Introduction

$L_{x=0}$ $\quad T\begin{bmatrix}0\\y\end{bmatrix} = \begin{bmatrix}0\\3y\end{bmatrix} = 3\cdot\begin{bmatrix}0\\y\end{bmatrix}$

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Question: Given a linear transformation T, are there any lines through the origin that are mapped to themselves by the linear transformation T?

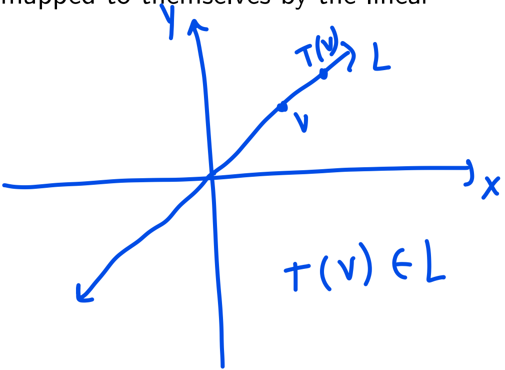
$n=2$

$A = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$

$T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x\\3y\end{bmatrix}$

$L_{y=0}$ $\quad T\begin{bmatrix}a\\0\end{bmatrix} = \begin{bmatrix}2a\\0\end{bmatrix} \in L_{y=0}$

$\begin{bmatrix}2a\\0\end{bmatrix} = 2\cdot\begin{bmatrix}a\\0\end{bmatrix}$



$T(v) \in L$

Introduction

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

Question: Given a linear transformation T, are there any lines through the origin that are mapped to themselves by the linear transformation T?

its scalar multiple

$3x+y=0$ $\quad A\begin{bmatrix}x\\-3x\end{bmatrix} = -\begin{bmatrix}+x\\-3x\end{bmatrix}$

For $A = \begin{bmatrix}2 & 1\\ 0 & -1\end{bmatrix}$, the line through origin and

$L_{y=0}: \; A\begin{bmatrix}x\\0\end{bmatrix} = \begin{bmatrix}2x\\0\end{bmatrix} = 2\cdot\begin{bmatrix}x\\0\end{bmatrix}$

$Av = c\cdot v$

$v = \begin{bmatrix}x\\y\end{bmatrix}$

$\times \; L_{x+y=0}: \; A\begin{bmatrix}x\\-x\end{bmatrix} = \begin{bmatrix}x\\x\end{bmatrix}$

$x-y=0 \quad \times \; A\begin{bmatrix}x\\x\end{bmatrix} = \begin{bmatrix}3x\\-x\end{bmatrix}$

$\times \; A\begin{bmatrix}0\\y\end{bmatrix} = \begin{bmatrix}y\\-y\end{bmatrix}$

LHS $= \begin{bmatrix}2x+y\\-y\end{bmatrix} = \cancel{c}\cdot\begin{bmatrix}cx\\cy\end{bmatrix}$

$\begin{bmatrix}2x+y\\-y\end{bmatrix} = \begin{bmatrix}-x\\-y\end{bmatrix}$

Introduction

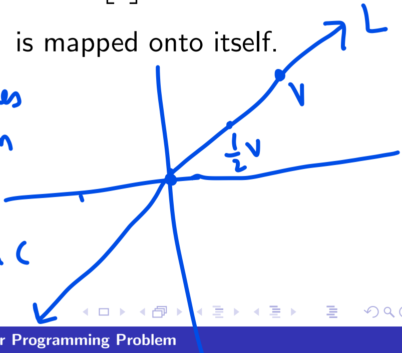
$T V \overset{?}{=} c \cdot V$ ⇒ line through V & 0 will ~~map~~ on itself

$\{\alpha V \mid \alpha \in \mathbb{R}\}$

Question: Given a linear transformation T, are there ~~any lines~~ points through the origin that are mapped to ~~themselves~~ their scalar multiple by the linear transformation T?

For $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, the line through origin and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is mapped onto itself and line through origin and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is mapped onto itself.

