The exact Darwin Lagrangian

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Darwin (1920) noted that when radiation can be neglected it should be possible to eliminate the radiation degrees-of-freedom from the action of classical electrodynamics and keep the discrete particle degrees-of-freedom only. Darwin derived his well known Lagrangian by series expansion in v/c keeping terms up to order $(v/c)^2$. Since radiation is due to acceleration the assumption of low speed should not be necessary. A Lagrangian is suggested that neglects radiation without assuming low speed. It cures deficiencies of the Darwin Lagrangian in the ultra-relativistic regime.

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When radiation can be neglected the Lagrangian of classical electrodynamics, putting $\beta = v/c$, can be written,

$$L = \sum_{a} \left\{ -m_a c^2 \sqrt{1 - \boldsymbol{\beta}_a^2} + \frac{e_a}{2} \left[\boldsymbol{\beta}_a \cdot \boldsymbol{A}(\boldsymbol{r}_a) - \phi(\boldsymbol{r}_a) \right] \right\}. \tag{1}$$

In 1920 Darwin [1] expanded the Liénard-Wiechert potentials to second order in $\beta = v/c$ and thus found that,

$$\phi(\mathbf{r}_a) = \sum_{b(\neq a)} \frac{e_b}{|\mathbf{r}_a - \mathbf{r}_b|} = \sum_{b(\neq a)} \frac{e_b}{r_{ba}},\tag{2}$$

and (hats are used for unit vectors),

$$\mathbf{A}(\mathbf{r}_a) = \sum_{b(\neq a)} \frac{e_b[\boldsymbol{\beta}_b + (\boldsymbol{\beta}_b \cdot \hat{\mathbf{r}}_{ba})\hat{\mathbf{r}}_{ba}]}{2r_{ba}}.$$
 (3)

give the correct Lagrangian to this order. More recent derivations can be found in a few textbooks [2–4]. In particular Jackson [4] notes that using the Coulomb gauge $(\nabla \cdot \mathbf{A} = 0)$ makes the electrostatic Coulomb potential ϕ exact and moves all approximation to the vector potential \mathbf{A} which obeys the inhomogeneous wave equation with the transverse (divergence free) current as source. The Darwin Lagrangian results when the term $c^{-2}\partial^2/\partial t^2$ in the d'Alembert operator is neglected so that the equation becomes a Poisson equation.

The Darwin Lagrangian has been shown to be a unique approximately relativistic Lagrangian (Woodcock and Havas [5], Kennedy [6]). It can be derived from the Fokker-Wheeler-Feynman action-at-a-distance theory (Anderson and Schiminovich [7]), and it is useful in various fundamental studies of electrodynamics [8–11]. The magnetic interaction described by the Darwin Lagrangian is essential in relativistic many-electron calculations as noted by Breit and others [12–15]. It has found applications in nuclear physics [16, 17], and especially in plasma physics, for numerical simulation [18–22], thermodynamics and kinetics [23–27], as well fundamental theory [28–30]. Barcons and Lapiedra [31] noted that the Darwin approach is not valid for a relativistic plasma

and therefore used a different approach to its statistical mechanics.

Corrections to the Darwin Lagrangian have been discussed. Since a system of particles with identical charge to mass ratio does not dipole radiate a higher order expansion should be meaningful for such systems [32–34]. To that order, however, acceleration inevitably enters and must be handled in some way. Others have argued that since radiation is due to acceleration, v/c expansion is irrelevant, and further that radiation can be negligible even if the particle speeds are considerable (Trubnikov and Kosachev [35], Frejlak [36]). We will pursue that lead here.

One frequently encounters the statement that the Darwin approach neglects retardation. This may be due to the fact that the, nowadays best known, elegant derivation by Jackson [4] hides the complications due to retardation. Nevertheless it is wrong. The derivations by Darwin [1] and by Landau and Lifshitz [2] show that the contribution of retardation to the Coulomb potential in the Lorenz gauge, is quite large. The main acceleration dependent part, however, vanishes either, as in Darwin's derivation, because it gives a total time derivative term in the Lagrangian, or, as in Landau and Lifshitz, because of a gauge transformation (to the Coulomb gauge). Both these derivations also show that the velocity dependent part of the retardation is handled exactly to order $(v/c)^2$.

