Orthogonality

Let $\{v_1, v_2, \ldots v_r\}$ be o.g.b. of Col(A)

$$\hat{b} = \frac{\langle v_1, b\rangle}{\langle v_1, v_1\rangle} \cdot v_1 + \frac{\langle v_2, b\rangle}{\langle v_2, v_2\rangle} \cdot v_2 + \cdots + \frac{\langle v_r, b\rangle}{\langle v_r, v_r\rangle} v_r$$

How to find $\{v_1, \ldots v_r\}$ ogb ?

$Ax = b$ has no sol^n

$b \notin Col(A)$

subspace

$\hat{b} = Proj_{Col(A)}\, b$

Then solve $Ax = \hat{b}$

Gram-Schmidt Process

$V_1, V_2 \neq 0$

$\langle V_1, V_2 \rangle = 0$

$C_1 V_1 + C_2 V_2 = \underline{0}$

$0 = \langle \underline{0}, V_1 \rangle = C_1 \langle V_1, V_1 \rangle + C_2 \langle V_2, V_1 \rangle$

Let $\{U_1, U_2, \ldots, U_r\}$ be a basis of $\mathrm{Col}(A)$

$\{V_1, V_2, \ldots, V_r\}$

$V_1 = U_1$

$$V_2 = U_2 - \frac{\langle U_2, V_1 \rangle}{\langle V_1, V_1 \rangle} \cdot V_1$$

$\mathrm{Proj}_{\langle V_1 \rangle} U_2$

$\Rightarrow \langle V_2, V_1 \rangle = 0$

$\Downarrow$

$\{V_2, V_1\}$ Lin. indep.

$$V_3 = U_3 - \frac{\langle U_3, V_1 \rangle}{\langle V_1, V_1 \rangle} \cdot V_1 - \frac{\langle U_3, V_2 \rangle}{\langle V_2, V_2 \rangle} \cdot V_2$$

$\vdots$

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is lin. dep but not o.g.

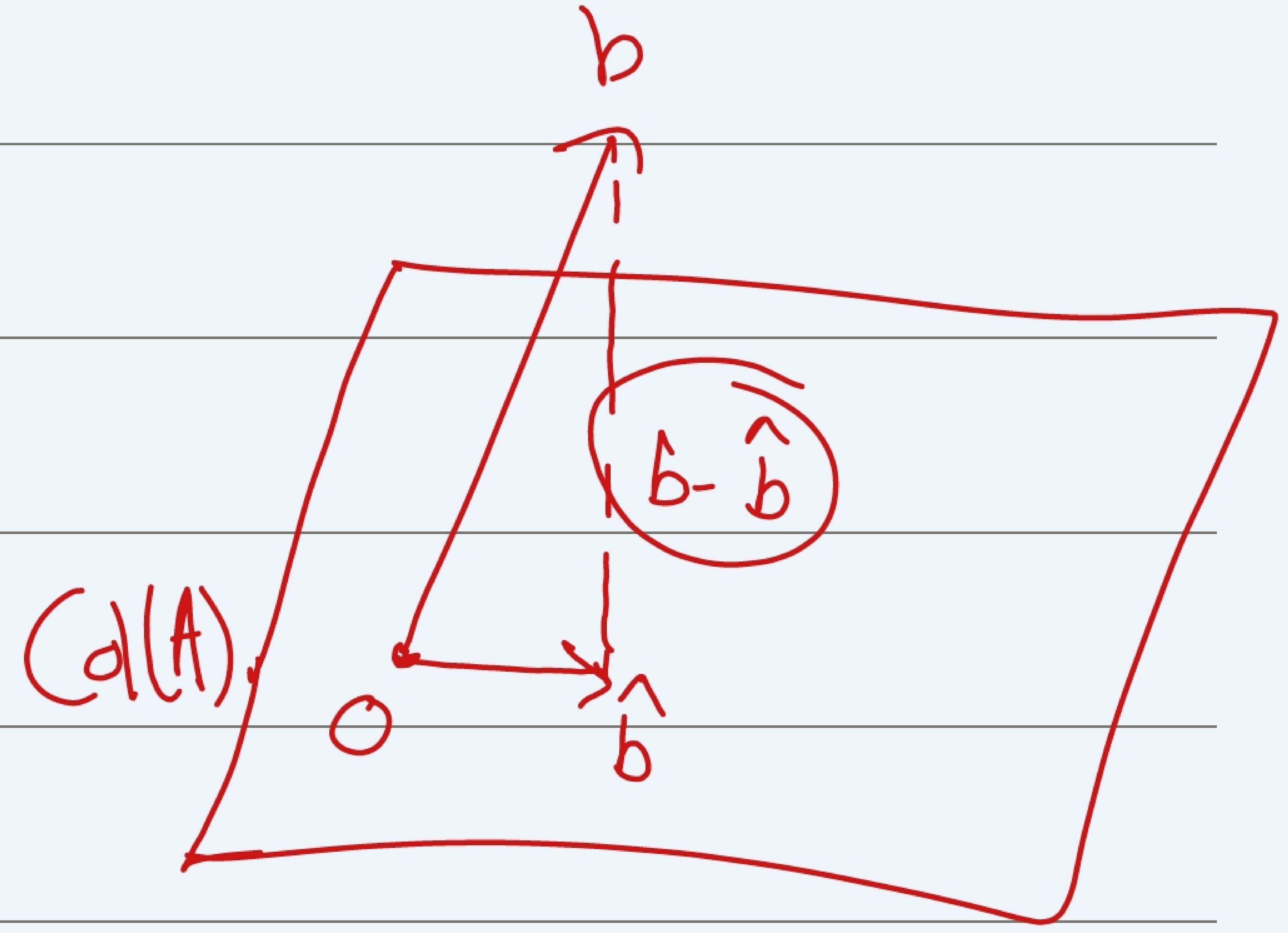
If $\hat{b} = \mathrm{Proj}_{\mathrm{Col}(A)}\, b$

