Subtleties of the Thomas precession

2015 Eur. J. Phys. 36 045007

(http://iopscience.iop.org/0143-0807/36/4/045007)

IOP Publishing European Journal of Physics
Eur. J. Phys. **36** (2015) 045007 (13pp) doi:10.1088/0143-0807/36/4/045007

Subtleties of the Thomas precession

Krzysztof Rębilas

Zakład Fizyki, Uniwersytet Rolniczy im. Hugona Kołłątaja w Krakowie, Al. Mickiewicza 21, 31-120 Kraków, Poland

E-mail: krzysztof.rebilas@ur.krakow.pl

Received 29 October 2014, revised 19 March 2015
Accepted for publication 1 April 2015
Published 30 April 2015

Abstract

The rarely mentioned fact that a pure boost in general distorts the axes of the boosted frame is shown to influence significantly the Thomas precession effect as observed in the laboratory frame. As a result the Thomas precession appears to be accompanied by a 'wobbling' motion of the axes of precessing rest frame connected with a moving particle. A simple method to get an exact solution for the development of the orientation of the frame axes in the case of uniform rotation of the particle is given. This is equivalent to finding the orientation of the spin vector used in the BMT theory. The discrepancy known from the literature in describing the spin, as performing a uniform rotation or as revealing in the same situation some additional oscillatory behaviour, is explained by pointing out two conceptually different approaches, 'hybrid' and fully consistent, in presenting the Thomas precession.

Keywords: Thomas precession, BMT theory, spin dynamics

1. Introduction

The Thomas precession is generally presented in undergraduate courses in the context of spin–orbit coupling as a phenomenon that introduces a correction to the energy of the electron–nucleus interaction. The problem is that while studying deeper this counterintuitive effect, the student encounters treatments in university textbooks that seem to be incompatible. As is explained in more detail below, some authors indicate that spin precesses in a uniform manner while others show that in the *same* physical situation we deal with a more complicated 'wobbling' motion and only after a procedure of averaging the two approaches seem to be equivalent. The discrepancy existing in the literature may cause a significant difficulty in grasping the meaning of the Thomas precession. The aim of this paper is to clarify the issue and point out some aspects of special relativity that are ignored in standard presentations of this topic but appear relevant and crucial for a full understanding of the Thomas precession.

The Thomas precession is a peculiar effect of special relativity [1]. To explain it one can consider the situation shown in figure 1 where system of reference S′ moves along the x-axis

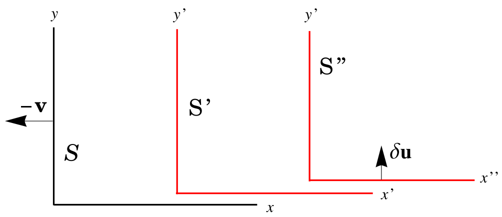

Figure 1. The configuration of frames S, S′ and S″. The velocities and the axes orientations of frames S, S′ and S″ are depicted *from the point of view of frame S′*.

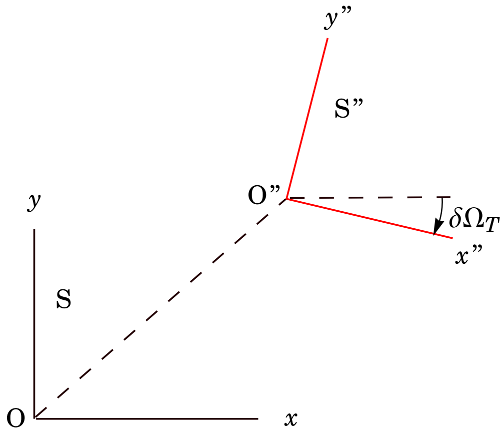

Figure 2. Are the axes of frame S″ actually rotated with respect to frame S by the angle d$\boldsymbol{\Omega}_T$? Line OO″ indicates a direction of relative velocity between frames S and S″.

with a velocity **v** with respect to frame S, and they both have mutually parallel axes. In turn, frame S″ moves along the y'-axis with a velocity $\delta\mathbf{u}$ with respect to frame S′, and the axes of S′ and S″ are parallel as well. To interconnect the coordinates of frames S and S″ one can perform a pure boost with the velocity $-\delta\mathbf{u}$ from frame S″ to frame S′, $L_{\mathrm{S}''\to\mathrm{S}'}$, and then a pure boost $L_{\mathrm{S}'\to\mathrm{S}}$ with the velocity $-\mathbf{v}$ from frame S′ to S. According to special relativity the two successive non-collinear boosts $L_{\mathrm{S}''\to\mathrm{S}'}$ and $L_{\mathrm{S}'\to\mathrm{S}}$ are equivalent to a pure boost $L_{\mathrm{S}''\to\mathrm{S}}$ directly from the frame S″ to the frame S, with the relative velocity between frames S and S″, and a spatial *rotation R*:

$$L_{\mathrm{S}'\to\mathrm{S}}L_{\mathrm{S}''\to\mathrm{S}'} \Longleftrightarrow RL_{\mathrm{S}''\to\mathrm{S}}. \tag{1}$$

