Grammars

Grammars

Grammars express languages

Example: the English language

$$\langle sentence \rangle \rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle$$

$$\langle noun\_phrase \rangle \rightarrow \langle article \rangle \ \langle noun \rangle$$

$$\langle predicate \rangle \rightarrow \langle verb \rangle$$

$$\langle article \rangle \rightarrow a$$

$$\langle article \rangle \rightarrow the$$

$$\langle noun \rangle \rightarrow cat$$

$$\langle noun \rangle \rightarrow dog$$

$$\langle verb \rangle \rightarrow runs$$

$$\langle verb \rangle \rightarrow walks$$

A derivation of "the dog walks":

$$\begin{aligned}
\langle sentence \rangle &\Rightarrow \langle noun_phrase \rangle \; \langle predicate \rangle \\
&\Rightarrow \langle noun_phrase \rangle \; \langle verb \rangle \\
&\Rightarrow \langle article \rangle \; \langle noun \rangle \; \langle verb \rangle \\
&\Rightarrow the \; \langle noun \rangle \; \langle verb \rangle \\
&\Rightarrow the \;\; dog \; \langle verb \rangle \\
&\Rightarrow the \;\; dog \;\; walks
\end{aligned}$$

A derivation of "a cat runs":

$$
\begin{aligned}
\langle sentence \rangle &\Rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle \\
&\Rightarrow \langle noun\_phrase \rangle \ \langle verb \rangle \\
&\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow a \ \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow a \ \ cat \ \ \langle verb \rangle \\
&\Rightarrow a \ \ cat \ \ runs
\end{aligned}
$$

Language of the grammar:

L = { "a cat runs",
"a cat walks",
"the cat runs",
"the cat walks",
"a dog runs",
"a dog walks",
"the dog runs",
"the dog walks" }

Notation

$$\overbrace{\begin{array}{l}\langle noun \rangle \rightarrow cat \\ \langle noun \rangle \rightarrow dog\end{array}}^{\text{Production Rules}}$$

Variable

Terminal

Another Example

Grammar: $S \to aSb$

$S \to \lambda$

Derivation of sentence ab:

$$S \Rightarrow aSb \Rightarrow ab$$

$S \to aSb$ $\qquad\qquad$ $S \to \lambda$

Grammar: $S \to aSb$

$S \to \lambda$

Derivation of sentence $aabb$:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

$S \to aSb$ $\qquad\qquad$ $S \to \lambda$

Other derivations:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb$$
$$\Rightarrow aaaaSbbbb \Rightarrow aaaabbbb$$

Language of the grammar

$$S \to aSb$$

$$S \to \lambda$$

$$L = \{a^n b^n : n \geq 0\}$$

More Notation

Grammar $G=(V,T,S,P)$

$V:$ Set of variables

$T:$ Set of terminal symbols Σ

$S:$ Start variable

$P:$ Set of Production rules

Example

Grammar G : $S \rightarrow aSb$

$S \rightarrow \lambda$

$$G = (V, T, S, P)$$

$V = \{S\}$ $T = \{a, b\}$

$P = \{S \rightarrow aSb,\ S \rightarrow \lambda\}$

More Notation

Sentential Form:

A sentence that contains variables and terminals

Example:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

Sentential Forms

sentence

We write: $S \overset{*}{\Rightarrow} aaabbb$

Instead of:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

In general we write: $$w_1 \overset{*}{\Rightarrow} w_n$$

If: $$w_1 \Rightarrow w_2 \Rightarrow w_3 \Rightarrow \cdots \Rightarrow w_n$$

By default: $w \overset{*}{\Rightarrow} w$

Example

Grammar

$$S \to aSb$$

$$S \to \lambda$$

Derivations

$$S \overset{*}{\Rightarrow} \lambda$$

$$S \overset{*}{\Rightarrow} ab$$

$$S \overset{*}{\Rightarrow} aabb$$

$$S \overset{*}{\Rightarrow} aaabbb$$

Example

Grammar

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

Derivations

$$S \overset{*}{\Rightarrow} aaSbb$$

$$aaSbb \overset{*}{\Rightarrow} aaaaaSbbbbb$$

Another Grammar Example

Grammar G :

$$S \to Ab$$
$$A \to aAb$$
$$A \to \lambda$$

Derivations:

$$S \Rightarrow Ab \Rightarrow b$$

$$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow abb$$

$$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow aaAbbb \Rightarrow aabbb$$

More Derivations

$$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow aaAbbb \Rightarrow aaaAbbbb$$

$$\Rightarrow aaaaAbbbbb \Rightarrow aaaabbbbb$$

$$S \overset{*}{\Rightarrow} aaaabbbbb$$

$$S \overset{*}{\Rightarrow} aaaaaabbbbbbb$$

$$S \overset{*}{\Rightarrow} a^n b^n b$$

Language of a Grammar

For a grammar G
with start variable S :

$$L(G) = \{w : S \overset{*}{\Rightarrow} w\}$$

String of terminals

Example

For grammar G :

$$S \rightarrow Ab$$

$$A \rightarrow aAb$$

$$A \rightarrow \lambda$$

$$L(G) = \{a^n b^n b : \quad n \geq 0\}$$

Since: $S \overset{*}{\Rightarrow} a^n b^n b$

A Convenient Notation

$$A \to aAb$$
$$A \to \lambda$$

$$\Longrightarrow \quad A \to aAb \mid \lambda$$

$$\langle article \rangle \to a$$
$$\langle article \rangle \to the$$

$$\Longrightarrow \quad \langle article \rangle \to a \mid the$$

Linear Grammars

Linear Grammars

Grammars with
at most one variable at the right side
of a production

Examples:

$$S \to aSb$$

$$S \to \lambda$$

$$S \to Ab \quad 1$$

$$A \to aAb \quad 1$$

$$A \to \lambda \quad 0$$

A Non-Linear Grammar

Grammar G:

$$S \to \underline{SS}$$ 2 X

$$S \to \lambda$$

$$S \to aSb$$

$$S \to bSa$$

bababa

S

↓

S S

↓

S S S

bSa bSa bSa

λ λ

What is L(G)?

$$L(G) = \{w: \; n_a(w) = n_b(w)\}$$

Number of a in string w

Another Linear Grammar

Grammar G :

$$S \to A$$

$$A \to aB \mid \lambda$$

$$B \to Ab$$

What is L(G)?

$$L(G) = \{a^n b^n : n \geq 0\}$$

Right-Linear Grammars

All productions have form:

$$A \to xB$$

or

$$A \to x$$

string of terminals

Example:

$$S \to abS$$

$$S \to a$$

$L(G)$?

