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* Convergent Sequence

A sequence $[a_n]$ is said to be convergent if $\lim_{n\to\infty} a_n$ is finite.

for eg - $\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \text{-----} \frac{1}{2^n} \text{-----}$

here $a_n = \frac{1}{2^n}$; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0$, which is finite.

* Divergent Sequence

A sequence $[a_n]$ is said to be divergent if $\lim_{n\to\infty}$ is not finite i.e.

$$\lim_{n\to\infty} a_n = +\infty \text{ or } -\infty$$

for eg - • let us consider a sequence $[n^2]$

here $a_n = n^2$, $\lim_{n\to\infty} n^2 = \infty$

• let $a_n = -2^n$

here, $\lim_{n\to\infty} -2^n = -\infty$

* Oscillatory Sequence

If a sequence $[a_n]$ neither converges to a finite number nor diverges to $-\infty$ or $+\infty$, it is called as an Oscillatory sequence.

for eg -

$$S_n = 1-1+1-1+1-1+1 \text{-----} n \text{ terms}$$

$$S_n = \begin{cases} 0 & ; n \text{ is even} \\ 1 & ; n \text{ is odd} \end{cases}$$

then, S_n is an Oscillatory sequence

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→ A bounded sequence that doesnot converges then it oscillates finitely.

for eg → $a_n = (-1)^n$

here, $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} (-1)^{2n} = 1$

$\lim_{n\to-\infty} a_{2n+1} = \lim_{n\to-\infty} (-1)^{2n+1} = -1$

Thus $\lim_{n\to\infty} a_n$ doesnot exist ⇒ the sequence doesnot converges.

Hence, the sequence oscillates finitely.

→ An unbounded sequence that doesnot diverges, oscillates infinitely.

for eg → $a_n = (-1)^n n$.

here, $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} (-1)^{2n} \cdot 2n = \infty$

$\lim_{n\to-\infty} a_{2n+1} = \lim_{n\to-\infty} (-1)^{2n+1}(2n+1) = -\infty$

* Monotonic Sequences

→ A sequence $\{a_n\}$ is said to be monotonically increasing if $a_{n+1} \geq a_n \ \forall\, n \in N$.

i.e. if $a_1 \leq a_2 \leq a_3 \cdots \leq a_n \leq a_{n+1} \cdots$

→ A sequence $\{a_n\}$ is said to be monotonically decreasing if $a_{n+1} \leq a_n \ \forall\, n \in N$

i.e. if $a_1 \geq a_2 \geq a_3 \geq \cdots a_n \geq a_{n+1} \cdots$

→ A sequence $\{a_n\}$ is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

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→ A sequence $\{a_n\}$ is said to be strictly monotonically increasing if $a_{n+1} > a_n \ \forall \ n \in N$

→ A sequence $\{a_n\}$ is said to be strictly monotonically decreasing if $a_{n+1} < a_n \ \forall \ n \in N$

⇒ **Convergence of Monotonic Sequences:-**

Theorem I → The necessary & sufficient condition for the convergence of monotic sequence is that it is bounded

A monotonic ↑ing sequence which is bounded above, converges

A monotonic ↓ing sequence which is bounded below, diverges.

Theorem II → If a monotonic ↑ing sequence is not bounded above, it diverges to $+\infty$.

Theorem III → If a monotonic ↓ing sequence is not bounded below it diverges to $-\infty$.

Theorem IV → If $\{a_n\}$ & $\{b_n\}$ are two convergent sequences, then sequence $\{a_n + b_n\}$ is also convergent

i.e. If $\lim a_n = A$ & $\lim b_n = B$ then $\lim(a_n + b_n) = A + B$

Theorem V → If $\{a_n\}$ & $\{b_n\}$ are two cgt sequences, then $\{a_n b_n\}$ and $\left\{\frac{a_n}{b_n}\right\}$ is also convergent.

Theorem VI → The sequence $\{|a_n|\}$ converges to 0, if & only if $\{a_n\}$ converges to 0.

Theorem VII → If a sequence $\{a_n\}$ converges to a & $a_n \geq 0 \ \forall n$, then $a \geq 0$

Theorem VIII → If $a_n \to a$, $b_n \to b$ & $a_n \leq b_n \ \forall n$, then $a \leq b$

Theorem IX → If $a_n \to l$, $b_n \to l$ & $a_n \leq c_n \leq b_n \ \forall n$, then $c_n \to l$

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Infinite Series :-

If $\{u_n\}$ is a sequence of real numbers, then the expression $u_1 + u_2 + u_3 + \cdots\cdots u_n \cdots\cdots$ (i.e., the sum of terms of sequence, which are infinite in number) is called an Infinite series.

→ The infinite series $u_1 + u_2 + \cdots\cdots$ is basically denoted as $\sum_{x=1}^{\infty} u_x$.

→ If all the terms of $\sum_{x=1}^{\infty} u_x$ are positive i.e. $u_x > 0 \ \forall n$. is called as positive series.

→ A series in which the terms are alternate positive & negative is called an alternating series.

Thus $\sum (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 \cdots\cdots$ where $u_n > 0 \ \forall n$ is an alternating series

→ Sum of first n terms of a series is called as Partial Sum.

$$S_n = 1 + x + x^2 + x^3 \cdots\cdots x^n \quad \text{(for eg)}$$

∴ To every infinite series $\sum u_n$, there corresponds a sequence $\{S_n\}$ of its partial sums.

i.e. if $\{S_n\}$ is cgt then $\sum u_n$ is cgt & same with divergence

→ for a GP → $1 + x + x^2 + \cdots\cdots \infty$

- Converges if $-1 < x < 1$, $|x| < 1$
- Diverges if $x \geq 1$
- Oscillates finitely if $x = -1$
- Oscillates infinitely if $x < -1$

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for AP, $S_n = \frac{n}{2}(2a+(n-1)d)$

for GP, $S_n = \frac{a(1-x^n)}{1-x}\ (x<1)$, $\frac{a(x^n-1)}{x-1}\ (x>1)$ Date

$\Rightarrow 1+x+x^2+ \cdots x^n + \cdots$

(for convergent) Proof when $|x|<1$, $-1<x<1$

$x^n \to 0$ as $n \to \infty$

$S_n = 1+x+x^2+ \cdots x^n$ (Partial sum)

$S_n = \frac{1(1-x^n)}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}$

$S_n = \lim_{n\to\infty}\left(\frac{1}{1-x} - \frac{x^n}{1-x}\right)$ $x^n \to 0$

$S_n = \lim_{n\to\infty} \frac{1}{1-x}$, which is finite.

So, $\{S_n\}$ is convergent & by theorm if partial sum is convergent then the given series is convergent if $-1<x<1$

Proved

$\Rightarrow 1+x+x^2+ \cdots x^n + \cdots$

(for divergent) Proof when $x \geq 1$

- case I when $x=1$

$S_n = 1+1+1+1+ \cdots$ n terms

$S_n = n$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$

therefore, $\{S_n\}$ is divergent & so is the given series. for $x=1$

- case II when $x>1$

$x^n \to \infty$ as $n \to \infty$

$S_n = 1+x+x^2+ \cdots x^n$

$S_n = \frac{1(x^n-1)}{x-1}$ as $x>1$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{x^n}{x-1} - \frac{1}{x-1}\right)$ $x^n \to \infty$

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$$\lim_{n\to\infty} S_n = \infty$$

$\therefore \{S_n\}$ diverges & so the given series for $x>1$

(for finite oscillation)

$\Rightarrow 1+x+x^2+x^3 - - - - - x^n - - - -$

when $x=-1$

$S_n = 1-1+1-1+1 - - - - (-1)^n$ (upto n terms)

$$S_n = \begin{cases} 0 & n \to \text{even} \\ 1 & n \to \text{odd} \end{cases}$$

Both value are finite

$\therefore \{S_n\}$ oscillates finitely & so the given series when $x=-1$

(for infinite oscillation)

$\Rightarrow 1+x+x^2+ - - - - x^n - - - -$

when $x<-1$

$-x>1$

let $r = -x$ → common ratio of G.P.

$r>1$

$r^n \to \infty$ as $n\to\infty$

Now, $S_n = 1+x+x^2+ - - - - - x^n$

$$S_n = \frac{1(1-x^n)}{1-x} = \frac{1-(-r)^n}{1-(-r)} = \frac{1+r^n}{1+r}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-(-r)^n}{1+r}$$

$$= \left\{ \lim_{n\to\infty} \left(\underset{+\infty}{\frac{1+r^n}{1+r}} \text{ or } \underset{-\infty}{\frac{1-r^n}{1+r}} \right) \right\}$$

odd, even

$\therefore \{S_n\}$ oscillates b/w $+\infty$ & $-\infty$ i.e. infinitely when $x<-1$ and so is the given series.

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Positive Term Series Tests

Date

⇒ Cauchy's Integral Test :-

If for $x \geq 1$, $f(x)$ is a non-negative monotonic decreasing function of x such that $f(n) = U_n$ (nth term of series) for all positive integral value of n then, the series ΣU_n and the integral $\int_1^\infty f(x)\,dx$ converge or diverge together.

Ques- Test the convergence of series -

$$\Sigma \frac{1}{n^2+1}$$

Sol- $f(n) = U_n = \frac{1}{n^2+1}$

$\therefore$

$$f(x) = U_x = \frac{1}{x^2+1}$$

for $x \geq 1$, $f(x)$ is +ve & monotonic decreasing

So,

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^2+1} = \left[\tan^{-1}x\right]_1^\infty$$

$$= \left(\tan^{-1}\infty - \tan^{-1}1\right) = \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}, \text{ which is finite.}$$

So, $\int_1^\infty f(x)\,dx$ is convergent and therefore ΣU_n is convergent (By Cauchy's Integral test)

Ques- Show that the series $\sum_1^\infty \frac{1}{n^p}$ converges if $p > 1$ & diverges if $0 < p \leq 1$

Sol- The given ~~test~~ question is based on p-test.

Here, $U_n = \frac{1}{n^p} = f(n)$

$$f(x) = U_x = \frac{1}{x^p}$$

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for $x \geq 1$, $f(x)$ is +ve & monotonic decreasing
so, we can apply Cauchy's Integral test

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx$$

<u>Case I</u> - if $p \neq 1$

$$\int_1^\infty \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^\infty$$

<u>Subcase I</u> - if $p > 1$
so, $1-p$ is -ve

$$\int_1^\infty f(x)\,dx = \frac{1}{1-p}\left[0-1\right] = \frac{1}{1-p}, \text{ which is finit}$$

$\therefore$ for $p > 1$, $\int_1^\infty f(x)\,dx$ converges & so is the ΣU_n

<u>Subcase II</u> - if $p < 1$
so, $1-p$ is +ve

$$\int_1^\infty f(x)\,dx = \frac{1}{1-p}\left[\infty - 1\right] = \infty$$

$\therefore$ for $p < 1$, $\int_1^\infty f(x)\,dx$ diverges & so is ΣU_n

<u>Case II</u> - if $p = 1$

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x}\,dx = \left[\log x\right]_1^\infty$$

$$= \infty - \log 1 = \infty - 0 = \infty$$

$\therefore$ U_n is divergent if $p = 1$

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→ Conclusion -

ΣU_n is convergent for $p>1$ and is divergent for $0<p\leq 1$

Hence proved

Ques- Discuss the convergence of the series-

$$\Sigma \frac{1}{n(\log n)^p}, \quad p>0$$

Sol- Here, $U_n = f(n) = \frac{1}{n(\log n)^p}$

$$U_x = f(x) = \frac{1}{x(\log x)^p}$$

for $\left[x \geq 2\right]$ or $\left[x>1\right]$, $f(x)$ is +ve & monotonic decreasing.

So, we can apply Cauchy's Integral test

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x(\log x)^p}\,dx$$

let $\log x = t$

$\frac{1}{x}dx = dt$

$$= \int \frac{1}{(t)^p}\,dt = \left[\frac{(\log x)^{-p+1}}{-p+1}\right]_2^\infty$$

Case I- for $p \neq 1$

Subcase I - if $p>1$

so, $1-p$ is -ve

$$\int_2^\infty f(x)\,dx = \frac{1}{-p+1}\left[0-(\log 2)^{-p+1}\right]$$, which is finite

So, ΣU_n is Convergent if $p>1$.

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Subcase II - if $p<1$
So, $1-p$ is +ve

$$\int_2^\infty f(x)dx = \frac{1}{1-p}\left[\infty - (\log 2)^{1-p}\right] = \infty$$

$\therefore \Sigma u_n$ is divergent/not convergent if $p<1$

Case II - for $p=1$

$$\int_2^\infty f(x)dx = \int_2^\infty \frac{1}{x\log x}dx = \left[\log(\log x)\right]_2^\infty = \infty$$

So, Σu_n is not convergent for $p=1$

→ Conclusion-
Σu_n is convergent for $p>1$ & divergent for $p\leq 1$.

⇒ Comparison Test:-

→ Test I: Let Σa_n and Σb_n (series) be two positive series & let K be a +ve constant independent of n and m (any fixed integers).

(i) If $a_n \leq Kb_n \ \forall \ n \geq m$ and Σb_n is convergent, then Σa_n is convergent.

(ii) If $a_n \geq Kb_n$ & Σb_n is divergent, then Σa_n is divergent.

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$\star \lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = e^a$

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Limit Comparison Test (Max Question)

$\star\star$ → Test II : Let Σa_n & Σb_n be two series.

(i) If $\lim_{n\to\infty} \frac{a_n}{b_n}$ = finite & non-zero, then Σa_n & Σb_n converge or diverge together.

(ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ & Σb_n is convergent, then Σa_n is convergent (Vice-Versa → not true)

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ & Σb_n is divergent, then Σa_n is divergent (Vice-Versa not true)

Ques- Discuss the convergence or divergence of -

$$\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \cdots\cdots$$

Sol- $\Sigma U_n = \Sigma \frac{1+(n-1)2}{n(n+1)(n+2)}$

$$U_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$U_n = \frac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{1}{n^2}\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

let $U_n = a_n = \frac{1}{n^2}\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$, $b_n = \frac{1}{n^2}$

Now, $\frac{a_n}{b_n} = \frac{\frac{1}{\cancel{n^2}}\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}}{\cancel{1/n^2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{2}{1\cdot1} = 2$, which is finite

∴ By Comparison, Σa_n & Σb_n converges or diverges together.

$b_n = \frac{1}{n^2}$, here $p = 2 > 1$

by p-test Σb_n is convergent, hence $\Sigma a_n (\Sigma U_n)$ is convergent.

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* Comparison test:-
Σu_n is cgt if its partial sum is cgt
Σu_n is dgt if its partial sum is dgt.

Date

Ques- Test the convergence or divergence of -

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}$$

Sol- let $a_n = \frac{(n+1)^n}{n^{n+1}}$

$$a_n = \frac{n^n\left(1+\frac{1}{n}\right)^n}{n^{n+1}} = \frac{1}{n}\left(1+\frac{1}{n}\right)^n$$

let $b_n = \frac{1}{n}$

Now, $\frac{a_n}{b_n} = \frac{\frac{1}{\cancel{n}}\left(1+\frac{1}{n}\right)^n}{\frac{1}{\cancel{n}}} = \left(1+\frac{1}{n}\right)^n$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,$$ which is finite

$\therefore$ By comparison, Σa_n & Σb_n converges or diverge together

$b_n = \frac{1}{n}$, $p = 1$

by p-test Σb_n is divergent, hence Σa_n is divergent

Ques- Test the convergence or divergence of -

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots$$

Sol- $a_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}\left(1+\sqrt{1+\frac{1}{n}}\right)}$

Take $b_n = \frac{1}{\sqrt{n}}$

Now, $\frac{a_n}{b_n} = \frac{1}{1+\sqrt{1+\frac{1}{n}}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{2},$$ which is finite

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$\therefore \Sigma a_n$ & Σb_n converges or diverges together.

here $b_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$

$p = \frac{1}{2}$; $0 < p \leq 1$

so, by p-test

Σb_n is divergent,

hence Σa_n is divergent.

2016-PTU

Ques- Define the term Absolute Convergence and use the concept to test the convergence of the given series-

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

Sol- The Σa_n is said be absolute convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

$a_n = \frac{\sin n}{n^2} \Rightarrow |a_n| = \left|\frac{\sin n}{n^2}\right|$

$|a_n| = \left|\frac{\sin n}{n^2}\right|$

Take $|b_n| = \left|\frac{1}{n^2}\right|$

$\left|\frac{\sin n}{n^2}\right| \leq \left|\frac{1}{n^2}\right|$, $a_n \leq K b_n$

and $b_n = \frac{1}{n^2}$, $p = 2 > 1$

so, Σb_n is convergent,

hence Σa_n is convergent.

