Subspaces of a Finite-dimensional vector Space

Let V be a vector space over $\mathbb{R}$. A subset W of V is said to be subspace of V if

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2. $u + v \in W$ for any elements $u, v \in W$.
3. $c.u \in W$ for any $u \in W$ and $c \in \mathbb{R}$.

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Examples:

1. The set of all polynomials of degree at most 3 is a subspace of the set of all polynomials with coeffiecients from $\mathbb{R}$.
2. The set of all 2×2 matices with trace zero is a subspace of the set of all 2×2 matrices over $\mathbb{R}$.

LINEARLY INDEPENDENT SETS and BASES

A set $\{v_1, v_2, \ldots, v_p\}$ of elements of a vector space V is said to be linearly independent if $c_1v_1 + c_2v_2 + \cdots c_pv_p = 0$ does not have any other solution than trivial solution, $(c_1 = c_2 = \cdots = c_p = 0)$. Consider a set $X = \{1 + x^2, x, 3\}$ in $\mathbb{R}_4[x]$, the vector space of all polynomials of degree less than or equal 4.

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$f(x) = c_1(1 + x^2) + c_2 x + c_3.3 = 0$
A polynomial $f(x)$ is zero iff for each value of $x = c$, $f(c) = 0$.
$0 = f(0) = c_1 + 3c_3, \quad 0 = f(1) = 2c_1 + c_2 + 3c_3, 0 = f(-1) = 2c_1 - c_2 + 3C_3$
$\Rightarrow c_2 = 0,$

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Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$$

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Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

Coordinate system

$V \xrightarrow{T} \mathbb{R}^n$ is bijective ~~lin. trans.~~

$x \mapsto \begin{bmatrix} c_1 \\ c_2 \\ c_n \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \sim c_1 b_1 + c_2 b_2 + \dots \dots c_n b_n$

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \dots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$ $\quad \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathcal{B}' = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is also a basis of V.

$\mathbb{R}^4 \ni \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{\mathcal{B}} =$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (b-a)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + (d-c)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\rightarrow (a, b-a, c, d-c)$

Co-ordinate vector of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \\ b-a \\ c \\ d-c \end{bmatrix}_{\mathcal{B}'}$

Coordinate system

$T'(A+B) = T'(A) + T'(B)$, $T'(cA) = cT'(A)$

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$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$-coordinates of x**) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a,b,c,d \in \mathbb{R} \right\}$, basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$\downarrow T'$$

$$\begin{bmatrix} a \\ b-a \\ c \\ d-c \end{bmatrix} \in \mathbb{R}^4$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

$\begin{bmatrix} a \\ b-a \\ c \\ d-c \end{bmatrix} = \begin{bmatrix} p \\ q-p \\ r \\ s-r \end{bmatrix}$

$a = p$, $c = r$, $b-a = q-p$, $b = q$

$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \xleftarrow{T'} \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix}$

T' is onto

$\begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \xleftarrow{T'} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Coordinate system

$T: V \to W$ is said to be L.T.

$T(u+v) = T(u) + T(v)$ for all $u, v \in V$

$T(\alpha u) = \alpha T(u)$ $\alpha \in \mathbb{R}$

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Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

$V \ni x = c_1b_1 + c_2b_2 + \dots + c_nb_n$

$y = d_1b_1 + d_2b_2 + \dots + d_nb_n$

$x + y = (c_1+d_1)b_1 + (c_2+d_2)b_2 + \dots + (c_n+d_n)b_n$

$\downarrow T$

$T(x)+T(y) \Leftarrow \begin{bmatrix} c_1+d_1 \\ c_2+d_2 \\ \vdots \\ c_n+d_n \end{bmatrix}$

$T(x) = T(y)$

If a vector space V has a basis $B = \{f_1, \dots, f_n\}$, then any set in V containing more than n vectors must be linearly dependent. L.D.

$V = \mathbb{R}^3$ $\{v_1, v_2, v_3, v_4\}$ in $\mathbb{R}^3$ is always L.D.

$\exists a, b, c, d$ (not all zero) s.t.

$av_1 + bv_2 + cv_3 + dv_4 = 0$

$\underbrace{[v_1\ v_2\ v_3\ v_3]}_{A}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$Ax = 0$ has a non-trivial sol^n

$A_{3\times 4}$ will have at least one free variable.

