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QUANTUM-MECHANICAL DESCRIPTION OF THE ELECTROMAGNETIC INTERACTION OF RELATIVISTIC PARTICLES WITH ELECTRIC AND MAGNETIC DIPOLE MOMENTS

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The Hamiltonian of relativistic particles with electric and magnetic dipole moments that interact with an electromagnetic field is determined in the Foldy–Wouthuysen representation. Transition to the semiclassical approximation is carried out. The quantum-mechanical and semiclassical equations of spin motion are derived.

Nowadays the electric dipole moments of particles, nuclei, and atoms are intensively investigated. The components comprising derivatives of the electric field strength cannot be neglected for atoms and nuclei, because they determine, in particular, the contact (Darwin) interaction. Relativistic effects are also important for particles, nuclei, and heavy atoms.

A very convenient method of description of the relativistic particle interaction with an external field and of transition to the semiclassical description is the Foldy–Wouthuysen transformation [1]. In the Foldy–Wouthuysen representation, the Hamiltonian and all operators have a block-diagonal structure (diagonal for two spinors). Relationships between the operators are equivalent to those between the corresponding classical parameters. The operators in the Foldy–Wouthuysen representation are the same as in nonrelativistic quantum theory. Only the Foldy–Wouthuysen representation has these properties, which considerably simplify the transition to the semiclassical description. The Foldy–Wouthuysen representation provides the best opportunity for the transition to the classical limit of relativistic quantum mechanics [1–3].

It is very important that in this representation, the operators of coordinates $\boldsymbol{r}$, momenta[1] $\boldsymbol{p} = -i\nabla$, and polarization

$$\boldsymbol{\Pi} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix},$$

have very simple forms. Here $\boldsymbol{\sigma}$ is the Pauli matrix. In other representations, these operators are described by much more cumbersome formulas [1, 3]. This makes the Foldy–Wouthuysen representation extremely convenient for obtaining equations of particle and spin motion. In particular, the operator equation of spin motion is given by the formula

$$\frac{d\boldsymbol{\Pi}}{dt} = i[H, \boldsymbol{\Pi}], \tag{1}$$

where H is the Hamiltonian, and [..., ...] designates the commutator. To derive the semiclassical equation of spin motion, averaging over wave functions [3] must be performed.

In [3] the Hamiltonian of relativistic particles with a spin of 1/2 interacting with the electromagnetic field was found in the Foldy–Wouthuysen representation. Calculations were carried out to within derivatives of electric and magnetic external field strengths (the fields were generally nonstationary) with allowance for the anomalous particle magnetic moments.

[1]The relativistic system of units with $h = c = 1$ is used in the present work.

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In the present work, the Foldy–Wouthuysen transformation is used for relativistic particles with anomalous magnetic and electric dipole moments interacting with an electromagnetic field. The transformation method was described in detail in [3].

The electric dipole moment can be considered by inclusion of terms characterising the electric dipole moment into the Dirac–Pauli Hamiltonian describing the interaction of particles having anomalous magnetic moments with the electromagnetic field. The Dirac–Pauli equation has the form

$$\left[\gamma^{\mu}\pi_{\mu}-m+\frac{\mu'}{2}\sigma^{\mu\nu}F_{\mu\nu}\right]\Psi=0,\qquad \pi_{\mu}=p_{\mu}-eA_{\mu},\tag{2}$$

where γ^{μ} are the Dirac matrices; $F_{\mu\nu}$ is the electromagnetic field tensor; p^{μ} and $A^{\mu}=(\Phi,\boldsymbol{A})$ are the four-dimensional particle momentum and external field potential, respectively; $\sigma^{\mu\nu}=i(\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu})/2$; m is the mass of the particle at rest; and μ' is the anomalous particle magnetic moment.

The corresponding Hamiltonian in the Foldy–Wouthuysen representation disregarding the electric dipole moment is characterized by the expression [3]

