Weighted Least Square fitting.

- Weighted Least Square fitting (WLS) is a generalization of ordinary least squares in which the knowledge of variable's variance is incorporated into the regression.
In Weighted Least squares method we give each data point its proper amount of influence over the parameter estimates.
In WLS all of the observables have their quality taken in account for better estimation.

- Moreover WLS method is used in the situations in which the data points are of varying quality.

Statistically speaking.

Consider a data set $\{(x_i, y_i)\}$ where x_i are known exactly & y_i are known with some uncertainity σ_i

let, $y = f(x: m, c)$ are set of parameters to be estimated

Then, From Centeral Limit theoram, distributi-on of measured y values about their ideal values is gaussian.
And Probablity of a Particular y_i for a given x_i is.

$$P(y_i ; m, c) = \frac{1}{\sigma_i \sqrt{2\pi}} e^{\left[-\frac{(y_i - f(x_i ; m, c))^2}{2\sigma_i^2}\right]}$$

then, Maximising Maximum Likelyhood function of the estimators $\hat{m}$ & $\hat{c}$ is similar to minimizing -

$$\sum_i \left[\frac{y_i - f(x_i ; m, c)}{\sigma_i}\right]^2$$

This term is called χ^2 or Chi-Squared
thus,

$$\chi^2 = \left[\frac{y_i - f(x_i ; m, c)}{\sigma_i}\right]^2$$

Then, we will minimize χ^2 w.r.t m & c for estimates $\hat{m}$ & $\hat{c}$

- Linear WLS

when,

$$y = mx + c$$

we will use χ^2 to find estimators of m & c

i.e,

$$\chi^2 = \sum w_i (y_i - mx_i - c)^2$$

here, $w_i = \dfrac{1}{\sigma_i^2}$

then,

$$\frac{\partial(\chi^2)}{\partial m} = 0$$

$$\sum \frac{\partial}{\partial m}\left[w_i (y_i - mx_i - c)^2\right] = 0$$

$$-2 \sum w_i x_i (y_i - mx_i - c)^2 = 0$$

$$\sum w_i x_i y_i - m \sum w_i x_i^2 - c \sum w_i x_i = 0 \qquad \text{—(1)}$$

when,

$$\frac{\partial(\chi^2)}{\partial c} = 0$$

$$-2 \sum w_i (y_i - mx_i - c) = 0$$

$$\sum w_i y_i - m \sum w_i x_i - c \sum w_i = 0$$

Date / /

$$\Rightarrow \quad c = \frac{\sum w_i y_i - m \sum w_i x_i}{\sum w_i}$$

$$\boxed{c = \bar{y} - m\bar{x}}$$

here,
$$\bar{y} = \frac{\sum w_i y_i}{\sum w_i} \quad \& \quad \bar{x} = \frac{\sum w_i x_i}{\sum w_i}$$

Putting c in (1)

$$\sum w_i x_i y_i - m \sum w_i x_i^2 - (\bar{y} - m\bar{x}) \sum w_i x_i = 0$$

$$m\left(\sum w_i x_i^2 - \bar{x} \sum w_i x_i\right) = \sum w_i x_i y_i - \bar{y} \sum w_i x_i$$

$$m = \frac{\sum w_i x_i y_i - \bar{y} \sum w_i x_i}{\sum w_i x_i^2 - \bar{x} \sum w_i x_i}$$

$$\boxed{m = \frac{S_{xy} - \bar{y} S_x}{S_{x^2} - \bar{x} S_x}} = \frac{\sum w_i (x_i - \bar{x})(y_i - \bar{y})}{\sum w_i (x_i - \bar{x})^2}$$

where, $S_{xy} = \sum w_i x_i y_i$, $S_x = \sum w_i x_i$

$S_{x^2} = \sum w_i x_i^2$

We can also write

$$m = \frac{\Sigma w_i \Sigma w_i x_i y_i - \Sigma w_i x_i \Sigma w_i y_i}{\Sigma w_i \Sigma w_i x_i^2 - (\Sigma w_i x_i)^2}$$

&

$$C = \frac{\Sigma w_i x_i^2 \Sigma w_i y_i - \Sigma w_i x_i \Sigma w_i x_i y_i}{\Sigma w_i \Sigma w_i x_i^2 - (\Sigma w_i x_i)^2}$$

let,

$$\Delta = \Sigma w_i \Sigma w_i x_i^2 - (\Sigma w_i x_i)^2$$

$\because$ the constant m depends on x_i & y_i but only y_i have uncertainity around therefore, by Propagation of error.

$$\sigma_m^2 = \Sigma \left(\frac{\partial m}{\partial y_i} \sigma_i \right)^2$$

i.e,

$$\frac{\partial m}{\partial y_i} = \frac{(\Sigma w_i) w_i x_i - (\Sigma w_i x_i) w_i}{\Delta}$$

then

$$\left(\frac{\partial m}{\partial y_i} \right) \sigma_i = \frac{(\Sigma w_i) x_i / \sigma_i - \Sigma w_i x_i / \sigma_i}{\Delta}$$

then,

$$\sigma_m^2 = \Sigma \frac{\left(\frac{(\Sigma w_i) x_i}{\sigma_i} - \frac{\Sigma w_i x_i}{\sigma_i} \right)^2}{\Delta^2}$$

$$\sigma_m^2 = \frac{\sum\left[\frac{(\Sigma w_i)^2 x_i^2}{\sigma_i^2} + \frac{(\Sigma w_i x_i)^2}{\sigma_i^2} - \frac{2(\Sigma w_i \Sigma w_i x_i) x_i}{\sigma_i^2}\right]}{\Delta^2}$$