$(ab)^* a$

Left-Linear Grammars

All productions have form:

$$A \to Bx$$

Σ / T

or

$$A \to x$$

string of terminals

Example:

$$S \to Aab$$

$$A \to Aab \mid B$$

$$B \to a$$

$S \to aAb$

Regular Grammars

Regular Grammars

A regular grammar is any
right-linear or left-linear grammar

Examples:

$$G_1$$

$$S \to abS$$

$$S \to a$$

$$G_2$$

$$S \to Aab$$

$$A \to Aab \mid B$$

$$B \to a$$

Observation

Regular grammars generate regular languages

Examples:

G_1

$$S \to abS$$

$$S \to a$$

G_2

$$S \to Aab$$

$$A \to Aab \mid B$$

$$B \to a$$

$$L(G_1) = (ab)^* a$$

$$L(G_2) = aab(ab)^*$$

$L(G_2) = aab(ab)^*$

Regular Grammars Generate Regular Languages

Theorem

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars}\end{array}\right\} = \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

Theorem - Part 1

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

Any regular grammar generates
a regular language

Theorem - Part 2

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

Any regular language is generated by a regular grammar

Proof - Part 1

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

The language $L(G)$ generated by any regular grammar G is regular

The case of Right-Linear Grammars

Let G be a right-linear grammar

We will prove: $L(G)$ is regular

Proof idea: We will construct NFA M with $L(M) = L(G)$

Grammar G is right-linear

Example:

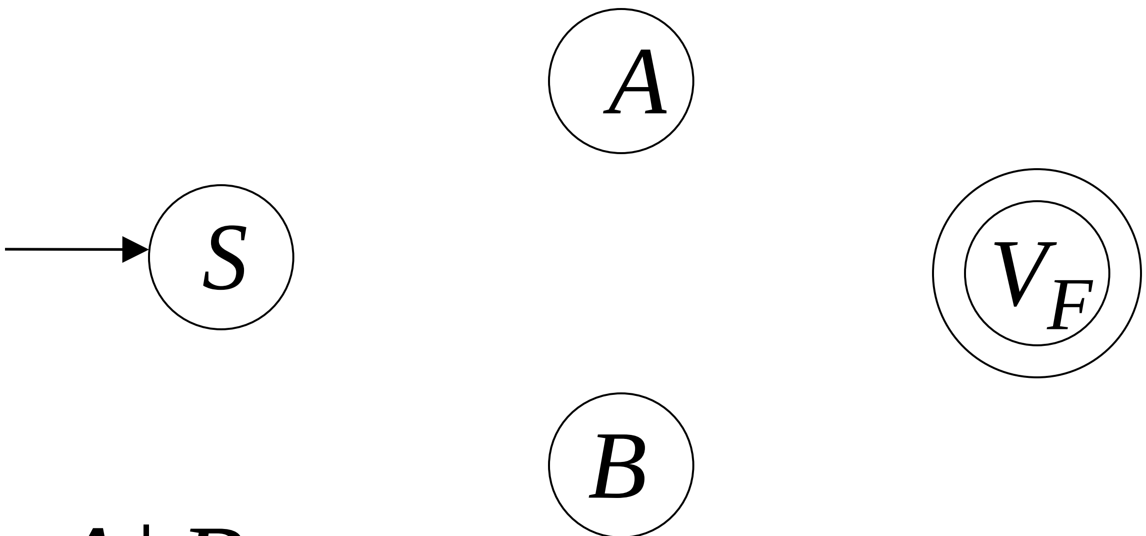
$$S \to aA \mid B$$

$$A \to aa\,B$$

$$B \to b\,B \mid a$$

L(G) ??

Construct NFA M such that every state is a grammar variable:



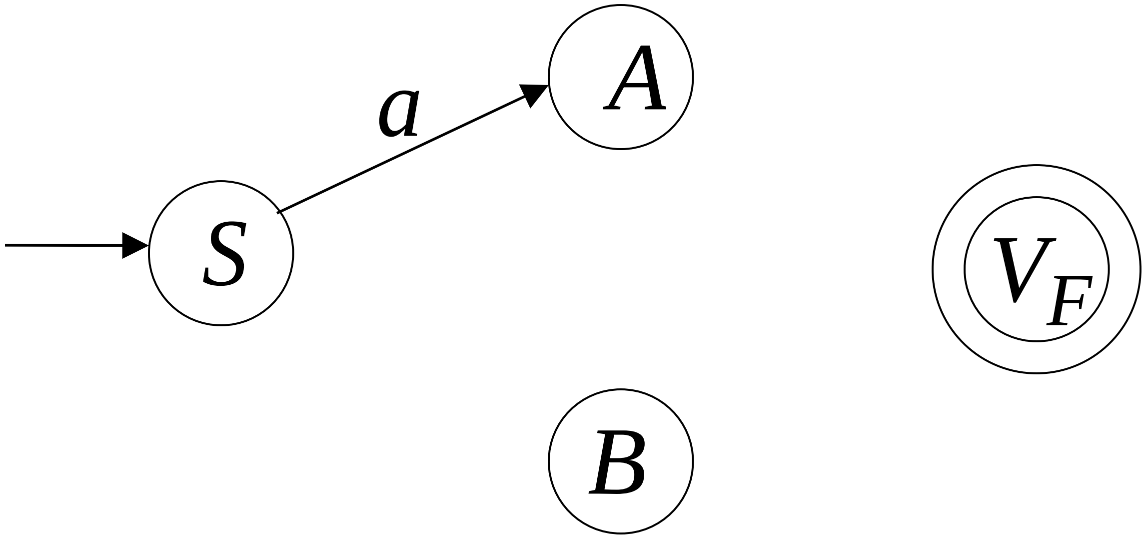
special final state

$$S \to aA \mid B$$

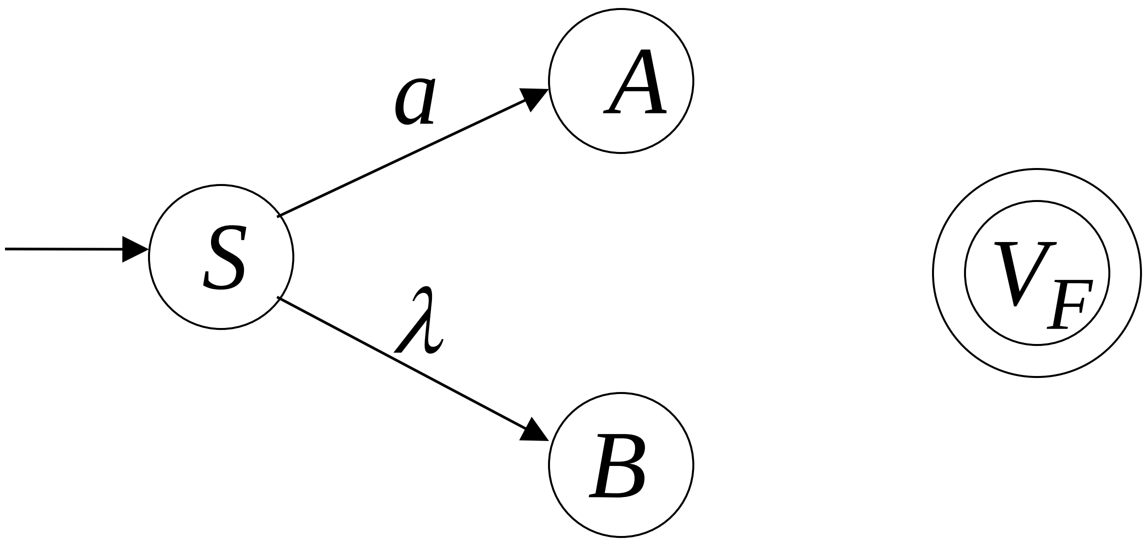
$$A \to aa\, B$$

$$B \to b\, B \mid a$$

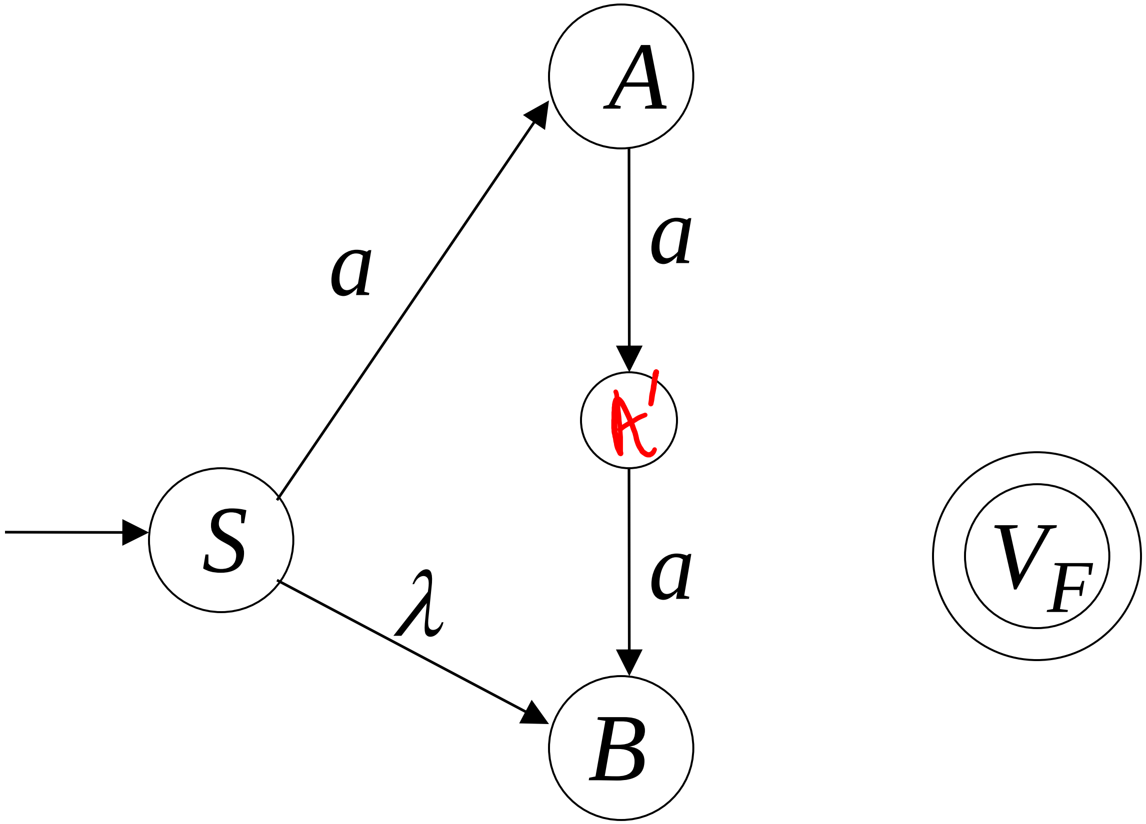
Add edges for each production:



$$S \rightarrow aA$$

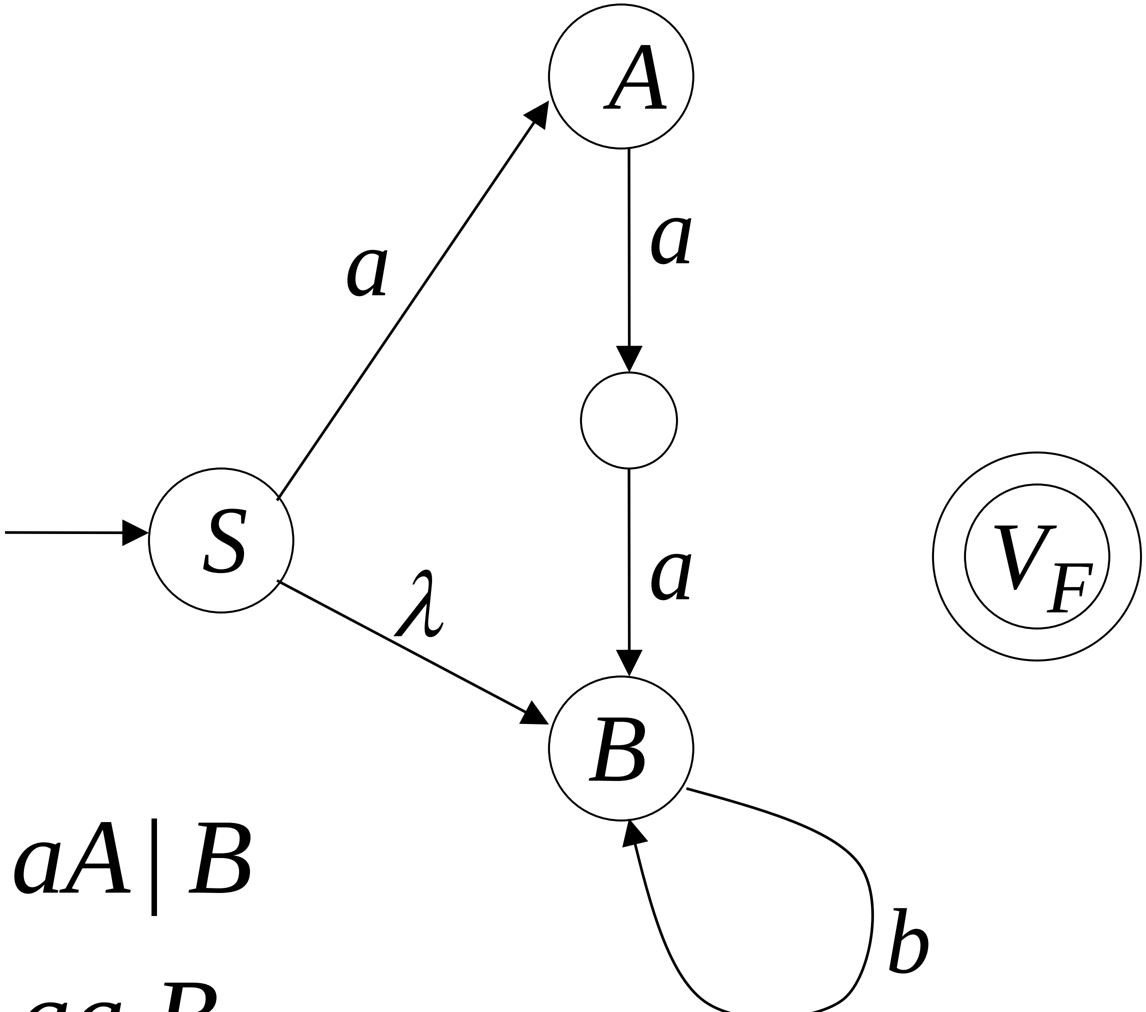


$$S \rightarrow aA \mid B$$



$$S \to aA \mid B$$

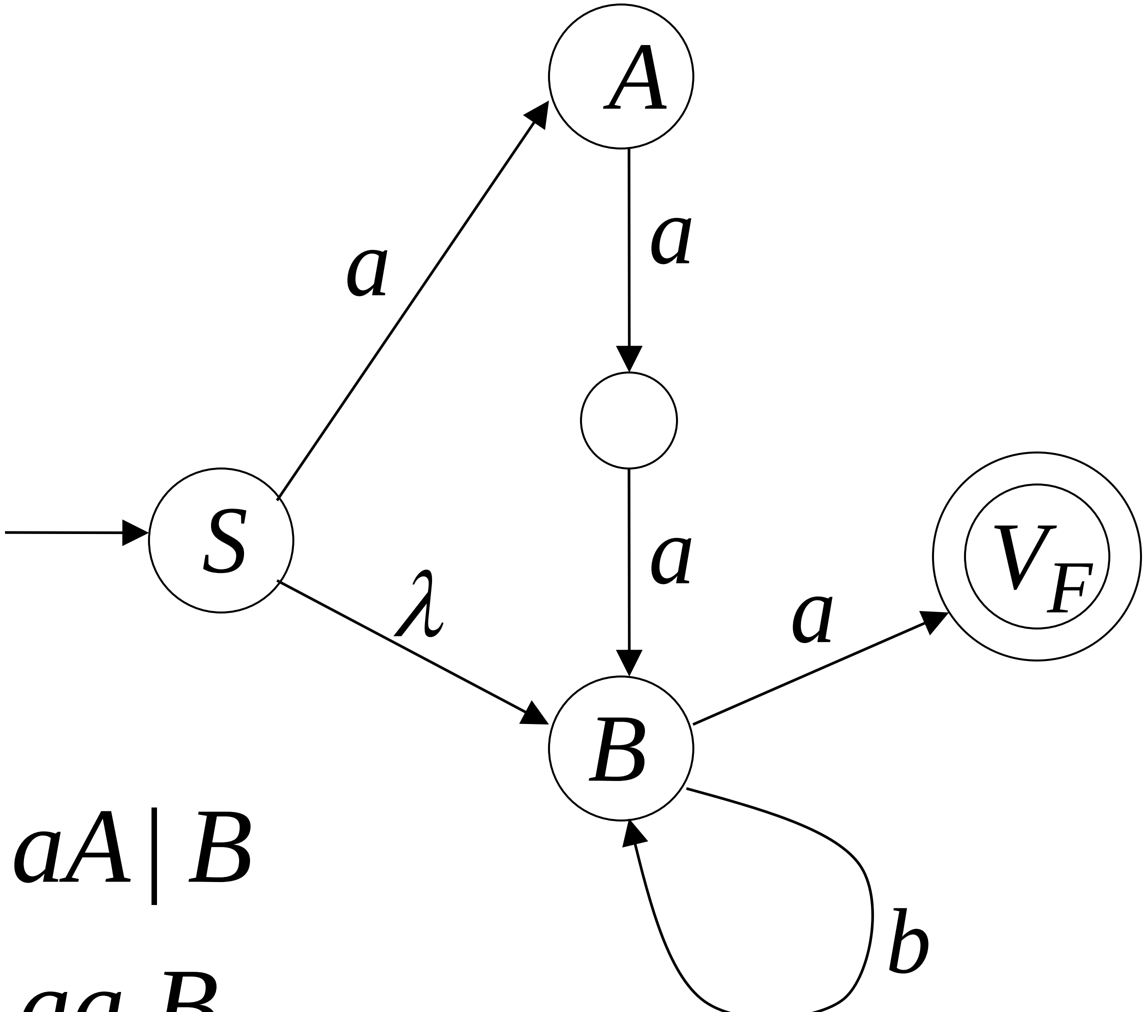
$$A \to aa\,B$$



$$S \to aA \mid B$$

$$A \to aa\ B$$

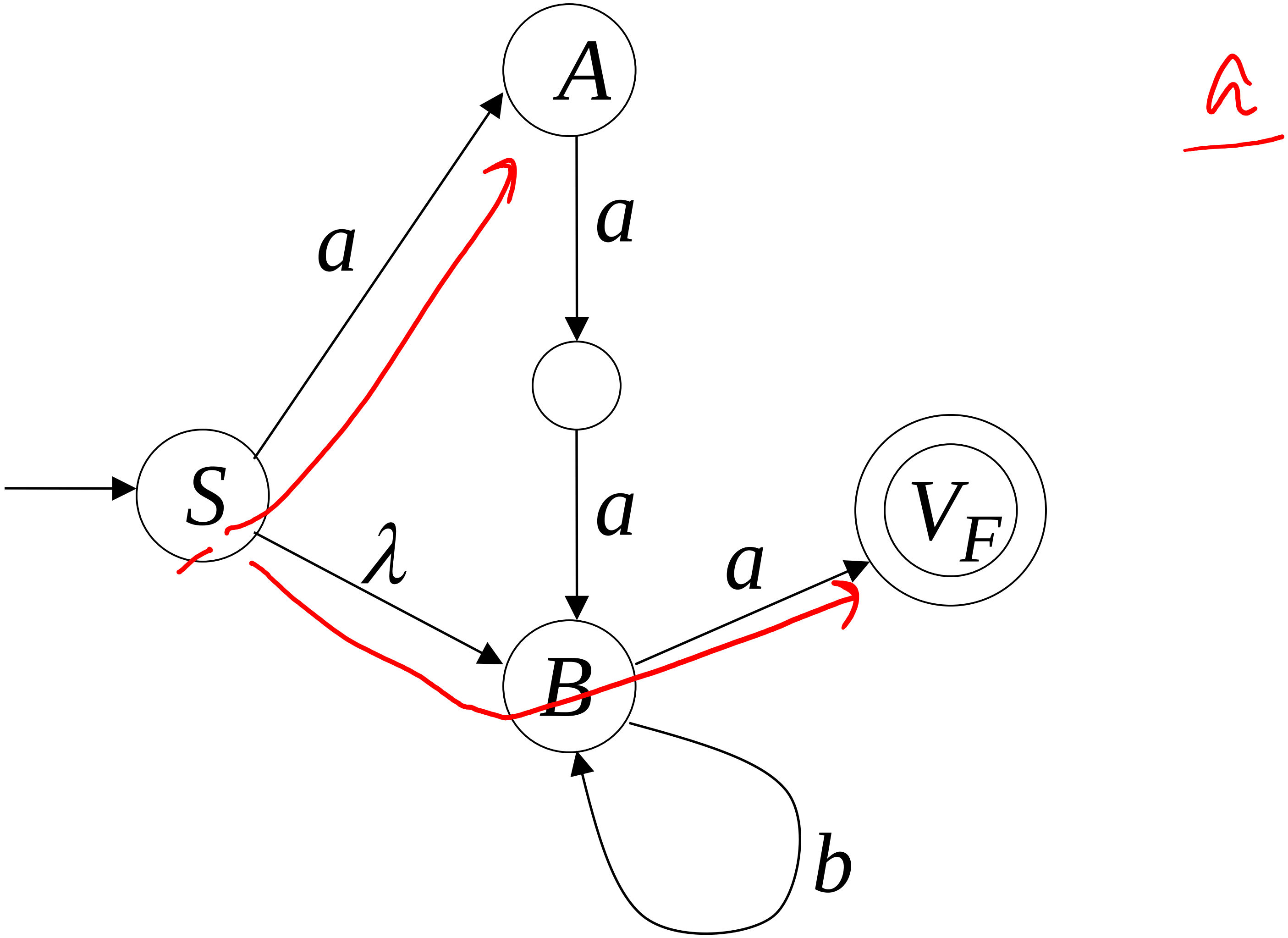
$$B \to bB$$



$$S \to aA \mid B$$

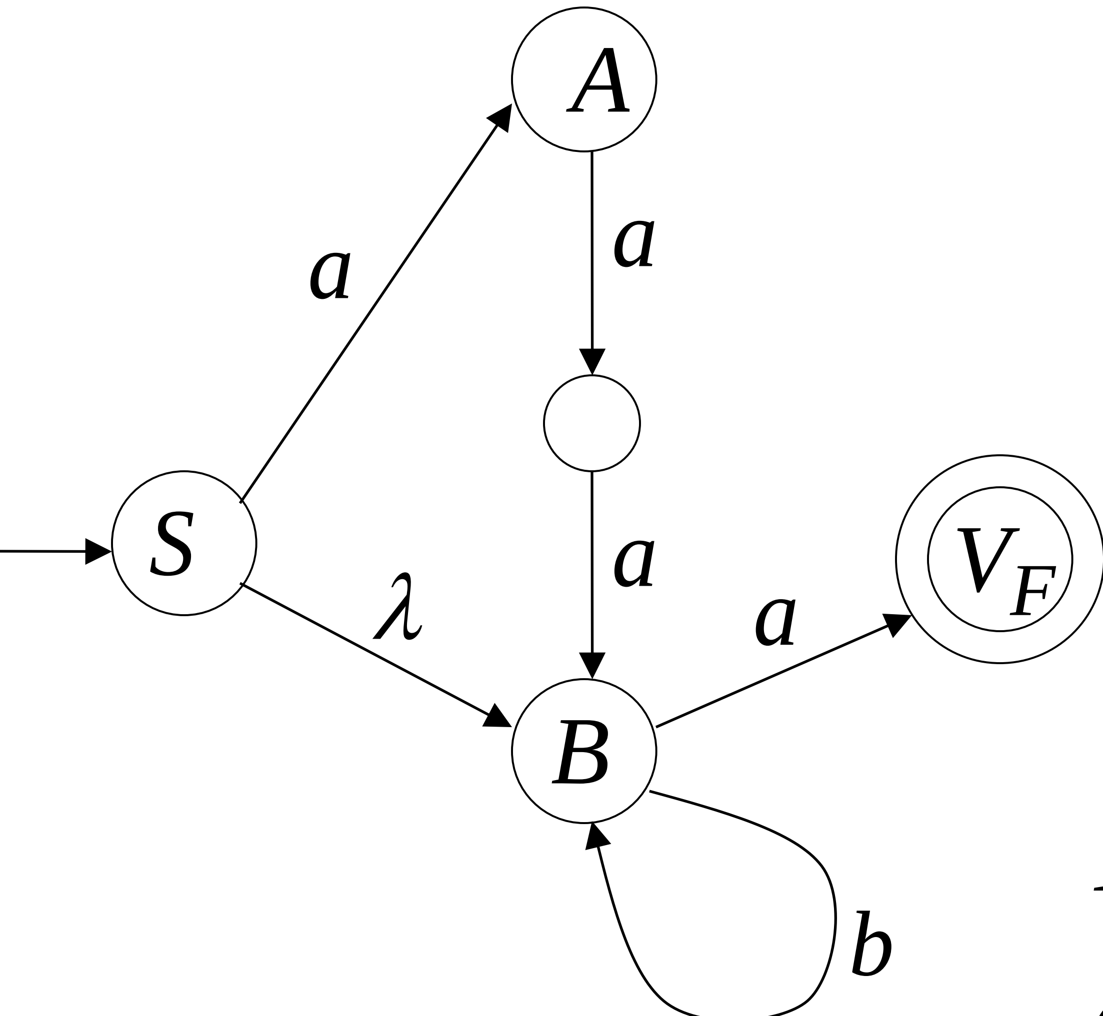
$$A \to aa\,B$$

$$B \to bB \mid a$$



$$S \Rightarrow aA \Rightarrow aaaB \Rightarrow aaabB \Rightarrow aaaba$$

NFA M



Grammar

G

$$S \rightarrow aA \mid B$$

