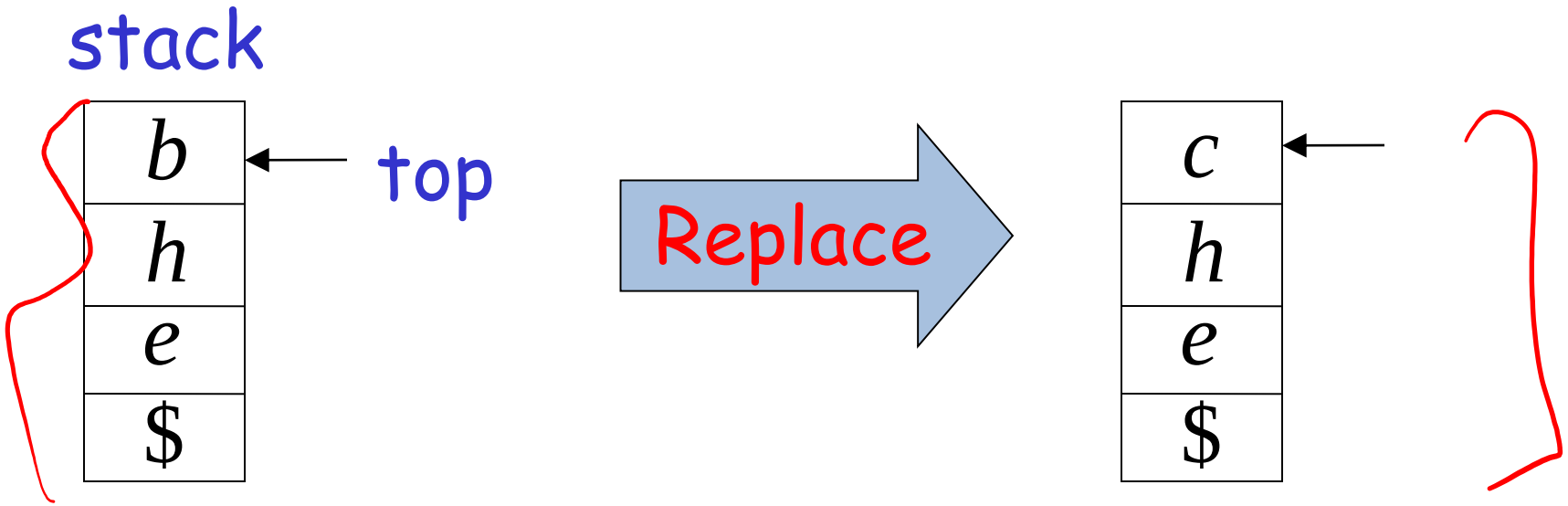
Push symbol

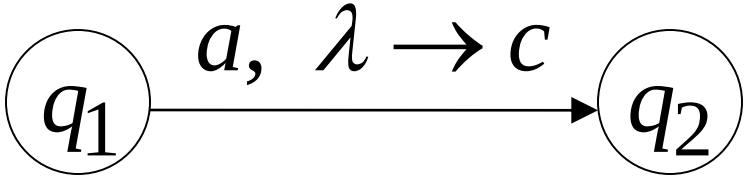




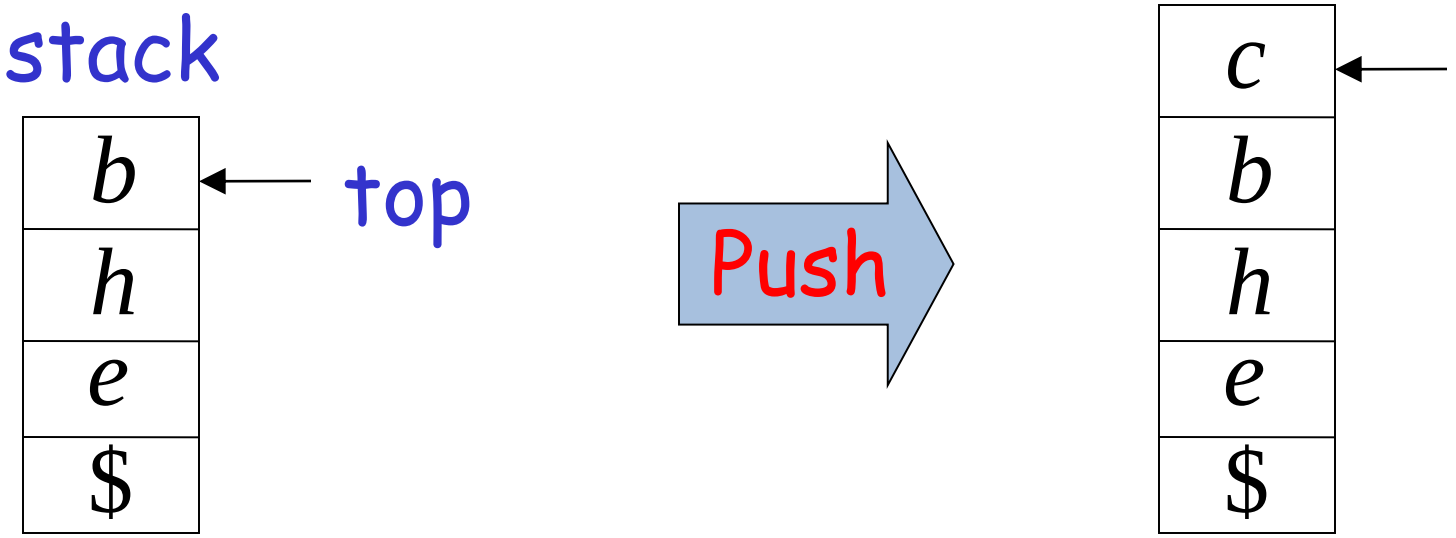

$a,\ b \to c$
q_1
q_2

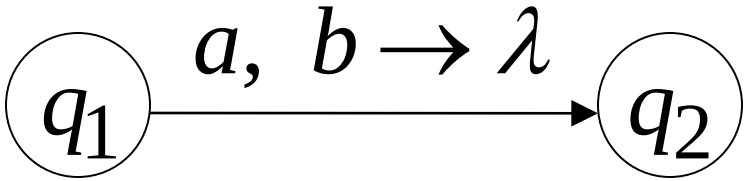

input
… a …
… a …


stack
b
h
e
$
top
Replace
c
h
e
$

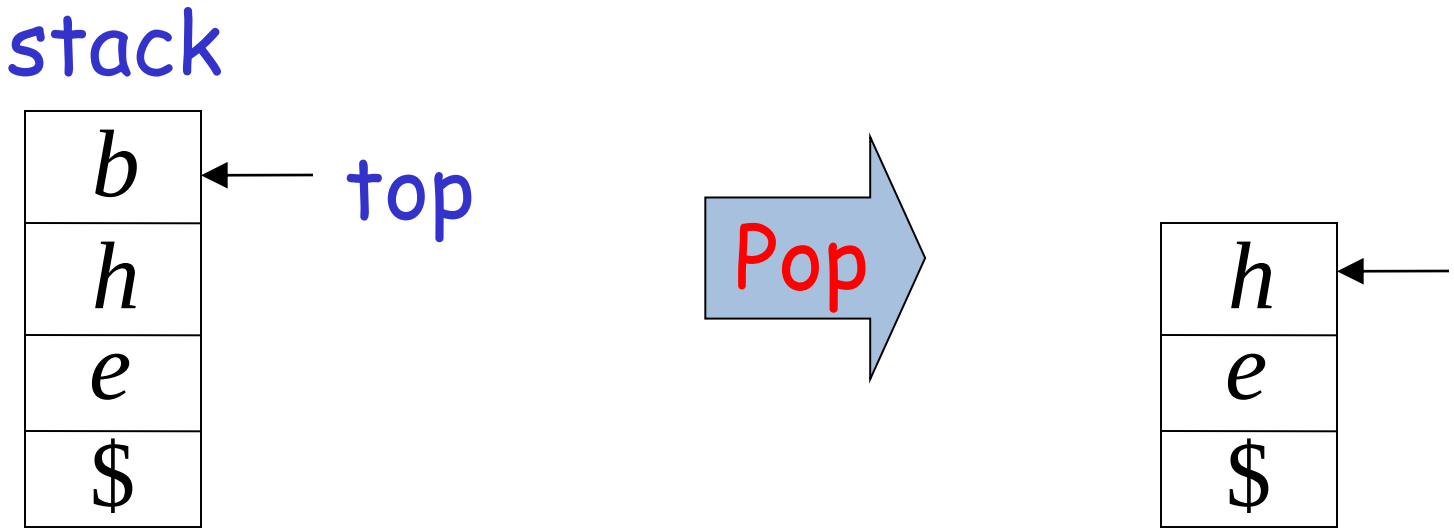

$a,\ \lambda \rightarrow c$
q_1
q_2

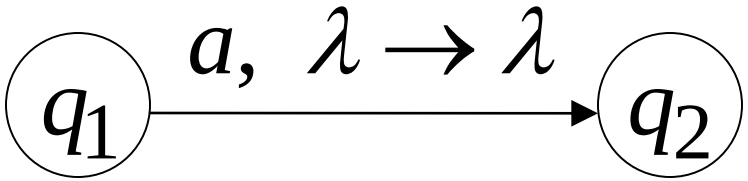

input
…
a
…
…
a
…


stack
b
top
h
e
$\$$
Push
c
b
h
e
$\$$

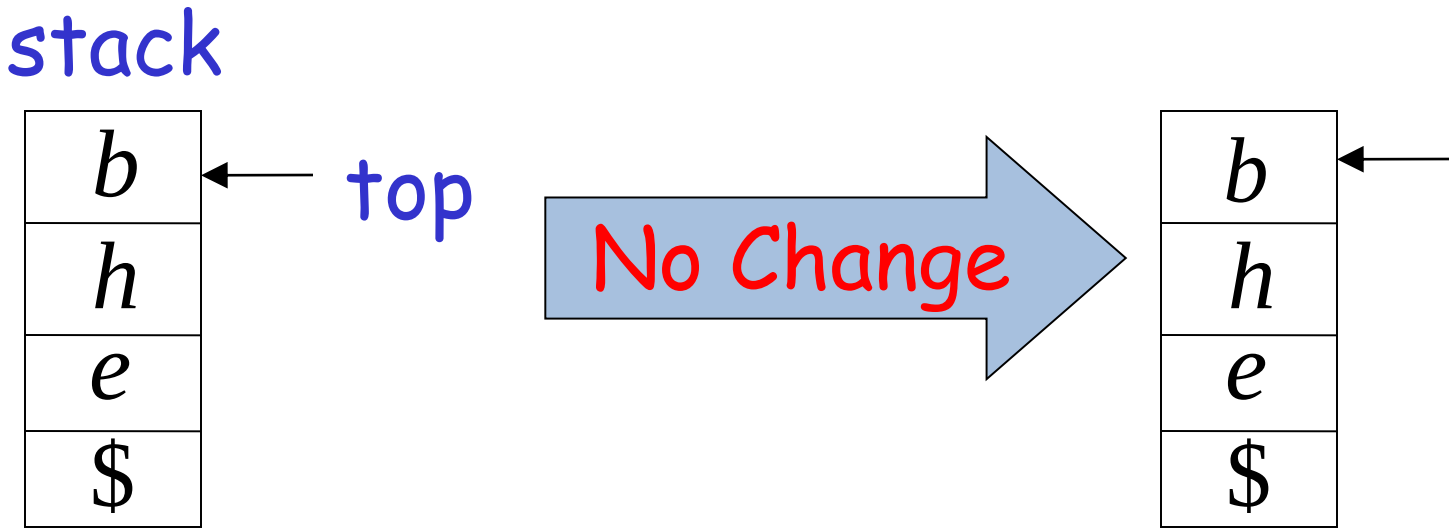

$a,\ b \to \lambda$
q_1
q_2


input
…
a
…
…
a
…

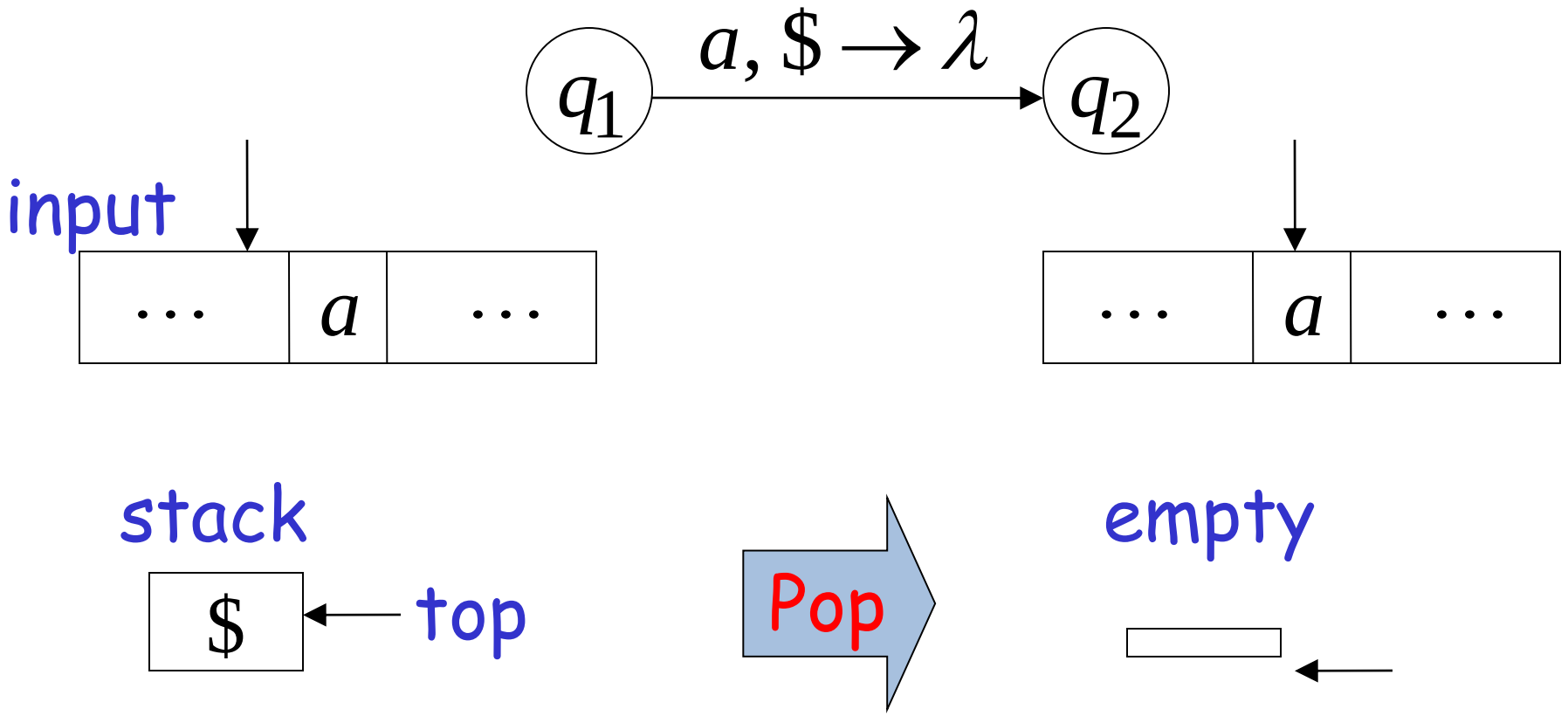

stack
b
top
h
e
$
Pop
h
e
$


$a,\ \lambda \rightarrow \lambda$
q_1
q_2

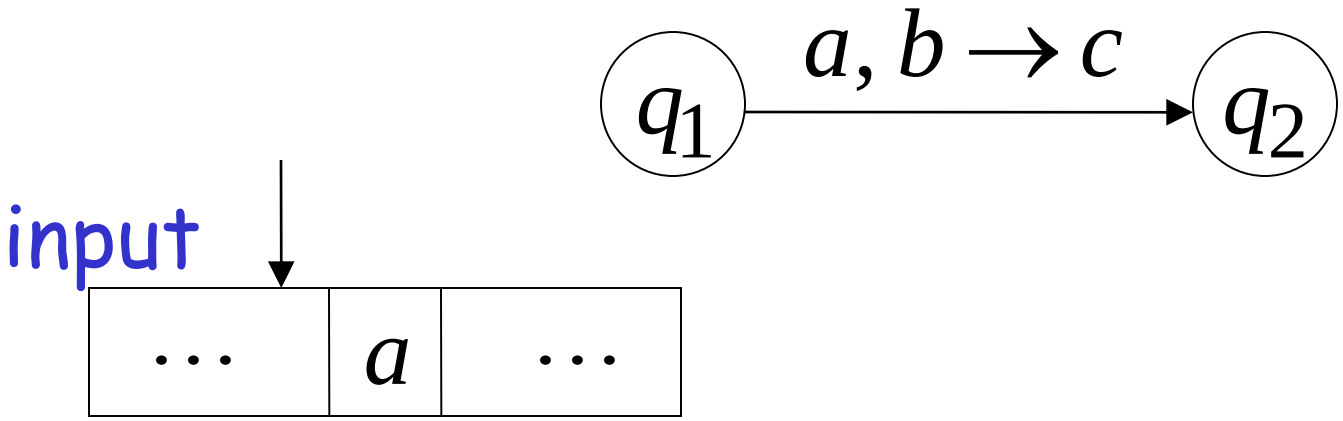

input
…
a
…
…
a
…


stack
b
h
e
$
top
No Change
b
h
e
$

A Possible Transition



A Bad Transition



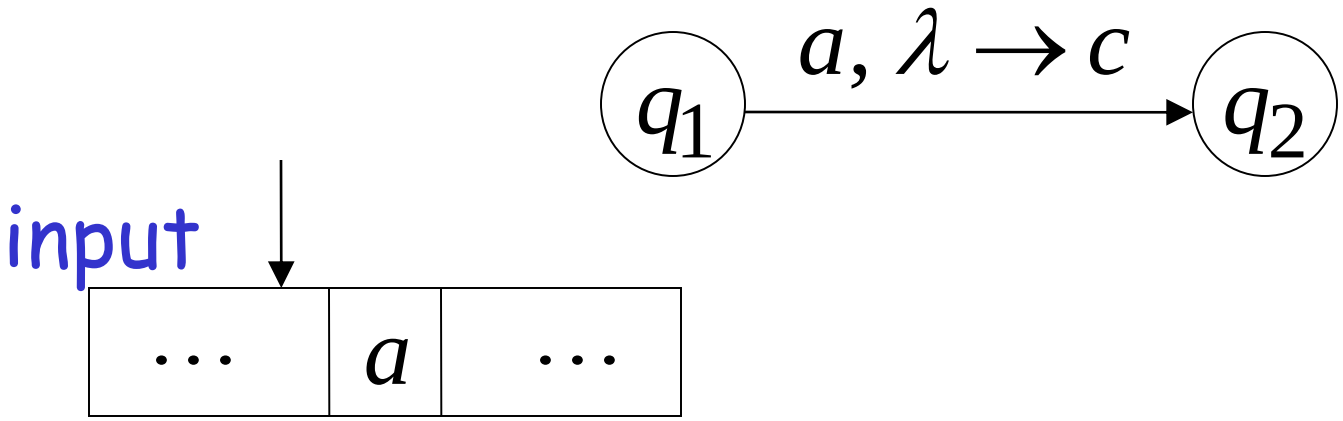
Empty stack

HALT

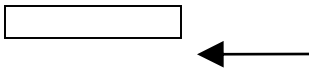
The automaton **Halts** in state q_1

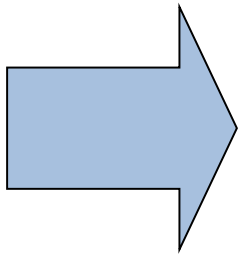
and **Rejects** the input string

A Bad Transition



Empty stack



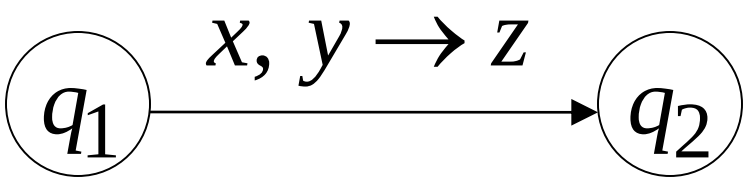


HALT

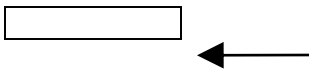
The automaton **Halts** in state q_1

and **Rejects** the input string

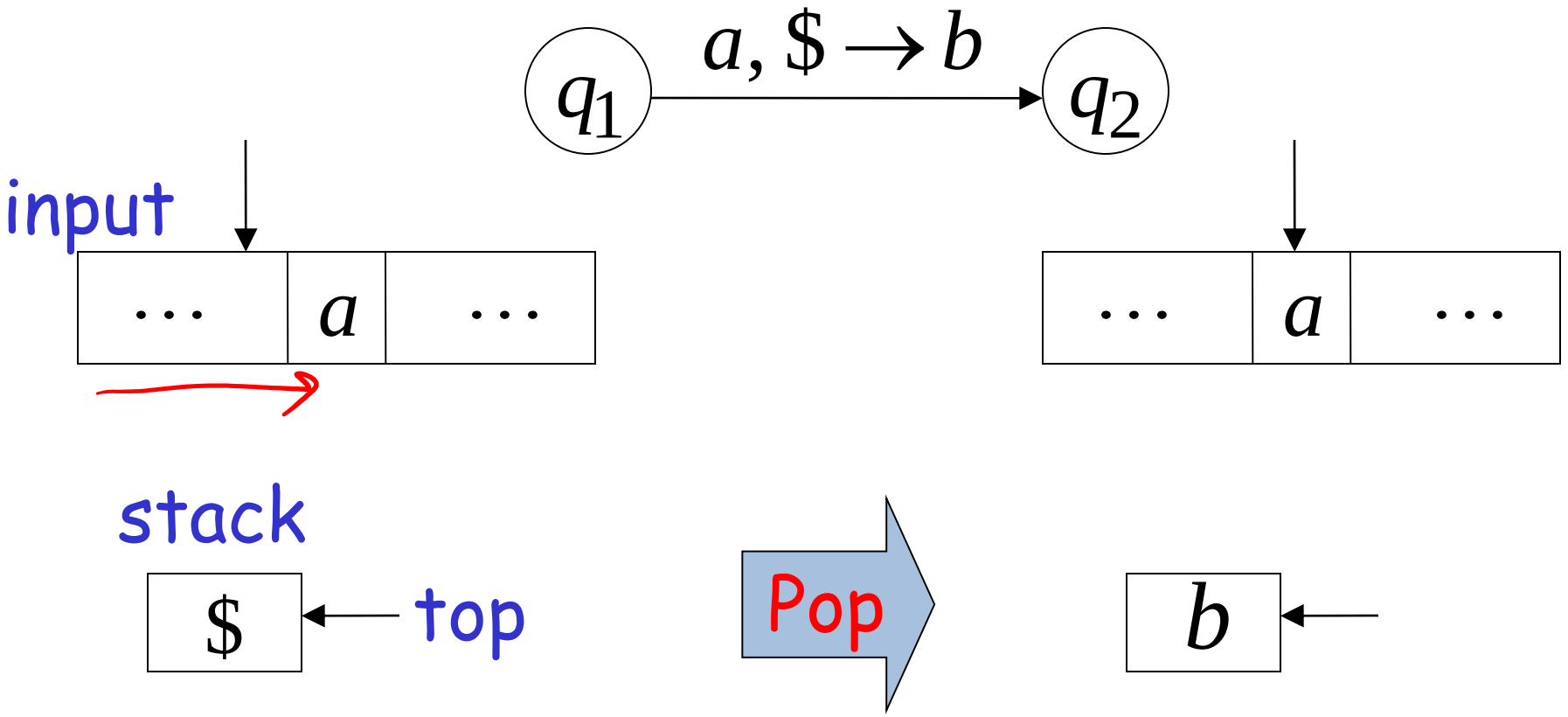
No transition is allowed to be followed
When the stack is empty



