Mathematical Preliminaries

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- Sets
- Functions
- Relations
- Graphs
- Proof Techniques

SETS

A set is a collection of elements

$$A = \{1, 2, 3\}$$

$$B = \{train, bus, bicycle, airplane\}$$

We write

$$1 \in A$$

$$ship \notin B$$

Set Representations

$C = \{ a, b, c, d, e, f, g, h, i, j, k \}$

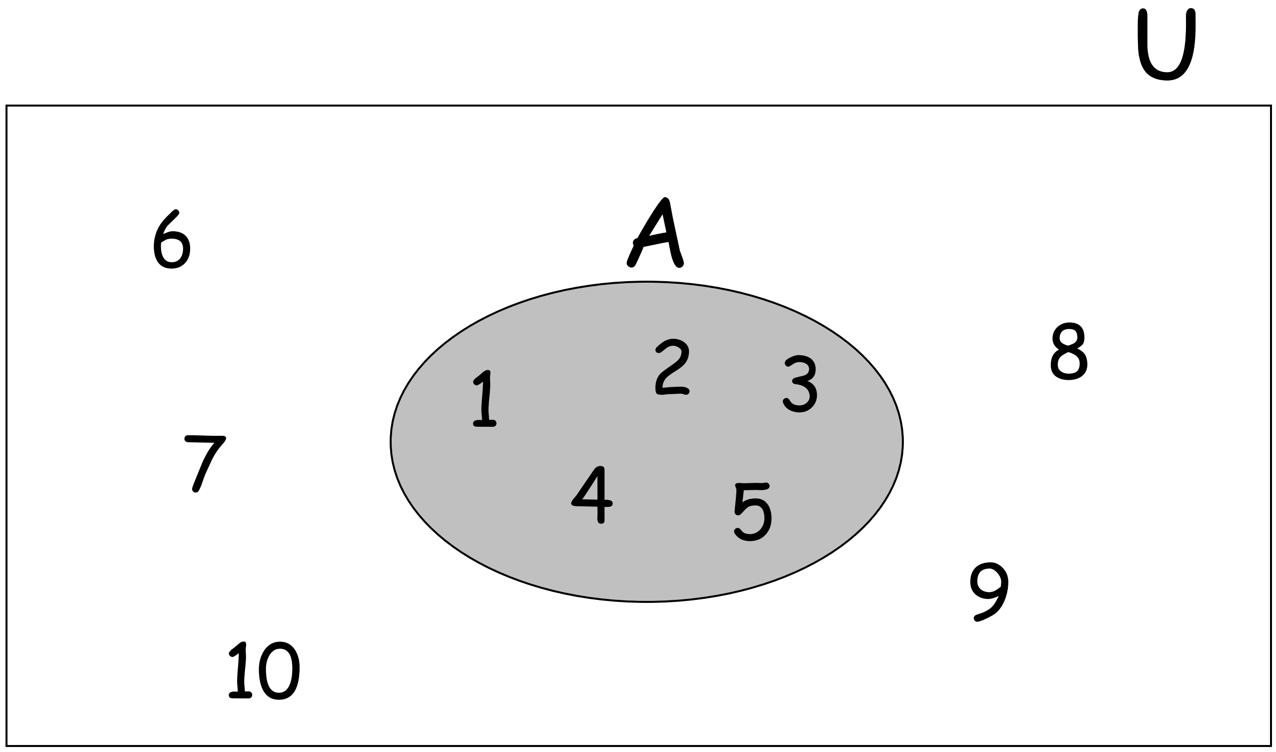
$C = \{ a, b, \ldots, k \} \longrightarrow$ *finite set*

