AMATH 562: Homework 4

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1.
$$dX_t = \kappa(\theta - X_t)dt + \delta\sqrt{X_t}dW_t$$
$$u(t,x) = \mathbb{E}\left[e^{-\int_t^T X_s ds}|X_t = x\right]$$

By Theorem 9.2.2, we can obtain the PDE

$$(\partial_t + \mathcal{A}(t))u + g = 0$$

where

$$g = 0$$

and

$$\mathcal{A}(t) = \kappa(\theta - x)\partial_x + \frac{\delta^2 x}{2}\partial_x^2 - x$$

then the boundary is given by

$$u(T,x) = 1$$

We use the following ansatz

$$u(t,x) = e^{-xA(t)-B(t)}$$

Plugging it into the PDE, we get

$$(-xA' - B')u - \kappa(\theta - x)Au + \frac{\delta^2 x A^2}{2}u - xu = 0$$

Thus,

$$-xA' - B' - \kappa(\theta - x)A + \frac{\delta^2 x A^2}{2} - x = 0$$
$$x(-A' + \kappa\theta A + \frac{\delta^2 A^2}{2} - 1) + (-B' - \kappa\theta A) = 0$$

Thus, we can get the following coupled odes.

$$A' = \kappa\theta A + \frac{\delta^2 A^2}{2} - 1$$
$$B' = -\kappa\theta A$$

Boundary conditions:

$$u(T,x) = 1$$
$$-xA(T) - B(T) = 0$$

Since this should hold for all x,

$$A(T) = B(T) = 0$$

$$A(t) = \frac{\sqrt{-2\delta^2 - \kappa^2\theta^2}\tan(\frac{1}{2}(c_1\sqrt{-2\delta^2 - \kappa^2\theta^2} + t\sqrt{-2\delta^2 - \kappa^2\theta^2})) - \kappa\theta}{\delta^2}$$

where

$$c_1 = \frac{2}{\sqrt{-2\delta^2 - \kappa^2\theta^2}}\tan^{-1}\left(\frac{\kappa\theta}{\sqrt{-2\delta^2 - \kappa^2\theta^2}}\right) - T$$

$$B(t) = \frac{2\log sech\frac{|c_1\sqrt{-2\delta^2-\kappa^2\theta^2}+t\sqrt{-2\delta^2-\kappa^2\theta^2})|}{|c_1\sqrt{-2\delta^2-\kappa^2\theta^2}+T\sqrt{-2\delta^2-\kappa^2\theta^2})|} - \kappa\theta(t-T)}{\delta^2}$$

2.

$$dX_t^i = -\frac{b}{2}X_t^i dt + \frac{1}{2}\sigma dW_t^i$$

$$B_t = \sum_i^d \int_0^t \frac{1}{\sqrt{R_s}} X_s^i dW_s^i$$

$$B_0 = \sum_i^d \int_0^0 \frac{1}{\sqrt{R_s}} X_s^i dW_s^i = 0$$

Since B_t has only dW_s terms , it is a martingale. Also, since B_t is an Ito process, it has continuous sample paths.

$$dB_t = \sum_i^d \frac{1}{\sqrt{R_s}} X_s^i dW_s^i$$

$$dB_t^2 = \sum_i^d \frac{1}{R_s} X_s^{2^i} dt$$

$$dB_t^2 = \frac{1}{R_s}\sum_i^d X_s^{2^i} dt$$

$$dB_t^2 = \frac{1}{R_s} R_s dt$$

$$dB_t^2 = dt$$

Thus, since the quadratic variation is t, using Levy's characterization theorem, B_t is a Brownian motion.

$$R_t = \sum_{i=1}^d (X_t^{(i)})^2$$

$$dR_t = \sum_i 2X_t^i dX_t^i + \frac{\sigma^2}{4}dt$$

$$= \sum_i 2X_t^i(-\frac{b}{2}X_t^i dt + \frac{1}{2}\sigma dW_t^i) + \frac{\sigma^2}{4}dt$$

$$= \sum_i (-bX_t^{2^i} + \frac{\sigma^2}{4})dt + X_t^i \sigma dW_t^i$$

$$= \sqrt{R_s}\sigma dB_t + \sum_i (-bX_t^{2^i} + \frac{\sigma^2}{4})dt$$

3.