$g_1 \to [\ ]w_1$ $\in \mathbb{R}^n$

$g_2 \to w_2$

$\vdots$

$g_{n+1} \to w_{n+1}$

$[w_1\ w_2 \dots w_{n+1}]_{n\times n+1}$ will have at least free var. (non-trivial homo sol^n)

$a_1w_1 + a_2w_2 + \dots a_{n+1}w_{n+1} = 0$

$\Rightarrow a_1g_1 + a_2g_2 + \dots a_{n+1}g_{n+1} = 0$

$-4.1857 v_1 - 0.2429 v_2 + 1.0429 v_3 + v_4 = 0$

$\Rightarrow -4.1857 f_1 + (-0.2429) f_2 + 1.0429 f_3 + f_4 = 0$

If a vector space V has a basis $B = \{f_1, \ldots, f_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Dimension of a vector space V is the no. of elements in any basis of V. If the no. of elements in a basis is finite then V is said to be finite dimensional vector space.

$V: \mathbb{R}_2[x] = \{a_0 + a_1 x + a_2 x^2 \mid a_i \in \mathbb{R}\} \to B = \{x^2, x, 1\}$

$f_1 = 2x^2 + 3x - 1$

$f_2 = -x^2 + 7x - 3$

$f_3 = 3x^2 + 6x + 2$

$f_4 = 5x^2 + 8x - 7$

$f_1 \to \underbrace{\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}}_{v_1}, \quad f_2 \to \underbrace{\begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}}_{v_2}, \quad f_3 \to \underbrace{\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}}_{v_3}, \quad f_4 \to \underbrace{\begin{bmatrix} 5 \\ 8 \\ -7 \end{bmatrix}}_{v_4}$

$$\begin{array}{c} \begin{bmatrix} 2 & -1 & 3 & 5 \\ 3 & 7 & 6 & 8 \\ -1 & -3 & 2 & -7 \end{bmatrix} \\ \quad v_1 \quad v_2 \quad v_3 \quad v_4 \end{array} \to \begin{bmatrix} 1 & 0 & 0 & 4.1857 \\ 0 & 1 & 0 & 0.2429 \\ 0 & 0 & 1 & -1.0429 \end{bmatrix}$$

$x_4 = 1$

$x_3 = 1.0429$

$x_2 = -0.2429$

$x_1 = -4.1857$

$\mathbb{R}[x] = \{ a_0 + a_1 x + \cdots + a_n x^n \mid n \in \mathbb{N}, a_i \in \mathbb{R} \}$ → infinite dim. v. s.

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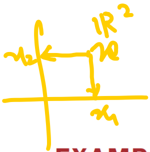
Theorem

The Basis Theorem Let V be a n-dimensional vector space, $n \geq 1$. Any linearly independent set of exactly n elements in V is automatically a basis for V. Any set of exactly n elements that spans V is automatically a basis for V.

Change of basis



$\{e_1, e_2\}$ $\begin{bmatrix}1\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\end{bmatrix}$

$x = x_1 e_1 + x_2 e_2$

$\{\underset{u_1}{\begin{bmatrix}-1\\0\end{bmatrix}}, \underset{u_2}{\begin{bmatrix}1\\1\end{bmatrix}}\} = \beta'$

$x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

vector

$x = \alpha_1 u_1 + \alpha_2 u_2$

$\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}$

EXAMPLE 1 Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \quad \text{and} \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2 \tag{1}$$

Suppose

$$\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2 \tag{2}$$

That is, suppose $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix}3\\1\end{bmatrix}$. Find $[\,\mathbf{x}\,]_{\mathcal{C}}$. $= \begin{bmatrix}6\\4\end{bmatrix}$

$\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} = \begin{bmatrix}-1 & +1\\0 & +1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}$

$= \begin{bmatrix}1\\3\end{bmatrix}$

$x = 3b_1 + b_2 = 3(4c_1 + c_2) + (-6c_1 + c_2)$

$= 6c_1 + 4c_2$

$\begin{bmatrix}1\\3\end{bmatrix}_{\beta'} = x = \begin{bmatrix}2\\3\end{bmatrix} = 1\cdot\begin{bmatrix}-1\\0\end{bmatrix} + 3\begin{bmatrix}1\\1\end{bmatrix}$