Legendre Polynomials
Lab Report for Assignment No. 2(b)

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Contents

1 Theory

1.1 Orthogonal Polynomials

We have seen that in $\mathbb{R}^2$ the length of a vector and the angle between two vectors can be expressed using the dot product. So in a sense the dot product is what gives rise to the geometry of vectors. It is certain properties of the dot product that make this work.
The generalization of the dot product to an arbitrary vector space is called an **inner product**.Just like the dot product, this is a certain way of putting two vectors together to get a number.
Hence for Polynomial Space we define the **inner product** for the polynomials p,q $\in C_{[a,b]}$ to be the following-

$$\langle p, q \rangle = \int_a^b p(x)q(x)\,dx$$

Since the inner product generalizes the dot product, it is reasonable to say that two vectors are **orthogonal** if their inner product is zero.
In standard form the orthogonalization condition for Polynomial space is -

$$\langle p, q \rangle = \int_{-1}^1 p(x)q(x)\,dx = 0$$

Then we can say that the Polynomials p and q are orhtogonal in interval [-1,1].

1.2 Legendre Polynomials as orthogonal basis

The Legendre polynomials form a complete orthogonal system over the interval [-1,1] with respect to the weighting function w(x)=1.
The orthogonality condition for the inner product of the two Legendre polynomials is given by -

$$\int_{-1}^1 P_n(x)q_m(x)\,dx = \frac{2}{2m+1}\delta_{mn}$$

where δ_{mn} is kronecker delta function which is given by -

$$\delta_{mn} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

Orthogonal basis from Gram-Schmidt process

$$\text{Original Basis} = \text{B} = \{a, b, c, d, ...\} = \{1, x, x^2, x^3, ...\}$$

Let the new orthogonal basis be -

$$\mathcal{L} = \{a_1, b_1, c_1, d_1, ...\}$$

$$a_1 = a = 1 \tag{1}$$

$$b_1 = b - \tfrac{\langle a_1, b\rangle}{\langle a_1, a_1\rangle} a_1$$

$$b_1 = b - \frac{\int_{-1}^{1} a_1(x) b(x)\, dx}{\int_{-1}^{1} [a_1(x)]^2\, dx} a_1$$

$$b_1 = x - \frac{\int_{-1}^{1} 1 \cdot x\, dx}{\int_{-1}^{1} [1]^2\, dx} \cdot 1$$

we know that for a odd function the integral in the numerartor turns out be 0 then -

$$b_1 = x \tag{2}$$

$$c_1 = c - \tfrac{\langle a_1, c\rangle}{\langle a_1, a_1\rangle} a_1 - \tfrac{\langle b_1, c\rangle}{\langle b_1, b_1\rangle} b_1$$

$$c_1 = c - \frac{\int_{-1}^{1} a_1(x) c(x)\, dx}{\int_{-1}^{1} [a_1(x)]^2\, dx} a_1 - \frac{\int_{-1}^{1} b_1(x) c(x)\, dx}{\int_{-1}^{1} [b_1(x)]^2\, dx} b_1$$

$$c_1 = x^2 - \frac{\int_{-1}^{1} 1 \cdot x^2\, dx}{\int_{-1}^{1} [1]^2\, dx} \cdot 1 - \frac{\int_{-1}^{1} x \cdot x^2\, dx}{\int_{-1}^{1} [x]^2\, dx} \cdot x$$

The third term is zero by the same argument.

$$\int_{-1}^{1} x^2\, dx = [\tfrac{x^3}{3}]_{-1}^{1} = \tfrac{2}{3}$$

$$c_1 = x^2 - \frac{1}{3} \tag{3}$$

The polynomials a_1, b_1 and c_1 are Monic Legendre polynomials.

1.3 Function as linear combination of Legendre polynomials

Since the Legendre polynomials form the complete set of orthogonal functions in the interval [-1,1].Then any function $f(x)$ can be represented as linear combination of Legendre polynomials,Thus we can write.

$$f(x) = \sum_0^\infty C_n \mathcal{P}_n$$

Determination of coefficient:
We can write the function expansion of a n^{th} degree polynomial as follows -

$$f(x) = C_0\mathcal{P}_0 + C_1\mathcal{P}_1 + C_2\mathcal{P}_2..... + C_n\mathcal{P}_n$$

