1.6 欧拉替换法 $\int G(x,\sqrt{ax^2+bx+c})\,dx$ 被积函数有理化

① 若 $a>0$：令 $\sqrt{ax^2+bx+c}=t\pm\sqrt{a}\cdot x$

② 若 $c>0$：令 $\sqrt{ax^2+bx+c}=xt\pm\sqrt{c}$

令 $x=\frac{1}{u}$

对于 $\int\frac{dx}{x+\sqrt{x^2-x+1}}$

令 $\sqrt{x^2-x+1}=t-x$

两边平方：$x^2-x+1=t^2-2tx+x^2$

得：$x=\frac{t^2-1}{2t-1}$，$dx=\frac{2(t^2-t+1)}{(2t-1)^2}dt$

$$\int\frac{dx}{x+\sqrt{x^2-x+1}}=\int\frac{\frac{2(t^2-t+1)}{(2t-1)^2}dt}{\frac{t^2-1}{2t-1}+t-\frac{t^2-1}{2t-1}}=\int\frac{2(t^2-t+1)}{t(2t-1)^2}dt$$

令 $\sqrt{x^2-x+1}=xt+1$

两边平方：$x^2-x+1=x^2t^2+2xt+1$

得：$x=\frac{2t+1}{1-t^2}$，$dx=-\frac{t^2-2t-2}{(1-t^2)^2}dt$

$$原式=\int\frac{-\frac{t^2-2t-2}{(1-t^2)^2}dt}{\frac{2t+1}{1-t^2}+\frac{2t+1}{1-t^2}\cdot t+1}$$

令 $\sqrt{x^2-x+1}=xt-1$，

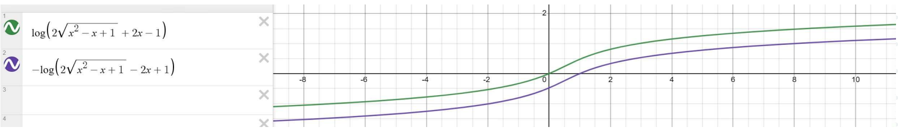
得 $x=\frac{2t-1}{t^2-1}$，$dx=-2\frac{t^2-t+1}{(t^2-1)}dt$

$$原式=\frac{-2\frac{t^2-t+1}{(t^2-1)}dt}{\frac{2t-1}{t^2-1}+\frac{2t-1}{t^2-1}\cdot t-1}$$

$\int\frac{dx}{\sqrt{x^2-x+1}}$，令 $\sqrt{x^2-x+1}=t-x$，得：$x=\frac{t^2-1}{2t-1}$，$dx=\frac{2(t^2-t+1)}{(2t-1)^2}dt$

$$原式=\int\frac{\frac{2(t^2-t+1)}{(2t-1)^2}dt}{t-\frac{t^2-1}{2t-1}}=\int\frac{2}{2t-1}dt=\ln|2t-1|+C=\ln\left|2\sqrt{x^2-x+1}+2x-1\right|+C$$

令 $\sqrt{x^2-x+1}=t+x\Longrightarrow 原式=-\ln\left|2\sqrt{x^2-x+1}-2x+1\right|+C$



③ 若 $\Delta=b^2-4ac>0$：令 $\sqrt{ax^2+bx+c}=t(x-\lambda)$，$\lambda$为任一实根

$\int \frac{dx}{x+\sqrt{x^2-5x+6}}$，有 $x^2-5x+6=(x-2)(x-3)$，令 $\sqrt{x^2-5x+6}=t(x-2)$

两边平方：$x^2-5x+6=t^2(x-2)^2=(x-2)(x-3)$，$t^2(x-2)=x-3$

得：$x=\frac{2t^2-3}{t^2-1}$，$dx=\frac{2t}{(t^2-1)^2}dt$，$\sqrt{x^2-5x+6}=-\frac{t}{t^2-1}$

$$原式=\int \frac{\frac{2t}{(t^2-1)^2}dt}{\frac{2t^2-3}{t^2-1}-\frac{t}{t^2-1}}=\int \frac{2t}{(t^2-1)(2t-3)(t+1)^2}dt$$

欧拉替换法总结：

①若 $a>0$：令 $\sqrt{ax^2+bx+c}=t\pm\sqrt{a}\cdot x$

②若 $c>0$：令 $\sqrt{ax^2+bx+c}=xt\pm\sqrt{c}$

令 $x=\frac{1}{u}$

③若 $\Delta=b^2-4ac>0$：令 $\sqrt{ax^2+bx+c}=t(x-\lambda)$，$\lambda$ 为任一实根

1.7 三角函数积分中的倍角法

棣莫弗公式 $(\cos x+i\sin x)^n=\cos nx+i\sin nx$

令 $\cos x+i\sin x=y$，有 $\cos x-i\sin x=\frac{1}{y}$ $\longrightarrow$ $2\cos x=y+\frac{1}{y}$，$2i\sin x=y-\frac{1}{y}$

则 $\cos nx+i\sin nx=y^n$，$\cos nx-i\sin nx=\frac{1}{y^n}$ $\longrightarrow$ $2\cos nx=y^n+\frac{1}{y^n}$，$2i\sin nx=y^n-\frac{1}{y^n}$

例38 $\int \sin^8 x\,dx$

$(2i\sin x)^8=2^8 i^8\sin^8 x=(y-\frac{1}{y})^8$

$$=\binom{8}{0}\underbrace{y^0\frac{1}{y^8}}_{\frac{1}{y^8}}-\binom{8}{1}\underbrace{y^1\frac{1}{y^7}}_{\frac{1}{y^6}}+\binom{8}{2}\underbrace{y^2\frac{1}{y^6}}_{\frac{1}{y^4}}-\binom{8}{3}\underbrace{y^3\frac{1}{y^5}}_{\frac{1}{y^2}}+\binom{8}{4}\underbrace{y^4\frac{1}{y^4}}_{1}-\binom{8}{5}\underbrace{y^5\frac{1}{y^3}}_{y^2}+\binom{8}{6}\underbrace{y^6\frac{1}{y^2}}_{y^4}-\binom{8}{7}\underbrace{y^7\frac{1}{y^1}}_{y^6}+\binom{8}{8}\underbrace{y^8\frac{1}{y^0}}_{y^8}$$

$$=\binom{8}{0}(y^8+\frac{1}{y^8})-\binom{8}{1}(y^6+\frac{1}{y^6})+\binom{8}{2}(y^4+\frac{1}{y^4})-\binom{8}{3}(y^2+\frac{1}{y^2})+\binom{8}{4}$$

$$=(y^8+\frac{1}{y^8})-8(y^6+\frac{1}{y^6})+28(y^4+\frac{1}{y^4})-56(y^2+\frac{1}{y^2})+70$$

$$=2\cos 8x-16\cos 6x+56\cos 4x-112\cos 2x+70$$