

Lab Assignment A2b

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(2020PHY1097)

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Project Report Submitted to

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"32221401 - Mathematical Physics"

1 THEORY

1.1 Orthogonal Polynomials

In mathematics, an orthogonal polynomial sequence is a family of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product.

Orthogonal polynomials are classes of polynomials ($p_n(x)$) defined over a range [a,b] that obey an orthogonality relation

$$\int_a^b w(x) p_n(x) q_m(x) dx = \delta_{mn} c_n$$

where $w(x)$ is a weighting function and δ_{mn} is the Kronecker delta. If $c_n = 1$, then the polynomials are not only orthogonal, but orthonormal.

Now we will define the inner product which will help us in orthogonal polynomials.

An inner product is a generalization of the dot product. In a vector space, it is a way to multiply vectors together, with the result of this multiplication being a scalar.

$$\langle p, q \rangle = \int_a^b p(x) q(x) dx$$

Simply we can say that the orthogonality relation is the inner product of two function and the weight function over the limit -1 to 1

1.2 Legendre Polynomials - An Orthogonal Basis

The Legendre polynomials form a complete orthogonal system over the interval [-1,1] with respect to the weighting function $w(x)=1$.

The orthogonality condition for the inner product of the two Legendre polynomials is given by -

$$\int_{-1}^1 P_n(x) q_m(x) dx = \frac{2}{2m+1} \delta_{mn}$$

where δ_{mn} is kronecker delta function.

Gram-Schmidt process for Orthogonal basis

Let A be the given basis

Given Basis = $A = \{1, x, x^2, x^3, \dots\} = \{a, b, c, d, \dots\}$

And the new orthogonal basis be $B = \{a_1, b_1, c_1, d_1, \dots\}$

such that $a_1 = a = 1$

Now by the use of Gram-Schmidt process

$$\begin{aligned} b_1 &= b - \frac{\langle a_1, b \rangle}{\langle a_1, a_1 \rangle} a_1 \\ b_1 &= b - \frac{\int_{-1}^1 a_1(x) b(x) dx}{\int_{-1}^1 [a_1(x)]^2 dx} a_1 \\ b_1 &= x - \frac{\int_{-1}^1 1 \cdot x dx}{\int_{-1}^1 [1]^2 dx} \cdot 1 \end{aligned}$$

As we know that for a odd function the integral turns out be 0 then -

$$b_1 = x \tag{1}$$

$$\begin{aligned} c_1 &= c - \frac{\langle a_1, c \rangle}{\langle a_1, a_1 \rangle} a_1 - \frac{\langle b_1, c \rangle}{\langle b_1, b_1 \rangle} b_1 \\ c_1 &= c - \frac{\int_{-1}^1 a_1(x) c(x) dx}{\int_{-1}^1 [a_1(x)]^2 dx} a_1 - \frac{\int_{-1}^1 b_1(x) c(x) dx}{\int_{-1}^1 [b_1(x)]^2 dx} b_1 \\ c_1 &= x^2 - \frac{\int_{-1}^1 1 \cdot x^2 dx}{\int_{-1}^1 [1]^2 dx} \cdot 1 - \frac{\int_{-1}^1 x \cdot x^2 dx}{\int_{-1}^1 [x]^2 dx} \cdot x \end{aligned}$$

Similarly, The third term will also be zero.

$$\int_{-1}^1 x^2 dx = \left[\frac{x^3}{3} \right]_{-1}^1 = \frac{2}{3}$$

$$c_1 = x^2 - \frac{1}{3} \tag{2}$$

The polynomials a_1, b_1 and c_1 are Monic Legendre polynomials.

1.3 Function as linear combination of Legendre polynomials

Since the Legendre polynomials form the complete set of orthogonal functions in the interval $[-1, 1]$. Then any function $f(x)$ can be represented as linear combination of Legendre polynomials, Thus we can write.

$$f(x) = \sum_0^{\infty} C_n \mathcal{P}_n$$

Determination of coefficient:

We can write the function expansion of a n^{th} degree polynomial as follows -

$$f(x) = C_0 \mathcal{P}_0 + C_1 \mathcal{P}_1 + C_2 \mathcal{P}_2 + \dots + C_n \mathcal{P}_n$$