$S = \{ 2, 4, 6, \ldots \} \longrightarrow$ *infinite set*

$S = \{ j : j > 0, \text{ and } j = 2k \text{ for some } k>0 \}$

$S = \{ j : j \text{ is nonnegative and even} \}$

$A = \{ 1, 2, 3, 4, 5 \}$



Universal Set: all possible elements

$U = \{ 1 , \dots , 10 \}$

Set Operations

$A = \{1, 2, 3\}$ $B = \{2, 3, 4, 5\}$

- Union

  $A \cup B = \{1, 2, 3, 4, 5\}$

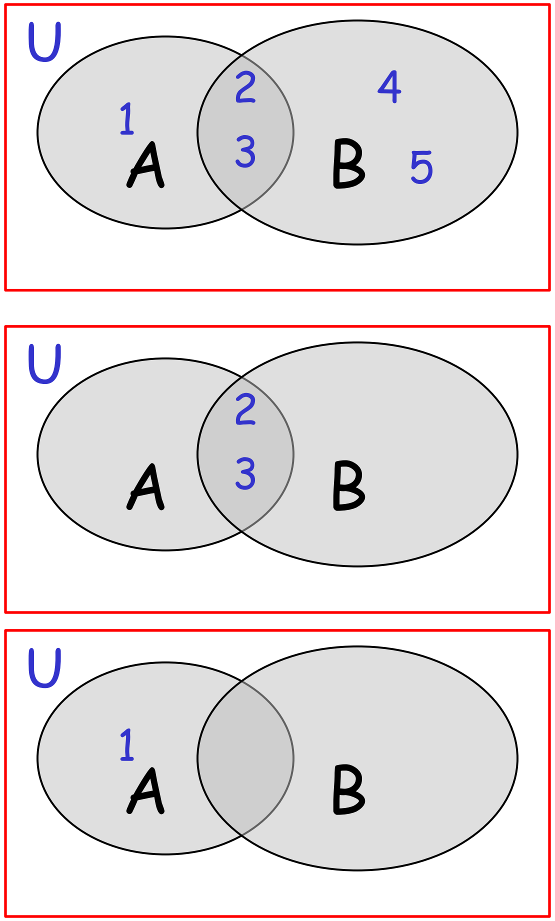
- Intersection

  $A \cap B = \{2, 3\}$

- Difference

  $A - B = \{1\}$

  $B - A = \{4, 5\}$

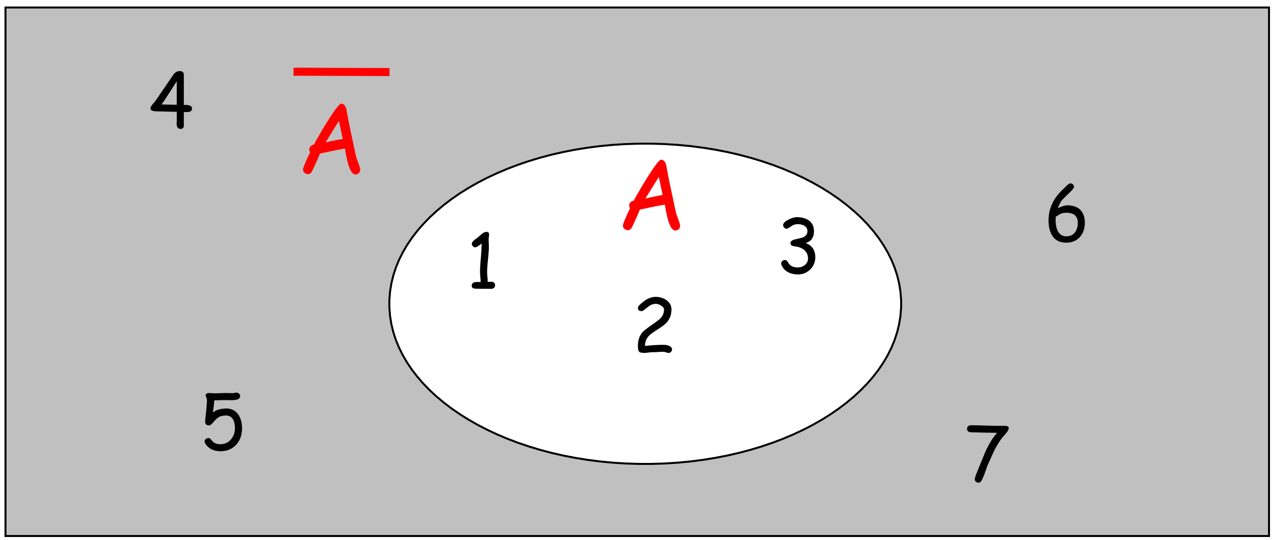


Venn diagrams

- Complement

Universal set = $\{1, \ldots, 7\}$

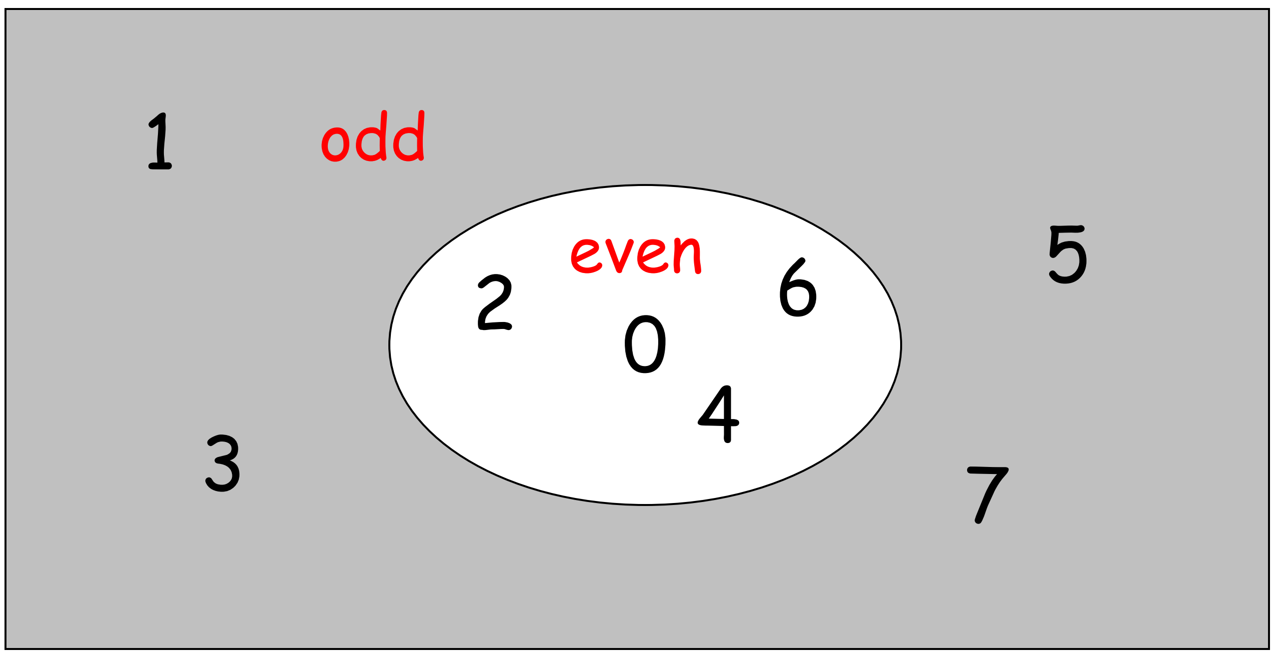
$A = \{ 1, 2, 3 \}$ $\Rightarrow$ $\overline{A} = \{ 4, 5, 6, 7\}$



$\overline{\overline{A}} = A$

$$\overline{\{\text{ even integers }\}} = \{\text{ odd integers }\}$$

Integers



DeMorgan's Laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Empty, Null Set: $\emptyset$

$\emptyset = \{\ \}$

$S \cup \emptyset = S$

$S \cap \emptyset = \emptyset$

$\overline{\emptyset}$ = Universal Set

$S - \emptyset = S$

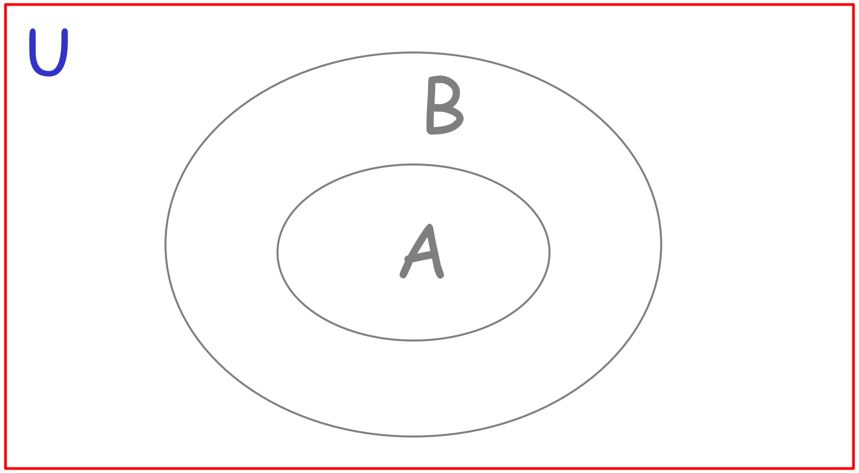
$\emptyset - S = \emptyset$

Subset

$A = \{1, 2, 3\}$ $B = \{1, 2, 3, 4, 5\}$

$$A \subseteq B$$

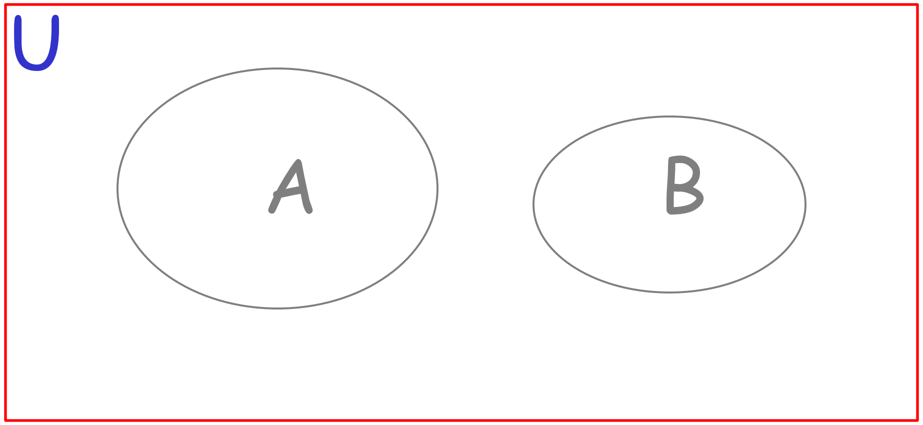
Proper Subset: $A \subset B$



Disjoint Sets

$A = \{ 1, 2, 3 \}$ $B = \{ 5, 6\}$

$$A \cap B = \emptyset$$



Set Cardinality

- For finite sets

$A = \{ 2, 5, 7 \}$

$|A| = 3$

(set size)

