MA-201 Tutorial

23/09/24

Properties of Gaussian Distributions :-

$$X \sim N(\mu, \sigma^2)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

cp of any X

(1) Normalizing X to have standard Normal

$$Y = \frac{X-\mu}{\sigma} \sim N(0,1)$$

(2) Linear transformation of $\underset{\sim N(\mu,\sigma^2)}{X} \rightarrow \sim N(\,,\,)$

i.e., $X \sim N(\mu, \sigma^2)$

then, $Y = \alpha X + \beta \sim N(\alpha\mu + \beta, \alpha^2\sigma^2)$

Proof

CDF $= F_Y(a) = P\{\alpha X + \beta \leq a\}$

let, $\alpha > 0$

$$= P\left\{X \leq \frac{a-\beta}{\alpha}\right\}$$

$$\therefore F_Y(a) = F_X\left(\frac{a-\beta}{\alpha}\right)$$

Differentiation of the CDF yield pdf of Y.

$$\therefore f_Y(a) = \frac{1}{\alpha} f_X\left(\frac{a-\beta}{\alpha}\right)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma\alpha} \exp\left\{-\left(\frac{a-\beta}{\alpha} - \mu\right)^2 / 2\sigma^2\right\}$$

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op^n on X

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$$= \frac{1}{\sqrt{2\pi}\cdot\sigma\alpha} \exp\left\{-\left(\frac{a-\beta}{\alpha} - \mu\right)^2 \Big/ 2\sigma^2\right\}$$

$$f_y(a) = \frac{1}{\underbrace{\sqrt{2\pi}\cdot a\alpha}_{\sqrt{Var}}} \exp\left\{-(\underbrace{a-\beta-\alpha\mu}_{\substack{\Downarrow \\ \text{mean} \\ \text{(Expectation)}}})^2/2(\underbrace{\sigma^2\alpha^2}_{\substack{\downarrow \\ Var}})\right\}$$

which shows that $Y \sim (\alpha\mu+\beta, \sigma^2\alpha^2)$

Similarly, it can be proved for $a < 0$.

(3) pdf of Standard normal $N(0,1)$ is denoted by $\phi_Z(z)$, $Z \sim N(0,1)$

$$\phi_Z(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}, \quad -\infty < z < \infty$$

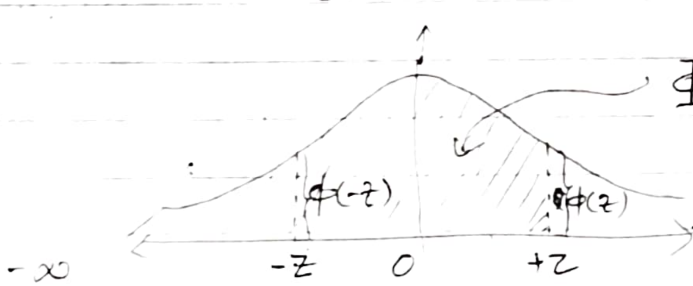
and $\int_{-\infty}^{\infty} \phi_Z(z) \cdot dz = 1$ (Area under std. Bell fun^n = 1)

(4) $\phi_Z(z) = \phi_Z(-z) \longrightarrow$ Symmetricity of pdf.

(5) Distribution function (CDF)

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \phi_Z(z) \cdot dz$$

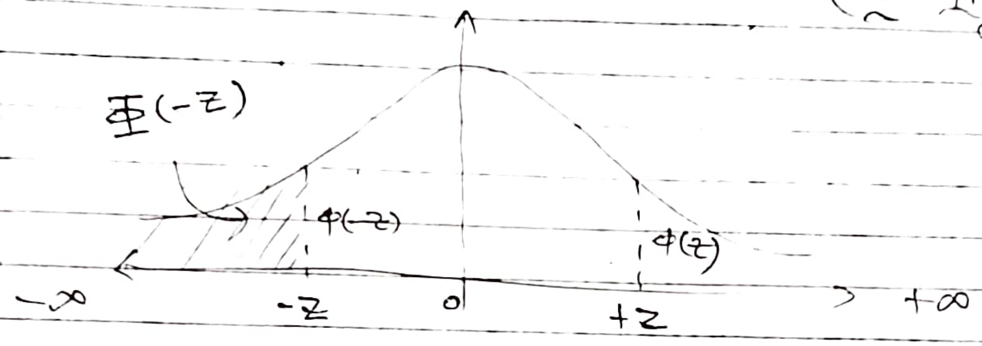
Cumulative the prob.