We can operate over the function expansion $f(x)$ with inner product with one of the basis polynomial say \mathcal{P}_1

$$\langle f, \mathcal{P}_1 \rangle = C_0 \langle \mathcal{P}_0, \mathcal{P}_1 \rangle + C_1 \langle \mathcal{P}_1, \mathcal{P}_1 \rangle + C_2 \langle \mathcal{P}_2, \mathcal{P}_1 \rangle + \dots + C_n \langle \mathcal{P}_n, \mathcal{P}_1 \rangle$$

We know that inner product of Legendre polynomials will be zero when $m \neq n$. Then,

$$C_1 = \frac{\langle f, \mathcal{P}_1 \rangle}{\langle \mathcal{P}_1, \mathcal{P}_1 \rangle}$$

In general,

$$C_n = \frac{\langle f, \mathcal{P}_n \rangle}{\langle \mathcal{P}_n, \mathcal{P}_n \rangle}$$

$$C_n = \frac{1}{2} \int_{-1}^1 f(x) \mathcal{P}_n dx$$

There will be $n+1$ no. of terms in the legendre series expansion of the polynomial of order n

Analytic determination of Coefficient:

→ (α) We know that since $f(x)$ is a 4th order polynomial we need the first 5 Legendre polynomials to represent it.

$$f(x) = C_0 P_0 + C_1 P_1 + C_2 P_2 + C_3 P_3 + C_4 P_4$$

Using the above the relation we can calculate the coefficients which come out to be:

$$C_2 = \frac{5}{2} \left(\frac{16}{35} \right)$$

$$C_0 = \frac{1}{2} \int_{-1}^1 (1)(2+3x+2x^4)dx$$

$$C_2 = \frac{8}{7}$$

$$C_0 = \frac{1}{2} \left(\frac{24}{5} \right)$$

$$C_0 = \frac{12}{5}$$

$$C_3 = \frac{7}{2} \int_{-1}^1 \frac{1}{2} (5x^3 - 3x)(2+3x+2x^4)dx$$

$$C_3 = \frac{7}{2}(0)$$

$$C_1 = \frac{3}{2} \int_{-1}^1 (x)(2+3x+2x^4)dx$$

$$C_3 = 0$$

$$C_1 = \frac{3}{2}(2)$$

$$C_1 = 3$$

$$C_4 = \frac{9}{2} \int_{-1}^1 \frac{1}{8} (35x^4 - 30x^2 + 3)(2+3x+2x^4)dx$$

$$C_4 = \frac{9}{2} \left(\frac{32}{315} \right)$$

$$C_4 = \frac{16}{35}$$

$$C_2 = \frac{5}{2} \int_{-1}^1 \frac{1}{2} (3x^2 - 1)(2+3x+2x^4)dx$$

$$f(x) = \left(\frac{12}{5} \right) P_0 + (3)P_1 + \left(\frac{8}{7} \right) P_2 + (0)P_3 + \left(\frac{16}{35} \right) P_4$$

→ (β) We have to find the first 5 terms of the expansion so,

$$f(x) = C_0 P_0 + C_1 P_1 + C_2 P_2 + C_3 P_3 + C_4 P_4$$

Using the above the relation we can calculate the coefficients which come out to be:

$$C_2=\frac{5}{2}(0)$$

$$C_0=\frac{1}{2}\int_{-1}^1(1)(cos(x)sin(x))dx \qquad \qquad C_2=0$$

$$C_0=\frac{1}{2}(0)$$

$$C_0 = 0$$

$$C_3=\frac{7}{2}\int_{-1}^1\frac{1}{2}(5x^3-3x)(cos(x)sin(x))dx$$

$$C_3=\frac{7}{2}(-0.060722)$$

$$C_1=\frac{3}{2}\int_{-1}^1(x)(cos(x)sin(x))dx \qquad \qquad C_3=-0.212527$$

$$C_1=\frac{3}{2}(0.43540)$$

$$C_1=0.6531$$

$$C_4=\frac{9}{2}\int_{-1}^1\frac{1}{8}(35x^4-30x^2+3)(cos(x)sin(x))dx$$

$$C_4=\frac{9}{2}(0)$$

$$C_2=\frac{5}{2}\int_{-1}^1\frac{1}{2}(3x^2-1)(cos(x)sin(x))dx \qquad \qquad C_4=0$$

