Distribution originating from fragmentation processes

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Two types of fragmentation processes were simulated: natural and artificial, in matlab. In 1941 mathematician Kolmogorov rigorously showed that fragmentation should lead to log-normal size distribution in rocks. However only the natural fragmentation process was found to yield the log-normal distribution. This work describes the simulation and provides an explanation as to why the artificial process yields a different distribution.

I. INTRODUCTION

In nature, over a long period of time due to weathering, pressure or other reasons, rocks break into random number of smaller rocks which further break into more number of smaller rocks and this cycle goes on randomly without any size restriction. In contrast, a human-made rock-crusher operates with a goal of reducing rocks to a desired commercial sized fragments: a rock-crusher will let go of rocks smaller than a certain demand-based critical size (s_{cr}); bigger rocks get crushed until it reaches that size and ultimately escape the crusher.

This work models the two fragmentation processes, obtains the rock-size distributions, and then provides an explanation for their differences.

A. Natural Fragmentation

A unit size rock is divided into random number of fragments. A fragment is generated by multiplying the volume of the mother rock with a positive random number $k < 1$. Since the fragments has to obey the law of conservation of mass/volume, the program makes sure that the sum of volume of fragments is equal to the mother rock. The number of fragments grows exponentially and thus a tree data structure is used to record the fragments size. A simple representative fragmentation process is shown in Fig. 1.

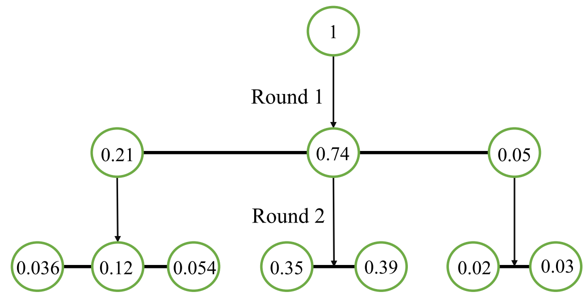

FIG. 1: A tree diagram representation of two rounds of sample natural fragmentation process. The numbers at each node represent a fragment's size.

Definition of few technical terms:
List: Collection of rocks that has gone through the same number of fragmentation process; for instance, 0.21,0.74, and 0.05 makes a list in Fig. 1; the initial list is made of a unit size rock
Round: Completion of fragmentation of all the rocks in a **List**
Experiment: Completion of the total (user-defined) fragmentation **Round** on a unit size rock

The natural fragmentation algorithm per **Experiment** per **Round** is as following:

1. select a rock from the **List**, let s be the size
2. set sum $= 0$
3. generate a random number $k \in (0, 1)$
4. update sum, i.e. sum $=$ sum $+ k \times s$
5. check: Is sum $< s$?
   Yes: set $k \times s$ as the fragment size and then repeat from step 3
   No: set $s - (\text{sum} - k \times s)$ as the fragment size and then repeat from step 1
6. exhausting the rock in the **List** completes one fragmentation **Round**
7. Record all the fragments in a new **List** and repeat from step 1

One **Experiment** is completed when all the fragmentation **Round**s are carried out. The simulation stops when all the **Experiment**s are exhausted. The final **List** comprises of fragments from all the **Experiment**s, which histogram represents the fragment size distribution. Note that defining a **Round** in the simulation needs some analysis of tree structure which is presented as comments in the source code.

B. Artificial Fragmentation

Algorithm for modeling artificial rock crushing is very similar to the preceding algorithm except that the fragmentation process is only applied to rocks bigger than a critical size; smaller rock directly gets enlisted in the final list. A tree-diagram representation is shown in Fig. 2.

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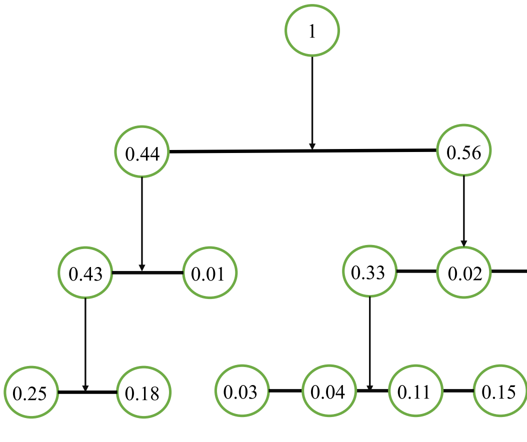

FIG. 2: A tree diagram representation of sample artificial rock crushing process. The numbers at each node represent a fragment's size. Here, a rock is fragmented until it is reduced to sizes smaller than the critical size of 0.3 unit size.

Note that there is no concept of **Round** in artificial fragmentation; as a rock is crushed until it can escape the crusher. The necessity of defining **Round** in natural fragmentation comes form the assumption that rocks in a neighbourhood experience similar external condition and therefore on average must go through equal number of fragmentation process.

