PUSHDOWN AUTOMATA

PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ 7-Tuple.

Instantaneous Description ID (q, x, α)
$q \in Q$, $x \in \Sigma^*$, $\alpha \in \Gamma^*$

Move relations $\vdash$ or $\succ$

Ex: $(q, a_1 a_2 \ldots a_n, Z_1 Z_2 \ldots Z_n) \vdash (q', a_2 \ldots a_n, \beta Z_2 \ldots Z_m)$
if $\delta(q, a_1, Z_1)$ contains (q', β).

$\vdash^*$ Reflexive - Transitive closure of $\vdash$.

Property 1: $(q_1, x, \alpha) \vdash^* (q_2, \wedge, \beta)$ then for every $y \in \Sigma^*$,
$(q_1, xy, \alpha) \vdash^* (q_2, y, \beta)$.

Property 2: If $(q, x, \alpha) \vdash^* (q', \wedge, \gamma)$, then for every $\beta \in \Gamma^*$
$(q, x, \alpha\beta) \vdash^* (q', \wedge, \gamma\beta)$.

Acceptance by PDA

Def^n: Let $A = (Q, \Sigma, \Gamma, q_0, Z_0, F)$ be a pda. The set accepted by pda by final state is defined by

$$T(A) = \{ w \in \Sigma^* \mid (q_0, w, Z_0) \vdash^* (q_f, \wedge, \alpha) \text{ for some } q_f \in F \text{ and } \alpha \in \Gamma^* \}.$$

Def^n: Let $A = (Q, \Sigma, \Gamma, q_0, Z_0, F)$ be a pda. The set $N(A)$ accepted by Null store/Empty store is defined by.

$$N(A) = \{ w \in \Sigma^* \mid (q_0, w, Z_0) \vdash^* (q, \wedge, \wedge) \text{ for some } q \in Q \}.$$

Theorem 1: If $A = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is a pda accepting L by empty store, we can find a pda
$B = (Q', \Sigma, \Gamma', \delta_B, q_0', Z_0', F')$ which accept L by final state, i.e. $L = N(A) = T(B)$.

Proof: B is constructed in a such a way that

a) by Initial move of B, it reaches an initial ID of A,

b) by final move of B, it reaches its final state, and

c) all intermediate moves of B are as in A.

$$B = (Q', \Sigma, \Gamma', \delta_B, q_0', Z_0', F').$$

where, q_0' is new state ($q_0' \notin Q$)

$F' = \{q_f\}$, q_f as new final state, $q_f \notin F$

$Q' = Q \cup \{q_0', q_f\}$

Z_0' = new start symbol for PDS B.

δ_B

$\boxed{Z_0'}$ R_1: $\underbrace{\delta_B(q_0', \Lambda, Z_0')}_{\text{initial ID of B}} = \underbrace{\{(q_0, Z_0 Z_0')\}}_{\text{initial ID of A}}$ $\begin{array}{|c|}\hline Z_0 \\ \hline Z_0' \\ \hline \end{array}$ $\Rightarrow$ Λ-move.

R_2: $\delta_B(q, a, Z) = \delta(q, a, Z)$ $\forall (q, a, Z) \in Q \times (\Sigma \cup \{\Lambda\}) \times F$

R_3: $\delta_B(q, \Lambda, Z_0') = \{q_f, \Lambda\}$ $\forall q \in Q$.

Λ-move

reach to final state

after processing according to A.

Simulate A

$\Rightarrow$ Thus, the behaviour of B and A are similar except Λ-moves in R_1 & R_3 rule.

NOTE $\Rightarrow$ from construction B, it is easy to check that B is Deterministic iff A is deterministic.

Ex: Let $A = (\{q_0, q_1\}, \{a, b\}, \{a, Z_0\}, \delta, q_0, Z_0, \phi)$.

δ:
$$\delta(q_0, a, Z_0) = \{(q_0, aZ_0)\}$$
$$\delta(q_0, a, a) = \{(q_0, aa)\}$$
$$\delta(q_0, b, a) = \{(q_1, \wedge)\}$$
$$\delta(q_1, b, a) = \{(q_1, \wedge)\}$$
$$\delta(q_1, \wedge, Z_0) = \{(q_1, \wedge)\}.$$

a) Determine $N(A)$?

b) Construct PDA B such that $T(B) = N(A)$.