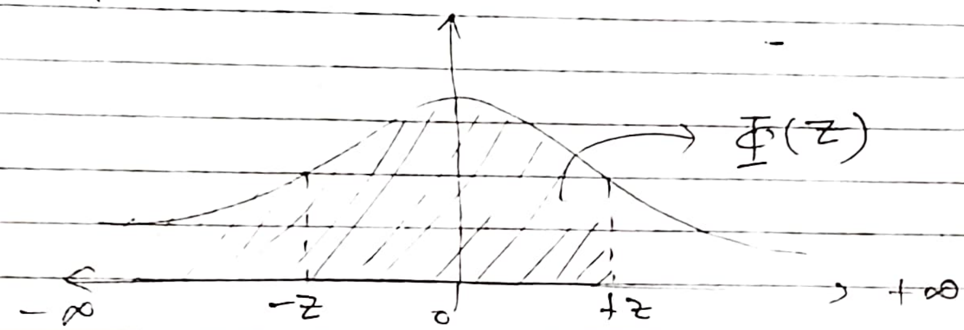


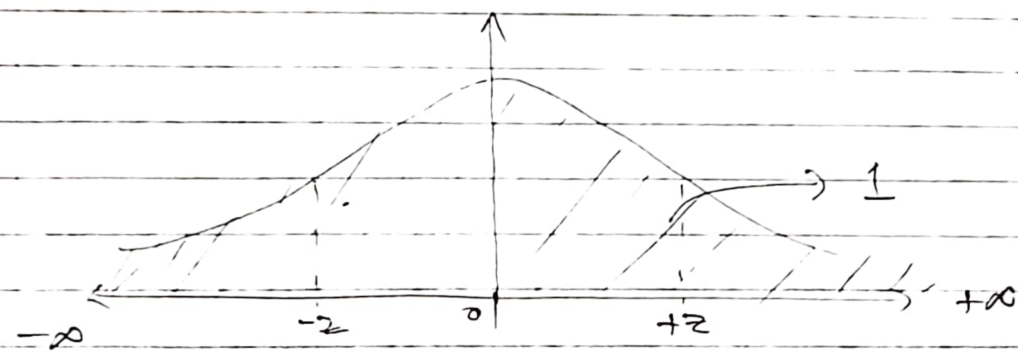
$\Phi(z) = P(Z \leq z)$
Area under curve till z from $-\infty$

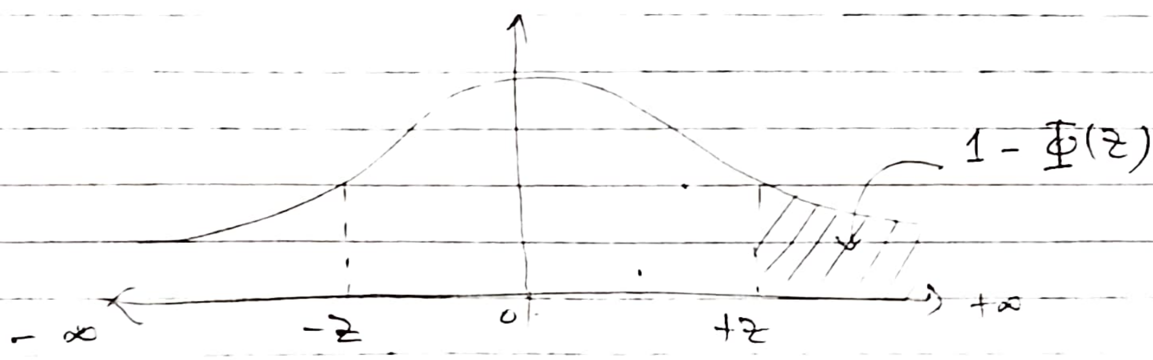
(6) $\Phi(-z) = 1 - \Phi(z)$ CDF property

