Convex Optimization Homework 13

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December 12, 2022

1 ADMM

(a) We know that

$$\begin{aligned}
&\arg\min_x \quad L_\alpha\left(x, y_k, u_k, v_k\right) \\
=&\arg\min_x \quad \|x\|_1 + u_k^T\left(x - y_k\right) + \frac{\alpha}{2}\|x - y_k\|_2^2 \\
=&\arg\min_x \quad \|x\|_1 + \frac{\alpha}{2}\|x - \left(y_k - \frac{1}{\alpha}u_k\right)\|_2^2 \\
=&\arg\min_x \quad \frac{1}{\alpha}\|x\|_1 + \frac{1}{2}\|x - \left(y_k - \frac{1}{\alpha}u_k\right)\|_2^2\,,
\end{aligned} \tag{1}$$

the minimizer of which is the soft threshold function

$$\begin{aligned}
x_{k+1} =& \mathcal{S}_{\frac{1}{\alpha}}\left(y_k - \frac{1}{\alpha}u_k\right) \\
=& \operatorname{sgn}\left(y_k - \frac{1}{\alpha}u_k\right)\cdot\max\left(|y_k - \frac{1}{\alpha}u_k| - \frac{1}{\alpha}, 0\right)\,.
\end{aligned} \tag{2}$$

Similarly we have

$$\begin{aligned}
&\arg\min_y \quad L_\alpha\left(x_{k+1}, y, u_k, v_k\right) \\
=&\arg\min_x \quad u_k^T\left(x_{k+1} - y\right) + v_k^T\left(Ay - b\right) + \frac{\alpha}{2}\|x_{k+1} - y\|_2^2 + \frac{\alpha}{2}\|Ay - b\|_2^2 \\
=&\arg\min_x \quad \|y - \left(x_{k+1} + \frac{1}{\alpha}u_k\right)\|_2^2 + \|Ay - b + \frac{1}{\alpha}v_k\|_2^2\,,
\end{aligned} \tag{3}$$

the partial derivative of (3) w.r.t y could be derived as

$$2\left(y - \left(x_{k+1} + \frac{1}{\alpha}u_k\right)\right) + 2A^T\left(Ay - b + \frac{1}{\alpha}v_k\right)\,. \tag{4}$$

Assign (4) to 0, it is obtained that

$$y_{k+1} = \left(I + A^TA\right)^{-1}\left(x_{k+1} + \frac{1}{\alpha}u_k + A^T\left(b - \frac{1}{\alpha}v_k\right)\right)\,. \tag{5}$$

Thus the update of u and v are respectively

$$u_{k+1} = u_k + \alpha\left(x_{k+1} - y_{k+1}\right) \tag{6}$$
$$v_{k+1} = v_k + \alpha\left(Ay_{k+1} - b\right). \tag{7}$$

Equations (2), (5), (6) and (7) are called ADMM iterations.

(b) Implement the above iteration and solve the problem for A, b provided in the attachment. The results are shown in Fig.1.

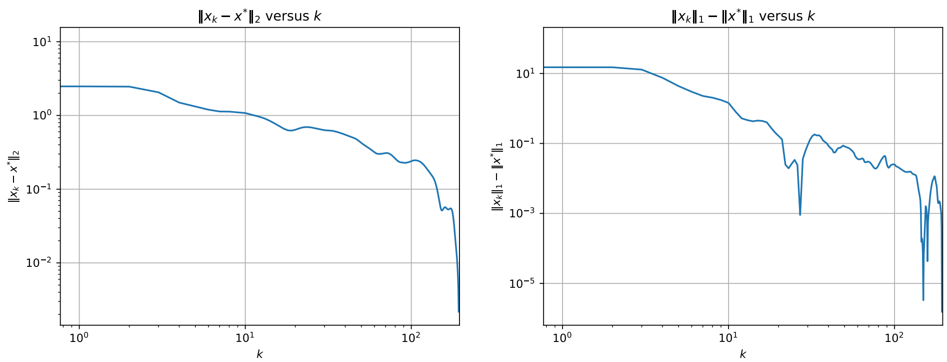

Figure 1: results