FINITE DIFFERENCE AND WAVE NUMBER MIGRATION *

BY

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ABSTRACT

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Finite difference migration has been developed and popularized by J. F. Claerbout of Stanford University and is now widely used in seismic processing. For most sections finite difference migration gives results comparable to those obtained by conventional Kirchhoff migration and, where events are not dipping too much, a cleaner appearance is often apparent. However, there are two practical limitations to the method, and these occur in regions of very steep dip and where there is a large variation of the velocity in the lateral direction.

It is possible to develop successively more accurate equations to deal with the steep dip problem, but above third order these schemes become prohibitively expensive to implement. The finite difference method itself introduces errors and so imposes further limitations on the angle of dip. For the effective treatment of steeply dipping beds there appears to be no method available in the time domain which does not suffer from dispersion inaccuracies. However, by developing wavenumber migration, an exact one-way wave equation can be used, and this eliminates any error except that caused by finite sampling.

The other difficulty with wave migration is the correct migration in regions with lateral velocity variation. A number of approaches are possible of which three are discussed here. The first uses an exact theory, the second is based on the deviation from a depth stratified model, and the third uses a transformation to a depth co-ordinate system. All methods are discussed with their advantages and limitations. Finally, some examples are shown of wave migration applied to synthetic and real data.

INTRODUCTION

There are a number of different options available for the migration of seismic data. Three of these methods are shown schematically in fig. 1. The first method is the Kirchhoff integral formulation of the wave equation, which gives rise to the "diffraction stack" process. This has been available for a number of years and offers a quite satisfactory method of migration. Essentially it uses summation along "NMO curves" of either $(\partial P/\partial t)^{1/2}$ or P, where P is the pressure recorded at the hydrophones. If one uses $(\partial P/\partial t)^{1/2}$ corrections for

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KIRCHOFF INTEGRAL (DIFFRACTION STACK)	—	SUMMATION ALONG NMO CURVES OF $(\frac{\partial P}{\partial t})^{1/2}$ OR P.
FINITE DIFFERENCE SOLUTION	—	MODIFICATION OF THE SCALAR WAVE EQUATION & SOLUTION BY THE FINITE DIFFERENCE METHOD.
WAVENUMBER MIGRATION	—	TRANSFORM TO (K,W) PLANE & USE EXACT ONE WAY WAVE EQUATION. INVERSE TRANSFORM.

Fig. 1. Three possible migration methods.

phase are automatically applied, whilst for algorithms based on P it is necessary to make a separate correction after summation (Gardner, French, and Matzuk 1974; French 1975; Larner and Hatton 1976; Newman 1975). Similarly, amplitude corrections can be made to account for variations in the number of traces over which the summation process is extended.

The finite difference solution to the wave equation is the second method listed and will be the main subject of this paper. A limited discussion of the third technique, which for brevity may be termed "wavenumber migration", will also be included. In "wavenumber migration" the migration is carried out entirely in the (spatial and temporal) frequency domain.

The use of finite difference techniques to migrate seismic data is so widely practised in the oil exploration industry that the subject requires little introduction. There are, however, two cases where finite difference wave migration can give poor results, namely in areas of steep dip or in regions where correct compensation for lateral velocity variation has not been applied. Although some think the best solution with a limited angle wave equation is probably to be obtained using slant stacked data (Claerbout et al. 1972-1977), a surprising degree of success can nevertheless be achieved with conventional CDP stacked data. In this paper, therefore, attention is focussed on migration of CDP stacked data and on the two problem areas: migration in regions with lateral velocity variation and migration of steeply dipping beds.

The subject of lateral velocity variation can fortunately be treated in isolation since, once a particular approach has been selected, it may be applied to a shallow or steep dip wave equation. The treatment of steep dip events, on the other hand, depends on the accuracy of the parabolic approximation to the wave equation used and on its numerical approximation. Both the parabolic wave equation and its approximation by finite differences introduce errors. In deriving a steep dip algorithm, errors from the two sources must be considered, and, as far as possible, made to cancel.

Unfortunately the very nature of the finite difference approximation means that errors can never be entirely eliminated, and for this reason migration in wavenumber space is now being considered. There appear to be a number of encouraging features with wavenumber migration and some preliminary results using this technique are discussed.

One Way Wave Equations

The starting point for wave migration is normally the scalar wave equation. In a coordinate system in which the shot and receiver are coincident (i.e. approximately in a CDP stack), the velocity must be halved (Loewenthal, Lu, Robinson, and Sherwood 1976), so that in these coordinates the wave equation is:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} = \frac{4}{c^2}\frac{\partial^2 P}{\partial t^2} \tag{1}$$

where $P(x, z, t)$ is the pressure recorded at the hydrophones, x, z, t are the horizontal, vertical, and time coordinate, respectively, and $c(x, z)$ is the speed of sound waves.

Following Claerbout (1970) the one-way wave equation which governs the propagation of waves along the direction of the z axis is (in the frequency domain):

$$\frac{\partial P}{\partial z} = \mathrm{i}\left(\frac{4\omega^2}{c^2} + \frac{\partial^2}{\partial x^2}\right)^{1/2} P = \mathrm{i}SP \tag{2}$$

where ω is the angular frequency.

The key to the success of various parabolic approximations to equation (2) lies in the accuracy to which the square root term S is approximated. Claerbout (1970) derives a set of approximations of various orders, and in fig. 2 the second and third order approximations to the square root are shown. These are called the 15° and 45° approximations, respectively. As can be seen the 45° approximation is a better approximation than the 15° approximation at all angles. Also shown is a different third order approximation which has been developed at Seismograph Service Limited. Here the emphasis is on achieving a closer fit at steeper angles, and, from about 40° dip and steeper, this approximation is better than the 45° approximation (with the effects of the numerical approximation included).

It is possible to use fourth order or higher approximations to S, but there are a number of computational drawbacks arising from the fact that the resulting finite difference equations no longer have a straightforward tri-diagonal form. This causes a progressive increase in the computational costs. Furthermore, errors introduced by the finite difference method can be more serious than those caused by the parabolic wave equation, so that, unless the

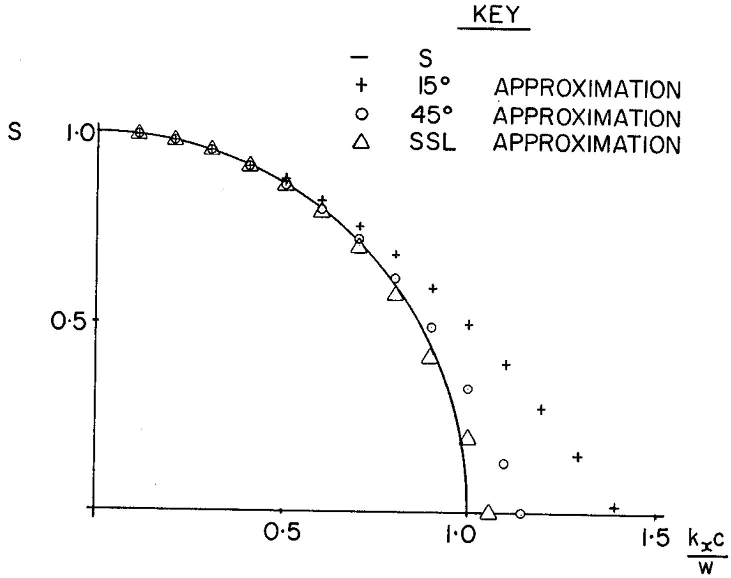

Fig. 2. Various approximations to S.

finite difference method is abandoned in favour of other techniques, there seems to be little prospect of high order approximations achieving any noticeable improvement.

In fig. 3 we show a synthetic section (generated using ray tracing techniques) of a model with beds dipping between 0° and 50° in steps of 10°. The geophone interval is 61 m and the velocity 3048 m/s with a sampling interval of 4 ms. The pulse is a zero phase wavelet centered at 15 Hz. Figs. 4a-c represent the migrated results using the 15°, 45°, and our current approximation, respectively, with a depth interval of 40 ms used in the migration. The beds should all be aligned after migration, and it is apparent that none of the methods works perfectly. All suffer from dispersion effects caused by the wave equation error and finite difference approximation. However, it must be noted that the gains on the plots are unrealistically large, and these dispersion effects, although existing near steep beds on real data, are not usually quite so apparent.

Returning now to equation (2), a 15° equation is derived for later reference.

Let $\bar{c}$ be a constant velocity. We have then

$$\frac{\partial P}{\partial z} = i\,\frac{2\omega}{\bar{c}}\left[1 - 1 + \left(\frac{\bar{c}}{c}\right)^2 + \left(\frac{\bar{c}}{2\omega}\right)^2 \frac{\partial^2}{\partial x^2}\right]^{1/2} P$$

$$\doteqdot i\,\frac{\omega}{\bar{c}}\left[1 + \left(\frac{\bar{c}}{c}\right)^2 + \left(\frac{\bar{c}}{2\omega}\right)^2 \frac{\partial^2}{\partial x^2}\right] P. \tag{3}$$

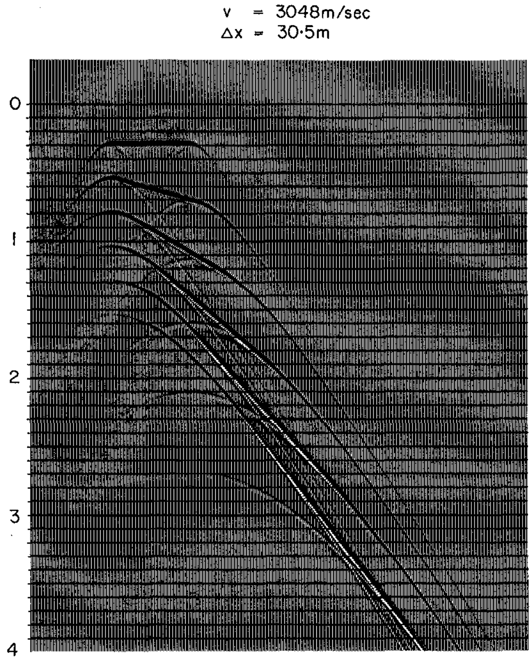

Fig. 3. Dipping planes 0-50°.

