Dirac Delta Function
Lab Report for Assignment No. 7

SHASHVAT JAIN
(2020PHY1114)

HARSH SAXENA
(2020PHY1162)

S.G.T.B. Khalsa College, University of Delhi, Delhi-110007, India.

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Contents

1 Theory

The Dirac delta function,is a generalized function or distribution over the real numbers, whose value is zero everywhere except at zero.
with the purpose of describing an extremely localized charge density, in 1930 **Paul Dirac** investigated the following **"function"**.

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \tag{1}$$

Further the Dirac delta function is defined its assigned properties.

$$f(0) = \int_{-\infty}^{+\infty} f(x)\delta(x)dx \tag{2}$$

where $f(x)$ is any well-behaved function and the integration includes the origin.The direct consequence of above property is that -

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1 \tag{3}$$

Unfortunately, this property of $\delta(x)$ is not compatible with the definition 1.In fact,If we choose our partition points to be ever left of zero and the function evaluation at these said points then the integral is **zero** further at the discontinuity for some width Δx to which my evaluated $\delta(x)$ at the said discontinuous point is ∞ then clearly the **limit diverges**.Thus it can be seen that dirac delta concept is **not a function** in general sense.
No matter what **finite** value we put in 1 in place of $+\infty$ we still get the integral in eqn 3 **zero**.
Instead of defining $\delta : \mathbb{R} \to \mathbb{R}$ where it is not a function.We define it by -
$\forall f \in \mathcal{F}$ where $\mathcal{F}$ is a certain vector space of functions.By definition δ is a function

$$\delta : \mathcal{F} \to \mathbb{R} \qquad\qquad , \delta(f) = f(0) \tag{4}$$

It is defined by mathematicians as a Linear function acting on a set of well behaved test functions $f(x)$

1.1 As limit of sequence of function

For the purpose of using these property of Dirac delta function can be developed rigorously as the limit of a sequence of functions, a distribution.
that is to interpret the integral in eq 3 as -

$$\int_{-\infty}^{+\infty} \delta(x)dx = \lim_{\epsilon \to 0} \int_{-\infty}^{+\infty} \delta_\epsilon(x)dx$$

we can change the dependence of δ from a real no. ϵ to a natural number n i.e, $n \in \mathbb{N}$,in doing so we can write.

$$\int_{-\infty}^{+\infty} \delta(x)dx = \lim_{n \to \infty} \int_{-\infty}^{+\infty} \delta_n(x)dx$$

Hence,this Dirac delta function can be approximated by the following sequence of functions.

1.

$$\delta_n(x) = \begin{cases} 0, & x < -\frac{1}{2n} \\ n, & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0, & x > \frac{1}{2n} \end{cases}$$

2.

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} exp\left(-n^2x^2\right)$$

3.

$$\delta_n(x) = \frac{n}{\pi}.\frac{1}{1+n^2x^2}$$

4.

$$\delta_n(x) = \frac{\sin nx}{\pi x}$$

5.

$$\delta_n(x) = \frac{n}{2} sech^2 nx$$

Further note that we can interpret the delta function as -

$$\lim_{n\to\infty} \delta_n(x) = \delta(x)$$
$$\lim_{\epsilon\to 0} \delta_\epsilon(x) = \delta(x)$$

But in more general mathematical sense this limit **does not exist** as $n \to \infty$ or $\epsilon \to 0$.
We know that main property of this Dirac delta function is how this function behaves under integral hence we focus our attention on it rather than at $\delta(x)$.All the above representations, $\forall n \in \mathbb{N}$ can be interpreted as **sequence of normalized functions**.
further it can be proven that.

$$\int_{-\infty}^{+\infty} \delta_n(x)dx = 1$$

Hence the following integral can be interpreted as the sequence of integrals $\{I_n\}$.

$$I_n = \int_{-\infty}^{+\infty} \delta_n(x)f(x)dx$$

Hence the integral in eq 2 can be wriitten as.

$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = \lim_{n\to\infty} I_n$$
$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = \lim_{n\to\infty} \int_{-\infty}^{+\infty} \delta_n(x)f(x)dx = f(0)$$

Note that the integral on the left is not a **Riemann integral instead a limit** of sequence.