$$A \rightarrow aa\,B$$

$$B \rightarrow bB \mid a$$

$$L(M) = L(G) =$$

$$aaab^*a + b^*a$$

In General

A right-linear grammar G

has variables: $V_0, V_1, V_2, \ldots$

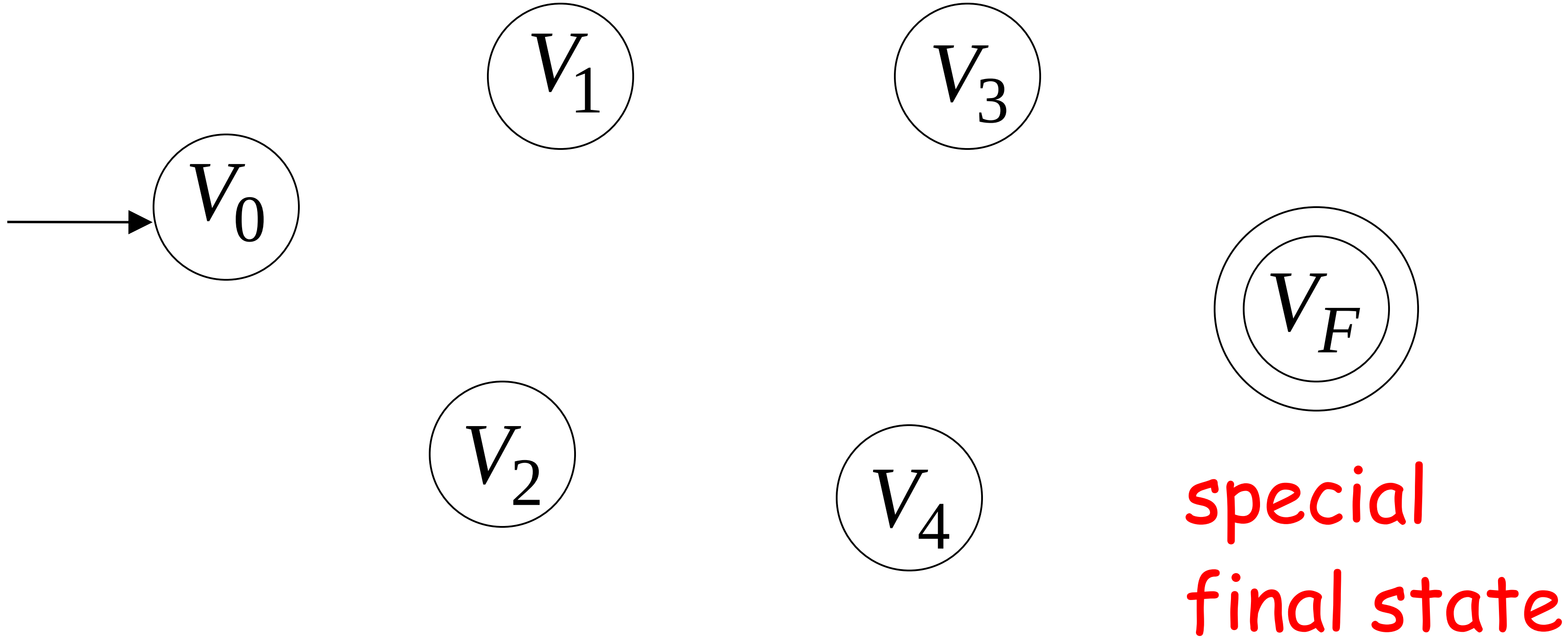
and productions: $V_i \rightarrow \underbrace{a_1 a_2 \cdots a_m}_{\Sigma} V_j$

or

$V_i \rightarrow \underbrace{a_1 a_2 \cdots a_m}_{\Sigma}$

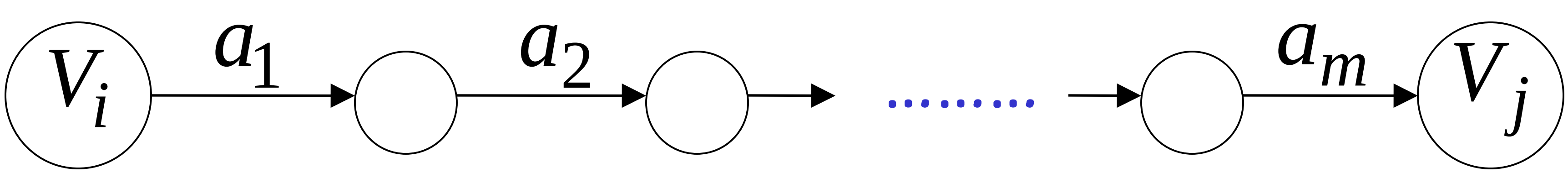
We construct the NFA M such that:

each variable V_i corresponds to a node:



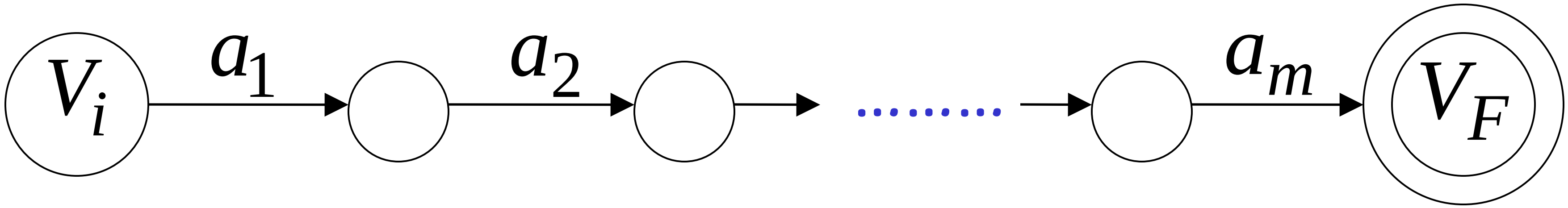
For each production: $V_i \rightarrow a_1 a_2 \cdots a_m V_j$

we add transitions and intermediate nodes

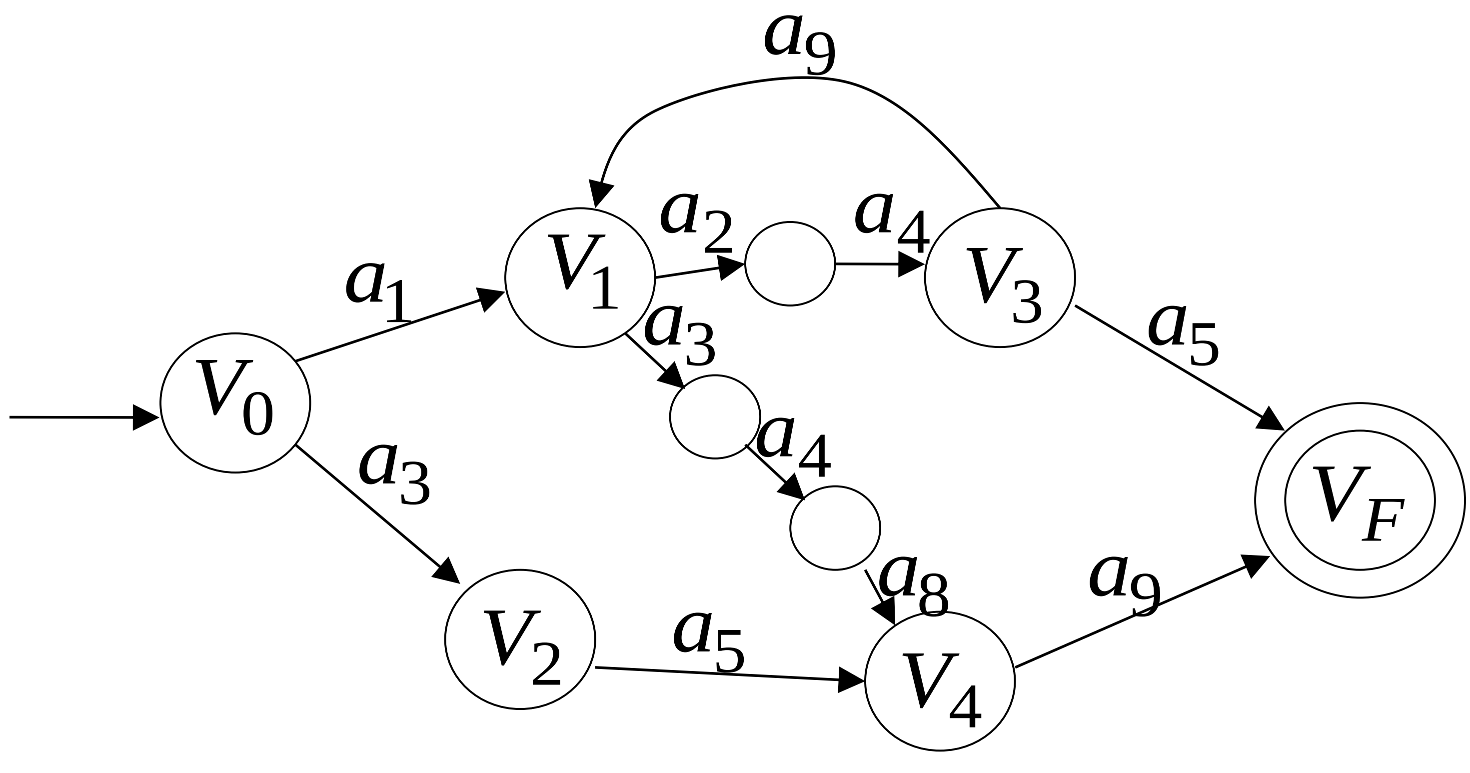


For each production: $V_i \to a_1 a_2 \cdots a_m$

we add transitions and intermediate nodes



Resulting NFA M looks like this:



It holds that: $L(G) = L(M)$

The case of Left-Linear Grammars

Let G be a left-linear grammar

We will prove: $L(G)$ is regular

Proof idea:

We will construct a right-linear grammar G' with $L(G) = L(G')^R$

Since G is left-linear grammar
the productions look like:

$$A \to Ba_1a_2 \cdots a_k$$

$$A \to a_1a_2 \cdots a_k$$

Construct right-linear grammar G'

Left linear G

$$A \to B a_1 a_2 \cdots a_k$$

$$A \to Bv$$

$$\Downarrow$$

Right linear G'

$$A \to a_k \cdots a_2 a_1 B$$

$$A \to v^R B$$

Construct right-linear grammar G'

Left linear G

$$A \to a_1 a_2 \cdots a_k$$

$$A \to v$$

Right linear G'

$$A \to a_k \cdots a_2 a_1$$

$$A \to v^R$$

It is easy to see that: $L(G) = L(G')^R$

Since G' is right-linear, we have:

$L(G') \Rightarrow L(G')^R \Rightarrow L(G)$

Regular Language | Regular Language | Regular Language

Proof - Part 2

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

Any regular language L is generated by some regular grammar G

Any regular language L is generated
by some regular grammar G

Proof idea:

Let M be the NFA with $L = L(M)$.

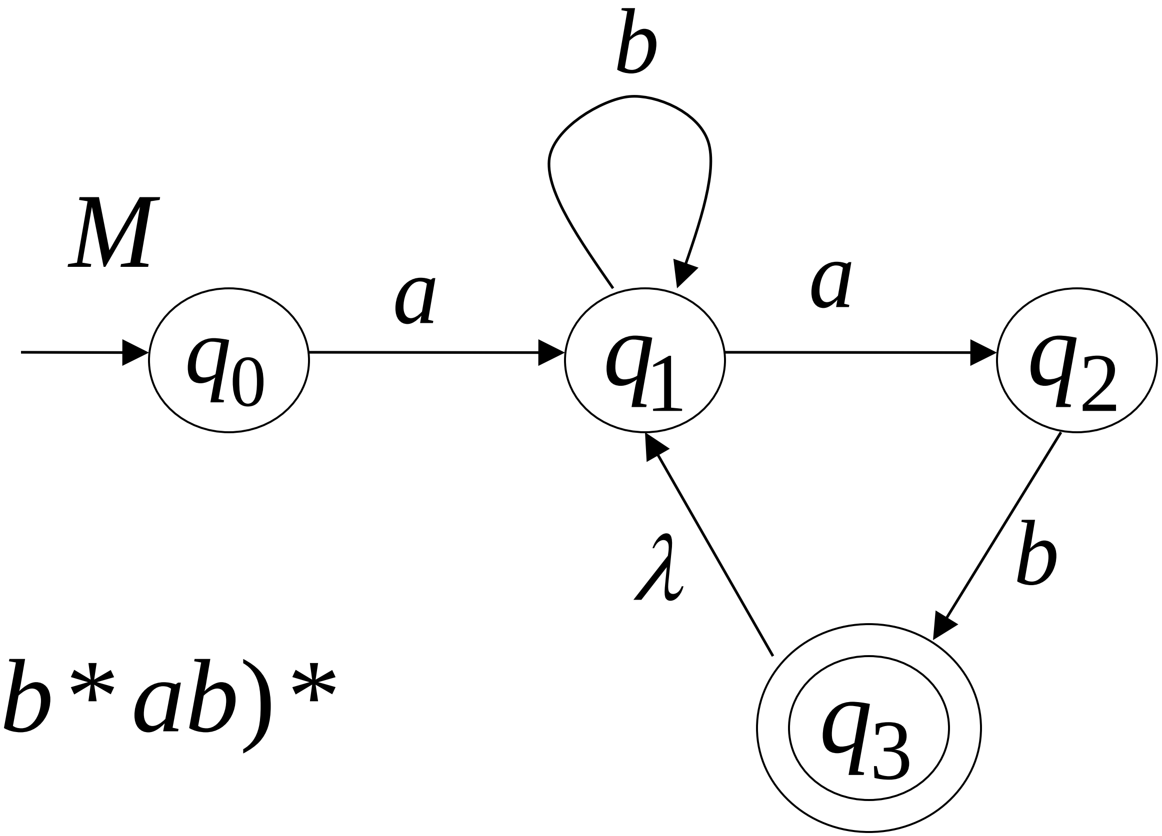
Construct from M a regular grammar G
such that