Powersets

A powerset is a set of sets

$S = \{ a, b, c \}$

Power set of S = the set of all the subsets of S

$2^S = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\},$
$\{a, b, c\} \}$

Observation: $| 2^S | = 2^{|S|}$ $(8 = 2^3)$

Cartesian Product

$A = \{ 2, 4 \}$ $\qquad$ $B = \{ 2, 3, 5 \}$

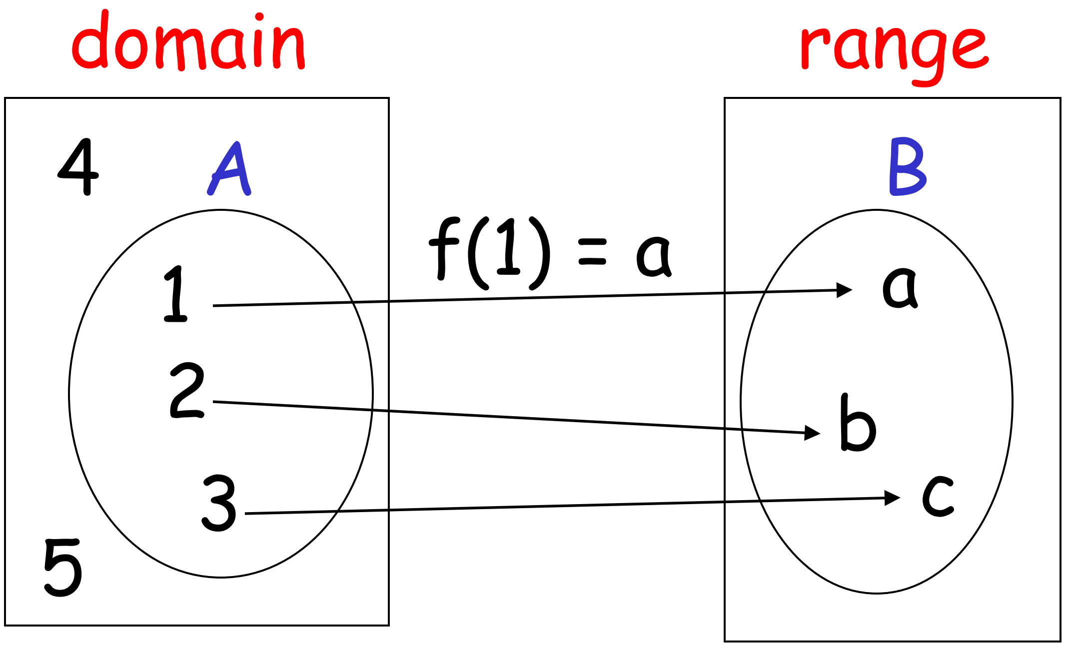
$$A \times B = \{ (2, 2), (2, 3), (2, 5), (4, 2), (4, 3), (4, 5) \}$$

$$|A \times B| = |A|\,|B|$$

Generalizes to more than two sets

$$A \times B \times \ldots \times Z$$

FUNCTIONS



f : A -> B

If A = domain

then f is a total function

otherwise f is a partial function

RELATIONS

$R = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots\}$

$x_i \, R \, y_i$

e. g. if R = '>': $2 > 1$, $3 > 2$, $3 > 1$

Equivalence Relations

- Reflexive: x R x
- Symmetric: x R y $\Rightarrow$ y R x
- Transitive: x R y and y R z $\Rightarrow$ x R z

Example: R = '='

- x = x
- x = y $\Rightarrow$ y = x
- x = y and y = z $\Rightarrow$ x = z

Equivalence Classes

For equivalence relation R

equivalence class of $x = \{y : x\ R\ y\}$

Example:

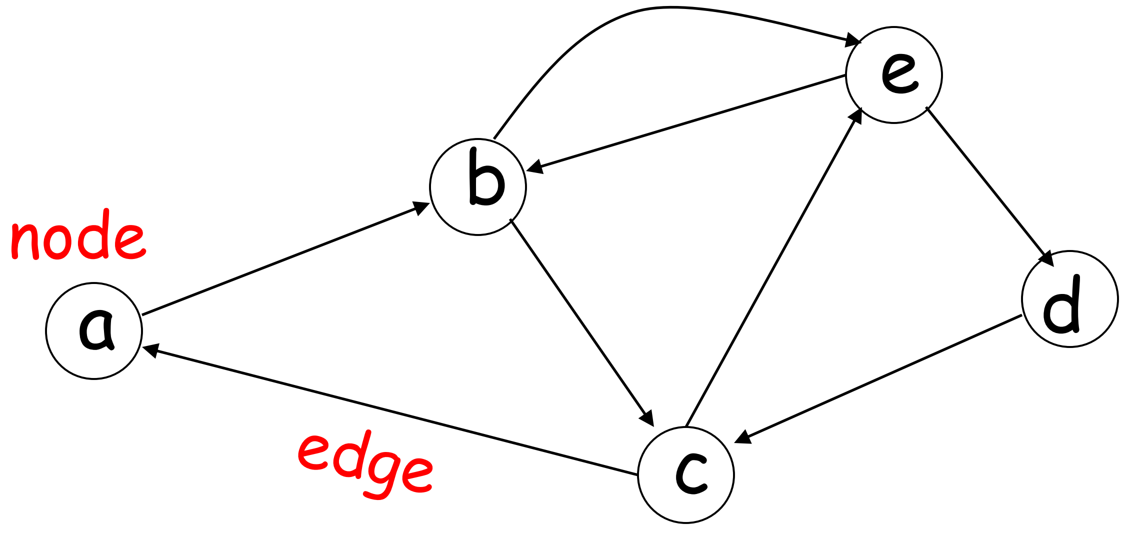
$R = \{ (1, 1), (2, 2), (1, 2), (2, 1),$
$(3, 3), (4, 4), (3, 4), (4, 3) \}$

Equivalence class of $1 = \{1, 2\}$

Equivalence class of $3 = \{3, 4\}$

GRAPHS

A directed graph



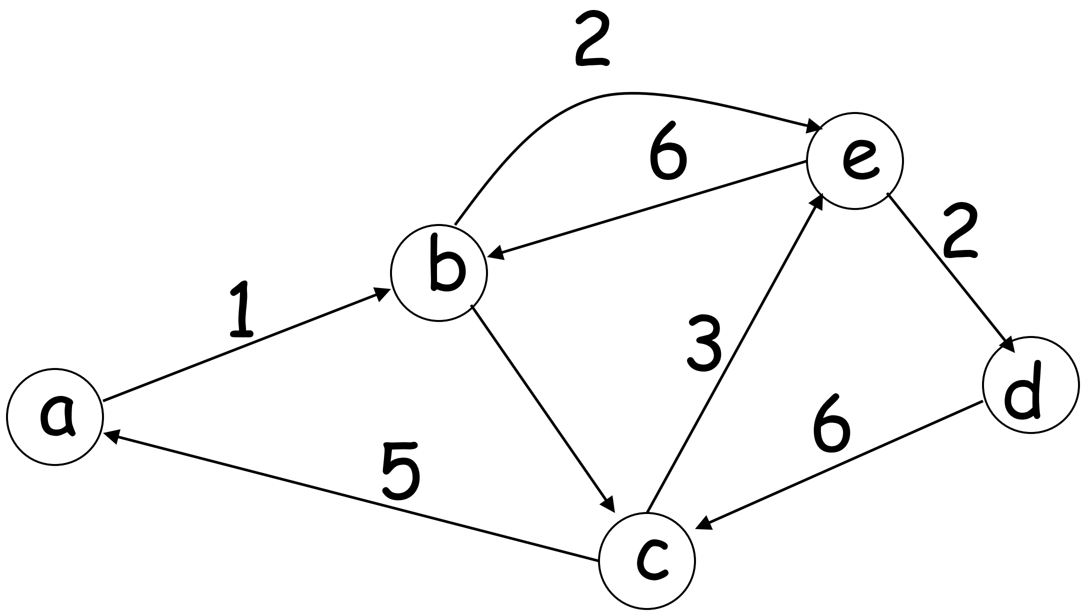
Nodes (Vertices)