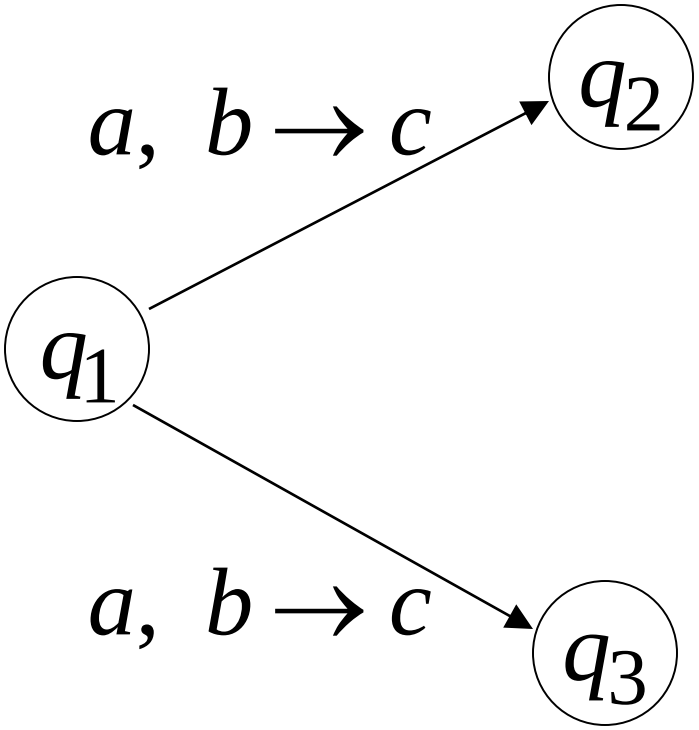
Empty stack

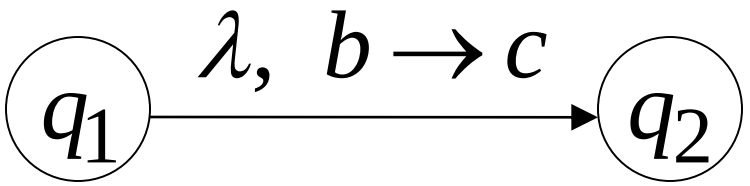


A Good Transition



Non-Determinism



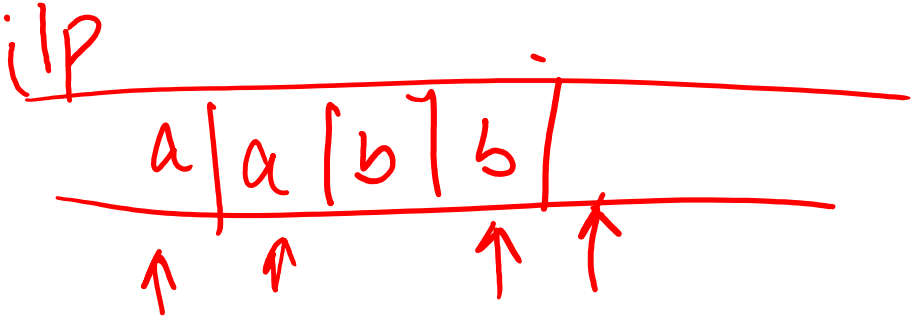


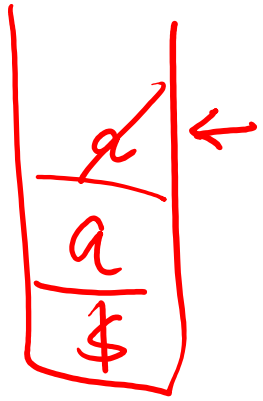
λ – transition

These are allowed transitions in a Non-deterministic PDA (NPDA)

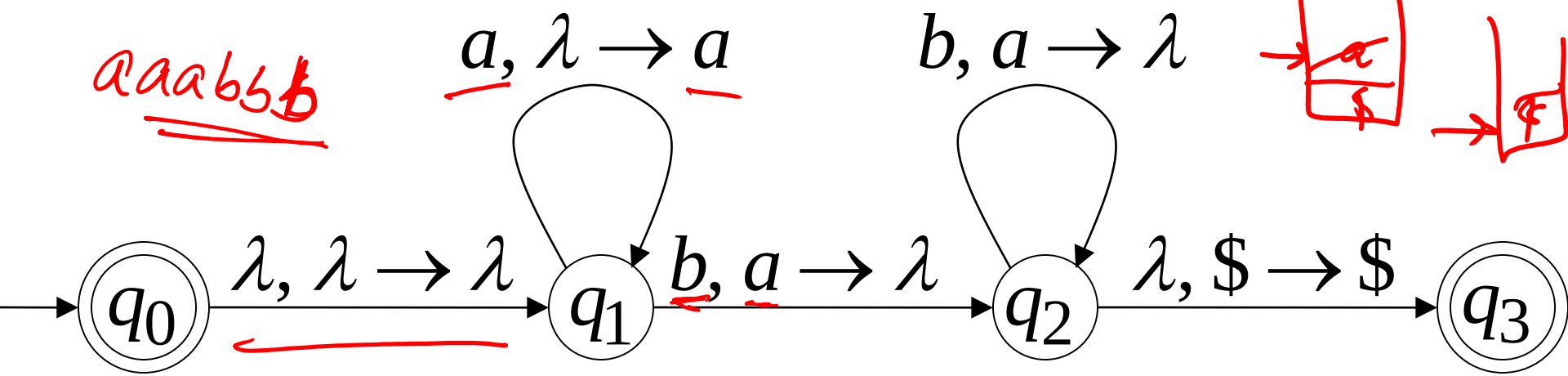
NPDA: Non-Deterministic PDA

Example:



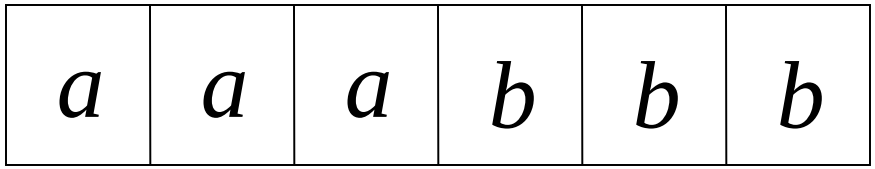


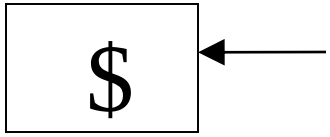
aaabbb



Execution Example: Time 0

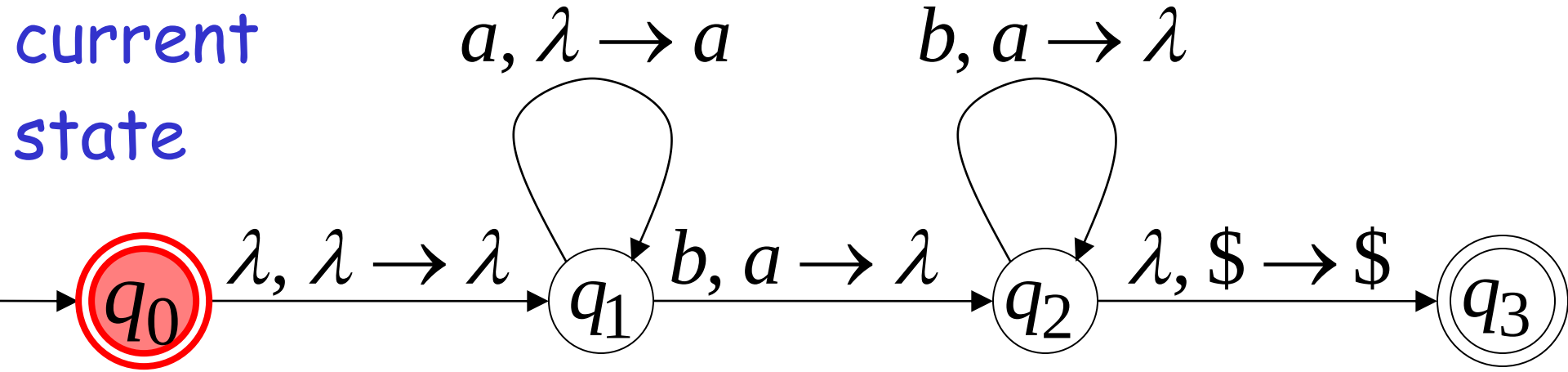
Input





Stack

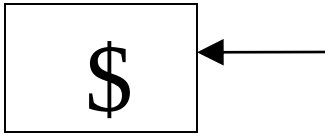
current state



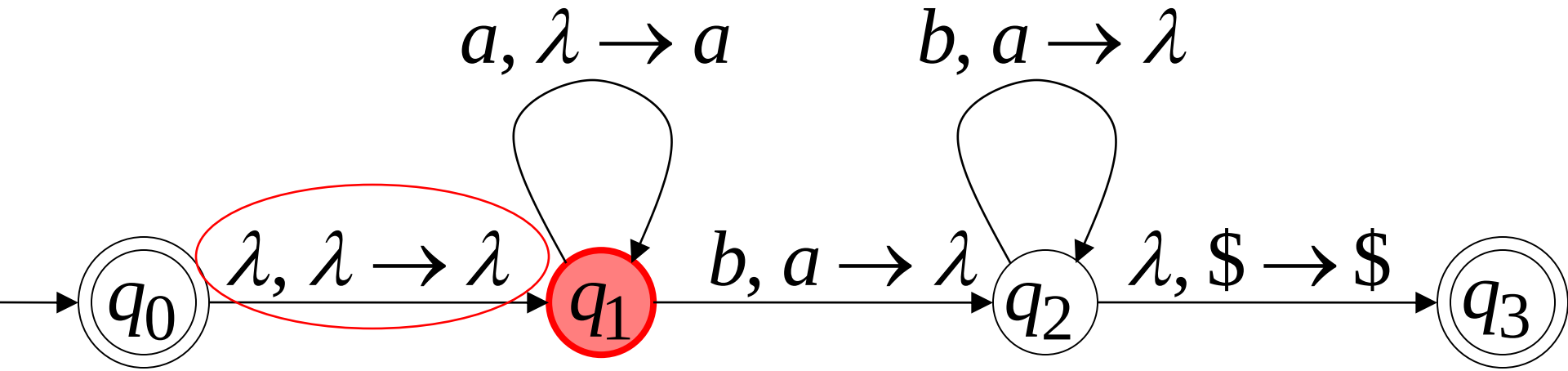
Time 1

Input

a	a	a	b	b	b

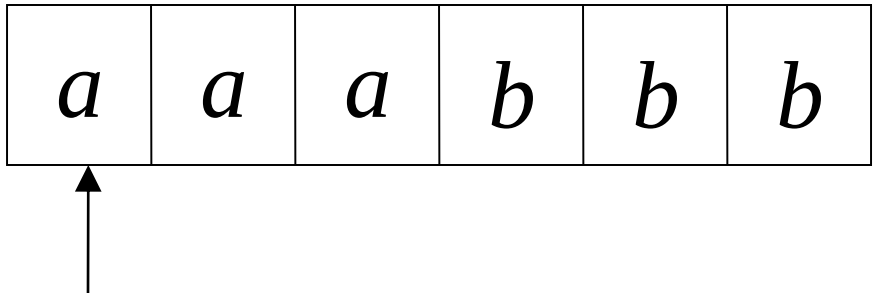


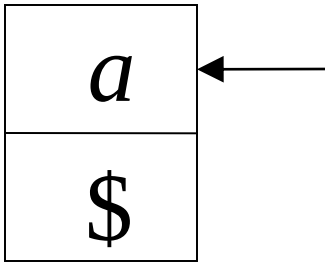
Stack



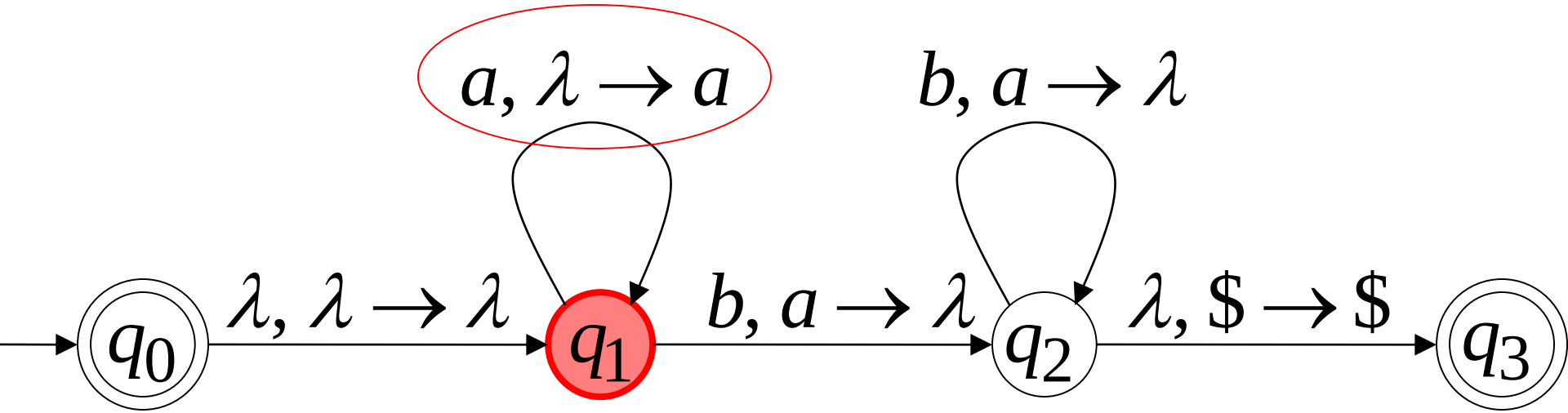
Time 2

Input





Stack



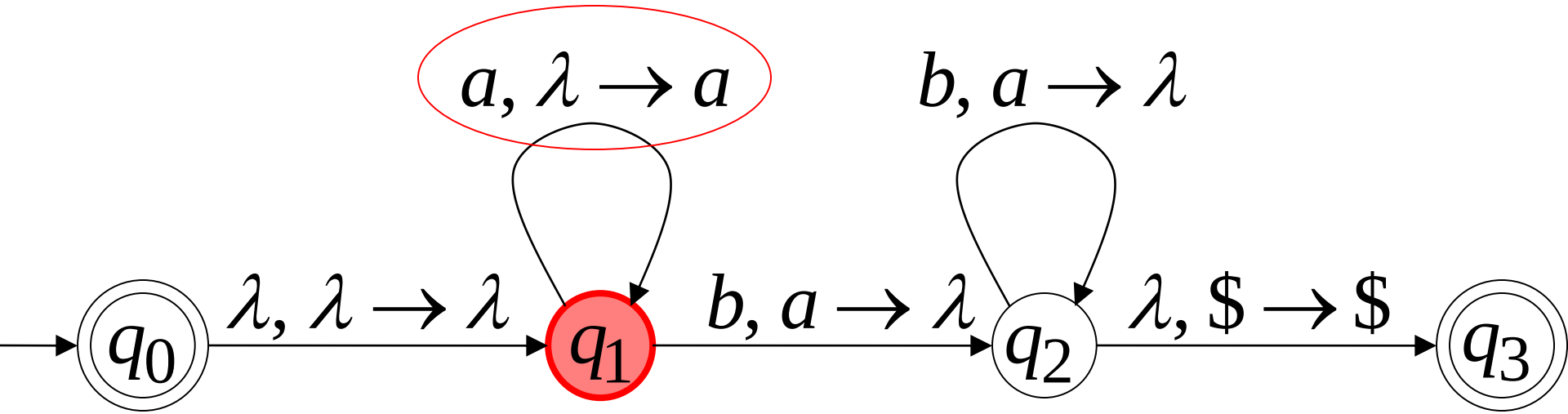
Time 3

Input

a	a	a	b	b	b

a
a
$\$$

Stack



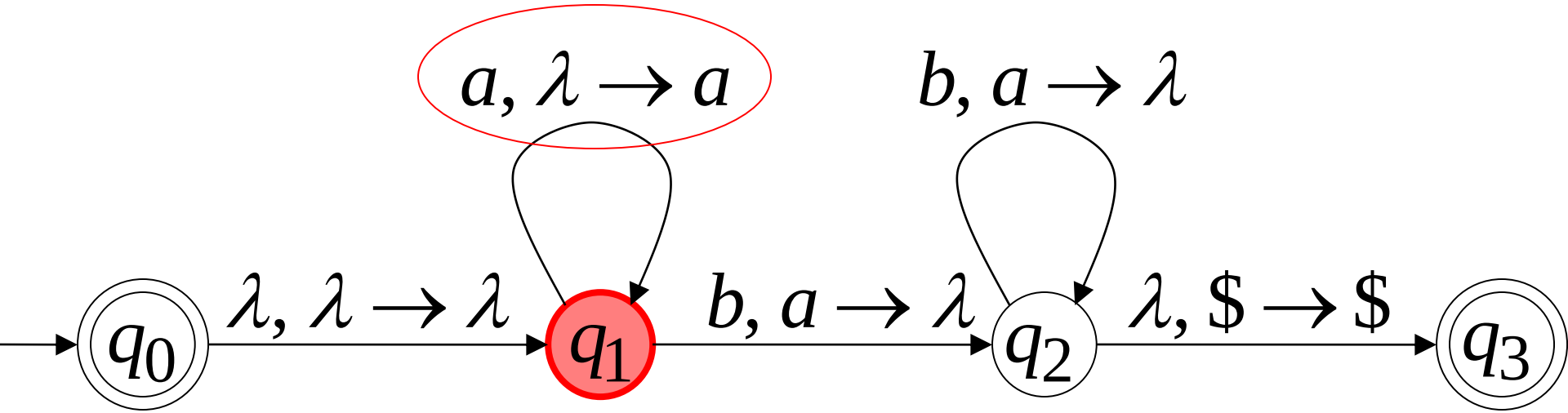
Time 4

Input

a	a	a	b	b	b

a
a
a
$\$$

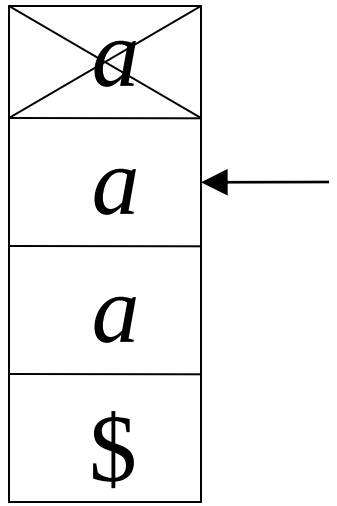
Stack



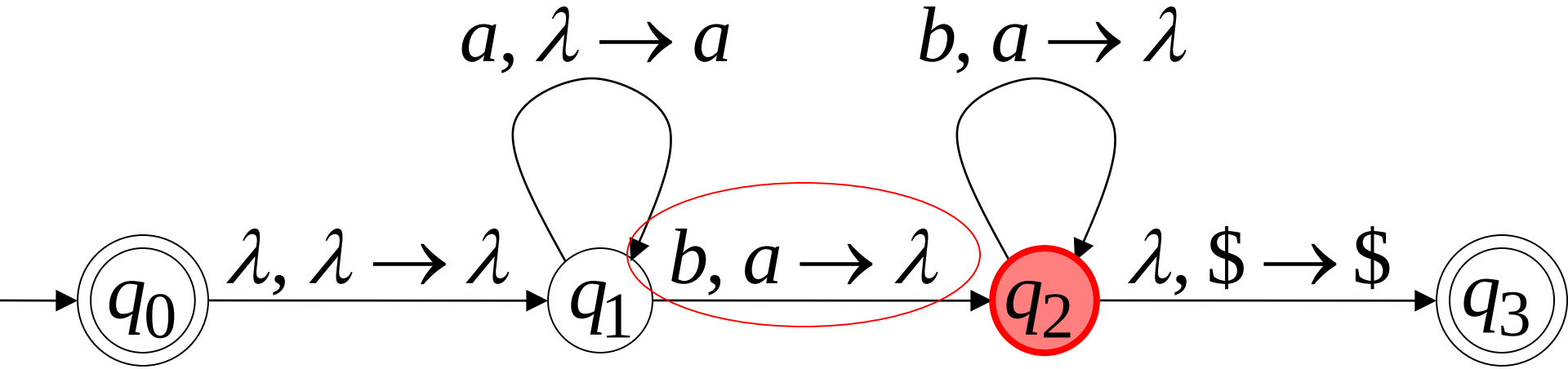
Time 5

Input

a	a	a	b	b	b

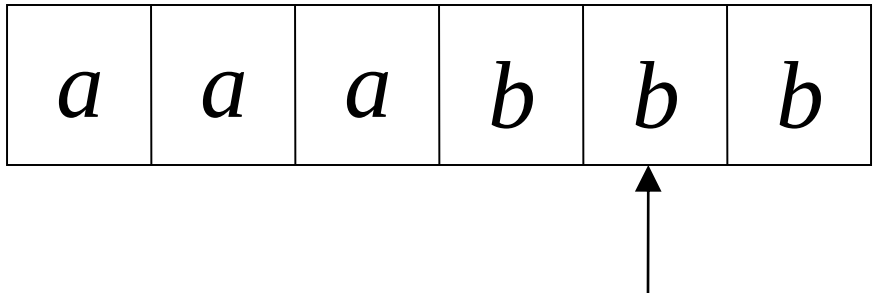


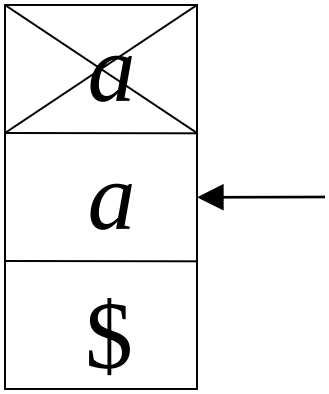
Stack



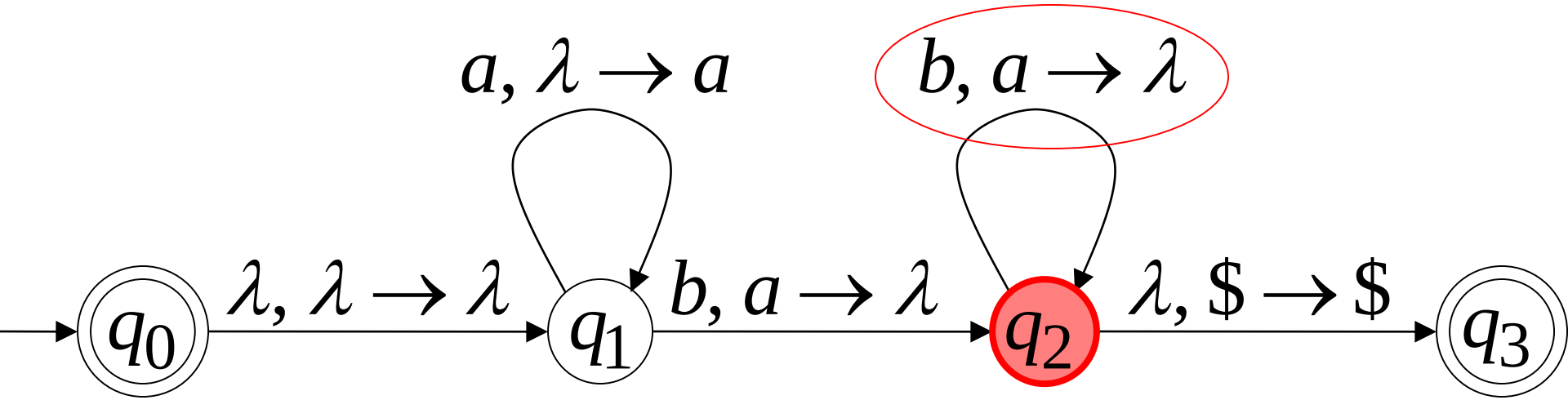
Time 6

Input



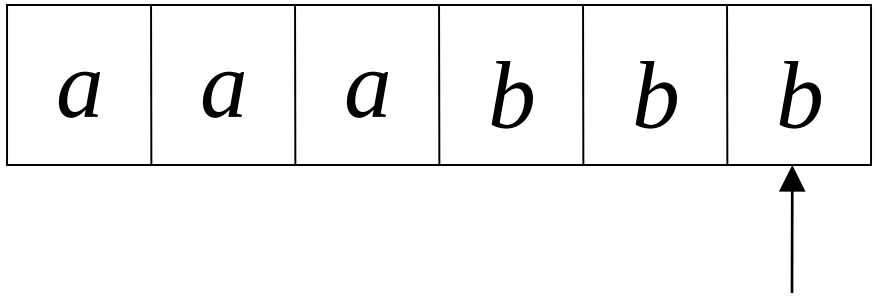


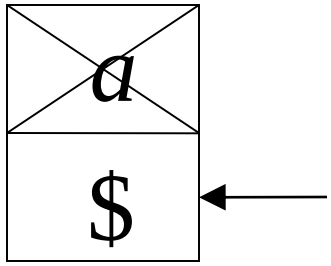
Stack



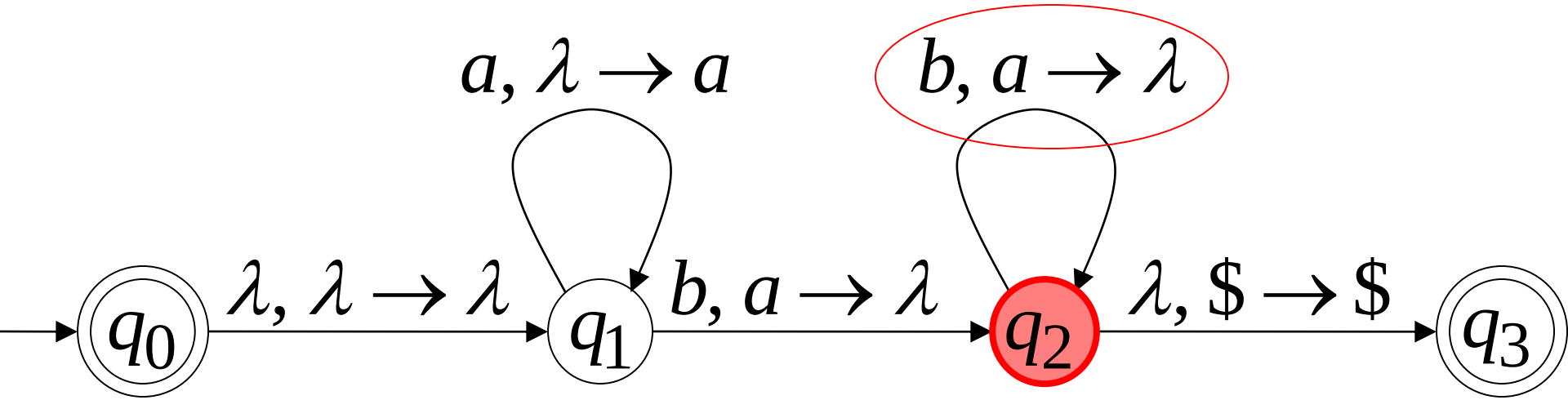
Time 7

Input



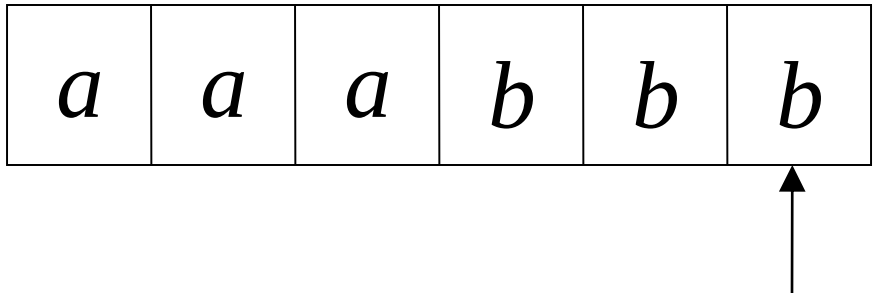


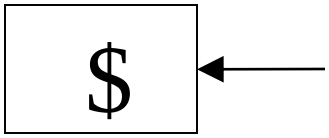
Stack



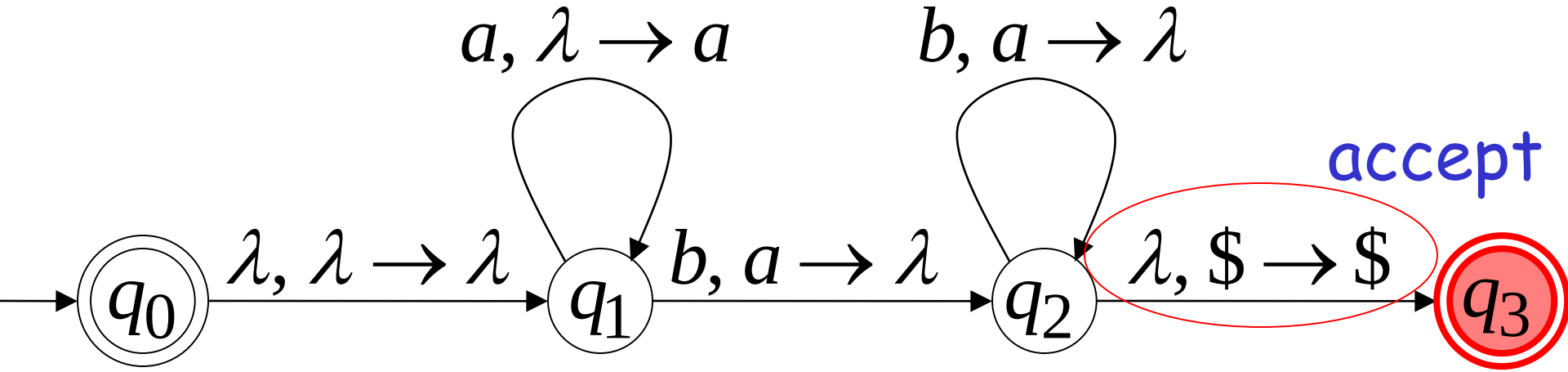
Time 8

Input





Stack



A string is accepted if there is
a computation such that:

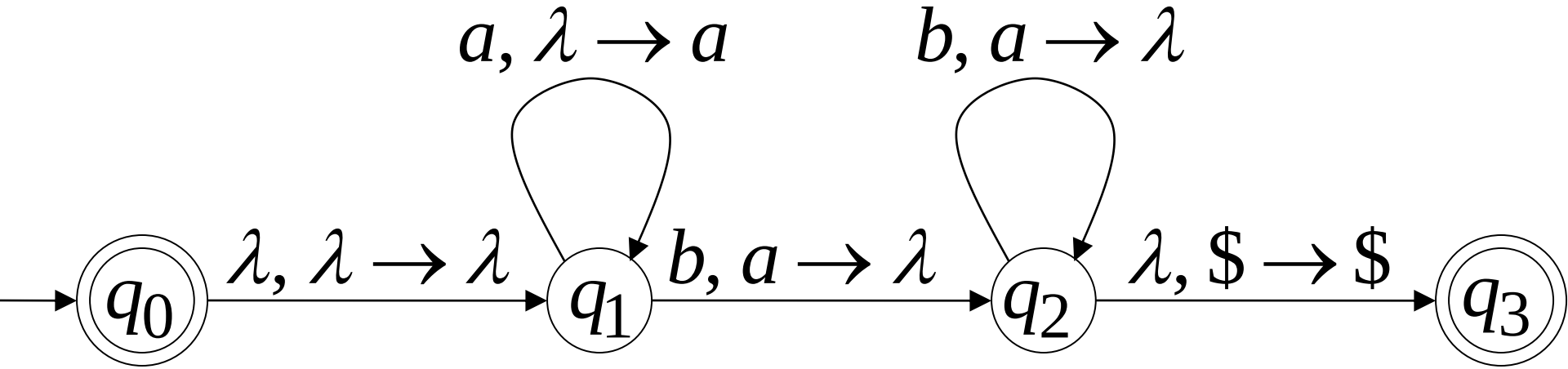
All the input is consumed

AND

The last state is a final state

At the end of the computation,
we do not care about the stack contents

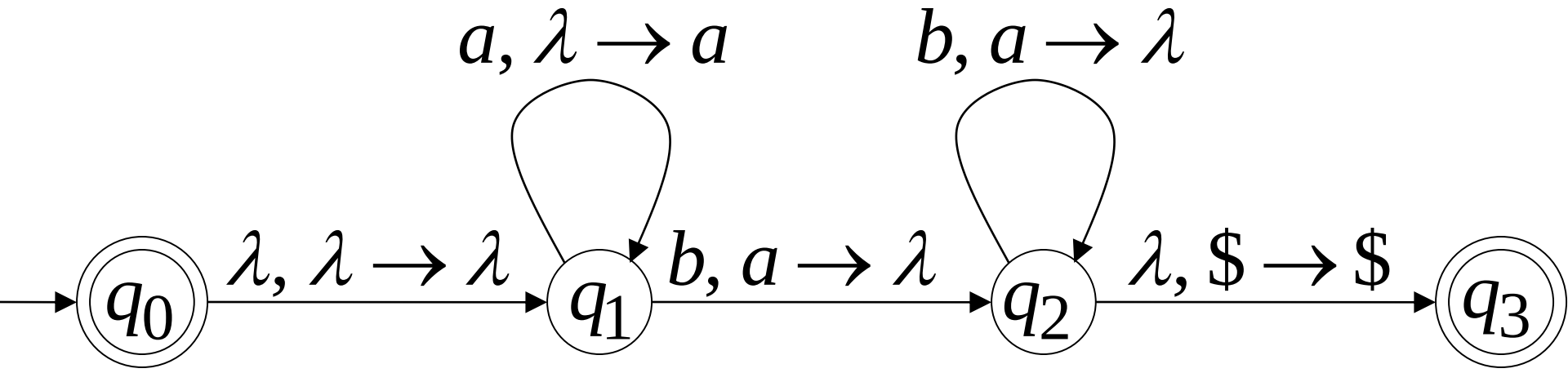
The input string $aaabbb$
is accepted by the NPDA:



In general,

$$L = \{a^n b^n : n \geq 0\}$$

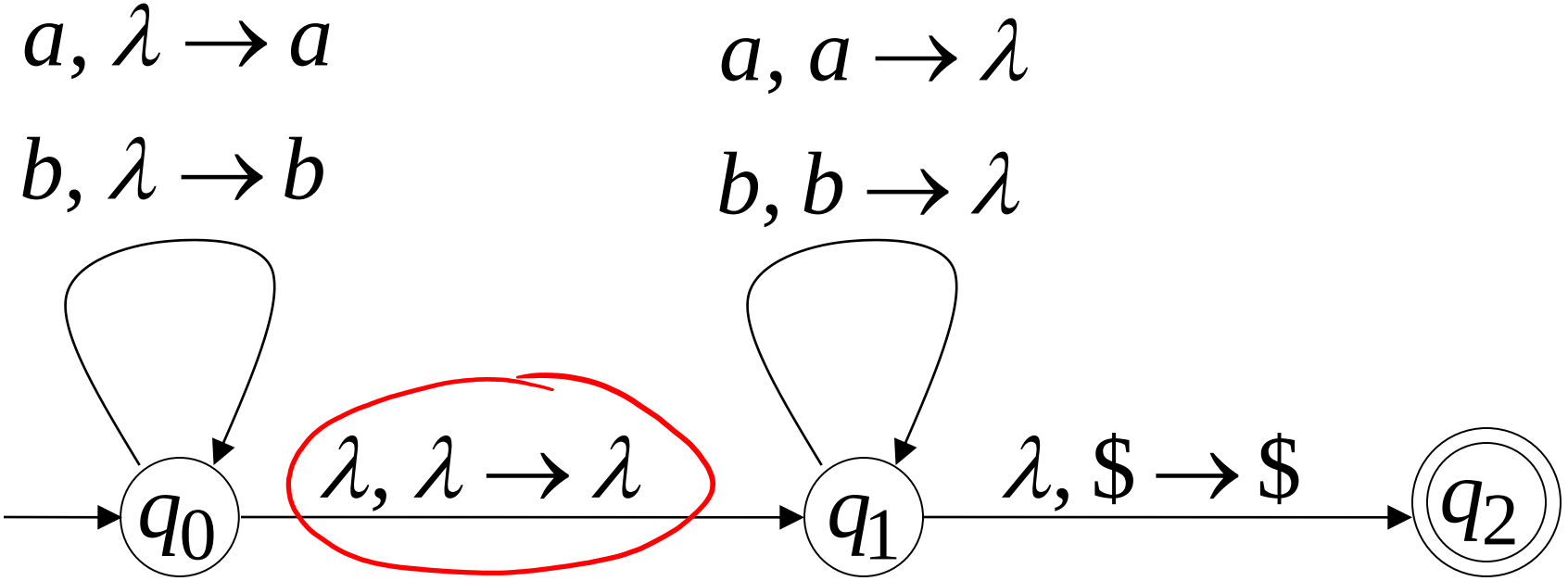
is the language accepted by the NPDA:



Another NPDA example

NPDA M

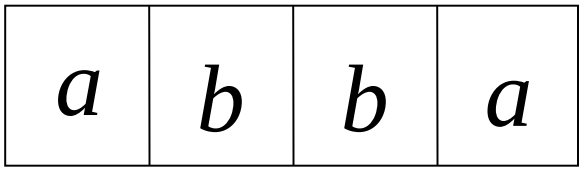
$$L(M) = \{ww^R\}$$

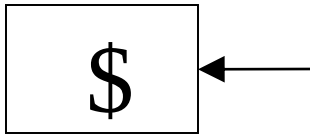


Execution Example:

Time 0

Input





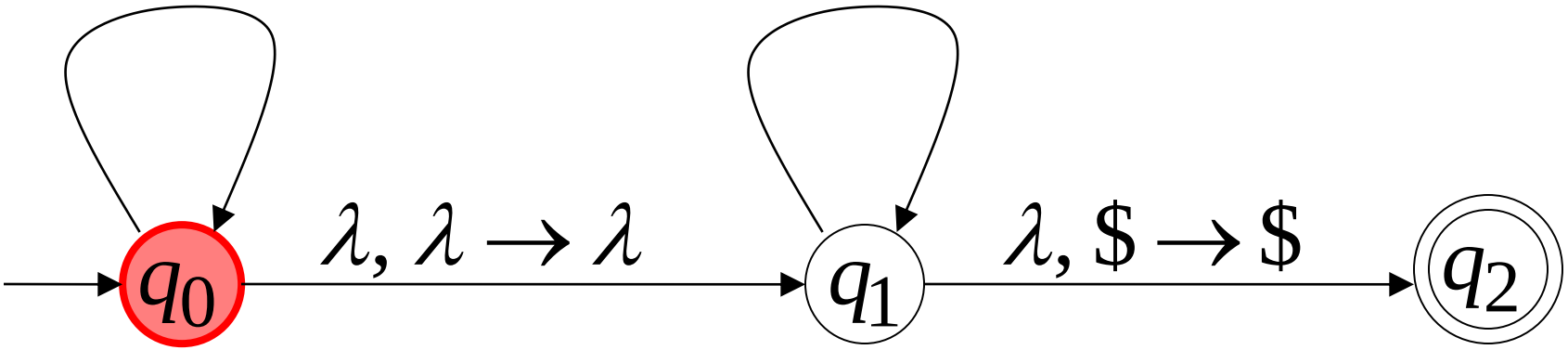
Stack

$a, \lambda \to a$

$b, \lambda \to b$

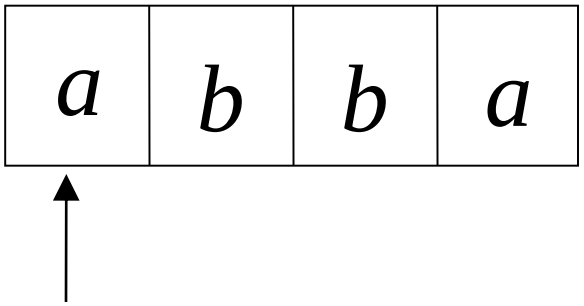
$a, a \to \lambda$

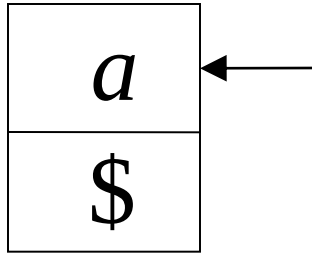
$b, b \to \lambda$



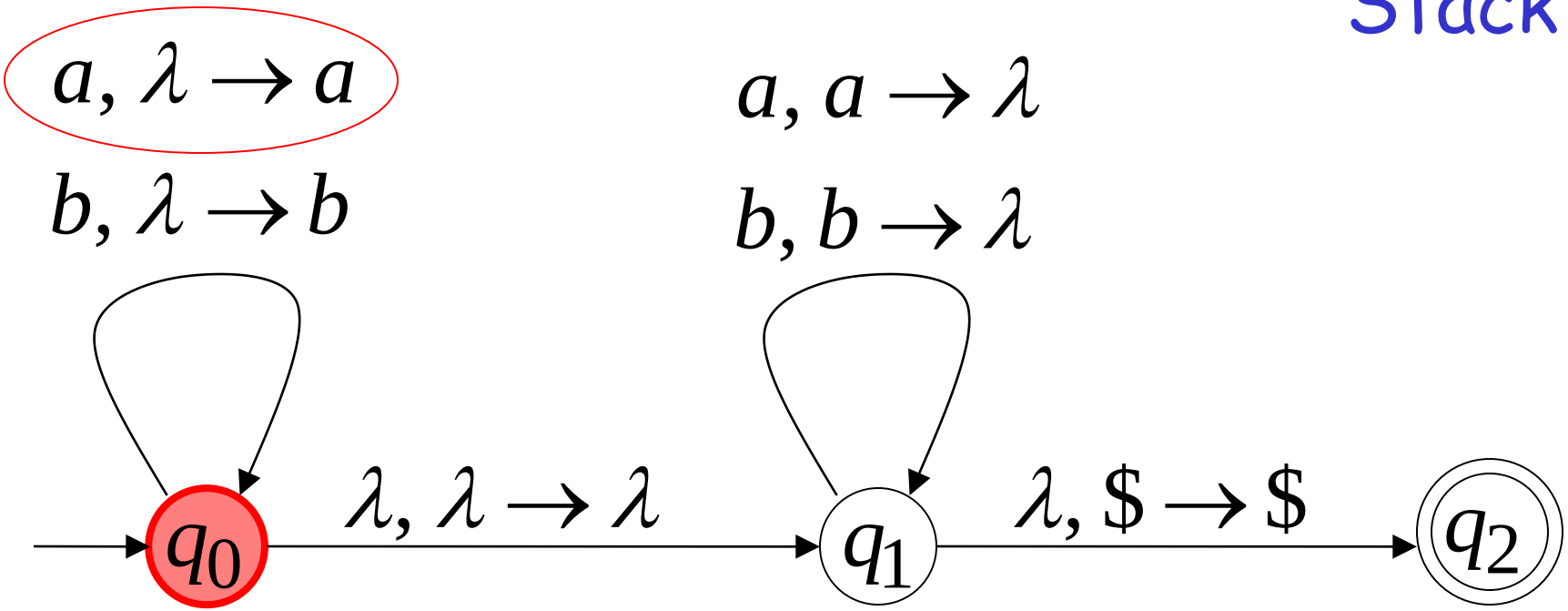
Time 1

Input

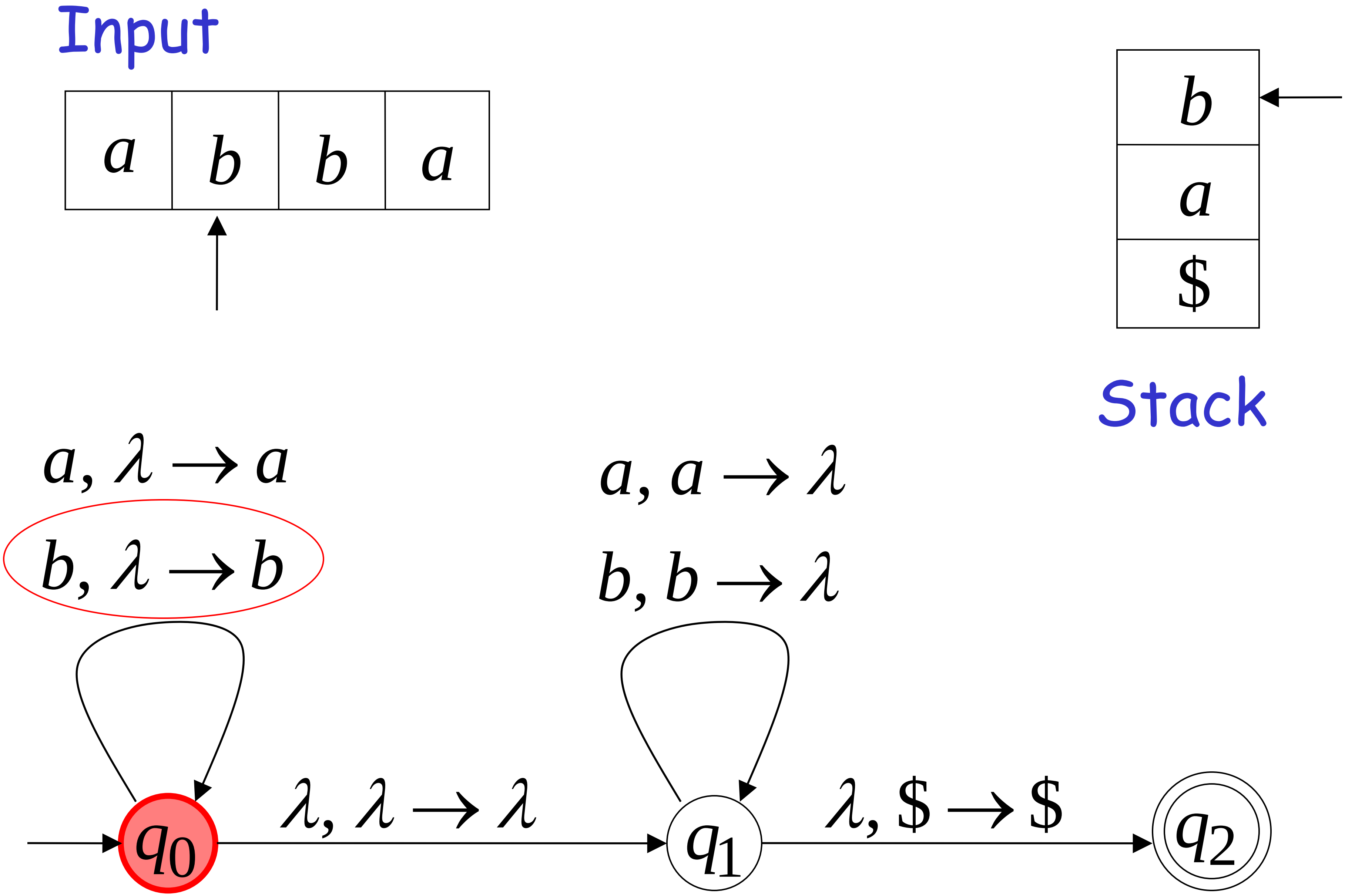




Stack



Time 2



Time 3

Input

a	b	b	a

Guess the middle of string

b
a
\$

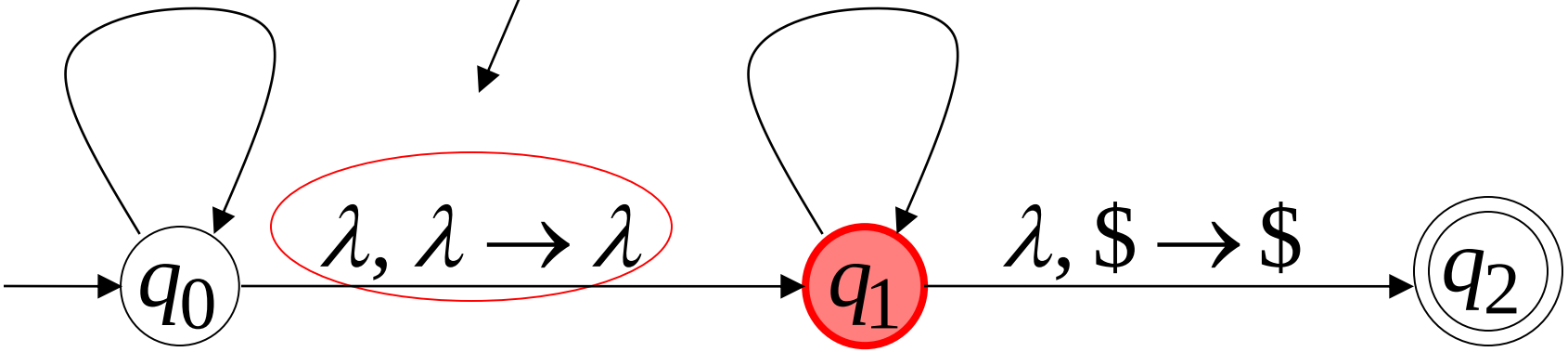
Stack

$a, \lambda \to a$

$b, \lambda \to b$

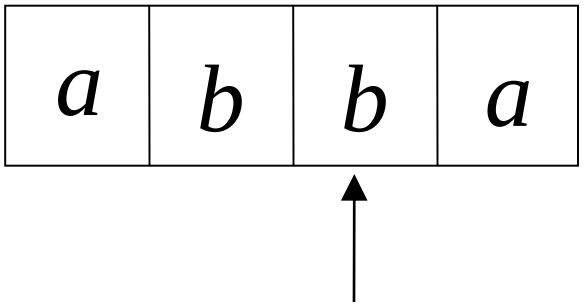
$a, a \to \lambda$

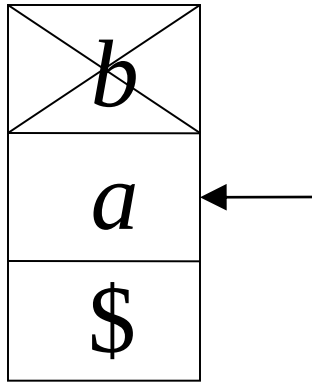
$b, b \to \lambda$



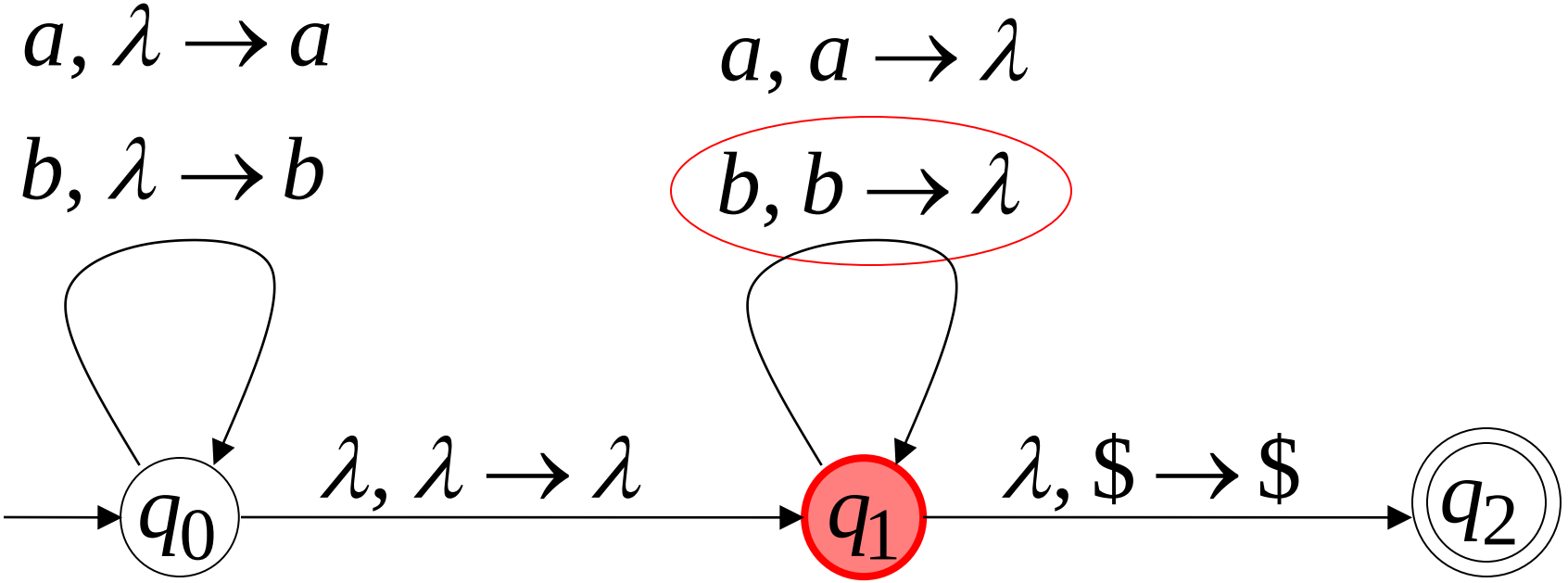
Time 4

Input



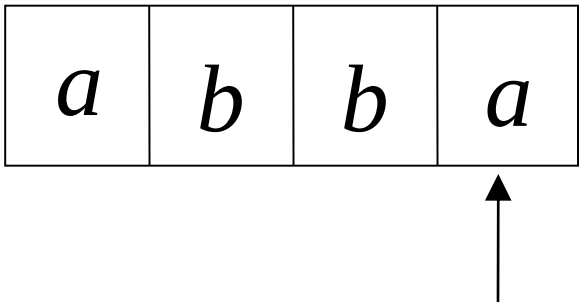


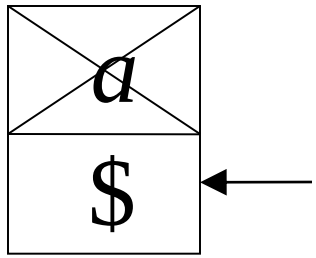
Stack



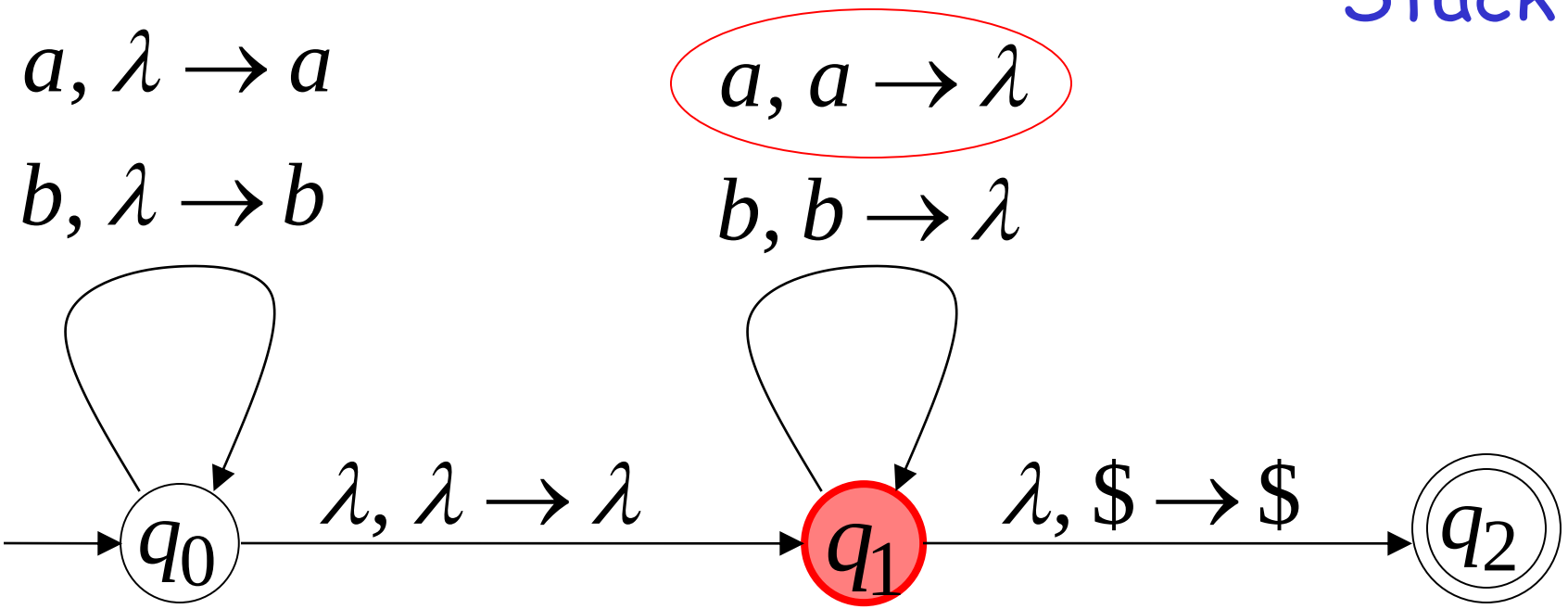
Time 5

Input



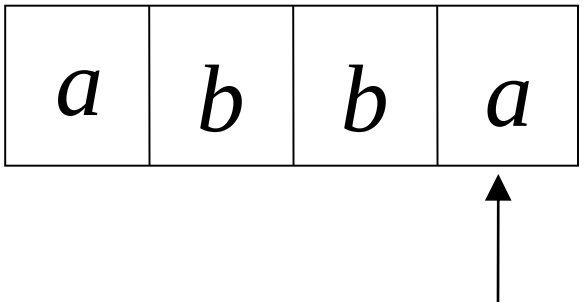


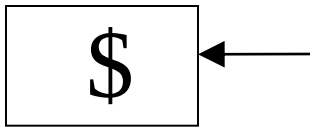
Stack



Time 6

Input





Stack

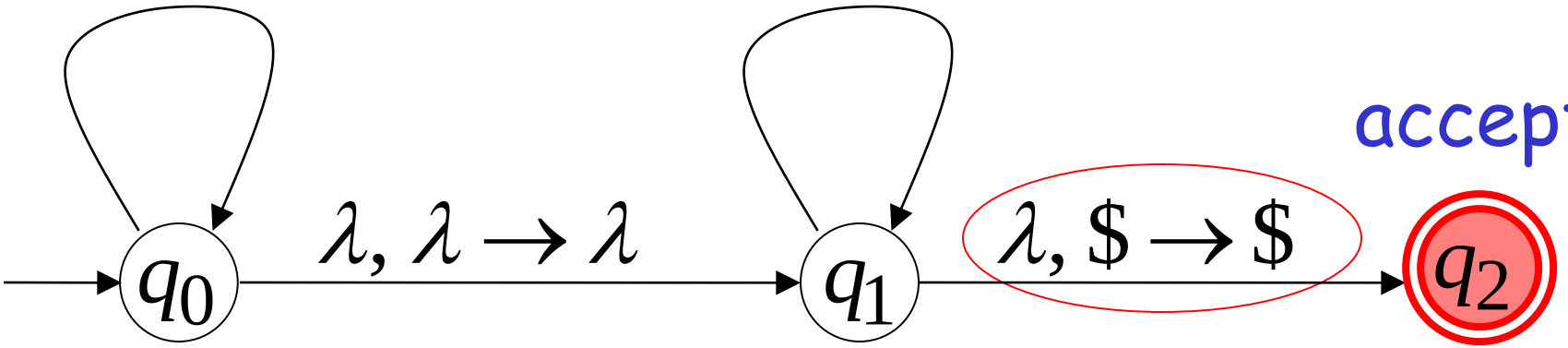
$a, \lambda \to a$

$b, \lambda \to b$

$a, a \to \lambda$

$b, b \to \lambda$

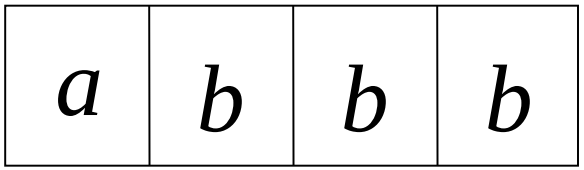
accept

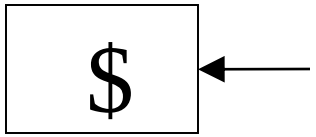


Rejection Example:

Time 0

Input





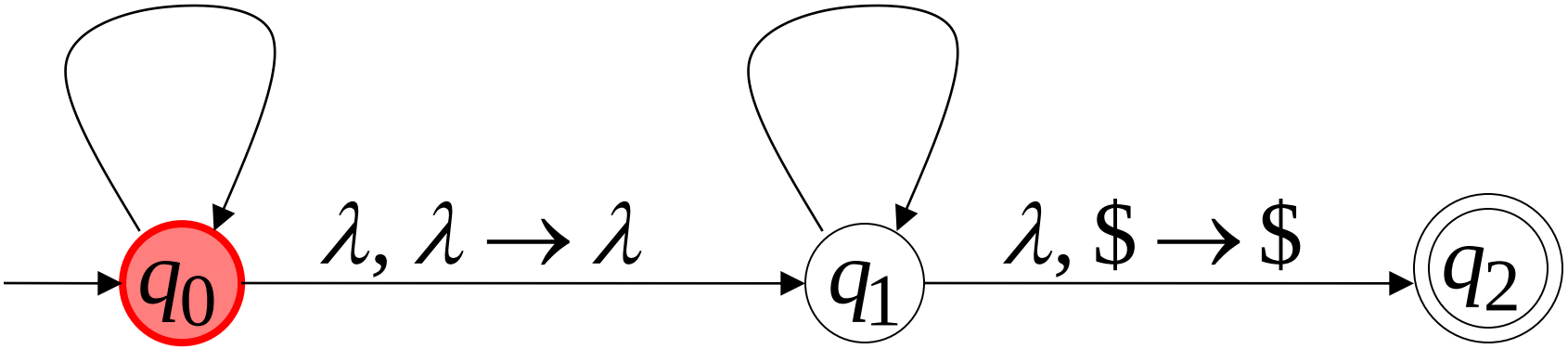
Stack

$a, \lambda \to a$

$b, \lambda \to b$

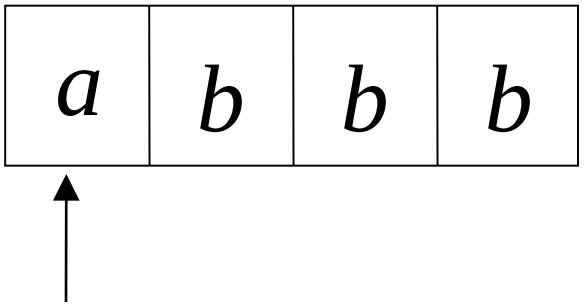
$a, a \to \lambda$

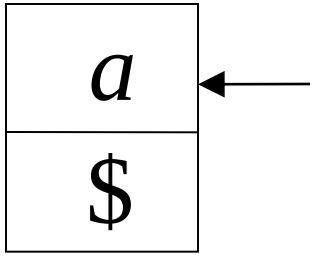
$b, b \to \lambda$



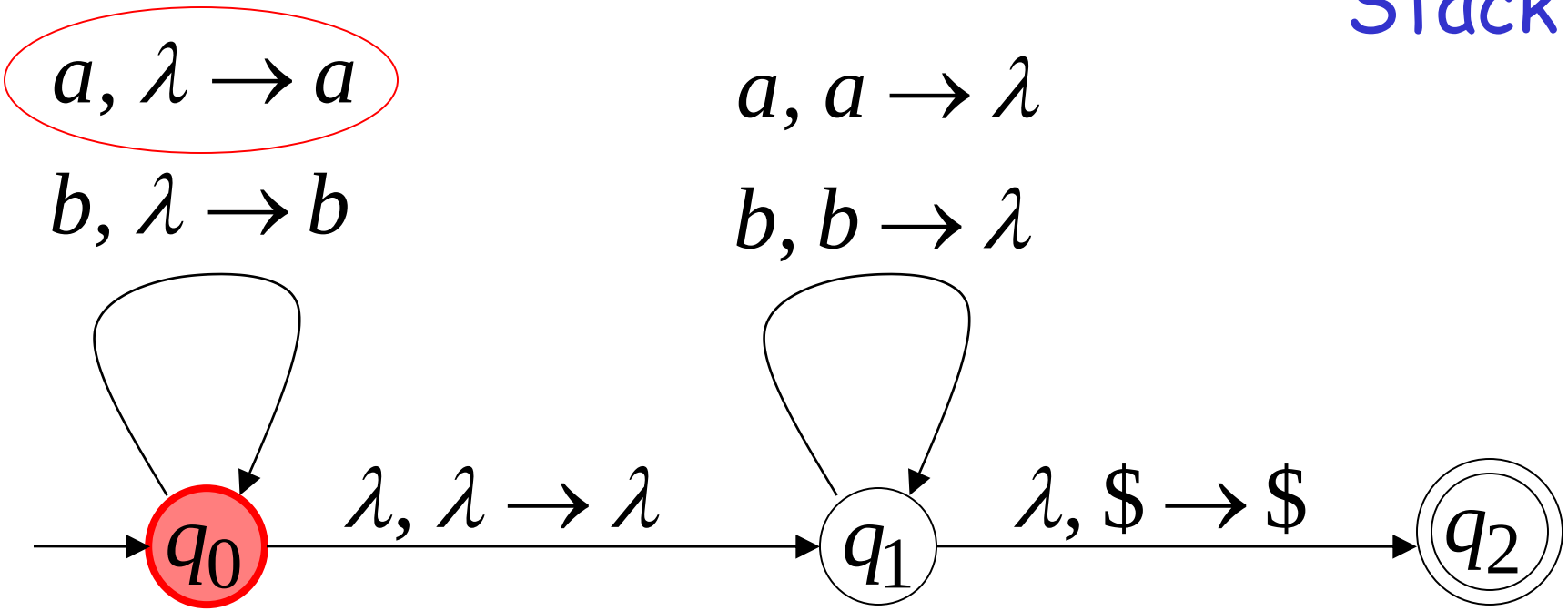
Time 1

Input



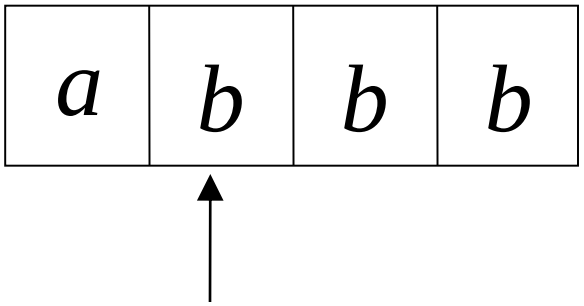


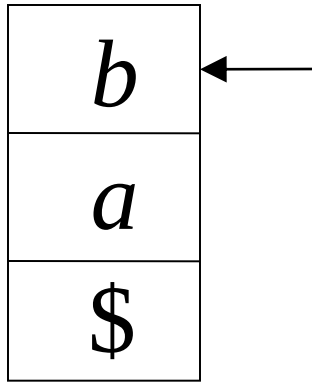
Stack



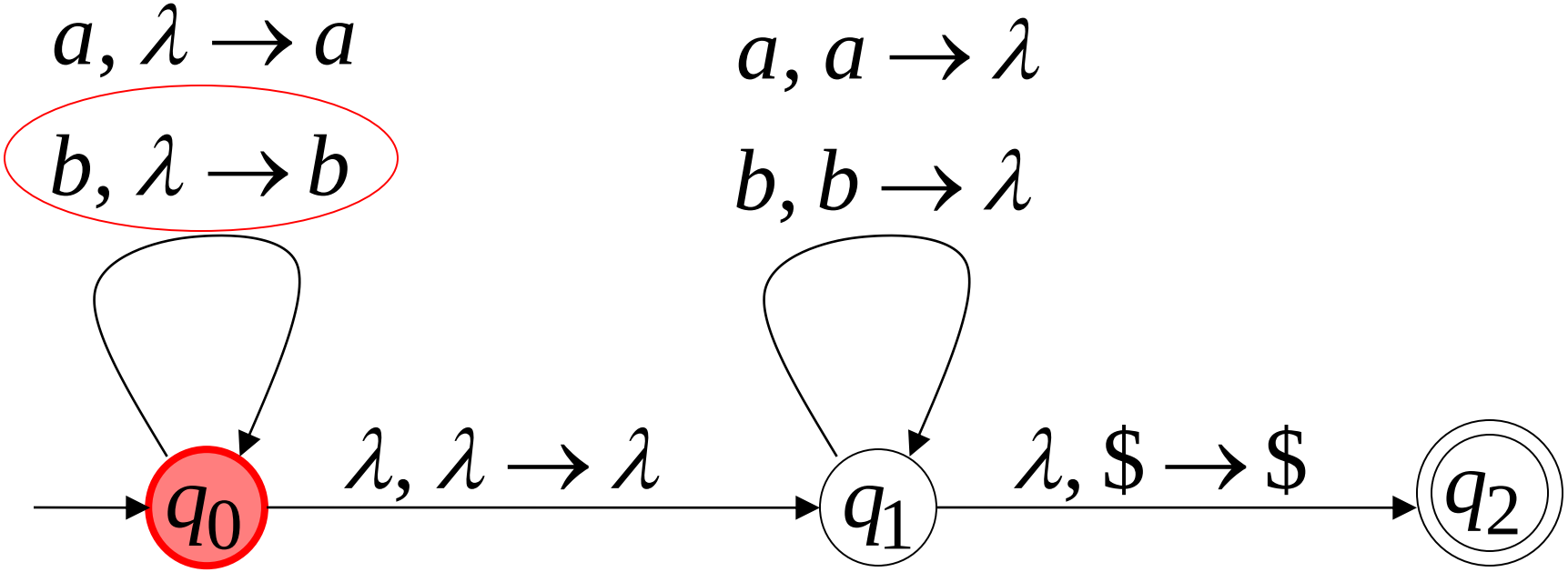
Time 2

Input



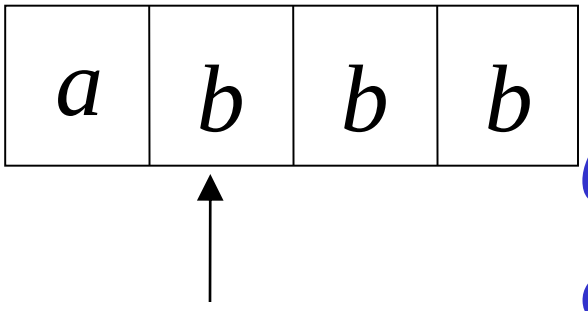


Stack

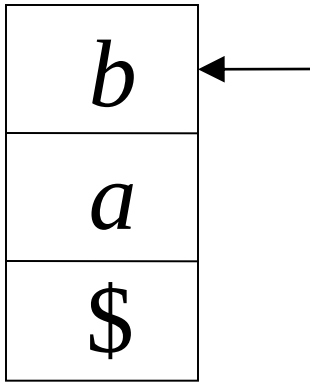


Time 3

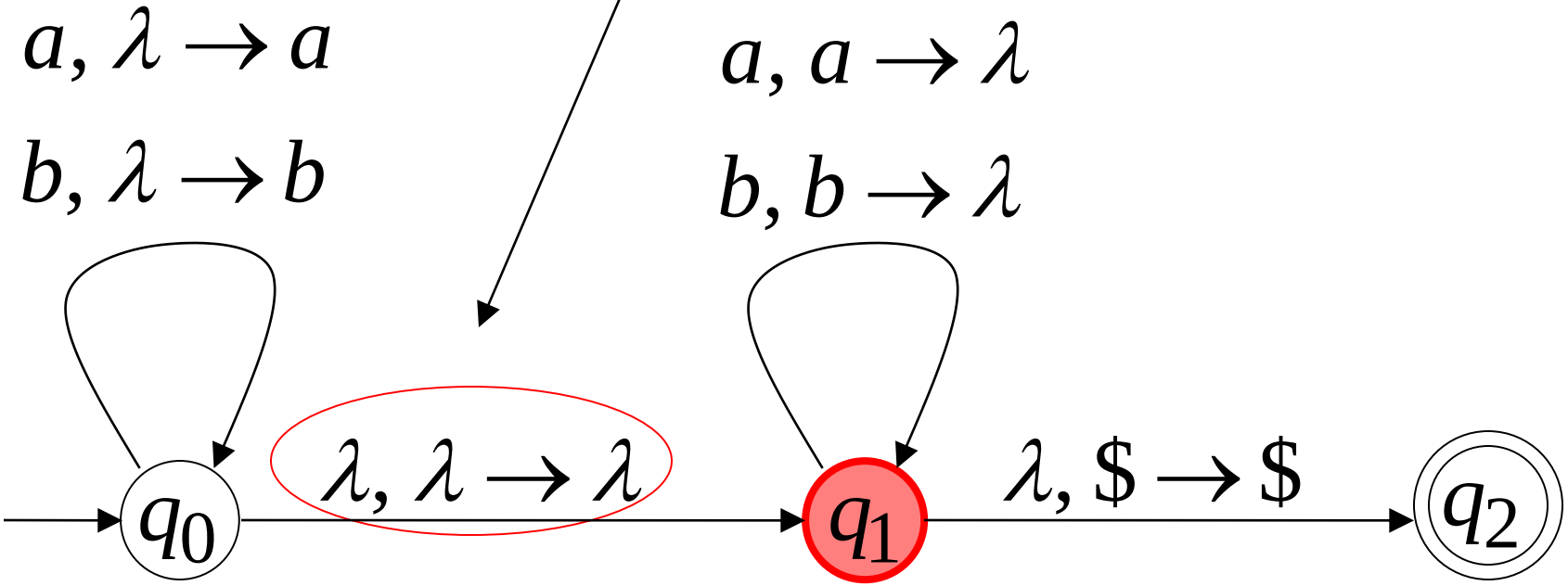
Input



Guess the middle of string

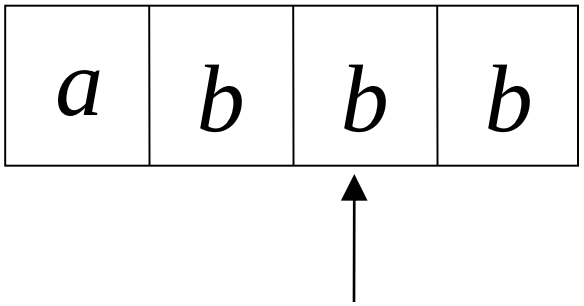


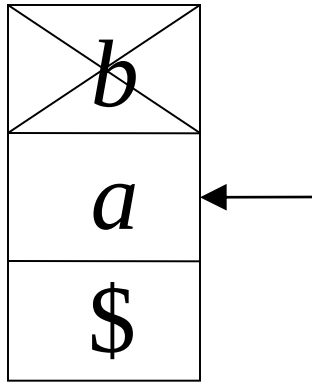
Stack



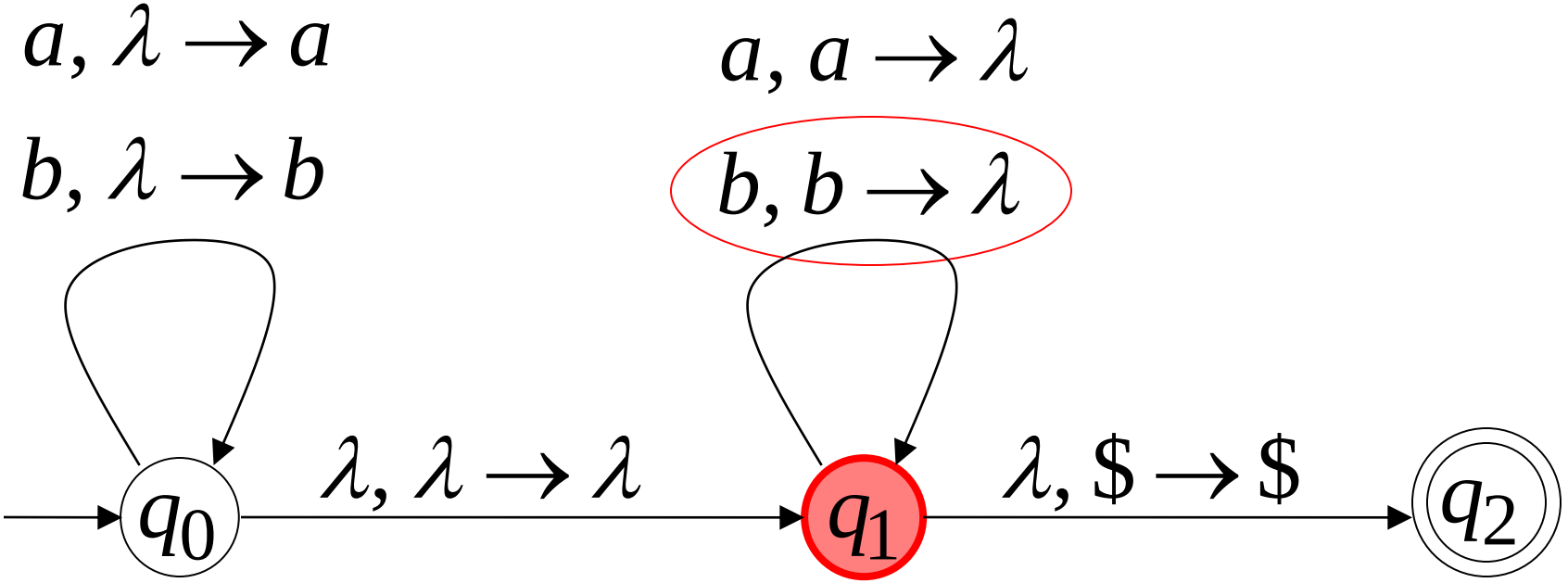
Time 4

Input





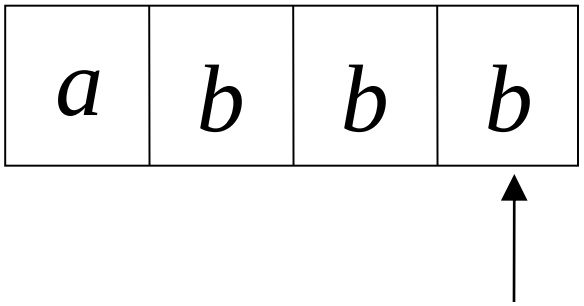
Stack



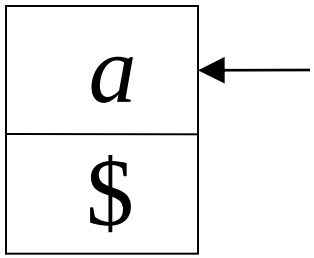
Time 5

There is no possible transition.

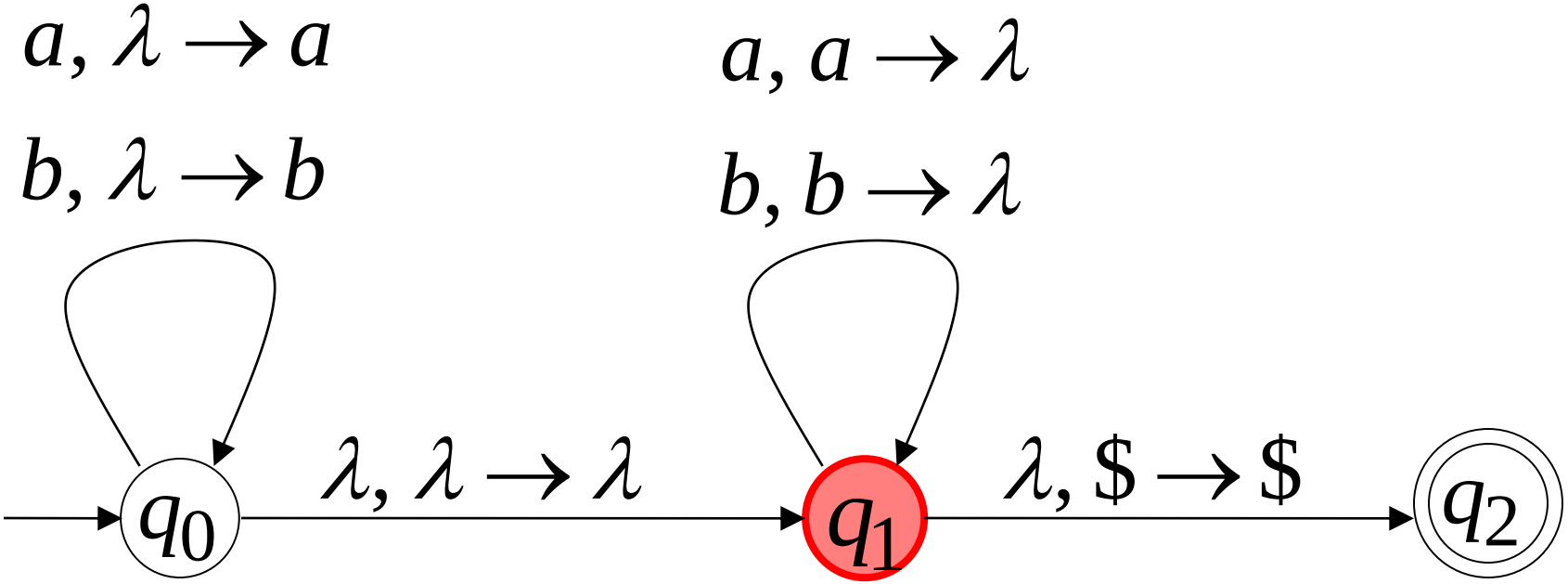
Input



Input is not consumed



Stack



Another computation on same string:

Input

Time 0

a	b	b	b

$\$$

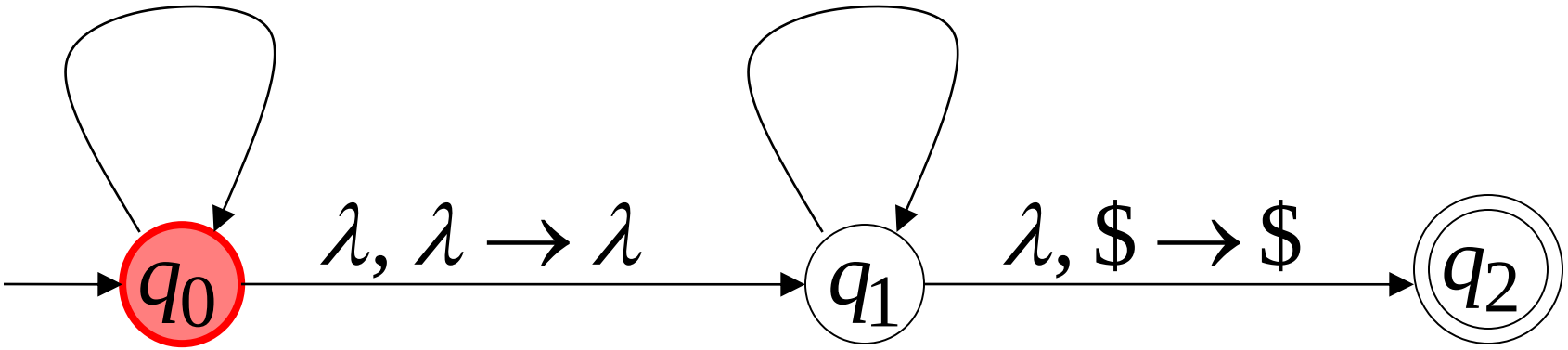
Stack

$a, \lambda \to a$

$b, \lambda \to b$

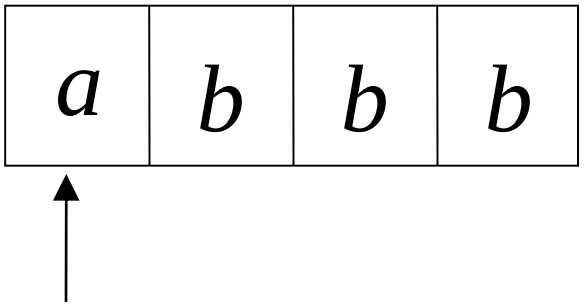
$a, a \to \lambda$

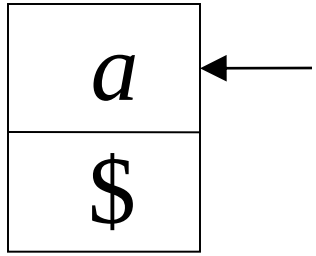
$b, b \to \lambda$



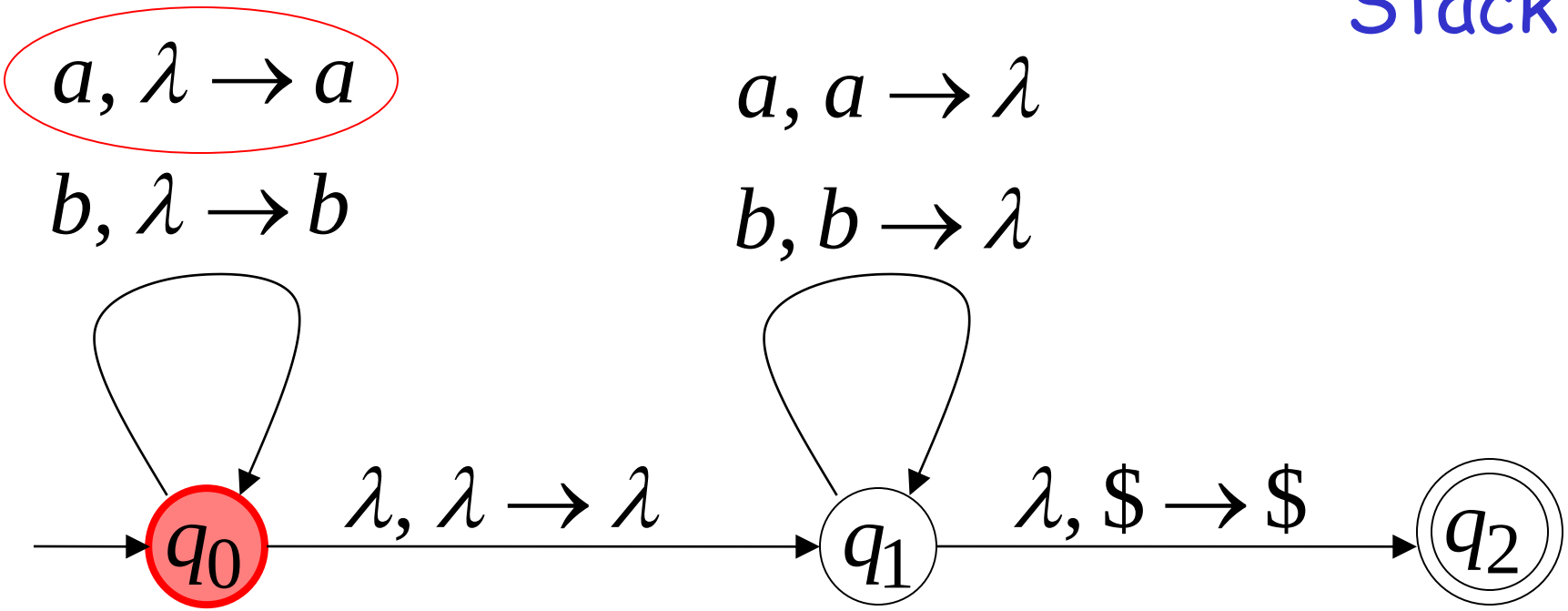
Time 1

Input



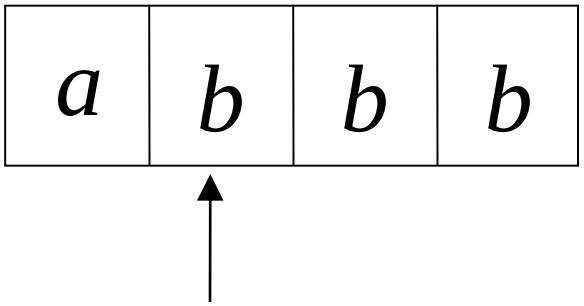


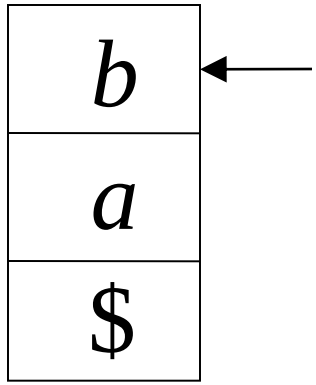
Stack



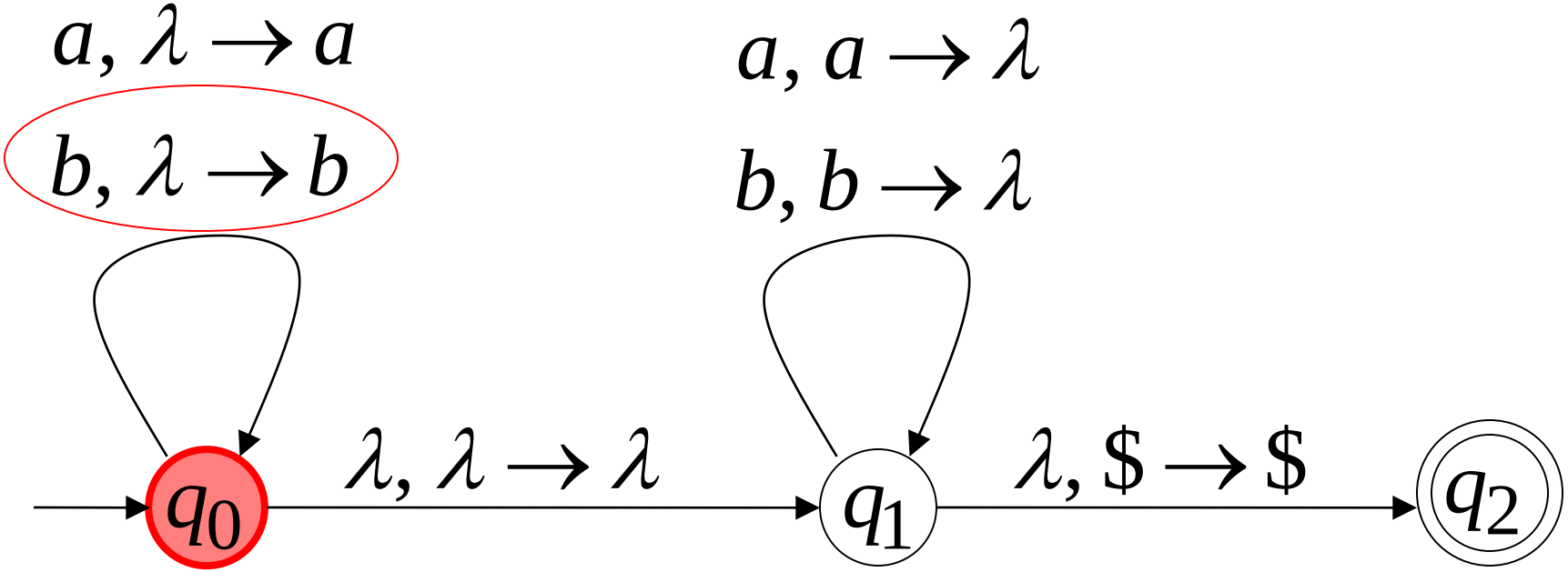
Time 2

Input



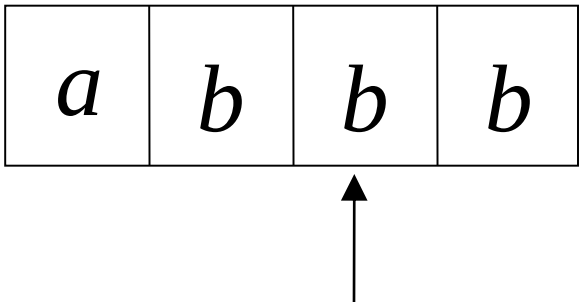


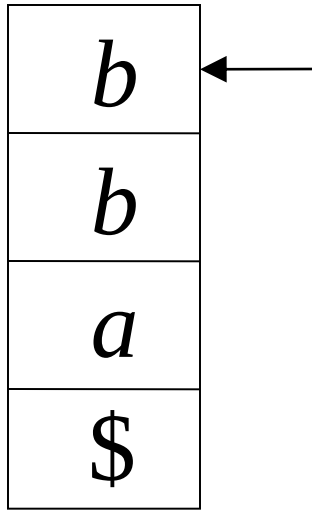
Stack



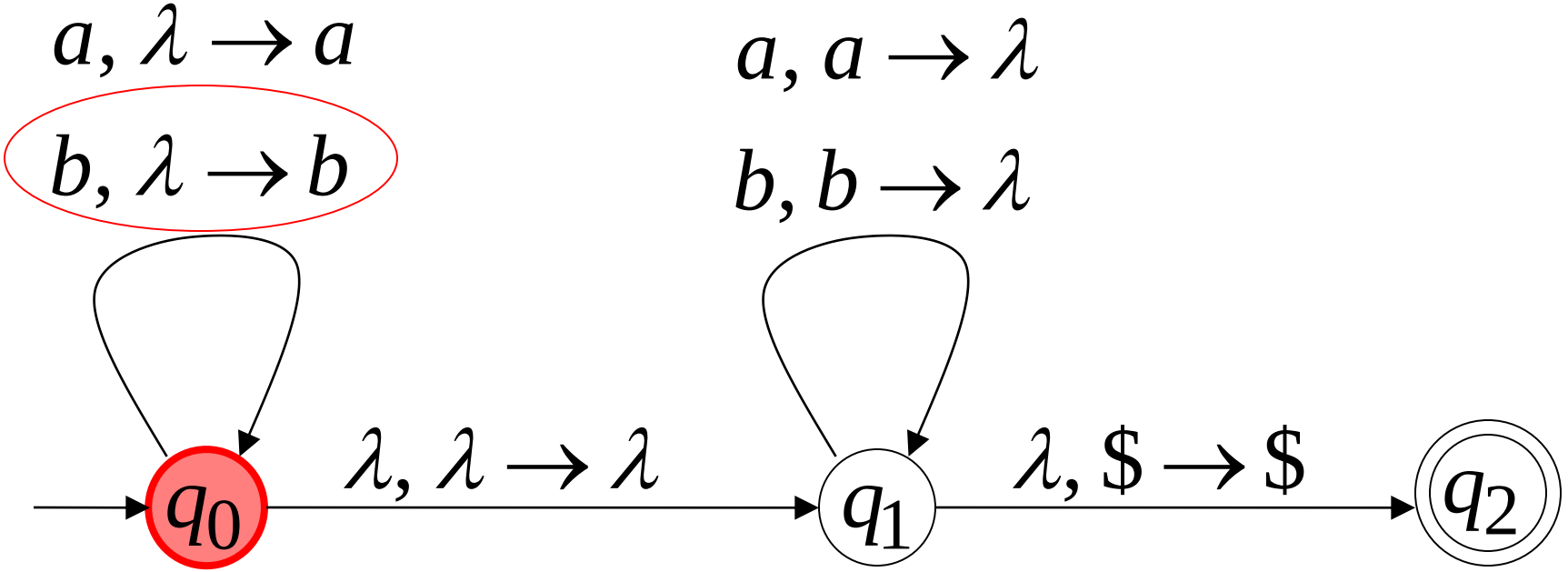
Time 3

Input



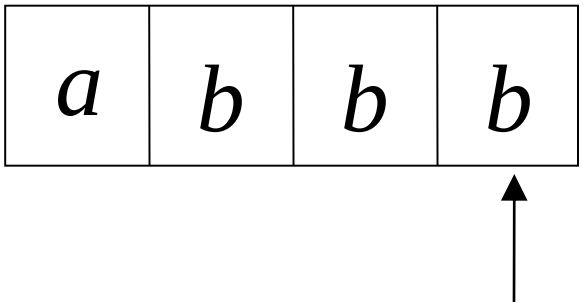


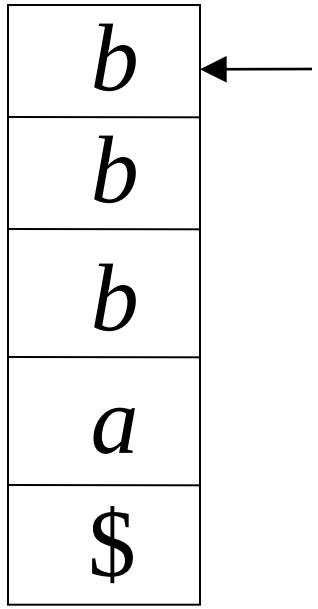
Stack



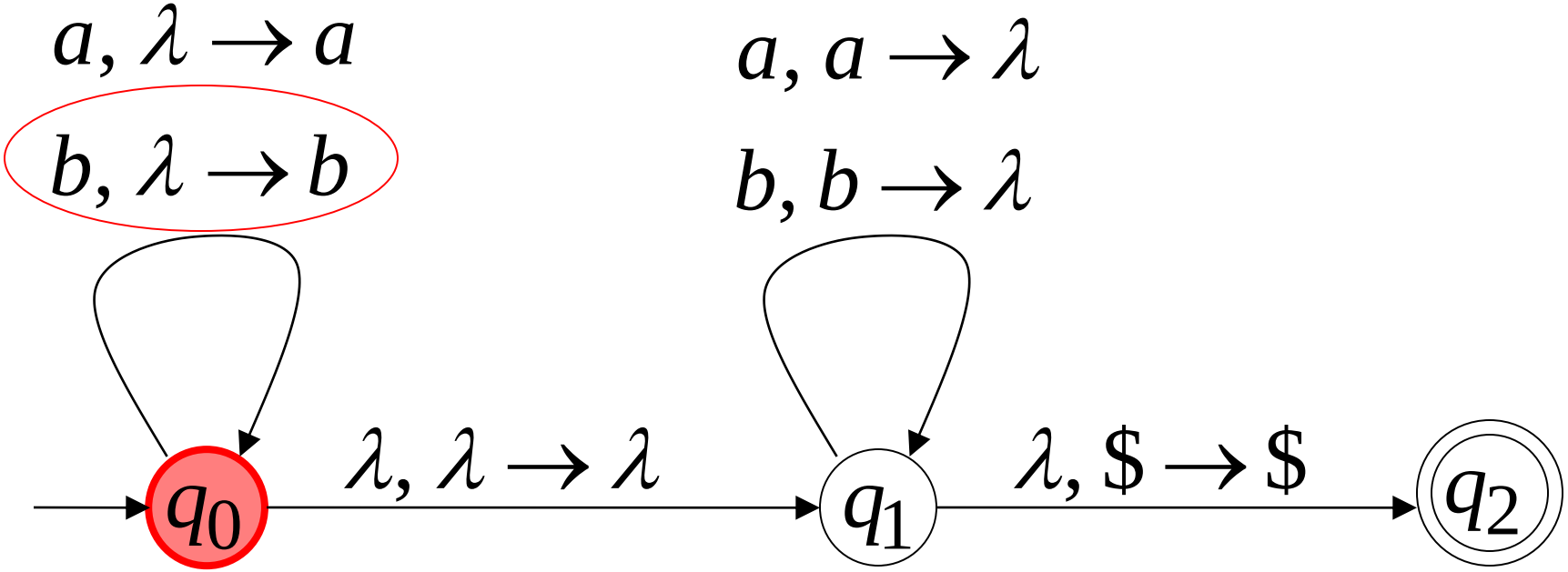
Time 4

Input





Stack



Time 5

Input

a	b	b	b

No final state is reached

b
b
b
a
$

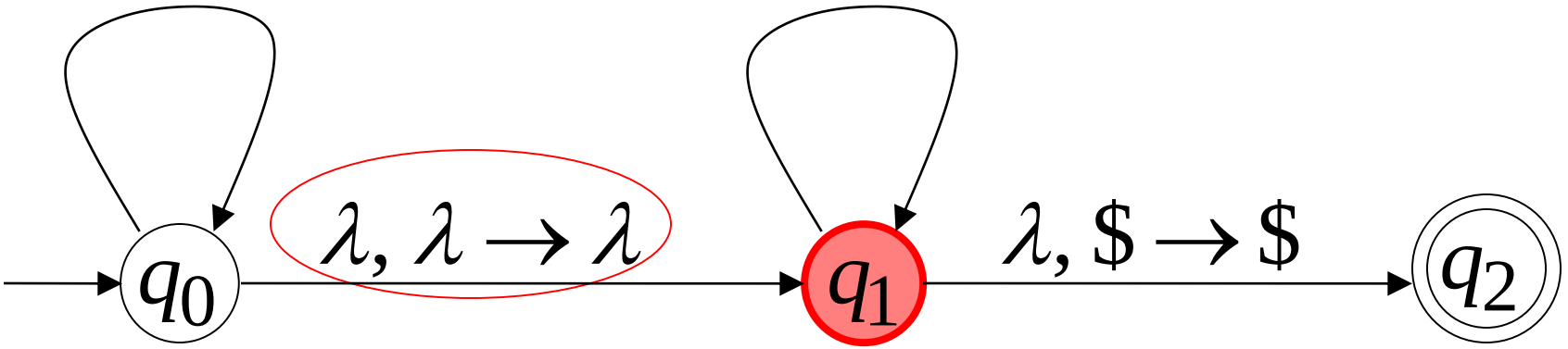
Stack

$a, \lambda \rightarrow a$

$b, \lambda \rightarrow b$

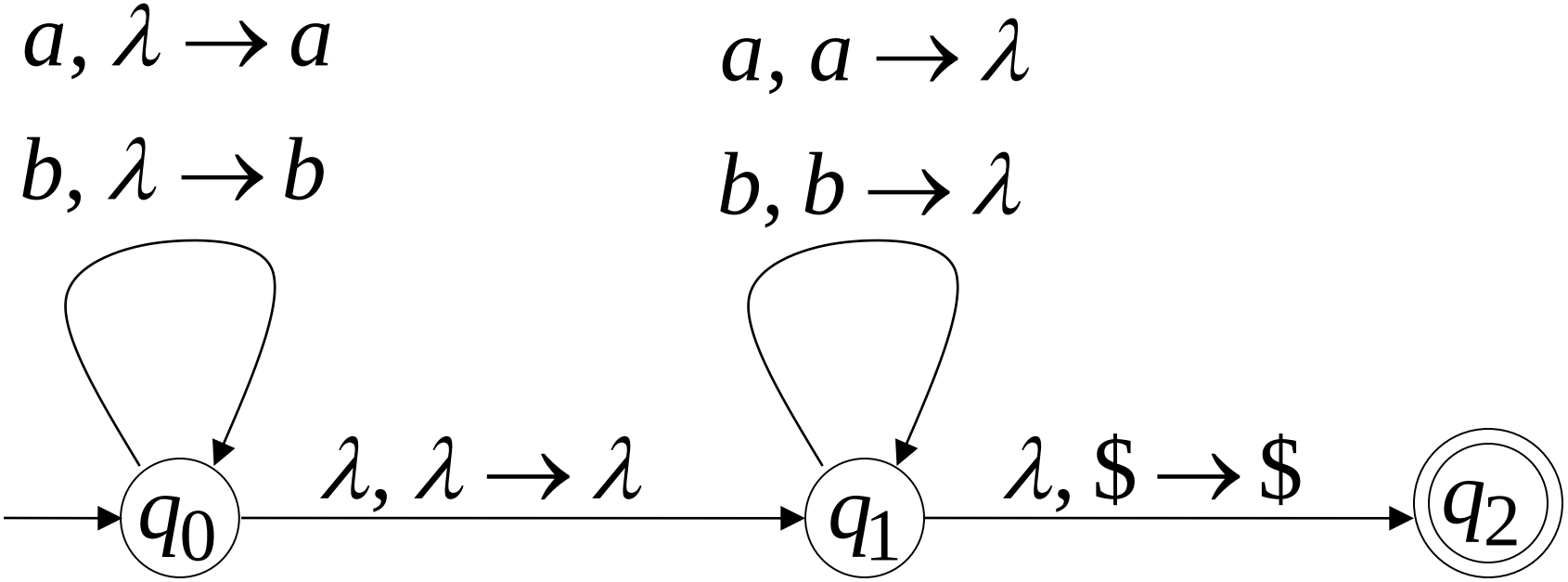
$a, a \rightarrow \lambda$

$b, b \rightarrow \lambda$



There is no computation that accepts string $abbb$

$$abbb \notin L(M)$$



A string is rejected if there is
NO computation such that:

All the input is consumed

AND

The last state is a final state

At the end of the computation,
we do not care about the stack contents

In other words, a string is rejected if in every computation with this string:

The input cannot be consumed

OR

The input is consumed and the last state is not a final state

OR

The stack head moves below the bottom of the stack

Another NPDA example

NPDA M

$$L(M) = \{a^n b^m \ : \ n \geq m-1\}$$

Another NPDA example

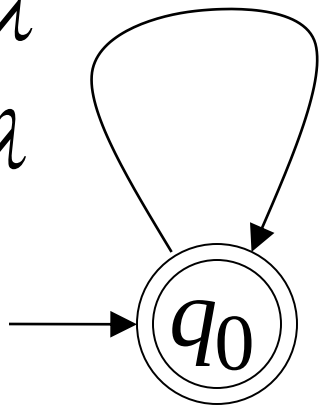
NPDA M

$$L(M) = \{a^n b^m \ : \ n \geq m-1\}$$

$a, \lambda \rightarrow a$

$b, a \rightarrow \lambda$

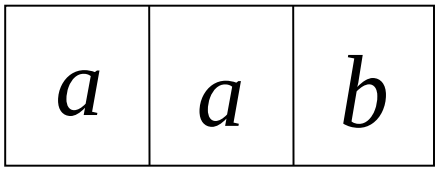
$b, \$ \rightarrow \lambda$



Execution Example:

Time 0

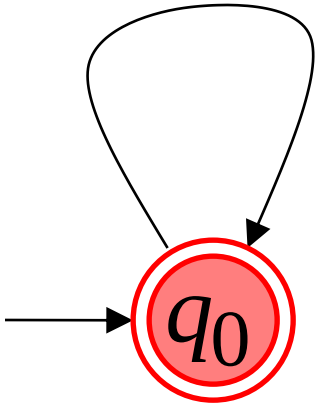
Input

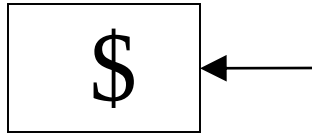


$a, \lambda \rightarrow a$

$b, a \rightarrow \lambda$

$b, \$ \rightarrow \lambda$

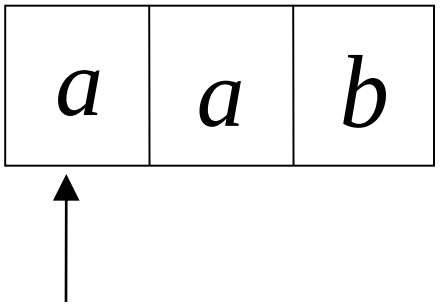


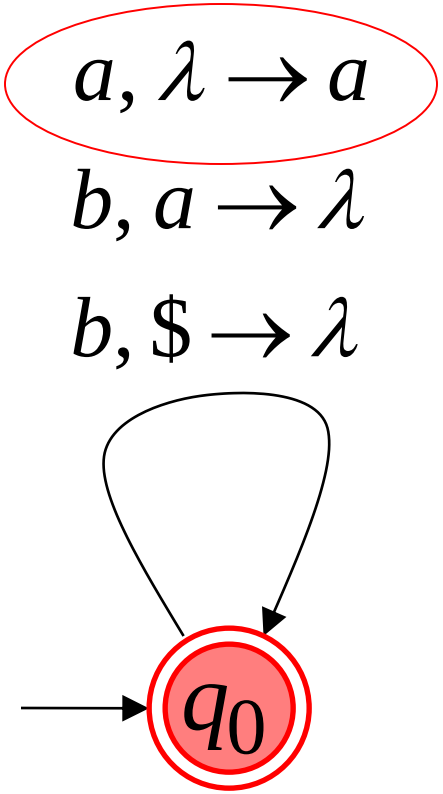


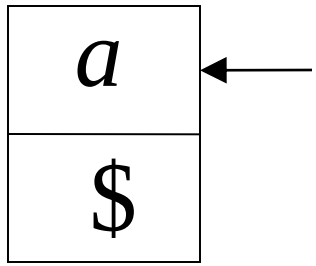
Stack

Time 1

Input



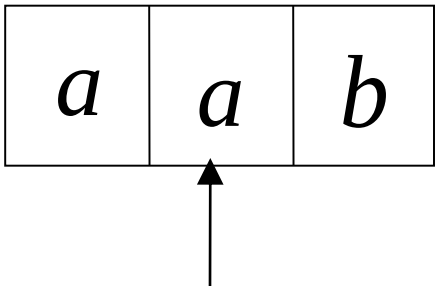


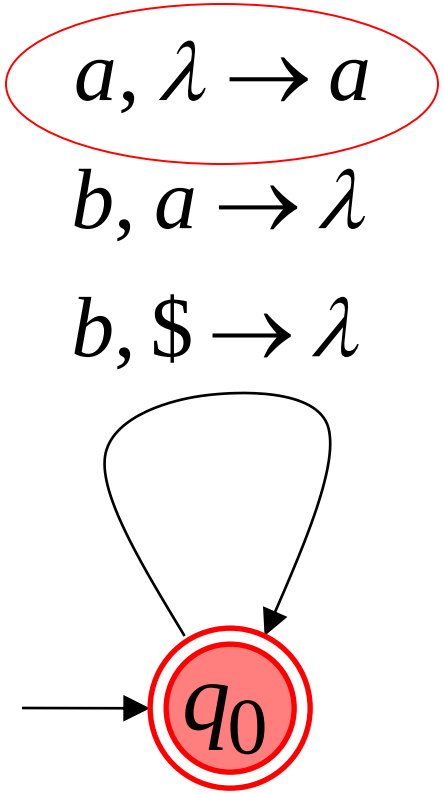


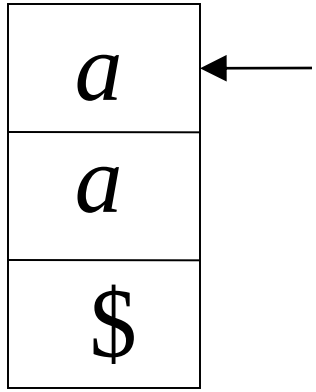
Stack

Time 2

Input



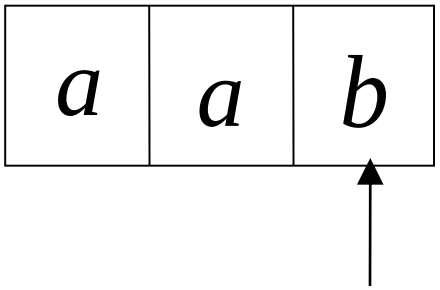


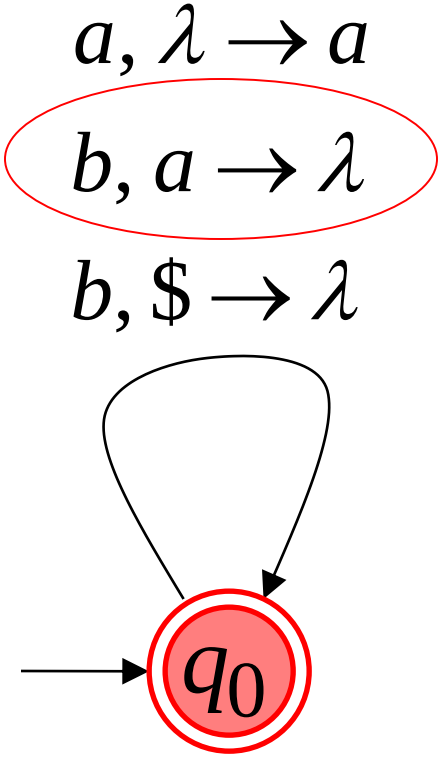


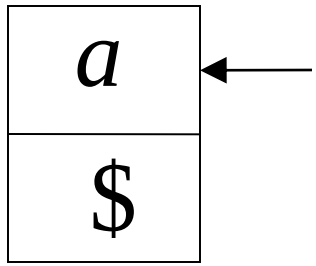
Stack

Time 3

Input







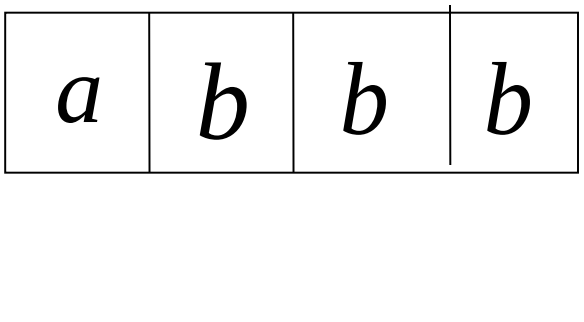
Stack

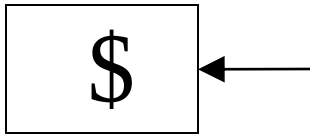
accept

Rejection example:

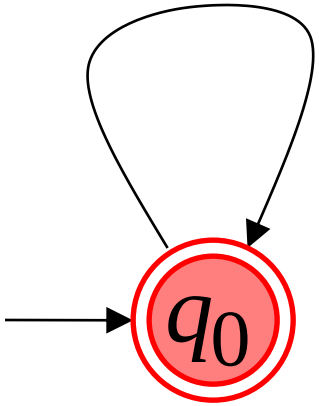
Time 0

Input



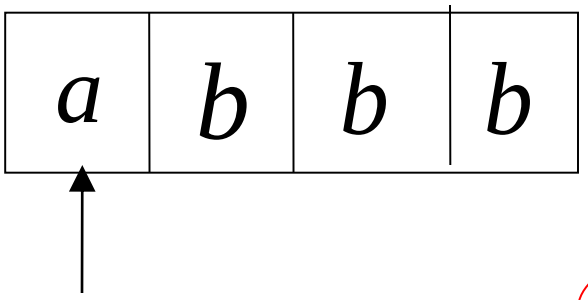


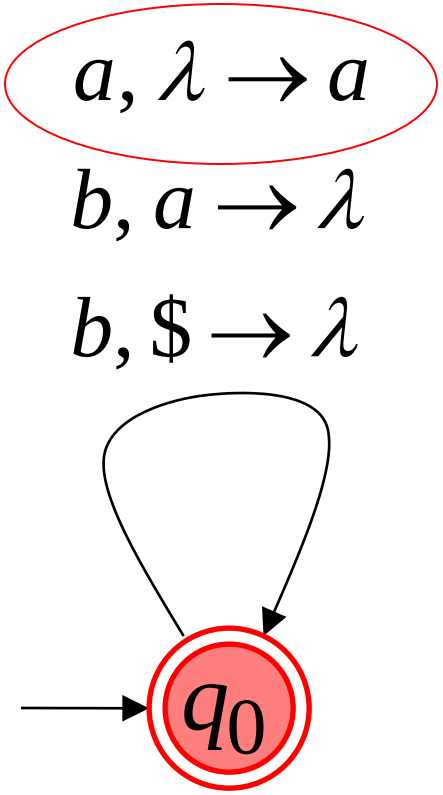
Stack

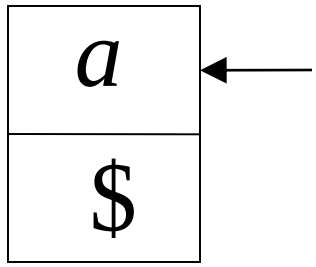


Time 1

Input



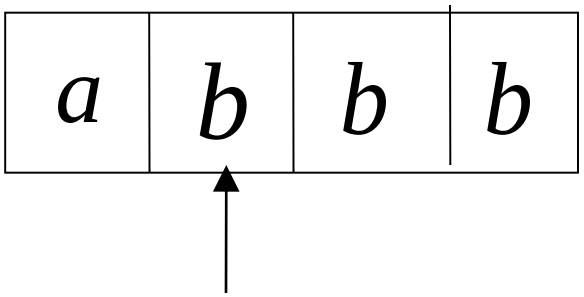


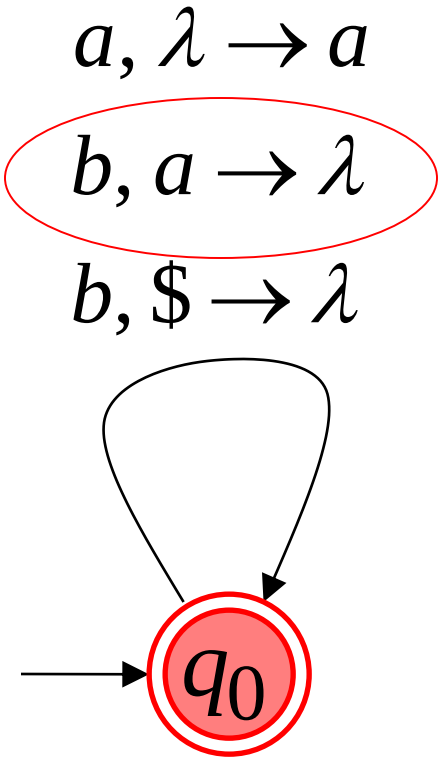


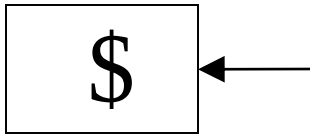
Stack

Time 2

Input



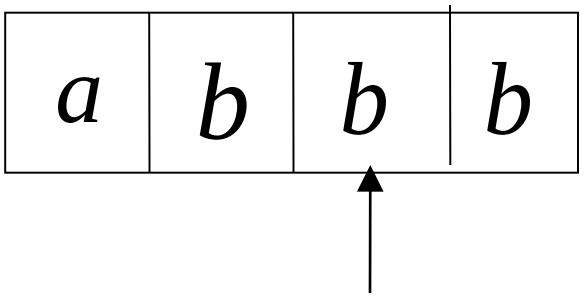




Stack

Time 3

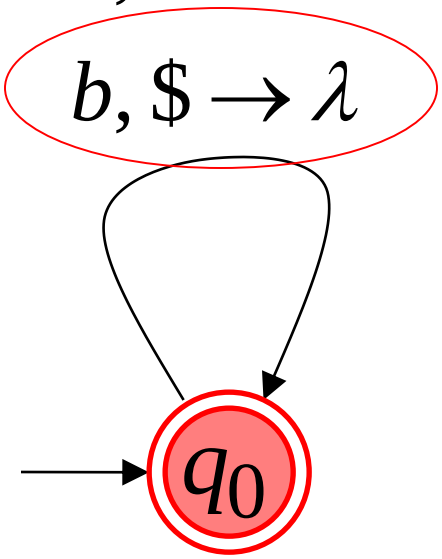
Input



$a, \lambda \to a$

$b, a \to \lambda$

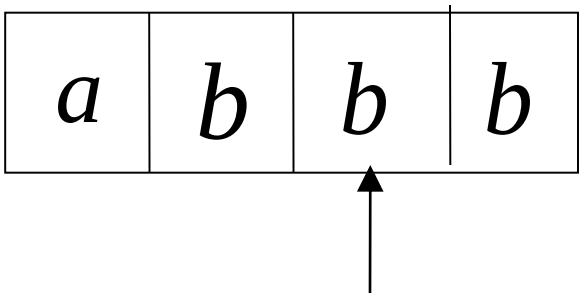
$b, \$ \to \lambda$



Stack

Time 4

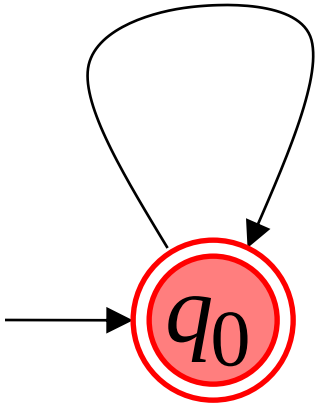
Input

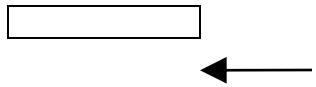


$a, \lambda \rightarrow a$

$b, a \rightarrow \lambda$

$b, \$ \rightarrow \lambda$

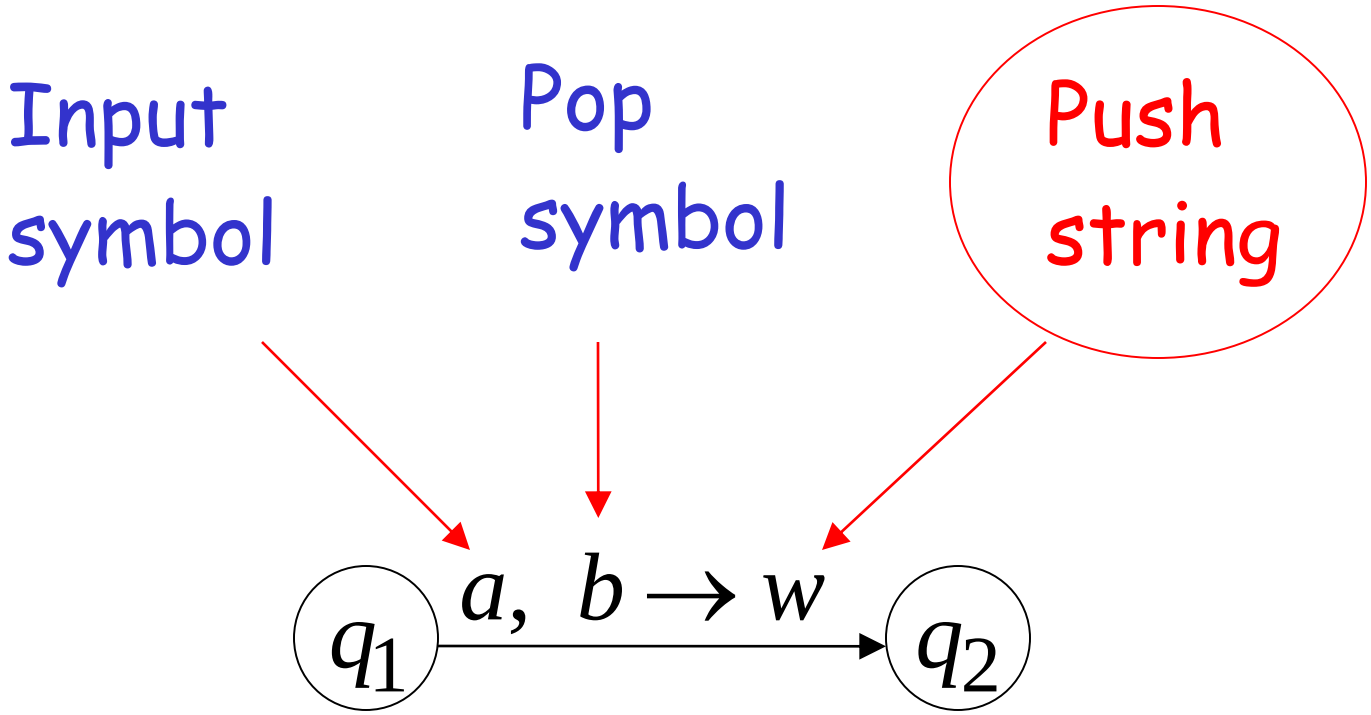




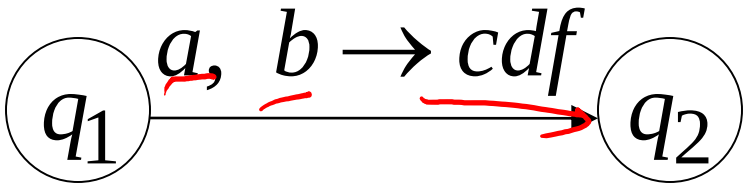
Stack

Halt and Reject

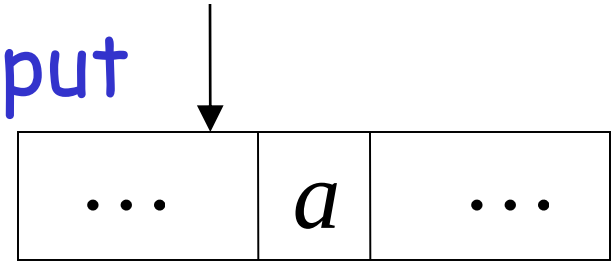
Pushing Strings

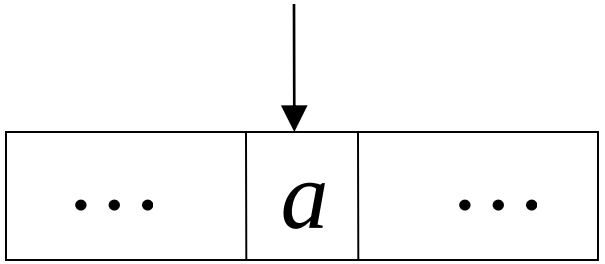


Example:

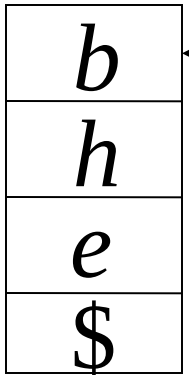


input



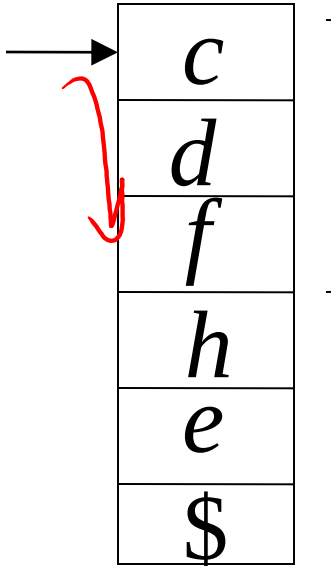


stack



top

Push



pushed string

Another NPDA example

NPDA M

$$L(M) = \{w: \quad n_a = n_b\}$$

Another NPDA example

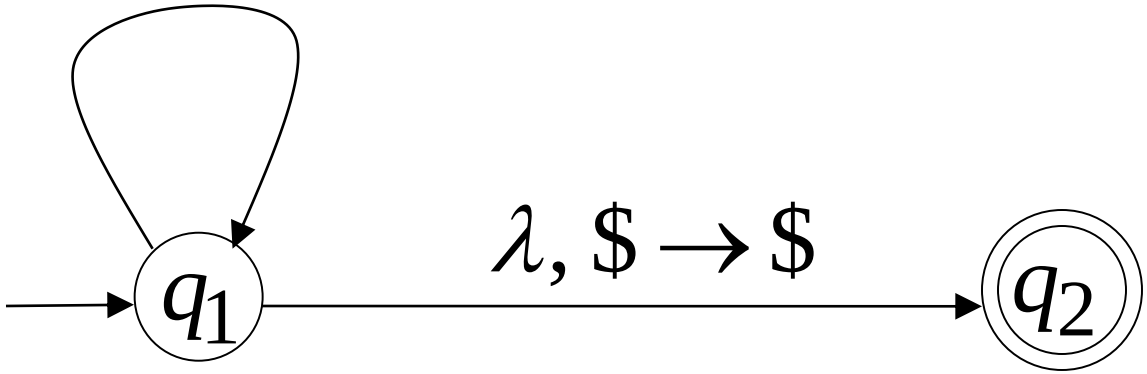
NPDA M

$$L(M) = \{w: \quad n_a = n_b\}$$

$a, \$ \rightarrow 0\$$ $\quad b, \$ \rightarrow 1\$$

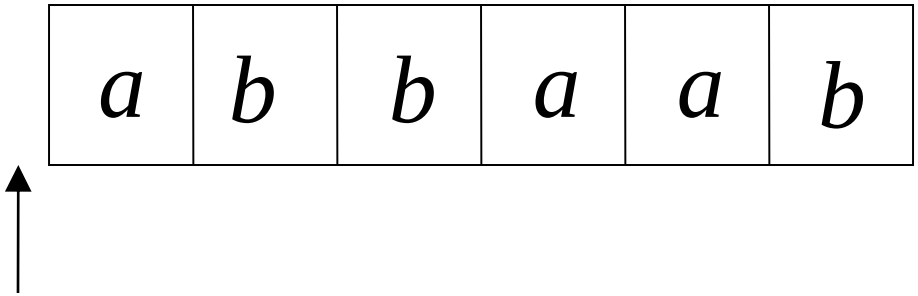
$a, 0 \rightarrow 00$ $\quad b, 1 \rightarrow 11$

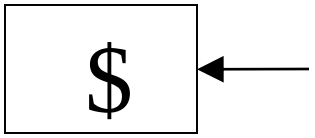
$a, 1 \rightarrow \lambda$ $\quad b, 0 \rightarrow \lambda$



Execution Example: Time 0

Input





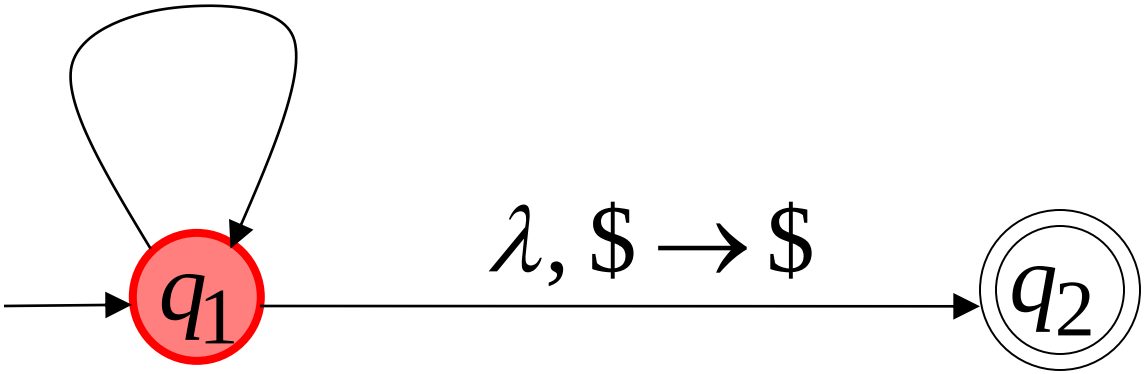
Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$

current state



Time 1

Input

a	b	b	a	a	b

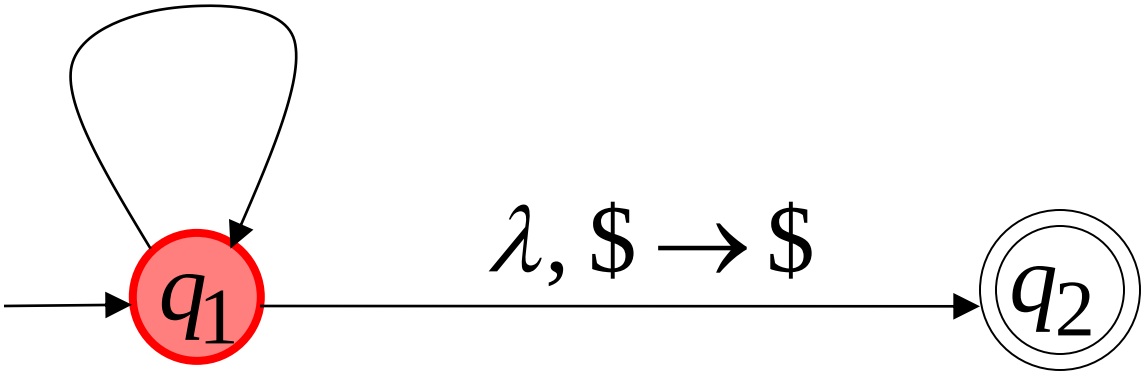
0
$

Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

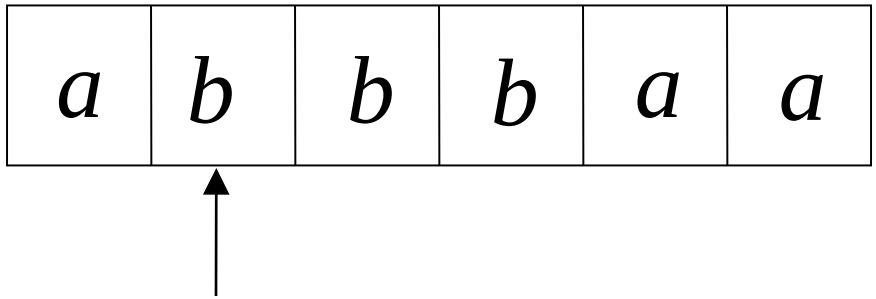
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

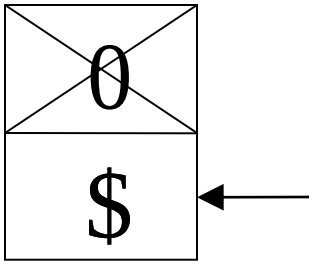
$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



Time 3

Input



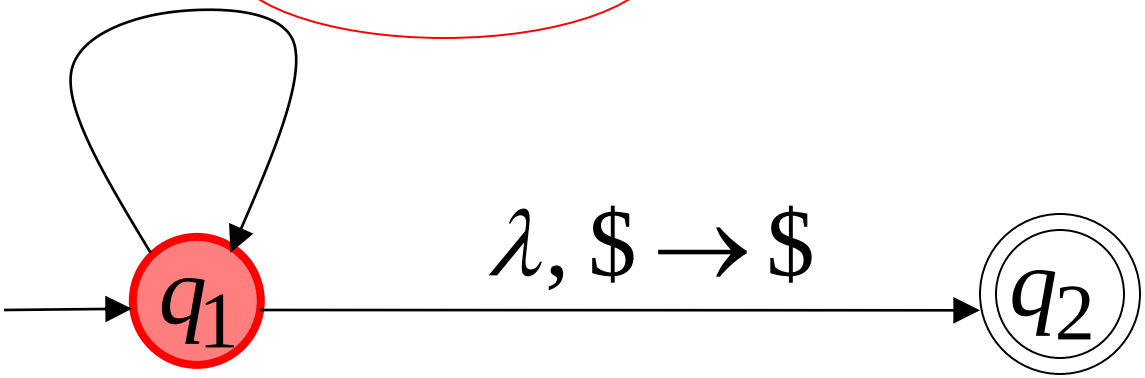


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

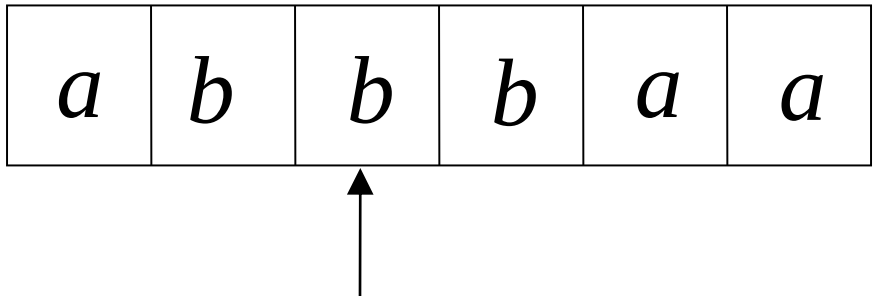
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

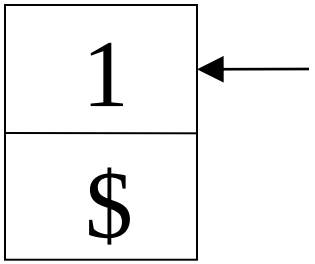
$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



Time 4

Input



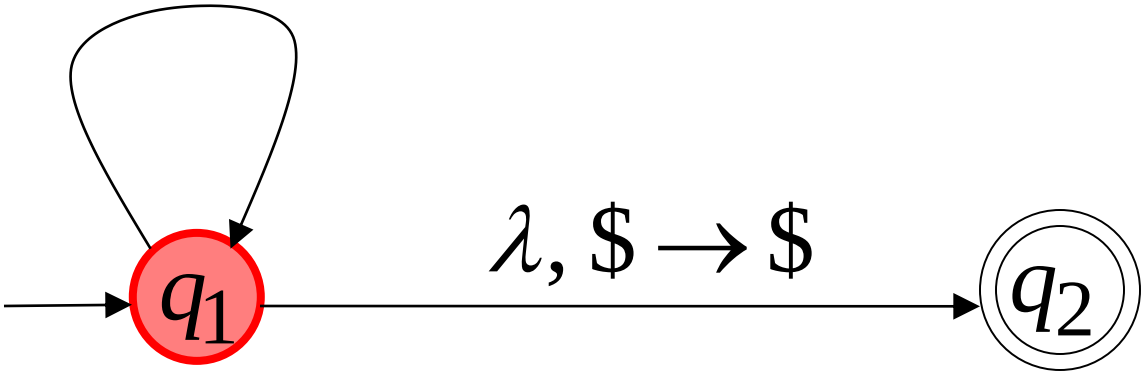


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

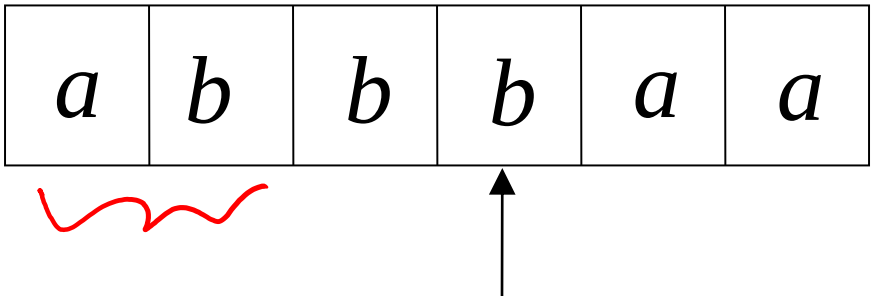
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

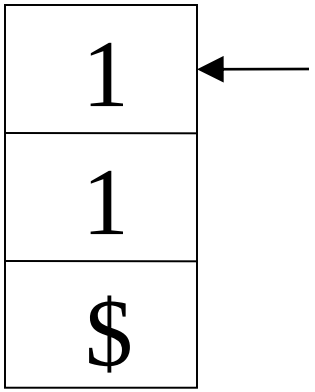
$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



Time 5

Input



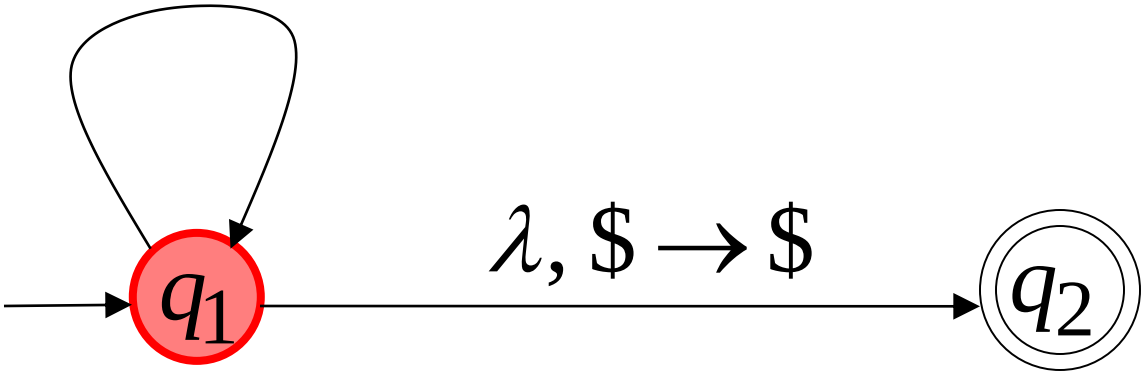


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

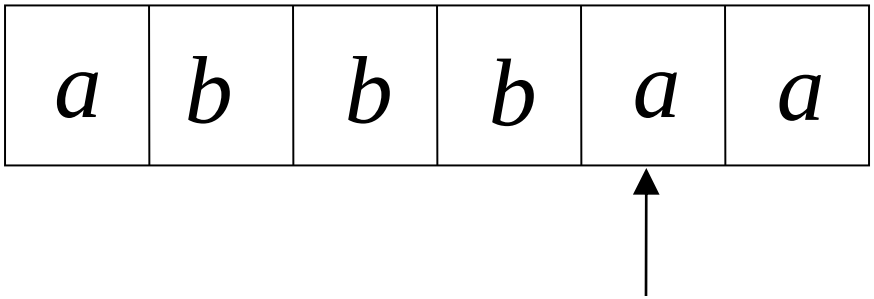
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

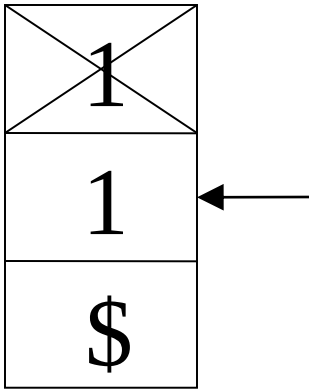
$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



Time 6

Input



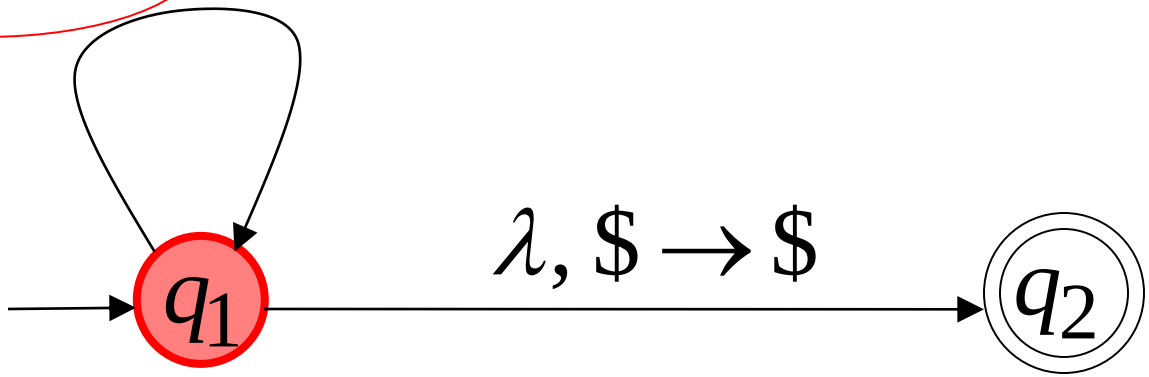


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

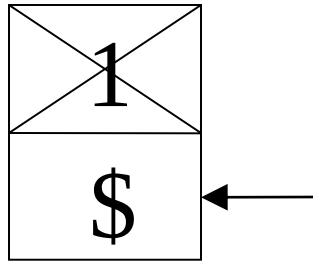
$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