Additionally, there are only two **List**s: **Initial** and **Final**, fragments smaller than the critical size are enlisted to **Final List**.

The algorithm per **Experiment** is as following:

1. select a rock from the **Initial List** of rocks, let s be the size
2. define a critical size $s_{\rm cr}$
3. check: Is $s > s_{\rm cr}$?
   Yes: proceed from step 4
   No: include s in the **Final List** and repeat from step 1
4. set sum $= 0$
5. generate a random number $k \in (0, 1)$
6. update sum, i.e. sum $=$ sum $+ k \times s$
7. check: Is sum $< s$?
   Yes: set $k \times s$ as the fragment size and then repeat from step 5
   No: set $s - (\text{sum} - k \times s)$ as the fragment size and then repeat from step 1
8. add fragments bigger than critical size at the end of the **Initial List**
9. exhausting the **Initial List** completes the fragmentation process

The simulation stops only after carrying out all the **Experiment**s. Histogram of the final list represents the fragment size distribution.

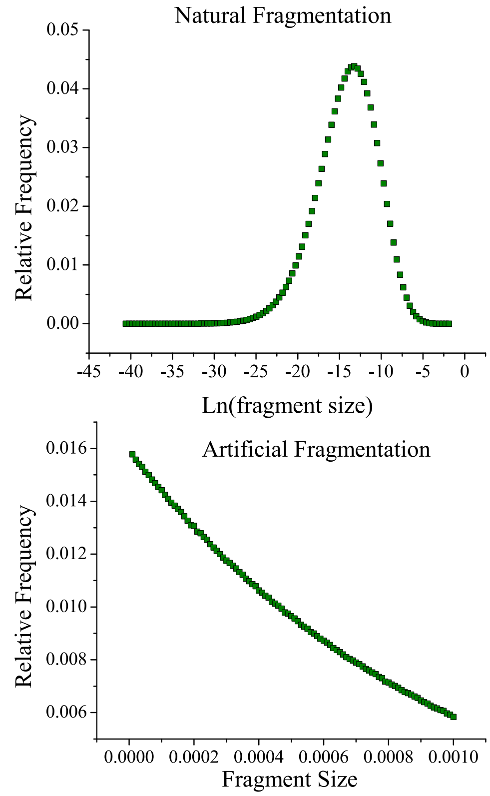

FIG. 3: Top: Fragment size distribution generated from 10 **Rounds** of natural fragmentation process of a unit sized rock. Bottom: Fragment size distribution generated from artificial fragmentation process of a unit sized rock. Both distributions resulted from 10000 **Experiments**.

II. RESULTS AND DISCUSSIONS

Size distribution obtained from the two simulations are shown in Fig. 3 which shows a log-normal distribution for natural fragmentation and monotonically decreasing distribution for artificial.

In nature, rocks exhibit log-normal size distribution[1, 2]. The size of a fragment at any given time s_t is a random fraction k_t of its mother fragment s_{t-1}, which in turn is a random fraction of its mother fragment s_{t-2}. Applying this recursive principle, the size of any fragment can be traced back to the original rock, s i.e. $s_t = k_t \times k_{t-1} \times \cdots \times k_1 \times s$. Now taking the log of both sides of the equation converts the product into sum, i.e, $\ln(s_t) = \ln(k_t) + \ln(k_{t-1}) + \cdots + \ln(k_1) + \ln(s)$. Since ks are i.i.d (independent and identically distributed) random numbers and thus $\ln k$s, Central Limit Theorem (CLT) guarantees that the sum $\ln(s_t)$ must be normally distributed. It follows then that s_t must be log-normally distributed [3, 4].

$$\rho(s_t) = \frac{1}{\sqrt{2\pi\sigma^2 t} s_t} \exp\left(\frac{-(\ln(s_t) - \mu t)^2}{2\sigma^2 t}\right) \qquad (1)$$

where the mean μt and variance $\sigma^2 t$ scales with time t

or in our case with the fragmentation round.

Similarly a fragment generated by artificial process s_t can also be written in terms of initial rock s_0, i.e. $s = k_1 \times k_2 \times \cdots \times k_t \times s_0$. However, it doesn't follow log-normal distribution which can be attributed to CLT. Application of CLT requires several assumption, particularly, finiteness of at least one moment higher than the second moment, and size-independent probability of breakage. The later assumption is known as Gibrat's law of proportionate effect related to economics, which states thats a firms's growth rate is independent of its size[5]. That assumption is not satisfied by the artificial fragmentation process where the probability of breakage is zero for rocks with size smaller than a critical size; essentially random fraction k are not a truly i.i.d variable. Thus, artificial fragmentation yields a completely different distribution than the natural fragmentation process even though fragments in both process follow the recursive principle.

III. CONCLUSION

Kolmogorov's approach of applying CLT to recursive fractions is valid only when a rock can disintegrate without a constraint. When a constraint is placed on the size, CLT is inapplicable and the size distribution is significantly different from log-normal.

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