V = { a, b, c, d, e }

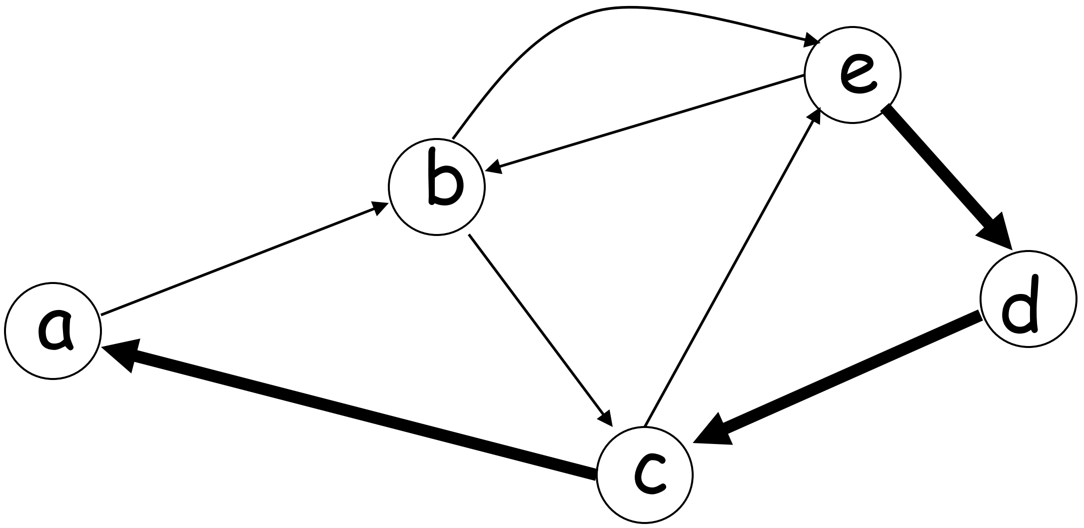
Edges

E = { (a,b), (b,c), (b,e),(c,a), (c,e), (d,c), (e,b), (e,d) }

Labeled Graph



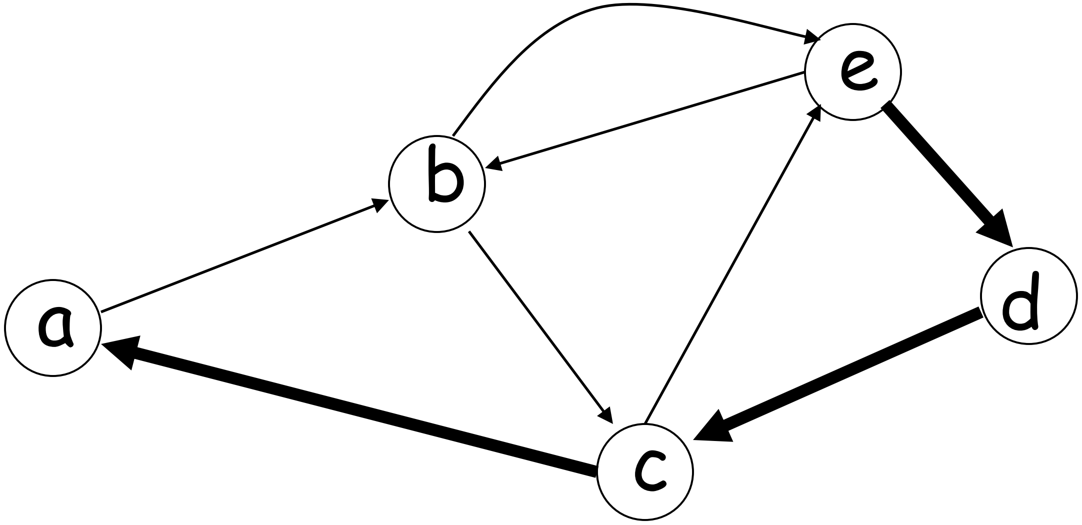
Walk



Walk is a sequence of adjacent edges

(e, d), (d, c), (c, a)