A natural idea that does *not* work is to assume constant velocities and use the known exact Liénard-Wiechert potentials for that case in (1). Darwin's original derivation shows that this does not give the electric interaction to sufficiently accuracy. It is important to note that gauge invariance (for a review, see Jackson and Okun [37]), which is valid for the exact theory, does not necessarily hold for approximations. We therefore impose the Coulomb gauge (for a recent discussion see Heras [38]) and then solve the inhomogeneous wave equation for \boldsymbol{A} assuming constant velocities in the transverse current density. In this way one treats the electric interaction exactly, neglects acceleration in the solution for \boldsymbol{A} , but do not assume low speeds.

The constant velocity exact Coulomb gauge vector potential does not seem to be well known. A special case was solved by Labarthe [39]. The explicit general solution has recently been published by Hnizdo [40] who used a gauge transformation function given by Jackson [41] to find it, starting from the corresponding Liénard-Wiechert potentials. Denote by, $\mathbf{R} = \mathbf{r} - \mathbf{r}'(t)$, the vector from the source particle at $\mathbf{r}'(t)$, with charge e, to the field point \mathbf{r} , so that $\mathbf{\beta} = \dot{\mathbf{r}}'(t)/c$. If we then put,

$$\boldsymbol{\eta} = \hat{\boldsymbol{R}} \times \boldsymbol{\beta},\tag{4}$$

Hnizdo's solution, which assumes the source particle to be at the origin at time t = 0 and to have constant velocity along the x-axis, i.e. $\mathbf{r}'(t) = c\beta t\hat{\mathbf{x}}$, can be written,

$$A_{Cx} = \beta \phi_L - (\phi_L - \phi_C)/\beta, \tag{5}$$

$$A_{Cy} = \frac{yx}{y^2 + z^2} (\phi_L - \phi_C)/\beta,$$
 (6)

$$A_{Cz} = \frac{zx}{y^2 + z^2} (\phi_L - \phi_C)/\beta,$$
 (7)

at t=0, so that x,y,z are the components of \mathbf{R} . Here, $\phi_C=e/R$, is the Coulomb potential, and, $\phi_L=\phi_C/\sqrt{1-\eta^2}$, its Lorenz gauge form. One notes the identity $1/(y^2+z^2)=(\beta/\eta)^2/R^2$. Using this, and that the only relevant vectors are \mathbf{R} and $\boldsymbol{\beta}$, one can, by expressing everything in terms of these, or scalar and vector products involving these, arrive at the coordinate independent form,

$$\mathbf{A}_{C}(\mathbf{r}) = \frac{e\left[g(\eta^{2})\boldsymbol{\beta} + h(\eta^{2})(\boldsymbol{\beta} \cdot \hat{\mathbf{R}})\hat{\mathbf{R}}\right]}{R},$$
 (8)

of Hnizdo's solution. Here we have introduced the notation, and the functions g and h are defined by,

$$g(x) \equiv \frac{1}{1 + \sqrt{1 - x}} \approx \frac{1}{2} + \frac{1}{8}x + \dots,$$
 (9)

and,

$$h(x) \equiv \frac{g(x)}{\sqrt{1-x}} \approx \frac{1}{2} + \frac{3}{8}x + \dots$$
 (10)

Note that g(1) = 1 but that h diverges for x = 1. From these expansions it is clear that the leading terms give the vector potentials (3) of the Darwin Lagrangian.

The vector potential of the original Darwin Lagrangian is thus recovered from (8) when $\eta=0$. One notes that in the derivation of (3) there was no need to assume that the velocity is constant since the solution to the Poisson equation does not require retardation, while it is necessary for solving the wave equation. Hence the assumption of constant velocity. One purpose of a Lagrangian is, after all, to find equations of motion that determine the accelerations. If it is necessary to know them beforehand a Lagrangian approach is pointless.