$$f(x)=(0)P_0+(0.6531)P_1+(0)P_2+(-0.212527)P_3+(0)P_4$$

2 Program

```
1 """
2 Kabir Sethi
3 Roll no. : 2020PHY1097
4 Examination Roll no. : 20068567031
5 """
6 import numpy as np
7 from MyIntegration import MyLegGauss
8 import matplotlib.pyplot as plt
9 import pandas as pd
10 from scipy.special import legendre, eval_legendre
11
12
13 plt.style.use("bmh") #For different style of plotting
14
15 def inner_pro(f1,f2,a,b,n): #Inner product function
16     f_new = lambda x: f1(x)*f2(x)
17     prod = MyLegGauss(f_new,a,b,n,200) #using Gauss-Legendre Quadrature
18     Integration
19     return prod
20
21
22 def Coeff(f,n,n_point): # n: no. of terms for the series , n_point:
23     The n_point formula to be used
24     Coef = np.zeros(n)
25     for i in range(n):
26         Coef[i] = (inner_pro(f,legendre(i),-1,1,n_point))/(inner_pro(
27             legendre(i),legendre(i),-1,1,n_point))
28     return Coef
29
30
31 def func1(x):
32     return 2 + 3*x + 2*x**4
33
34 def func2(x):
35     return np.cos(x)*np.sin(x)
36
37 poly_coeff = Coeff(func1, 5,4)
38 csin_coeff = Coeff(func2, 10,4)
39
40 df1 = pd.DataFrame({"Coefficient of Legendre Polynomial":['C0','C1','',
41                 'C2','C3','C4'],"Value of Coefficient":poly_coeff})
42 print(df1)
43 df1.to_csv('tab1.csv')
44
45 df2 = pd.DataFrame({"Coefficient of Legendre Polynomial":['C0','C1','',
46                 'C2','C3','C4','C5','C6','C7','C8','C9'],"Value of Coefficient":csin_coeff})
47 print(df2)
48 df2.to_csv('tab2.csv')
49
50
51 def partial_series(f,x,n,n_point):
52     coeff = Coeff(f,n,n_point)
53     n_arr = np.arange(0,len(coeff))
```

```

47     return eval_legendre(n_arr,x).dot(coeff)
48
49 partial_series = np.vectorize(partial_series)
50
51 def Comp(f,n_max,d):
52     x = np.linspace(-np.pi,np.pi)
53
54     for i in range(2,n_max):
55         old = partial_series(f,x,i-1,4)
56         new = partial_series(f,x,i,4)
57         if max(abs((new - old)/new)) <= 0.5/10**d:
58             return i
59
60 print('No. of terms required for accuracy of 6 significant digits :',
61       Comp(func2,100,6))
62
63 x_new = np.linspace(-2,2,50)
64 n1 = 6
65 n2 = 10
66 fig,(ax1,ax2) = plt.subplots(1,2)
67 for i in range(1,n1):
68     z = partial_series(func1,x_new,i,4)
69     ax1.plot(x_new,z,'2--',label = f'for n = {i} terms')
70     ax1.legend()
71 ax1.plot(x_new,func1(x_new),label = 'Analytical')
72 ax1.set_xlabel('value of x')
73 ax1.set_ylabel('Series calculated')
74 ax1.set_title('Series calculated for polynomial  $2+3x+2x^4$ ')
75 for j in range(2,12,2):
76     m = partial_series(func2,x_new,j,4)
77     ax2.plot(x_new,m,'1--',label = f'for n = {j} terms')
78     ax2.legend()
79 ax2.plot(x_new,func2(x_new),label = 'Analytical')
80 ax2.set_xlabel('value of x')
81 ax2.set_ylabel('Series calculated')
82 ax2.set_title('Series calculated for function  $\cos(x)\sin(x)$ ')
83 plt.show()

```

Source Code 1: Python Program

3 Result

The Terminal output is

```
[Running] python -u "e:\python\python files\sem 4\2020PHY1097_A2b.py"
| Coefficient of Legendre Polynomial Value of Coefficient
0 C0 2.400000e+00
1 C1 3.000000e+00
2 C2 1.142857e+00
3 C3 4.145035e-14
4 C4 4.571429e-01
| Coefficient of Legendre Polynomial Value of Coefficient
0 C0 5.474679e-16
1 C1 6.530967e-01
2 C2 1.756679e-15
3 C3 -2.125273e-01
4 C4 3.339126e-15
5 C5 1.449343e-02
6 C6 4.932415e-15
7 C7 -4.207242e-04
8 C8 6.694298e-15
9 C9 6.751457e-06
No. of terms required for accuracy of 6 significant digits : 3
```