$$\begin{aligned}H&=\beta\varepsilon'+e\Phi+\frac{1}{4}\left\{\left(\frac{\mu_0 m}{\varepsilon'+m}+\mu'\right)\frac{1}{\varepsilon'},(\boldsymbol{\Sigma}[\boldsymbol{\pi}\boldsymbol{E}]-\boldsymbol{\Sigma}[\boldsymbol{E}\,\boldsymbol{\pi}]-\nabla\cdot\boldsymbol{E})\right\}_{+}\\&+\frac{\mu_0 m}{16}\left\{\frac{2\varepsilon'^2+2\varepsilon' m+m^2}{\varepsilon'^4(\varepsilon'+m)^2},\boldsymbol{\pi}\nabla(\boldsymbol{\pi}\cdot\boldsymbol{E}+\boldsymbol{E}\boldsymbol{\pi})\right\}_{+}-\frac{1}{2}\left\{\left(\frac{\mu_0 m}{\varepsilon'}+\mu'\right),\boldsymbol{\Pi}\cdot\boldsymbol{H}\right\}_{+}\\&+\frac{\mu'}{4}\left\{\frac{1}{\varepsilon'(\varepsilon'+m)},[(\boldsymbol{H}\boldsymbol{\pi})(\boldsymbol{\Pi}\boldsymbol{\pi})+(\boldsymbol{\Pi}\boldsymbol{\pi})(\boldsymbol{\pi}\boldsymbol{H})+2\pi(\boldsymbol{\pi}\cdot\boldsymbol{j}+\boldsymbol{j}\boldsymbol{\pi})]\right\}_{+},\end{aligned}\tag{3}$$

where $\{\ldots,\ldots\}_{+}$ designates the anticommutator, $\mu_0=e/(2m)$ is the Dirac magnetic moment, and

$$\varepsilon'=\sqrt{m^2+\boldsymbol{\pi}^2}.\tag{4}$$

The anomalous magnetic and electric dipole moments are closely related, because they determine the real and imaginary parts of the same physical quantity [4, 5]. The contributions of the anomalous magnetic and electric dipole moments to the Lagrangian are the following [4]:

$$L_{\text{AMM}}=\frac{\mu'}{2}\sigma^{\mu\nu}F_{\mu\nu},\qquad L_{\text{EDM}}=-i\frac{d}{2}\sigma^{\mu\nu}\gamma^5F_{\mu\nu},\qquad \gamma^5=\begin{pmatrix}0&-1\\-1&0\end{pmatrix},\tag{5}$$

where d is the electric dipole moment of the particle, and 0 and -1 designate the corresponding 2×2 matrices.

A consideration of the electric dipole moment of the particle consists in the inclusion of the term proportional to d into the Dirac–Pauli equation. As a result, this equation assumes the form

$$\left[\gamma^{\mu}\pi_{\mu}-m+\frac{\mu'}{2}\sigma^{\mu\nu}F_{\mu\nu}-i\frac{d}{2}\sigma^{\mu\nu}\gamma^5F_{\mu\nu}\right]\Psi=0.\tag{6}$$

The Hamiltonian in the Dirac representation is described by the expression

$$H_{\text{D}}=\beta m+\boldsymbol{\alpha}\cdot\boldsymbol{\pi}+e\Phi+\mu'(-\boldsymbol{\Pi}\cdot\boldsymbol{H}+i\boldsymbol{\gamma}\cdot\boldsymbol{E})-id(-\boldsymbol{\Pi}\cdot\boldsymbol{H}+i\boldsymbol{\gamma}\cdot\boldsymbol{E})\gamma^5,\tag{7}$$

where $\boldsymbol{\pi} = -i\nabla - e\boldsymbol{A}$ and $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic field strengths. Hereinafter, we use the following standard designations:

$$\gamma = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta \equiv \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\alpha} = \beta\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$
$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad \boldsymbol{\Pi} = \beta\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}.$$

The Hamiltonian given by Eq. (7) is reduced to the form

$$H_{\mathrm{D}} = \beta m + \boldsymbol{\alpha}\cdot\boldsymbol{\pi} + e\Phi + \mu'(-\boldsymbol{\Pi}\cdot\boldsymbol{H} + i\boldsymbol{\gamma}\cdot\boldsymbol{E}) - d(\boldsymbol{\Pi}\cdot\boldsymbol{E} + i\boldsymbol{\gamma}\cdot\boldsymbol{H}). \tag{8}$$

Formula (8) demonstrates that terms describing contributions of anomalous magnetic and electric dipole moments to the Hamiltonian are transformed into each other using the substitutions

$$\boldsymbol{H} \to \boldsymbol{E}, \qquad \boldsymbol{E} \to -\boldsymbol{H}, \qquad \mu' \to d. \tag{9}$$

The same relationships between these terms take place in the classical description.