1.2 Properties of Dirac delta

More general form of Dirac delta function is where we shift the center of the distribution from 0 to some arbitrary point a, in doing so we write our Dirac delta function as $\delta(x-a)$ where $a \in \mathbb{R}$.
Since we know that the Dirac delta function has significance only as part of an integrand.Hence we can write that the linear operator $\int \delta(x-a)dx$ acts on $f(x)$ and yields $f(a)$.

$$\int_{-\infty}^{+\infty} \delta(x-a)f(x)dx = f(a)$$

Note that upper and lower limit on the integral can be any numbers {u,v} such that $u, v \in \mathbb{R}$ but $a \in [u, v]$ we can write.

$$\int_{u}^{v} \delta(x-a)f(x)dx = \begin{cases} f(a), & a \in [u, v] \\ 0, & otherwise \end{cases}$$

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

<u>3 dimensional Dirac delta function</u>

It is easy to see that we can formulate a three dimensional form of this delta function by imposing that.

$$\int_{allspace} \delta^3(\vec{r}-\vec{a})d\tau = 1$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector and $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ is the position where the "peak" of the delta function occurs.In rectangular coordinates it is just the product of three one dimensional delta function.

$$\delta^3(\vec{r}-\vec{a}) = \delta(x-a_x)\delta(y-a_y)\delta(z-a_z)$$

So that

$$\int_{allspace} \delta^3(\vec{r})d\tau = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(x)\delta(y)\delta(z)dxdydz$$

One more property that is similar to one dimensional delta function is -

$$\int_{allspace} f(\vec{r})\delta^3(\vec{r}-\vec{a})d\tau = f(\vec{a})$$

1.3 Integrals

$\int_{-\infty}^{+\infty} \delta(x-2)(x+1)^2 dx$

here we have the function $f(x) = (x+1)^2$ under the integral hence.

$$\begin{aligned}\int_{-\infty}^{+\infty} \delta(x-2) f(x) dx &= f(2) \\ &= (2+1)^2 \\ \int_{-\infty}^{+\infty} \delta(x-2)(x+1)^2 dx &= 9\end{aligned}$$

$\int_{-\infty}^{+\infty} \delta(3x+1) 9x^2 dx$

here we have the function $f(x) = x^2$ under the integral hence.

$$\begin{aligned}9\int_{-\infty}^{+\infty} \delta(3x+1) f(x) dx &= 9\int_{-\infty}^{+\infty} \delta\left(3\left(x+\frac{1}{3}\right)\right) f(x) dx \\ &= \frac{9}{3}\int_{-\infty}^{+\infty} \delta\left(x-\left(-\frac{1}{3}\right)\right) f(x) dx \\ &= 3f\left(-\frac{1}{3}\right) \\ \int_{-\infty}^{+\infty} \delta(3x+1) 9x^2 dx &= \frac{1}{3}\end{aligned}$$

$\int_{-\infty}^{+\infty} 5e^{t^2} \cos(t) \delta(t-3) dx$

here we have the function $f(x) = 5e^{t^2} \cos t$ under the integral hence.

$$\begin{aligned}\int_{-\infty}^{+\infty} \delta(t-3) f(x) dx &= f(3) \\ &= 5e^9 cos(3)\end{aligned}$$

1.4 Programming

```python
import numpy as np
from Myintegration import *
import matplotlib.pyplot as plt

plt.style.use("bmh")

delta_lorr = np.vectorize(lambda x,e,a=0: e/(((x-a)**2 + e**2)*np.pi))
delta_sin = np.vectorize(lambda x,e,a=0: np.sin((x-a)/e)/((x-a)*np.pi))
delta_decay1 = np.vectorize(lambda x,e,a=0: np.exp(-np.abs((x-a))/e)/(2*e))
delta_decay2 = np.vectorize(lambda x,e,a=0: np.exp(-(x-a)**2/(4*e))/(2*np.sqrt(e*np.pi)))
delta_sech = np.vectorize(lambda x,e,a=0: 1/(np.cosh((x-a)/e)*2*e) )

def f_e(f,e):
    return(lambda x,a=0: f(x,e,a))

x_space = np.linspace(-1,1,1000)

def makePlots(f,x_space,f_title = "" ):
    dat= np.zeros((3,2))
    herm_vardat = np.zeros((3,10))
    herm_nvar = 2**np.arange(1,11)
    fig,ax = plt.subplots(1,1)
    e_arr = 0.4*2**(-1*np.arange(1,6,1,dtype=float))
    for e in e_arr:
        y = f_e(f,e)(x_space)
        plt.plot(x_space,y,label="$\epsilon$="+str(e))
    print(f_title)

    f_lst1 = [f_e(f,e) ,lambda x: f_e(f,e)(x)*(x+1)**2,lambda x: f_e(f,e)(3*x+1)*9*x**2]
    dat[:,0] = MyLegQuadrature(f_lst1,-1e5,1e5,n=80,m=int(2e3))
    dat[:,1] = MyHermiteQuad(f_lst1,n=80)
    ##
    np.savetxt(f"1162_{f_title}.dat",dat)
    plt.legend()

makePlots(delta_lorr,x_space,f_title = "delta_lorr")
makePlots(delta_sin,x_space,f_title= "delta_sin")
plt.show()
```

