1.3 分部积分法

$$d(uv) = u\,dv + v\,du$$

$$\int u\,dv = \int d(uv) - \int v\,du = uv - \int v\,du$$

推广：$\int uv^{(n+1)}dx = \int u\,dv^{(n)} = uv^{(n)} - \int v^{(n)}du = uv^{(n)} - \int u'v^{(n)}dx$

$$\int u'v^{(n)}dx = \int u'\,dv^{(n-1)} = u'v^{(n-1)} - \int v^{(n-1)}du' = u'v^{(n-1)} - \int u''v^{(n-1)}dx$$

$$\int u''v^{(n-1)}dx = u''v^{(n-2)} - \int u'''v^{(n-2)}dx$$

$\cdots$

$$\Rightarrow \int uv^{(n+1)}dx = uv^{(n)} - u'v^{(n-1)} + u''v^{(n-2)} - \cdots$$

例 9　求积分 $\int \underbrace{(2x^3 + 3x^2 + 4x + 5)}_{u}\underbrace{e^x}_{v}dx$.

求导　$u = 2x^3+3x^2+4x+5$　　$6x^2+6x+4$　　$12x+6$　　12　　0

　　　$+$　　$-$　　$+$　　$-$

求积分　$v = e^x$　　e^x　　e^x　　e^x　　e^x

$$\int(2x^3+3x^2+4x+5)e^x dx = (2x^3+3x^2+4x+5)e^x - (6x^2+6x+4)e^x + (12x+6)e^x - 12e^x + C$$

例 10　求积分 $\int \underbrace{\cos x}_{v}\underbrace{(x^3 + 2x^2 + 3x + 4)}_{u}dx$.

求导　$u = x^3+2x^2+3x+4$　　$3x^2+4x+3$　　$6x+4$　　4　　0

　　　$+$　　$-$　　$+$　　$-$

求积分　$v = \cos x$　　$\sin x$　　$-\cos x$　　$-\sin x$　　$\cos x$

$\int P(x)e^{ax}dx$, $\int P(x)\sin bx\,dx$, $\int P(x)\cos bx\,dx$　反对幂指三（关于 u 的选择）

例 11 求积分$\int \underbrace{x^3}_{v} \underbrace{(\ln x)^2}_{u} dx$.

$$\begin{array}{llll} u=(\ln x)^2 & \boxed{\dfrac{2\ln x}{x}} & \dfrac{2(1-\ln x)}{x^2} \\ v=x^3 & \boxed{\frac{1}{4}x^4} & \frac{1}{20}x^5 \end{array}$$

①对 u 求导明显不能化为 0 的

②对 u 求导出现可以约分的

$$\int x^3(\ln x)^2dx = \frac{1}{4}x^4(\ln x)^2 - \int \frac{1}{4}x^4 \cdot \frac{2\ln x}{x}dx$$

$$= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{2}\int \underbrace{x^3}_{v}\underbrace{\ln x}_{u}dx$$

$u=\ln x \quad \frac{1}{x}$

$v=x^3 \quad \frac{1}{4}x^4$

$$\int x^3\ln x\,dx = \frac{1}{4}x^4\ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x}dx$$

$$= \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3dx$$

$$= \frac{1}{4}x^4\ln x - \frac{1}{16}x^4$$

$$= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{2}\left(\frac{1}{4}x^4\ln x - \frac{1}{16}x^4\right) + C$$

$$= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{8}x^4\ln x + \frac{1}{32}x^4 + C$$

求$\int x^k(\ln x)^n dx$递推公式

$u=(\ln x)^n \quad \dfrac{n\boxed{(\ln x)^{n-1}}}{x}$

$v=x^k \quad \frac{1}{k+1}x^{k+1}$

$$\int x^k(\ln x)^n dx = x^k(\ln x)^n - \frac{n}{k+1}\int x^k \underline{(\ln x)^{n-1}}dx$$

补充例题:

$$\int \underbrace{e^{ax}}_{v}\underbrace{\cos bx}_{u}dx = e^{ax}\frac{a\cos bx + b\cos bx}{a^2+b^2} + C$$

$$\int e^{ax}\sin bx\,dx = e^{ax}\frac{a\sin bx - b\cos bx}{a^2+b^2} + C$$

其他类似的:

$$\int x^n e^{ax}\cos bx\,dx = x^n e^{ax}\frac{a\cos bx + b\cos bx}{a^2+b^2} - \frac{na}{a^2+b^2}\int x^{\boxed{n-1}} e^{ax}\cos bx\,dx$$

$$- \frac{nb}{a^2+b^2}\int x^{\boxed{n-1}} e^{ax}\sin bx\,dx$$

$$\int x^n e^{ax}\sin bx\,dx = x^n e^{ax}\frac{a\cos bx - b\cos bx}{a^2+b^2} - \frac{na}{a^2+b^2}\int x^{\boxed{n-1}} e^{ax}\sin bx\,dx$$

$$- \frac{nb}{a^2+b^2}\int x^{\boxed{n-1}} e^{ax}\cos bx\,dx$$

$$\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{2(n-1)a^2(x^2+a^2)^{\boxed{n-1}}} + \frac{2n-3}{2(n-1)a^2}\int \frac{dx}{(x^2+a^2)^{\boxed{n-1}}}$$

$n=2 \sim \frac{1}{a}\arctan\frac{x}{a}$

分部积分法总结: 反对幂指三(关于u的选择)

$$\left.\begin{array}{l}\int (2x^3+3x^2+4x+5)e^x\,dx \\ \int (x^3+2x^2+3x+4)\cos x\,dx\end{array}\right\} \int P(x)e^{ax}\,dx,\ \int P(x)\sin bx\,dx,\ \int P(x)\cos bx\,dx$$

$\int \ln x\,dx$, $\int \arctan x\,dx$, $\int \arcsin x\,dx$:把1作为因式

$\int x^3(\ln x)^2\,dx$ ①对u求导明显不能化为0的 写一步算一步

②对u求导出现可以约分的

$$\left\{\begin{array}{l}\int x^k(\ln x)^n\,dx,\ \int x^n e^{ax}\cos bx\,dx,\ \int x^n e^{ax}\sin bx\,dx \qquad n=1 \\ \int \frac{1}{(x^2+a^2)^n}\,dx \qquad n=2 \sim \frac{1}{a}\arctan\frac{x}{a}\end{array}\right.$$