We note here that to correctly include the electric dipole moment into the Dirac–Pauli equation, we can obviate the need for Eqs. (5) and (6). A more natural form is used in classical electrodynamics [6]. In this case, the interaction of the electric dipole moment with the electromagnetic field is described by the tensor $G^{\mu\nu} = (-\boldsymbol{H}, -\boldsymbol{E})$ dual with respect to the electromagnetic field $F^{\mu\nu} = (-\boldsymbol{E}, \boldsymbol{H})$. When the tensor $G^{\mu\nu}$ is used to describe the electric dipole moment, the generalized Dirac–Pauli equation assumes the form

$$\left[\gamma^{\mu}\pi_{\mu} - m + \frac{\mu'}{2}\sigma^{\mu\nu}F_{\mu\nu} - \frac{d}{2}\sigma^{\mu\nu}G_{\mu\nu}\right]\Psi = 0, \tag{10}$$

and the Lagrangian L_{EDM} is reduced to the form

$$L_{\mathrm{EDM}} = -\frac{d}{2}\sigma^{\mu\nu}G_{\mu\nu}. \tag{11}$$

The Hamiltonian of particles with anomalous magnetic and electric dipole moments, calculated by the method suggested in [3], has the form

$$\begin{aligned} H = {} & \beta\varepsilon' + e\Phi + \frac{1}{4}\left\{\left(\frac{\mu_0 m}{\varepsilon' + m} + \mu'\right)\frac{1}{\varepsilon'}, (\boldsymbol{\Sigma}[\boldsymbol{\pi}{\cdot}\boldsymbol{E}] - \boldsymbol{\Sigma}[\boldsymbol{E}\cdot\boldsymbol{\pi}] - \nabla{\cdot}\boldsymbol{E})\right\}_{+} \\ & + \frac{\mu_0 m}{16}\left\{\frac{2\varepsilon'^2 + 2\varepsilon' m + m^2}{\varepsilon'^4(\varepsilon' + m)^2}, \boldsymbol{\pi}\nabla(\boldsymbol{\pi}{\cdot}\boldsymbol{E} + \boldsymbol{E}{\cdot}\boldsymbol{\pi})\right\}_{+} - \frac{1}{2}\left\{\left(\frac{\mu_0 m}{\varepsilon'} + \mu'\right), \boldsymbol{\Pi}{\cdot}\boldsymbol{H}\right\}_{+} \\ & + \frac{\mu'}{4}\left\{\frac{1}{\varepsilon'(\varepsilon' + m)}, [(\boldsymbol{H}{\cdot}\boldsymbol{\pi})(\boldsymbol{\Pi}{\cdot}\boldsymbol{\pi}) + (\boldsymbol{\Pi}{\cdot}\boldsymbol{\pi})(\boldsymbol{\pi}{\cdot}\boldsymbol{H}) + 2\pi(\boldsymbol{\pi}{\cdot}\boldsymbol{j} + \boldsymbol{j}{\cdot}\boldsymbol{\pi})]\right\}_{+} - d\boldsymbol{\Pi}{\cdot}\boldsymbol{E} \\ & + \frac{d}{4}\left\{\frac{1}{\varepsilon'(\varepsilon' + m)}, [(\boldsymbol{E}{\cdot}\boldsymbol{\pi})(\boldsymbol{\Pi}{\cdot}\boldsymbol{\pi}) + (\boldsymbol{\Pi}{\cdot}\boldsymbol{\pi})(\boldsymbol{\pi}{\cdot}\boldsymbol{E})]\right\}_{+} - \frac{d}{4}\left\{\frac{1}{\varepsilon'}, (\boldsymbol{\Sigma}[\boldsymbol{\pi}\times\boldsymbol{H}] - \boldsymbol{\Sigma}[\boldsymbol{H}\times\boldsymbol{\pi}])\right\}_{+}. \end{aligned} \tag{12}$$

A comparison between Eqs. (3) and (12) demonstrates that substitution (9) is allowable for Hamiltonian (3). Despite analogous descriptions of the anomalous magnetic and electric dipole moments, the interaction of these two moments with the electromagnetic field differs in principle. In the expression for the Hamiltonian in the Foldy–Wouthuysen representation, there are no terms proportional to the electric dipole moment and involving the first derivatives of the field strengths. The corresponding terms proportional to the anomalous magnetic moment characterize the contact interaction with external charges and currents. This result, caused by the absence of magnetic charges and currents, is very important, because it simplifies the estimation of the electric dipole moment contribution to relativistic expression (12) for the Hamiltonian.

An analysis of Eq. (12) demonstrates that in the examined approximation, the last but one term gives the main relativistic correction to the operator describing the interaction of the electric dipole moment with the external field for atoms and nuclei:

$$\Delta H = \frac{d}{4}\left\{\frac{1}{\varepsilon'(\varepsilon'+m)},[(\boldsymbol{E}\boldsymbol{\pi})(\boldsymbol{\Pi}\boldsymbol{\pi})+(\boldsymbol{\Pi}\boldsymbol{\pi})(\boldsymbol{\pi}\boldsymbol{E})]\right\}_+. \tag{13}$$

Let us emphasize that the relativistic corrections for nuclei and heavy atoms are not small and can reach several tens of percent [7]. Their calculations in the Foldy–Wouthuysen representation (rather than in the Dirac representation commonly used) can be more convenient.

