HOMEWORK-3 Diptendra Nath Bagchi

I. Given

$A_{n \times n}$ = orthonormal (rows) matrix of size n

we have to show that columns of matrix A are also orthonormal

Def: Orthonormal: Two vectors are said to be orthonoral if they are orthogonal to each other and each of length 1.

∴ If A is orthonormal (as rows) then,

$$A^T A = A A^T = I_{n \times n} \quad \text{— (A)}$$

∴ if $r_i \cdot r_j \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

∴ Using eq (A) we can also show that

$$c_i \cdot c_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{therwise.} \end{cases}$$

Hence, if rows are orthonormal then columns are also orthonormal.

2. For matrix A, of rank r

SVD is given by $\sum_i \sigma_i u_i v_i^T$

Given,

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$ is a k-rank approximations for some $k \leq r$

1. $\|A_k\|_F^2$ = Square of the singular values.

$$= \sum_{i=1}^{k} \sigma_i^2(A)$$ where σ_i = singular values

2. $\|A_k\|_2^2$ = Two norm = σ_1^2 [where σ_1 = first singular value which also means the largest singular value]

3. $\|A - A_k\|_F^2$

$$\|A\|_F^2 = \sum_{i=1}^{r} \sigma_i^2 \quad \text{(for rank r)}$$

$$\|A_k\|_F^2 = \sum_{i=1}^{k} \sigma_i^2$$

$$\|A - A_F\|_F^2 = \sum_{i=1}^{r} \sigma_i^2 - \sum_{i=1}^{k} \sigma_i^2$$

$$= \sum_{i=k+1}^{r} \sigma_i^2$$

(iv) $\|A - A_k\|_2^2$

$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$ be the singular value decomposition of A

$A_k = \sum \sigma_i u_i v_i^T$

$\Rightarrow A - A_k = \sum_{i=k+1}^{r} \sigma_i u_i v_i^T$

let v be the top singular value of $A - A_k$, where v is the linear combination of $v_1, v_2 \dots v_r$

$\therefore v = \sum_{i=1}^{r} \alpha v_i$, then.

$|(A - A_k) v| = \sqrt{\sum_{i=k+1}^{r} \sigma_i^2 \alpha_i^2}$.

$$\boxed{\|A - A_k\|_2^2 = \sigma_{k+1}^2}$$

3) A : symmetric matrix

$\Rightarrow \quad A = A^T \quad$ — (1)

Performing SVD on A gives,

$A = UDV^T$ — (A) where D is a diagonal matrix of singular value

Now, let transpose A,

$A^T = (UDV^T)^T$

$= V^{T^T} D^T U^T$

$= V D U^T$ — (B) $\left[\because \begin{array}{l} V^{T^T} = V \\ D^T = D \end{array} \right]$

using (A) and (B) in eqn. (1) we get,

$UDV^T = VDU^T$

$\Rightarrow \quad U = V$ and $V^T = U^T$

$\therefore$ Hence, replacing U with V in eq (A) we get,

$$\boxed{A = VDV^T}$$

4 Power method to compute SVD

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = A^T A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$

How, to find B^k, we need k to be large.

fro our purposes. we have take $k=10$ and took the normalized vector.

$$v_1 = [0.576 \quad 0.817]$$

let $x = Av_1$

$$x = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.576 \\ 0.817 \end{bmatrix} = \begin{bmatrix} 0.576 + 1.634 \\ 1.728 + 3.268 \end{bmatrix}$$

$$= \begin{bmatrix} 2.21 \\ 4.996 \end{bmatrix}$$

$$\therefore u_1 = \frac{Av_1}{\|Av_1\|} = [0.404 \quad 0.914]$$

where $\sigma_1 = \|Av_1\| = 5.464$

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Now.

$$B_2 = B - \sigma_1 V_1 V_2 = \begin{bmatrix} 0.089 & -0.063 \\ -0.063 & 0.044 \end{bmatrix}$$

Now choosing the second vector, we get,

$$V_2 = \begin{bmatrix} -0.817 & 0.576 \end{bmatrix}$$

$$\therefore \sigma_2 = \|AV_2\| = 0.365$$

$$\therefore U_2 = \frac{\|AV_2\|}{\sigma_2} = \begin{bmatrix} 0.914 \\ -0.404 \end{bmatrix}$$

Hence,

$$A = \begin{bmatrix} 0.404 & 0.914 \\ 0.914 & -0.404 \end{bmatrix} \begin{bmatrix} 5.46 & 0 \\ 0 & 0.365 \end{bmatrix} \begin{bmatrix} 0.576 & 0.817 \\ -0.817 & 0.576 \end{bmatrix}$$

5a $$\underset{n\times d}{A} = \begin{bmatrix} \underset{n_1\times d_1}{A_1} & \underset{n_1\times d_2}{A_2} \\ \underset{n_2\times d_1}{A_3} & \underset{n_2\times d_2}{A_4} \end{bmatrix}$$

where $n = n_1 + n_2$, $d = d_1 + d_2$

How,

$$\underset{n\times d}{A} = \begin{bmatrix} \underset{n_1\times d_1}{A_1} & \underset{n_1\times d_2}{0} \\ \underset{n_2\times d_1}{0} & \underset{n_2\times d_2}{0} \end{bmatrix} + \begin{bmatrix} \underset{n_1\times d_1}{0} & \underset{n_1\times d_2}{A_2} \\ \underset{n_2\times d_1}{0} & \underset{n_2\times d_2}{0} \end{bmatrix} + \begin{bmatrix} \underset{n_1\times d_1}{0} & \underset{n_1\times d_2}{0} \\ \underset{n_2\times d_1}{A_3} & \underset{n_2\times d_2}{0} \end{bmatrix} + \begin{bmatrix} \underset{n_1\times d_1}{0} & \underset{n_1\times d_2}{0} \\ \underset{n_2\times d_1}{0} & \underset{n_2\times d_2}{A_4} \end{bmatrix}$$

$$= A_1' + A_2' + A_3' + A_4'$$

$$\begin{aligned} rank(A_1'+A_2'+A_3'+A_4') &\le dim(A_1') + dim(A_2') + dim(A_3') + dim(A_4') \\ &\le rank(A_1') + ran(A_2') + rank(A_3') + ran(A_4') \\ &\le rank(A_1) + rank(A_2) + rank(A_3) + rank(A_4) \end{aligned}$$

because, $rank(A_1') = rank(A_1)$ [$\because$ block matrix with all 0 columns]

$$\therefore \boxed{rank(A) \le rank(A_1) + rank(A_2) + rank(A_3) + rank(A_4)}$$

5b Given, $\|A_i - B_i\|_F \le \varepsilon$ — (1)

$$\left\| A - \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \right\| \le 4\varepsilon$$

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

$$[A - B] = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} - \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} A_1 - B_1 & A_2 - B_2 \\ A_3 - B_3 & A_4 - B_4 \end{bmatrix}$$

$$A - B = \begin{bmatrix} A_1 - B_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & A_2 - B_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A_3 - B_3 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & A_4 - B_4 \end{bmatrix}$$

— (2)

we know that $\|A + B\|_F \le \|A\|_F + \|B\|_F$

— (3)

using (1) (2) and (3)

$$\|A - B\|_F \le \|A_1 - B_1\|_F + \|A_2 - B_2\|_F + \|A_3 - B_4\|_F + \|A_4 - B_4\|_F$$

$$\|A - B\|_F \le \varepsilon + \varepsilon + \varepsilon + \varepsilon$$

$$\|A - B\|_F \le 4\varepsilon$$

where $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$