Convex Optimization Homework 14

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1 Least Square on a Sphere

(a) The original problem

$$\begin{aligned} \min_{x \in R^n} \quad & \|Ax - b\|_2^2 \\ \text{s.t.} \quad & \|x\|_2 = 1\,, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^n$ can be written in the SDP form

$$\begin{aligned} \min_{X} \quad & tr\,(Q_0 X) \\ \text{s.t.} \quad & tr\,(Q_1 X) = 0 \\ & X_{n+1,n+1} = 1, X \geq 0 \\ & rank\,(X) = 0\,, \end{aligned} \tag{2}$$

where $X = \begin{bmatrix} x \\ 1 \end{bmatrix} \begin{bmatrix} x^T & 1 \end{bmatrix}$, $Q_0 = \begin{bmatrix} A^T A & -A^T b \\ -b^T A & b^T b \end{bmatrix}$ and $Q_1 = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}$.

Its SDP relaxation can be derived as

$$\begin{aligned} \min_{X} \quad & tr\,(Q_0 X) \\ \text{s.t.} \quad & tr\,(Q_1 X) = 1 \\ & X_{n+1,n+1} = 1, X \geq 0\,, \end{aligned} \tag{3}$$

(b) There is only $m = 1$ linear constraint on X, we can find an optimal solution $\hat{X}$ with rank one, thus the SDP relaxation (3) provides the exact value of the original problem (1).

2 Max Complete Subgraph Problem

(a) The subgraph is defined as $G' = (V', E')$, s.t. $V' \subseteq V, E' \subseteq E$. We can categorize V into two subsets $V_0 = \{i \in V | x_i = 0\}\,, V_1 = \{i \in V | x_i = 1\}$, thus $V_0 \cup V_1 = V$ and $V_0 \cap V_1 = \varnothing$. The resulting subgraphs are $G_0 = (V_0, E_0), G_1 = (V_1, E_1)$. If G_1 is complete, then the partition satisfies the constraints, and the function value is $|G_1|$. If G_1 is incomplete, then the partition violates the constraints. By moving some vertices in G_1 to G_0, we can make G_1 complete.

The original problem is equivalent to maximizing over all partitions $V = V_0 \cup V_1$, s.t. G_1 is complete. Thus the optimal value $\alpha(G)$ is the size of the max complete subgraph of G.

(b) The original problem

$$\begin{aligned} \max \quad & \sum_{i \in V} x_i \\ \text{s.t.} \quad & x_i x_j = 0 \;\; if \;\; (i,j) \notin E \\ & x_i \in \{0,1\}, \forall i \in V \,, \end{aligned} \tag{4}$$

is equivalent to

$$\begin{aligned} \max \quad & \sum_{i \in V} x_i^2 \\ \text{s.t.} \quad & x_i x_j = 0 \;\; if \;\; (i,j) \notin E \\ & 0 \leq x_i \leq 1, \forall i \in V \,, \end{aligned} \tag{5}$$

which can be further converted to

$$\begin{aligned} \max \quad & \sum_{i \in V} x_i^2 \\ \text{s.t.} \quad & x_i x_j = 0 \;\; if \;\; (i,j) \notin E \\ & x_i^2 - x_i \leq 0, \forall i \in V \,, \end{aligned} \tag{6}$$

We can rewrite (6) in SDP form

$$\begin{aligned} \max_X \quad & tr\,(Q_0 X) \\ \text{s.t.} \quad & tr\,(P_{i,j} X) = 0 \;\; if \;\; (i,j) \notin E \\ & tr\,(R_i X) \leq 0, \forall i \in V \\ & X_{n+1,n+1} = 1, X \geq 0 \\ & rank\,(X) = 0 \,, \end{aligned} \tag{7}$$

where $n = |G|$, $x = \begin{bmatrix} x_0 \; x_1 \cdots x_n \end{bmatrix}^T$, $X = \begin{bmatrix} x \\ 1 \end{bmatrix} \begin{bmatrix} x^T \; 1 \end{bmatrix}$, $Q_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $P_{i,j} = \begin{bmatrix} E_{i,j} + E_{j,i} & 0 \\ 0 & 0 \end{bmatrix}$, $R_i = \begin{bmatrix} E_{i,i} & -\frac{1}{2} e_i \\ -\frac{1}{2} e_i^T & 0 \end{bmatrix}$.

The relaxation of (7) can be derived as

$$\begin{aligned} \max_X \quad & tr\,(Q_0 X) \\ \text{s.t.} \quad & tr\,(P_{i,j} X) = 0 \;\; if \;\; (i,j) \notin E \\ & tr\,(R_i X) \leq 0, \forall i \in V \\ & X_{n+1,n+1} = 1, X \geq 0 \,. \end{aligned} \tag{8}$$