The particle spin motion in the electromagnetic field is also very important. The operator equation of spin motion for relativistic particles with anomalous magnetic and electric dipole moments, derived with the help of Eqs. (1) and (12), has the form

$$\begin{aligned}\frac{d\boldsymbol{\Pi}}{dt} &= \left\{\left(\frac{\mu_0 m}{\varepsilon'+m}+\mu'\right)\frac{1}{\varepsilon'},[\boldsymbol{\Pi}\times[\boldsymbol{E}\times\boldsymbol{\pi}]]\right\}_+ + \left\{\left(\frac{\mu_0 m}{\varepsilon'}+\mu'\right),[\boldsymbol{\Sigma}\times\boldsymbol{H}]\right\}_+ \\ &-\frac{\mu'}{2}\left\{\frac{1}{\varepsilon'(\varepsilon'+m)},([\boldsymbol{\Sigma}\times\boldsymbol{\pi}](\boldsymbol{\pi}\cdot\boldsymbol{H})+(\boldsymbol{H}\cdot\boldsymbol{\pi})[\boldsymbol{\Sigma}\times\boldsymbol{\pi}])\right\}_+ + 2d[\boldsymbol{\Sigma}\times\boldsymbol{E}] \\ &-\frac{d}{2}\left\{\frac{1}{\varepsilon'(\varepsilon'+m)},([\boldsymbol{\Sigma}\times\boldsymbol{\pi}](\boldsymbol{\pi}\cdot\boldsymbol{E})+(\boldsymbol{E}\cdot\boldsymbol{\pi})[\boldsymbol{\Sigma}\times\boldsymbol{\pi}])\right\}_+ - d\left\{\frac{1}{\varepsilon'},[\boldsymbol{\Pi}\times[\boldsymbol{H}\times\boldsymbol{\pi}]]\right\}_+.\end{aligned} \tag{14}$$

Transition to the semiclassical approximation was described in [3]. Introducing the factor $\eta = 4dm/e$, we obtain the semiclassical equation of spin motion:

$$\begin{aligned}\frac{d\boldsymbol{\xi}}{dt} &= (\boldsymbol{\Omega}_{\text{TBMT}}+\boldsymbol{\Omega}_{\text{EDM}})\times\boldsymbol{\xi}, \\ \boldsymbol{\Omega}_{\text{TBMT}} &= -\frac{e}{2m}\left\{\left(g-2+\frac{2}{\gamma}\right)\boldsymbol{H}-\frac{(g-2)\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{H})-\left(g-2+\frac{2}{\gamma+1}\right)[\boldsymbol{\beta}\times\boldsymbol{E}]\right\}, \\ \boldsymbol{\Omega}_{\text{EDM}} &= -\frac{e\eta}{2m}\left(\boldsymbol{E}-\frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{E})+\boldsymbol{\beta}\times\boldsymbol{H}\right),\end{aligned} \tag{15}$$

where $\boldsymbol{\Omega}_{\text{TBMT}}$ is determined by the Thomas–Bargmann–Michel–Telegdi equation [8]. Equation (15) coincides with the corresponding classical equation [6].

The influence of the electric dipole moment on the particle motion is negligibly small; however, it influences significantly the spin motion. The character of spin motion caused by the interaction of the electric and magnetic dipole moments with the external field differs radically. This circumstance allows experiments on measuring the electric dipole moment in rings with memory to be carried out [9]. The spin rotation about the momentum vector in the horizontal plane can be compensated with the help of application of a radial electric field having the strength

$$\boldsymbol{E} = \frac{a\gamma^2}{1-a\beta^2\gamma^2}[\boldsymbol{\beta}\times\boldsymbol{H}], \qquad a = \frac{g-2}{2}.$$

In this case, the contribution of the electric dipole moment to the spin motion is characterized by the angular velocity[2]

$$\boldsymbol{\Omega}_{\text{EDM}} = -\frac{e\eta}{2m}\frac{1+a}{1-a\beta^2\gamma^2}[\boldsymbol{\beta}\times\boldsymbol{H}]. \tag{16}$$

Thus, the Foldy–Wouthuysen transformation has been successively used to determine the Hamiltonian and the equation of spin motion for relativistic particles with anomalous magnetic and electric dipole moments. Relativistic corrections to the interaction operator can be used to calculate the electric dipole moments of nuclei and heavy atoms.

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[2] The difference in signs with [9] is explained by the fact that in the present work, the angular velocity vector was defined with the opposite sign.