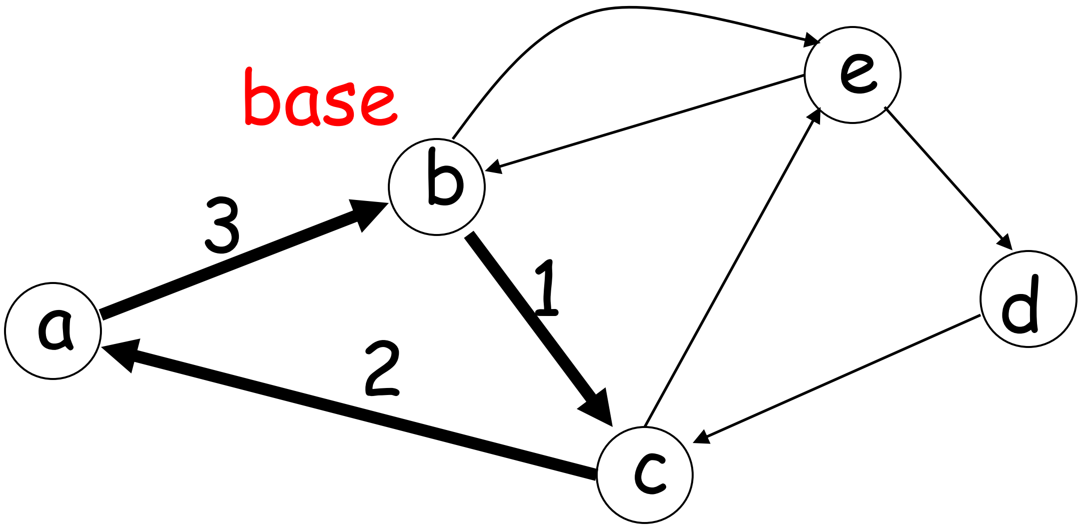
Path



Path is a walk where no edge is repeated

Simple path: no node is repeated

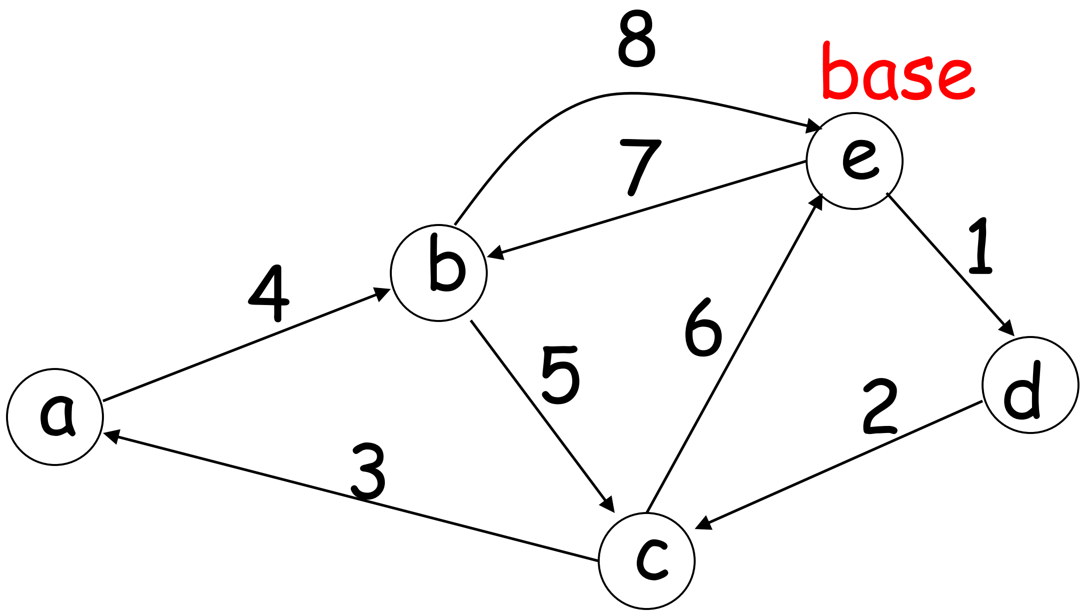
Cycle



Cycle: a walk from a node (base) to itself

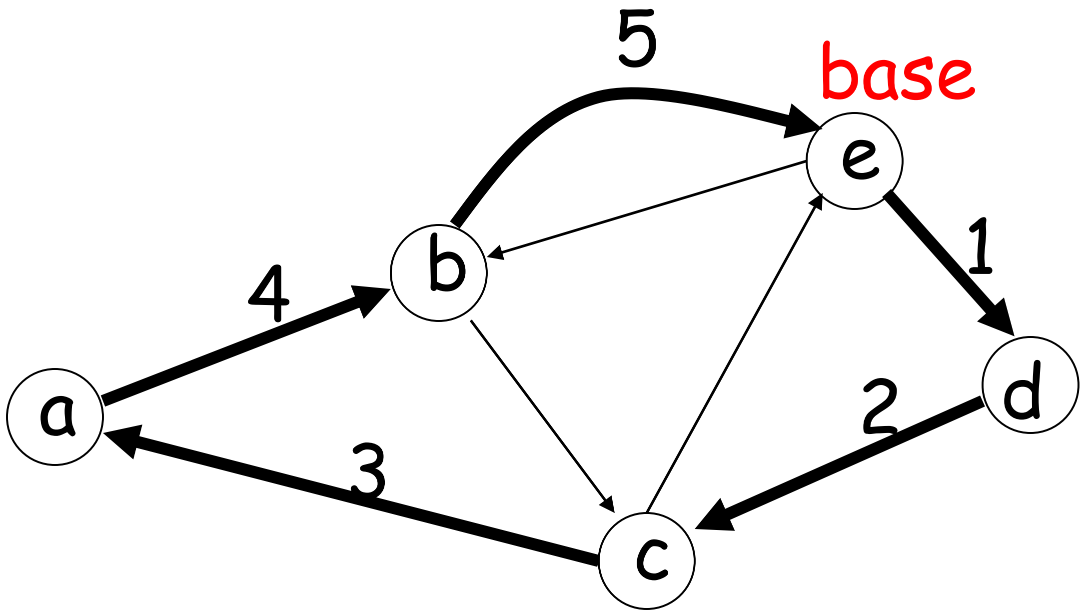
Simple cycle: only the base node is repeated

Euler Tour



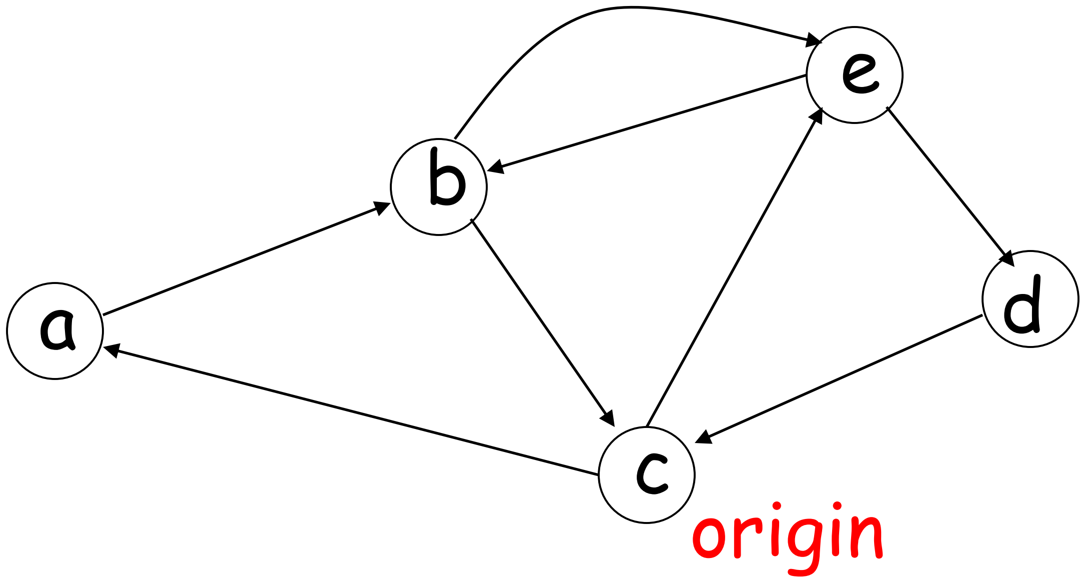
A cycle that contains each edge once

Hamiltonian Cycle

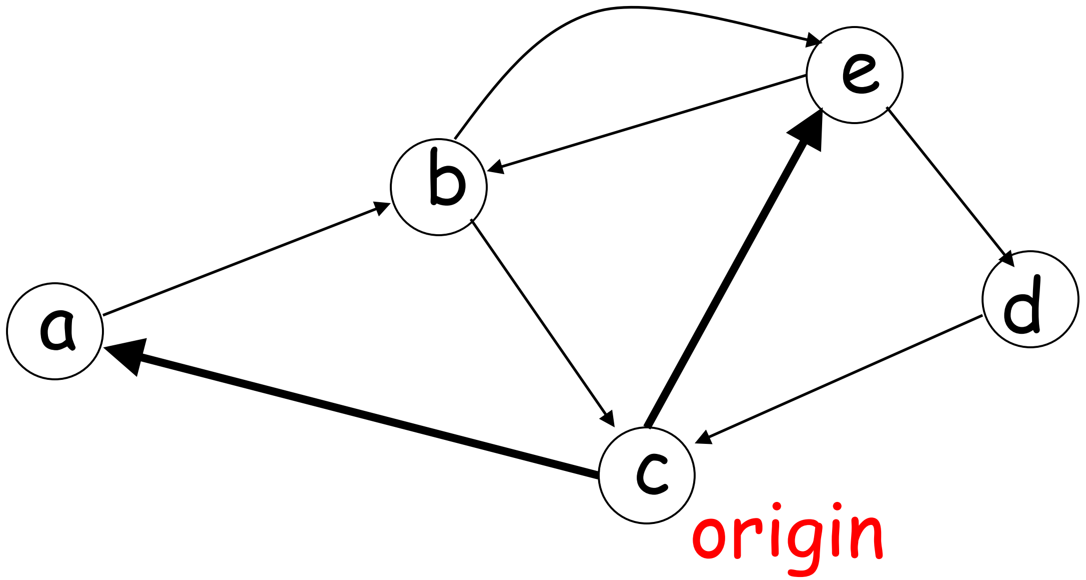


A simple cycle that contains all nodes

Finding All Simple Paths



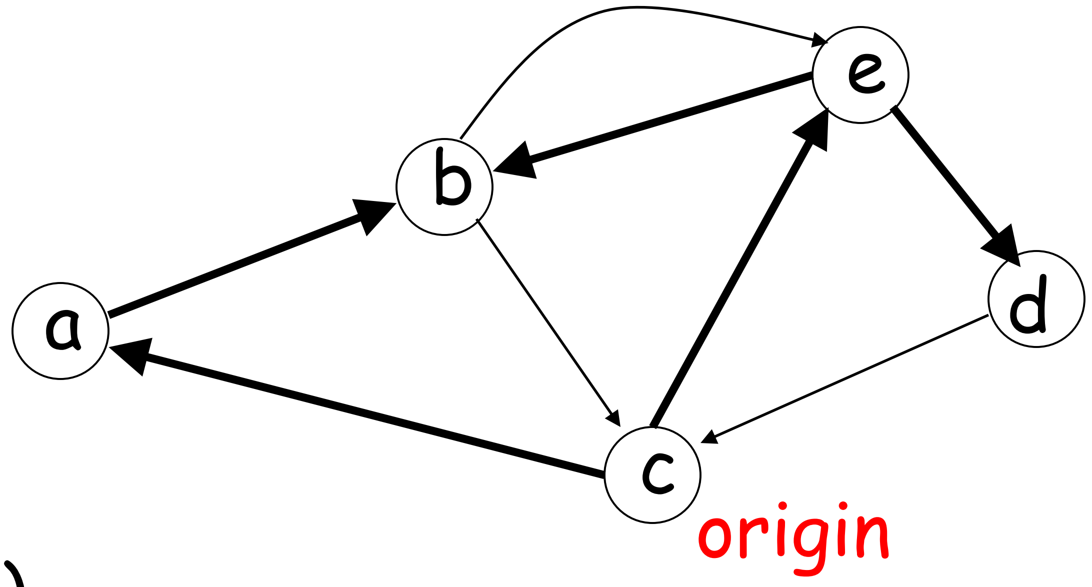
Step 1



(c, a)

(c, e)

Step 2



(c, a)