$-\infty < z < \infty$ ($\because$ Symmetricity of pdf)







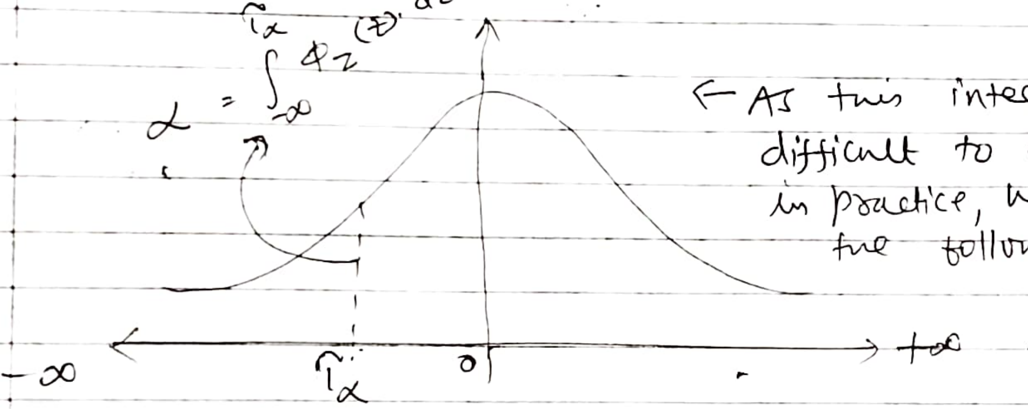


(7) For any α, $0 < \alpha < 1$

$$1 - \alpha = \Phi(\tilde{z}_\alpha)$$

The Value of $\tilde{z}_\alpha$ Could be obtained from the Table. ($\Phi(z)$ area under the Std. Normal Curve to the left of z)

$$\alpha = \int_{-\infty}^{\tilde{z}_\alpha} \phi_Z(z)\, dz$$



← As this integrals is difficult to calculate in practice, we use the following Table

Table

	+Z				
Z	0.00	0.01	0.02	- - -	0.09
.0	0.5				
.1					
.2					
3.4					0.9998

← be'coz the curve is symmetric about 0, so ½ on both the sides at 0.

$\Phi(Z)$ Areas

NB (1) → In the Table, $\Phi(z)$ are given.
So, $\Phi(-z)$ can be calculated by property (6)

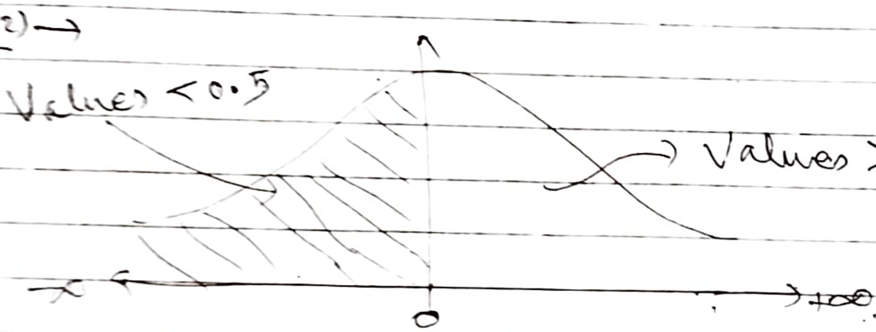
$$\Phi(-z) = 1 - \Phi(z)$$

e.g., $z = 0.64$ the value $\Phi(z) = \Phi(0.64)$
$= 0.7389$
(Table entry)