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

$$Z_t = \log(X_t)$$

$$dZ_t = \frac{1}{X_t}dX_t - \frac{1}{2X_t^2}\sigma^2 X_t^2 dt$$

$$dZ_t = \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt$$

$$dZ_t = \underbrace{(\mu - \frac{1}{2}\sigma^2)}_{a} dt + \sigma dW_t$$

The transition probability for Z , Γ_z satisfies the following pde.

$$(\partial_t + \mathcal{A}_z)\Gamma_z = 0$$

$$\mathcal{A}_z = (\mu - \frac{1}{2}\sigma^2)\partial_z + \frac{\sigma^2}{2}\partial_z^2$$

Usually, we would expect eigenfunctions of the form e^{wz} but that wouldn't work at the boundaries where we need

$$\psi_n(\log(l)) = \psi_n(\log(r)) = 0$$

So, we say

$$\psi_n(z) = Ce^{wz}\sin\left(\frac{n\pi(z-\log(l))}{\log(r)-\log(l)}\right)$$

where C is the normalization constant

$$\mathcal{A}_z\psi_n(z) = aw\psi_n(z) + \frac{an\pi}{\log(r/l)}e^{wz}cos\left(\frac{n\pi(z-\log(l))}{\log(r/l)}\right)$$

$$+\frac{1}{2}\sigma^2\left(w^2\psi_n(z) + \frac{2n\pi w}{\log(r/l)}e^{wz}cos\left(\frac{n\pi(z-\log(l))}{\log(r/l)}\right) - \left(\frac{n\pi}{\log(r/l)}\right)^2\psi_n(z)\right)$$

For $\mathcal{A}_z\psi_n(z) = \lambda\psi_n(z)$,

$$a + \sigma^2 w = 0$$

$$w = \frac{-a}{\sigma^2} = \frac{\frac{1}{2}\sigma^2 - \mu}{\sigma^2}$$

and

$$\lambda_n = aw + \frac{1}{2}\sigma^2(w^2 - \left(\frac{n\pi}{\log(r/l)}\right)^2)$$

Since $aw = -\sigma^2 w^2$,

$$\lambda_n = -\frac{1}{2}\sigma^2(w^2 + \left(\frac{n\pi}{\log(r/l)}\right)^2)$$

$$m(y) = \frac{2}{\sigma^2} e^{\int dy \frac{2\mu - \sigma^2}{\sigma^2}} = \frac{2}{\sigma^2} e^{y \frac{2\mu - \sigma^2}{\sigma^2}} = \frac{2}{\sigma^2} e^{-2wy}$$

$$\langle \psi_n, \psi_n \rangle_m = 1$$
$$\int_{\log l}^{\log r} \psi_n(z)\psi_n(z)m(z) = 1$$
$$C^2 \int_{\log l}^{\log r} e^{wz} \sin\left(\frac{n\pi(z - \log(l))}{\log(r) - \log(l)}\right) e^{wz} \sin\left(\frac{n\pi(z - \log(l))}{\log(r) - \log(l)}\right) \frac{2}{\sigma^2} e^{-2wz}$$
$$= \frac{2C^2}{\sigma^2} \int_{\log l}^{\log r} \sin\left(\frac{n\pi(z - \log(l))}{\log(r) - \log(l)}\right) \sin\left(\frac{n\pi(z - \log(l))}{\log(r) - \log(l)}\right) = \frac{C^2}{\sigma^2} \log(r/l) = 1$$
$$C = \frac{\sigma}{\sqrt{\log(r/l)}}$$