It is remarkable that an equivalent vector potential has been found by Crater and Lusanna [9] in a canonical formalism. When the momenta (denoted $\vec{\kappa}$) of Eq. (5.28) of [9] are replaced by $m\beta/\sqrt{1-\beta^2}$ the expression (8) is recovered. The authors of [9] use a relativistic phase space formalism and assume that charges are anticommuting Grassmann variables. In this way they treat the Pauli exclusion principle semiclassically. Their Hamiltonian formalism, which must entail the neglect of acceleration in an indirect way, is mainly intended for treatment of bound states. The Hamiltonian based on the ordinary Darwin Lagrangian (1) is discussed in [25].

The explicit expression for the interaction Lagrangian of two particles that results when (8) replaces (3) in (1) is.

$$L_{12} = \frac{e_1 e_2}{r_{21}} \left[\frac{g(\eta_1^2) + g(\eta_2^2)}{2} \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2 + \frac{h(\eta_1^2) + h(\eta_2^2)}{2} (\boldsymbol{\beta}_1 \cdot \hat{\boldsymbol{r}}_{21}) (\boldsymbol{\beta}_2 \cdot \hat{\boldsymbol{r}}_{21}) - 1 \right], \tag{11}$$

where, $\eta_a^2 = (\hat{r}_{ab} \times \boldsymbol{\beta}_a)^2$. We now consider two special cases. If the velocity of a particle is parallel to the interparticle vector to another particle, $\boldsymbol{\eta} = \mathbf{0}$, so the Darwin interaction needs no correction in these cases. Assuming that two particles have equal velocities $\boldsymbol{v}_1 = \boldsymbol{v}_2 = \boldsymbol{v}$ parallel to \hat{r}_{21} we find the the interaction term in (1) gives,

$$L_{12} = \frac{e_1 e_2}{r_{21}} \frac{v^2}{c^2} - \frac{e_1 e_2}{r_{21}},\tag{12}$$

and (11) gives the same result. One sees that this term, and the corresponding force, goes to zero in the ultrarelativistic limit $v \to c$. Now consider instead the interaction of two particles that move with equal velocities, $v_1 = v_2 = v$, but side by side, so that, $\hat{r}_{21} \perp v$. The interaction part of the Lagrangian (1) is then,

$$L_{12} = \frac{1}{2} \frac{e_1 e_2}{r_{21}} \frac{v^2}{c^2} - \frac{e_1 e_2}{r_{21}}.$$
 (13)

One sees that even in the limit $v \to c$ the Coulomb interaction dominates; the magnetic interaction can only compensate for half of it. This is clearly wrong, however. It is well known that in an ultra-relativistic beam the transverse Lorentz force cancels the transverse Coulomb repulsion (see e.g. [42]).

Let us instead use the vector potential (8) and the corresponding interaction (11). We first note that in this, side by side, case $\eta^2 = v^2/c^2$ and thus $\eta^2 = 1$ in the limit $v \to c$. In this limit g(1) = 1 and h diverges. The two scalar products in the second term will, however, be zero and a simple investigation shows that this compensates for the divergence of h, so that term does not contribute.

Finally we get,

$$L_{12} = g(v^2/c^2) \frac{e_1 e_2}{r_{21}} \frac{v^2}{c^2} - \frac{e_1 e_2}{r_{21}},$$
 (14)

and in the limit $v \to c$ this term and the corresponding force, are zero, as they should, when (11) is used.

In conclusion, the Lagrangian obtained by using the exact constant velocity Coulomb gauge vector potential (8), instead of the \mathbf{A} used in (3), has been derived without assuming that v/c is small, only that accelerations are not needed in estimating the Coulomb gauge vector potential. In this way all velocity dependent retardation, and, as discussed above, also the main part of the acceleration dependent retardation, is accounted for. We have also shown that using this Lagrangian we account correctly for the pinching of an ultra-relativistic beam, something the original Darwin Lagrangian does not do.

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