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2013-PTU

Ques- Examine the converge or divergence of series-

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$$

Sol- Using $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots$

$$a_n = \frac{1}{n} \sin \frac{1}{n}$$

$$a_n = \frac{1}{n}\left[\frac{1}{n} - \frac{\left(\frac{1}{n}\right)^3}{3!} + \frac{\left(\frac{1}{n}\right)^5}{5!} - \cdots\right]$$

$$a_n = \frac{1}{n}\left[\frac{1}{n} - \frac{1}{3!n^3} + \frac{1}{5!n^5} - \cdots\right]$$

$$a_n = \frac{1}{n^2}\left[1 - \frac{1}{3!n^2} + \frac{1}{5!n^4} - \cdots\right]$$

Take $b_n = \frac{1}{n^2}$

Now $\frac{a_n}{b_n} = \left[1 - \frac{1}{3!n^2} + \frac{1}{5!n^4} - \cdots\right]$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left[1 - \frac{1}{3!n^2} + \frac{1}{5!n^4} - \cdots\right]$$

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, which is finite.

$\therefore \Sigma a_n$ & Σb_n converges or diverges together.

Now, $b_n = \frac{1}{n^2}$, $p = 2 > 1$

So Σb_n is convergent,
hence Σa_n is convergent.

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* Ratio Test or D'Alembert's Ratio test:-

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$$

then Σu_n is convergent
if $l > 1$
& Σu_n is divergent
if $l < 1$
$\Rightarrow$ test fails when $l = 1$

* Raabe's Test:-

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$$

then Σu_n is convergent
if $l > 1$
& Σu_n is divergent
if $l < 1$
$\Rightarrow$ test fails when $l = 1$.

* Logrithmic test:-

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = l$$

then Σu_n is convergent
if $l > 1$
& Σu_n is divergent
if $l < 1$
$\Rightarrow$ test fails when $l = 1$

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Ques- Test the convergence of :-

$$\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$$

Sol- here, $U_n = \frac{5^n}{4^n+3}$

$$U_{n+1} = \frac{5^{n+1}}{4^{n+1}+3}$$

$$\frac{U_n}{U_{n+1}} = \frac{5^n}{4^n+3} + \frac{4^{n+1}+3}{5^{n+1}}$$

$$\frac{U_n}{U_{n+1}} = \frac{4^{n+1}+3}{5(4^n+3)}$$

$$\frac{U_n}{U_{n+1}} = \frac{\cancel{4^n}\left(4+\frac{3}{4^n}\right)}{5\left[\cancel{4^n}\left(1+\frac{3}{4^n}\right)\right]}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{1}{5}\left[\frac{4+\frac{3}{4^n}}{1+\frac{3}{4^n}}\right]$$

$$= \frac{1}{5}[4] = \frac{4}{5} < 1$$

So, $l < 1$

$\therefore$ By Ratio test, $\sum U_n$ is divergent.

Ques- Discuss the convergence of -

$$\frac{1}{2} + \frac{1\cdot 3}{2\cdot 4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \cdots$$

Sol- here, $U_n = \frac{1\cdot 3\cdot 5\cdots\cdots(2n-1)}{2\cdot 4\cdot 6\cdots\cdots(2n)}$ $\rightarrow a+(n-1)d$, $1+2n-2$, $2n-1$; $\rightarrow 2+(n-1)2$, $2+2n-2$

$$U_{n+1} = \frac{1\cdot 3\cdot 5\cdots\cdots(2n-1)(2n+1)}{2\cdot 4\cdot 6\cdots\cdots(2n)(2n+2)}$$

Now, $\frac{U_n}{U_{n+1}} = \frac{2n+2}{2n+1}$

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$$\frac{U_n}{U_{n+1}} = \frac{n}{n}\left[\frac{2+2/n}{2+1/n}\right]$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{2+2/n}{2+1/n}$$

$$= \frac{2}{2} = 1 = l.$$

$\therefore$ Ratio test fails.

Now, Acc. to Raabe's test-

$$\lim_{n\to\infty} n\left[\frac{U_n}{U_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\frac{2n+2}{2n+1} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\cancel{2n}+2-\cancel{2n}-1}{2n+1}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{1}{2n+1}\right]$$

$$= \lim_{n\to\infty} \frac{n}{n}\left[\frac{1}{2+1/n}\right] = \frac{1}{2} < 1$$

$\therefore$ By Raabe's test ΣU_n is divergent.

Ques- Test the convergence of the series-

neglect it (not in domain of sol)

$$\frac{x^0 1}{1^0} + \frac{x 1!}{2^1} + \frac{2!\,x^2}{3^2} + \frac{3!\,x^3}{4^3} + \frac{4!\,x^4}{5^4} + \cdots$$

Sol- neglecting 1st term.

$$U_n = \frac{n!\,x^n}{(n+1)^n}$$

$$U_{n+1} = \frac{(n+1)!\,x^{n+1}}{(n+2)^{n+1}}$$

$$\frac{U_n}{U_{n+1}} = \frac{n!\,x^n}{(n+1)^n} \times \frac{(n+2)^{n+1}}{(n+1)!\,x^{n+1}} = \frac{(n+2)^{n+1}}{(n+1)^n (n+1)\,x}$$

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$$\frac{U_n}{U_{n+1}} = \frac{(n+2)^n (n+2)}{(n+1)^n \cdot (n+1)x}$$

$$\frac{U_n}{U_{n+1}} = \frac{(n+2)^n (n+2)}{(n+1)^n (n+1)x}$$

$$\frac{U_n}{U_{n+1}} = \frac{\cancel{n^n}(1+2/n)^n \cancel{n}(1+2/n)}{\cancel{n^n}(1+1/n)^n \cancel{n}(1+1/n)x}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{(1+2/n)^n (1+2/n)}{(1+1/n)^n (1+1/n)x}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{e^2 (1)}{e\,(1)\,x} = \frac{e}{x}$$

Now, By Ratio Test, ΣU_n is Convergent if $\frac{e}{x} > 1$ i.e. $e > x$, ΣU_n is divergent if $\frac{e}{x} < 1$ i.e. $e < x$, & test fails if $\frac{e}{x} = 1$ i.e. $e = x$

→ when $e = x$

$$\frac{U_n}{U_{n+1}} = \frac{(1+2/n)^{n+1}}{(1+1/n)^{n+1}} \cdot \frac{1}{e}$$

Since $\frac{U_n}{U_{n+1}}$ involves $\frac{1}{e}$, we will take log both sides & perform logarithmic test instead of Raabe's test.

taking log both sides

$$\log \frac{U_n}{U_{n+1}} = (n+1)\log(1+2/n) - (n+1)\log(1+1/n) - \log e$$

$$= (n+1)\left[\log(1+2/n) - \log(1+1/n)\right] - 1$$

As we knew, $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

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$$\log \frac{u_n}{u_{n+1}} = (n+1)\left[\left(\frac{2}{n} - \frac{4}{2n^2} + \frac{8}{3n^3} - \cdots\right) - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right)\right] - 1$$

$$\log \frac{u_n}{u_{n+1}} = (n+1)\left[\frac{1}{n} - \frac{3}{2n^2} + \frac{7}{3n^3} - \cdots\right] - 1$$

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lim_{n\to\infty}$$

$$\log \frac{u_n}{u_{n+1}} = \left[1 - \frac{3}{2n} + \frac{7}{3n^2} + \frac{1}{n} - \frac{3}{2n^2} + \frac{7}{3n^3} - \cdots\right] - 1$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty}$$

$$n \log \frac{u_n}{u_{n+1}} = -\frac{3}{2} + \frac{7}{3n} + 1 - \frac{3}{2n} + \frac{7}{3n^2} + \cdots$$

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left(-\frac{1}{2} + \frac{7}{3n} - \frac{3}{2n} + \cdots\right)$$

$$= -\frac{1}{2} < 1$$

$\therefore$ By logarithmic test, Σu_n is divergent if $l < 1$ i.e. when $e = x$.

Concluding that, Σu_n is convergent if $e > x$ and Σu_n is divergent if $e \leq x$.

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★ Gauss Test :-

If ΣU_n is positive term series

$$\frac{U_n}{U_{n+1}} = 1 + \frac{\lambda}{n} + O(n^2)$$

then ΣU_n is convergent if $\lambda > 1$ and
ΣU_n is divergent if $\lambda \leq 1$.

Ques:- Discuss the convergence or divergence of the following series:-

$$\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \text{-------}$$

Sol:-

$$U_n = \frac{1^2 \cdot 3^2 \cdot 5^2 \text{------} (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \text{-------} (2n)^2}$$

$\rightarrow a + (n-1)d$, $1 + (n-1)2 = 2n-1$

$\rightarrow 2 + (n-1)2 = 2n$

$$U_{n+1} = \frac{1^2 \cdot 3^2 \cdot 5^2 \text{------} (2n-1)^2 (2n+1)^2}{2^2 \cdot 4^2 \cdot 6^2 \text{-------} (2n)^2 (2n+2)^2}$$

$$\frac{U_n}{U_{n+1}} = \frac{1^2 \cdot 3^2 \cdot 5^2 \cancel{\text{---}} (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cancel{\text{---}} (2n)^2} \times \frac{2^2 \cdot 4^2 \cdot 6^2 \cancel{\text{---}} (2n)^2 (2n+2)^2}{1^2 \cdot 3^2 \cdot 5^2 \cancel{\text{---}} (2n-1)^2 (2n+1)^2}$$

$$\frac{U_n}{U_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}$$

$$\lim_{n \to \infty} \frac{U_n}{U_{n+1}} = \lim_{n \to \infty} \frac{4n^2(1 + 1/n)^2}{4n^2(1 + 1/2n)^2} = 1$$

$\therefore$ Ratio test fails

→ Now, Acc. to Raabe's test -

$$\frac{U_n}{U_{n+1}} - 1 = \frac{(2n+2)^2 - (2n+1)^2}{(2n+1)^2} = \frac{(4n^2 + 8n + 4) - (4n^2 + 4n + 1)}{(4n^2 + 4n + 1)}$$

$$\frac{U_n}{U_{n+1}} - 1 = \frac{4n + 3}{4n^2 + 4n + 1}$$

$$\lim_{n \to \infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n \to \infty} \frac{4n^2(1 + 3/4n)}{4n^2(1 + 1/n + 1/4n^2)} = 1$$

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$* (1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots$

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Raabe's test also failed.

→ Now, Acc. to Gauss test-

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{4n^2(1+1/n)^2}{4n^2(1+1/2n)^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{(1+1/n)^2}{(1+1/2n)^2} = (1+1/n)^2(1+1/2n)^{-2}$$

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n^2}+\frac{2}{n}\right)\left(1+(-2)\left(\frac{1}{2n}\right)+\frac{(-2)(-3)}{2!}\left(\frac{1}{2n}\right)^2+\cdots\right)$$

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{1}{n}+\frac{3}{4n^2}+\cdots\right)$$

$$\frac{u_n}{u_{n+1}} = 1-\frac{1}{n}+\frac{2}{n}+\frac{3}{4n^2}-\frac{2}{n^2}+\frac{1}{n^2}+\cdots$$

$$\frac{u_n}{u_{n+1}} = 1+\frac{1}{n}-\frac{1}{n^2}+\cdots$$

here, it is in the form

$$\frac{u_n}{u_{n+1}} = 1+\frac{\lambda}{n}+O(n^2)$$

$\lambda = 1$

∴ By Gauss test, Σu_n is divergent.

Ques- Discuss the convergence or divergence of -

neglecting this term $\frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^7}{7} + \cdots$

Sol- $u_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1}$ → $3+(n-1)2$, $3+2n-2 = 2n+1$

$\Rightarrow \frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \cdots$

we neglected this term, bcz it is of the form $\frac{1}{1}\cdot\frac{x}{1}$

then series would be - $\frac{1}{1}\cdot\frac{x}{1} + \frac{1}{1}\cdot\frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{1}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \cdots$

not forming either AP or GP

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$$U_{n+1} = \frac{1\cdot3\cdot5\cdots\cdots(2n-1)(2n+1)}{2\cdot4\cdot6\cdots\cdots(2n)(2n+2)} \cdot \frac{x^{2n+3}}{2n+3}$$

$$\frac{U_n}{U_{n+1}} = \frac{(2n+2)}{(2n+1)} \cdot \frac{x^{2n+1}}{(2n+1)} \cdot \frac{(2n+3)}{x^{2n+3}}$$

$$\frac{U_n}{U_{n+1}} = \frac{(2n+2)\,(2n+3)}{(2n+1)\,(2n+1)\cdot x^2}$$

$$\frac{U_n}{U_{n+1}} = \frac{\cancel{2n}(1+1/n)}{\cancel{2n}(1+1/2n)} \; \frac{\cancel{2n}(1+3/2n)}{\cancel{2n}(1+1/2n)\cdot x^2}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{(1+1/n)\,(1+3/2n)}{(1+1/2n)\,(1+1/2n)\,x^2}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{1}{x^2}$$

$\therefore$ By Ratio test, ΣU_n is convergent if $\frac{1}{x^2} > 1$ i.e. $x^2 < 1$, ΣU_n is divergent if $\frac{1}{x^2} < 1$ i.e. $x^2 > 1$ & test fails if $\frac{1}{x^2} = 1$ i.e. $x^2 = 1$

Now, Acc. to Raabe's test-

when $x^2 = 1$

$$\frac{U_n}{U_{n+1}} - 1 = \frac{(2n+2)\,(2n+3) - 1}{(2n+1)\,(2n+1)}$$

$$\frac{U_n}{U_{n+1}} - 1 = \frac{(2n+2)(2n+3) - (2n+1)^2}{(2n+1)^2}$$

$$\frac{U_n}{U_{n+1}} - 1 = \frac{\cancel{4n^2} + 6n + \cancel{4n} + 6 - \cancel{4n^2} - \cancel{4n} - 1}{(4n^2+4n+1)}$$

$$\frac{U_n}{U_{n+1}} - 1 = \frac{6n+5}{4n^2+4n+1}$$

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{\cancel{n^2}(6+5/n)}{\cancel{n^2}(4+4/n+1/n^2)} = \frac{6}{4} > 1$$

$\therefore$ By Raabe's test, ΣU_n is convergent.

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Cauchy Root Test :-

If ΣU_n is a positive term series and $\lim_{n\to\infty} (U_n)^{1/n} = l$

then, ΣU_n is convergent if $l<1$

ΣU_n is divergent if $l>1$

$\Rightarrow$ test fails when $l=1$

PTU-2016

Ques- Test the convergence of -

$$\Sigma\left(\frac{n}{n+1}\right)^{n^2} \text{ or } \Sigma\left(1+\frac{1}{n}\right)^{-n^2}$$

Sol- $U_n = \left(\frac{n}{n+1}\right)^{n^2} \Rightarrow (U_n)^{1/n} = \left(\frac{n}{n+1}\right)^{n}$

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \frac{n^n}{n^n(1+1/n)^n} = \frac{1}{1^\infty} = \frac{1}{e} < 1$$

$\therefore$ By Root test,

ΣU_n is convergent.

PTU-2016

Ques- Examine the convergence or divergence of series-

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \cdots$$

Sol- $U_n = \left(\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{(n+1)}{n}\right)^{-n}$

Applying Root test

$$U_n^{1/n} = \left(\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right)^{-1}$$

$$U_n^{1/n} = \left(\left(\frac{n+1}{n}\right)^{n+1} - \frac{\cancel{n}(1+1/n)}{\cancel{n}}\right)^{-1}$$

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* $\lim_{n\to\infty} (1+1/n)^n = e$

* $\lim_{n\to\infty} n^{1/n} = 1$

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$$u_n^{1/n} = \left(\frac{\cancel{n^{n+1}}(1+\frac{1}{n})^{n+1}}{\cancel{n^{n+1}}} - (1+1/n)\right)^{-1}$$

$$u_n^{1/n} = \left((1+1/n)^n \cdot (1+1/n) - (1+1/n)\right)^{-1}$$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left((1+1/n)^n \cdot (1+1/n) - (1+1/n)\right)^{-1}$$

$$= [e \cdot 1 - 1]^{-1} = (e-1)^{-1} = \frac{1}{e-1} < 1$$

$\therefore$ By Cauchy Root Test, Σu_n is convergent.

Ques- Discuss the convergence or divergence of-

$1 \cdot x^1 + 2 \cdot x^2 + 3x^3 + \text{-------}$

Sol- $u_n = nx^n$

Applying Root test

$$u_n^{1/n} = (nx^n)^{1/n} = n^{1/n} \cdot x$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} n^{1/n} \cdot x = 1 \cdot x = x.$$

$\therefore$ By Cauchy root test, if $x < 1$, Σu_n is cgt. if $x > 1$ then Σu_n is dgt and if $x = 1$, then test fails.

→ when $x = 1$

$u_n = n$

$S_n = 1 + 2 + 3 + \text{---} n \text{ terms}$

$S_n = \frac{n(n+1)}{2}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n^2(1+1/n)}{2} = \infty$$

$\therefore S_n$ is divergent, so by comparison test → Σu_n is dgt at $x = 1$

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Concluding that - Σu_n is convergent when $x > 1$ & Σu_n is divergent when $x \leq 1$.