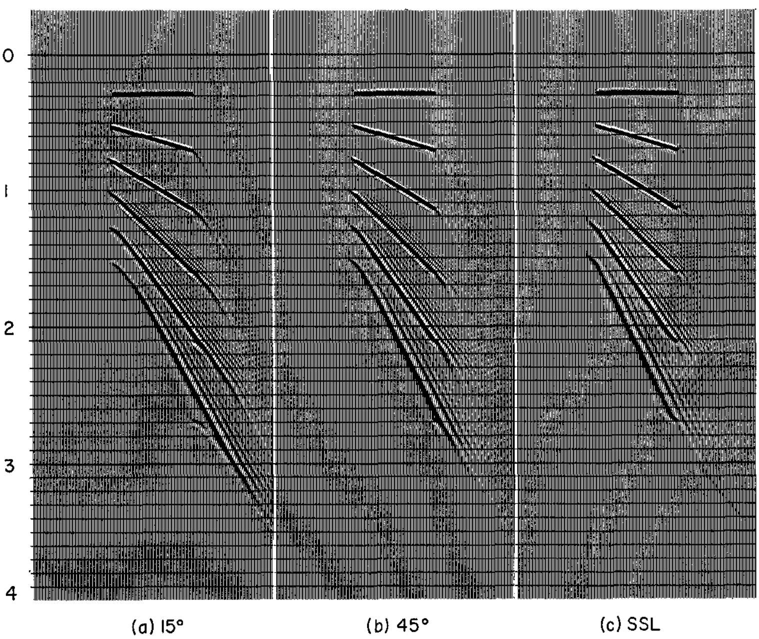

Fig. 4. Migration with various parabolic approximations.

This may be placed in a retarded time frame, setting $P = P' \exp(\mathrm{i}\, 2\omega z/\bar{c})$, thus:

$$\frac{\partial P'}{\partial z} = \mathrm{i}\,\frac{\omega}{\bar{c}}\left[\left(\frac{\bar{c}}{c}\right)^2 - 1 + \left(\frac{\bar{c}}{2\omega}\right)^2 \frac{\partial^2}{\partial x^2}\right] P'. \tag{4}$$

Restoring (4) to the time domain and dropping the primes we obtain:

$$\frac{\partial^2 P}{\partial t\,\partial z} = \frac{\partial^2 P}{\partial t^2}\left(\frac{\bar{c}}{c^2} - \frac{1}{\bar{c}}\right) - \frac{\bar{c}}{4}\,\frac{\partial^2 P}{\partial z^2}. \tag{5}$$

A special case of this arises if lateral velocity variations are insignificant, in which case, neglecting transmission effects, the following equation can be derived:

$$\frac{\partial^2 P}{\partial z\,\partial t} = -\frac{c(z)}{4}\,\frac{\partial^2 P}{\partial x^2}. \tag{6}$$

Finite Difference Approximation

The application of the finite difference method to parabolic wave equations is fairly standard and has, for example, been discussed by Claerbout and Johnson (1971). However, one or two parameters are of interest and so a brief discussion of the equations will be given in this section. Let $P_{j,k,n}$ be the value of P at the mesh point $P(t_j, x_k, z_n)$. The following difference operators may be defined:

$$D_x P_{j,k,n} = (P_{j,k+1,n} - P_{j,k,n})\,\frac{1}{\Delta x} \qquad \left(\approx \frac{\partial P}{\partial x}\right),$$

$$D_{xx} P_{j,k,n} = (P_{j,k+1,n} - 2\,P_{j,k,n} + P_{j,k-1,n})\,\frac{1}{\Delta x^2} \qquad \left(\approx \frac{\partial^2 P}{\partial x^2}\right),$$

and let:

$$A_1 P_{j,k,n} = \tfrac{1}{2}\left[(1-\theta)\,P_{j,k,n} + \theta\,P_{j,k,n+1} + (1-\theta)\,P_{j+1,k,n} + \theta\,P_{j+1,k,n+1}\right],$$

and

$$A_2 P_{j,k,n} = \alpha\,P_{j,k+1,n} + (1-2\alpha)\,P_{j,k,n} + \alpha\,P_{j,k-1,n}.$$

Equation (6) may be expressed in difference notation as:

$$\left(A_2 D_z D_t + \frac{c(z)}{4}\,D_{xx} A_1\right) P_{j,k,n} = 0. \tag{7}$$

For stability of equation (7) $0 \leqslant \alpha \leqslant \frac{1}{4}$ and $0.5 \leqslant \theta \leqslant 1$ is required. Values of α in the range $\frac{1}{12} \leqslant \alpha \leqslant \frac{1}{6}$ appear to be widely accepted (Claerbout 1976; Loewenthal et al. 1976).

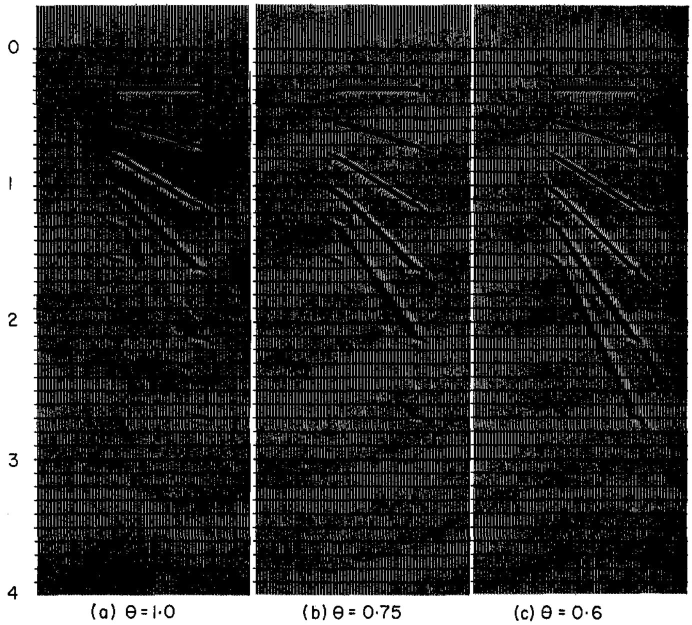

Fig. 5. Various values of θ (synthetic data).

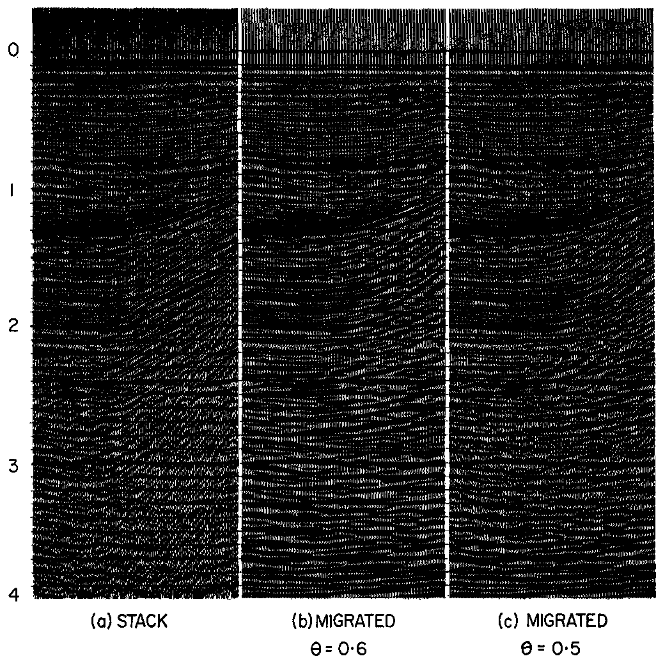

Fig. 6. Various values of θ (North Sea data).

The parameter θ is interesting in that its value determines whether or not the difference scheme is implicit or explicit. The migration results in fig. 4 were obtained with $\theta = 0.52$ in all three examples. By increasing θ towards unity waves at large angles to the axis are successively attenuated with the high frequencies being first affected. In fig. 5a-c a 45° approximation is used with $\theta = 1$, $\theta = 0.75$, and $\theta = 0.6$, respectively. It is clear that at $\theta = 0.6$ we have practically eliminated the dispersion from the dipping planes yet still retained all beds up to 40° dip. Therefore it might be considered attractive to use an off-centred difference scheme as a kind of dip filter to remove precursors from steeply dipping planes. Unfortunately, on real data the results are disappointing, giving diffuse and blurred sections. Fig. 6a shows part of a stacked line before migration and fig. 6b-c after migration using $\theta = 0.6$ and 0.5, respectively.

Once α and θ are selected, the question arises: what are the possible choices of sampling step size which can be chosen to permit steeply dipping events to migrate accurately? The sampling interval in the x direction is fixed by the geophone interval, and resampling to a coarser Δx will generally degrade the results totally (Larner and Hatton 1976). Similarly Δt, the time increment, would normally correspond to the actual recording sampling interval; however, resampling to a coarser Δt can sometimes give tolerable results. In the z direction the choice of step size is less critical and varies widely in company practice, the accuracy is usually higher with a small Δz. Step size in this direction is conventionally expressed not as Δz, but $\Delta\tau$, the two-way travel time corresponding to the depth step. In all cases an analysis of the errors arising for a particular frequency, velocity, dip, and migration equation can be used to assess the optimum difference scheme and increments necessary for acceptable results.

Lateral Velocity Variation

We have identified three main methods of including the effects of lateral velocity variation. Without doubt the list is incomplete, and each seismic data processing company will have its own particular preference for inclusion of these effects.

The first and most obvious method is to solve equation (5) directly by a finite difference procedure. The stability of the solution depends on the range of velocity departure from the frame velocity $\bar{c}$ and on the recording parameters. For normal values there appear to be certain stability difficulties. An alternative solution suggested by various authors is to treat the effects of lateral velocity variation in two parts. The first part of equation (5) deals with diffraction and wave propagation effects, the other with a time shift associated with the velocity fluctuations. This method is unconditionally stable.

The second method of treating lateral velocity variations is based on the deviation from a depth stratified model (equation (6)). It is founded upon the idea that the wave migration method is simply a method for projecting the seismogram downwards. The section is migrated when all events have returned to their point of origin. In one sense then, it is immaterial what migration velocity is chosen in each depth step, provided that the migration is continued to a depth sufficient to encompass all sources. For example, suppose a reflecting interface is located at a depth of 3000 m. We could arrive at this depth either by migrating over 1 s with a velocity of 3000 m/s or over 2 s using a migration velocity of 1500 m/s. This may be demonstrated more rigorously by considering what migration does to each Fourier component of the wavefield. Propagation through a depth z in a medium with velocity c is achieved by multiplication of each component by $\exp\left\{i\frac{2\omega}{c}z\left[\left(1-\left(\frac{ck}{2\omega}\right)^2\right)^{1/2}-1\right]\right\}$ using a retarded time frame (Maginness 1972). If instead the migration uses a velocity c_1 then the component is multiplied by $\exp\left\{i\frac{2\omega}{c_1}z_1\left[\left(1-\left(\frac{c_1 k}{2\omega}\right)^2\right)^{1/2}-1\right]\right\}$ and is projected through a distance z_1. In order to achieve the same migration results with the velocity c_1, the phase in both expressions must be equal. Thus dropping third order terms in the expansion of the square root and equating coefficients gives:

$$z_1 = z\frac{c}{c_1}.$$

So, to second order accuracy, the results using the incorrect velocity in any depth interval may be corrected by adjusting the depth of migration in the ratio of the true/migration velocities.

The third method is analogous in many ways to the second method but involves a conversion to a "depth" section. Thus instead of using the usual time shifted co-ordinate system (Claerbout and Johnson 1971):

$$t' = t + \frac{z}{c},$$

a depth co-ordinate d is used defined by:

$$d = \frac{c_a t}{2} + z,$$

where $c_a t = \int_0^t c_{int}\,dt$.

c_a is commonly referred to as the average velocity and c_{int} as the interval velocity. Provided that the interval velocity varies slowly as a function of x

and z, several simplifications can be made to the equations and migration through depth layers proceeds in the normal manner.

Of all the methods outlined only the first method makes no assumption about the velocity variation, but it is difficult to apply since the migration technique requires data in multiplexed format. The second method assumes that third order effects are negligible which is consistent with a 15° approximation, whilst the third method assumes definite bounds on velocity changes and introduces further inaccuracy due to resampling. However, since velocity control is normally only approximate, probably any of the inexact methods could be used without introduction of serious error, and the choice will largely depend on individual practice.

In fig. 7 a depth model is shown consisting of two point scatterers located in regions with interval velocities of 1524 m/s and 3048 m/s, respectively. Fig. 8a shows the time section with superimposed background noise of two types—

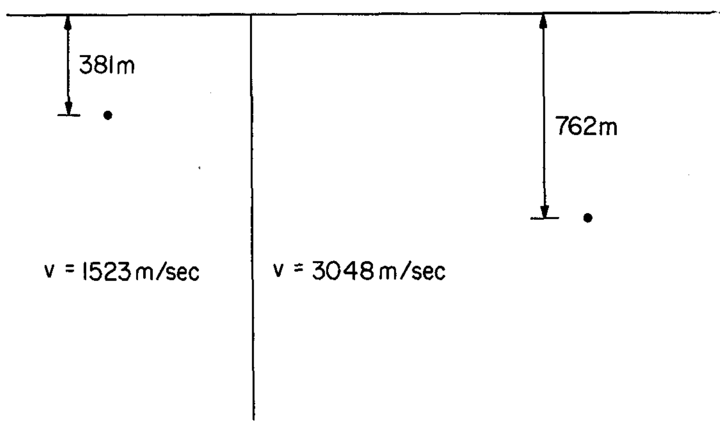

Fig. 7. Depth model lateral velocity synthetic.

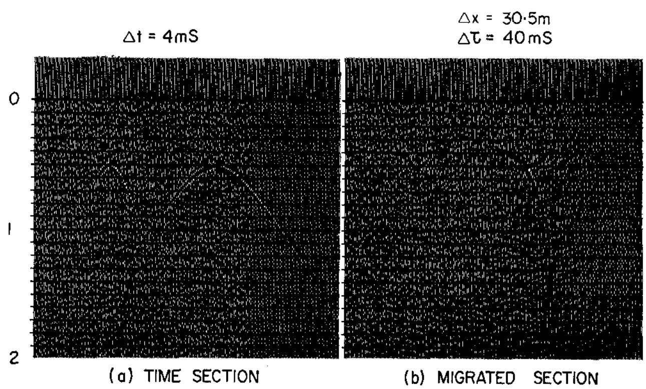

Fig. 8. Lateral velocity synthetic.

random and partially organized. In fig. 8b a migrated section using the second method is shown. Both point scatterers are now fairly clear, with the diffractions largely collapsed, and with the noise spread out over the wavefronts.

Wavenumber Migration

The idea that migration may be carried out entirely in the frequency domain is not new. Maginness (1972) presented a scheme for reconstruction of wavefields at successive planes parallel to $z = 0$ using surface recorded information. The procedure he discussed may be applied in the following manner to seismic data:

1. From $P(x, t; 0)$ form the Fourier transform
   $\hat{P}(k, \omega; 0) = \iint P(x, t; 0) \exp[i(kx - \omega t)]\, dx\, dt;$
2. modify the phase of each Fourier component by multiplication by the term $\exp\left[i\Delta z \sqrt{\frac{4\omega^2}{c^2} - k^2}\right]$; and
3. form the inverse Fourier transform and obtain $P(x, t; \Delta z)$, the seismogram at a depth Δz.

The migration procedure in the wavenumber domain thus follows exactly the same procedure as finite difference migration in that the seismogram is projected downwards a distance Δz at each stage. To avoid circular convolution

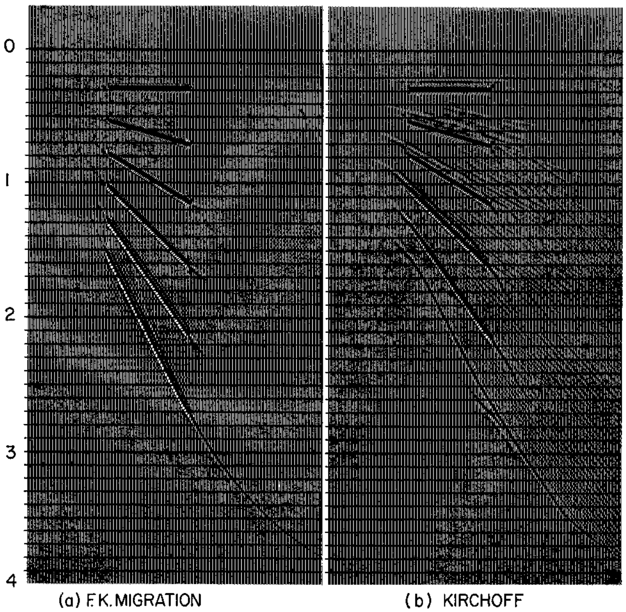

Fig. 9. Migration of synthetic, data by $f - k$ Kirchhoff method.

it is necessary to multiply components by an additional shifting term exp $(-i2\omega\Delta z/c)$ in step 2.

As presented above, this procedure is inefficient for migration purposes in that only one particular time is of interest at any given depth z—the one at which the upgoing waves were generated. A modification is thus performed in which the projection downwards yields only the results at the required depths and times. Results for the migration of the synthetic section in fig. 3 are shown in fig. 9a. In fig. 9b the same synthetic section is migrated using Kirchhoff summation. It is clear that wavenumber migration suffers from none of the frequency dispersion of the finite difference method, nor the noise generated by the Kirchhoff summation method. Migration of the lowest bed is incomplete since some time section data for this bed appeared at times later than the 4 s used for this migration. Although only limited experience has been gained using wavenumber migration, these early results are encouraging.

Migration of Real Data

After the synthetic sections some examples of migration with real data are now discussed. The first example is a section from the Yorkshire coastal area and includes a salt piercement dome which is typical of the area (fig. 10a). The section is a forty-eight fold stack and has been corrected for instrument phase distortion. The migrated results are shown in fig. 10b using finite difference migration with a depth interval $\Delta\tau = 40$ ms. Migration clearly improves the right-hand edge of the section where the overlapping beds are returned to their correct locations, although in this particular case the two-dimensional assumption is not sufficient. In fig. 10c wavenumber migration was used to migrate the section. The improvement of wavenumber migration over conventional finite difference migration is noticeable after detailed inspection of the sections. In particular, the migration noise is much less if the wavenumber method is used.

The next example is also from North Sea data and shows a buried focus at about 4 s on a thirty fold stack (fig. 11a). Fig. 11b shows the migrated section again using finite difference migration, demonstrating the simplification in interpretation which can be achieved through migration.

Conclusions

The migration of seismic data may be achieved in either the time or wavenumber domain. Results based on the finite difference method can be very good up to 45° dip or steeper provided that a suitable choice of one-way wave equation and difference approximation is made. However, dispersion errors from various sources can spoil results near steeply dipping beds. For precise migration at all dips, wavenumber migration offers possibly the ultimate in migration accuracy and represents a very promising new development.

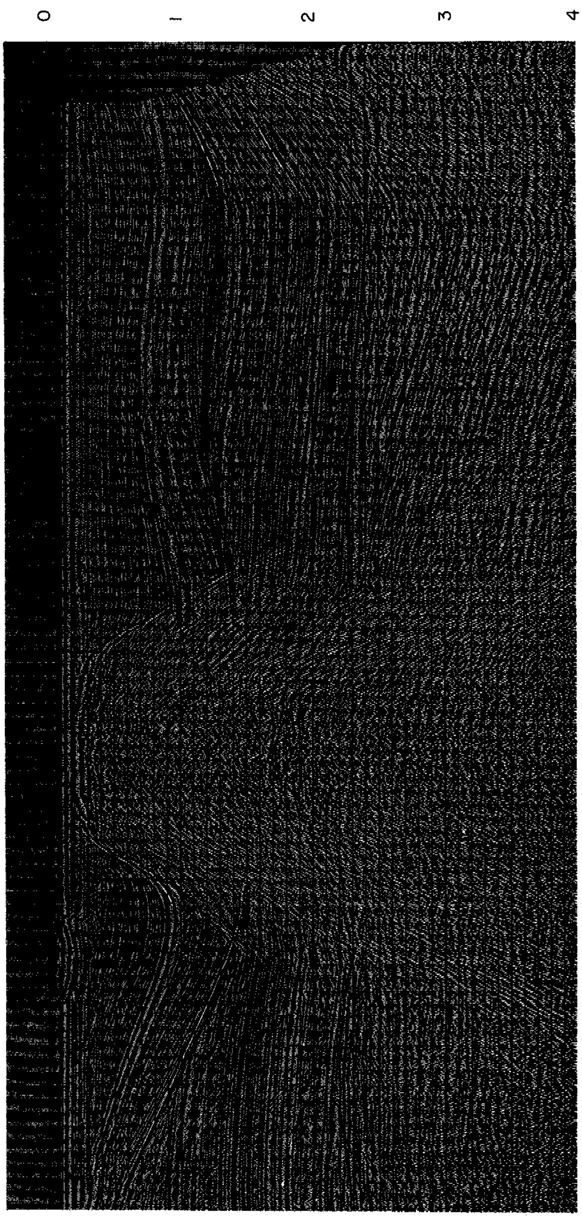

Fig. 10(a) 48-fold stack North Sea area.

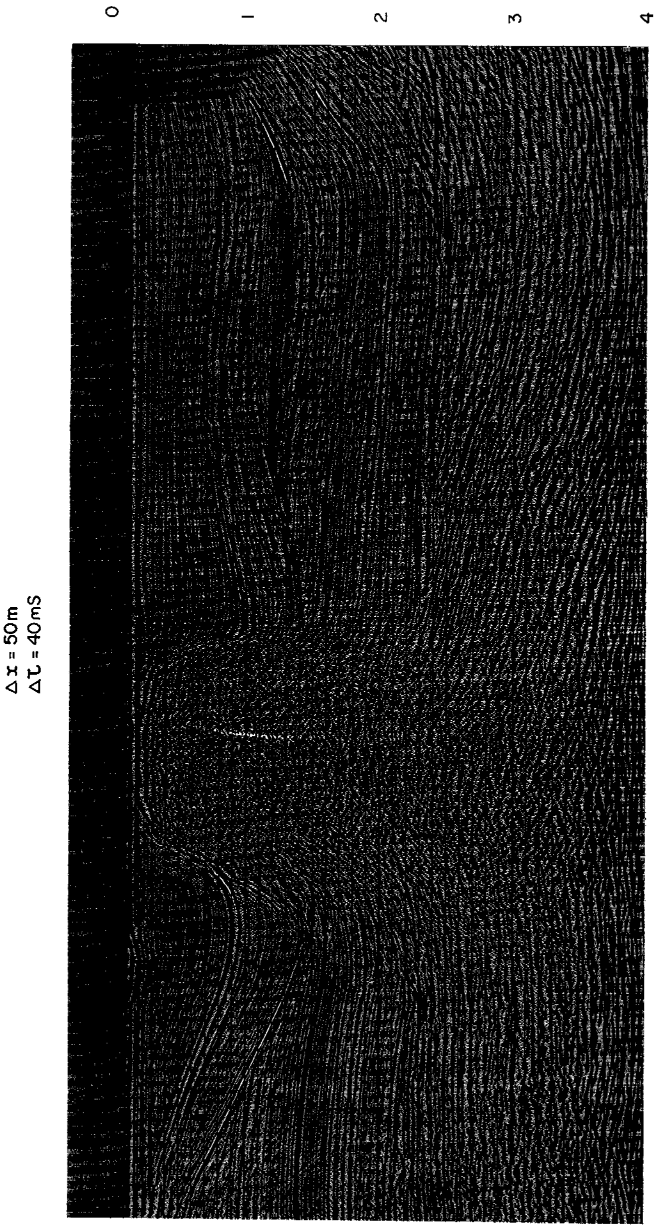

Fig. 10(b) Wave equation migrated stack.

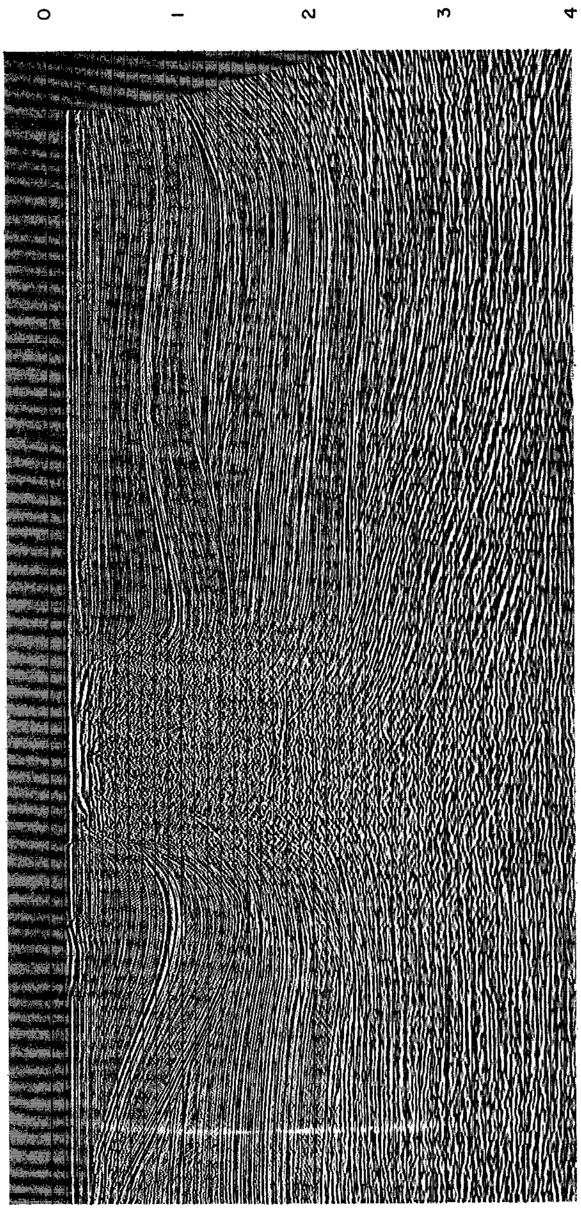

Fig. 10(c) f—k migrated stack.

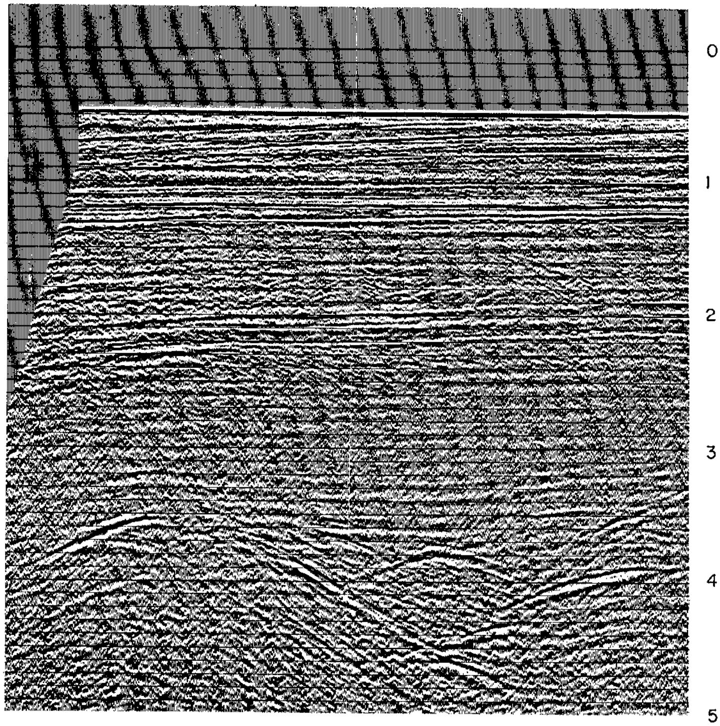

Fig. 11(a) Buried focus 30-fold stack North Sea.

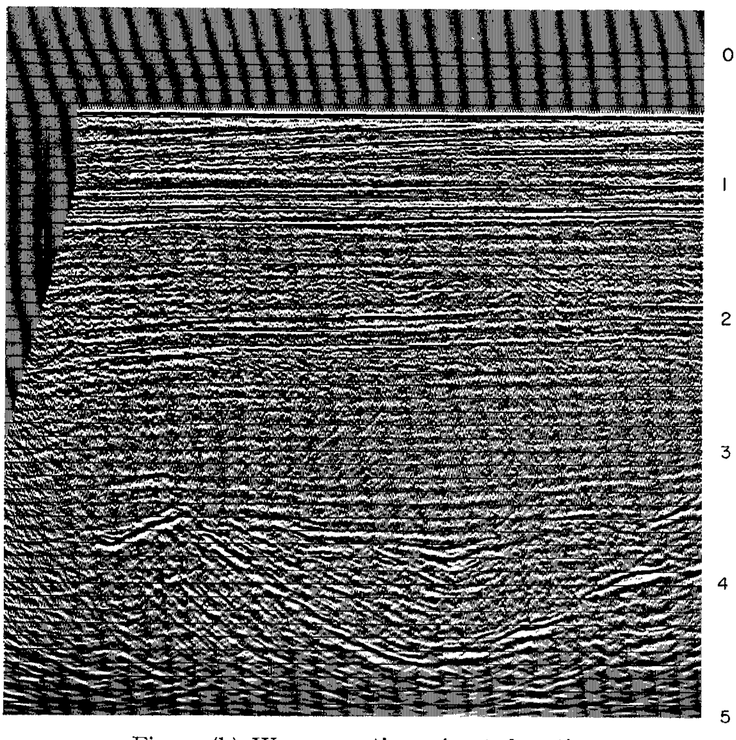

Fig. 11(b) Wave equation migrated section.

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