1.5 Discussion

For lorentzian representation the delta function emerges as we reduce ϵ

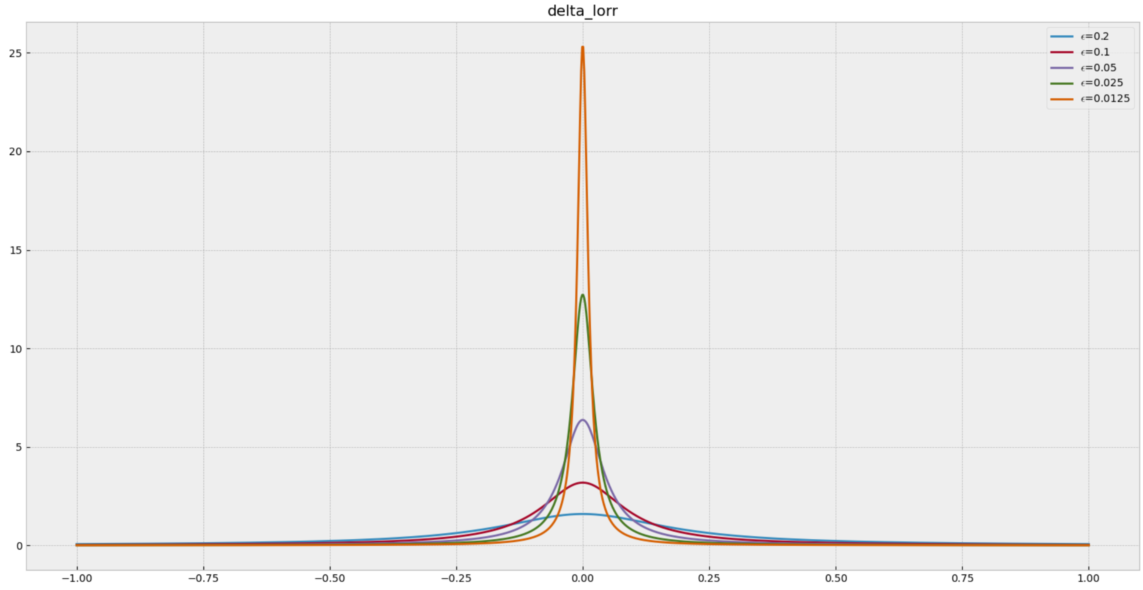

Figure 1: Lorentzian representation

Similarly for sine representation the delta function emerges as we reduce ϵ

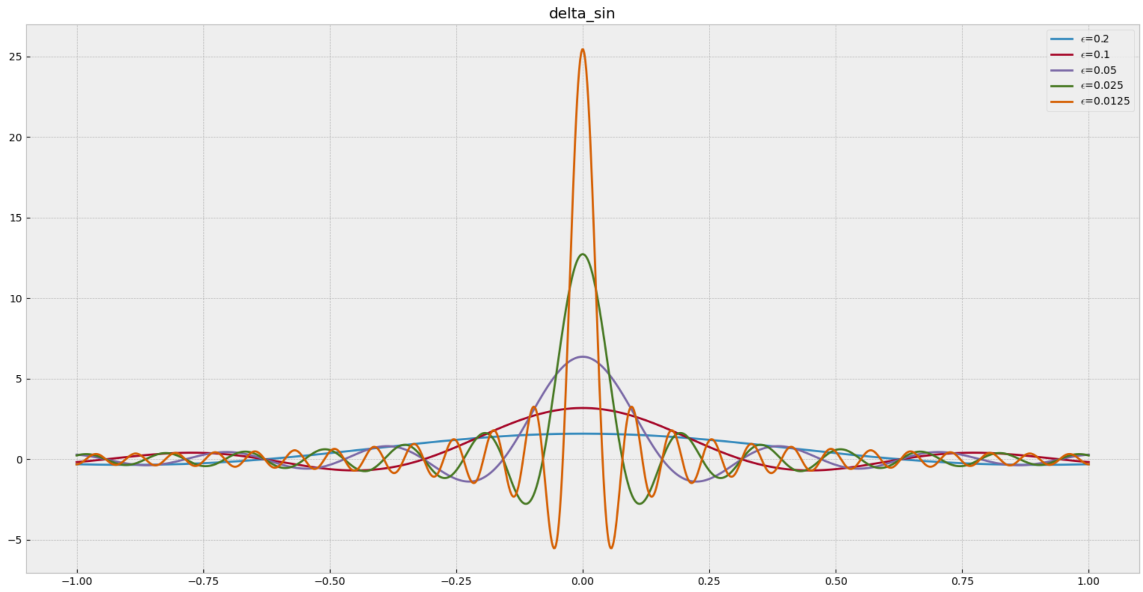

Figure 2: sine representation

For Sine representation

	Legendre Quad	Hermite Quad
f1	4.83E-01	-1.02E+00
f2	1.18E+05	2.97E-01
f3	2.73E+05	-2.57E-02

Table 1: Data for integral

For Lorentzian representation

	Legendre Quad	Hermite Quad
f1	8.12E-01	1.57E-01
f2	7.97E+02	2.54E-01
f3	7.96E+02	1.94E-01

Table 2: Data for integral