Alternating Term Series Test

* Leibnitz's Test :-

The alternating series -

$\Sigma (-1)^{n-1} u_n = u_1 - u_2 + u_3 - - - - - -$

where $u_n > 0 \; \forall n$

converges if -

(i) $u_n > u_{n+1} \; \forall n$

(ii) $\lim_{n \to \infty} u_n = 0$

Ques:- Examine the convergence of -

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - - - - - -$

Sol:- The given series is alternating series.

$u_n = \frac{1}{n}$

$u_{n+1} = \frac{1}{n+1}$

Here, $u_n > u_{n+1} \; \forall n$. — condition 1

Now, $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = 0$ — condition 2.

$\therefore$ both the conditions of leibnitz's test are satisfied, hence the given series is convergent.

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Ques:- Examine the convergence of-

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$$

Sol- The given is alternating series-

$$U_n = \frac{n+1}{n}$$

$$U_{n+1} = \frac{n+2}{n+1}$$

Now, $U_n - U_{n+1} = \frac{n+1}{n} - \frac{n+2}{n+1}$

$$= \frac{(n+1)^2 - n(n+2)}{n(n+1)}$$

$$= \frac{\cancel{n^2} + \cancel{2n} + 1 - \cancel{n^2} - \cancel{2n}}{n^2+n} = \frac{1}{n^2+n} > 0$$

$\therefore U_n > U_{n+1}$ — condition 1

$$\rightarrow \lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{\cancel{n}(\frac{1}{n}+1)}{\cancel{n}} = 1 \neq 0$$

So, condition 2 of leibnitz's test is not satified, therefore the given series is not convergent.

Ques- Test the convergence of-

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2n-1} \Rightarrow \text{given is alternating series.}$$

Sol-

$$U_n = \frac{n}{2n-1}$$

$$U_{n+1} = \frac{n+1}{2n+1}$$

Now, $U_n - U_{n+1} = \frac{n}{2n-1} - \frac{n+1}{2n+1} = \frac{2n^2 + n - (n+1)(2n-1)}{(2n-1)(2n+1)}$

$$= \frac{\cancel{2n^2} + \cancel{n} - \cancel{2n^2} - \cancel{n} + 2n - 1}{(2n-1)(2n+1)} = \frac{1}{2n+1} > 0$$

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$\therefore U_n > U_{n+1}$ — condition 1

$\rightarrow \lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{n}{2n-1}$

$= \lim_{n\to\infty} \frac{n}{n(2-1/n)} = \frac{1}{2} \neq 0$

So, condition 2 is not satisfied and therefore given series is not convergent.

$\Rightarrow$ Absolute Convergence of a series:-

Alternating series cgt if all the make it is also cgt then it is Absolute cgt.

If a convergent series whose terms are not all positive remains convergent when all its terms are made positive, than it is called Absolute Convergent series.

$\rightarrow$ if the series made is not cgt then it is conditionally cgt.

Ques- Test whether the series is absolutely convergent or conditionally convergent.

$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Sol- The given series is alternating series-

$U_n = \frac{1}{n^2}$, $U_{n+1} = \frac{1}{(n+1)^2}$

here, $U_n > U_{n+1}$ — condition 1

$\rightarrow \lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$ — condition 2

$\therefore$ both conditions of leibnitz's test are satisfied so, given series is cgt

Now, in order to check its absolute convergence-

$\Sigma|U_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

$U_n = \frac{1}{n^2}$, which is cgt by p-test.

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$\therefore$ the given series is absolutely convergent.

Ques- Examine whether the given is absolutely cgt or conditionally cgt- $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$

Sol- The given series is alternating series-

$u_n = \frac{1}{2n-1}$, $u_{n+1} = \frac{1}{2n+1}$

here, $u_n > u_{n+1}$ — condition 1

$\rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n(2-1/n)} = 0$ — condition 2

$\therefore$ both the conditions of leibnitz's test are satisfied so, given series is convergent.

Now, in order to check the absolute convergence-

$\Sigma|u_n| = \sum_{n=1}^{\infty} \left|\frac{1}{2n-1}\right|$

$|u_n| = \frac{1}{2n-1} = \frac{1}{n(2-1/n)}$

$|v_n| = \frac{1}{n}$

$\frac{u_n}{v_n} = \frac{1}{2-1/n}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{2-1/n} = \frac{1}{2}$, which is finite.

$\therefore$ Σu_n & Σv_n converges or diverges together.

Now, $v_n = 1/n$ is divergent by p-test as $p = 1$

so, $\Sigma|u_n|$ is divergent.

$\therefore$ the given series is conditionally convergent.

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⇒ Uniformly Convergent:-

A series $\sum_{n=1}^{\infty} U_n(x)$ converges uniformly to $S(x)$ if for given $\epsilon > 0$, there exists a positive integer m depending upon ϵ & independent of x such that for every $x \in I$ (interval)

$$|S_n(x) - S(x)| < \epsilon \quad \forall n > m$$

☆ Weierstrass's M Test :- (test to check uniform convergence)

A series $\sum_{n=1}^{\infty} U_n(x)$ of functions converges uniformly and absolutely on an interval I if $\exists$ a convergent series $\sum_{n=1}^{\infty} M_n$ of +ve constants such that $|U_n(x)| \leq M_n \; \forall n \in N$ and $\forall x \in I$

Ques:- find whether the series is uniformly convergent or not - $\sum_{n=1}^{\infty} \frac{1}{n^p + n^q x^2}$ for all real x & $p > 1$

Sol:- $U_n(x) = \frac{1}{n^p + n^q x^2}$ → always +ve

$x^2 \geq 0$

Multiply both sides by n^q

$n^q x^2 \geq 0$

Add n^p both sides

$n^p + n^q x^2 \geq n^p$

$$\frac{1}{n^p} \geq \frac{1}{n^p + n^q x^2} \Rightarrow M_n = \frac{1}{n^p}$$

Now, $\frac{1}{n^p}$ is cgt if $p > 1$ and dgt if $0 < p \leq 1$ by p-test

But, $p > 1$ (given) & ∴ $\sum M_n$ is cgt

∴ By M test $\sum U_n(x)$ is uniformly cgt.

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Ques- Check whether the given series is uniformly cgt or not.
$$\sum_{n=1}^{\infty} \frac{\sin(x^2+nx)}{n(n+2)} \quad \forall \text{ real } x.$$

Sol- $U_n(x) = \dfrac{\sin(x^2+nx)}{n(n+2)}$

$$\frac{\sin(x^2+nx)}{n(n+2)} \leq \frac{1}{n(n+2)}$$ (bcz max. value of sin is 1)

$$\frac{1}{n(n+2)} < \frac{1}{n^2} = M_n$$

$$\frac{1}{n^2+2n}$$

Now, $M_n = \dfrac{1}{n^2}$ is convergent by p-test as $p > 1$

$\therefore$ By M test, $\sum_{n=1}^{\infty} U_n(x)$ is uniformly convergent.

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Ques- Check whether the given series is uniformly cgt or not - $\sum_{n=1}^{\infty} \dfrac{1}{n^4+n^3x^2}$

Sol- $U_n(x) = \dfrac{1}{n^4+n^3x^2}$

$x^2 \geq 0$

$n^3x^2 \geq 0$

$n^4+n^3x^2 \geq n^4$

$$\frac{1}{n^4+n^3x^2} \leq \frac{1}{n^4} = M_n$$

Now, $M_n = \dfrac{1}{n^4}$ is convergent by p-test as $p > 1$

$\therefore$ By M test,
$\sum_{n=1}^{\infty} U_n(x)$ is uniformly convergent

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⇒ Subsequence :-
A sequence $\{y_n\}$ is called a subsequence of sequence $\{x_n\}$ if there exists a sequence $\{n_k\}$ of +ve integers such that $n_1 < n_2 < n_3 - - - -$ and $y_k = x_{n_k}$.

* Botzano Weirotrass Theorem :-

Statement: Every bounded sequence has a convergent subsequence.

Proof: Let $\{x_n\}$ be a bounded sequence.
$\because \{x_n\}$ is a sequence, then $\{x_n\}$ have a monotonic subsequence $\{x_{n_k}\}$

$\{x_n\}$ is bounded $\Rightarrow \{x_{n_k}\}$ is also bounded

[Every subsequence of a bounded sequence is bounded]

Now, $\{x_{n_k}\}$ is monotonic bounded sequence.

$\therefore \{x_{n_k}\}$ is convergent

($\because$ Every monotonic bounded sequence is convergent)

⇒ Cauchy Sequence :-
A sequence $\{x_n\}$ is said to be a cauchy sequence if given $\epsilon > 0$, however small $\exists$ a +ve integer (depends upon ϵ) such that $|x_n - x_m| < \epsilon \ \forall \ n, m \geq K$

- Every Cauchy sequence is bounded.
- Every Cauchy sequence is convergent.
(vice-versa is also true).

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Cauchy's General Principle of Convergence:-

The necessary and sufficient condition for convergence of a sequence $\{x_n\}$ of real number is that it is a cauchy sequence.

Ques- Prove that the sequence $\left\{\frac{n^3}{n^3+1}\right\}$ is a cauchy sequence.

Sol- Here, $a_n = \frac{n^3}{n^3+1}$, $a_m = \frac{m^3}{m^3+1}$

Take $n > m$, let $\epsilon > 0$, however small, then-

$$|a_n - a_m| < \epsilon$$

if $\left|\frac{n^3}{n^3+1} - \frac{m^3}{m^3+1}\right| < \epsilon$

$$\left|\left(1 - \frac{1}{n^3+1}\right) - \left(1 - \frac{1}{m^3+1}\right)\right| < \epsilon$$

$$\left|-\frac{1}{n^3+1} + \frac{1}{m^3+1}\right| < \epsilon$$

$$\frac{1}{m^3+1} - \frac{1}{n^3+1} < \epsilon$$

$$\frac{1}{m^3+1} < \epsilon + \frac{1}{n^3+1}$$

$$\frac{1}{m^3+1} < \epsilon$$

$$m^3+1 > 1/\epsilon \Rightarrow m^3 > 1/\epsilon - 1$$

$$m > \left(\frac{1}{\epsilon} - 1\right)^{1/3} = K$$

Hence, the given sequence $\{a_n\}$ is a cauchy sequence $\therefore$ the given sequence is convergent.

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Ques:- Prove that the sequence $\{a_n\}$ where $a_n = 8 + \frac{1}{n^3}$ is a cauchy sequence and find its limit.

Sol:- $a_n = 8 + \frac{1}{n^3}$, $a_m = 8 + \frac{1}{m^3}$

Take $n > m$, let $\epsilon > 0$, however small, then-

$$|a_n - a_m| < \epsilon$$

$$\left|\left(8 + \frac{1}{n^3}\right) - \left(8 + \frac{1}{m^3}\right)\right| < \epsilon$$

$$\left|\frac{1}{n^3} - \frac{1}{m^3}\right| < \epsilon, \text{ here } n > m$$

$$\frac{1}{m^3} - \frac{1}{n^3} < \epsilon$$

$$\frac{1}{m^3} < \epsilon + \frac{1}{n^3}$$

$$\frac{1}{m^3} < \epsilon$$

$$m^3 > \frac{1}{\epsilon} \Rightarrow m > \left(\frac{1}{\epsilon}\right)^{1/3} = K$$

Hence, the given sequence $\{a_n\}$ is cauchy sequence.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 8 + \frac{1}{n^3}$$

$$= 8.$$

Revision Questions :-

Cauchy Int. Test

Ques:- Test the convergence of series $\frac{1}{n(n+1)}$

Sol:- $U_n = \frac{1}{n(n+1)} = f(n)$

$$f(x) = \frac{1}{x(x+1)}$$

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for $x \geq 1$, $f(x)$ is a non-negatic monotonic decreasing series.

Now, $\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x(x+1)}\,dx$

By Partial fractioning method-

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$1 = A(x+1) + B(x)$$

At $x = 0$, At $x = -1$

$A = 1$ $B = -1$

$$\therefore \int_1^\infty f(x)\,dx = \int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= [\log x]_1^\infty - [\log(x+1)]_1^\infty$$

$$= (\infty - 0) - (\infty - \log 2)$$

$= \log 2$, which is finite.

So, $\int_1^\infty f(x)\,dx$ is convergent $\Rightarrow \sum u_n$ is convergent. (By Cauchy integral test)

Ques- Test the convergence of $\sum ne^{-n^2}$.

Sol- $u_n = f(n) = \sum ne^{-n^2}$

$f(x) = \sum xe^{-x^2}$

for $x \geq 1$, $f(x)$ is a non-negative monotonic decreasing series.

Now, $\int_1^\infty f(x)\,dx = -\frac{1}{2}\int_1^\infty -2x\, e^{-x^2}\,dx$

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let $-x^2 = t$

$-2x\,dx = dt$

$$\int_1^\infty f(x)\,dx = -\frac{1}{2}\int_1^\infty e^{-t}\,dt$$

$$= -\frac{1}{2}\left[e^{-\infty} - e^{-1}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{e^\infty} - \frac{1}{e}\right] = -\frac{1}{2}\left[\frac{1}{\infty} - \frac{1}{e}\right]$$

$$= -\frac{1}{2}\left(0 - \frac{1}{e}\right) = \frac{1}{2e}, \text{ which is finite.}$$

$\therefore \int_1^\infty f(x)\,dx$ is convergent & by cauchy integral test, Σu_n is convergent.

Comparison Test :-

Test III - If $\frac{a_{n+1}}{a_n} < \frac{b_{n+1}}{b_n}$ and Σb_n is convergent,

then Σa_n is convergent.

- If $\frac{a_{n+1}}{a_n} > \frac{b_{n+1}}{b_n}$ and Σb_n is divergent,

then Σa_n is also divergent.

Test IV - If $H < \frac{a_n}{b_n} < K$ then,

Σa_n & Σb_n behaves alike or converges/diverges together.

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Ques= Test the convergence of series -

$$\frac{1}{1\cdot 4}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 6}+\cdots\cdots\infty$$

Sol- $U_n = \frac{1}{n(n+3)}$

$$U_n = \frac{1}{n^2(1+3/n)}$$

Now, take $V_n = \frac{1}{n^2}$

$$\frac{U_n}{V_n} = \frac{1}{\cancel{n^2}(1+3/n)}\cdot \cancel{n^2}$$

$$\frac{U_n}{V_n} = \frac{1}{1+3/n}$$

$$\lim_{n\to\infty}\frac{U_n}{V_n} = \lim_{n\to\infty}\frac{1}{1+3/n} = 1,$$ which is finite

$\therefore$ ΣU_n & ΣV_n behaves alike.

Now, $\Sigma V_n = \frac{1}{n^2}$ is convergent by p-test

So, ΣU_n is also convergent.

Ques- Test the convergence of series -

$$\Sigma\left(\sqrt[3]{n^3+1}-n\right)$$

Sol- $U_n = (n^3+1)^{1/3} - n$

$$= n\left(1+\frac{1}{n^3}\right)^{1/3} - n$$

$(1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\cdots\cdots$

$$= n\left[1+\frac{1}{3}\left(\frac{1}{n^3}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{n^3}\right)^2+\cdots\cdots\right]-n$$

$$= n\left[1+\frac{1}{3n^3}-\frac{1}{9n^6}+\cdots\cdots\right]-n$$

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$$U_n = \left[\cancel{x} + \frac{1}{3n^2} - \frac{1}{9n^5} + \cdots\right] - \cancel{x}$$

$$U_n = \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9n^3} + \cdots\right]$$

take $V_n = \frac{1}{n^2}$

Now,

$$\frac{U_n}{V_n} = \left(\frac{1}{3} - \frac{1}{9n^3} + \cdots\right)$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{9n^3} + \cdots\right) = \frac{1}{3}$$, which is finite.

So, ΣU_n & ΣV_n behaves alike.

here $\Sigma V_n = \frac{1}{n^2}$ is convergent by p-test.

$\therefore$ ΣU_n is also convergent.