{set of all non-zero pts} ⟶ set of all lines through origin



$$L_{x=y} = \left\{ c \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle/ c \in \mathbb{R} \right\}$$

$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for some c

Introduction

Question: Given a linear transformation T, are there any lines through the origin that are mapped to themselves by the linear transformation T?

For $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, the line through origin and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is mapped onto itself and line through origin and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is mapped onto itself.

Question: In general, given a linear transformation T, are there subspaces that are mapped to themselves by T?

invariant subspace!

$A\begin{bmatrix}1\\-3\end{bmatrix} = -1\begin{bmatrix}+1\\-3\end{bmatrix} \Rightarrow \begin{bmatrix}1\\-3\end{bmatrix}$ is an e.vector corresponding to e.value -1

$\begin{bmatrix}2\\-6\end{bmatrix}$

2 unknowns.

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $\boxed{A\mathbf{x} = \lambda\mathbf{x}}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

$$\begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix}$$

$L_{x,0} : T(L_{x,0}) \subseteq L_{x,0}$

$\Rightarrow \begin{bmatrix}1\\0\end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 2$

$\begin{bmatrix}3\\0\end{bmatrix}$

$\begin{bmatrix}-5\\0\end{bmatrix}$

Eigenvalues and Eigenvectors

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

Hence for $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue 2 and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue -1.

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Question Is $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ also an eigenvector?

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Finding Eigenvalues and Eigenvectors (for square matrices)

Question: How to find eigenvalue/vector?

Finding Eigenvalues and Eigenvectors (for square matrices)

$T : \mathbb{R}^n \to \mathbb{R}^n$ $\qquad$ $A \cdot 0 = \lambda \cdot 0 \quad \forall \lambda \in \mathbb{R}$

Question: How to find eigenvalue/vector?

$A_{n \times n}$

$$Ax = \lambda x$$

We have to find $0 \neq x$ & scalar λ.

$$(A - \lambda I) x = 0$$

$Ax - \lambda x = 0$

$\lambda \cdot I \cdot x$

$\exists\ x \in \text{Null}(\underbrace{A - \lambda I}_{B})$

$\exists\ 0 \neq x$ s.t. $\quad Bx = 0 \iff$ col^n of B are lin. dep. i.e. $\det(B) = 0$

involves only $\lambda \leftarrow \det(A - \lambda I) = 0 \rightarrow$ characteristic eq^n

Finding Eigenvalues and Eigenvectors (for square matrices)

$Ay = -y$

Question: How to find eigenvalue/vector?

$$Ax = \lambda x$$

$$Ax - \lambda . x = 0 \Rightarrow (A - \lambda . I)x = 0$$

$\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ $\det(A-\lambda I)=0$ $\Rightarrow \det\begin{bmatrix} 2-\lambda & 1 \\ 0 & -1-\lambda \end{bmatrix} = 0$

Case 2) $\lambda = -1$

$A-\lambda I = A+I = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -\frac{1}{3}c \\ c \end{bmatrix} = c\begin{bmatrix} -1/3 \\ 1 \end{bmatrix} = y$

$(2-\lambda)(+1+\lambda) - 0 = 0$

$2+\lambda-\lambda^2=0 \Rightarrow \lambda^2-\lambda-2=0$

LHS is poly of deg 2

$\Rightarrow 2, -1$

Case 1) $\lambda = 2$

$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}$

$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$(A-2I)x = 0$

$x = \begin{bmatrix} c \\ 0 \end{bmatrix}$ $Ax = 2x$

Finding Eigenvalues and Eigenvectors (for square matrices)

Question: How to find eigenvalue/vector?

$$Ax = \lambda x$$

$Ax - \lambda.x = 0 \Rightarrow (A - \lambda.I)x = 0$
$\Rightarrow x \in Null(A - \lambda.I)$.

i be the last digit of your id

$$A = \begin{bmatrix} 1 & 0 \\ i & 2 \end{bmatrix}$$

Find e.values & e.vectors of A

Finding Eigenvalues and Eigenvectors (for square matrices)

Question: How to find eigenvalue/vector?

$$Ax = \lambda x$$

$Ax - \lambda.x = 0 \Rightarrow (A - \lambda.I)x = 0$
$\Rightarrow x \in Null(A - \lambda.I)$.
$Null(A - \lambda.I) \neq \{0\} \Leftrightarrow \boxed{det(A - \lambda.I) = 0}$ — polyin λ of deg n.
Given a matrix A, the polynomial $det(A - \lambda.I) = 0$ is called characteristic polynomial of A(here λ is treated as a variable). Its roots are called the eigenvalues of A.