$$L(M) = L(G)$$

Since L is regular

there is an NFA M such that $L = L(M)$

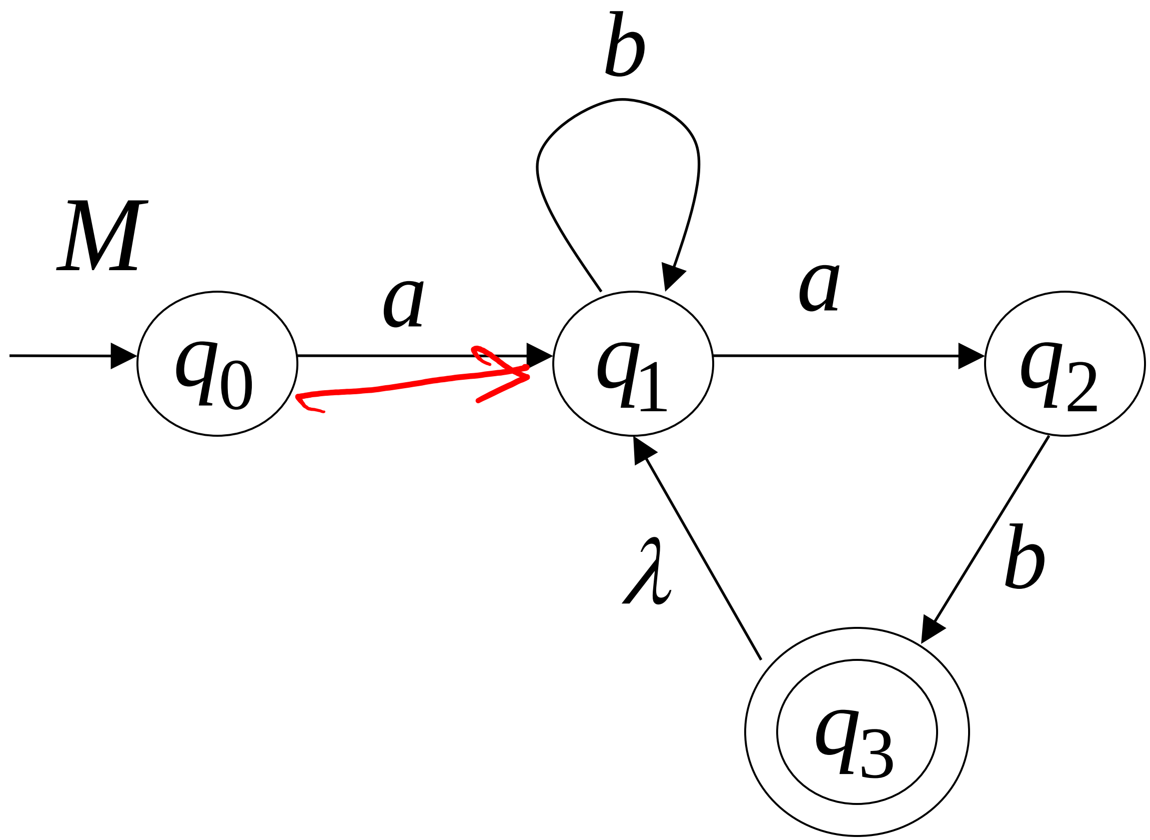
Example:



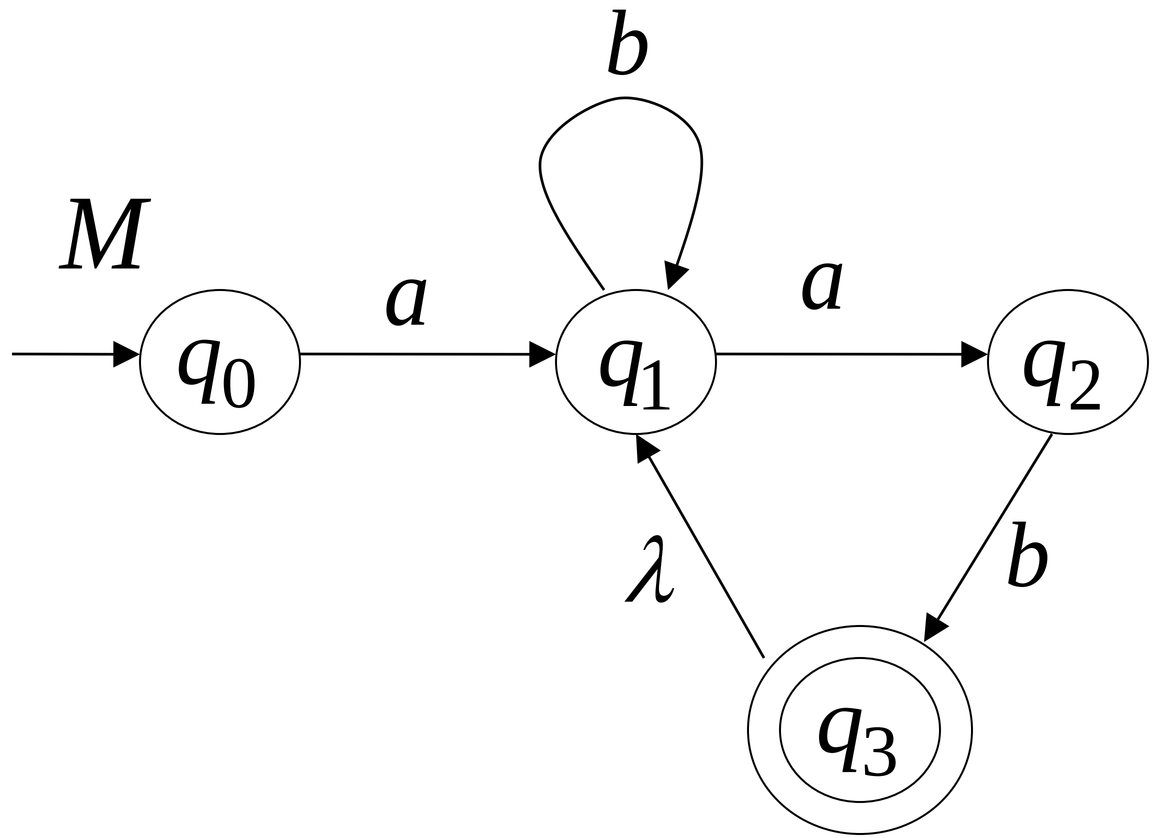
$$L = ab^*ab(b^*ab)^*$$

$$L = L(M)$$

Convert M to a right-linear grammar



$$q_0 \rightarrow aq_1$$



$$q_0 \to aq_1$$

$$q_1 \to bq_1$$

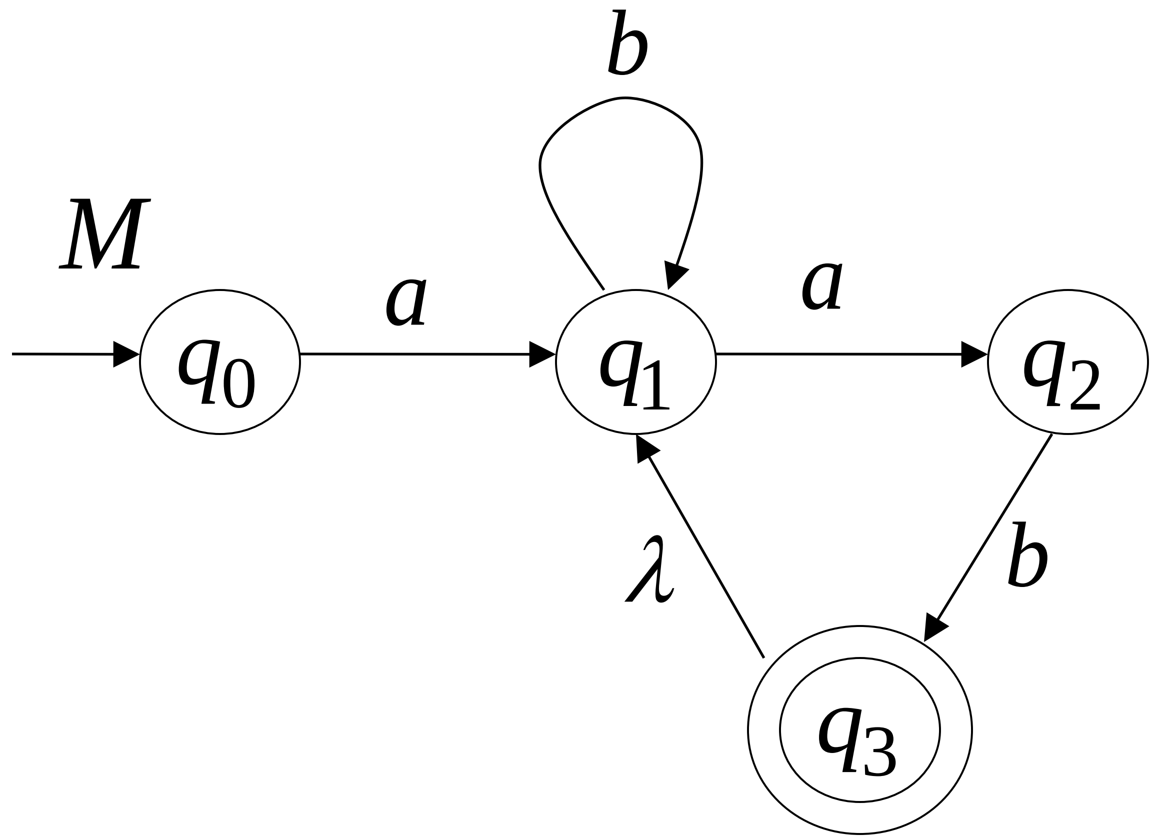
$$q_1 \to aq_2$$

$$q_0 \rightarrow aq_1$$

$$q_1 \rightarrow bq_1$$

$$q_1 \rightarrow aq_2$$

$$q_2 \rightarrow bq_3$$



$$L(G) = L(M) = L$$

G

$q_0 \rightarrow aq_1$

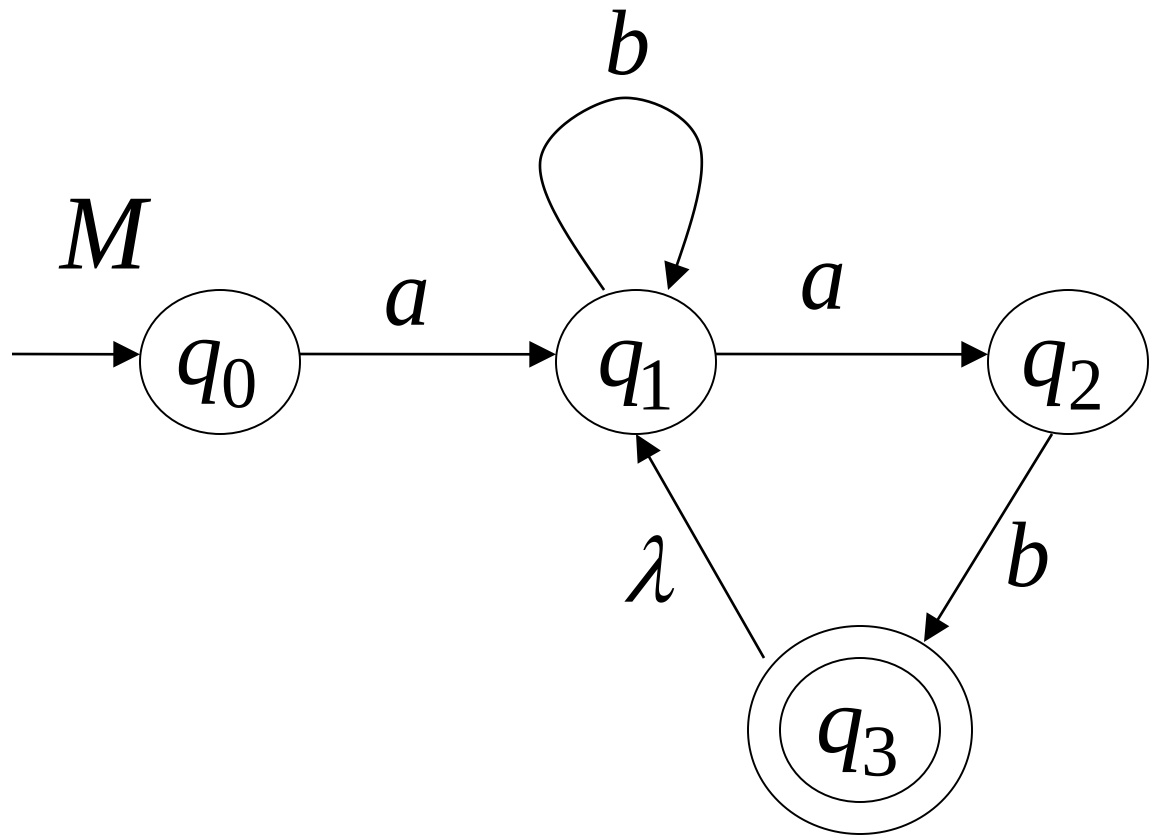
$q_1 \rightarrow bq_1$

$q_1 \rightarrow aq_2$

$q_2 \rightarrow bq_3$

$q_3 \rightarrow q_1$

$q_3 \rightarrow \lambda$

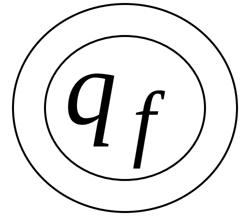


In General

For any transition: $q \xrightarrow{a} p$

Add production: $q \to ap$

variable terminal variable

For any final state: 

Add production: $q_f \to \lambda$

Since G is right-linear grammar

G is also a regular grammar

with $L(G) = L(M) = L$