We can operate over the function expansion f(x) with inner product with one of the basis polynomial say $\mathcal{P}_1$

$$\langle f, \mathcal{P}_1\rangle = C_0 \langle \mathcal{P}_0, \mathcal{P}_1\rangle + C_1 \langle \mathcal{P}_1, \mathcal{P}_1\rangle + C_2 \langle \mathcal{P}_2, \mathcal{P}_1\rangle ..... + C_n \langle \mathcal{P}_n, \mathcal{P}_1\rangle$$

We know that inner product of Legendre polynomials will be zero when $m \neq n$,Then.

$$C_1 = \frac{\langle f,\mathcal{P}_1\rangle}{\langle \mathcal{P}_1,\mathcal{P}_1\rangle}$$

In general,

$$C_n = \frac{\langle f,\mathcal{P}_n\rangle}{\langle \mathcal{P}_n,\mathcal{P}_n\rangle}$$

$$\boxed{C_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)\mathcal{P}_n \, dx}$$

There will be n+1 no. of terms in the legendre series expansion of the polynomial of order n
Analytic determination of Coefficient:
$\rightarrow$ (α) We know that since $f(x)$ is a 4th order polynomial we need the first 5 Legendre polynomials to represent it.

$f(x) = C_0P_0 + C_1P_1 + C_2P_2 + C_3P_3 + C_4P_4$

Using the above the relation we can calculate the coefficients which come out to be:

$$C_0 = \frac{1}{2}\int_{-1}^{1}(1)(2 + 3x + 2x^4)dx$$
$$C_0 = \frac{1}{2}\left(\frac{24}{5}\right)$$
$$C_0 = \frac{12}{5}$$

$$C_1 = \frac{3}{2}\int_{-1}^{1}(x)(2 + 3x + 2x^4)dx$$

$$C_1 = \frac{3}{2}(2)$$
$$C_1 = 3$$

$$C_2 = \frac{5}{2}\int_{-1}^{1}\frac{1}{2}(3x^2-1)(2+3x+2x^4)dx$$
$$C_2 = \frac{5}{2}\left(\frac{16}{35}\right)$$
$$C_2 = \frac{8}{7}$$

$$C_3 = \frac{7}{2}\int_{-1}^{1}\frac{1}{2}(5x^3-3x)(2+3x+2x^4)dx$$
$$C_3 = \frac{7}{2}(0)$$
$$C_3 = 0$$

$$C_4 = \frac{9}{2}\int_{-1}^{1}\frac{1}{8}(35x^4-30x^2+3)(2+3x+2x^4)dx$$
$$C_4 = \frac{9}{2}\left(\frac{32}{315}\right)$$
$$C_4 = \frac{16}{35}$$

$f(x) = \left(\frac{12}{5}\right)P_0 + (3)P_1 + \left(\frac{8}{7}\right)P_2 + (0)P_3 + \left(\frac{16}{35}\right)P_4$

$\rightarrow (\beta)$ We have to find the first 5 terms of the expansion so,

$f(x) = C_0P_0 + C_1P_1 + C_2P_2 + C_3P_3 + C_4P_4$

Using the above the relation **??** we can calculate the coefficients which come out to be:

$$C_0 = \frac{1}{2}\int_{-1}^{1}(1)(cos(x)sin(x))dx$$
$$C_0 = \frac{1}{2}(0)$$
$$C_0 = 0$$