$(A - \lambda I)x = 0$
$\Downarrow$
$Ax = \lambda x$

Finding Eigenvalues and Eigenvectors (for square matrices)

Question: How to find eigenvalue/vector?

$$Ax = \lambda x$$

$V_\lambda = \{ x \mid Ax = \lambda x \}$

becomes of $\mathbb{R}^n$

$Ax_1 = \lambda x_1$ & $Ax_2 = \lambda x_2$

$A(x_1 + x_2) = \lambda(x_1 + x_2)$

$A(c.x_1) = \lambda(cx_1)$

eigenspace

$Ax - \lambda.x = 0 \Rightarrow (A - \lambda.I)x = 0$
$\Rightarrow x \in Null(A - \lambda.I)$.
$Null(A - \lambda.I) \neq \{0\} \Leftrightarrow det(A - \lambda.I) = 0$

Given a matrix A, the polynomial $det(A - \lambda.I) = 0$ is called characteristic polynomial of A(here λ is treated as a variable). Its roots are called the eigenvalues of A.

Theorem

The eigenvalues of a triangular matrix are the entries on its main diagonal.

$$\underset{\text{triangular}}{A - \lambda I} = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & \ddots & \\ 0 & & & a_{nn}-\lambda \end{bmatrix}$$

$$det(A - \lambda I) = \prod_{i=1}^{n} (a_{ii} - \lambda)$$

Determinant and eigenvalues

Recall for A an $n \times n$ matrix, let U be any row echelon form of A obtained by row replacements and row interchanges (without scaling), and let r be the number of such row interchanges. Then the determinant of A,

$$\boxed{det(A) = (-1)^r u_{11} u_{22} \cdots u_{nn}}$$

where u_{ii} are diagonal entries of U.

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where u_{ii} are diagonal entries of U.

Theorem (Invertible matrix theorem)

Let A be an $n \times n$ matrix. Then A is invertible (i.e., det $(A) \neq 0$) iff 0 is not an eigenvalue of A. A is not invertible $\Leftrightarrow$ 0 is an e.value.

A not invertible $\Rightarrow$ REF will have less pivot entries, i.e. some free variable $\Rightarrow$ $Ax = 0$ will have non-zero sol^n

$Ax = 0 = 0 \cdot x$ $\Rightarrow$ x e.vector corresp. e.v. 0

$Ax = 0$ $\Rightarrow$ $x \in Null(A) \Rightarrow dim(Null(A)) > 0$

Determinant and eigenvalues

Recall for A an $n \times n$ matrix, let U be any row echelon form of A obtained by row replacements and row interchanges (without scaling), and let r be the number of such row interchanges. Then the determinant of A,

$$det(A) = (-1)^r u_{11} u_{22} \cdots u_{nn}$$

where u_{ii} are diagonal entries of U.

ch. poly of A

$\det(A - \lambda I) = \det(PBP^{-1} - P\lambda I P^{-1})$

$= \det(P(B - \lambda I)P^{-1})$

$= \det(P)\, \det(B - \lambda I)\, \det(P^{-1})$

ch. poly of B

Theorem (Invertible matrix theorem)

Let A be an $n \times n$ matrix. Then A is invertible (i.e., $\det(A) \neq 0$) iff 0 is not an eigenvalue of A.

$B \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ PBP^{-1}.

Theorem

If matrices A and B are similar ($A = PBP^{-1}$), then they have the same characteristic polynomial and hence the same eigenvalues.

$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

PBP^{-1}

$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & +2 \\ +1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix}$

ch poly of $B = (1-\lambda)(4-\lambda) - 6$,

$= \lambda^2 - 5\lambda - 2$

$\boxed{A = PBP^{-1}}$

A & B are similar.