For an infinitesimally small $\delta\mathbf{u}$, the rotation R is infinitesimal as well (the Thomas precession) and its angle is [4–6]

$$\mathrm{d}\boldsymbol{\Omega}_T = \frac{\gamma^2}{\gamma+1}\frac{\delta\mathbf{v}\times\mathbf{v}}{c^2} = \frac{\gamma-1}{\beta^2}\frac{\delta\mathbf{v}\times\mathbf{v}}{c^2}, \tag{2}$$

where $\gamma = (1-\beta^2)^{-1/2}$, $\boldsymbol{\beta} = \mathbf{v}/c$ and $\delta\mathbf{v}$ is the velocity between S′ and S″, as measured in frame S. The magnitude of $\delta\mathbf{v}$ is connected with δu throughout the Lorentz transformation, and in our configuration it is given as $\delta v = \delta u/\gamma$.

As the pure boost $L_{\mathrm{S}''\to\mathrm{S}}$ by definition does not involve any spatial rotation, the right-hand side of relation (1) seems to suggest that the composition of transformations

$L_{S'\to S}L_{S''\to S'}$ yields the axes of frame S″ rotated by the angle $\mathrm{d}\boldsymbol{\Omega}_T$ with respect to the axes of frame S (see figure 2). We want to show however that, unexpectedly, this is not the case. And if we use figure 2 to present graphically the Thomas precession, one should remember that it does *not* represent orientations of the axes of S″ in the frame S. We are also going to explain the true meaning of this kind of figure [3].

Another problem is how the Thomas precession influences spin dynamics. If a particle, endowed with spin $\boldsymbol{\sigma}$, moves around a circle with speed v, one can regard frames S′ and S″ as an instantaneous rest frame of the particle at successive moments t and $t + \mathrm{d}t$. If in an instantaneous rest frame the rate of change of the spin is $\left(\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t}\right)_{\mathrm{Rest}}$, then, taking into account that the rest frame performs the Thomas precession, special relativity provides us the equation that determines the spin dynamics as measured in the laboratory frame [4–6]:

$$\left(\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t}\right)_{\mathrm{Lab}} = \left(\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t}\right)_{\mathrm{Rest}} + \boldsymbol{\omega}_T \times \boldsymbol{\sigma}, \tag{3}$$

where $\boldsymbol{\omega}_T = \mathrm{d}\boldsymbol{\Omega}_T/\mathrm{d}t$ is the Thomas precession rate. For a uniform circular motion, $\boldsymbol{\omega}_T$ is constant and its magnitude is given by

$$\omega_T = (\gamma - 1)\omega, \tag{4}$$

where ω is the angular velocity of the particle. In the case when the spin remains at rest in the instantaneous rest frame of the particle, i.e. $\left(\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t}\right)_{\mathrm{Rest}} = 0$, equation (3) becomes

$$\left(\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t}\right)_{\mathrm{Lab}} = \boldsymbol{\omega}_T \times \boldsymbol{\sigma}. \tag{5}$$

This seems to imply that the spin $\boldsymbol{\sigma}$, as observed in the laboratory frame, rotates uniformly with the constant angular velocity ω_T. On the other hand one can find in the literature [8–10] that spin in this case performs a kind of oscillatory motion. We want to show how to resolve this discrepancy and point out that the problem is connected with two conceptually different methods of presentation of the Thomas rotation as observed in the laboratory frame.

2. What is rotated due to the Thomas rotation?

Referring to figure 2, our first task is to find the true orientation of the axes of frame S″ as measured in frame S. The direction of the x''-axis in S″ is determined in this frame by a unit four-vector along this axis:

$$\mathbf{e}''_x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{6}$$

To find its coordinates in frame S, we perform successively the transformations $L_{S''\to S'}$ and $L_{S'\to S}$. The first boost from S″ to S′ is given in a matrix form as

$$L_{\mathrm{S}''\to\mathrm{S}'} = \begin{pmatrix} 1 & 0 & \dfrac{\delta u}{c} & 0 \\ 0 & 1 & 0 & 0 \\ \dfrac{\delta u}{c} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

(to first order in δu), and the next boost, from S′ to S, is

$$L_{\mathrm{S}'\to\mathrm{S}} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

As a result, we get coordinates of e''_x in frame S as

$$\mathbf{e}''_{x(\downarrow\leftarrow)} = \begin{pmatrix} \beta\gamma \\ \gamma \\ 0 \\ 0 \end{pmatrix}. \tag{9}$$

Any result achieved throughout the two perpendicular boosts $L_{\mathrm{S}''\to\mathrm{S}'}$ and $L_{\mathrm{S}'\to\mathrm{S}}$ will be denoted by the index $(\downarrow\leftarrow)$. Similarly, a direction of the y''-axis of the frame S″ is determined in this frame by a unit vector along this axis,

$$\mathbf{e}''_y = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{10}$$

which, when transformed by means of $L_{\mathrm{S}''\to\mathrm{S}'}$ and $L_{\mathrm{S}'\to\mathrm{S}}$ to frame S, is

$$\mathbf{e}''_{y(\downarrow\leftarrow)} = \begin{pmatrix} \gamma\dfrac{\delta u}{c} \\ \beta\gamma\dfrac{\delta u}{c} \\ 1 \\ 0 \end{pmatrix}. \tag{11}$$

The time coordinates of unit vectors $\mathbf{e}''_x$ and $\mathbf{e}''_y$ determining directions of axes in S″ are no longer zero in frame S (equations (9) and (11)). Because any space-time point lying on the x''-axis in S″ is given as $x''\mathbf{e}''_x$ and now, in the frame S, its coordinates are $x''\mathbf{e}''_{x(\downarrow\leftarrow)}$, we see that the simultaneous points forming the axes in frame S″ are no longer simultaneous in frame S. Nevertheless they still determine a spatial direction in frame S. The situation is similar to a trajectory that consists of non-simultaneous points occupied successively by a moving object. Yet these points define a direction when traced back to be in synchronicity with an observer. Formally then, in our case, one has to project the points given in S as $x''\mathbf{e}''_{x(\downarrow\leftarrow)}$ on the hypersurface of constant time (say $t = 0$) in the frame S. And similarly for the points $y''\mathbf{e}''_{y(\downarrow\leftarrow)}$ forming the y''-axis as observed in the frame S. Certainly, the directions of the projected axes are determined now by the unit vectors (9) and (11) projected on the hypersurface $t = 0$, i.e. given in the frame S as (see figure 3):

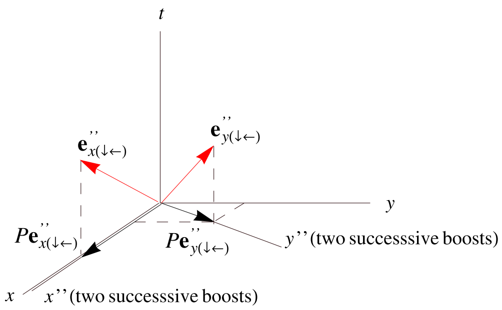

Figure 3. Projection of $\mathbf{e}''_{x(\downarrow\leftarrow)}$ and $\mathbf{e}''_{y(\downarrow\leftarrow)}$ on the hypersurface of constant time in frame S gives us vectors $P\mathbf{e}''_{x(\downarrow\leftarrow)}$ and $P\mathbf{e}''_{y(\downarrow\leftarrow)}$ determining spatial orientations of the x'' and y'' axes of S$''$ in the frame S.

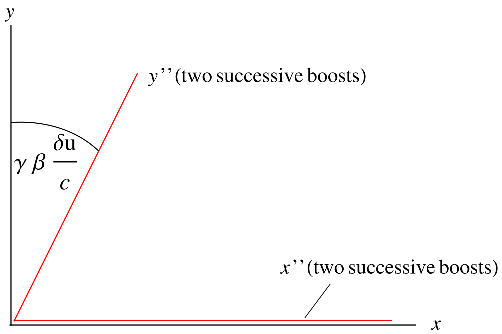

Figure 4. The spatial orientation of the axes x'' and y'' of frame S$''$ from the point of view of frame S when we use the two successive boosts $L_{S''\to S'}$ and $L_{S'\to S}$ to transform S$''$ to S.

$$P\mathbf{e}''_{x(\downarrow\leftarrow)} = \begin{pmatrix} 0 \\ \gamma \\ 0 \\ 0 \end{pmatrix}, \tag{12}$$

$$P\mathbf{e}''_{y(\downarrow\leftarrow)} = \begin{pmatrix} 0 \\ \beta\gamma\dfrac{\delta u}{c} \\ 1 \\ 0 \end{pmatrix}. \tag{13}$$

Using them one can easily establish the spatial orientation of the axes of S$''$ as observed in S. Because the y component of vector $P\mathbf{e}''_{x(\downarrow\leftarrow)}$ is zero (equation (12)), the x''-axis of S$''$ makes with the x-axis in S an angle equal to zero (see figure 4):

$$\alpha_{x(\downarrow\leftarrow)} = 0. \tag{14}$$

In turn, $P\mathbf{e}''_{y(\downarrow\leftarrow)}$ (equation(13)), determining the spatial direction of the y''-axis of S$''$ as measured in frame S, is inclined with respect to the y-axis of frame S by an angle (figure 4):

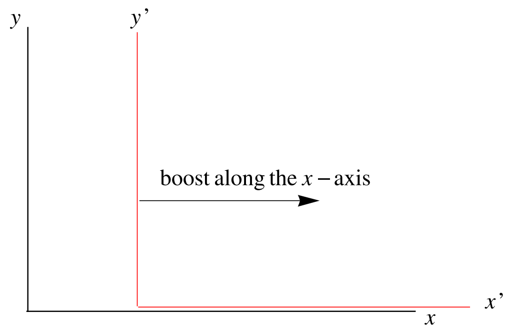

Figure 5. Typically, a boost is chosen to be performed along the x-axis. This choice does not influence the orientation of the frame axes.

$$\alpha_{y(\downarrow\leftarrow)} = \beta\gamma\frac{\delta u}{c}. \tag{15}$$

Surprisingly, the axes of the frame S″ are not Thomas-rotated with respect to the axes of frame S, as we could expect on the basis of relation (1). One of them is not rotated at all and the second one is rotated but by an angle that does not agree with equation (2). The effect is then a distortion of the frame rather than a rotation (precession). Does therefore our result contradict special relativity?

To solve this puzzle we notice something that is almost never mentioned in the literature, namely that a pure boost in an arbitrary direction in general distorts the system of coordinates[1]. Normally, if a boost is performed along, say, an x-axis, the axes of the boosted frame preserve their orientation (figure 5). And this is how a boost is usually depicted in the textbooks. Let us however perform a single boost $L_{\mathrm{S}''\to\mathrm{S}}$ directly from frame S″ to frame S. In this case, the velocity of frame S with respect to frame S″, to lowest order in δu, is equal to $-v\mathbf{e}''_x - \delta u\mathbf{e}''_y$, and is not oriented along any of the axes of the frame S″. The corresponding matrix form of the Lorentz transformation is

$$L_{\mathrm{S}''\to\mathrm{S}} = \begin{pmatrix} \gamma & \gamma\beta & \gamma\frac{\delta u}{c} & 0 \\ \gamma\beta & \gamma & \frac{(\gamma-1)\delta u}{\beta c} & 0 \\ \gamma\frac{\delta u}{c} & \frac{(\gamma-1)\delta u}{\beta c} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

Using $L_{\mathrm{S}''\to\mathrm{S}}$ we can transform the unit vectors of frame S″ (given by equations (6) and (10)) to frame S. As a result, we obtain

[1] To our knowledge the fact that the pure boost in general distorts the axes of the boosted frame is mentioned only in [2] p 47.

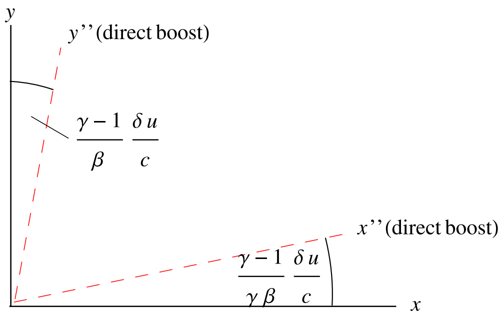

Figure 6. The orientation of axes x'' and y'' of frame S$''$ from the point of view of frame S in the case when we use the direct boost $L_{S''\to S}$ to transform S$''$ to S.

$$\mathbf{e}''_{x(\swarrow)} = \begin{pmatrix} \beta\gamma \\ \gamma \\ \dfrac{(\gamma-1)}{\beta}\dfrac{\delta u}{c} \\ 0 \end{pmatrix}, \tag{17}$$

$$\mathbf{e}''_{y(\swarrow)} = \begin{pmatrix} \gamma\,(\delta u/c) \\ \dfrac{(\gamma-1)}{\beta}\dfrac{\delta u}{c} \\ 1 \\ 0 \end{pmatrix}, \tag{18}$$

where the symbol $(\swarrow)$ indicates that we deal with a single direct boost from S$''$ to S obliquely to the frame axes. Projecting $\mathbf{e}''_{x(\swarrow)}$ and $\mathbf{e}''_{y(\swarrow)}$ on the hypersurface of constant time we get

$$P\mathbf{e}''_{x(\swarrow)} = \begin{pmatrix} 0 \\ \gamma \\ \dfrac{(\gamma-1)}{\beta}\dfrac{\delta u}{c} \\ 0 \end{pmatrix}, \tag{19}$$

$$P\mathbf{e}''_{y(\swarrow)} = \begin{pmatrix} 0 \\ \dfrac{(\gamma-1)}{\beta}\dfrac{\delta u}{c} \\ 1 \\ 0 \end{pmatrix}. \tag{20}$$

Vectors $P\mathbf{e}''_{x(\swarrow)}$ and $P\mathbf{e}''_{y(\swarrow)}$ indicate spatial orientations of the axes of frame S$''$ transformed to S by means of pure boost $L_{S''\to S}$.

Contrary to the situation shown in figure 5, equations (19) and (20) point out that the pure direct boost $L_{S''\to S}$ affects the spatial part of both x'' and y'' axes, making them *not* parallel to the axes of frame S. According to equation (19) the x''-axis is skewed in S with respect to the x-axis at an angle

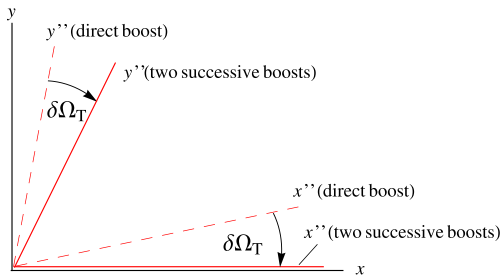

Figure 7. The Thomas rotation by an angle $\delta\Omega_T$ observed in frame S refers to the mutual orientation of the axes of frame S″ in two cases: when they are transformed to frame S by means of the two successive boosts and when they are achieved in frame S due to the direct boost.

$$\alpha_{x(\swarrow)} = \frac{\gamma - 1}{\beta\gamma}\frac{\delta u}{c} \tag{21}$$

(which is shown in figure 6), and from equation (20) we get that the y''-axis makes with the y-axis in S an angle

$$\alpha_{y(\swarrow)} = \frac{\gamma - 1}{\beta}\frac{\delta u}{c}. \tag{22}$$

The fact that the (spatial) directions of the axes of S″ are no longer perpendicular to each other when transformed to frame S is an unintuitive feature of special relativity, but it enables us to restore the Thomas precession effect which is veiled in figure (4). The question now is: where is the Thomas rotation? Let us compose together figures 4 and 6; the result is shown in figure 7.

As can be seen, the axes x'' and y'' of the frame S″ transformed to S by means of two successive perpendicular boosts are actually rotated, but not with respect to the axes x and y. The rotation of the axes of S″ is with respect to the skewed directions of the axes of this same frame S″ but achieved in S due to the direct boost $L_{S''\to S}$. And one can verify directly that the both axes are actually rotated by the same angle equal to

$$\begin{aligned}\delta\Omega_T &= \left|\alpha_{x(\downarrow\leftarrow)} - \alpha_{x(\swarrow)}\right| = \left|\alpha_{y(\downarrow\leftarrow)} - \alpha_{y(\swarrow)}\right| \\ &= \frac{\gamma - 1}{\gamma\beta}\frac{\delta u}{c} = \frac{\gamma - 1}{\beta}\frac{\delta v}{c} = \frac{\gamma - 1}{\beta^2}\frac{|\delta\mathbf{v}\times\mathbf{v}|}{c^2},\end{aligned} \tag{23}$$

which is just the Thomas rotation angle in accordance with equation (2). All these certainly agree with relation (1). But the surprise is that although $L_{S''\to S}$ in (1) itself does not contain a rotation, it changes direction of the transformed system axes, so that the overall rotation included in the product $RL_{S''\to S}$ is not simply a rotation R by the angle $\delta\Omega_T$.

Coming back to figure 2, its true meaning can now be explained. In fact it consists of two merged, *independent* descriptions that are performed in the different frames S and S″. To be more specific, figure 2 presents simultaneously the orientation of the axes of frame S with respect to the direction OO″, *as seen in the frame* S and, secondly, the orientation of the axes of frame S″ with respect to the same direction OO″, but *seen in frame* S″. Figure 2 cannot then be regarded as depicting the orientation of frame S″ as measured in S. Rather, the orientation of frame S″ from the point of view of frame S is correctly represented in figure 4.

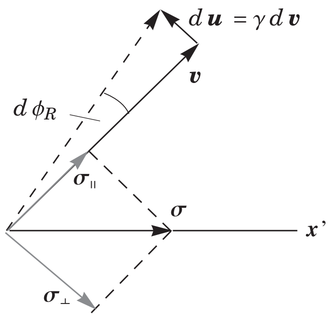

Figure 8. While in the laboratory system the velocity changes its direction by an angle $\mathrm{d}v/v$ then, with respect to the axes of the instantaneous rest frame of particle, the velocity rotates by an angle $\mathrm{d}\phi_R = \gamma \mathrm{d}v/v$.

3. Time development of spin orientation in the laboratory system

In the BMT theory[7] the spin of a particle is identified with a four-vector that in the rest frame of particle always has the form of $(0, \boldsymbol{\sigma})$. Because its time component in the rest frame is always zero, in the case when $\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)_{\mathrm{Rest}} = 0$, one can regard $\boldsymbol{\sigma}$ as still aligned with an axis of the rest frame, i.e. as representing the axis direction. Because frames S′ and S″ can be identified with an instantaneous rest frame of the particle at successive moments t and $\mathrm{d}t$, the results of the previous section must be relevant to the spin dynamics as well. The fact that a pure boost distorts the frame axes entails non-trivial behaviour of spin from the point of view of the laboratory frame.

3.1. Hybrid description

Consider a circular motion of a particle having spin. During the whole motion the axes of the particle rest frame[2] may assume arbitrary orientation with respect to the particle velocity $\mathbf{v}$. To obtain dynamics of the axes of the rest frame (and the spin aligned with one of the axis) from the point of view of the laboratory frame S one could in essence repeat the procedure from section 2. However, the method is not so straightforward in this more general case. To deal with the task we choose another way. Let the motion proceed in the x–y plane in the frame S and $\boldsymbol{\sigma}$ be aligned with the x- axis of the rest frame (figure 8). The crucial observation is that if in the laboratory system the particle changes its velocity by δv, in the instantaneous rest frame of the particle, according to the Lorentz transformation, the velocity change is $\mathrm{d}u = \gamma \mathrm{d}v$. It follows that while the particle velocity rotates in the laboratory frame by an angle

$$\mathrm{d}\phi = \mathrm{d}v/v, \tag{24}$$

then in the rest frame, i.e. *with respect to its axes*, the velocity vector appears to be rotated by

$$\mathrm{d}\phi_R = \gamma \mathrm{d}v/v. \tag{25}$$

In other words, while the velocity vector rotates in the laboratory frame with an angular velocity

[2] The rest frame $R_{t+\mathrm{d}t}$ at a moment $t + \mathrm{d}t$ is achieved by means of an appropriate *pure boost* from the rest frame R_t given at a moment t.

$$\omega \mathrm{d}\phi/\mathrm{d}t, \tag{26}$$

then with respect to the axes of the instantaneous rest frame of the particles the rate of rotation of velocity is

$$\omega_R = \mathrm{d}\phi_R/\mathrm{d}t = \gamma\omega. \tag{27}$$

But because the spin $\boldsymbol{\sigma}$ is assumed to lie on the axis of the rest frame, it means that ω_R is just the rate of rotation of $\boldsymbol{\sigma}$ *with respect to the velocity*.

Unlike in classical physics, ω_R is not equal to ω. This fact is just the source of the Thomas effect. The difference

$$\omega_R - \omega = (\gamma - 1)\omega \equiv \omega_T \tag{28}$$

characterizes an excessive, compared to classical physics, lagging of the spin $\boldsymbol{\sigma}$ behind the velocity. In this way then we have derived equations (4) and (5). Notice however that these equations, although being true, are a bit peculiar because they mix the quantity $\boldsymbol{\sigma}$ defined in the *rest* frame with a description performed from the point of view of the *laboratory* frame. This hybrid approach is commonly used because the state of spin interacting with external fields is most easily characterized in its rest frame. As we can see, in this picture the spin of circulating particle performs *uniform* Thomas rotation.

3.2. Fully consistent description

It is possible to develop further our reasoning in order to obtain a fully consistent description of the evolving spin. The additional thing we have to do is to transform the four-vector $(0, \boldsymbol{\sigma})$ to the laboratory frame in which the spin behaviour is to characterized. As a result we get a four-vector Z (called spin four-vector) with the spatial part equal to

$$\mathbf{Z} \equiv \mathbf{Z}_{\|} + \mathbf{Z}_{\perp} = \gamma\boldsymbol{\sigma}_{\|} + \boldsymbol{\sigma}_{\perp}, \tag{29}$$

where the indices refer to the components parallel and perpendicular to the velocity. Notice that the vector $\mathbf{Z}$ is just the spatial orientation of the axis $(0, \boldsymbol{\sigma})$ of the rest frame, as measured in the laboratory frame. Therefore, finding $\mathbf{Z}$ is a direct counterpart, and a generalization, of discovering how axes of S″ are represented in the frame S, which was performed in section 2 for the specific orientation of the frame axes.

For the more general case the development of the axis represented in the laboratory frame by the vector $\mathbf{Z}$ can be found as follows. Because the particle velocity is given as $\mathbf{v} = v(\cos\omega t, \sin\omega t)$, the components $\mathbf{Z}_{\|}$, and $\mathbf{Z}_{\perp}$ can be written as

$$\begin{aligned} \mathbf{Z}_{\|} &= Z_{\|}(\cos\omega t, \sin\omega t), \\ \mathbf{Z}_{\perp} &= Z_{\perp}(\sin\omega t, -\cos\omega t). \end{aligned} \tag{30}$$

It follows then that the components of $\mathbf{Z}$ with respect to the x and y axes of the laboratory frame are

$$\begin{aligned} Z_x &= Z_{\|}\cos\omega t + Z_{\perp}\sin\omega t, \\ Z_y &= Z_{\|}\sin\omega t - Z_{\perp}\cos\omega t. \end{aligned} \tag{31}$$

Taking into account that

$$\begin{aligned} \sigma_{\|} &= \sigma\cos(\omega_R t) = \sigma\cos(\gamma\omega t) \\ \sigma_{\perp} &= \sigma\sin(\omega_R t) = \sigma\sin(\gamma\omega t), \end{aligned} \tag{32}$$

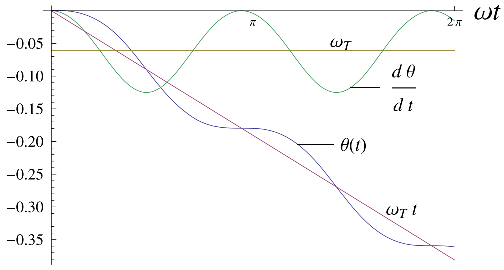

Figure 9. Plot of the angle $\theta(t)$ the vector $\mathbf{Z}$ makes with the x-axis in the laboratory frame. It is compared to the angle $\omega_T t$ describing the spin $\boldsymbol{\sigma}$ orientation in its uniform precession according to equation (5). Additionally, the angular velocity $\mathrm{d}\theta/\mathrm{d}t$ is shown with the Thomas precession rate ω_T in the background.

and recalling equation (29) we get the x and y components of $\mathbf{Z}$:

$$\begin{aligned} Z_x &= \sigma[\gamma\cos(\gamma\omega t)\cos\omega t + \sin(\gamma\omega t)\sin\omega t],\\ Z_y &= \sigma[\gamma\cos(\gamma\omega t)\sin\omega t - \sin(\gamma\omega t)\cos\omega t]. \end{aligned} \tag{33}$$

This is the desired fully consistent description of the behaviour of spin made in the laboratory frame. Our result (33) agrees with an outcome presented in [8] (notice that the authors of [8] chose different initial conditions).

Equations (33) show that the spin vector $\mathbf{Z}$, representing in the laboratory frame the axis $(0, \boldsymbol{\sigma})$ of the rest frame, exhibits a quite non-trivial behaviour in the laboratory system. To visualize this result, we present it in figure 9. The plot shows the time dependence of the angle $\theta(t)$ defined as

$$\theta(t) = \mathrm{Arctan}\left(\frac{Z_y}{Z_x}\right), \tag{34}$$

representing the direction of $\mathbf{Z}$ measured in the laboratory frame with respect to the x-axis. For a comparison, it is imposed on the plot of $\omega_T t$, i.e. the angle of the uniform rotation of spin $\boldsymbol{\sigma}$ given by equation (5). Additionally, we have plotted the rate of change of the orientation of spin, $\mathrm{d}\theta/\mathrm{d}t$, to compare it with the uniform Thomas precession rate ω_T. As can be seen, $\mathrm{d}\theta/\mathrm{d}t$ is not uniform but reveals a kind of oscillatory development. We can check that there are moments when the rotation of $\mathbf{Z}$ stops, $\mathrm{d}\theta/\mathrm{d}t = 0$, which agrees with the above discussion of the case of the axis momentarily aligned with the velocity (see figure 4). Accordingly, the time dependence of the function $\theta(t)$ shows that $\mathbf{Z}$ wobbles around the direction established by the approximate (or 'averaged', see below) time dependence $\omega_T t$.

3.3. Approximate description

It is instructive to present our results also in an approximate form for small β. In this case we can substitute $\gamma = 1 + \frac{1}{2}\beta^2$ and, leaving only the terms of order of β^2, find that

$$\begin{aligned} Z_x &= \sigma\left[1 + \frac{1}{2}\beta^2\cos^2(\omega t)\right],\\ Z_y &= \frac{1}{2}\sigma\beta^2\left[\frac{1}{2}\sin(2\omega t) - \omega t\right]. \end{aligned} \tag{35}$$

Then we get

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Arctan}\left(\frac{Z_y}{Z_x}\right) = -\omega\beta^2\,\sin^2(\omega t). \tag{36}$$

This agrees with the approximate result obtained by Muller [9] (note that in Muller's paper $\mathrm{d}\theta/\mathrm{d}t = \omega\beta^2\,\cos^2(\omega t)$ due to different initial conditions). Similar time dependence of the rate of the spin $\mathbf{Z}$ rotation can be found also in [10] (although the derivation there does not seem free from errors). Compared to [9] and [10], the advantage of our approach is its interpretational and technical simplicity (we need to refer merely to the difference between ω_R and ω). This does not, however, hinder us in obtaining the exact result (33). What is more, we have introduced the important distinction between spin $\boldsymbol{\sigma}$ and its laboratory representation $\mathbf{Z}$, which entails that the spin development can be analysed in two significantly different manners. To our knowledge, this fact was never clearly stated in the literature.

Finally let us verify that when we average (36) over a whole cycle of the particle motion, we get

$$\left\langle \mathrm{d}\theta \middle/ \mathrm{d}t \right\rangle = -\omega\beta^2\left\langle \sin^2(\omega t) \right\rangle = -\frac{1}{2}\omega\beta^2 \equiv \omega_T, \tag{37}$$

which is the common expression for the Thomas precession rate in the approximate approach for $\beta \to 0$ and when the vectors $\boldsymbol{\sigma}$ and $\mathbf{Z}$ are not differentiated.

4. Conclusions

We have explained the origin of two seemingly incompatible descriptions of evolving spin as connected with the use of two different representations of spin: the proper spin $\boldsymbol{\sigma}$ defined in the rest frame of the particle and the spin four-vector Z established in the laboratory frame. The fact that in the case of uniform rotation of $\boldsymbol{\sigma}$ the four-vector Z performs some additional 'wobbling' motion is pointed out to be related to the relativistic effect of distortion the boosted frame axes.

The approach in which we deal with the proper spin is much preferred in experimental practice due to its mathematical simplicity and clear physical meaning of the vector $\boldsymbol{\sigma}$. The particles' polarization with respect to their velocity, measured just in the particles' *rest* frame, is the basis for analysing outcomes in such experiments as, for example, those making use of the Mott scattering asymmetry [11] or where the asymmetric distribution of the decay electrons from polarized muons is investigated [12].

Acknowledgments

The author wishes to thank P Prawda for his invaluable assistance in the realization of this work and Vladimir Hnizdo for an inspiring discussion and numerous remarks that improved the presentation.

References

[1] Thomas L H 1927 The kinematics of an electron with an axis *Phil. Mag.* **3** 1–22
[2] Møller C 1952 *The Theory of Relativity* (London: Oxford University Press) pp 54–5
[3] This kind of figure is sometimes used to find the angle of the Thomas rotation, see: Eisberg E and Resnick R 1974 *Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles* (New York: Wiley) appendix J

Rębilas K 2002 Comment on 'The Thomas rotation' by John P Costella *et al* [*Am J. Phys.* 69 (8), 837–847 (2001)] *Am. J. Phys.* **70** 1163–5

Rębilas K 2013 Comment on 'Elementary analysis of the special relativistic combination of velocities, Wigner rotation and Thomas precession' *Eur. J. Phys.* **34** 55–61

[4] Jackson J D 1975 *Classical Electrodynamics* (New York: Wiley) section 11.8

[5] Rębilas K 2011 Simple approach to relativistic spin dynamics *Am. J. Phys.* **79** 1064–7

[6] Rębilas K 2011 Thomas precession and the Bargmann–Michel–Telegdi equation *Found. Phys.* **41** 1800–9

[7] Bargmann V, Michel L and Telegdi V L 1959 Precession of the polarization of particles moving in a homogeneous electromagnetic field *Phys. Rev. Lett.* **2** 435–6

[8] Misner C, Thorne K and Wheeler J 1973 *Gravitation* (San Francisco: Freeman) p 175

[9] Muller R A 1992 Thomas precession: where is the torque? *Am. J. Phys.* **60** 313–7

[10] Taylor E F and Wheeler J A 1966 *Spacetime Physics* (San Francisco: Freeman) section 103

[11] Wesley J C and Rich A 1971 High-field electron g–2 measurement *Phys. Rev.* A **4** 1341–63 Compare also the references in [7]

[12] Rich A and Wesley J C 1972 The current status of the lepton g factors *Rev. Mod. Phys.* **44** 250–83