Ques- Test the convergence of series -

$$U_n = \frac{n^n x^n}{n!}$$

Sol- $U_n = \frac{n^n x^n}{n!}$

$$U_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$$

$$\frac{U_n}{U_{n+1}} = \frac{n^n \cancel{x^n}}{\cancel{n!}} \times \frac{\cancel{(n+1)!}}{(n+1)^{n+1} \cancel{x^{n+1}}}$$

$$\frac{U_n}{U_{n+1}} = \frac{n^n (n+1)}{(n+1)^{n+1} \cdot x} = \frac{n^n \cancel{(n+1)}}{(n+1)^n \cancel{(n+1)} \cdot x}$$

$$\frac{U_n}{U_{n+1}} = \cancel{\frac{n^{n+1}\left(1+\frac{1}{n}\right)}{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1} \cdot x}} \quad \frac{\cancel{n^n}}{\cancel{n^n}\left(1+\frac{1}{n}\right)^n \cdot x}$$

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$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n \cdot x} = \frac{1}{xe}$$

$\therefore$ By Ratio test, Σu_n is convergent if $\frac{1}{xe} > 1$
$\Rightarrow xe < 1$, Σu_n is divergent if $xe > 1$
and test fails if $\frac{1}{xe} = 1 \Rightarrow x = \frac{1}{e}$

Now, when $xe = 1$

$$\frac{u_n}{u_{n+1}} = e\left(1+\frac{1}{n}\right)^{-n}$$

take log both sides

$$\log \frac{u_n}{u_{n+1}} = \log e + (-n)\log\left(1+\frac{1}{n}\right) \quad \log(1+x) = 1 - \frac{x^2}{2} + \frac{x^3}{3}$$

$$= 1 - n\left(\frac{1}{n} - \left(\frac{1}{n}\right)^2 \frac{1}{2} + \left(\frac{1}{n}\right)^3 \frac{1}{3} - \cdots\right)$$

$$= \cancel{1} - \cancel{1} + \frac{1}{2n} - \frac{1}{3n^2} + \cdots$$

Now multiply by n

$$n\log \frac{u_n}{u_{n+1}} = \frac{1}{2} - \frac{1}{3n} + \cdots$$

$$\lim_{n\to\infty} n\log \frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{3n} + \cdots\right) = \frac{1}{2},$$

which is finite

$\therefore$ By logarithmic test, Σu_n is convergent when $x = \frac{1}{e}$

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Integration formulae

Trignometric-

→ $\int \sin x \, dx = -\cos x$

→ $\int \cos x \, dx = \sin x$

→ $\int \tan x \, dx = -\log|\cos x|$ or $\log|\sec x|$

→ $\int \sec x \, dx = \log|\sec x + \tan x|$

→ $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x|$

→ $\int \cot x \, dx = \log|\sin x|$

→ $\int \sec^2 x \, dx = \tan x$

→ $\int \operatorname{cosec}^2 x \, dx = -\cot x$

Algebric-

→ $\int x^n \, dx = \frac{x^{n+1}}{n+1}, \quad n \neq -1$

→ $\int \frac{1}{x} \, dx = \log_e x$

→ $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$

→ $\int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right|$

→ $\int \frac{1}{a^2-x^2} \, dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right|$

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$\rightarrow \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}$

$\rightarrow \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log|x+\sqrt{x^2-a^2}|$ or $\cosh^{-1}\frac{x}{a}$

$\rightarrow \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log|x+\sqrt{x^2+a^2}|$ or $\sinh^{-1}\frac{x}{a}$

$\rightarrow \int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|$

or

$\frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$

$\rightarrow \int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|$

or

$\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$

$\rightarrow \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

$\rightarrow \int e^{ax}\,dx = \frac{e^{ax}}{a}$

$\rightarrow \int a^x\,dx = \frac{a^x}{\log a}$

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Chapter → Multiple Integration

The integral of a function of more than one variable is known as Multiple Integral. As the integral of a function of one variable over an interval results in an area, the double integral of a function of two variables calculated over a region results in a volume.

for eg - $\int_1^2 \int_1^3 1\, dx\, dx = \int_1^2 [x]_1^3\, dy = \int_1^2 (3-1)\, dy$

$= \int_1^2 2\, dy$

$= [2y]_1^2 = 4 - 2 = 2$

Ques= Solve- $\int_2^3 \int_1^y xy\, dx\, dy$

Sol- $\Rightarrow \int_2^3 \int_1^y y\,(x\,dx)\, dy = \int_2^3 y \left[\frac{x^2}{2}\right]_1^y dy$

$= \int_2^3 y\left(\frac{y^2}{2} - \frac{1}{2}\right) dy$

$= \int_2^3 \left(\frac{y^3}{2} - \frac{y}{2}\right) dy$

$= \left[\frac{y^4}{8} - \frac{y^2}{4}\right]_2^3$

$= \left(\frac{3^4}{8} - \frac{2^4}{8}\right) - \left(\frac{3^2}{4} - \frac{2^2}{4}\right)$

$= \frac{81}{8} - \frac{16}{8} - \frac{9}{4} + 1 = \frac{65}{8} - \frac{5}{4}$

$= \boxed{\frac{55}{8}}$ Ans

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Ques- Solve - $\int_2^3 \int_1^2 y \, dy \, dx$

Sol- $\Rightarrow \int_2^3 \left[\frac{y^2}{2}\right]_1^2 dx = \int_2^3 \left(\frac{4}{2} - \frac{1}{2}\right) dx$

$= \int_2^3 \frac{3}{2} dx$

$= \left[\frac{3}{2}x\right]_2^3$

$= \frac{9}{2} - \frac{3}{1} = \frac{9-6}{2}$

$= \boxed{\frac{3}{2}}$ Ans

Ques- Verify -

$\int_3^4 \int_1^2 (x^2+y^2) \, dy \, dx = \int_1^2 \int_3^4 (x^2+y^2) \, dx \, dy$

Sol- LHS $= \int_3^4 \left[x^2 y + \frac{y^3}{3}\right]_1^2 dx = \int_3^4 \left[\left(2x^2 + \frac{8}{3}\right) - \left(x^2 + \frac{1}{3}\right)\right] dx$

$= \int_3^4 \left(x^2 + \frac{7}{3}\right) dx = \left[\frac{x^3}{3} + \frac{7}{3}x\right]_3^4$

$= \left(\frac{4^3}{3} + \frac{28}{3}\right) - \left(\frac{3^3}{3} + \frac{21}{3}\right)$

$= \frac{64}{3} + \frac{28}{3} - \frac{27}{3} - \frac{21}{3} = \boxed{\frac{44}{3}}$

RHS $= \int_1^2 \left[\frac{x^3}{3} + y^2 x\right]_3^4 dy = \int_1^2 \left[\left(\frac{64}{3} + 4y^2\right) - \left(\frac{27}{3} + 3y^2\right)\right] dy$

$= \int_1^2 \left(\frac{37}{3} + y^2\right) dy = \left[\frac{37}{3}y + \frac{y^3}{3}\right]_1^2$

$= \left(\frac{74}{3} + \frac{8}{3}\right) - \left(\frac{37}{3} + \frac{1}{3}\right)$

$= \frac{74}{3} + \frac{8}{3} - \frac{37}{3} - \frac{1}{3} = \boxed{\frac{44}{3}}$

LHS = RHS

Hence, Verified

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Ques- Evaluate - $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$

Sol- $\int_0^1 x \left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx = \int_0^1 x\left(\frac{(2-x)^2}{2} - \frac{x^4}{2}\right) dx$

$= \int_0^1 \frac{x}{2}\left(4+x^2-4x-x^4\right) dx$

$= \frac{1}{2}\int_0^1 \left(4x+x^3-4x^2-x^5\right) dx$

$= \frac{1}{2}\left[\frac{4x^2}{2} + \frac{x^4}{4} - \frac{4x^3}{3} - \frac{x^6}{6}\right]_0^1$

$= \frac{1}{2}\left[\left(2+\frac{1}{4}-\frac{4}{3}-\frac{1}{6}\right) - 0\right]$

$= \frac{1}{2}\left(\frac{9}{12}\right) = \boxed{\frac{9}{24}}$ Ans

Ques- Evaluate - $\int_1^{\log 8} \int_0^{\log y} e^{x+y} \, dx \, dy$

Sol- $\cancel{\int_1^{\log 8} [e^{x+y}}$ $\int_1^{\log 8} \int_0^{\log y} e^x \cdot e^y \, dx \, dy$

$\Rightarrow \int_1^{\log 8} e^y \left[\frac{e^x}{1}\right]_0^{\log y} dy = \int_1^{\log 8} e^y (y-1) \, dy$

$= \int_1^{\log 8} ye^y \, dy - \int_1^{\log 8} e^y \, dy$

$= \left[ye^y - \int e^y dy\right]_1^{\log 8} - \left[e^y\right]_1^{\log 8}$

$= \left[ye^y - e^y\right]_1^{\log 8} - \left[e^y\right]_1^{\log 8}$

$= (8\log 8 - 8 - \cancel{e} + \cancel{e}) - (8 - e)$

$= 8\log 8 - 8 - 8 + e$

$= \boxed{8\log 8 - 16 + e}$ Ans

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Ques- Evaluate $\iint xy\, dx\, dy$ over the +ve ~~surface~~ quadrant of circle $x^2+y^2=1$

Sol- $x^2+y^2=1$

$x^2 = 1-y^2$

$x = \pm\sqrt{1-y^2}$

we have to find the limit over +ve quadrant

so, $x \Rightarrow 0$ to $\sqrt{1-y^2}$

$\therefore x$ varies from 0 to $\sqrt{1-y^2}$

Now, $\underset{0}{x^2} + y^2 = 1$

$y^2 = 1$

$y = \pm 1$

$\therefore y$ varies from 0 to 1

So, integral is $\int_0^1 \int_0^{\sqrt{1-y^2}} xy\, dx\, dy = \int_0^1 \int_0^{\sqrt{1-y^2}} y\,(x\,dx)\, dy$

$= \int_0^1 y \left[\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$

$= \int_0^1 y \left(\frac{1-y^2}{2} - 0\right) dy$

$= \frac{1}{2}\int_0^1 (y - y^3)\, dy$

$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1$

$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) - 0$

$= \boxed{\frac{1}{8}}$ Ans

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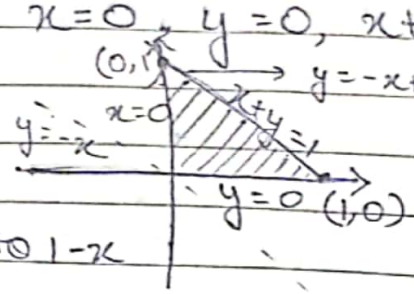
Ques Evaluate $\iint e^{2x+3y}\, dy\, dx$ over the triangle bounded by the lines $x=0$, $y=0$, $x+y=1$

Sol:

$x+y=1$

$y=-x+1$

$y=1-x$



$\therefore$ y varies from 0 to $1-x$

Now, $x+y=1$

$x=1$

$\therefore$ x varies from 0 to 1

$$\int_0^1\int_0^{1-x} e^{2x}\cdot e^{3y}\, dy\, dx = \int_0^1 e^{2x}\left[\frac{e^{3y}}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 e^{2x}\cdot\left(\frac{e^{3-3x}}{3}-\frac{1}{3}\right) dx$$

$$= \frac{1}{3}\int_0^1 \left(e^{3-x}-e^{2x}\right) dx$$

$$= \frac{1}{3}\left[\frac{e^{3-x}}{-1}-\frac{e^{2x}}{2}\right]_0^1$$

$$= \frac{1}{3}\left[\left(\frac{e^2}{-1}-\frac{e^2}{2}\right)-\left(\frac{e^3}{-1}-\frac{1}{2}\right)\right]$$

$$= \frac{1}{3}\left(\frac{e^3}{1}-\frac{e^2}{2}-e^2+\frac{1}{2}\right)$$

$$= \frac{1}{3}\left(e^3-\frac{3e^2}{2}+\frac{1}{2}\right)$$

$$= \boxed{\frac{e^3}{3}-\frac{e^2}{2}+\frac{1}{6}}$$ Ans

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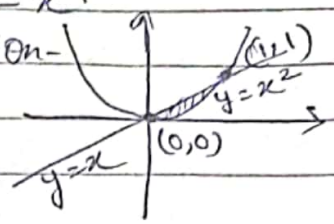
Ques- Evaluate $\iint xy(x+y)\,dx\,dy$ bounded by the curve $y = x^2$ & the line $y = x$.

Sol- To evaluate point of intersection-



of $y = x$ & $y = x^2$

$y = x$ — ①

$y = x^2$ — ②

from ① & ②

$x = x^2$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0,\ x = 1$

when $x = 0,\ y = 0$ $(0,0)$

& when $x = 1,\ y = 1$ $(1,1)$

$$\int_0^1 \int_y^{\sqrt{y}} (x^2y + xy^2)\,dx\,dy \quad \text{or} \quad \int_0^1 \int_{x^2}^{x} (x^2y + xy^2)\,dy\,dx$$

$$= \int_0^1 \left[\frac{x^3}{3}y + \frac{x^2}{2}y^2\right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 \left(\frac{y^{5/2}}{3} + \frac{y^3}{2}\right) - \left(\frac{y^4}{3} + \frac{y^4}{2}\right) dy$$

$$= \int_0^1 \left(\frac{y^{5/2}}{3} + \frac{y^3}{2} - \frac{5}{6}y^4\right) dy$$

$$= \left[\frac{2y^{7/2}}{21} + \frac{y^4}{8} - \frac{y^5}{6}\right]_0^1$$

$$= \left(\frac{2}{21} + \frac{1}{8} - \frac{1}{6}\right) - (0)$$

$$= \boxed{\frac{3}{56}} \text{ Ans}$$

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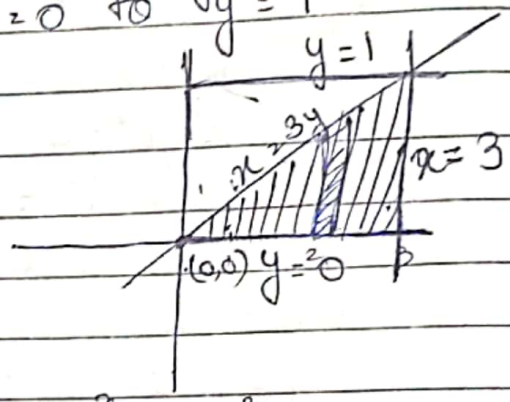
Change of Order of Integration :-

Ques- Evaluate - $\int_0^1 \int_{3y}^{3} e^{x^2} dx\, dy$

Sol- Here x varies from $x = 3y$ to $x = 3$ and y varies from $y = 0$ to $y = 1$



$$\int_0^3 \int_0^{x/3} e^{x^2} dy\, dx$$

$$\Rightarrow \int_0^3 e^{x^2} [y]_0^{x/3} dx = \frac{1}{3} \int_0^3 x e^{x^2} dx$$

Put $x^2 = t$ $\quad$ $x \to 0, t \to 0$

$2x\, dx = dt$ $\quad$ $x \to 3, t \to 9$

$x\, dx = \frac{dt}{2}$

$$= \frac{1}{6} \int_0^9 e^t dt$$

$$= \frac{1}{6} [e^t]_0^9 = \frac{1}{6}(e^9 - e^0)$$

$$= \boxed{\frac{1}{6}(e^9 - 1)} \text{ Ans}$$

Ques- Change the order of integration and evaluate -

$$\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$$

Sol- Here y varies from $y = x^2$ to $y = 2 - x$ & x varies from $x = 0$ to $x = 1$

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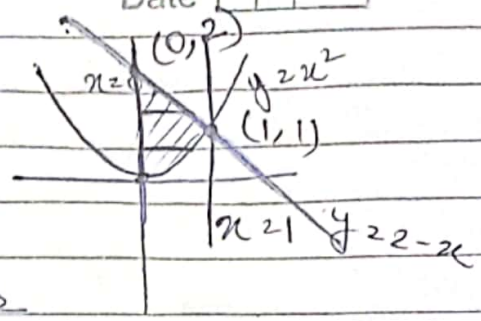
To find point of intersection of $y = 2-x$ & $y = x^2$



$\Rightarrow 2-x = x^2$

$x^2+x-2 = 0$

$x = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$

$x = 1, 2$

when $x=1$, $y=1$

$$\Rightarrow \int_0^1\int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2\int_0^{2-y} xy\,dx\,dy$$

$$= \frac{1}{2}\int_0^1 y\left[x^2\right]_0^{\sqrt{y}} dy + \frac{1}{2}\int_1^2 y\left[x^2\right]_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 (y^3+4y-4y^2)\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[\frac{y^4}{4} + \frac{4y^2}{2} - \frac{4y^3}{3}\right]_1^2$$

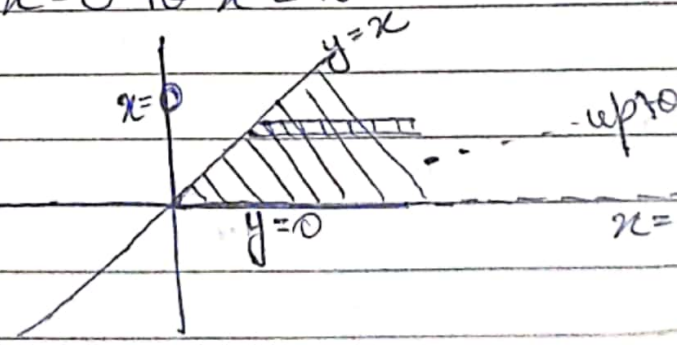
$$= \frac{1}{6}(1-0) + \frac{1}{2}\left[\left(4+8-\frac{32}{3}\right) - \left(\frac{1}{4}+2-\frac{4}{3}\right)\right]$$

$$= \frac{1}{6} + \frac{5}{12} = \frac{2+5}{12} = \boxed{\frac{7}{12}}$$ Ans

Ques. Evaluate $\int_0^\infty\int_0^x xe^{-x^2/y}\,dy\,dx$ by changing the order of integration.

