$$\begin{cases} \dfrac{\partial}{\partial t}\vec{S}_\eta\big|_{\text{Thomas}} = \gamma^2\vec{S}_\mu\times\dfrac{\partial\vec{\beta}}{\partial t}+\dfrac{\gamma^3}{\gamma+1}\vec{S}_\mu\times\left(\vec{\beta}\times\left(\vec{\beta}\times\dfrac{\partial\vec{\beta}}{\partial t}\right)\right)+\dfrac{\gamma^2}{\gamma+1}\left(\vec{\beta}\times\dfrac{\partial\vec{\beta}}{\partial t}\cdot\vec{S}_\mu\right)\vec{\beta} \\ \dfrac{\partial}{\partial t}\vec{S}_\mu\big|_{\text{Thomas}} = -\gamma^2\left(\vec{S}_\mu\times\vec{\beta}\right)\times\dfrac{\partial\vec{\beta}}{\partial t}-\dfrac{\gamma^3}{\gamma+1}\left(\vec{S}_\mu\times\vec{\beta}\right)\times\left(\vec{\beta}\times\left(\vec{\beta}\times\dfrac{\partial\vec{\beta}}{\partial t}\right)\right)+\dfrac{\gamma^2}{\gamma+1}\vec{S}_\mu\times\left(\vec{\beta}\times\dfrac{\partial\vec{\beta}}{\partial t}\right) \end{cases}$$

$$-\gamma^2(\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}}-\frac{\gamma^3}{\gamma+1}(\vec{S}_\mu\times\vec{\beta})\cdot(\vec{\beta}\times\dot{\vec{\beta}})\vec{\beta}+\frac{\gamma^3}{\gamma+1}(\vec{S}_\mu\times\vec{\beta})\cdot\vec{\beta}(\vec{\beta}\times\dot{\vec{\beta}})$$

$$=-\gamma^2(\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}}-\frac{\gamma^3}{\gamma+1}(\vec{S}_\mu\times\vec{\beta})\cdot(\vec{\beta}\times\dot{\vec{\beta}})\vec{\beta}$$

$$=-\frac{\gamma^3}{\gamma+1}\left[(\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}}+\underbrace{(\vec{S}_\mu\times\vec{\beta})\cdot(\vec{\beta}\times\dot{\vec{\beta}})\vec{\beta}}_{\text{混合积}}\right]-\frac{\gamma^2}{\gamma+1}(\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}}$$

$$=-\frac{\gamma(\gamma-1)}{\beta^2}\left[(\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}}-((\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}})\cdot\vec{\beta}\,\vec{\beta}\right]-\frac{\gamma^2}{\gamma+1}(\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}}$$

$$=-(\gamma^2-\gamma)\frac{|(\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}}|}{\beta^2}(\vec{e}-\cos\langle\vec{e},\vec{e}_\beta\rangle\vec{e}_\beta)-\frac{\gamma^2}{\gamma+1}(\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}}$$

$$=-\gamma^2(\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}}_{\parallel}+\frac{\gamma^2+\gamma}{\gamma+1}(\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}}_{\parallel}-\frac{\gamma}{\gamma+1}(\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}}-\frac{\gamma^2}{\gamma+1}(\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}}$$

$$=-\gamma^2(\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}}_{\parallel}-\gamma(\vec{S}_\mu\times\vec{\beta})\times\dot{\vec{\beta}}_{\perp}$$

$$\gamma^2\vec{S}_\mu\times\dot{\vec{\beta}}+\frac{\gamma^3}{\gamma+1}\vec{S}_\mu\times(\vec{\beta}\times(\vec{\beta}\times\dot{\vec{\beta}}))+\frac{\gamma^3}{\gamma+1}(\vec{\beta}\times\dot{\vec{\beta}}\cdot\vec{S}_\mu)\vec{\beta}$$

$$=\gamma^2\vec{S}_\mu\times\dot{\vec{\beta}}+\frac{\gamma^3}{\gamma+1}\vec{S}_\mu\cdot(\vec{\beta}\times\dot{\vec{\beta}})\vec{\beta}-\frac{\gamma^3}{\gamma+1}\vec{S}_\mu\cdot\vec{\beta}(\vec{\beta}\times\dot{\vec{\beta}})+\frac{\gamma^3}{\gamma+1}(\vec{S}_\mu\cdot\vec{\beta}\times\dot{\vec{\beta}})\vec{\beta}$$

$$=\gamma^2\vec{S}_\mu\times\dot{\vec{\beta}}+\gamma^2\vec{S}_\mu\cdot(\vec{\beta}\times\dot{\vec{\beta}})\vec{\beta}-\frac{\gamma^3}{\gamma+1}\vec{S}_\mu\cdot\vec{\beta}(\vec{\beta}\times\dot{\vec{\beta}})$$

$$=\gamma^2\vec{S}_\mu\times\dot{\vec{\beta}}-\gamma^2\vec{S}_\mu\times\dot{\vec{\beta}}\cdot\vec{\beta}\,\vec{\beta}-\frac{\gamma^3}{\gamma+1}\vec{S}_\mu\cdot\vec{\beta}(\vec{\beta}\times\dot{\vec{\beta}})$$

$$=\gamma^2|\vec{S}_\mu\times\dot{\vec{\beta}}|(\vec{e}-\beta^2\vec{e}_{\parallel})-\frac{\gamma^3}{\gamma+1}\vec{S}_{\mu\parallel}\cdot\vec{\beta}(\vec{\beta}\times\dot{\vec{\beta}}_{\perp})$$

$$=\gamma^2(\vec{S}_\mu\times\dot{\vec{\beta}})_{\perp}+(\vec{S}_\mu\times\dot{\vec{\beta}})_{\parallel}-\frac{\gamma^3}{\gamma+1}\beta^2|\vec{S}_{\mu\parallel}|\dot{\vec{\beta}}_{\perp}\vec{e}$$

$$=\gamma^2(\vec{S}_\mu\times\dot{\vec{\beta}})_{\perp}+(\vec{S}_\mu\times\dot{\vec{\beta}})_{\parallel}-\gamma^2|\vec{S}_{\mu\parallel}|\dot{\vec{\beta}}_{\perp}\vec{e}+\gamma|\vec{S}_{\mu\parallel}|\dot{\vec{\beta}}_{\perp}\vec{e}$$

$$=\gamma^2(\vec{S}_\mu\times\dot{\vec{\beta}})_{\perp}+(\vec{S}_\mu\times\dot{\vec{\beta}})_{\parallel}-\gamma^2(\vec{S}_{\mu\parallel}\times\dot{\vec{\beta}}_{\perp})+\gamma(\vec{S}_{\mu\parallel}\times\dot{\vec{\beta}}_{\perp})$$

$$=\gamma^2(\vec{S}_{\mu\perp}\times\dot{\vec{\beta}})-\gamma^2\beta^2(\vec{S}_\mu\times\dot{\vec{\beta}})_{\parallel}-\gamma^2(\vec{S}_{\mu\parallel}\times\dot{\vec{\beta}}_{\perp})+\gamma(\vec{S}_{\mu\parallel}\times\dot{\vec{\beta}})$$

$$=\gamma^2(\vec{S}_{\mu\perp}\times\dot{\vec{\beta}})-\gamma^2\vec{\beta}\cdot(\vec{S}_\mu\times\dot{\vec{\beta}})\vec{\beta}+\gamma(\vec{S}_{\mu\parallel}\times\dot{\vec{\beta}})$$

$$\frac{1}{\beta^2}(\vec{e}-\cos\langle\vec{e},\vec{e}_\beta\rangle\vec{\beta}$$
$$=\vec{e}-\vec{e}_{\parallel}+\frac{1}{\gamma^2-1}\vec{e}$$
$$=\vec{e}_{\perp}+\frac{\gamma^2}{\gamma^2-1}\vec{e}$$

$$\vec{S}_\mu\times\dot{\vec{\beta}}=\vec{S}_{\mu\parallel}\times\dot{\vec{\beta}}_{\parallel}+\vec{S}_{\mu\perp}\times\dot{\vec{\beta}}_{\parallel}+\vec{S}_{\mu\parallel}\times\dot{\vec{\beta}}_{\perp}+\vec{S}_{\mu\perp}\times\dot{\vec{\beta}}_{\perp}$$
$$\vec{S}_{\mu\parallel}\times\dot{\vec{\beta}}_{\parallel}=0$$