$A = \begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix}$

same

ch poly of $A = (3-\lambda)(2-\lambda) - 8$

$= \lambda^2 - 5\lambda - 2$

$\frac{5 \pm \sqrt{25+8}}{2} = \frac{5 \pm \sqrt{33}}{2} =$

$\lambda_1 = \frac{5+\sqrt{33}}{2}$, $\lambda_2 = \frac{5-\sqrt{33}}{2}$

$\in \mathbb{R}$

$\lambda_1\lambda_2 = \left(\frac{5}{2}\right)^2 - \left(\frac{\sqrt{33}}{2}\right)^2 = -2 = \det(A)$

$\lambda_1 + \lambda_2 = 5 = \operatorname{trace}(A)$

$A : \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{3\times 3}$ $\det(A - \lambda I) = (r_1 - \lambda)(r_2 - \lambda)(a+ib-\lambda)(a-ib-\lambda)$

$p(x) = 0$

$a+ib$

$p(a+ib) = 0 \Rightarrow p(a-ib) = 0$

Theorem

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $\det(A) = \prod_i \lambda_i$ and trace of $(A) = \sum_i \lambda_i$

$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$

$= \lambda^n - \lambda^{n-1}\left(\text{sum of all } \lambda_i\right) + \cdots + (-1)^n \text{ product of all } \lambda_i\text{s}$

$\det(A - \lambda I)$

$= (-1)^n \lambda^n + \operatorname{trace}(A)\lambda^{n-1} + \cdots + \det(A).$

matrix P is invertible ⟺ all its columns are l. indep.

$\begin{bmatrix} +1 \\ 3 \end{bmatrix}$ $AV_2 = 7V_2$ v_2

$P = \begin{bmatrix} -2 & +1 \\ 1 & 3 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$ PDP^{-1}

$A = \frac{-1}{7}\begin{bmatrix} -2 & +1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -7 & -14 \end{bmatrix} = \frac{1}{-7}\begin{bmatrix} -2 & +1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix}$

Definition

A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix, i.e.,

$$A = PDP^{-1}$$

$\iff AP = PD$, $[v_1 \dots v_n] = P$, $D = \text{diag}(a_1 \dots a_n)$

$Av_i = a_i v_i$

for some invertible matrix P and diagonal matrix D.

e. values are 0, 7

1st column of $AP = Av_1$

1st column of $PD = a_1 v_1$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$A - 0.I = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

P will be matrix of invertible eigenvectors

$(1-\lambda)(6-\lambda) - 6 = 0$

$\begin{bmatrix} -2c \\ c \end{bmatrix} = c\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda^2 - 7\lambda = 0$

$\lambda(\lambda - 7) = 0$

$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $Av_1 = 0 \cdot v_1$

$A - 7I = \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix} \to \begin{bmatrix} -6 & 2 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -d/3 \\ d \end{bmatrix} = d\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$

D will be diag. matrix of eigenvalues.

Diagonalization

$T_A(v) = Av$

New basis of $\mathbb{R}^n$: $\{v_1, \dots, v_n\} = \beta$

$[T_A]_\beta = D$ $\quad T_A(v_i) = \lambda_i v_i$ $\quad \begin{bmatrix} 0 \\ \lambda_i \\ 0 \end{bmatrix}$

Definition

A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix, i.e.,

$$A = PDP^{-1} \Rightarrow AP = PD \Rightarrow$$

Colums of P are eigenvectors of A

for some invertible matrix P and diagonal matrix D.

$\Rightarrow$ Columns of P form a basis of $\mathbb{R}^n$

Theorem

An $n \times n$ matrix A is diagonalizable ($A = PDP^{-1}$) iff A has n linearly independent eigenvectors. $\equiv$ ~~A has n l.i.~~ $A = PDP^{-1}$

In this case eigenvectors will form a basis of $\mathbb{R}^n$.