Sol. Here y varies from $y=0$ to $y=x$ and x varies from $x=0$ to $x=\infty$

$$\Rightarrow \int_0^\infty\int_y^\infty xe^{-x^2/y}\,dx\,dy$$



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let $x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$$\Rightarrow \int_0^\infty \int_y^\infty \frac{1}{2} e^{-t/y}\, dt\, dy$$

$$= \int_0^\infty \frac{1}{2}\left[\frac{e^{-t/y}}{-1/y}\right]_y^\infty dy = \frac{1}{2}\int_0^\infty \left[\frac{e^{-x^2/y}}{-1/y}\right]_y^\infty dy$$

$$= -\frac{y}{2}\int_0^\infty \left(e^{-\infty} - e^{-y}\right) dy$$

$$= -\frac{y}{2}\int_0^\infty \left(0 - e^{-y}\right) dy$$

$$= \frac{1}{2}\int_0^\infty \underset{I}{y}\,\underset{II}{e^{-y}}\, dy$$

By using by parts integration

$$= \frac{1}{2}\left[y\int e^{-y}dy - \int 1 \cdot \int e^{-y}dy\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{ye^{-y}}{-1} + \int \frac{e^{-y}}{-1}\, dy\right]_0^\infty$$

$$= \frac{1}{2}\left[-ye^{-y} + \frac{e^{-y}}{-1}\right]_0^\infty$$

$$= -\frac{1}{2}\left[ye^{-y} + e^{-y}\right]_0^\infty$$

$$= -\frac{1}{2}\left[(0+0) - (\cancel{y}^{0} + 1)\right]$$

$$= -\frac{1}{2}(-1) = \boxed{\frac{1}{2}}$$ Ans

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Triple Integration :-

$$\Rightarrow \int_{c}^{c}\int_{cv}^{cv}\int_{cv}^{cv} dx\,dy\,dz$$

here c ← constant
v ← variable

Ques- Evaluate $\int_0^2\int_1^2\int_0^{yz} xyz\,dx\,dy\,dz$

Sol- $\int_0^2\int_1^2\int_0^{yz} xyz\,dx\,dy\,dz = \int_0^2\int_1^2 \left[\frac{x^2}{2}yz\right]_0^{yz} dy\,dz$

$$= \frac{1}{2}\int_0^2\int_1^2 y^3z^3\,dy\,dz$$

$$= \frac{1}{2}\int_0^2 \left[z^3\frac{y^4}{4}\right]_1^2 dz$$

$$= \frac{1}{8}\int_0^2 (16-1)z^3\,dz$$

$$= \frac{1}{8}\int_0^2 15z^3\,dz$$

$$= \frac{1}{8}\left[\frac{15z^4}{4}\right]_0^2$$

$$= \frac{1}{8}\left[15(4-0)\right]$$

$$= \frac{1}{\cancel{8}_{42}} \times \cancel{60}^{\cancel{30}\,15} = \boxed{\frac{15}{2}}$$ Ans

Ques- Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by planes :- $x=0$, $y=0$, $z=0$, $x+y+z=1$

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Sol- $x=0,\ y=0,\ z=0 \longrightarrow$ lower limit is fixed i.e 0.

$x+y+z=1$ —— (1)

$\boxed{x=1-y-z} \rightarrow$ upper limit of x

Now, Put $x=0$ in (1)

$\boxed{y=1-z} \rightarrow$ upper limit of y

Put $x=0,\ y=0$ in (1)

$\boxed{z=1} \rightarrow$ upper limit of z

$$\Rightarrow \int_0^1\int_0^{1-z}\int_0^{1-y-z} (x+y+z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^{1-z} \left[\frac{x^2}{2} + yx + zx\right]_0^{1-y-z} dy\,dz$$

$$= \int_0^1\int_0^{1-z} \left[\frac{(1-y-z)^2}{2} + y(1-y-z) + z(1-y-z)\right] dy\,dz$$

$$= \int_0^1\int_0^{1-z} \left[\frac{1}{2}\left((1-y-z)(1-y-z)\right) + y - y^2 - yz + z - yz - z^2\right] dy\,dz$$

$$= \int_0^1\int_0^{1-z} \left[\frac{1}{2}\left(1-y-z-y+y^2+yz-z+yz+z^2\right) + y - y^2 - yz + z - yz - z^2\right] dy\,dz$$

$$= \int_0^1\int_0^{1-z} \left[\frac{1}{2}\left(1-2y-2z+2yz+y^2+z^2\right) + y+z-y^2 - z^2 - 2yz\right] dy\,dz$$

$$= \int_0^1\int_0^{1-z} \left(\frac{1}{2} - \cancel{y} - \cancel{z} + yz + \frac{y^2}{2} + \frac{z^2}{2} + \cancel{y} + \cancel{z} - y^2 - z^2 - 2yz\right) dy\,dz$$

$$= \int_0^1\int_0^{1-z} \left(\frac{1}{2} - yz - \frac{y^2}{2} - \frac{z^2}{2}\right) dy\,dz$$

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$$= \int_0^1 \left[\frac{y}{2} - \frac{y^2}{2} z - \frac{y^3}{6} - \frac{z^3}{6} \right]_0^{1-z} dz$$

$$= \int_0^1 \left[\frac{1-z}{2} - \frac{(1-z)^2}{2} z - \frac{(1-z)^3}{6} - \frac{(1-z)^3}{6} \right] dz$$

$$= \int_0^1 \left[\left(\frac{1-z}{2}\right) - \left(\frac{1+z^3-2z^2}{2}\right) - \left(\frac{1-z^3-3z(1-z)}{6}\right) - \left(\frac{1-z^3-3z(1-z)}{6}\right) \right] dz$$

$$= \int_0^1 \left[\frac{1}{2} - \frac{z}{2} - \frac{z}{2} - \frac{z^3}{2} + z^2 - \frac{1}{6} + \frac{z^3}{6} + \frac{z}{2}(1-z) - \frac{1}{6} + \frac{z^3}{6} + \frac{z}{2}(1-z) \right] dz$$

$$= \int_0^1 \left(\frac{1}{2} - \frac{1}{6} - \frac{1}{6} - z + z(1-z) + z^2 - \frac{z^3}{2} + \frac{z^3}{6} + \frac{z^3}{6} \right) dz$$

$$= \int_0^1 \left(\frac{1}{6} - \cancel{z} + \cancel{z} - \cancel{z^2} + \cancel{z^2} - \frac{z^3}{6} \right) dz$$

$$= \int_0^1 \left(\frac{1}{6} - \frac{z^3}{6} \right) dz = \left[\frac{z}{6} - \frac{z^4}{24} \right]_0^1$$

$$= \left(\frac{1}{6} - \frac{1}{24} \right) - (0) = \frac{4-1}{24} = \frac{\cancel{3}^{1}}{\cancel{24}_{8}}$$

$$= \boxed{\frac{1}{8}}$$ Ans

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* $\int f(z)\,dz = 0$, if $f(z)$ is odd.

Date

Ques- Evaluate $\iiint (z^5+z)\,dx\,dy\,dz$ bounded by the curve $x^2+y^2+z^2 \leq 1$

Sol- $\Rightarrow \iiint (z^5+z)\,dz\,dy\,dx$

To find limit, $x^2+y^2+z^2 \leq 1$

$z^2 \leq 1-x^2-y^2 \Rightarrow z^2 = 1-x^2-y^2$

$z = \pm(1-x^2-y^2)^{1/2}$

So, $-\sqrt{1-x^2-y^2} \leq z \leq \sqrt{1-x^2-y^2}$

Now, Put $z=0$, $x^2+y^2 \leq 1 \Rightarrow y^2 \leq 1-x^2$

$-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$

Now, Put $z=0$, $y=0$, $x^2 \leq 1 \Rightarrow$ $x=1$, $x=\pm 1$

$-1 \leq x \leq 1$

$$\Rightarrow \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (z^5+z)\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[\frac{z^6}{6}+\frac{z^2}{2}\right]_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \cancel{\frac{(1-x^2-y^2)^3}{6}} + \cancel{\frac{(1-x^2-y^2)^2}{2}} - \cancel{\frac{(1-x^2-y^2)^3}{6}} - \cancel{\frac{(1-x^2-y^2)^2}{2}}\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 0\,dy\,dx$$

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$\star \int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$ or $\int_0^{\frac{\pi}{2}} \cos^n\theta \, d\theta = \frac{(n-1)(n-3)(n-5)\cdots 2}{n(n-2)(n-4)\cdots 1}$, when n is odd

$= \frac{\pi}{2}\frac{(n-1)(n-3)(n-5)\cdots 1}{n(n-2)(n-4)\cdots 2}$, when n is even

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Ques- Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\int_0^{a^2-r^2} r \, dz \, dr \, d\theta$

Sol- $\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta} r\,[z]_0^{a^2-r^2} \, dr \, d\theta = \int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta} (a^2r - r^3) \, dr \, d\theta$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_0^{a\sin\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{a^2}{2}(a^2\sin^2\theta) - \frac{a^4\sin^4\theta}{4}\right) d\theta$$

$$= \frac{a^4}{2}\int_0^{\frac{\pi}{2}} \left(\sin^2\theta - \frac{\sin^4\theta}{2}\right) d\theta$$

$$= \frac{a^4}{2}\left[\int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta\right]$$

$$= \frac{a^4}{2}\left[\frac{(2-1)}{2}\frac{\pi}{2} - \frac{1}{2}\left(\frac{(4-1)(4-3)}{4(4-2)}\right)\frac{\pi}{2}\right]$$

$$= \frac{a^4}{2}\left(\frac{1}{2} - \frac{3}{16}\right)\frac{\pi}{2} = \boxed{\frac{5a^4\pi}{64}}$$ Ans

Ques- Evaluate the following after changing the order of integration.

$$\int_0^a \int_{y^2/a}^{y} \frac{y \, dx \, dy}{(a-x)\sqrt{ax-y^2}}$$

Sol- Here, x varies from $x = y^2/a$ to $x = y$

& y varies from $y = 0$ to $y = a$

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To find intersection point of

$x = y$ & $x = y^2/a$

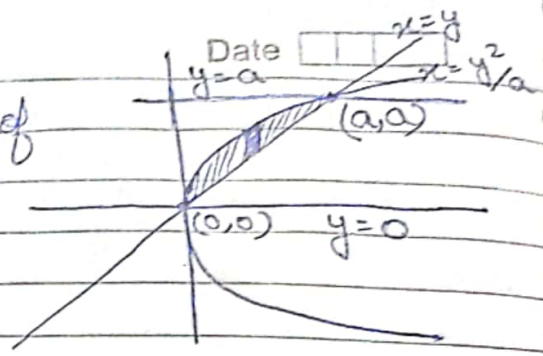
$y = y^2/a$

$y^2 = ay$

$y^2 - ay = 0$

$y(y-a) = 0$

$y = 0$ or $y = a$



Now, y varies from $y = x$ to $y = \sqrt{ax}$

& x varies from $x = 0$ to $x = a$

$$\Rightarrow \int_0^a \int_x^{\sqrt{ax}} \frac{y \, dy \, dx}{(a-x)\sqrt{ax-y^2}}$$

$$= \int_0^a \frac{1}{(a-x)} \int_x^{\sqrt{ax}} \frac{y \, dy}{\sqrt{ax-y^2}} \, dx$$

Now, Put $ax - y^2 = t$

$-2y \, dy = dt$

$y \, dy = -\frac{dt}{2}$

$$= \int_0^a \frac{-1}{2(a-x)} \int_x^{\sqrt{ax}} \frac{dt}{\sqrt{t}} \, dx = -\frac{1}{2} \int_0^a \frac{1}{a-x} \left[\frac{t^{1/2}}{1/2} \right]_x^{\sqrt{ax}} dx$$

$$= -\frac{1}{2} \int_0^a \frac{1}{a-x} \left[\frac{(ax-y^2)^{1/2}}{1/2} \right]_x^{\sqrt{ax}} dx$$

$$= \frac{-1}{2} \int_0^a \frac{-\sqrt{ax-x^2}}{\frac{1}{2}(a-x)} \, dx$$

$$= \int_0^a \frac{1}{a-x} \frac{\sqrt{x}\sqrt{a-x}}{\sqrt{a-x}} \, dx$$

$$= \int_0^a \frac{\sqrt{x}}{\sqrt{a-x}} \, dx$$

Ques

Sol

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Put $x = a\sin^2\theta$ $\qquad x \to 0, \theta \to 0$

$dx = 2a\sin\theta\cos\theta\, d\theta$ $\qquad x \to a, \theta \to \frac{\pi}{2}$

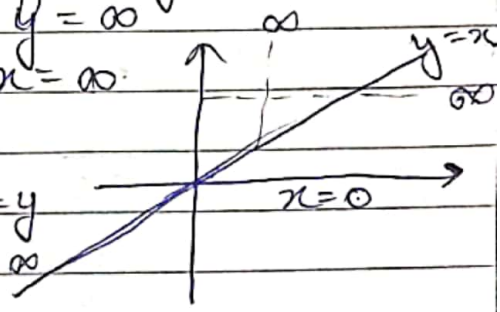
$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{a}\sin\theta}{\sqrt{a}\cos\theta}\, 2a\sin\theta\cos\theta\, d\theta$$

$$= 2a\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$$

$$= \cancel{2}a\,\frac{(2-1)}{\cancel{2}}\cdot\frac{\pi}{2} = \boxed{\frac{a\pi}{2}}$$ Ans

Ques- Evaluate $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ by changing the order of integration.

Sol- Here y varies from $y = x$ to $y = \infty$
& x varies from $x = 0$ to $x = \infty$.



Now, x varies from $x = 0$ to $x = y$
& y varies from $y = 0$ to $y = \infty$

$$\Rightarrow \int_0^\infty\int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\,[x]_0^y\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{\cancel{y}}\cdot \cancel{y}\, dy = \left[\frac{e^{-y}}{-1}\right]_0^\infty$$

$$= -(e^{-\infty} - e^0) = -(0-1)$$

$$= \boxed{1}$$ Ans

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* Change of Variables :-

① To change cartesian (x,y) co-ordinates to polar (r,θ) co-ordinates :-

$$\iint f(x,y)\,dx\,dy = \iint f(r\cos\theta, r\sin\theta)\,|J|\,dr\,d\theta$$

Here, $x \to r\cos\theta$, $y \to r\sin\theta$

$$dx\,dy = |J|\,dr\,d\theta$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$x = r\cos\theta$

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$y = r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = \boxed{r}$$

So,

$$\Rightarrow \iint f(x,y)\,dx\,dy = \iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

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② To change cartesian co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ) :-

Let $x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

so, $dx\,dy\,dz = |J|\, dr\, d\theta\, d\phi$

where $J = \dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$

$$= \begin{vmatrix} \sin\theta \cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\,(0 + r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi\,(0 - r\sin\theta\cos\theta\cos\phi)$$
$$- r\sin\theta\sin\phi\,(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)$$

$$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin\theta\sin^2\phi(1)$$

$$= r^2\sin\theta\sin^2\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin\theta\sin^2\phi$$

$$= r^2\sin\theta\cos^2\phi\,(\sin^2\theta + \cos^2\theta) + r^2\sin\theta\sin^2\phi$$

$$= r^2\sin\theta\,(\cos^2\phi + \sin^2\phi)$$

$$= r^2\sin\theta.$$

so,

$$\iiint f(x, y, z)\,dx\,dy\,dz = \iiint f(r, \theta, \phi)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

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③ To change cartesian co-ordinates (x, y, z) to cylendrical co-ordinates (r, ϕ, z)

here, $x = r\cos\phi$

$y = r\sin\phi$

$z = z$

& $dx\,dy\,dz = |J|\,dr\,d\phi\,dz$

where $J = \dfrac{\partial(x,y,z)}{\partial(r,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix}$

$= \begin{vmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= \cos\phi\,(r\cos\phi) + r\sin\phi\,(\sin\phi)$

$= r\cos^2\phi + r\sin^2\phi$

$= r$

So,

$$\iiint f(x,y,z)\,dx\,dy\,dz = \iiint f(r,\phi,z)\,r\,dr\,d\theta\,dz$$

Ques:- Evaluate $\iint \sqrt{a^2-x^2-y^2}\,dx\,dy$ over the semi-circle $x^2+y^2 = ax$ in +ve quadrant by changing the variable.