Theorem 2: If $A = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ accepts L by final state, we can find pda B accepting L by Empty store i.e.
$$L = T(A) = N(B).$$

Proof: B is constructed from A in such a way that

a) By the initial move of B an initial ID of A is reached.

b) Once B reaches to initial ID of A, it behaves like A until a final state of A is reached,

c) when B reaches a final state of A, it guesses whether the i/p string is exhausted. Then B simulates A or it erases all the symbols in PDS.

$$B = (Q \cup \{q_0', d\}, \Sigma, \Gamma \cup \{Z_0'\}, \delta_B, q_0', Z_0', \phi)$$

where, q_0' = new state $(q_0' \notin Q)$ $\quad$ Z_0' = new start symbol for PDS of B.

d = dead state (new)

δ_B:

R_1: $\delta_B(q_0', \wedge, Z_0') = \{(q_0, Z_0 Z_0')\}$

R_2: $\delta_B(q, a, Z) = \delta(q, a, Z) \quad \forall a \in \Sigma, q \in Q, Z \in \Gamma$

R_3: $\delta_B(q, \wedge, Z) = \delta(q, \wedge, Z) \cup \{(d, \wedge)\} \quad \forall Z \in \Gamma \cup \{Z_0'\}$ and $q \in F$

R_4: $\delta_B(d, \wedge, Z) = \{(d, \wedge)\} \quad \forall Z \in \Gamma \cup \{Z_0'\}$.

PDA & CFG

Theorem 3: If L is a CFL, then we can construct a PDA A accepting L by empty store, i.e. $L = N(A)$.

Construction: $L = L(G)$, $G = (V, \Sigma, P, S)$ is CFG.

The A can be defined as

$$A = (Q, \Sigma, V \cup \Sigma, \delta, q, S, \phi), \quad Q = \{q\}.$$

δ:

R_1: $\delta(q, \wedge, A) = \{(q, \alpha) \mid A \to \alpha \text{ is in } P\}$

R_2: $\delta(q, a, a) = \{(q, \wedge)\} \quad \forall\, a \in \Sigma.$

Ex: G: $S \to 0BB$, $B \to 0S \mid 1S \mid 0$ $\longleftarrow 01a^4$

Now,
$$A = (\{q\}, \{0, 1\}, \{S, B, 0, 1\}, \delta, q, S, \phi)$$

δ:

R_1: $\delta(q, \wedge, S) = \{(q, 0BB)\}$

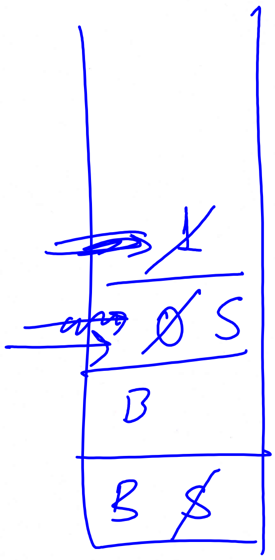
R_2: $\delta(q, \wedge, B) = \{(q, 0S), (q, 1S), (q, 0)\}$

R_3: $\delta(q, 0, 0) = \{(q, \wedge)\}$

R_4: $\delta(q, 1, 1) = \{(q, \wedge)\}$

Now check for $010^4 \in N(A)$ or not!

$$
\begin{aligned}
(q, 010^4, S) &\vdash (q, 010^4, 0BB) && R_1\\
&\vdash (q, 10^4, BB) && R_3\\
&\vdash (q, 10^4, 1SB) && R_2\\
&\vdash (q, 0^4, SB) && R_4\\
&\vdash (q, 0^4, 0BBB) && R_1\\
&\vdash (q, 0^3, BBB) && R_3\\
&\vdash^* (q, 0^3, 000) && R_2\\
&\vdash^* (q, \wedge, \wedge)
\end{aligned}
$$



Thus $010^4 \in N(A)$.

Theorem 4: If $A = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is a pda, then there exists a CFG G such that $L(G) = N(A)$.

Construction: Let G be (V, Σ, P, S)

$\underline{V}$ $\quad V = \{S\} \cup \{[q, Z, q'] \mid q, q' \in Q, Z \in \Gamma\}$

$\underline{P}$ R_1: S-prod^n are given by $S \to [q_0, Z_0, q]$ $\forall q \in Q$.