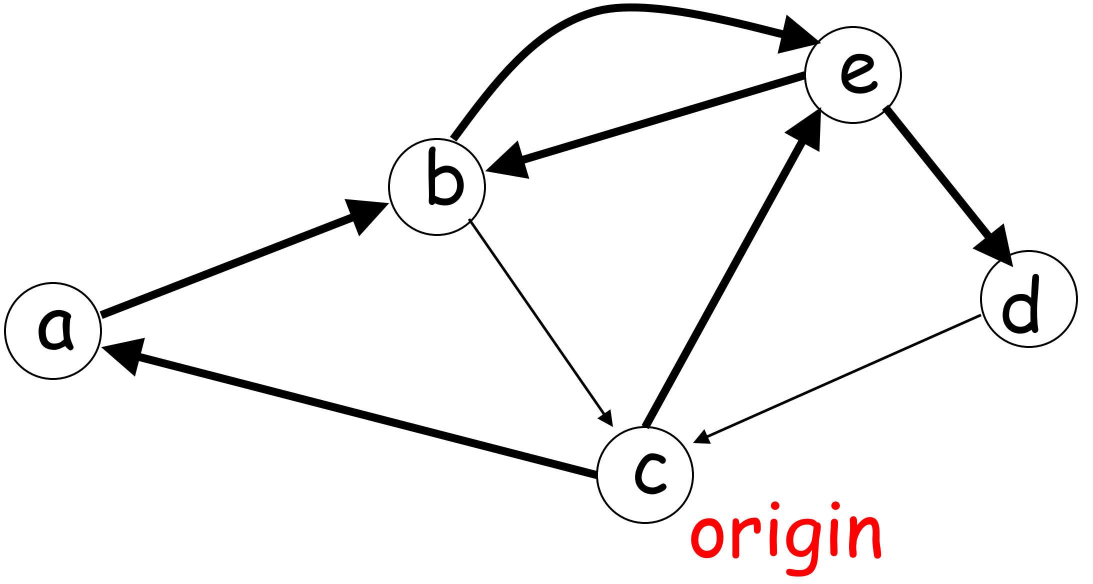
(c, a), (a, b)

(c, e)

(c, e), (e, b)

(c, e), (e, d)

Step 3



(c, a)

(c, a), (a, b)

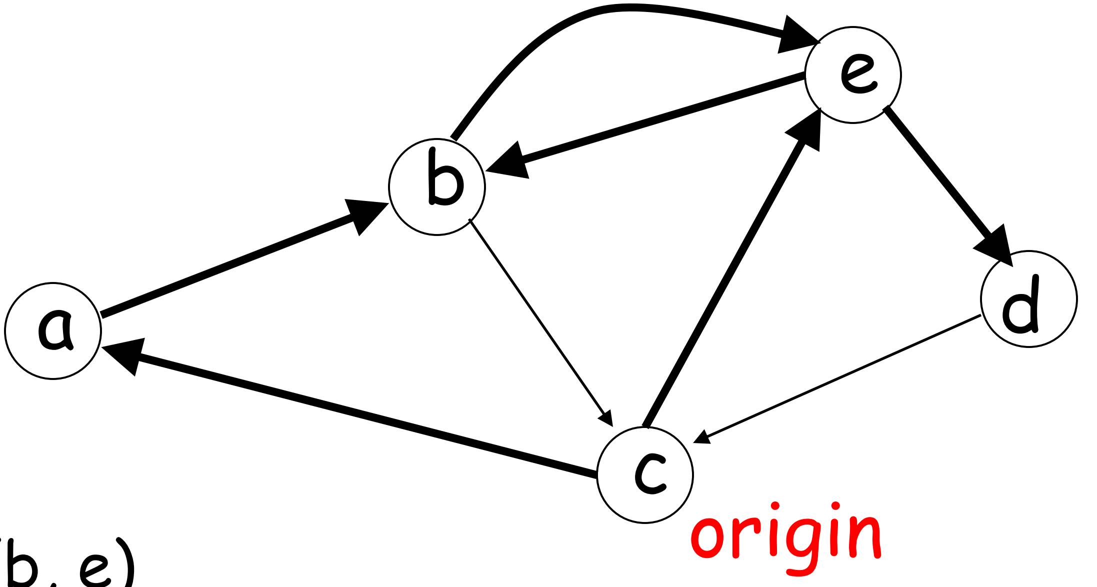
(c, a), (a, b), (b, e)

(c, e)

(c, e), (e, b)

(c, e), (e, d)

Step 4



(c, a)

(c, a), (a, b)

(c, a), (a, b), (b, e)

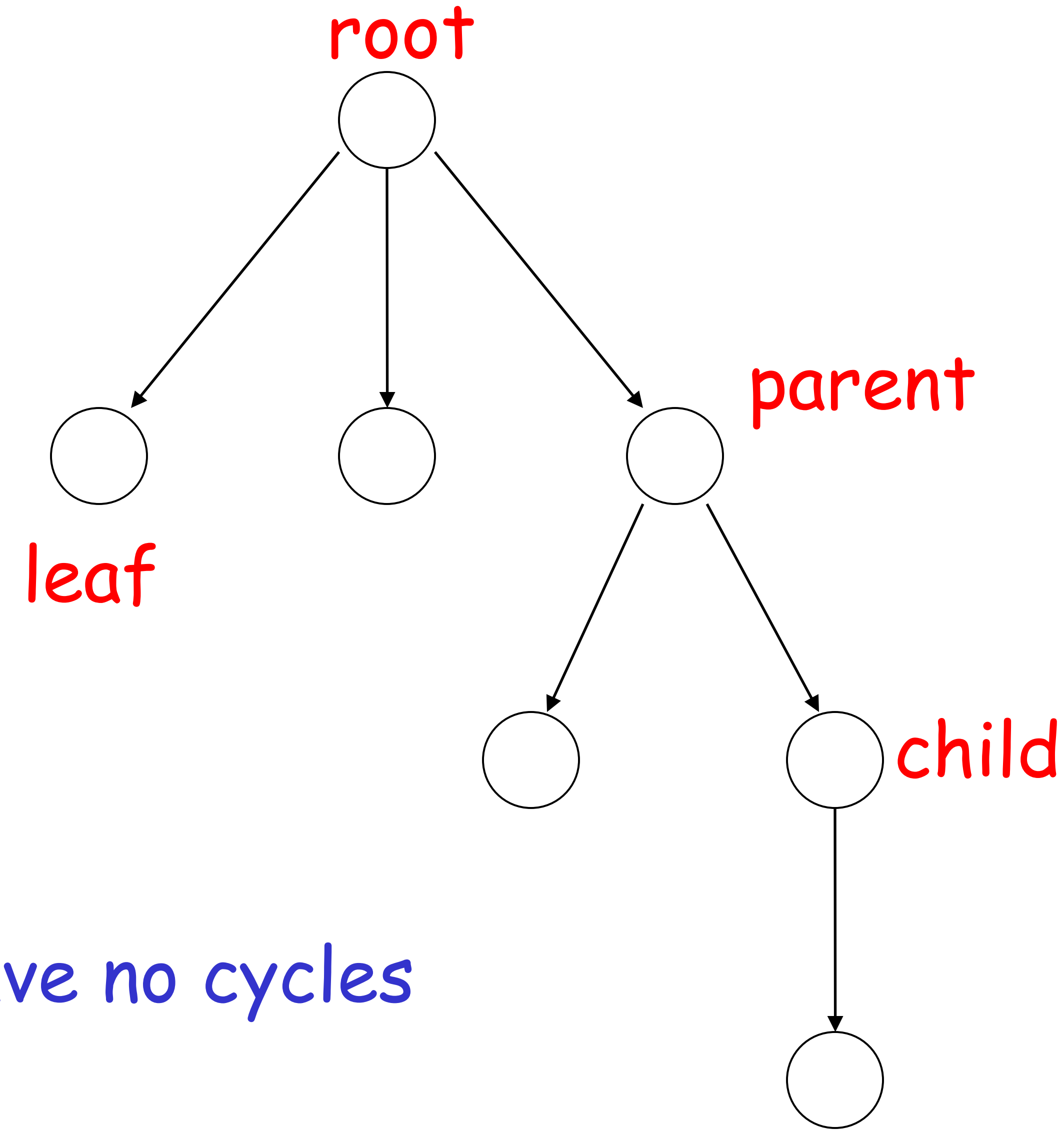
(c, a), (a, b), (b, e), (e,d)

