AMATH 568: Problem Set 3

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1.

$$r^3 - r + \epsilon = 0$$

$$r = r_0 + r_1\epsilon + r_2\epsilon^2 + O(\epsilon^3)$$
$$r^3 = r_0^3 + 3r_0^2 r_1\epsilon + (3r_0^2 r_2 + 3r_0 r_1^2)\epsilon^2$$
$$r^3 - r + \epsilon = r_0^3 - r_0 + (3r_0^2 r_1 - r_1 + 1)\epsilon + (3r_0^2 r_2 + 3r_0 r_1^2 - r_2)\epsilon^2$$

The exact solutions are

$$r = 0, 1, -1$$

For r_0=0,

$$r_1 = 1$$
$$r_2 = 0$$
$$r = \epsilon + O(\epsilon^3)$$

For r_0=1,

$$r_1 = -\frac{1}{2}$$
$$r_2 = -\frac{3}{8}$$
$$r = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 + O(\epsilon^3)$$

For r_0=-1,

$$r_1 = -\frac{1}{2}$$
$$r_2 = \frac{3}{8}$$
$$r = -1 - \frac{1}{2}\epsilon - \frac{3}{8}\epsilon^2 + O(\epsilon^3)$$

The first figure is for ϵ between -1 and 1. The second is between -2 and 2. The red circles represent the exact root. The first series around 0 is the starred line. The second series around 1 is the regular line and the third one around -1 is dashed line.

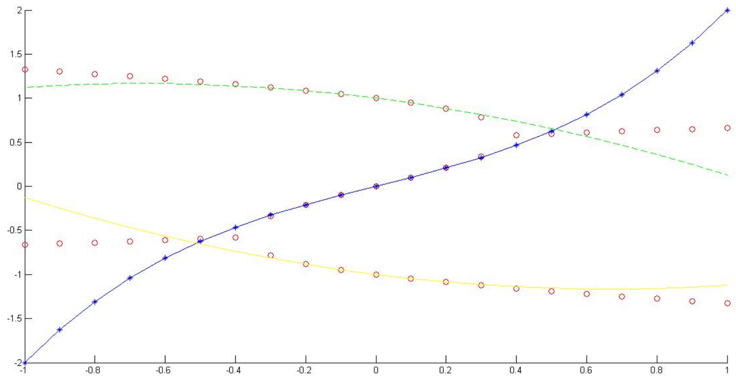

Figure 1: From -1 to 1

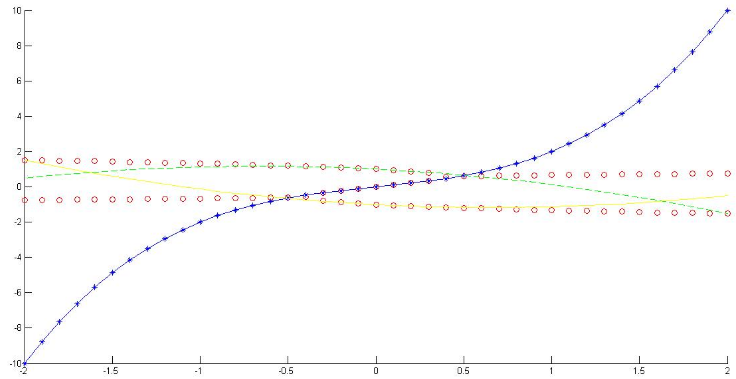

Figure 2: From -2 to 2

Thus we see that the series around the root 0 works well between -0.5 and 0.5. The series around the root 1 works well between -1 and 0.5. The series around the root -1 works well between -0.5 and 1.

2. The main equation is

$$(1+\epsilon)r^2 - (4+\epsilon)r + 4 = 0$$

$$r = r_0 + r_1\epsilon + O(\epsilon^2)$$

$$(1+\epsilon)r^2 = (1+\epsilon)(r_0+r_1\epsilon)^2 = (1+\epsilon)(r_0^2+r_1^2\epsilon^2+2r_0r_1\epsilon) = r_0^2+r_1^2\epsilon^2+2r_0r_1\epsilon+r_0^2\epsilon+r_1^2\epsilon^3+2r_0r_1\epsilon^2$$
$$= r_0^2 + \epsilon(2r_0r_1 + r_0^2) + \epsilon^2(r_1^2 + 2r_0r_1) + r_1^2\epsilon^3$$

$$(4+\epsilon)r = (4+\epsilon)(r_0 + r_1\epsilon) = 4r_0 + (4r_1 + r_0)\epsilon + r_1\epsilon^2$$

We get

$$r_0 = 2$$

Equating the coefficients of ϵ, we get

$$2r_0r_1 + r_0^2 - 4r_1 - r_0 = 0$$

$$4 = 2$$

This is inconsistent. So, this kind of asymptotic expansion is not valid.