Time 7

Input

a	b	b	b	a	a

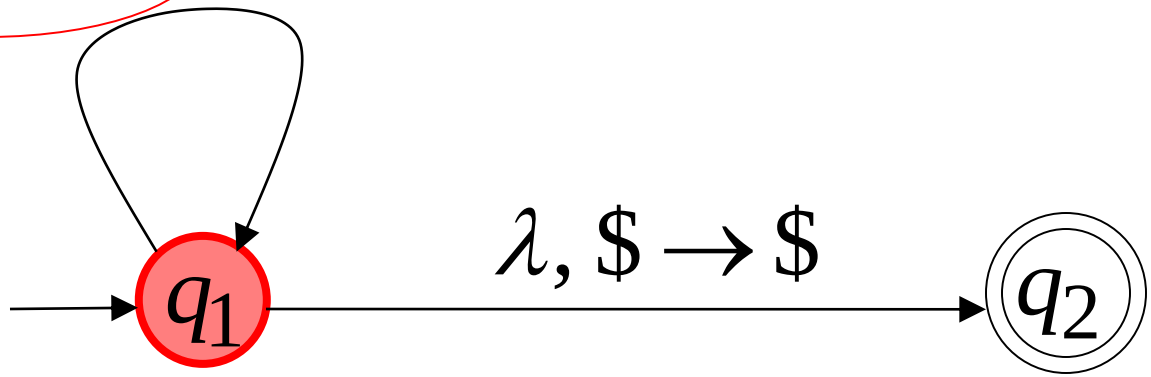


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

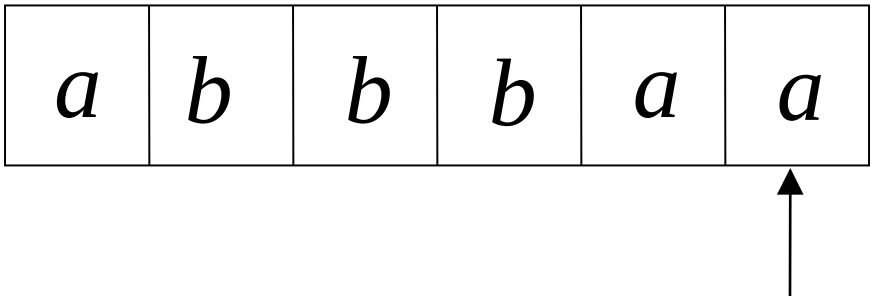
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

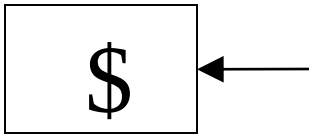
$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



Time 8

Input



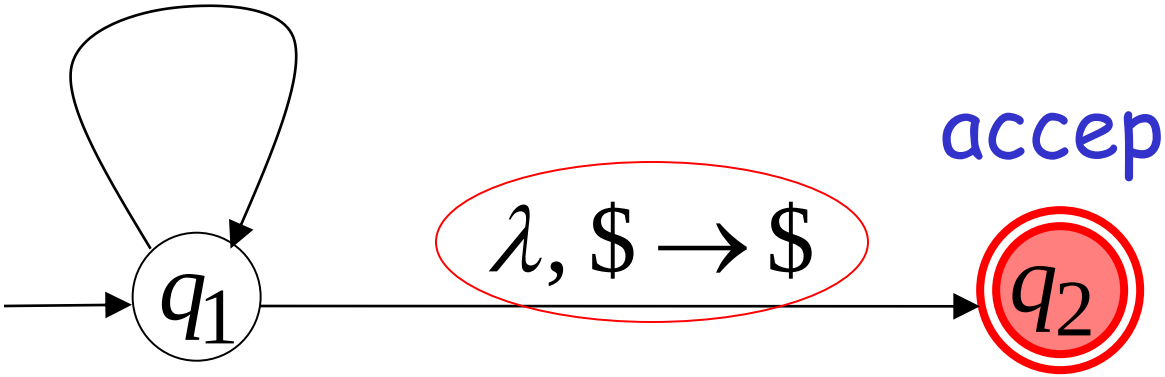


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

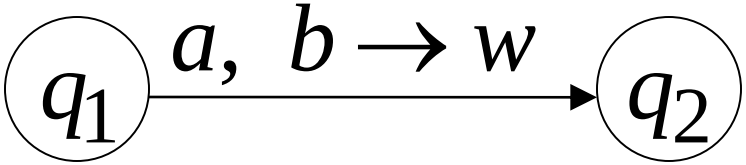
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



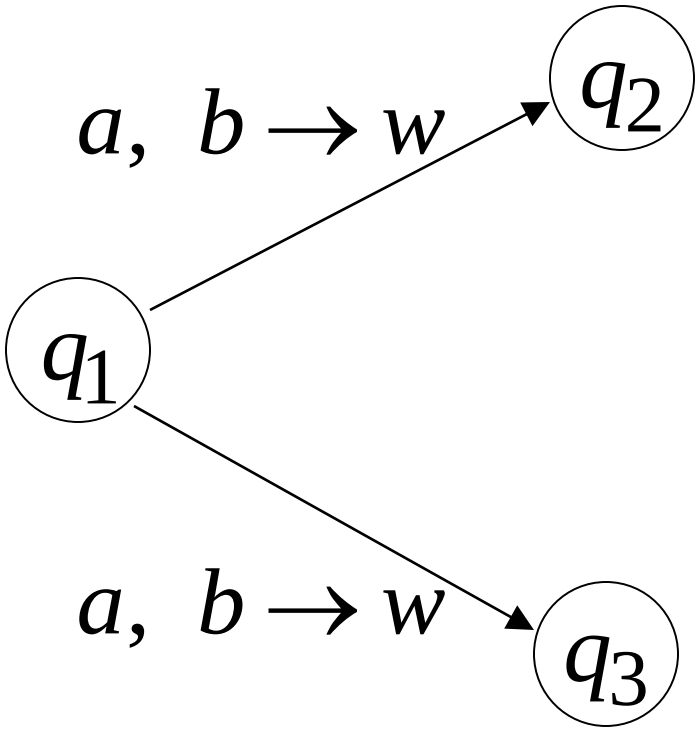
accept

Formalities for NPDAs



Transition function:

$$\delta(q_1, a, b) = \{(q_2, w)\}$$



Transition function:

$$\delta(q_1, a, b) = \{(q_2, w),\ (q_3, w)\}$$

Formal Definition

Non-Deterministic Pushdown Automaton

NPDA

$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$

States

Input alphabet

Stack alphabet

Transition function

Initial state

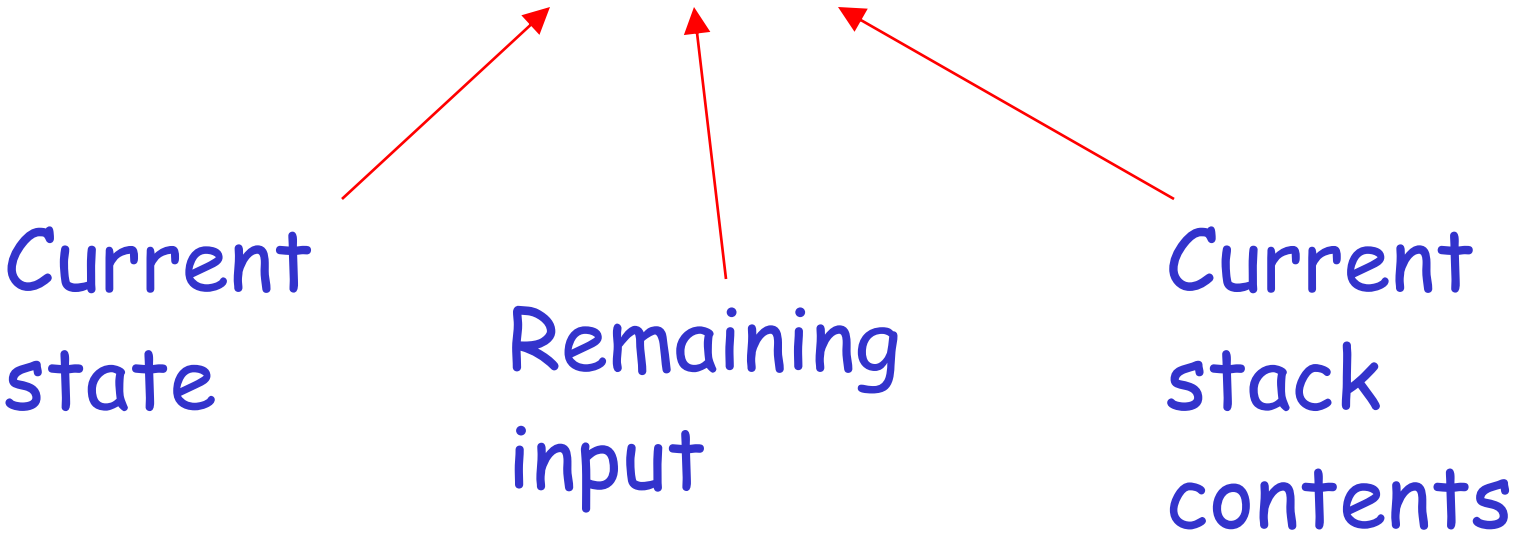
Stack start symbol

Final states

$$\delta: Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \rightarrow \text{finite subsets of } Q \times \Gamma^*$$

Instantaneous Description

$$(q,\ u,\ s)$$

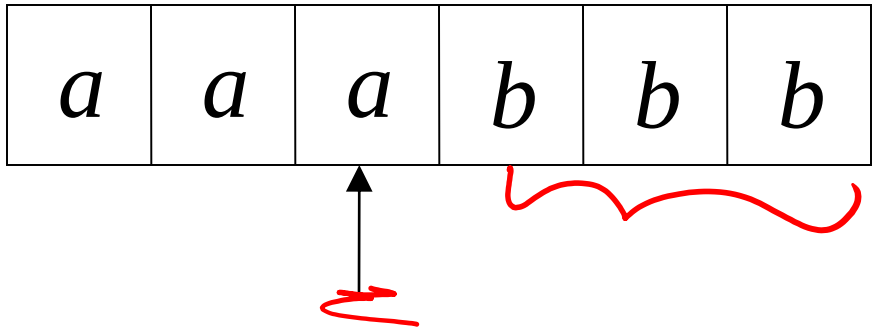


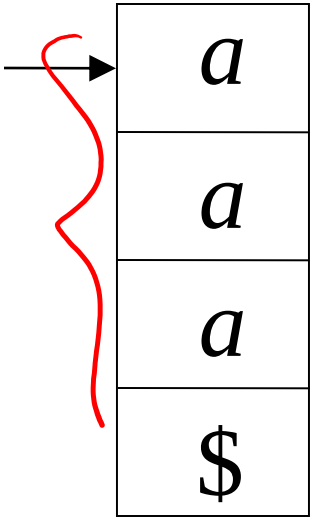
Example: Instantaneous Description

$$(q_1, bbb, aaa\$)$$

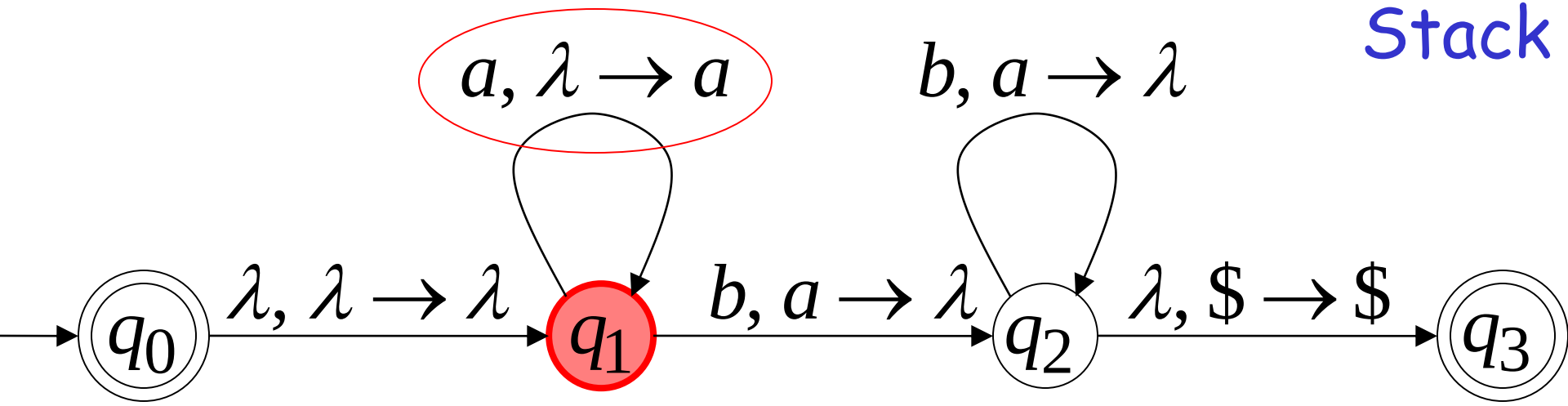
Time 4:

Input





Stack

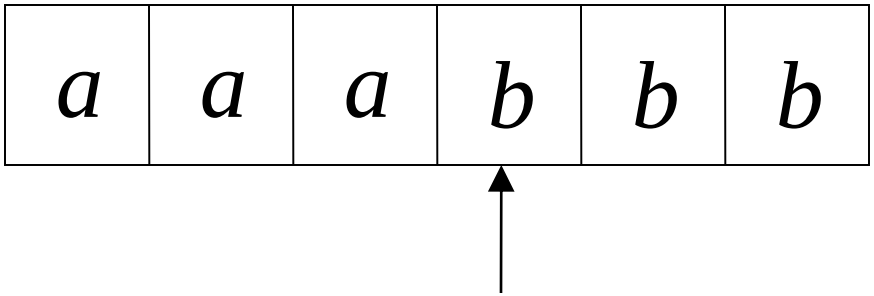


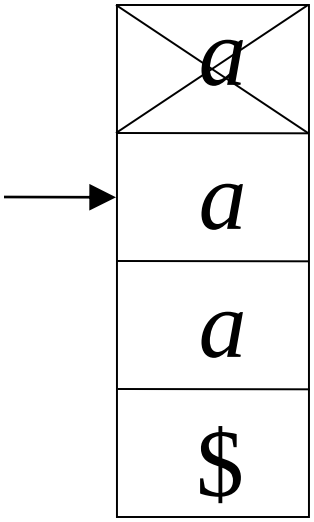
Example: **Instantaneous Description**

$$(q_2, bb, aa\$)$$

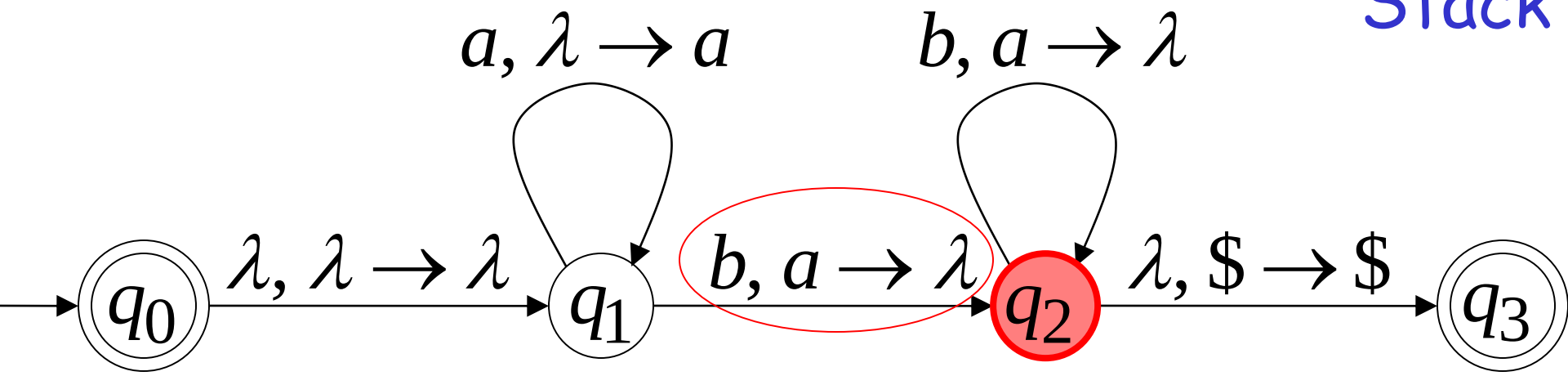
Time 5:

Input





Stack



We write:

$$(q_1, bbb, aaa\$) \succeq (q_2, bb, aa\$)$$

Time 4

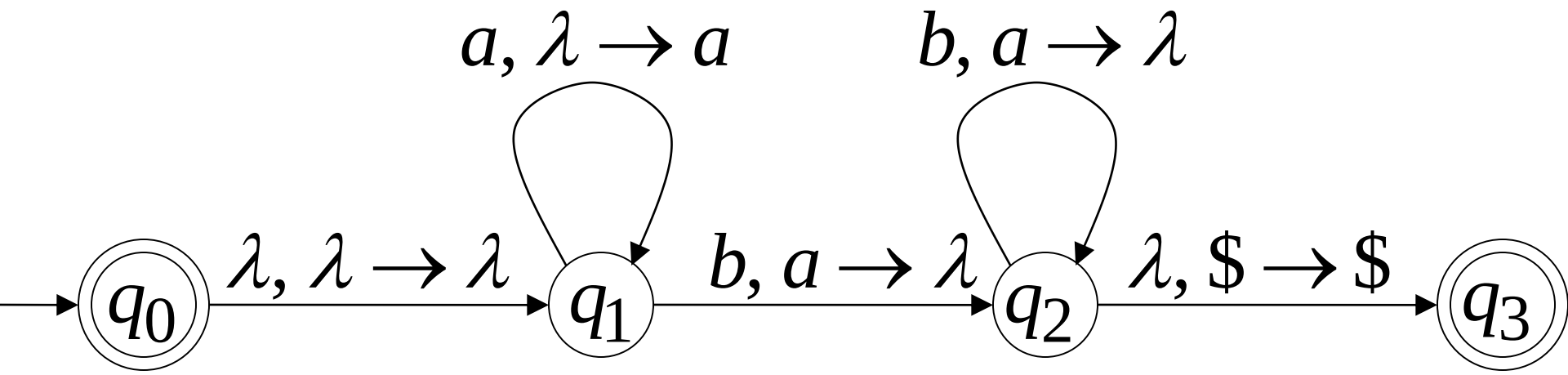
Time 5

A computation:

$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$

$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$

$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \lambda, \$) \succ (q_3, \lambda, \$)$$



$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$

$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$

$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \lambda, \$) \succ (q_3, \lambda, \$)$$

For convenience we write:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

Formal Definition

Language $L(M)$ of NPDA M

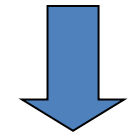
$$L(M) = \{w: \ (q_0, w, s) \overset{*}{\succ} (q_f, \lambda, s')\}$$

Initial state

Final state

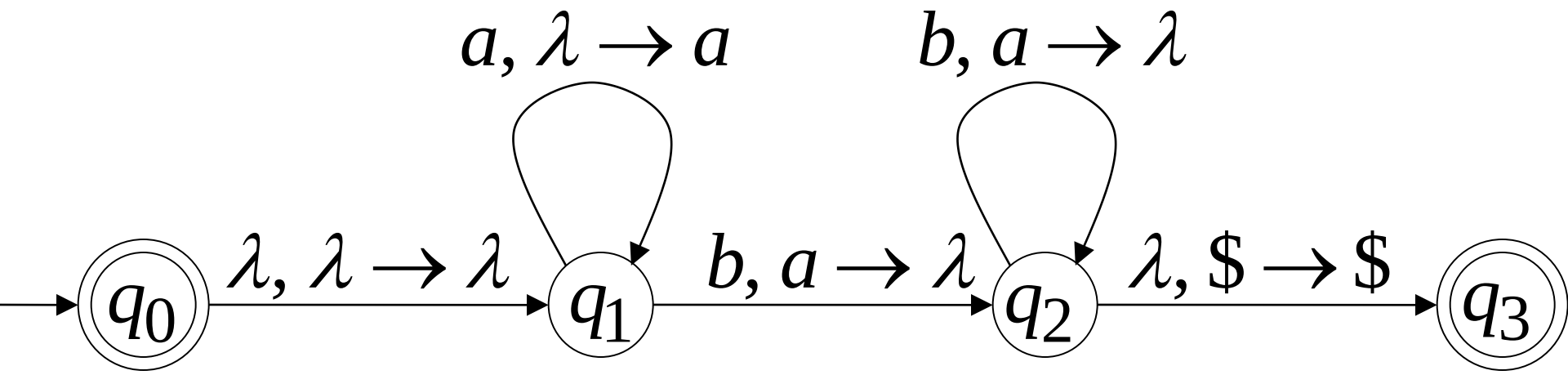
Example:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$



$$aaabbb \in L(M)$$

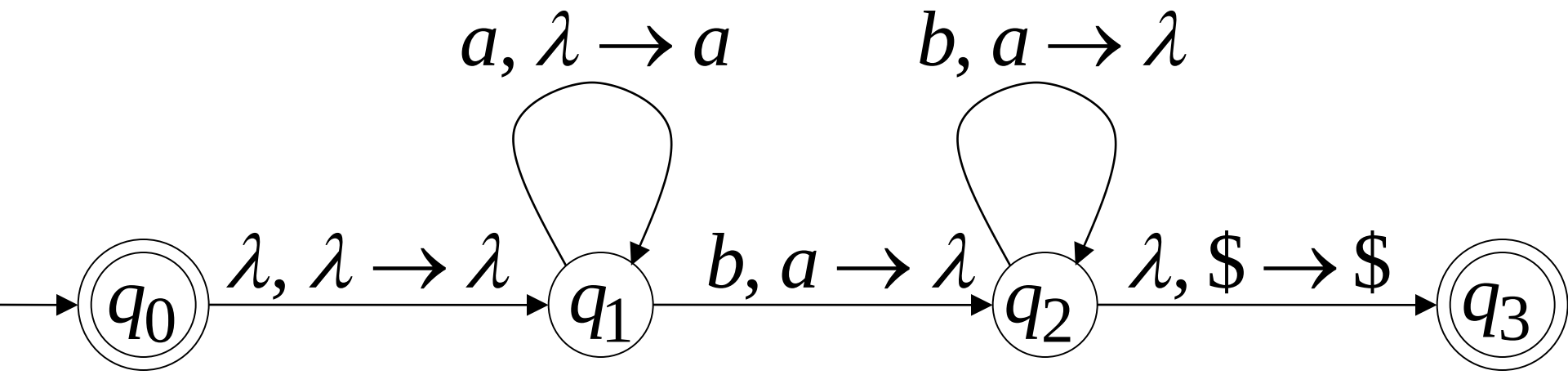
NPDA M:



$$(q_0, a^n b^n, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

$$a^n b^n \in L(M)$$

NPDA M:



Therefore: $L(M) = \{a^n b^n : n \geq 0\}$

NPDA M: