凸优化第 12 周作业

1 预习内容

下周无预习内容。

2 作业题

1. (Proximal Gradient Descent) Consider the following LASSO problem:

$$\min_x \frac{1}{2}\|Ax-b\|_2^2 + \|x\|_1.$$

Let $f(x) = \frac{1}{2}\|Ax-b\|_2^2, g(x) = \|x\|_1$ and the objective function $h(x) = f(x)+g(x)$. Since f has Lipschitz gradient, we have

$$\begin{aligned} & x^* = \arg\min_x f(x) + g(x) \\ \Leftrightarrow & 0 \in (\nabla f + \partial g)x^* \\ \Leftrightarrow & 0 \in (I - \alpha\nabla f)x^* - (I + \alpha\partial g)x^* \\ \Leftrightarrow & (I - \alpha\nabla f)x^* \in (I + \alpha\partial g)x^* \\ \Leftrightarrow & (I + \alpha\partial g)^{-1}(I - \alpha\nabla f)x^* = x^* \\ \Leftrightarrow & x^* = \mathrm{Prox}_{\alpha g} \circ (I - \alpha\nabla f)x^*, \end{aligned}$$

where $\circ$ is the composition of operators.

   (a) Fix data A, b, prove that $\nabla f(x)$ is Lipschitz and show the smallest positive constant scalar M such that $\|\nabla f(x) - \nabla f(y)\|_2 \le M\|x-y\|, \forall x, y$. (show M as an expression of A, b).

   (b) Write out the exact form of iteration

$$x_{k+1} = \mathrm{Prox}_{\alpha g} \circ (I - \alpha\nabla f)x_k.$$

   Choose $\alpha = 1/M$, achieve the iteration by coding.

   (c) Given two groups of A, b in files A1.csv, b1.csv (path ./problem1data/) where $A1$ is full-rank and A2.csv, b2.csv where $A2$ is not full-rank, implement the iteration to solve the LASSO problem. Start with $x_0 = \mathbf{0}$. Stop when $|h(x_{k+1}) - h(x_k)| < 10^{-5}$. Plot the corresponding figure of $\log(k)$ vs $\log(\|x_k - x^*\|_2)$ and $\log(k)$ vs $\log(h(x_k))$ and $\log(k)$ vs $\log(h(x_{k-1}) - h(x_k))$. (6 figures in total).

   (d) Discuss the convergence rate difference between two groups of data and explain by referring to convergence theories introduced in lecture.

2. (选做题)(Douglas-Rachford Splitting (DRS)) Consider the following optimization problem

$$\min_x \|x\|_1, \text{ s.t. } Ax = b.$$

- The data A, b is given in files A.csv, b.csv, in path ./problem2data/
- To plot the curve, all the algorithms stop when $\|x_{k-1}\|_1 - \|x_k\|_1 < 10^{-5}$.

Reformulate the problem as

$$\min_x \|x\|_1 + \mathbb{I}_{\{Ax=b\}}(x).$$

Define $f(x) = \|x\|_1, g(x) = \mathbb{I}_{\{Ax=b\}}(x)$. The iteration given by DRS is

$$\begin{aligned} x_{k+1} &= \text{Prox}_{\alpha g}(z_k) \\ y_{k+1} &= \text{Prox}_{\alpha f}(2x_{k+1} - z_k), \\ z_{k+1} &= z_k + y_{k+1} - x_{k+1}. \end{aligned}$$

Write the exact form of this iteration and achieve the iteration by coding. Choose appropriate α. Given A, b in files, implement the iteration to solve the optimization problem. Plot the corresponding figure of $\log(k)$ vs $\log(\|x_k - x^*\|_2)$ and $\log(k)$ vs $\log(\|x_k\|_1)$.

3. (选做题) (Dual Ascent) Consider the following optimization problem

$$\min f(x, y) = \frac{1}{2}(x^2 + y^2), \text{ s.t. } x - 3y = 10.$$

The iteration given by dual ascent is

$$\begin{aligned} \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} &= \arg\min_{(x,y)}, L(x, y, u_k), \\ u_{k+1} &= u_k + \alpha(x_{k+1} - 3y_{k+1} - 10), \end{aligned}$$

where $L(x, y, u)$ is the Lagrangian.

(a) Please give the exact form of this iteration above, where the initial dual variable u_0 is set to 1.

(b) Choose an appropriate α such that dual ascent method converges and give the optimal solution of the original problem.

3 作业说明

1. 编程作业需要撰写报告（包含推导步骤和程序运行结果）。请将报告（pdf 电子版）和代码（编程语言不限）一起打包提交至网络学堂。

2. **请大家务必在截止时间之前提交作业**，迟交一周以内的作业得分是卷面分的 50%，迟交超过一周的作业不得分。

3. 每次作业的满分是 25 分，做选做题有额外加分，但每次作业总分不超过 25 分。