Representations of Dirac Equation in General Relativity (*).

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Summary. — After a reformulation of the General Relativistic equations of motion of a point mass in a gravitational field in terms of « fourlegs », the generalized Dirac equation is written in the Schrödinger representation and the equations of motion of classical observable quantities compared with the previous one. The Foldy-Wouthuysen representation is obtained in presence of gravitational fields. The most interesting result is that the gravitational « gyromagnetic » factor is 1 instead of 2 as the electromagnetic one. The interpretation of this fact is that the spinor field (and probably all fields) describes rotating particles where the gravitational mass has exactly the same space distribution as the inertial mass. This is not true in Moshinsky-Birkhoff linear theory. Finally the red shift of hydrogen atoms levels in presence of gravitational field is obtained and found to coincide with the usual prediction of General Relativity.

Introduction.

The equations of General Relativity can be understood either as describing the laws of Physics (including gravitation) in arbitrary co-ordinate system or as a descritpion of these laws in a priviledged frame corresponding to an underlying flat space (1). In the last case the whole difference $g_{\mu\nu} - \overset{\circ}{g}_{\mu\nu}$ is ascribed to the gravitational potential. In field theory this is the most appropriate interpretation. We shall adopt this point of view and thus deal with the equations as non-linear equations in Minkowsky space.

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(1) V. A. FOCK: *Theory of Space, Time and Gravitation* (London, 1959).

In this paper we are mainly interested in the Dirac equation. However for the interpretation of these equations where the gravitational potentials appear in the combination given by the « fourlegs » $h^{(\alpha)}{}_{\mu}$ and not $g_{\mu\nu}$, it is convenient to analyse the classical equations of a moving point mass in terms of $h^{(\alpha)}{}_{\mu}$; this is done in Sections **1-2**.

In Section **3**, Dirac equation is written in an obviously self-adjoint form. In Section **4** it is written in hermitian or Schrödinger form and comparison with the non quantum formulation is made. In Section **5** the Foldy-Wouthuysen representation of generalized Dirac equation is obtained. Finally, in Section **6** the red shift of hydrogen atom in a gravitational field is examined in the lines of the used interpretation and the same result is obtained as in the metric interpretation. The red shift corresponds to the fact that the rate of vibration of an atomic clock referred to the underlying « flat » time is changed in presence of a gravitational potential. The fact that the size of the atom is also changed as directly examined in Moshinsky analysis, is implied here by the fact that comparison of the hydrogen atoms in presence and in absence of gravitational field is much simpler when different length scales are used.

1. – The classical (non quantum) equations of motion for a point mass in a gravitational field.

The equations of motion of a point mass m in a gravitational-metric field $g_{\mu\nu}$ are given by (*),

$$m\frac{\mathrm{d}u^{\varrho}}{\mathrm{d}s} = -\,m\Gamma^{\varrho}_{\lambda\sigma}u^{\lambda}u^{\sigma} = -\frac{m}{2}\,g^{\varrho\nu}\left(\frac{\partial g_{\nu\lambda}}{\partial x^{\sigma}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\lambda}} - \frac{\partial g_{\lambda\sigma}}{\partial x^{\nu}}\right)u^{\lambda}u^{\sigma}\,, \tag{1}$$

where,

$$u^{\varrho} = \frac{\mathrm{d}x^{\varrho}}{\mathrm{d}s}\,; \qquad \mathrm{d}s^{2} = g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}\,, \tag{2a}$$

$$g^{\mu\nu}g_{\nu\lambda} = \delta^{\mu}_{\lambda}\,. \tag{2-a}$$

From the variational principle which leads to eq. (1), we find the four-momentum p_{μ} of the point-mass in the gravitational field

$$p_{\mu} = m g_{\mu\nu}u^{\nu}\,. \tag{3}$$

(*) Throughout this paper we will use natural units, $c=\hbar=1$. Greek suffixes run from 0 to 3; Latin suffixes from 1 to 3.

The equation of motion for p_μ is:

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}s} = \frac{m}{2}\frac{\partial g_{\varrho\sigma}}{\partial x^\mu} u^\varrho u^\sigma . \tag{4}$$

It is known that the relativistic quantum equations for a point particle with spin $\frac{1}{2}$ in a gravitational field (Dirac equation) is more conveniently written in terms of «local frames» or four-legs ([2]) $h^{(\alpha)}{}_\mu$ and their reciprocal $h^{(\alpha)\mu}$, defined by the relations:

$$h^{(\alpha)}{}_\mu h^{(\beta)}{}_\nu \mathring{g}_{\alpha\beta} = g_{\mu\nu} , \tag{5-a}$$

$$h^{(\alpha)}{}_\mu h^{(\beta)\mu} = \mathring{g}_{\alpha\beta} , \tag{5-b}$$

$$h^{(\alpha)\mu} h^{(\beta)\nu} \mathring{g}_{\alpha\beta} = g^{\mu\nu} , \tag{5-c}$$

where, $\mathring{g}_{\alpha\beta}$ is the Galilean «local metric tensor».

$$\mathring{g}_{00} = -\mathring{g}_{ii} = 1 , \qquad \mathring{g}_{ij} = 0 , \qquad i \neq j. \tag{5-d}$$

It is convenient, for a future comparison, to write the eqs. (4) in terms of the four-legs components. We find by a straightforward calculation ($x^0 = t$)

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}t} = -\frac{\partial h_{(\alpha)}{}^\varrho}{\partial x^\mu} p_\varrho v^{(\alpha)} , \tag{6a}$$

where,

$$h_{(\alpha)}{}^\varrho = \mathring{g}_{\alpha\beta} h^{(\beta)\varrho} , \tag{6-b}$$

and $v^{(\alpha)}$ is the «local velocity» given by:

$$v^{(\alpha)} = h^{(\alpha)}{}_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}t} = h^{(\alpha)}{}_\varrho u^\varrho \frac{\mathrm{d}s}{\mathrm{d}t} . \tag{7}$$

It is also convenient to eliminate p_0 from the second member of (6-a). This is obtained if we use a special set of four-legs which we shall call the *canonical frame*, with the property:

$$h^{(i)0} = -h_{(i)}{}^0 = 0 . \tag{8}$$

These restrictions and even three new ones, can be imposed in view of the fact that only ten out of the sixteen numbers $h^{(\alpha)}{}_\mu$ are independent. Thus

([2]) H. Weyl: *Zeit. Phys.*, **56**, 330 (1929); V. A. Fock: *Zeit. Phys.*, **57**, 261 (1929); J. A. Wheeler and D. R. Brill: *Rev. Mod. Phys.*, **29**, 466 (1957).

we take the canonical frame as:

$$(h^{(\alpha)\mu}) = \begin{pmatrix} h^{(0)0} & h^{(0)1} & h^{(0)2} & h^{(0)3} \\ 0 & h^{(1)1} & h^{(1)2} & h^{(1)3} \\ 0 & 0 & h^{(2)2} & h^{(2)3} \\ 0 & 0 & 0 & h^{(3)3} \end{pmatrix}. \tag{9}$$

This is a unique definition of the $h^{(\alpha)}{}_{\mu}$, if we fix the signs of the diagonal elements. We have:

$$h_{(0)}{}^{0} = h^{(0)0} = \sqrt{g^{00}}\,, \tag{10a}$$

$$\tilde{h}_{(0)}{}^{i} = \tilde{h}^{(0)i} = \frac{g^{0i}}{g^{00}}\,, \tag{10b}$$

$$\tilde{h}_{(k)}{}^{i}\tilde{h}^{(k)j} = \frac{g^{ij}}{g^{00}} - \frac{g^{0i}g^{0j}}{(g^{00})^2}\,, \tag{10c}$$

where,

$$\tilde{h}_{(\nu)}{}^{\mu} = \frac{h_{(\nu)}{}^{\mu}}{h_{(0)}{}^{0}}\,.$$

In general only (10-a, b) will be used. It is easy to conclude from (8) and (5) that:

$$h^{(0)}{}_{i} = 0\,. \tag{10-d}$$

Thus we obtain:

$$v^{(0)} = h^{(0)}{}_{\mu}v^{\mu} = h^{(0)}{}_{0}v^{0} = h^{(0)}{}_{0} = (h_{(0)}{}^{0})^{-1}\,, \tag{11}$$

where,

$$v^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\,.$$

Equation (6-a) takes now the form,

$$\frac{\mathrm{d}p_{\mu}}{\mathrm{d}t} = m h_{(0)}{}^{0}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\partial}{\partial x^{\mu}}(h_{(0)}{}^{0})^{-1} - \frac{\partial \tilde{h}_{(j)}{}^{k}}{\partial x^{\mu}}\,p_k v^{(j)} h_{(0)}{}^{0} - \frac{\partial \tilde{h}_{(0)}{}^{j}}{\partial x^{\mu}}\,p_j\,. \tag{12}$$

In derivation of the eq. (12) we used the property

$$p_{\nu}u^{\nu} = m\,. \tag{13}$$

Equation (13) which is linear in p_{ν} can be used to obtain the expression of the energy p_0 in terms of the momenta p_i and the local velocity $v^{(j)}$. We find, using eqs. (7) and (11):

$$p_0 = m\frac{\mathrm{d}s}{\mathrm{d}t} - p_i h_{(k)}{}^i v^{(k)} - p_i \tilde{h}_{(0)}{}^i . \tag{14}$$

This is the classical equation which will be compared with the Schrödinger representation of Dirac equation.

2. – Slow motions and weak gravitational fields.

The approximation correspondent to the weak field is,

$$g^{\mu\nu} = \mathring{g}^{\mu\nu} + \varkappa(\Phi^{\mu\nu} + \Phi^{\nu\mu}) \qquad (\varkappa \text{ small}), \tag{15-a}$$

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} - \varkappa(\Phi_{\dot{\mu}\dot{\nu}} + \Phi_{\dot{\nu}\dot{\mu}}) , \tag{15-b}$$

where a dotted index is an index lowered with $\mathring{g}_{\mu\nu}$ instead of $g_{\mu\nu}$.

For the four-legs we have, in this approximation:

$$h^{(\mu)\nu} = \mathring{g}^{\mu\nu} + \varkappa\Phi^{\mu\nu} ; \qquad h^{(\mu)}{}_\nu = \delta^\mu{}_\nu + \varkappa\Phi^\mu{}_\nu . \tag{15-c}$$

In the canonical representation (eq. (9)),

$$\Phi^{\mu\nu} = 0 , \qquad \text{if } \nu > \mu. \tag{15-d}$$

For a moving point source of velocity $\boldsymbol{V}$ and mass M, the time component $\varkappa\Phi^{00} = \Phi$, represents the newtonian potential, given in weak field approximation by:

$$\Phi = \frac{kM}{r} . \tag{16a}$$

The components Φ^{0i} are linear in V^i,

$$\varkappa\Phi^{0i} = \frac{4kM}{r} V^i = a^i , \tag{16b}$$

and, the remaining components are,

$$\varkappa\Phi^{ij} = \frac{kM}{r}\delta^{ij} + \chi^{ij} , \tag{16c}$$

χ^{ij} being quadratic in V^i, will be neglected for slow moving sources.

In what follows we shall be concerned only with $\Phi^{\mu\nu}$'s of the form:

$$(\varkappa\Phi^{\mu\nu}) = \begin{pmatrix} \Phi & a^1 & a^2 & a^3 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{pmatrix}. \tag{17}$$

The equation of motion (12) for slow moving bodies (velocity $\boldsymbol{v}$) in the weak field may be written:

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = m\boldsymbol{e} + m\boldsymbol{v}\wedge\boldsymbol{b}\,, \tag{18}$$

where,

$$\boldsymbol{P} = \boldsymbol{p} - m\boldsymbol{a} \simeq \frac{m\boldsymbol{v}}{\sqrt{1-v^2}}\,,$$

$$\left\{\begin{aligned} &\boldsymbol{p} = (p^1, p^2, p^3) = (-p_1, -p_2, -p_3)\,, \\ &\boldsymbol{a} = (a^1, a^2, a^3)\,; \qquad a^i = \tilde{h}_{(0)}{}^i \simeq g^{0i}\,, \\ &\boldsymbol{b} = \operatorname{rot}\boldsymbol{a}\,; \qquad \boldsymbol{e} = \nabla\Phi\,. \end{aligned}\right. \tag{19}$$

In the case of a moving point source we have from eqs. (16):

$$\left\{\begin{aligned} &\boldsymbol{e} = \nabla\Phi = kM\nabla\left(\frac{1}{r}\right), \\ &\boldsymbol{b} = -\,4kM\,\boldsymbol{V}\wedge\nabla\left(\frac{1}{r}\right). \end{aligned}\right. \tag{20}$$

Therefore we see that the « magnetic » field $\boldsymbol{b}$ as compared to the « electric » field $\boldsymbol{e}$ is four-times greater in the gravitational case than in the electromagnetic case (for charges of the same sign) and the relative signs are opposite:

$$\boldsymbol{b}:\boldsymbol{B} = -\,4\boldsymbol{e}:\boldsymbol{E}\,.$$

The minus sign correspond to the fact that the « electric » force is attractive in the gravitational case and repulsive in the electromagnetic case.

3. – The Dirac equation in general relativity.

The four-component spinor field $\psi(x)$ which describes a spin $\frac{1}{2}$ particle is invariant (scalar) under co-ordinates transformations, and transforms under local infinitesimal rotations of the four-legs, as [3]:

$$\psi(x) \to \psi(x) - \frac{i}{4} S^{(\varrho)(\sigma)} \varepsilon_{(\varrho)(\sigma)}(x)\, \psi(x) \,, \tag{21a}$$

under,

$$h^{(\varrho)}{}_{\mu} \to h^{(\varrho)}{}_{\mu} + \varepsilon^{(\varrho)}{}_{(\sigma)}(x)\, h^{(\sigma)}{}_{\mu} \,; \qquad x^{\mu} = \text{const.} \tag{21-b}$$

where $S^{(\varrho)(\sigma)}$ are the « flat » spinor matrices [8],

$$S^{(\varrho)(\sigma)} = \frac{i}{2} [\gamma^{(\varrho)}, \gamma^{(\sigma)}] \,, \tag{22a}$$

$$\mathring{g}^{\varrho\sigma} = \tfrac{1}{2} \{\gamma^{(\varrho)}, \gamma^{(\sigma)}\} \,, \tag{22-b}$$

for any quantities a, b we use the usual conventions,

$$[a, b] = ab - ba \,, \qquad \{a, b\} = ab + ba \,.$$

Here, and in the following sections we shall use the vierbein-formalism, in which are used only the « flat » spin-matrices $\gamma^{(\alpha)}$ identical to the usual Dirac matrices [4].

The wave-function ψ satisfies the generalized Dirac equation [2],

$$i h_{(\varrho)}{}^{\mu} \gamma^{(\varrho)} \Delta_{\mu} \psi = m \psi \,, \tag{23}$$

where,

$$\Delta_{\mu} = \frac{\partial}{\partial x^{\mu}} - \frac{i}{4} S_{(\varrho)(\sigma)} h^{(\varrho)\nu} \left(\frac{\partial h^{(\sigma)}{}_{\nu}}{\partial x^{\mu}} - \Gamma^{\lambda}_{\mu\nu} h^{(\sigma)}{}_{\lambda} \right) . \tag{24}$$

Now, in view of the relations:

$$\gamma_{(\lambda)} S_{(\varrho)(\sigma)} = \tfrac{1}{2} [\gamma_{(\lambda)}, S_{(\varrho)(\sigma)}] + \tfrac{1}{2} \{\gamma_{(\lambda)}, S_{(\varrho)(\sigma)}\} \,, \tag{25-a}$$

$$[\gamma_{(\lambda)}, S_{(\varrho)(\sigma)}] = 2i (\mathring{g}_{\lambda\varrho} \gamma_{(\sigma)} - \mathring{g}_{\lambda\sigma} \gamma_{(\varrho)}) \,, \tag{25-b}$$

$$\{\gamma_{(\lambda)}, S_{(\varrho)(\sigma)}\} = -2i \varepsilon_{(\lambda)(\varrho)(\sigma)(\alpha)} \gamma^{(\alpha)} \gamma_5 \,, \tag{25-c}$$

$$\gamma_5 = \gamma_{(0)} \gamma_{(1)} \gamma_{(2)} \gamma_{(3)} \,, \tag{25-d}$$

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[3] F. J. BELINFANTE: *Physica*, **7**, 305 (1940); see also J. A. WHEELER and D. R. BRILL [1].

[4] Throughout the following sections we will use the « flat »-spin matrices $\gamma^{(\alpha)}$ in the notation given in the book of S. SCHWEBER, H. BETHE and F. DE HOFFMAN: *Mesons and Fields*, vol. I (New York, 1954).

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$\varepsilon_{(\lambda)(\varrho)(\sigma)(\alpha)}$ being the full antisymmetric unit tensor constructed with the fundamental permutation 0123, it is possible to write eq. (23) after some substitutions as:

$$\frac{i}{2}\gamma^{(\lambda)}\left[\left\{h_{(\lambda)}{}^{\mu}, \frac{\partial}{\partial x^{\mu}}\right\} + h_{(\lambda)}{}^{\varrho}\Gamma^{\mu}_{\mu\varrho}\right]\psi + \frac{i}{2}\gamma^{(\lambda)}\gamma_5\mathscr{B}_{(\lambda)}\psi = m\psi\,, \tag{26}$$

where,

$$\mathscr{B}_{(\lambda)} = \frac{1}{2}\,\varepsilon_{(\lambda)(\beta)(\varrho)(\sigma)}\,h^{(\beta)\mu}h^{(\varrho)\nu}\,\frac{\partial h^{(\sigma)}{}_{\nu}}{\partial x^{\mu}}\,. \tag{27}$$

The dual-equation satisfied by

$$\overline{\psi} = \psi^{\dagger}\beta\,, \qquad \beta = \gamma^{(0)}\,,$$

has the following expression:

$$\frac{i}{2}\left[\left\{h_{(\lambda)}{}^{\mu}, \frac{\partial}{\partial x^{\mu}}\right\} + h_{(\lambda)}{}^{\varrho}\Gamma^{\mu}_{\mu\varrho}\right]\overline{\psi}\gamma^{(\lambda)} - \frac{i\overline{\psi}}{2}\gamma^{(\lambda)}\gamma_5\mathscr{B}_{(\lambda)} = -\,m\overline{\psi}\,. \tag{28}$$

From (26), (28) are obtained immediately the known relations for the law of conservation correspondent to the generalized four-vector density of current,

$$\frac{\partial j^{\mu}}{\partial x^{\mu}} = 0\,, \tag{29a}$$

$$j^{\mu} = \sqrt{-g}\,h_{(\varrho)}{}^{\mu}\,\overline{\psi}\gamma^{(\varrho)}\psi\,, \tag{29b}$$

where, $g = \det(g_{\mu\nu})$.

This justifies the use of the normalization condition,

$$\int j^0\,\mathrm{d}v = \int\sqrt{-g}\,h_{(0)}{}^{0}\,\psi^{+}\psi\,\mathrm{d}v = 1\,.$$

4. – Schrödinger representation of the Dirac equations in presence of gravitational field.

In the next section we shall apply the Foldy-Wouthuysen transformation [5] to the Dirac eq. (26). As a preliminary stage to this method it is necessary to write (26) in the Schrödinger form:

$$i\,\frac{\partial\chi}{\partial t} = H\chi\,, \tag{30}$$

[5] L. L. Foldy and S. A. Wouthuysen: *Phys. Rev.*, **78**, 29 (1950).

where, H is the hermitian hamiltonian operator. Thus the probability density must be

$$j^0 = \chi^+\chi = \bar{\chi}\gamma^{(0)}\chi \,. \tag{31}$$

If we multiply eq. (26) by β, it takes a form similar to (30) where H does not involve time-differentiation (in our canonical frame); the unique difference appears in the first member of these equations, since now we obtain

$$\frac{i}{2}\left\{h_{(0)}{}^0, \frac{\partial}{\partial t}\right\}\psi = -\frac{i}{2}\alpha^{(\mu)}\left\{h_{(\mu)}{}^i, \frac{\partial}{\partial x^i}\right\}\psi - \frac{i}{2}\alpha^{(\lambda)}h_{(\lambda)}{}^{\varrho}\Gamma^{\mu}_{\mu\varrho}\psi - \\ -\frac{i}{2}\alpha_{(\lambda)}\gamma_5\mathscr{B}_{(\lambda)}\psi + m\beta\psi \,, \tag{32}$$

where

$$\alpha^{(\lambda)} = \beta\gamma^{(\lambda)} \,.$$

However this « hamiltonian » is not hermitian; in other words, j^0 does not take the form (31) but is (by (29-*b*)),

$$j^0 = \sqrt{-g}\, h_{(0)}{}^0 \psi^+\psi \,. \tag{33}$$

Comparison of (31) and (33) suggests that we must make the nonunitary transformation (*),

$$\chi = (-g)^{\frac{1}{4}}(h_{(0)}{}^0)^{\frac{1}{2}}\psi \,. \tag{34}$$

Indeed, after this transformation we obtain,

$$i\frac{\partial\chi}{\partial t} = \left[\tilde{m}\beta - \frac{1}{2}\alpha^{(k)}\{\tilde{h}_{(k)}{}^i, p_i\} - \frac{1}{2}\{h_{(0)}{}^i, p_i\} - \frac{i}{2}\gamma_5\alpha^{(\lambda)}\tilde{\mathscr{B}}_{(\lambda)}\right]\chi \,, \tag{35}$$

where

$$p_i = i\frac{\partial}{\partial x^i} \,,$$

and, for any quantity F,

$$\tilde{F} = \frac{F}{h_{(0)}{}^0} \,.$$

Taking $i(\partial/\partial t) = p_0$, we see that eq. (35) is just the symmetrized Schrödinger equation corresponding to (14), except for the « quantum term » in-

(*) Clearly the same conclusion is obtained if we try to bring the first member of (32) to the usual form involving only time derivatives of the wave function.

volving $\mathscr{B}_{(\lambda)}$, with the following correspondence:

$$\alpha^{(j)} \rightarrow v^{(j)} h_{(0)}{}^0 , \tag{36-a}$$

$$\beta \rightarrow \frac{\mathrm{d}s}{\mathrm{d}t} h_{(0)}{}^0 . \tag{36-b}$$

The « quantum » term,

$$-\frac{i}{2}\gamma_5 \alpha^{(\lambda)} \widetilde{\mathscr{B}}_{(\lambda)} = \frac{1}{2}\sigma^{(\lambda)} \widetilde{\mathscr{B}}_{(\lambda)} ,$$

in eq. (35) has the nature of a « magnetic » moment interaction. Indeed, in the weak field approximation we find from eqs. (27) and (15), (16),

$$\mathscr{B}_0 = 0 ; \qquad \sigma^{(i)} \widetilde{\mathscr{B}}_{(i)} = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathrm{rot}\, \boldsymbol{a} = \boldsymbol{b} \cdot \frac{\boldsymbol{\sigma}}{2} .$$

We may now obtain the equation of motion for p_i; from the general relation:

$$\frac{\mathrm{d}O}{\mathrm{d}t} = \frac{1}{i}[O, H] .$$

We find,

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\widetilde{m}\beta}{h_{(0)}{}^0}\frac{\partial h_{(0)}{}^0}{\partial x^i} - \frac{1}{2}\left\{\frac{\partial h_{(0)}{}^j}{\partial x^i}, p_j\right\} - \frac{\alpha^{(k)}}{2}\left\{\frac{\partial \widetilde{h}_{(k)}{}^j}{\partial x^i}, p_j\right\} + \frac{1}{2}\sigma^{(\lambda)}\frac{\partial \widetilde{\mathscr{B}}_{(\lambda)}}{\partial x^i} , \tag{37}$$

$\sigma^{(\lambda)}$ being the hermitian four-vector,

$$\sigma^{(\lambda)} = -i\alpha^{(\lambda)}\gamma_5 . \tag{38}$$

Using again the correspondence given by eqs. (36) we see that eq. (37) is identical to the classical eq. (12), except for the symmetrization of non-commuting products and for the « quantum » term involving $\sigma^{(\lambda)}$.

5. – The Foldy-Wouthuysen representation (5).

We shall consider the Foldy-Wouthuysen approximation of the Dirac equation for a charged particle in presence of gravitational and electromagnetic fields.

The Schrödinger eq. (35), in the weak-static approximation, after the usual substitution,

$$p_i \rightarrow \pi_i = p_i + eA_i ; \qquad \boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A} ,$$

$$i\frac{\partial}{\partial t} \rightarrow i\frac{\partial}{\partial t} + e\varphi ,$$

takes the form:

$$i\,\frac{\partial\chi}{\partial t} = \tilde{m}\beta + \mathscr{E} + \mathscr{O}\,, \tag{39}$$

where, in the present approximation:

$$\tilde{m} = m(1-\Phi)\,, \tag{40-a}$$

$$\mathscr{E} = -\,e\varphi + \tfrac{1}{2}\{\boldsymbol{a},\,\cdot\boldsymbol{\pi}\} + \tfrac{1}{4}\boldsymbol{\sigma}\cdot\boldsymbol{b}\,, \tag{40-b}$$

$$\mathscr{O} = \tfrac{1}{2}\{\boldsymbol{\alpha}\cdot\boldsymbol{\pi},\,(1-2\Phi)\}\,, \tag{40-c}$$

and,

$$\boldsymbol{p} = \frac{1}{i}\boldsymbol{\nabla} = \left(\frac{1}{i}\,\frac{\partial}{\partial x^j}\right);\qquad \boldsymbol{\sigma} = (\sigma^{23},\,\sigma^{31},\,\sigma^{12})\,;\qquad \boldsymbol{\alpha} = (\alpha^1,\,\alpha^2,\,\alpha^3)\,. \tag{40d}$$

Following the F-W procedure [5] we find, after the transformation:

$$\chi \to \chi' = \exp[iS]\,\chi\,,$$

$$S = -\,\frac{i}{4m}\,\beta\{\mathscr{O},\,(1+\Phi)\} \simeq -\,\frac{i}{2m}\,\beta\boldsymbol{\alpha}\cdot\left(\boldsymbol{\pi} - \frac{1}{2}\,\{\Phi,\,\boldsymbol{\pi}\}\right),$$

$$i\,\frac{\partial\chi'}{\partial t} = \tilde{m}\beta + \mathscr{E}' + \mathscr{O}',$$

where $\mathscr{E}'$ is an even operator and $\mathscr{O}'$ is an odd operator of the orders $(1/m)$ and $(1/m)^2$. As we are interested in the approximation up to the order $(1/m)^2$, we proceed with two new successive transformations which eliminate the odd terms up to $(1/m)^2$ (inclusive) and produce only even terms from $(1/m)^3$ up. Thus, neglecting these terms, we find: (using also the usual transformation: $\chi'' = \exp[-imt]\Psi$)

$$\begin{aligned} i\,\frac{\partial\Psi}{\partial t} = \Bigg[&-m\Phi - e\varphi + \frac{1}{4m}\,\{\boldsymbol{\pi}^2,\,1-3\Phi\} - \frac{e}{2m}\,\boldsymbol{\sigma}\cdot\boldsymbol{B}(1-3\Phi) + \\ &+ \frac{1}{2}\,\{\boldsymbol{a},\,\cdot\boldsymbol{\pi}\} + \frac{1}{4}\,\boldsymbol{\sigma}\cdot\boldsymbol{b} - \frac{e}{8m^2}\,\{\boldsymbol{\sigma}\cdot\boldsymbol{E}\wedge\boldsymbol{\pi},\,1-2\Phi\} + \\ &+ \frac{e}{8m^2}\,(1-2\Phi)\,\operatorname{div}\boldsymbol{E} - \frac{3}{4m}\,\boldsymbol{\sigma}\cdot\boldsymbol{e}\wedge\boldsymbol{\pi} - \frac{3}{8m}\,\operatorname{div}\boldsymbol{e} - \frac{e}{8m^2}\,\boldsymbol{E}\cdot\boldsymbol{e}\Bigg]\,\Psi\,. \end{aligned} \tag{41}$$

In eq. (41) terms quadratic in φ, and those of order $(1/m)^2$ involving $\boldsymbol{a}$ or $\boldsymbol{b}$ (thus the small velocity of the gravitational source) were also neglected. Finally the condition $\beta\Psi = \Psi$ was used.

We see that to the order $(1/m)^2$, besides the usual terms of the electromagnetic case ([5]) with corrections coming from the gravitational field, terms were obtained of pure gravitational nature. Let us examine these firstly.

i) $\frac{1}{2}\{\boldsymbol{a}, \cdot\boldsymbol{\pi}\} + \frac{1}{4}\boldsymbol{\sigma}\cdot\boldsymbol{b}$.

The first term would already have been obtained in the Moshinsky-Birkhoff theory ([6]) which coincides with the present one in the weak field approximation, except for the term $-(i/2)\gamma^{(\lambda)}\gamma_5\mathscr{B}_{(\lambda)}$. The second term however, did not show up in his work, as it comes out precisely and exclusively from the refered additional term which is characteristic of general relativity.

If the atom is in a region where $\boldsymbol{b} = \operatorname{rot}\boldsymbol{a}$ is uniform we can write:

$$\boldsymbol{a} = \tfrac{1}{2}\boldsymbol{b}\wedge\boldsymbol{r}\,,$$

$$\tfrac{1}{2}\{\boldsymbol{a}, \cdot\boldsymbol{\pi}\} + \tfrac{1}{4}\boldsymbol{\sigma}\cdot\boldsymbol{b} = \tfrac{1}{2}\boldsymbol{b}\cdot(\boldsymbol{r}\wedge\boldsymbol{p} + \tfrac{1}{2}\boldsymbol{\sigma}) - e\boldsymbol{A}\cdot\boldsymbol{a} = \tfrac{1}{2}\boldsymbol{b}\cdot\boldsymbol{J} - e\boldsymbol{A}\cdot\boldsymbol{a}\,.$$

Thus we see that the « magnetic » gravitational field $\boldsymbol{b}$ interacts with the total angular momentum $\boldsymbol{J}$. In other words, the gravitational « gyromagnetic » factor is equal to 1, even for spin $\frac{1}{2}$ particles, in contradistinction with the electromagnetic case. Therefore the general relativistic equation assures that the intrinsic angular momentum behaves as if the particle was a gyroscope. We believe that this result holds true for any value of the spin, as the value $G = 1$ implies that also for the internal structure of the particle the distribution of gravitational mass (in the interacting term) is the same as that for the inertial mass (in the angular momentum).

ii) The « spin orbit term » $(-(3/4m)\boldsymbol{\sigma}\cdot\boldsymbol{e}\wedge\boldsymbol{\pi})$.

This term, together with the previous one, gives for the precession equation of the spin ($\boldsymbol{s} = \frac{1}{2}\boldsymbol{\sigma}$)

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = (\boldsymbol{\Omega}_e + \boldsymbol{\Omega}_g)\wedge\boldsymbol{s}\,, \tag{42}$$

where $\boldsymbol{\Omega}_e$ is the usual precession angular velocity and $\boldsymbol{\Omega}_g$ is the correspondent gravitational one:

$$\boldsymbol{\Omega}_g = -\frac{3}{2m}\,\boldsymbol{e}\wedge\boldsymbol{\pi} + \frac{1}{2}\,\boldsymbol{b}\,. \tag{43}$$

If the gravitational field is produced by a rotating sphere we find that (43) is identical to the equation for the angular precession of a spinning body in

([6]) M. Moshinsky: *Phys. Rev.*, **80**, 514 (1950).

a gravitational field as given by SCHIFF [7] (eq. (3) of the reference. The term $\boldsymbol{F}\wedge\boldsymbol{v}$, is of higher order then those in (43), so would not show up in the present approximation).

iii) The divergence term $(-(3/8m)\,\mathrm{div}\,\boldsymbol{e})$, similar to the divergence term occurring in the electromagnetic problem, and the very small term in $\boldsymbol{E}\cdot\boldsymbol{e}$.

6. – Red shift in a uniform gravitational field.

If a hydrogen atom is in presence of gravitation its levels will be shifted. MOSHINSKY [6], using a version of Birkhoff theory, found this shift to coincide with the usual prediction of General Relativity.

Let us show that for the present purpose Moshinsky's equations are just the linearized General Relativistic equations.

First, our Schrödinger equation for the electron in the Coulomb field coincides with his one to the order of his approximation ($\boldsymbol{a}=\boldsymbol{b}=\boldsymbol{e}=0$),

$$(44)\qquad i\,\frac{\partial\Psi}{\partial t}=\left\{-m\Phi-e\varphi+(1-3\Phi)\,\frac{p^2}{2m}-\frac{e}{4m^2}\,(1-2\Phi)\left(\boldsymbol{\sigma}\cdot\boldsymbol{E}\wedge\boldsymbol{p}-\frac{1}{2}\,\mathrm{div}\,\boldsymbol{E}\right)\right\}\Psi\,.$$

Second, his equations for the electromagnetic field in presence of gravitation are just the linear approximation of Maxwell equations in General Relativity. In the weak field approximation the Maxwell equations

$$(45)\qquad \frac{\partial}{\partial x^\mu}\left[\sqrt{-g}\,g^{\mu\varrho}g^{\nu\sigma}\left(\frac{\partial A_\varrho}{\partial x^\sigma}-\frac{\partial A_\sigma}{\partial x^\varrho}\right)\right]=j^\nu\,,$$

become

$$(46)\qquad \frac{\partial}{\partial x^\alpha}\left(F^{\dot\beta\dot\alpha}-2l^{\beta\varrho}F_\varrho{}^{\dot\alpha}-2F^{\dot\beta}{}_\varrho l^{\varrho\alpha}+l^\varrho{}_{\dot\varrho}F^{\dot\beta\dot\alpha}\right)=j^\beta\,,$$

where

$$l^{\mu\nu}=-\,\Phi\delta^{\mu\nu}$$

(we used eq. (15-a, b)), and

$$F^{\dot\mu\dot\nu}=\mathring{g}^{\mu\varrho}\,\mathring{g}^{\nu\sigma}F_{\varrho\sigma}\,;\qquad l^\varrho{}_{\dot\varrho}=\mathring{g}_{\varrho\sigma}\,l^{\varrho\sigma}\,.$$

Equation (46) is indeed identical with Moshinsky's (10-a) since our $l^{\beta\alpha}$ are identical to his $h^{\beta\alpha}$ (when $\alpha,\beta=0,1,2,3$) and our φ is the same as his f.

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[7] L. SCHIFF: *Phys. Rev. Lett.*, **4**, 215 (1960).

Thus if we keep only the terms up to order $1/m$ in eq. (44) we obtain the same result of reference [6]. We shall show now that the same expression for the red shift results when the $1/m^2$ terms are included. Instead of going into detailed calculation we shall reduce eq. (44) to a form similar to the one for the free gravitation case:

$$(47)\qquad i\,\frac{\partial \Psi_0}{\partial t} = E_0 \Psi_0 = \left\{ -e\varphi_0 + \frac{p^2}{2m} - \frac{e}{4m^2}\left(\boldsymbol{\sigma}\cdot \boldsymbol{E}_0 \wedge \boldsymbol{p} - \frac{1}{2}\,\mathrm{div}\,\boldsymbol{E}_0 \right) \right\} \Psi_0 = H_0(\boldsymbol{r}, \boldsymbol{p}, \varphi_0)\Psi_0 \,.$$

Here as in ref. [6]

$$\varphi(\boldsymbol{r}) \simeq (1 - 2\Phi)\varphi_0(\boldsymbol{r})\,; \qquad \varphi_0(\boldsymbol{r}) = \frac{e}{4\pi r}\,; \qquad \boldsymbol{E} = -\nabla\varphi\,; \qquad \boldsymbol{E}_0 = -\nabla\varphi_0 \,.$$

We find indeed that if we make the change of variable

$$(48a)\qquad \boldsymbol{r} \to \boldsymbol{r}' = \boldsymbol{r}(1+\Phi)\,; \qquad \boldsymbol{p} \to \boldsymbol{p}' = \frac{1}{i}\nabla'\,,$$

we obtain

$$(48b)\qquad E\Psi'(\boldsymbol{r}', t) = i\,\frac{\partial \Psi'}{\partial t} + m\Phi\Psi' = H\Psi'\,,$$

$$(48\text{-}c)\qquad H = (1-\Phi)\,H_0(\boldsymbol{r}', \boldsymbol{p}', \varphi')\,,$$

where

$$\varphi' = \varphi_0(\boldsymbol{r}')\,, \qquad \Psi'(\boldsymbol{r}', t) = \Psi(\boldsymbol{r}, t)\,.$$

Thus we find, by comparison of (47) and (48):

$$(49)\qquad \Psi'(\boldsymbol{r}', t) = (1 + \tfrac{3}{2}\Phi)\,\Psi_0(\boldsymbol{r}', t)\exp\left[-i(E - E_0)t\right],$$

$$(50)\qquad E = h\nu = (1-\Phi)E_0 = (1-\varphi)h\nu_0 \,.$$

The factor $(1+\frac{3}{2}\varphi)$ in (49) comes from the normalization conditions

$$\int \Psi^\dagger \Psi\,\mathrm{d}v = \int \Psi_0^\dagger \Psi_0\,\mathrm{d}v = 1\,.$$

Therefore we have proved that to the present approximation not only the Coulomb potential contribution to the energy [6] but all electric terms are shifted to the red by the known relativistic equation

$$\frac{\Delta\lambda}{\lambda} = \Phi\,.$$

Actually this result can be proved to be exact to all orders straightforward from Dirac equation in the form (35) for gravitational fields of the form [8],

$$g_{\mu\nu}=0\,,\qquad \mu\neq\nu\,;\qquad g_{11}=g_{22}=g_{33}=g_{ii}\,,$$

or,

$$(51)\qquad \begin{cases} h_{(0)}{}^{\mu}=\sqrt{g^{00}}\,\delta_0{}^{\mu}=\dfrac{1}{\sqrt{g_{00}}}\,\delta_0{}^{\mu}\,;\\[2ex] h_{(r)}{}^{\mu}=\sqrt{g^{ii}}\,\delta_{(r)}{}^{\mu}=\dfrac{1}{\sqrt{g_{ii}}}\,\delta_{(r)}{}^{\mu}\,.\end{cases}$$

In this case we fined from eq. (45)

$$\varphi(\boldsymbol{r})=\sqrt{\frac{g_{00}}{g_{ii}}}\,\varphi_0(\boldsymbol{r})\,.$$

Thus, after the change of variables:

$$(52)\qquad \boldsymbol{r}'=\sqrt{g_{ii}}\,\boldsymbol{r}\,,\qquad \boldsymbol{p}'=\frac{1}{i}\boldsymbol{\nabla}'\,,$$

eq. (35) becomes ($\boldsymbol{A}=\boldsymbol{B}=\boldsymbol{e}=0$):

$$(53)\qquad i\,\frac{\partial\Psi'}{\partial t}=\sqrt{g_{00}}\{m\beta-\alpha^{(k)}p'_k-e\varphi'\}\Psi'=E\Psi'\,,$$

where

$$\varphi'=\varphi_0(\boldsymbol{r}')\,,$$

from which we obtain, by comparison with the free gravitational case:

$$(54)\qquad \Psi'=(g_{ii})^{\frac{3}{4}}\,\Psi_0(\boldsymbol{r}',t)\exp\left[i(E-E_0)t\right]\,,$$

$$(55)\qquad E=h\nu=\sqrt{g_{00}}\,E_0=\sqrt{g_{00}}\,h\nu_0\,,$$

which is the exact prediction of General Relativity in the usual interpretation.

Finally we shall consider the more general case when magnetic fields are present. From Maxwell eqs. (45) we find for stationary magnetic fields in presence of gravitation:

$$(55)\qquad \boldsymbol{A}(\boldsymbol{r})=\frac{1}{4\pi}\sqrt{\frac{g_{ii}}{g_{00}}}\int\frac{\boldsymbol{J}(\boldsymbol{r}_1)}{|\boldsymbol{r}-\boldsymbol{r}_1|}\,\mathrm{d}v_1\,.$$

However, for comparison with the expression in absence of gravitational field we need more information about the current $\boldsymbol{J}(r)$ which is altered by

[8] W. PAULI: *Theory of Relativity* (London, 1958), p. 164.

gravitation. Here we assume that in general the relation of $\boldsymbol{J}$ to $\boldsymbol{J}_0$ (in absence of gravitation) is the same as for the atomic current $(j^i = (h_{(k)}{}^i / h_{(0)}{}^0)\bar{\psi}\gamma^{(k)}\psi)$:

$$\boldsymbol{j}(\boldsymbol{r}) = e\sqrt{\frac{g^{ii}}{g^{00}}}\,\bar{\psi}'\boldsymbol{\gamma}\psi'\,; \qquad \boldsymbol{j}_0(\boldsymbol{r}) = e\,\bar{\psi}_0\boldsymbol{\gamma}\psi_0\,,$$

Thus

$$\boldsymbol{j}(\boldsymbol{r}) = e\,\frac{g_{ii}}{\sqrt{g_{00}}}\,\bar{\psi}_0(\boldsymbol{r}')\boldsymbol{\gamma}\psi_0(\boldsymbol{r}') = e\,\frac{g_{ii}}{\sqrt{g_{00}}}\,\boldsymbol{j}_0(\boldsymbol{r}')\,.$$

Therefore:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{e}{4\pi}\int\frac{\boldsymbol{J}_0(\boldsymbol{r}_1')}{|\boldsymbol{r}-\boldsymbol{r}_1|}\,\mathrm{d}v_1(g_{ii})^{\frac{3}{2}} = \sqrt{g_{ii}}\,\boldsymbol{A}_0(\boldsymbol{r}')\,.$$

Equation (35) becomes now ($\boldsymbol{e}=0$)

$$i\,\frac{\partial\Psi'}{\partial t} = \sqrt{g_{00}}\{m\beta - \alpha^{(k)}\pi_k' - e\varphi'\}\Psi',$$

with,

$$\pi_k' = p_k' - eA_{0,k}(\boldsymbol{r}')\,.$$

Thus we obtain again, by comparison with the free gravitational case, exactly the results (54), (55).

It is interesting to mention that the change of variables $\boldsymbol{r}\to\boldsymbol{r}'$, given by eq. (52) corresponds to the fact that the size of the atom in presence of gravitation is reduced by the factor $1/\sqrt{g_{ii}}$ as could be found by direct computation.

RIASSUNTO (*)

Dopo aver riformulato le equazioni relativistiche generalizzate dal moto di una massa puntiforme in un campo gravitazionale in termini di « vierbein », si scrive l'equazione di Dirac generalizzata nella rappresentazione di Schrödinger e le equazioni del moto delle quantità osservabili classiche vengono confrontate con quella. Si ottiene la rappresentazione di Foldy-Wouthuysen in presenza di campi gravitazionali. Il risultato più interessante è che il fattore « giromagnetico » gravitazionale è 1 invece che 2, come quello elettromagnetico. L'interpretazione di questo fatto è che il campo spinoriale (come probabilmente tutti i campi) descrive particelle rotanti, mentre la massa gravitazionale ha esattamente la stessa distribuzione spaziale della massa inerziale. Questo non è vero nella teoria lineare di Moshinsky-Birkhoff. Infine si ottiene lo spostamento verso il rosso dei livelli degli atomi di idrogeno in presenza di campi gravitazionali e si trova che coincidono con le predizioni usuali della relatività generale.

(*) *Traduzione a cura della Redazione.*