$$C_1 = \frac{3}{2}\int_{-1}^{1}(x)(cos(x)sin(x))dx$$

$$C_1 = \frac{3}{2}(0.43540)$$

$$C_1 = 0.6531$$

$$C_2 = \frac{5}{2}\int_{-1}^{1}\frac{1}{2}(3x^2-1)(cos(x)sin(x))dx$$

$$C_2 = \frac{5}{2}(0)$$

$$C_2 = 0$$

$$C_3 = \frac{7}{2}\int_{-1}^{1}\frac{1}{2}(5x^3-3x)(cos(x)sin(x))dx$$

$$C_3 = \frac{7}{2}(-0.060722)$$

$$C_3 = -0.212527$$

$$C_4 = \frac{9}{2}\int_{-1}^{1}\frac{1}{8}(35x^4-30x^2+3)(cos(x)sin(x))dx$$

$$C_4 = \frac{9}{2}(0)$$

$$C_4 = 0$$

$f(x) = (0)P_0 + (0.6531)P_1 + (0)P_2 + (-0.212527)P_3 + (0)P_4$

2 Programming

```
import numpy as np
from Myintegration import *
import matplotlib.pyplot as plt
import pandas as pd
from scipy.special import legendre,eval_legendre
import sympy as sym

plt.style.use("bmh")

def inner_prod(f1,f2,a,b,n):
    new_f = lambda x: f1(x)*f2(x)
    prod = MyLegQuadrature(new_f,a,b,n,100)
    return prod

def leg_fourier(f,n): # n is no. of terms we want in our series
    Coeff = np.zeros(n)
    for i in range(n):
        Coeff[i] = (inner_prod(f,legendre(i),-1,1,10))/(inner_prod(legendre(i),le
    return Coeff


def func1(x):
    return 2 + 3*x + 2*x**4

def func2(x):
    return np.cos(x)*np.sin(x)

poly_coeff = leg_fourier(func1, 5)
csin_coeff = leg_fourier(func2, 10)

df1 = pd.DataFrame({"Coefficient corresponding to nth Legendre Polynomial":['C0',
print(df1)

df2 = pd.DataFrame({"Coefficient corresponding to nth Legendre Polynomial":['C0',
print(df2)

df1.to_csv('tab1.csv')
df2.to_csv('tab2.csv')

def partial_series(f,x,n):
    coeff = leg_fourier(f,n)
    n_arr = np.arange(0,len(coeff))
    return eval_legendre(n_arr,x).dot(coeff)

partial_series = np.vectorize(partial_series)

def Compare_original(f,n_max,d):
    x = np.linspace(-np.pi,np.pi)
    for i in range(2,n_max):
```

```
        old = partial_series(f,x,i-1)
        new = partial_series(f,x,i)
        if max(abs((new - old)/new)) <= 0.5/10**d:
            return i

print('No. of terms in expansion of Cos(x)Sin(x) which result in accuracy of 6 si

x_new = np.linspace(-2,2,50)
n1 = 6
n2 = 10
fig,(ax1,ax2)  = plt.subplots(1,2)
for i in range(1,n1):
    z = partial_series(func1,x_new,i)
    ax1.plot(x_new,z,'2--',label = f'for n = {i} terms')
    ax1.legend()
ax1.plot(x_new,func1(x_new),label = 'Analytical')
ax1.set_xlabel('value of x')
ax1.set_ylabel('Series calculated')
ax1.set_title('Series calculated for polynomial $2 + 3x + 2x^{4}$')
for j in range(2,12,2):
    m = partial_series(func2,x_new,j)
    ax2.plot(x_new,m,'1--',label = f'for n = {j} terms')
    ax2.legend()
ax2.plot(x_new,func2(x_new),label = 'Analytical')
ax2.set_xlabel('value of x')
ax2.set_ylabel('Series calculated')
ax2.set_title('Series calculated for function $cos(x)sin(x)$')
```

3 Discussion

	Coefficient corresponding to nth Legendre Polynomial	Value of Coefficient
0	C0	2.3999999999999995
1	C1	3.000000000000001
2	C2	1.142857142857143
3	C3	9.714451465470118e-17
4	C4	0.4571428571428581

Table 1: Coefficient of Legendre series expansion

- Note that the Polynomial $2 + 3x^2 + 2x^4$ has no cubic and linear term hence the coefficient C3 corresponding to $mathcal P_3 = \frac{1}{2}(5x^3 - 3x)$ is zero.
- Further the numerically calculated value of the integral by 10 point gauss quadrature formula is very close to analytically derived value.

	Coefficient corresponding to nth Legendre Polynomial	Value of Coefficient
0	C0	0.0
1	C1	0.6530966624699879
2	C2	-2.1684043449710098e-17
3	C3	-0.21252734182006158
4	C4	-2.341876692568689e-16
5	C5	0.014493433736342922
6	C6	-2.311519031739095e-16
7	C7	-0.0004207241777510246
8	C8	-1.3270634591222564e-16
9	C9	6.751457332559559e-06

Table 2: Coefficient of Legendre series expansion

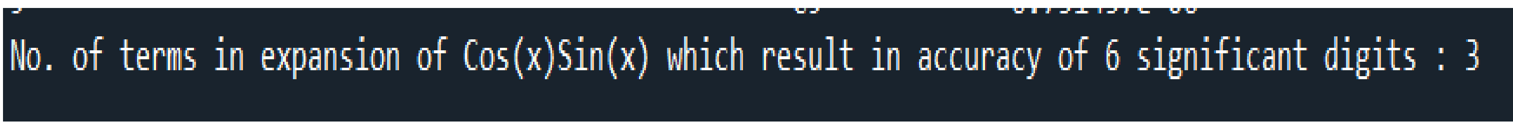

Figure 1: Correct value of cos(x)sin(x) in interval [-1,1]