Then $b - \hat{b} \perp \mathrm{Col}(A)$

If $A = \begin{bmatrix} u_1 \mid u_2 \mid \cdots \mid u_m \end{bmatrix}_{m \times n}$

then $\langle u_i, b - \hat{b} \rangle = 0 \quad \forall i$

$\| \; u_i^T \cdot (b - \hat{b})$



$$A = \left[u_1 \mid u_2 \mid \cdots \mid u_n \right]$$

$$\left[\quad A^T \quad \right] \left[b - \hat{b} \right] = \begin{bmatrix} u_1^T (b - \hat{b}) \\ u_2^T (b - \hat{b}) \\ \vdots \\ u_n^T (b - \hat{b}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A^T (b - \hat{b}) = 0$$

$$A^T b - A^T \hat{b} = 0$$

$$A^T b = A^T A \hat{x}$$

$Ax = b$

best approx. sol^n

$$A \hat{x} = \hat{b}$$

$$A^T A \hat{x} = A^T b$$

$\hat{x}$ is a sol^n of $A^T A x = A^T b$

Normal equation

$R_1 \to R_1 - 4R_3$

$$\left[\begin{array}{cc|c} \cancel{4} & \cancel{0} & \cancel{2} \\ 0 & 2 & 0 \\ 1 & 1 & 11 \end{array}\right] \qquad \begin{array}{ccc} 0 & -4 & -42 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 11 \\ 0 & 2 & 0 \\ 0 & \cancel{-4}\; 0 & -42 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

No sol^n

Note that $A^T A X = A^T b$ is always consistent

$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

$\boxed{A^T b \in \text{Col}(A^T A)}$? $\cancel{\in \text{Col}(A^T)}$

$A^T b$ = lin. comb of col^{ns} of A^T

$A = [u_1\ u_2\ \cdots\ u_n]$

$A^T A = [A^T u_1 \mid A^T u_2 \mid \cdots \mid A^T u_n]$

lin. comb. of Columns of A^T

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} X = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$X = \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \begin{bmatrix} 19 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\hat{X}$ best approximate sol^n

$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} -2 \\ -4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A\hat{X} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} = \hat{b}$$

$$b - \hat{b} = \begin{bmatrix} -2 \\ -4 \\ +8 \end{bmatrix}$$

QR factorization

$$* = \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle}$$

$$\delta = \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle}$$

$$A_{m\times n} = [u_1 \; u_2 \; \cdots \; u_n]$$

Let $\operatorname{rank}(A) = n$

By G.S. $\quad v_i = u_i - \left(\sum_{j=1}^{i-1} \frac{\langle u_i, v_j \rangle}{\langle v_j, v_j \rangle} \cdot v_j \right)$

$$u_2 = v_2 + \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} \cdot v_1$$

$$u_2' = \frac{1}{\|u_2\|} \cdot u_2$$

$$u_i = v_i + \sum_{j=1}^{i-1} \frac{\langle u_i, v_j \rangle}{\langle v_j, v_j \rangle} \cdot v_j$$

for $1 \le i \le n$

$PA = LU$

$A = PDP^{-1}$ diag matrix

$$A_{m\times n} = QR$$

if col^{ns} of A are lin indep

$\operatorname{Rank}(A) = n$

$$A = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} & & \\ & i^{th} \text{ col}^n & \\ & & \end{bmatrix} = \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}_{Q}_{m\times n} \begin{bmatrix} 1 & * & * \\ 0 & 1 & \delta \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_{m\times n} = Q_{m\times n} R_{n\times n}$$

$$A = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}_{m\times n} \qquad Q_{m\times n} \qquad (Q^T Q)_{n\times n} = I_{n\times n}$$

Apply G. S. & normalize to get $v_1 \cdots v_n$

$$Q = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

$$\Rightarrow \boxed{Q^T Q = I \;\&\; R = Q^T A}$$

$A = QR$

$Q^T A = R$

$Q_3 Q_2 Q_1 A = R$

$Ax = b$ inconsistent

$A^T A x = A^T b$ Normal eqn

Soln of $\rightarrow$ is called (best approx. soln of $Ax = b$) least square soln

Thm

$$\| \underbrace{A\hat{x}}_{\hat{b}} - b \| \leq \| Ax - b \| \quad \forall x \in \mathbb{R}^n$$

Find least square sol^n of $Ax = b$

wh $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$ & $b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$

Orthogonal matrix $\quad QQ^T = I$

$\Rightarrow$ Columns of Q are orthonormal.

$$U = \begin{bmatrix} 1 & -1/2 \\ 1 & +1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1/2 & +1/2 \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 1 & +1/2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

If $u_1, \dots u_n$ are orthogonal vectors in $\mathbb{R}^n$

$$U = \begin{bmatrix} u_1 \,|\, u_2 \,|\, \cdots \,|\, u_n \end{bmatrix}_{n \times n} \qquad U^T U = \begin{bmatrix} \|u_1\|^2 & 0 & & \\ 0 & \|u_2\|^2 & & \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & & \\ 0 & 0 & & \|u_n\|^2 \end{bmatrix}$$