Sol- let $x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2 = r^2$

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$* \int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1}$

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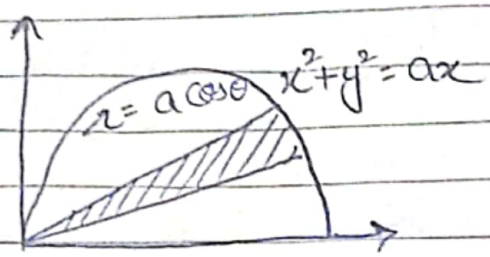
$x^2+y^2=ax$

So, $r^2 = ax$

$r^2 = ar\cos\theta$

$r = a\cos\theta$



So, r varies from 0 to $a\cos\theta$ &
θ varies from 0 to $\pi/2$

$$\Rightarrow \int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2-r^2}\; r\,dr\,d\theta$$

Put $a^2-r^2 = t$

$-2r\,dr = dt$

$r\,dr = \frac{-dt}{2}$

$$= \frac{-1}{2}\int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{t}\,dt\,d\theta = \frac{-1}{2}\int_0^{\pi/2}\left[\frac{t^{3/2}}{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{-1}{3}\int_0^{\pi/2}\left[(a^2-r^2)^{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{-1}{3}\int_0^{\pi/2}(a^2-a^2\cos^2\theta)^{3/2}-(a^2)^{3/2}\,d\theta$$

$$= \frac{-1}{3}\int_0^{\pi/2}(a^3\sin^3\theta - a^3)\,d\theta$$

$$= \frac{-1}{3}\int_0^{\pi/2} a^3(\sin^3\theta - 1)\,d\theta$$

$$= \frac{-a^3}{3}\left[\int_0^{\pi/2}\sin^3\theta\,d\theta - \int_0^{\pi/2} 1\,d\theta\right]$$

$$= \frac{-a^3}{3}\left[\frac{2}{3\cdot 1} - [\theta]_0^{\pi/2}\right]$$

$$= \frac{-a^3}{3}\left(\frac{2}{3}-\frac{\pi}{2}\right) = \boxed{\frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)}$$

Ans

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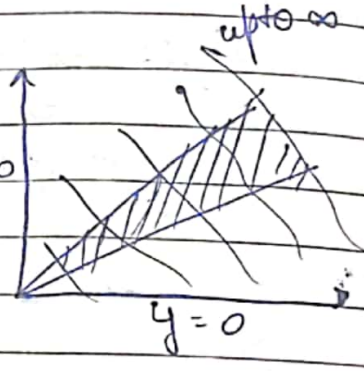
Ques- Change into polar co-ordinates & evaluate-

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dy\, dx$$

Sol- Here, y varies from $y=0$ to $y=\infty$
& x varies from $x=0$ to $x=\infty$



Changing into polar co-ordinates-
$x = r\cos\theta$, $y = r\sin\theta$
so, $x^2+y^2 = r^2$

$\therefore$ r varies from 0 to ∞ &
θ varies from 0 to $\frac{\pi}{2}$

$$\Rightarrow \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\, r\, dr\, d\theta =$$

Put $r^2 = t$ $\quad$ $r=0,\ t=0$
$2r\,dr = dt$ $\quad$ $r=\infty,\ t=\infty$
$r\,dr = \frac{dt}{2}$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-t}\, dt\, d\theta = \frac{1}{2}\int_0^{\pi/2}\left[\frac{e^{-t}}{-1}\right]_0^\infty d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\frac{e^{-\infty}}{-1} - \frac{e^{0}}{-1}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1\, d\theta = \frac{1}{2}\left[\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \boxed{\frac{\pi}{4}}$$ Ans

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Ques- Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ by changing into polar co-ordinates.

Sol- Here, $y = 0$ to $y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2$

& $x = 0$ to $x = 2$ $\qquad x^2+y^2 = 2x$

Changing into polar co-ordinates-

$x = r\cos\theta, \quad y = r\sin\theta$

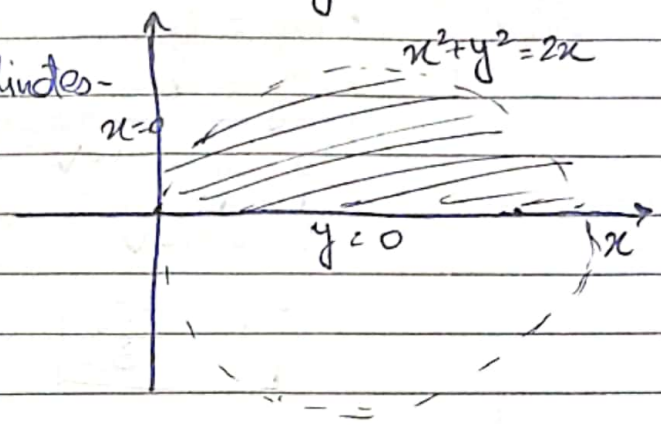
& $x^2+y^2 = r^2$

also, $x^2+y^2 = 2x$

$r^2 = 2x$

$r^2 = 2r\cos\theta$

$\underline{\underline{r = 2\cos\theta}}$



Now, r varies from 0 to $2\cos\theta$ &

θ varies from 0 to $\pi/2$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{\cancel{r}\cos\theta}{\cancel{r}}\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_0^{2\cos\theta} \cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2\cos^3\theta\, d\theta$$

$$= 2\left[\frac{2}{3\cdot 1}\right] = \boxed{\frac{4}{3}} \text{ Ans}$$

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Ques:- Evaluate $\iint_R (x+y)^2\,dx\,dy$ where R is a ||gm with vertices $(1,0), (3,1), (2,2), (0,1)$, using the transformation $u = x+y$ & $v = x-2y$

Sol:-

$x+y = u$

$x-2y = v$

$y+2y = u-v$

$3y = u-v$

$\boxed{y = \frac{u-v}{3}}$

$x + \frac{u-v}{3} = u$

$3x + u - v = 3u$

$3x = 2u - v$

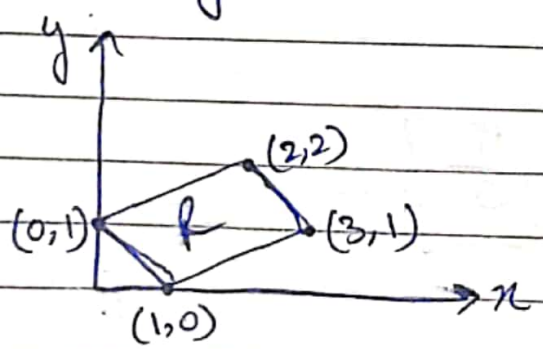
$\boxed{x = \frac{2u-v}{3}}$

$$\Rightarrow \iint f(x,y)\,dx\,dy = \iint f(u,v)\,|J|\,du\,dv$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{vmatrix} = -2/9 - 1/9 = -3/9 = \boxed{-1/3}$$

$x-y$ curve:-



$u = x+y$

$v = x-2y$

at $(0,1) \Rightarrow u=1, v=-2$

at $(1,0) \Rightarrow u=1, v=1$

at $(2,2) \Rightarrow u=4, v=-2$

at $(3,1) \Rightarrow u=4, v=1$

So, vertices for $u-v$ curve are -

$(1,-2), (1,1), (4,-2), (4,1)$

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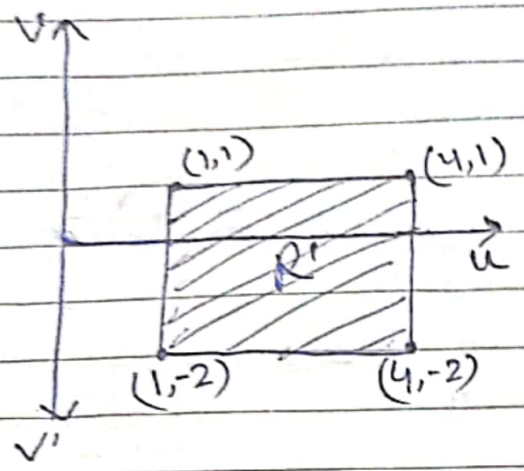
u-v curve :-



$$\Rightarrow \iint_{R'} u^2 |J| \, du \, dv \qquad \left(\begin{array}{l} J = -\frac{1}{3} \\ |J| = \frac{1}{3} \end{array} \right)$$

$$= \int_{-2}^{1} \int_{1}^{4} u^2 \left(\frac{1}{3}\right) du \, dv$$

$$= \frac{1}{3} \int_{-2}^{1} \left[\frac{u^3}{3}\right]_1^4 dv = \frac{1}{9} \int_{-2}^{1} (64-1) \, dv$$

$$= \frac{1}{9} \int_{-2}^{1} 63 \, dv$$

$$= \frac{63}{9} \left[v \right]_{-2}^{1} = \frac{\cancel{63}^{\cancel{21}\,7}}{\cancel{9}_{\cancel{3}\,1}} (1+2)$$

$$= 7(3) = \boxed{21} \text{ Ans}$$

Ques- Show that $\iiint (x^2+y^2+z^2)^m \, dx \, dy \, dz = \frac{4\pi}{2m+3}$

over the region $x^2+y^2+z^2 \leq 1$, $m > 0$

Sol- Changing into spherical polar co-ordinates

$x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$

$z = r \cos\theta$

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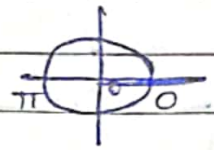
So, $x^2+y^2+z^2 = r^2$ & $dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$

Now, the given curve is-

$$x^2+y^2+z^2 \leq 1$$
$$r^2 \leq 1$$



So, $r = 0$ to $r = 1$

$\phi = 0$ to $\phi = 2\pi$] → standard.

$\theta = 0$ to $\theta = \pi$ acc to curve

LHS $\Rightarrow \int_0^1 \int_0^{2\pi} \int_0^{\pi} (r^2)^m r^2 \sin\theta \, d\theta\, d\phi\, dr$

$$= \int_0^1 \int_0^{2\pi} \int_0^{\pi} r^{2m+2} \sin\theta \, d\theta\, d\phi\, dr$$

$$= \int_0^1 \int_0^{2\pi} \left[-\cos\theta\right]_0^{\pi} r^{2m+2}\, d\phi\, dr$$

$$= \int_0^1 \int_0^{2\pi} (1+1)\, r^{2m+2}\, d\phi\, dr$$

$$= 2\int_0^1 \int_0^{2\pi} r^{2m+2}\, d\phi\, dr = 2\int_0^1 r^{2m+2} \left[\phi\right]_0^{2\pi} dr$$

$$= 4\pi \int_0^1 r^{2m+2}\, dr = 4\pi \left[\frac{r^{2m+3}}{2m+3}\right]_0^1$$

$$= \frac{4\pi}{2m+3} \rightarrow \text{RHS}$$

LHS = RHS Hence proved

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Ques- Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$ by changing into spherical polar coordinates

Sol- let $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

& $x^2+y^2+z^2 = r^2$, $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

here, $z = 0$ to $z = \sqrt{1-x^2-y^2} \Rightarrow z^2 = 1-x^2-y^2$

$x^2+y^2+z^2 = 1$

$r^2 = 1$

$\therefore 0 \le r \le 1$

$y = 0$ to $y = \sqrt{1-x^2}$

$\Rightarrow y^2 = 1-x^2$

$x^2+y^2 = 1$

$x^2+y^2+z^2 = a^2$

$0 \le r \le a,\ 0 \le \theta \le \pi$

$0 \le \phi \le 2\pi$

Now, r varies from 0 to 1,

ϕ varies from 0 to $\pi/2$

θ varies from 0 to $\pi/2$

$$\Rightarrow \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2\sin\theta}{\sqrt{1-r^2}}\,dr\,d\theta\,d\phi$$

$$= -\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{1-r^2-1}{\sqrt{1-r^2}}\sin\theta\,dr\,d\theta\,d\phi$$

$$= -\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \sin\theta\left[\underbrace{\sqrt{1-r^2}}_{\sqrt{a^2-x^2}} - \underbrace{\frac{1}{\sqrt{1-r^2}}}_{\frac{1}{\sqrt{a^2-x^2}}}\right]dr\,d\theta\,d\phi$$

$$= -\int_0^{\pi/2}\int_0^{\pi/2}\sin\theta\left[\frac{r}{2}\sqrt{1-r^2}+\frac{1}{2}\sin^{-1}r - \sin^{-1}r\right]_0^1 d$$

$$= -\int_0^{\pi/2}\int_0^{\pi/2}\sin\theta\left(\frac{\pi}{4}-\frac{\pi}{2}\right)d\theta\,d\phi$$

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$$= \frac{\pi}{4}\int_0^{\pi/2} \left[-\cos\theta\right]_0^{\pi/2} d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/2} 1\, d\phi = \frac{\pi}{4}\left[\phi\right]_0^{\pi/2} = \boxed{\frac{\pi^2}{8}}$$ Ans

Ques:- Evaluate $\iiint z(x^2+y^2+z^2)\,dx\,dy\,dz$, $x^2+y^2=a^2$ intercepted by the plane $z=0$ & $z=h$

Sol:- let $x = r\cos\phi$, $y = r\sin\phi$, $z = z$

& $x^2+y^2 = r^2$ & $dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = a^2$

$r^2 = a^2 \Rightarrow 0 \le r \le a$

$$\Rightarrow \int_0^h\int_0^{2\pi}\int_0^a z(r^2+z^2)\, r\, dr\, d\phi\, dz$$

$$= \int_0^h\int_0^{2\pi}\int_0^a (zr^3 + z^3 r)\, dr\, d\phi\, dz$$

$$= \int_0^h\int_0^{2\pi} \left[z\frac{r^4}{4} + z^3\frac{r^2}{2}\right]_0^a d\phi\, dz$$

$$= \int_0^h\int_0^{2\pi} \left(z\frac{a^4}{4} + z^3\frac{a^2}{2}\right) d\phi\, dz$$

$$= \int_0^h \left[\phi\right]_0^{2\pi}\left(\frac{za^4}{4} + z^3\frac{a^2}{2}\right) dz$$

$$= \int_0^h \cancel{2}\pi\left(\frac{za^4}{\cancel{4}_2} + z^3\frac{a^2}{\cancel{2}_1}\right) dz = \pi\int_0^h\left(\frac{za^4}{2} + z^3a^2\right) dz$$

$$= \pi\left[\frac{z^2}{4}a^4 + \frac{z^4}{4}a^2\right]_0^h = \pi\left(\frac{h^2a^4}{4} + \frac{h^4a^2}{4}\right)$$

$$= \boxed{\frac{\pi}{4}\left(h^2a^4 + h^4a^2\right)}$$ Ans

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Area and Volume by Double Integration :-

Area

(i) for cartesian coordinates (x, y)

$$\text{Area} = \iint dy\,dx \quad \text{or} \quad \iint dx\,dy$$

(ii) for polar coordinates (r, θ)

$$\text{Area} = \iint r\,dr\,d\theta$$

Volume

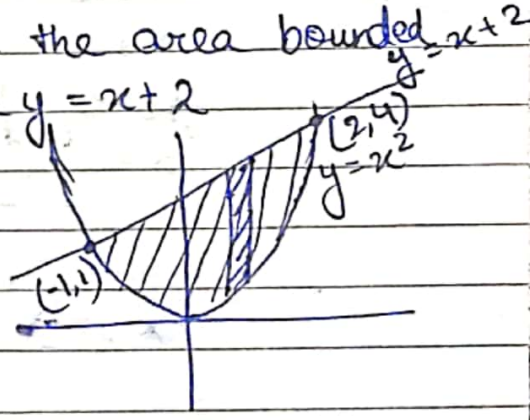
(i) for cartesian,

$$\text{Volume} = \iint z\,dx\,dy$$

(ii) for cylendrical polar,

$$\text{Volume} = \iint z\,dr\,d\phi$$

Ques- Using double integration, find the area bounded b/w curve $y = x^2$ and the line $y = x + 2$



Sol- To find point of intersection

Put $y = x+2$ in $y = x^2$

so, $x + 2 = x^2$

$$x^2 - x - 2 = 0$$

$$x = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

$$x = -1, 2$$

Now, when $x = -1$, $y = 1$ $\quad (-1, 1)$

& when $x = 2$, $y = 4$ $\quad (2, 4)$

So, $$\text{Area} = \int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx$$

$$A = \int_{-1}^{2} [y]_{x^2}^{x+2}\,dx = \int_{-1}^{2} (x + 2 - x^2)\,dx$$

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$$A = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= \left(+6 - \frac{8}{3}\right) - \left(-\frac{3}{2} + \frac{1}{3}\right)$$

$$= \left(\frac{+10}{3}\right) - \left(\frac{-7}{6}\right)$$

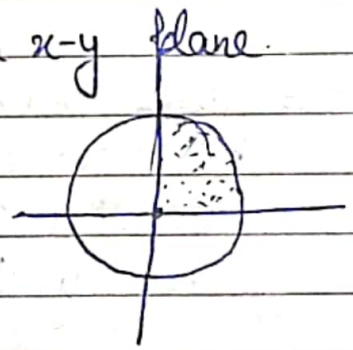
$$= \frac{+20+7}{6} = \frac{27}{6} = \boxed{\frac{9}{2}} \text{ Ans}$$

Ques = Find the volume of the sphere $x^2+y^2+z^2=9$ using double integration.