(c, e)

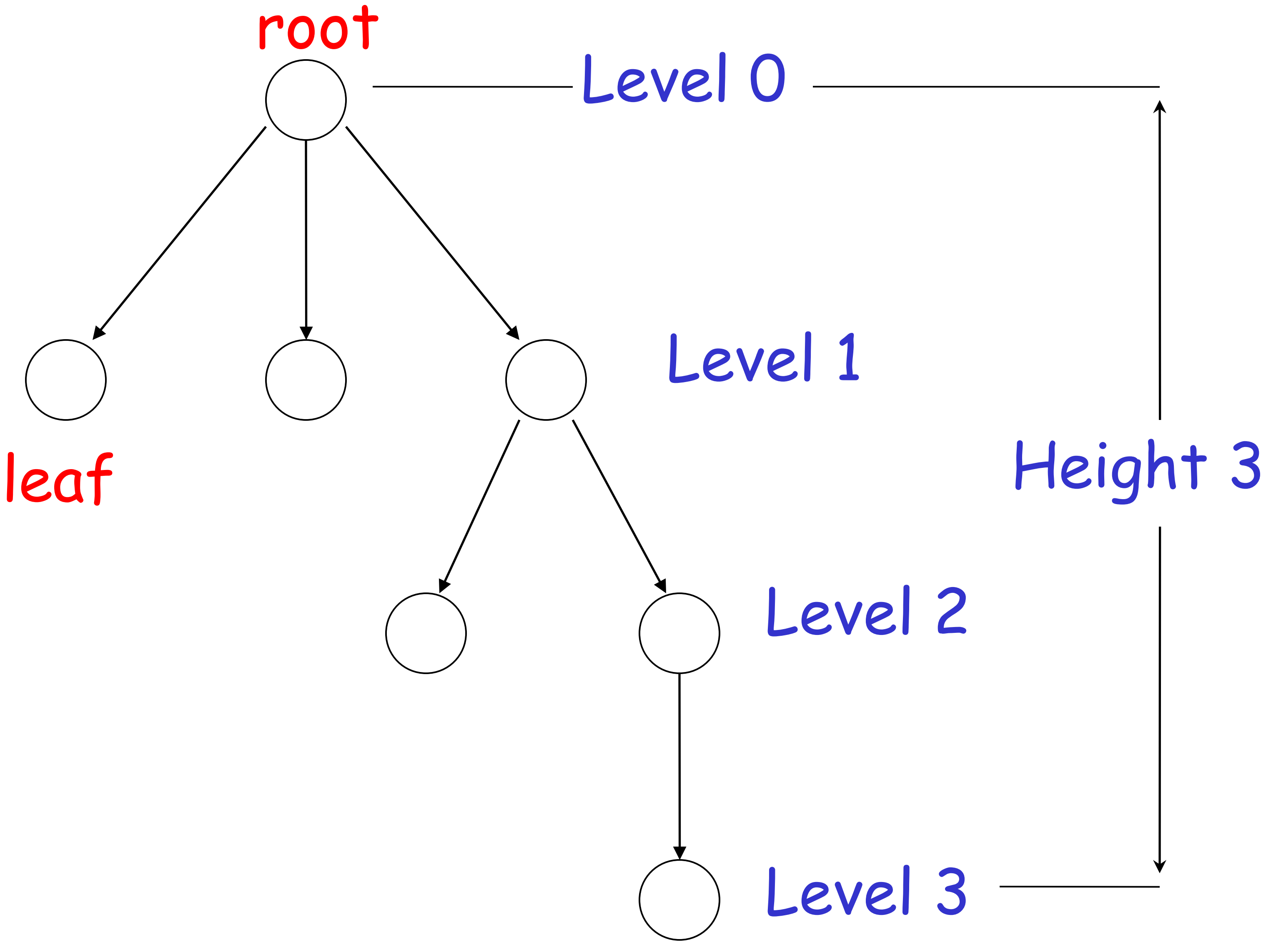
(c, e), (e, b)

(c, e), (e, d)