We try an asymptotic expansion with non-integer powers.

$$r = r_0 + r_1\epsilon^{\alpha} + r_2\epsilon^{2\alpha} + O(\epsilon^{2\alpha+1})$$

where $\alpha < 1$

$$(1+\epsilon)r^2 = (1+\epsilon)(r_0+r_1\epsilon^{\alpha}+r_2\epsilon^{2\alpha}+\ldots)^2 = (1+\epsilon)(r_0^2+r_1^2\epsilon^{2\alpha}+2r_0r_1\epsilon^{\alpha}+2r_0r_2\epsilon^{2\alpha}+r_1^3\epsilon^{3\alpha}+2r_1r_2\epsilon^{3\alpha}+\ldots)$$
$$= r_0^2 + (r_1^2 + 2r_0r_2)\epsilon^{2\alpha} + 2r_0r_1\epsilon^{\alpha} + r_0^2\epsilon + (r_1^2 + 2r_0r_2)\epsilon^{2\alpha+1} + 2r_0r_1\epsilon^{\alpha+1} + r_2^2\epsilon^{4\alpha} + 2r_1r_2\epsilon^{3\alpha} + \ldots$$

$$(4+\epsilon)r = (4+\epsilon)(r_0 + r_1\epsilon^{\alpha} + r_2\epsilon^{2\alpha}) = 4r_0 + 4r_1\epsilon^{\alpha} + 4r_2\epsilon^{2\alpha} + r_0\epsilon + r_1\epsilon^{\alpha+1} + 4r_2\epsilon + \ldots$$

Equating terms of ϵ^{α}, we get

$$4r_1 = 4r_1$$

Equating the coefficients of ϵ, we find

$$r_0^2 = r_0$$

This is again inconsistent. The only way this series is valid if some $\epsilon^{n\alpha}$ is ϵ. Since α is less that zero, only $\epsilon^{2\alpha}$ is a candidate.

$$\alpha = \frac{1}{2}$$

Equating terms of ϵ now,

$$r_1^2 + 4r_2 + r_0^2 = r_0 + 4r_2$$
$$r_1^2 + 4 = 2$$
$$r_1 = i\sqrt{2}, -i\sqrt{2}$$

Equating terms of $\epsilon^{\frac{3}{2}}$,

$$2r_1r_2 + 2r_0r_1 = r_1$$
$$r_2 = -\frac{3}{2}$$

So, the two perturbed solutions are

$$r = 2 + i\sqrt{2}\sqrt{\epsilon} - \frac{3}{2}\epsilon + O(\epsilon^{3/2})$$

$$r = 2 - i\sqrt{2}\sqrt{\epsilon} - \frac{3}{2}\epsilon + O(\epsilon^{3/2})$$

The quadratic formula gives us

$$\frac{4+\epsilon+\sqrt{\epsilon^2-8\epsilon}}{2(1+\epsilon)}, \frac{4+\epsilon-\sqrt{\epsilon^2-8\epsilon}}{2(1+\epsilon)}$$

Taylor series of these two (through Mathematica) give us

$$r = 2 + i\sqrt{2}\sqrt{\epsilon} - \frac{3}{2}\epsilon + O(\epsilon^{3/2})$$

and

$$r = 2 - i\sqrt{2}\sqrt{\epsilon} - \frac{3}{2}\epsilon + O(\epsilon^{3/2})$$

This matches with our result.

3. The taylor expansion of $\sqrt{x+\epsilon}$ around $\epsilon = 0$ is

$$\sqrt{x} + \frac{1}{2\sqrt{x}}\epsilon - \frac{1}{8x^{3/2}}\epsilon^2 + \dots$$

$$= \sqrt{x}(1 + \frac{\epsilon}{2x} - \frac{\epsilon^2}{8x^2} + \dots)$$

We can see that this cannot be valid when x is close to zero. Furthermore, the radius of convergence always depends on x. So, this series is not uniform with respect to x.

4.

$$y'' = y^2$$

B.Cs are

$$y(0) = \epsilon, y'(0) = 0$$

We take

$$y = y_0 + y_1\epsilon + y_2\epsilon^2 + y_3\epsilon^3 + O(\epsilon^4)$$

Plugging this in, we get

$$y_0'' + y_1''\epsilon + y_2''\epsilon^2 + y_3''\epsilon^3 = y_0^2 + 2y_0^2y_1\epsilon + (y_1^2 + 2y_0y_2)\epsilon^2 + (y_0^3 + 2y_1y_2 + 2y_0y_3)\epsilon^2$$

We get ,

$$y_0'' = y_0^2, y_0(0) = 0, y_0'(0) = 0$$

The trivial equation $y_0 = 0$ satisfies this.

$$y_1'' = 2y_0^2y_1 = 0, y_1(0) = 1, y_2'(0) = 0$$

$$y_1 = 1$$

$$y_2'' = y_1^2 + 2y_0^2y_2 = 1, y_2(0) = 0, y_2'(0) = 0$$

$$y_2 = \frac{1}{2}x^2 + C_1x + C_2$$

$$y_2 = \frac{1}{2}x^2$$

$$y_3'' = y_0^3 + 2y_1y_2 + 2y_0y_3 = x^2, y_3(0) = 0, y_3'(0) = 0$$

$$y_3 = \frac{1}{12}x^4$$

So, the perturbed solution is

$$y = \epsilon + \frac{1}{2}x^2\epsilon^2 + \frac{1}{12}x^4\epsilon^3 + O(\epsilon^4)$$