Sol- We have to find its volume in x-y plane.

so, $z = 0$

Eqn of sphere in x-y plane is - $\boxed{x^2+y^2=9}$



$\therefore$ $y^2 = 9 - x^2 \Rightarrow y = \pm\sqrt{9-x^2}$

$x^2 = 9 \Rightarrow x = \pm 3$

Now, the sphere is symmetrical about its axis -

$\therefore$ its volume = 8 times volume in Ist quadrant.

so, $V = 8\iint z\,dy\,dx$

$$= 8\int_0^3\int_0^{\sqrt{9-x^2}} z\,dy\,dx$$

from given eqn, $x^2+y^2+z^2=9$

$z = \sqrt{9-x^2-y^2}$

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$$V = 8\int_0^3\int_0^{\sqrt{9-x^2}} \sqrt{9-x^2-y^2}\, dy\, dx$$

Now, put $x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2 = r^2$, $dy\,dx = r\,d\theta\,dr$

$0 \le r \le 3$ & $0 \le \theta \le \pi/2$

$$V = 8\int_0^{\pi/2}\int_0^3 \sqrt{9-r^2}\, r\, dr\, d\theta$$

$$V = \frac{-8}{2}\int_0^{\pi/2}\int_0^3 (9-r^2)^{1/2}(-2r)\, dr\, d\theta$$

& we know that

$$\int (f(x))^n f'(x)\, dx = \frac{(f(x))^{n+1}}{n+1}$$

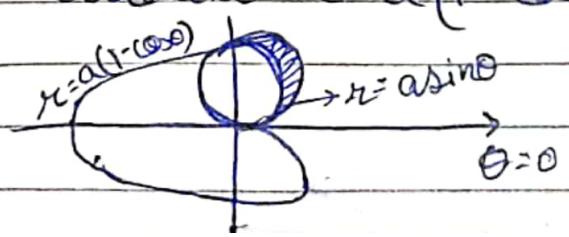
$$V = -4\int_0^{\pi/2}\left[\frac{(9-r^2)^{3/2}}{3/2}\right]_0^3 d\theta$$

$$= +4\int_0^{\pi/2} +\frac{\cancel{27}^{9}\times 2}{\cancel{3}}\, d\theta$$

$$= 4\left[18\theta\right]_0^{\pi/2}$$

$$= 4\left(\frac{\cancel{18}^{9}\pi}{2} - 0\right) = \boxed{36\pi}$$ Ans

Ques- find the area ~~and inside~~ lying outside the circle $r = a\sin\theta$ and ~~outside~~ inside the cardioid $r = a(1-\cos\theta)$



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$r = a(1 - \cos\theta)$

$r = a\sin\theta$

$\cancel{a}\sin\theta = \cancel{a}(1 - \cos\theta)$

$\sin\theta = 1 - \cos\theta$

$\sin\theta + \cos\theta = 1$

Squaring both sides,

$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1$

$1 + \sin 2\theta = 1$

$\sin 2\theta = 0 \Rightarrow 2\theta = 0, \pi$

$\theta = 0, \frac{\pi}{2}$

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~~Ques. find the area lying inside the cardioid $r = a(1-\cos\theta)$ and outside~~

$$A = \int_0^a \int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$A = \int_0^a \frac{1}{2}\left[r^2\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

Ques:- find the area by double integration bounded by the curve $x = 2y - y^2$ and $x = y^2$.

Sol:- Point of intersection

$2y - y^2 = y^2$

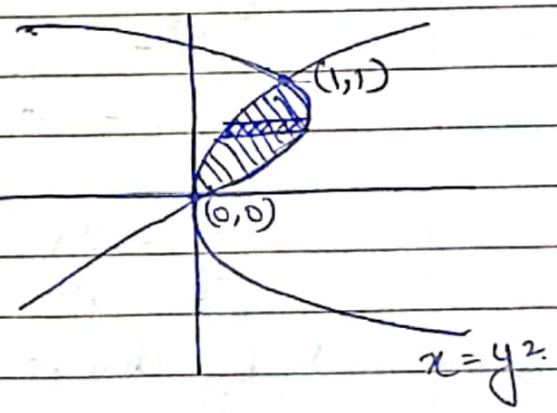
$2y^2 - 2y = 0$

$2y(y-1) = 0$

$y = 0, 1$

when $y = 0$, $x = 0$ $(0,0)$

when $y = 1$, $x = 1$ $(1,1)$



$$A = \int_0^1 \int_{y^2}^{2y-y^2} 1\,dx\,dy$$

$$= \int_0^1 \left[x\right]_{y^2}^{2y-y^2} dy = \int_0^1 (2y - y^2 - y^2)\,dy$$

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$$= \int_0^1 (2y - 2y^2)\,dy$$

$$= \left[y^2 - \frac{2}{3}y^3\right]_0^1$$

$$= \left(1 - \frac{2}{3}\right) = \boxed{\frac{1}{3}}$$ Ans

Ques- find the area enclosed by $y^2 = x^3$ and $y = x$.

Sol- Point of intersection

$y^2 = x^2$

$x^3 = x^2$

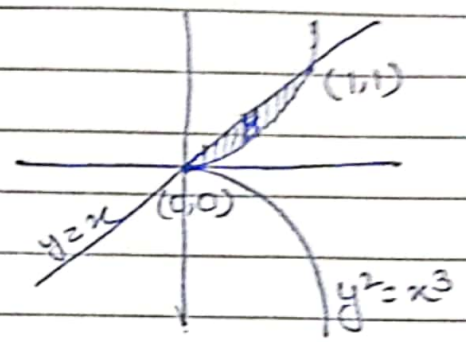
$x^3 - x^2 = 0$

$x^2(x-1) = 0$

$x = 0, 1$

when $x = 0$, $y = 0$ $(0,0)$

when $x = 1$, $y = 1$ $(1,1)$



$$A = \int_0^1 \int_{x^{3/2}}^{x} 1\,dy\,dx = \int_0^1 [y]_{x^{3/2}}^{x}$$

$$= \int_0^1 (x - x^{3/2})\,dx = \left[\frac{x^2}{2} - \frac{x^{5/2}}{5/2}\right]_0^1$$

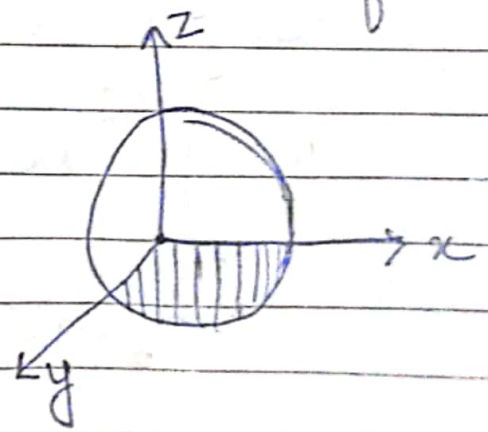
$$= \left(\frac{1}{2} - \frac{2}{5}\right) = \frac{5-4}{10} = \boxed{\frac{1}{10}}$$ Ans

Ques- find by double integration, the volume of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Sol- $$z = \pm c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

Volume = 8 times the vol. in +ve quadrant



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Eqn

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \; ; \; z = 0$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) \Rightarrow y = \pm b\sqrt{1 - x^2/a^2}$$

$\therefore$ y will vary from 0 to $b\sqrt{1 - x^2/a^2}$

Now, Put $y = 0 \Rightarrow \frac{x^2}{a^2} = 1$

$$x^2 = a^2 \Rightarrow x = \pm a$$

$\therefore$ x will vary from 0 to a

$$V = 8 \iint z \, dy \, dx$$

$$V = 8 \int_0^a \int_0^{b\sqrt{1 - x^2/a^2}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \, dy \, dx.$$

Put $1 - \frac{x^2}{a^2} = t^2$

$$V = 8 \int_0^a \int_0^{b\sqrt{1 - x^2/a^2}} c\sqrt{t^2 - \frac{y^2}{b^2}} = 8 \int_0^a \int_0^{b\sqrt{1 - x^2/a^2}} c\sqrt{(bt)^2 - y^2} \, dy \, dx$$

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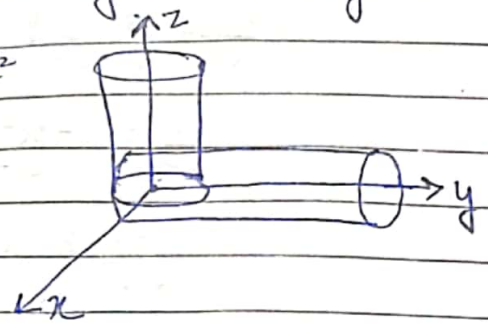
Ques- find the volume common to cylender $x^2+y^2=a^2$ and $x^2+z^2=a^2 \Rightarrow z^2=a^2-x^2$, $z=\pm\sqrt{a^2-x^2}$

Sol- $V = \iint z\,dy\,dx$



As we Know,
Volume = 8 times the vol. in Ist octant

Eqn

$y^2=a^2-x^2 \Rightarrow y=\pm\sqrt{a^2-x^2}$

$\therefore$ y will vary from 0 to $\sqrt{a^2-x^2}$

Now, Put $y=0$

$x^2=a^2 \Rightarrow x=\pm a$

$\therefore$ x will vary from 0 to a

$$V = 8\int_0^a\int_0^{\sqrt{a^2-x^2}} (a^2-x^2)^{1/2}\,dy\,dx$$

$$= 8\int_0^a (a^2-x^2)^{1/2}\,[y]_0^{\sqrt{a^2-x^2}}\,dx$$

$$= 8\int_0^a (a^2-x^2)^{1/2}(a^2-x^2)^{1/2}\,dx$$

$$= 8\int_0^a (a^2-x^2)\,dx$$

$$= 8\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$= 8\left(a^3-\frac{a^3}{3}\right)$$

$$= \frac{8}{3}(3a^3-a^3) = \boxed{\frac{16a^3}{3}}$$ Ans

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* Volume by triple integration :-

(i) for cartesian co-ordinates

$$V = \iiint dx\,dy\,dz$$

(ii) for spherical polar co-ordinates

$$V = \iiint r^2 \sin\theta \, dr\,d\theta\,d\phi$$

(iii) for cylendrical co-ordinates

$$V = \iiint r\,dr\,d\phi\,dz$$

Ques- find (volume) by triple integration of sphere with radius a.

Sol- Eq^n of sphere with radius a

$$x^2+y^2+z^2 = a^2$$

→ Changing into spherical polar co-ordinates.

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\Rightarrow x^2+y^2+z^2 = r^2$$
$$r^2 = a^2 \Rightarrow r = \pm a$$

Since, the sphere is symmetrical about axis,

∴ volume of sphere = 8 times volume in Ist octant

$$V = 8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{a} r^2 \sin\theta \, dr\,d\theta\,d\phi$$

$$= 8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin\theta \left[\frac{r^3}{3}\right]_0^a d\theta\,d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2} \frac{a^3}{3}\sin\theta \, d\theta\,d\phi$$

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$$= \frac{8a^3}{3}\int_0^{\pi/2} \left[-\cos\theta\right]_0^{\pi/2} d\phi$$

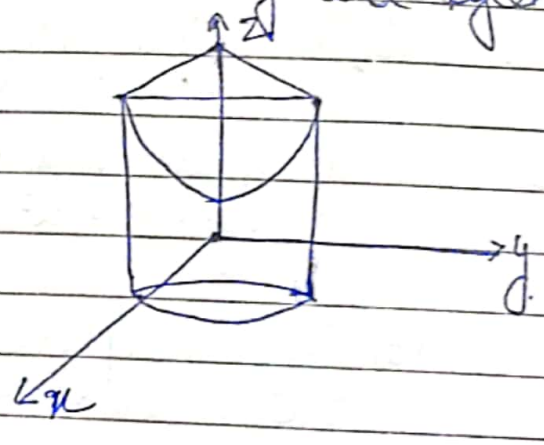
$$= \frac{8a^3}{3}\int_0^{\pi/2} 1\, d\phi$$

$$= \frac{8a^3}{3}\left[\phi\right]_0^{\pi/2}$$

$$= \frac{\cancel{8}^{4}a^3}{3} \times \frac{\pi}{\cancel{2}} = \boxed{\frac{4a^3\pi}{3}}$$ Ans

Ques= find by triple integration the volume of region bounded by paraboloid $az = x^2+y^2$ and cylender $x^2+y^2 = R^2$.

Sol- Changing to cylendrical co-ordinates.



$x = r\cos\phi$, $y = r\sin\phi$,

$z = z$

$x^2+y^2 = r^2$.

$az = r^2 \Rightarrow z = \frac{r^2}{a}$.

$\rightarrow r^2 = R^2 \Rightarrow r = \pm R$

On account of symmetry, the curve is symmetrical about its axis.

Required volume = 4 times vol in Ist octant.

$$V = 4\int_0^{r^2/a}\int_0^{\pi/4}\int_0^{R} r\, dr\, d\phi\, dz$$

$$V = 4\int_0^{r^2/a}\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{R} d\phi\, dz$$

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$$V = 2\int_0^{r^2/a} R^2[\phi]_0^{\pi/2}\, dz$$

$$= 2\int_0^{r^2/a} R^2 \cdot \frac{\pi}{2}\, dz$$

$$= \pi R^2 [z]_0^{r^2/a} = \pi R^2 \cdot \frac{r^2}{a}$$

$$V = 4\int_0^R\int_0^{\pi/2}\int_0^{r^2/a} r\, dz\, d\phi\, dr$$

$$= 4\int_0^R\int_0^{\pi/2} r[z]_0^{r^2/a}\, d\phi\, dr$$

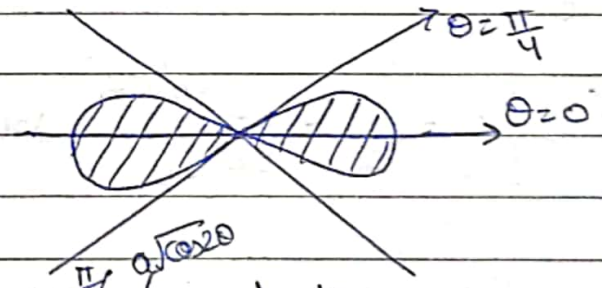
$$= 4\int_0^R\int_0^{\pi/2} \frac{r^3}{a}\, d\phi\, dr$$

$$= 4\int_0^R \frac{r^3}{a}[\phi]_0^{\pi/2}\, dr$$

$$= \cancel{4}^{2}\int_0^R \left(\frac{r^3}{a}\cdot\frac{\pi}{\cancel{2}}\right) dr$$

$$= \frac{2\pi}{a}\left[\frac{r^4}{\cancel{4}_2}\right]_0^R = \boxed{\frac{\pi R^4}{2a}}$$ Ans

Ques- find the area enclosed by Leminscate $r^2 = a^2\cos 2\theta$



$$\text{Area} = 4\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\, dr\, d\theta.$$

$$= \cancel{4}^{2}\int_0^{\pi/4}\left[\frac{r^2}{\cancel{2}}\right]_0^{a\sqrt{\cos 2\theta}} d\theta = 2\int_0^{\pi/4} a^2\cos 2\theta\, d\theta$$

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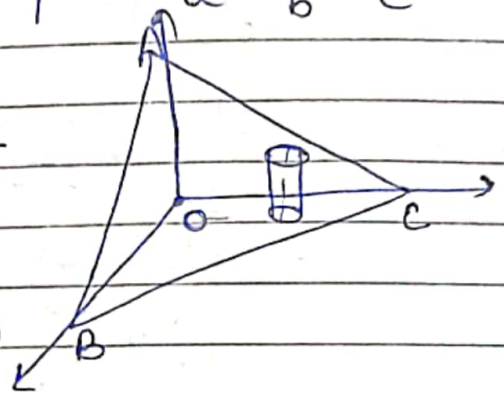
$$A = 2a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= a^2 \left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= a^2(1-0) = \boxed{a^2} \text{ Ans}$$

Ques- Find the volume of the tetrahedron bounded by co-ordinate plan and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Sol- $z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$



In x-y plane, the eq^n is -

$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow y = b\left(1 - \frac{x}{a}\right)$

$\therefore$ y varies from 0 to $b\left(1 - \frac{x}{a}\right)$

Now, put $y = 0$

$\frac{x}{a} = 1 \Rightarrow x = a$

$\therefore$ x varies from 0 to a

$$V = \iint z \, dy \, dx$$

$$V = \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy \, dx$$

$$V = \int_0^a c\left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b(1-\frac{x}{a})} dx$$

$$V = \int_0^a c\left[b\left(1-\frac{x}{a}\right) - \frac{x\,b\left(1-\frac{x}{a}\right)}{a} - \frac{b^{\cancel{2}}\left(1-\frac{x}{a}\right)^2}{2\cancel{b}}\right] dx$$

$$= \int_0^a c\left[b - \frac{bx}{a} - \frac{b\cancel{x}}{\cancel{a}} + \frac{bx^2}{a^2} - \frac{b}{2} - \frac{bx^2}{2a^2} + \frac{\cancel{2}b\cancel{x}}{\cancel{2}\cancel{a}}\right] dx$$

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$$V = \int_0^a c\left[\frac{b}{2} - \frac{bx}{a} + \frac{bx^2}{2a^2}\right] dx$$

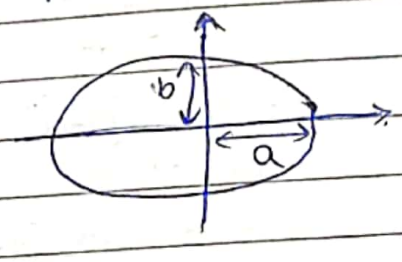
$$V = c\left[\frac{bx}{2} - \frac{bx^2}{2a} + \frac{bx^3}{6a^2}\right]_0^a$$

$$V = c\left[\frac{ab}{2} - \frac{a^2b}{2a} + \frac{a^3b}{6a^2}\right]$$

$$V = c\left[\frac{ab}{2} - \frac{ab}{2} + \frac{ab}{6}\right] = \boxed{\frac{abc}{6}}$$ Ans

Ques- Using double integration, find the area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol- $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$



$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

$\therefore$ y varies from 0 to $b\sqrt{1 - x^2/a^2}$

Now, Put $y = 0$

$\frac{x^2}{a^2} = 1$

$x^2 = a^2$

$x = \pm a$

[Req. area = 4 times the area in Ist octant]

$\therefore$ x varies from 0 to a

So, $$A = 4\int_0^a \int_0^{b\sqrt{1 - x^2/a^2}} dy\, dx$$

$$A = 4\int_0^a [y]_0^{b\sqrt{1 - x^2/a^2}} dx$$

$$A = 4\int_0^a b\sqrt{1 - x^2/a^2}\, dx$$

$$A = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[\frac{a^2\pi}{4}\right] = \boxed{ab\pi}$$ Ans

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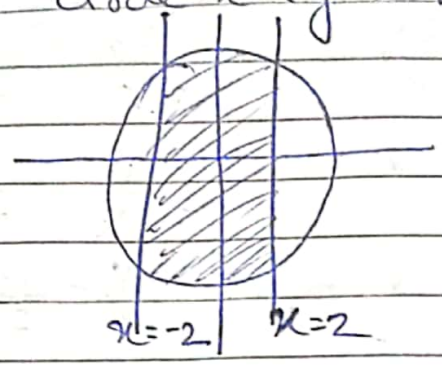
Ques:- find the area of the region bounded by lines $x=2$ & $x=-2$ and the circle $x^2+y^2=9$.