$B' = \{e_1, e_2\}$

$$[T]_{B'} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

$T(e_1)$, $T(e_2)$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1e_1 + 3e_2$

$B = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ basis of $\mathbb{R}^2$

$v_1 \quad v_2$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x+6y \end{bmatrix}$$

$$[T]_B = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix} = D$$

$$[T]_B = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix} = D$$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_B = Tv_1 = 0 \cdot v_1 = \underline{0} \cdot v_1 + \underline{0} \cdot v_2$

$\begin{bmatrix} 0 \\ 7 \end{bmatrix}_B = Tv_2 = 7v_2 = 0 \cdot v_1 + 7 \cdot v_2$

$\Rightarrow \exists\, n$ lin. indep. eigenvectors $\rightarrow P$

$D \rightarrow$ eigenvalues $A = PDP^{-1}$

Theorem

An $n \times n$ matrix with n **distinct eigenvalues** *is diagonalizable*

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \rightarrow \begin{array}{l} \lambda_1 = 1 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \lambda_2 = 2 \rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} \neq d \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ for any } d. \end{array}$$

$$A - 2I = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$$

Two eigenvectors v_1, v_2 corresponding distinct eigenvalues λ_1, λ_2 $(\lambda_1 \neq \lambda_2)$ are lin. indep.

$c_1 v_1 + c_2 v_2 = 0 \quad \times \lambda_1$

Let $A \times c_1 v_1 + c_2 v_2 = 0 \quad \times A$

$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0$

$$c_1 = 0 \Leftarrow \underset{\substack{\| \\ 0}}{c_2} \underset{\substack{\neq \\ 0}}{(\lambda_2 - \lambda_1)} \underset{\substack{\neq \\ 0}}{v_2} = 0$$

$\lambda_1 \quad \times \quad c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + c_3 \lambda_3 v_3 = 0$

$c_2(\lambda_1 - \lambda_2) v_2 + c_3(\lambda_1 - \lambda_3) v_3 = 0$

$c_2 = 0 = c_3$

$\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_n$

$v_1 \quad v_2 \quad \quad v_n$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow$ e.values 1

$Ix = 1 \cdot x \quad \forall x \in \mathbb{R}^2$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Theorem

An $n \times n$ matrix with n **distinct eigenvalues** *is diagonalizable*

Proof: Let $v_1, \ldots, v_n$ be eigenvectors corresponding to the n distinct eigenvalues of a matrix A. Then $\{v_1, \ldots, v_n\}$ is linearly independent set, hence basis of $\mathbb{R}^n$.

Claim: $\{v_1 \ldots v_n\}$ is lin indep.

$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ — (1)

Claim: $c_i = 0 \quad \forall i$

$A \times (1) \Rightarrow c_1 A v_1 + c_2 A v_2 + \cdots + c_n A v_n = 0$

$c_1 \lambda_1 v_1 + \cdots + c_n \lambda_n v_n = 0$ — (2)

$\lambda_1 \times (1) \quad c_1 \lambda_1 v_1 + c_2 \lambda_1 v_2 + \cdots + c_n \lambda_1 v_n = 0$ — (3)

(2) − (3)

ch. poly $= \det(A-\lambda I) = (\lambda_0 - \lambda)^r \cdot p(\lambda)$

r max. possible

$\mathrm{Null}(A-\lambda_0 I) = V_{\lambda_0} \to$ eigenspace corresp to λ

$\dim(V_{\lambda_0}) =$ Nullity of $(A-\lambda_0 I) =$ geometric multiplicity of λ_0

Theorem

alg. multiplicity

An $n \times n$ matrix with n **distinct eigenvalues** *is diagonalizable*

Proof: Let $v_1, \ldots, v_n$ be eigenvectors corresponding to the n distinct eigenvalues of a matrix A. Then $\{v_1, \ldots, v_n\}$ is linearly independent set, hence basis of $\mathbb{R}^n$.

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

geo multiplicity

a. For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

alg.

$\sum$ geo. mult. of $\lambda = n$

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k. alg. mult = geo mult.

c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \text{ch. poly} = (1-\lambda)^1 (3-\lambda)^2$$

alg. mult of 3.

1 is called alg. mult. of 1