So, at $-z$ i.e., -0.64

$$\Phi(-z) = 1 - \Phi(z) = 1 - 0.7389$$
$$= 0.2611$$

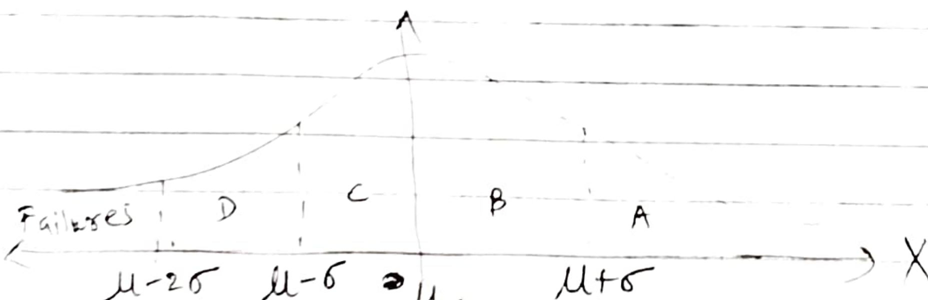
NB (2) →



(See in the Table all the Values are > 0.5)

Ex: Grading of marks on 4 pointer scale.
(Find μ and σ^2 from data.)

Strategy:
$X > \mu + \sigma$: A Grade.
$\mu \leq X < \mu + \sigma$: B
$\mu - \sigma \leq X < \mu$: C
$\mu - 2\sigma \leq X < \mu - \sigma$: D



→ Find % of students getting A Grade
Note that, irrespective of values of μ and σ^2 the % remains same.

$P(X \geq \mu+\sigma)$

$= P\left(\frac{X-\mu}{\sigma} \geq \frac{\mu+\sigma-\mu}{\sigma}\right)$ NORMALIZATION

So that we can use the Table.

$= P(Z \geq 1)$ Now $Z \sim N(0,1)$ Std. normal

$= 1 - P(Z \leq 1)$

$= 1 - \Phi(1)$

$= 1 - 0.8413$ (from table, $\Phi(1) = 0.8413$)

$= 0.1587$

$\therefore \approx 15\%$ Students will get A

→ Find % of students getting B

$P(\mu \leq X < \mu+\sigma)$

$= P\left(\frac{\mu-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} < \frac{\mu+\sigma-\mu}{\sigma}\right)$

$= P(0 \leq Z < 1)$

$= \Phi(1) - \Phi(0)$

$= 0.843 - 0.5$

$= 0.343$

$\therefore \approx 34\%$

→ Find % of students getting C

$P(\mu - \sigma \leq X < \mu)$

$= P\left(\frac{\mu-\sigma-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} < \frac{\mu-\mu}{\sigma}\right)$

$= P(-1 \leq Z < 0)$

$= \Phi(0) - \Phi(-1)$

$= \Phi(0) - \{1 - \Phi(1)\}$

$= \Phi(0) - 1 + \Phi(1)$

$= 0.5 - 1 + 0.8413$

$= 0.3413$

$\therefore \approx 34\%$ will get C

→ Find % of students getting D

$P(\mu - 2\sigma \leq X < \mu - \sigma)$

$= P\left(\frac{\mu-2\sigma-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} < \frac{\mu-\sigma-\mu}{\sigma}\right)$

$= P(-2 \leq Z < -1)$

$= \Phi(-1) - \Phi(-2)$

$= \{1 - \Phi(1)\} - \{1 - \Phi(2)\}$

$= 1 - \Phi(1) - 1 + \Phi(2)$

$= \Phi(2) - \Phi(1)$

$= 0.9772 - 0.8413$

$= 0.1359$

$\therefore \approx 14\%$ will get D

→ Find % of failures

$$P(X < \mu - 2\sigma)$$
$$= P\left(\frac{X-\mu}{\sigma} < \frac{\mu - 2\sigma - \mu}{\sigma}\right)$$
$$= P(Z < -2)$$
$$= \Phi(-2)$$
$$= 1 - \Phi(2)$$
$$= 1 - 0.9772$$
$$= 0.0228$$

$\therefore \approx 2.28\%$ will F

Verify

NB:

$$\overset{A}{0.1587} + \underbrace{\overset{B}{0.343} + \overset{C}{0.343}}_{\text{Symmetricity of pdf}} + \overset{D}{0.1359} + \overset{F}{0.0228} \overset{\checkmark}{\cong} 1.0000$$

NB: All Competitive Exams uses the Gaussian distribution for comparative results.

Also, note that NORMALIZATION has been used EXTENSIVELY to Analyze the data.