Sol:- $x^2+y^2=9$

$y = \pm\sqrt{9-x^2}$



Req. Area = 4 times area in Ist octant

$$A = 4\int_0^2\int_0^{\sqrt{9-x^2}} dy\,dx$$

$$A = 4\int_0^2 [y]_0^{\sqrt{9-x^2}}\,dx$$

$$A = 4\int_0^2 \sqrt{9-x^2}\,dx$$

$$A = 4\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^2$$

$$A = 4\left[\sqrt{5} + \frac{9}{2}\sin^{-1}\frac{2}{3}\right]$$

$$\boxed{A = 4\sqrt{5} + 18\sin^{-1}\frac{2}{3}}$$ Ans

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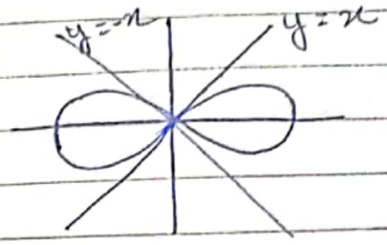
Some Standard Curves

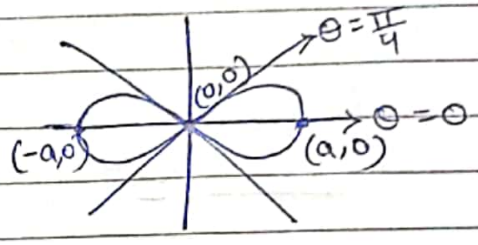
1. $a^2y^2 = x^2(a^2 - x^2)$

Leminscate → Cartesian



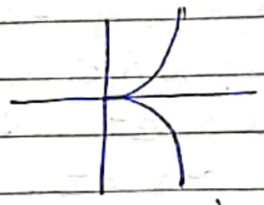
2. $r^2 = a^2\cos 2\theta$

Leminscate → polar



3. $y^2(2a - x) = x^3$

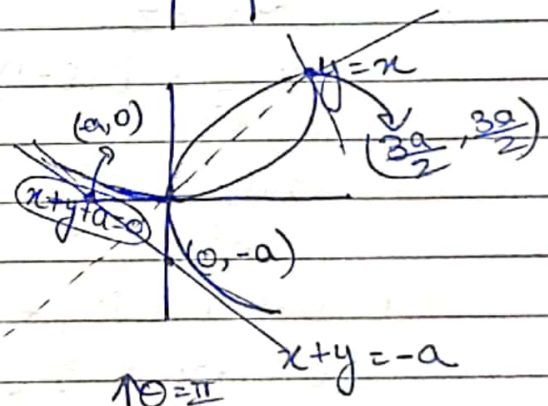
Cissoid



4. $x^3 + y^3 = 3axy$

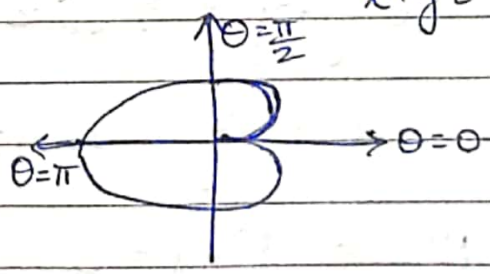
Folium of Descartes

$$\left[\begin{array}{l} x = \frac{3at}{1+t^3}, \\ y = \frac{3at^2}{1+t^3} \end{array}\right]$$

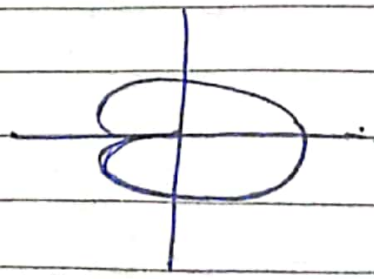


5. $r = a(1 - \cos\theta)$

Cardeoid



6. $r = a(1 + \cos\theta)$



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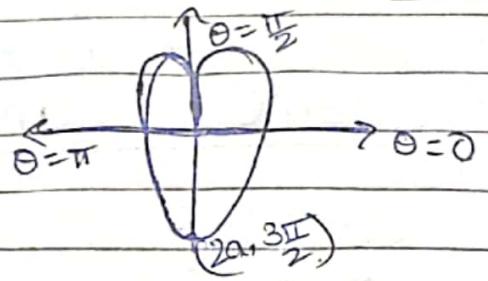
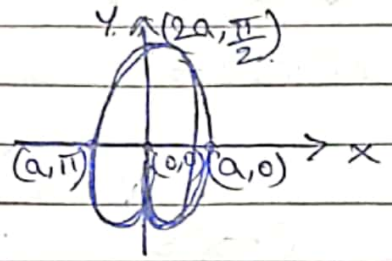
(7) $r = a(1-\sin\theta)$



(8) $r = a(1+\sin\theta)$



Revision:-

Ques- Solve- $x^2y'' + xy' + y = \log x \sin(\log x)$

Sol- We know that given eqn is cauchy's linear eqn

Put $x = e^z$

$\log x = z$

$x^2 \frac{d^2y}{dx^2} = D(D-1)y$

$x\frac{dy}{dx} = Dy$ where $D = \frac{d}{dz}$

$\Rightarrow D(D-1)y + Dy + y = z\sin z$

$D^2y + y = z\sin z$

$(D^2+1)y = z\sin z$

AE $\Rightarrow D^2+1 = 0$

$D^2 = -1$

$D = \pm i$

Cf $\Rightarrow e^{0z}(c_1\cos z + c_2 \sin z)$

$= c_1\cos z + c_2 \sin z$

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$$PI = \frac{1}{D^2+1} z \sin z \qquad \left[xV = \left(x - \frac{f'(D)}{f(D)}\right) \cdot \frac{1}{f(D)} V \right]$$

$$\Rightarrow \frac{1}{(D+i)(D-i)} z \sin z$$

Now, let $\frac{1}{(D+i)(D-i)} = \frac{A}{(D+i)} + \frac{B}{(D-i)}$

$$1 = A(D-i) + B(D+i)$$

Put $D = i$, $B = \frac{1}{2i}$

Put $D = -i$, $A = \frac{-1}{2i}$

So, $PI = \frac{1}{2i}\left[\frac{-1}{D+i} + \frac{1}{D-i}\right] z \sin z$

$$= \frac{1}{2i}\left[\underbrace{-\frac{1}{D+i} z \sin z}_{-I_1} + \underbrace{\frac{1}{D-i} z \sin z}_{I_2}\right]$$

$$I_1 = \frac{1}{D+i}(z \sin z)$$

$$= e^{-iz}\int z \sin z \cdot e^{iz} dz \qquad \left[\because \frac{1}{D+a} x = e^{-ax}\int x e^{ax} dx\right]$$

$$= e^{-iz}\int z \sin z (\cos z + i \sin z) dz$$

$$= e^{-iz}\int (z \sin z \cos z + iz \sin^2 z) dz$$

$$= e^{-iz}\left[\int z \frac{\sin 2z}{2} dz + i\int z \left(\frac{1-\cos 2z}{2}\right) dz\right]$$

$$= e^{-iz}\left[\frac{1}{2}\left[-z\frac{\cos 2z}{2} + \int \frac{1+\cos 2z}{2} dz\right] + \frac{i}{2}\left[\int z dz - \int z \cos 2z dz\right]\right]$$

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$$I_1 = e^{-iz}\left[\frac{1}{2}\left(\frac{-z\cos 2z}{2} + \frac{\sin 2z}{4}\right) + \frac{i}{2}\left(\frac{z^2}{2} - \left(\frac{z\sin 2z}{2} - \int \frac{1\sin 2z\,dz}{2}\right)\right)\right]$$

$$= e^{-iz}\left[\frac{-z\cos 2z}{4} + \frac{\sin 2z}{8} + \frac{i}{2}\left(\frac{z^2}{2} - \left(\frac{z\sin 2z}{2} + \frac{\cos 2z}{4}\right)\right)\right]$$

$$= e^{-iz}\left[\left(\frac{-z\cos 2z}{4} - i\,\frac{z}{4}\sin 2z\right) + \left(\frac{\sin 2z}{8} - \frac{i\cos 2z}{8}\right) + \frac{iz^2}{4}\right]$$

$$= e^{-iz}\left[\frac{-z}{4}\left(\cos 2z + i\sin 2z\right) + \frac{1}{8}\left(\sin 2z - i\cos 2z\right) + \frac{iz^2}{4}\right]$$

$$I_1 = e^{-iz}\left[\frac{-z}{4}e^{i2z} + \frac{1}{8}e^{-i\left(\frac{\pi}{2}-\theta\right)} + \frac{iz^2}{4}\right]$$

$$I_2 = \frac{1}{D-i}\,z\sin z$$

$$= e^{iz}\int z\sin z\, e^{-iz}\,dz \qquad \left[\because \frac{1}{D-a}X = e^{az}\int X e^{-ax}\,dx\right]$$

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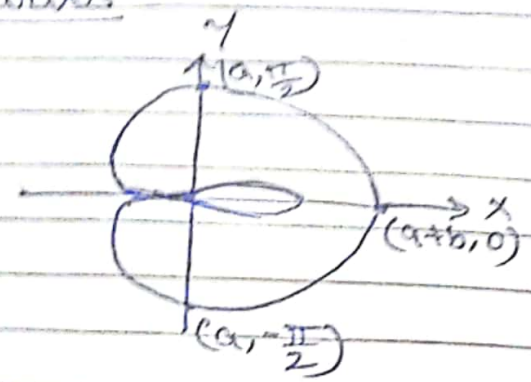
Some More Curves-

① Lamicon

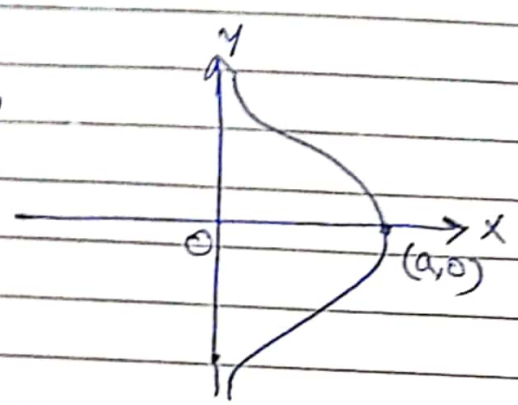
$r = a + b\cos\theta$

$a < b$



② Witch of Agensi

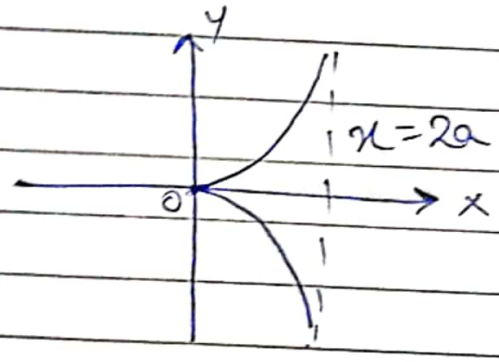
$xy^2 = a^2(a-x), a>0$



③ Cissoid.

$x(x^2+y^2) = 2ay^2$

or $y^2(2a-x) = x^3$



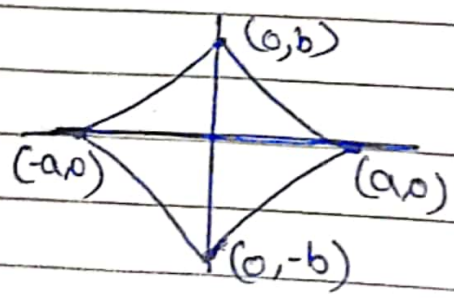
④ Asteroid or Hypocycloid with four cusps.

$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$

or

$x = a\cos^3 t$

$y = b\sin^3 t$



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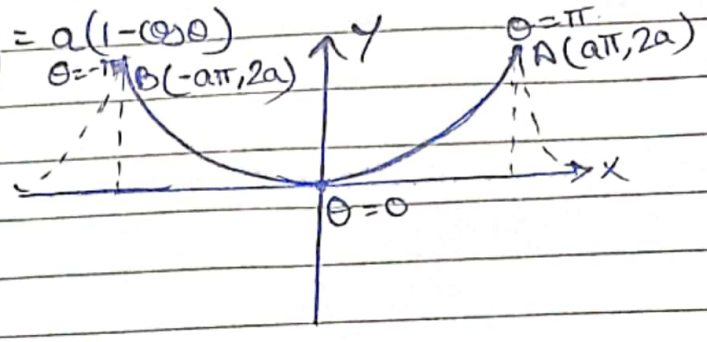
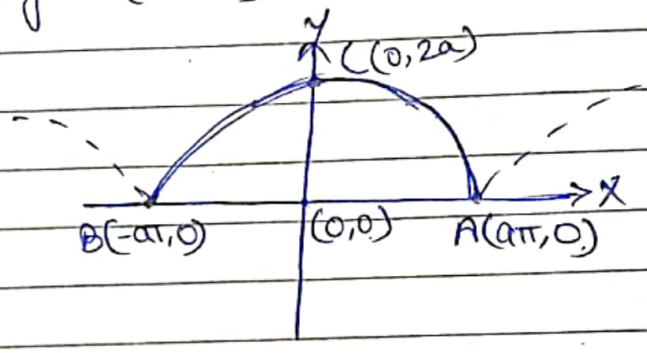
(3.) Cycloid

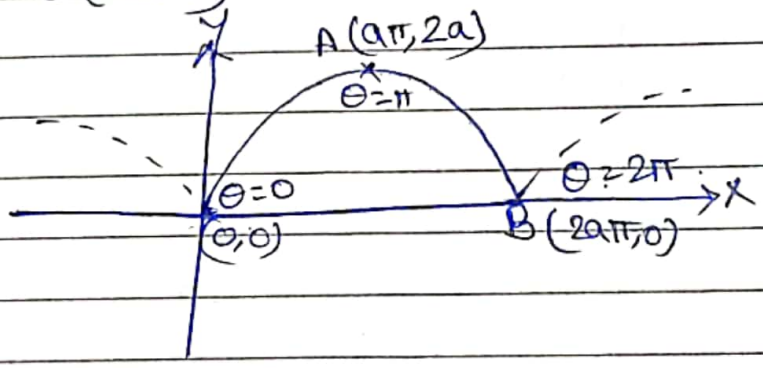
(i) $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$
$a > 0$



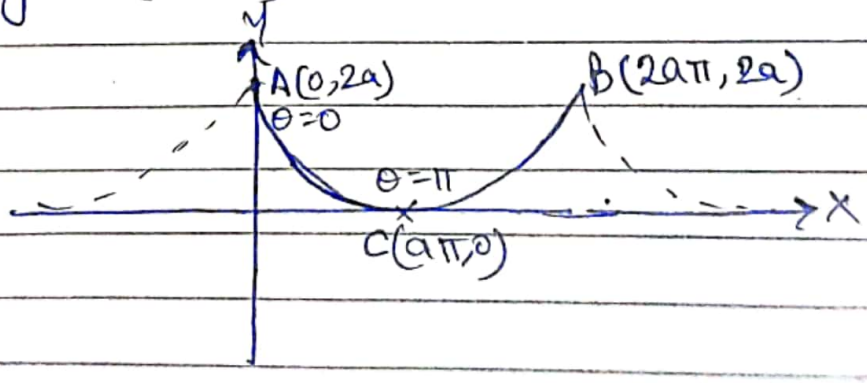
(ii) $x = a(\theta + \sin\theta),\ y = a(1 + \cos\theta)$
$a > 0$



(iii) $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$
$a > 0$



(iv) $x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$
$a > 0$



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