Figure 1: Terminal Output

The graph is

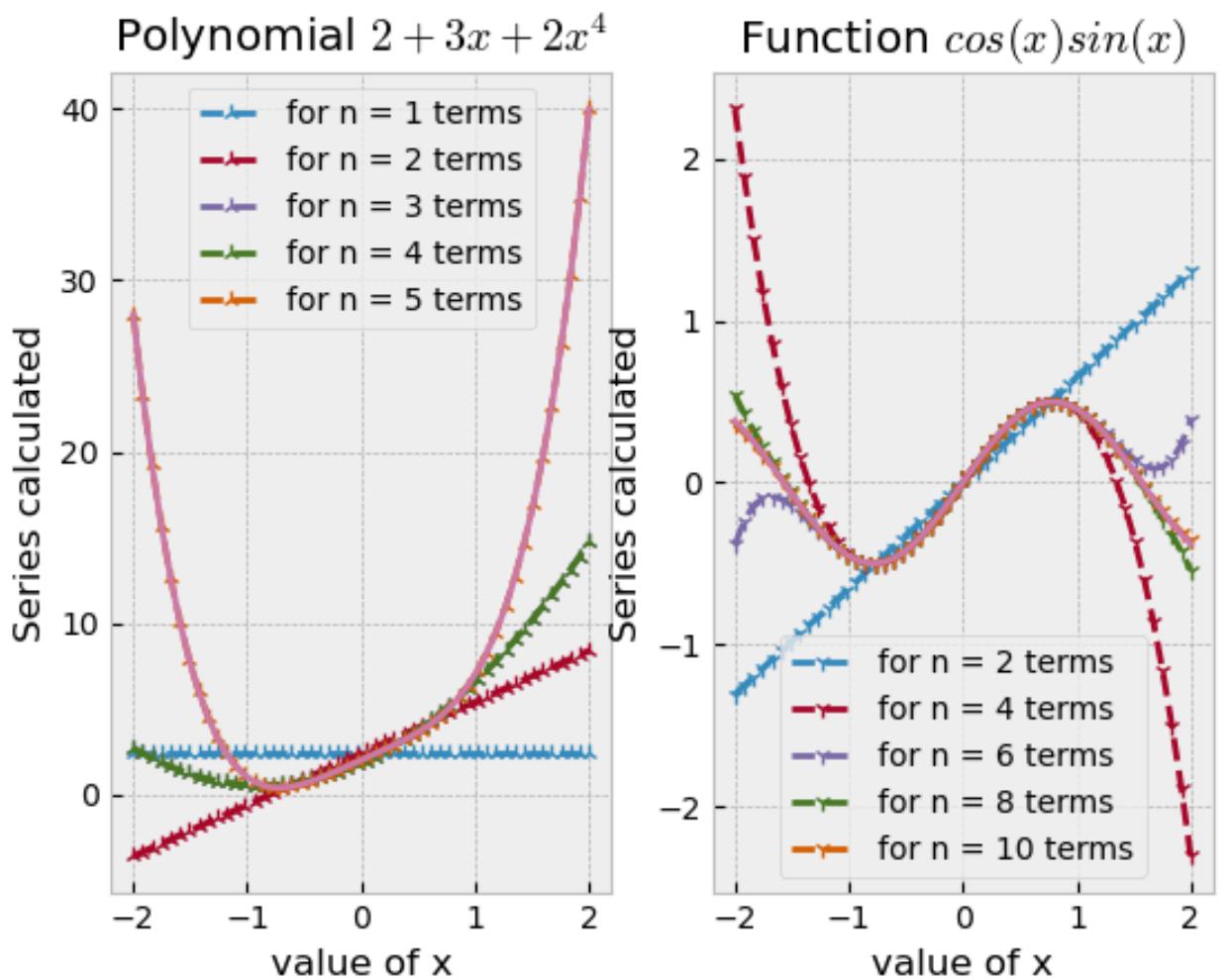


Figure 2: Graph

4 Discussion

As we can see from the programm and the graphs, we can calculate the coefficients for the given polynomials and as from the graph we can see that as we are increasing the number of terms we are getting the results almost equal to the analytic result.