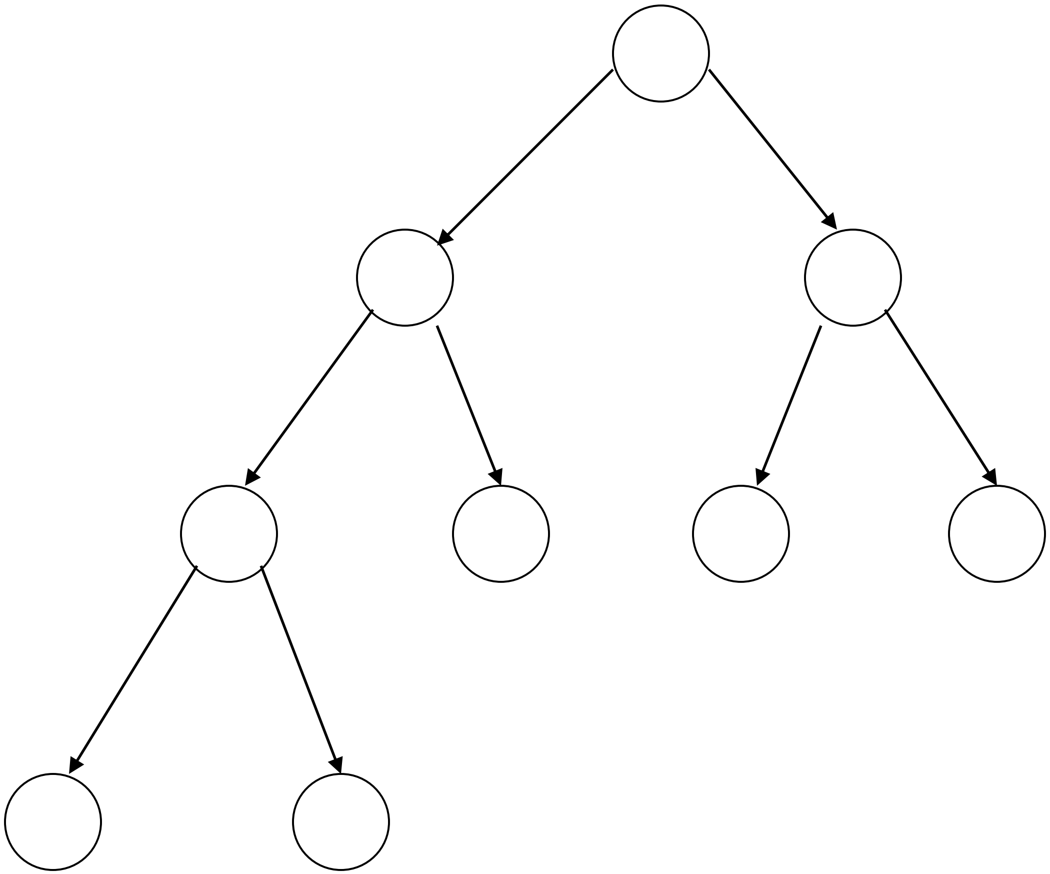
Trees



Trees have no cycles


root
Level 0
Level 1
leaf
Height 3
Level 2
Level 3

Binary Trees



PROOF TECHNIQUES

- Proof by induction

- Proof by contradiction

Induction

We have statements $P_1, P_2, P_3, \ldots$

If we know

- for some b that $P_1, P_2, \ldots, P_b$ are true
- for any k >= b that

$P_1, P_2, \ldots, P_k$ imply P_{k+1}

Then

Every P_i is true

Proof by Induction

- Inductive basis

  Find $P_1, P_2, \ldots, P_b$ which are true

- Inductive hypothesis

  Let's assume $P_1, P_2, \ldots, P_k$ are true,

  for any k >= b

- Inductive step

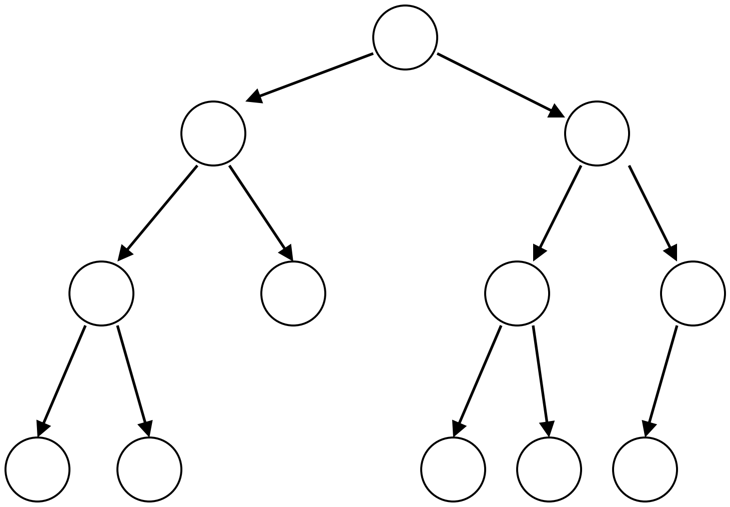
  Show that P_{k+1} is true

Example

Theorem: A binary tree of height n has at most 2^n leaves.

Proof by induction:

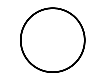
let $L(i)$ be the maximum number of leaves of any subtree at height i



We want to show: $L(i) <= 2^i$

- Inductive basis

  $L(0) = 1$ (the root node)



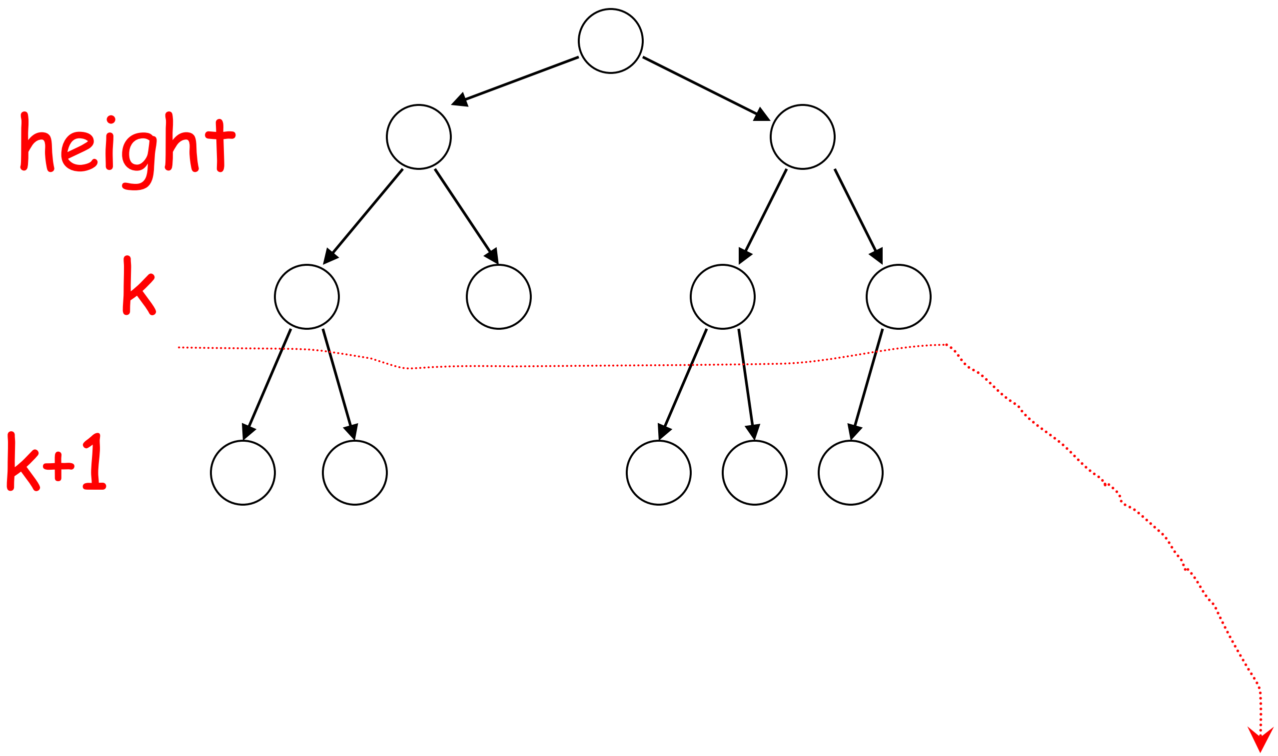
- Inductive hypothesis

  Let's assume $L(i) <= 2^i$ for all $i = 0, 1, \ldots, k$

- Induction step

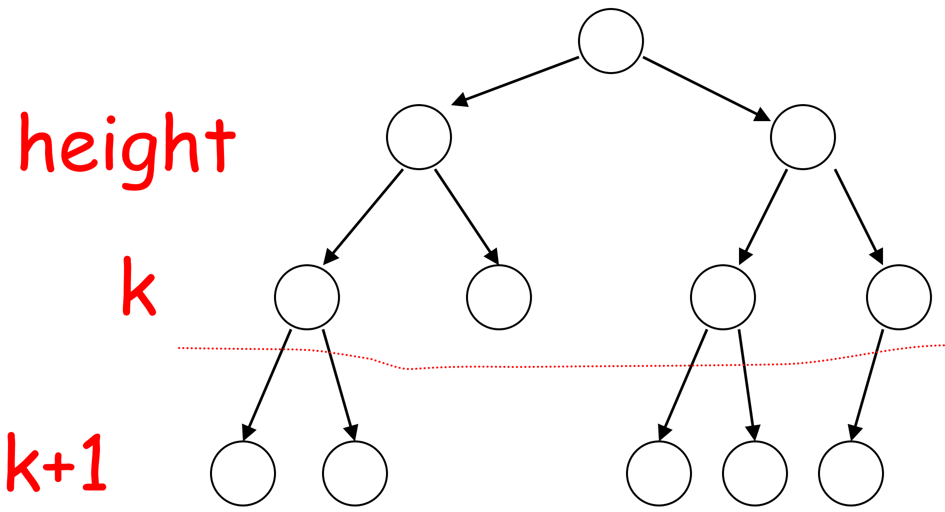
  we need to show that $L(k + 1) <= 2^{k+1}$

Induction Step



From Inductive hypothesis: $L(k) <= 2^k$

Induction Step



$L(k) <= 2^k$

$L(k+1) <= 2 * L(k) <= 2 * 2^k = 2^{k+1}$

(we add at most two nodes for every leaf of level k)

Remark

Recursion is another thing

Example of recursive function:

$$f(n) = f(n-1) + f(n-2)$$

$$f(0) = 1, \quad f(1) = 1$$

Proof by Contradiction

We want to prove that a statement P is true

- we assume that P is false
- then we arrive at an incorrect conclusion
- therefore, statement P must be true

Example

Theorem: $\sqrt{2}$ is not rational

Proof:

Assume by contradiction that it is rational

$$\sqrt{2} = n/m$$

n and m have no common factors

We will show that this is impossible

$\sqrt{2} = n/m \Rightarrow 2m^2 = n^2$

Therefore, n^2 is even $\Rightarrow$ n is even

$n = 2k$

$2m^2 = 4k^2 \Rightarrow m^2 = 2k^2 \Rightarrow$ m is even

$m = 2p$

Thus, m and n have common factor 2

Contradiction!