R_2: Each move erasing* a pushdown symbol given by $(q', \wedge) \in \delta(q, a, Z)$ induces the production $[q, Z, q'] \to a$.

R_3: Each move not erasing a pushdown symbol given by $(q_1, Z_1 Z_2 \ldots Z_m) \in \delta(q, a, Z)$ induces many prod^{n}s of form

$$[q, Z, q'] \to a[q_1, Z_1, q_2][q_2, Z_2, q_3] \ldots [q_m, Z_m, q']$$

where each of state $q', q_2, \ldots q_m$ can be any state in Q.

$\underline{Ex}$. $A = (\{q_0, q_1\}, \{a, b\}, \{Z_0, Z\}, \delta, q_0, Z_0, \phi)$

$\underline{\delta}$ $\quad \delta(q_0, b, Z_0) = \{(q_0, ZZ_0)\}$ $\qquad \delta(q_0, \wedge, Z_0) = \{(q_0, \wedge)\}$

$\delta(q_0, b, Z) = \{(q_0, ZZ)\}$ $\qquad \delta(q_0, a, Z) = \{(q_1, Z)\}$

$\delta(q_1, b, Z) = \{(q_1, \wedge)\}$ $\qquad \delta(q_1, a, Z_0) = \{(q_0, Z_0)\}$

Let $G = (V, \{a, b\} P, S)$ $\qquad \boxed{L(A) = ?}$

$V \in \{S, [q_0, Z_0, q_0], [q_0, Z_0, q_1], [q_0, Z, q_0], [q_0, Z, q_1],$
$[q_1, Z_0, q_0], [q_1, Z_0, q_1], [q_1, Z, q_0], [q_1, Z, q_1]\}$

$\underline{P}$

P_1: $S \to [q_0, Z_0, q_0]$

P_2: $S \to [q_0, Z_0, q_1]$

S-Prod^n

* Erasing means that A and all the successive strings by which it is replaced are removed from the stack, bringing the symbol originally below A to the top. $\quad [q_0, Z_0, q_t] \xRightarrow{*} w$

$\delta(q_0, b, Z_0) = \{(q_0, ZZ_0)\}$ yields $\rightarrow R_3$

$P_3 : [q_0, Z_0, q_0] \rightarrow b[q_0, Z, q_0][q_0, Z_0, q_0]$

$P_4 : [q_0, Z_0, q_0] \rightarrow b[q_0, Z, q_1][q_1, Z_0, q_0]$

$P_5 : [q_0, Z_0, q_1] \rightarrow b[q_0, Z, q_0][q_0, Z_0, q_1]$

$P_6 : [q_0, Z_0, q_1] \rightarrow b[q_0, Z, q_1][q_1, Z_0, q_1]$

$\delta(q_0, \wedge, Z_0) = \{(q_0, \wedge)\}$ yields R_2

$P_7 : [q_0, Z_0, q_0] \rightarrow \wedge$

$\delta(q_0, b, Z) = \{(q_0, ZZ)\}$ gives R_3

$P_8 : [q_0, Z, q_0] \rightarrow b[q_0, Z, q_0][q_0, Z, q_0]$

$P_9 : [q_0, Z, q_0] \rightarrow b[q_0, Z, q_1][q_1, Z, q_0]$

$P_{10} : [q_0, Z, q_1] \rightarrow b[q_0, Z, q_0][q_0, Z, q_1]$

$P_{11} : [q_0, Z, q_1] \rightarrow b[q_0, Z, q_1][q_1, Z, q_1]$

$\delta(q_1, b, Z) = \{(q_1, \wedge)\}$ R_2

$P_{12} : [q_1, Z, q_1] \rightarrow b$

$\delta(q_0, a, Z) = \{(q_1, Z)\}$ R_3

$P_{13} : [q_0, Z, q_0] \rightarrow a[q_1, Z, q_0]$

$P_{14} : [q_0, Z, q_1] \rightarrow a[q_1, Z, q_1]$

$\delta(q_1, a, Z_0) = \{(q_0, Z_0)\}$ R_3

$P_{15} : [q_1, Z_0, q_0] \rightarrow a[q_0, Z_0, q_0]$

$P_{16} : [q_1, Z_0, q_1] \rightarrow a[q_0, Z_0, q_1]$.

$\Rightarrow$ Reduce the numbes of variables and productions if they are useless with the procedure discussed in the past.