凸优化第 13 周作业

1 预习

下节课讲授应用讲义的第一章 Semidefinite Programming，没有小测，但请提前预习。

2 必做题

1. (ADMM) Consider the optimization problem

$$\text{minimize } \|x\|_1, \text{ subject to } Ax = b. \tag{1}$$

The problem can be reformulated as

$$\text{minimize } \|x\|_1, \text{ subject to } x = y, Ay = b. \tag{2}$$

The corresponding ADMM iteration is:

$$x_{k+1} \in \arg\min_x L_\alpha(x, y_k, u_k, v_k) \tag{3}$$

$$y_{k+1} \in \arg\min_y L_\alpha(x_{k+1}, y, u_k, v_k) \tag{4}$$

$$u_{k+1} = u_k + \alpha(x_{k+1} - y_{k+1}) \tag{5}$$

$$v_{k+1} = v_k + \alpha(Ay_{k+1} - b), \tag{6}$$

where L_α is the augmented Lagrangian defined as:

$$L_\alpha(x, y, u, v) = \|x\|_1 + u^\top(x - y) + v^\top(Ay - b) + \frac{\alpha}{2}\|x - y\|_2^2 + \frac{\alpha}{2}\|Ay - b\|_2^2.$$

(a) Derive the exact form of the ADMM iteration (3)-(6).

(b) Implement the above iteration and solve the problem for A, b provided in the attachment. Choose a proper α by yourself. Stop the iteration when $\|x_{k-1}\|_1 - \|x_k\|_1 < 10^{-5}$, and treat the last x_k as x^*. Plot: i) $\log(\|x_k - x^*\|_2)$ vs. $\log(k)$; ii) $\log(\|x_k\|_1 - \|x^*\|_1)$ vs. $\log(k)$.

3 选做题

1. (Another form of ADMM) Consider the problem (1) with another reformulation below:

$$\text{minimize } \|x\|_1, \text{ subject to } Ax = b, y = Px, \tag{7}$$

where P is a matrix of proper dimensions.

(a) Derive the ADMM iteration.

(b) Choose P, such that x can be updated via soft thresholding. (You can refer to note §5.2 for the definition of soft thresholding)

(c) Implement the above iteration and solve the problem for A, b provided in the attachment. Choose a proper step size by yourself. Stop the iteration when $\|x_{k-1}\|_1 - \|x_k\|_1 < 10^{-5}$, and treat the last x_k as x^*. Plot: i) $\log(\|x_k - x^*\|_2)$ vs. $\log(k)$; ii) $\log(\|x_k\|_1 - \|x^*\|_1)$ vs. $\log(k)$.

(Hint: you can refer to note §6.5.2.)

2. (Uniform polynomial approximation) Consider the problem of approximating x^n on the interval $[-1, 1]$ using a polynomial of degree at most $n-1$. Let

$$p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1},$$

then the above approximation problem can be formulated as the semi-infinite programming problem

$$\begin{aligned} \underset{a_0,\ldots,a_{n-1},e}{\text{minimize}} \quad & e \\ \text{subject to} \quad & |x^n - p(x)| \le e, \quad \forall x \in [-1, 1]. \end{aligned}$$

Find approximate solutions to the above problem for $n = 3, 4$ respectively, using scenario-based optimization:

(a) Suppose that x is drawn from the uniform distribution $\mathcal{U}_{[-1,1]}$, and that it is required that the constraint $|x^n - p(x)| \le e$ is satisfied for 99% of x's with probability at least 99%. Find the number of samples needed to satisfy the requirement for $n = 3, 4$ respectively.

(b) Solve the problem for $n = 3, 4$ respectively. Report the polynomial coefficients you find, and plot x^n and $p(x)$ in the same graph over $[-1, 1]$.

You are allowed to use existing solvers (e.g., CVX) for this problem; you may also use your own LP solver written in previous homeworks.

4 作业说明

1. 编程作业部分需要撰写报告，包含推导步骤和运行结果。报告提交电子版，和代码一起打包提交至网络学堂。

2. 编程语言不限，但不能使用现成的优化器（选做最后一题除外）。

3. **请大家务必在截止时间之前提交作业**，迟交一周以内的作业得分是卷面分的 50%，迟交超过一周的作业不得分。

4. 每次作业的满分是 25 分，做选做题有额外加分，但每次作业总分不超过 25 分。