1.13 分母含有三角函数的积分

$$\int\frac{dx}{\sin x},\ \int\frac{dx}{\cos x},\ \int\frac{dx}{a+b\cos x},\ \int\frac{dx}{a+b\sin x},\ \int\frac{dx}{a\sin x+b\cos x},\ \int\frac{dx}{a+b\cos x+c\sin x}$$

$$\int\frac{dx}{\sin^n x},\ \int\frac{dx}{\cos^n x},\ \int\frac{dx}{(a+b\cos x)^n},\ \int\frac{dx}{(a+b\sin x)^n},\ \int\frac{dx}{(a\sin x+b\cos x)^n},\ \int\frac{dx}{(a+b\cos x+c\sin x)^n}$$

微分积分法

(1) $\int\frac{dx}{\sin x}$

$$\int\frac{1}{1-x^2}dx=\operatorname{artanh}x=\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$$

$$\int\frac{dx}{\sin x}=\int\frac{\sin x}{\sin^2 x}dx=-\int\frac{d(\cos x)}{1-\cos^2 x}=\frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right|+C=\frac{1}{2}\ln\left|\tan\frac{x}{2}\right|+C$$

(2) $\int\frac{dx}{\cos x}$

$$\int\frac{dx}{\cos x}=\int\frac{\cos x}{\cos^2 x}dx=\int\frac{d(\sin x)}{1-\sin^2 x}=\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+C$$

(3) $\int\frac{dx}{a+b\cos x}$

① 若 $a>b$，有 $a+b\cos x=a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)$

$$=(a+b)\cos^2\frac{x}{2}+(a-b)\sin^2\frac{x}{2}$$

$$=(a-b)\cos^2\frac{x}{2}\left(\frac{a+b}{a-b}+\tan^2\frac{x}{2}\right)$$

得 $\int\frac{dx}{a+b\cos x}=\int\frac{dx}{(a-b)\cos^2\frac{x}{2}\left(\frac{a+b}{a-b}+\tan^2\frac{x}{2}\right)}$

$$=\frac{2}{a-b}\int\frac{d\left(\tan\frac{x}{2}\right)}{\frac{a+b}{a-b}+\tan^2\frac{x}{2}}\qquad\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\arctan\frac{x}{a}+C$$

② 若 $a<b$，有 $a+b\cos x=a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)$

$$=(a+b)\cos^2\frac{x}{2}-(b-a)\sin^2\frac{x}{2}$$

$$=(b-a)\cos^2\frac{x}{2}\left(\frac{b+a}{b-a}-\tan^2\frac{x}{2}\right)$$

得 $\int\frac{dx}{a+b\cos x}=\int\frac{dx}{(b-a)\cos^2\frac{x}{2}\left(\frac{b+a}{b-a}-\tan^2\frac{x}{2}\right)}$

$$= \frac{2}{b-a}\int \frac{d(\tan\frac{x}{2})}{\frac{b+a}{b-a}-\tan^2\frac{x}{2}} \qquad \int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

(4) $\int \frac{dx}{a+b\sin x}$

法1：令 $x = \frac{\pi}{2} + y$，则 $\sin x = \cos y$，$dx = dy$，得 $\int \frac{dx}{a+b\sin x} = \int \frac{dy}{a+b\cos y}$

法2：$a+b\sin x = a(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}) + 2b\sin\frac{x}{2}\cos\frac{x}{2}$

$= a\cos^2\frac{x}{2}\left(1+\tan^2\frac{x}{2} + \frac{2b}{a}\tan\frac{x}{2}\right)$

$= a\cos^2\frac{x}{2}\left[(\tan\frac{x}{2}+\frac{b}{a})^2 - \frac{b^2}{a^2} + 1\right]$

得 $\int \frac{dx}{a+b\sin x} = \int \frac{dx}{a\cos^2\frac{x}{2}\left[(\tan\frac{x}{2}+\frac{b}{a})^2 + \frac{a^2-b^2}{a^2}\right]}$

$= \frac{2}{a}\int \frac{d(\tan\frac{x}{2}+\frac{b}{a})}{(\tan\frac{x}{2}+\frac{b}{a})^2 + \frac{a^2-b^2}{a^2}}$

$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\frac{x}{a} + C \quad (a^2 > b^2)$

$\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C \quad (a^2 < b^2)$

(5) $\int \frac{dx}{a\sin x + b\cos x}$

$a\sin x + b\cos x = \sqrt{a^2+b^2}\sin(x+\alpha)$，$\alpha = \arctan\frac{b}{a}$

得 $\int \frac{dx}{a\sin x + b\cos x} = \int \frac{dx}{\sqrt{a^2+b^2}\sin(x+\alpha)} = \frac{1}{\sqrt{a^2+b^2}}\int \frac{d(x+\alpha)}{\sin(x+\alpha)}$

(6) $\int \frac{dx}{a+b\cos x + c\sin x}$

$a+b\cos x + c\sin x = a + \sqrt{b^2+c^2}\sin(x+\alpha)$，$\alpha = \arctan\frac{b}{c}$

$a+b\cos x + c\sin x = a + \sqrt{b^2+c^2}\cos(x+\beta)$，$\beta = \arctan\frac{c}{b}$