- The function $cos(x)sin(x)$ can be represented by legendre series in interval [-1,1] accurately by using only **3 terms**.
- Note that C0 and C2 are 0 i.e, The expansin of the function contains no constant term
- If we use more than 3 terms in the expansion we can extend the domain from [-1,1] to some interval [-1-ϵ,1+ϵ] where $\epsilon > 0$.

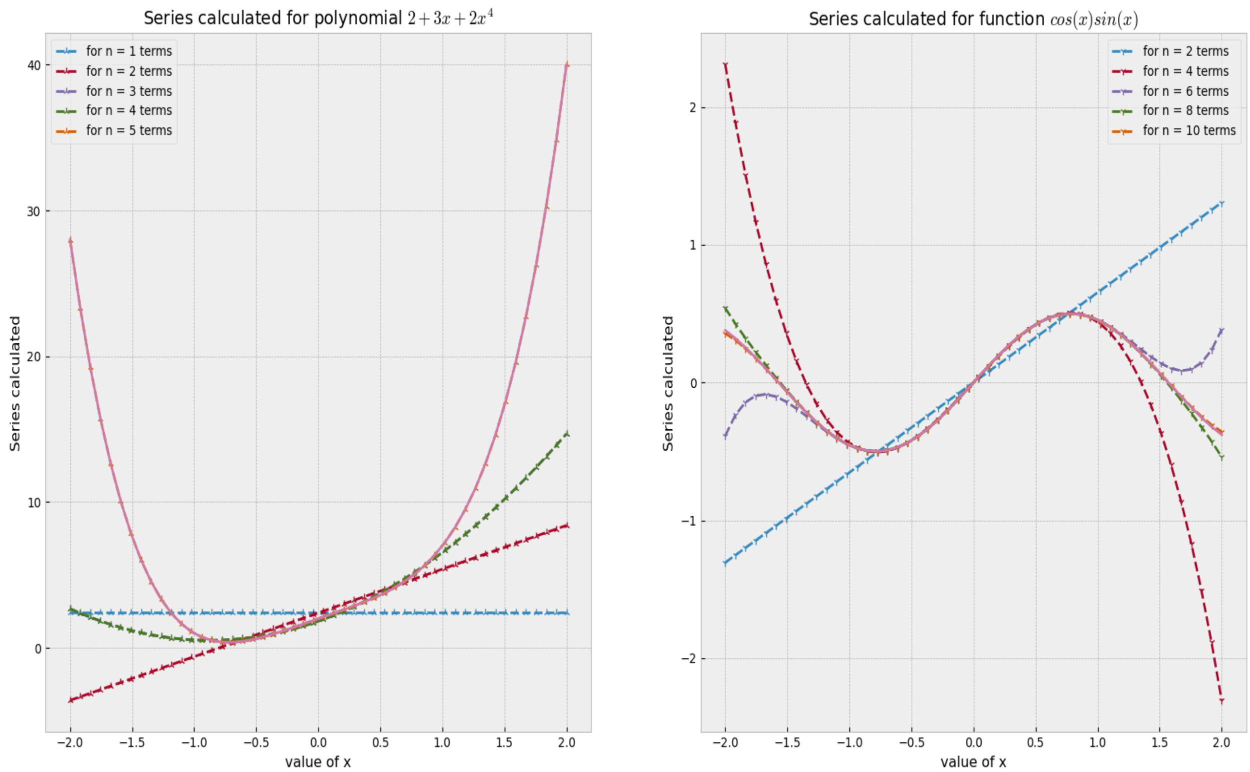

Figure 2: Graphing of functions for different no. of terms

- Note that for the polynomial $2 + 3x^2 + 2x^4$ when we include 5 terms in the series expansion i.e, the Legendre polynomial $\mathcal{P}_4$ has degree 4 hence the given polynomial is exactly calculated by the series expansion,then we can extend the domain of the series from [-1,1] to the domain of the original Polynomial.
- For the function $cos(x)sin(x)$ The series expansion including 10 terms can approximate the function in the interval [-2,2],as we reduce the no. of terms the interval in which series approximate the function reduces