$$\sigma_m^2 = \frac{\sum w_i \left((\Sigma w_i)^2 x_i^2 + (\Sigma w_i x_i)^2 - 2(\Sigma w_i \Sigma w_i x_i) x_i \right)}{\Delta^2}$$

$$\sigma_m^2 = \frac{\Sigma w_i \left[(\Sigma w_i)^2 x_i^2 + [(\Sigma w_i)\bar{x}]^2 - 2[(\Sigma w_i)^2 \bar{x} x_i] \right]}{\Delta^2}$$

$$\sigma_m^2 = \frac{\Sigma w_i \left[(\Sigma w_i)^2 (x_i^2 + \bar{x}^2 - 2\bar{x} x_i) \right]}{\Delta^2}$$

$$\sigma_m^2 = \Sigma w_i \left[(\Sigma w_i)^2 (x_i - \bar{x})^2 \right]$$

$$\sigma_m^2 = \frac{(\Sigma w_i)^2 \Sigma w_i (x_i - \bar{x})^2}{\Delta^2}$$

$$\sigma_m^2 = \frac{(\Sigma w_i)^2 \Sigma w_i (x_i - \bar{x})^2}{\Delta \left[\Sigma w_i (\Sigma w_i x_i^2 - \bar{x} \Sigma w_i x_i) \right.}$$

$$\boxed{\sigma_m^2 = \frac{\Sigma w_i}{\Delta}}$$

$$\Rightarrow \boxed{\sigma_m = \sqrt{\frac{\Sigma w_i}{\Delta}}}$$

//2

In similar fashion

$$\sigma_c^2 = \sum \left(\frac{\partial c}{\partial y_i} \sigma_i \right)^2$$

here,

$$\frac{\partial c}{\partial y_i} = \frac{(\sum w_i x_i^2) w_i - (\sum w_i x_i) w_i x_i}{\Delta}$$

Using similar algebra, it can be shown that,

$$\sigma_c^2 = \frac{\sum w_i x_i^2}{\Delta} = \frac{S_{x^2}}{\Delta}$$

$$\boxed{\sigma_c = \sqrt{\frac{S_{x^2}}{\Delta}}}$$

- Correlation coefficient

we know that,

$$\rho = r = \hat{m} \frac{\sigma_x}{\sigma_y}$$

$$\Rightarrow \boxed{\rho = \frac{\sum w_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum w_i (x_i - \bar{x})^2 \sum w_i (y_i - \bar{y})^2}}}$$

Date / /

- Simplification to ordinary Least square fit.

when our data points are equally distributed along their best estimates, i.e

$$\sigma_i = \sigma \quad \forall i$$

then, our data points Provides equally Precise information about deterministic part of the Process.
i.e for our all the values of explanatory Variables Standard deviation is constant.
then,

$$\omega =$$

i.e, ω being constant can be written out of the summation
then, our Parameters for best values of y becomes

$$m = \frac{\cancel{\omega}\sum(x_i-\bar{x})(y_i-\bar{y})}{\cancel{\omega}\sum(x_i-\bar{x})^2}$$

$$\boxed{m = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sum(x_i-\bar{x})^2}}$$

$$\boxed{c = \bar{y} - m\bar{x}}$$

Date / /

PROCEDURE

(I) Resolving Power of Prism.

1. Find the Angle of Prism (used), using the Procedure in Lab report A1b.

2. Place a adjustable micrometer slit on the objective of the eye-piece, before the teliscope.
At this Point keep slit wide open.

3. We will see two exactly distinguishable lines of Blue & Purple for Mercury Spectra.

4. Now slowly decrease the width of the slit with the help of Micrometer screw till the two lines of Mercury spectra are just seen separate.
Note the reading on the scale at this Point.
Now close the slit completely & note this reading.
The difference of these two readings is the width of our rectangular aperture a, of telescope.

Date / /

5. Repeat Step 4, while opening the slit & measure a
our total aperture width $\bar{a}$ will be Mean of these two.

6. Now determine the angle of Minimum deviation for light of smaller wavelength i.e, Purple & then for Blue, the Net angle of Minimum will be mean of these two.

Now, we can find t using

$$t = \frac{2\bar{a}\,\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A+\bar{\delta}_m}{2}\right)}$$

7. Use the value of cachy's Constant, derived from weighted Least square fitting
i.e,
then,

$$\frac{\lambda}{d\lambda} = -\frac{2Bt}{\lambda^3}$$

where,

$$\lambda = \frac{\lambda_B + \lambda_P}{2}$$

We can use λ_B & λ_P from Lab report A2

Date / /

(II) Dispersive Power of Prism

1. Find angle of Prism using Procedure discussed in lab report A1b.

2. Determine δ_m for Red Purple & yellow lines of Mercury Spectra.

3. Determine μ_r, μ_p, μ_y, & use formula discussed in theory to determine ω.

- Using WLS

1. Determine corresponding best fitted values of μ for $\lambda_1 (\lambda_p)$ & $\lambda_2 (\lambda_r)$

2. find μ_m for $\lambda = \frac{\lambda_1 + \lambda_2}{2}$, from that fitted curve.

3. Determine ω using μ_1, μ_2 & μ_m.

4. Find corresponding error associated with it

Date / /

PRECAUTIONS

1. Always move the screw on the Micrometer in one rotation, no back & forth motion should happen, if not so will include a backlash error.

2. The measurements for δm for various lines must be taken with extra care, because they are very close.

3. The width of the aperture should be noted carefully.

4. Statistical & Reading errors in the observations should be handled properly.