Thus, the normalized eigenfunctions are

$$\psi_n(z) = \frac{\sigma}{\sqrt{\log(r/l)}} e^{wz} \sin\left(\frac{n\pi z - \log(l)}{\log(r) - \log(l)}\right)$$

$$\Gamma_z(t, x, T, y) = m(y) \sum_n e^{(T-t)\lambda_n} \psi_n(y)\psi_n(x)$$
$$= \frac{2}{\sigma^2} e^{-2wy} \sum_n e^{(T-t)\lambda_n} \psi_n(y)\psi_n(x)$$
$$\Gamma_z(t, x, T, y) = \frac{2}{\sigma^2} e^{-2wy} \sum_n e^{(T-t)\lambda_n} \psi_n(\log(y))\psi_n(\log(x))$$
$$\Gamma_z(T, x, T, y) = \delta_y$$
$$P(X_T \le y | X_t = x) = \int_l^y \Gamma_x(t, x, T, s) ds$$
$$P(X_T \le y | X_t = x) = P(Z_T \le \log y | Z_t = \log x) = \int_{\log l}^{\log y} \Gamma_z(t, \log x, T, s) ds$$

$$\int_l^y \Gamma_x(t, x, T, s) ds = \int_{\log l}^{\log y} \Gamma_z(t, \log x, T, s) ds$$

By Fundamental Theorem of Calculus,

$$\Gamma_x(t, x, T, y) = \frac{1}{y} \Gamma_z(t, \log x, T, \log y) = \frac{2}{\sigma^2 y} e^{-2wy} \sum_n e^{(T-t)\lambda_n} \psi_n(\log y)\psi_n(\log x)$$

4. (a) We can see that

$$\lambda = g = 0$$

Thus, u satisfies

$$(\partial_t + \mathcal{A})u = 0$$

where

$$\mathcal{A} = \frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_y^2$$

and on the boundary

$$u(x,y) = u(X_\tau, a)$$

(b)

$$\phi(x) = \frac{1}{2\pi}\int e^{iwx}\hat{\phi}(w)dw$$

$$u(x,y) = \mathbb{E}[\phi(X_\tau)|X_t = x, Y_t = y] = \mathbb{E}[\frac{1}{2\pi}\int e^{iwX_\tau}\hat{\phi}(w)dw|X_t = x, Y_t = y]$$

$$= \mathbb{E}[\frac{1}{2\pi}\int e^{iwX_\tau}\hat{\phi}(w)dw|X_t = x, Y_t = y]$$

$$= \frac{1}{2\pi}\int \hat{\phi}(w)\mathbb{E}[e^{iwX_\tau}dw|X_t = x, Y_t = y]$$

$$= \frac{1}{2\pi}\int \hat{\phi}(w)\mathbb{E}[\mathbb{E}[e^{iwX_\tau}dw|X_t = x, Y_t = y, \tau]|X_t = x, Y_t = y]$$

$$= \frac{1}{2\pi}\int \hat{\phi}(w)\mathbb{E}[\mathbb{E}[e^{iwX_\tau}dw|X_t = x, Y_t = y, \tau]|X_t = x, Y_t = y]$$

X_τ is normally distributed with mean x (starting value) and variance $(\tau - t)$.

$$= \frac{1}{2\pi}\int \hat{\phi}(w)\mathbb{E}[e^{iwx-\frac{1}{2}w^2(\tau-t)}|X_t = x, Y_t = y]$$

$$= \frac{1}{2\pi}\int \hat{\phi}(w)e^{iwx}\mathbb{E}[e^{-\frac{1}{2}w^2(\tau-t)}|X_t = x, Y_t = y]$$

For hitting time τ_m, we know

$$\mathbb{E}[e^{-\lambda\tau_a}] = e^{-|a|\sqrt{2\lambda}}$$

since we start from y at time t,

$$\mathbb{E}[e^{-\lambda(\tau_a - t)}] = e^{-|a-y|\sqrt{2\lambda}}$$

Thus,

$$u(x) = \frac{1}{2\pi}\int \hat{\phi}(w)e^{iwx}e^{-|a-y||w|}dw$$

For u to be the transition probability P, $\phi(x)$ should be the indicator function $\mathbb{I}_{dz}$.

$$\hat{\phi}(w) = \int_{-\infty}^{\infty}\mathbb{I}_{dz}e^{-iwx}dx$$

$$= \int_{z}^{z+dz}\mathbb{I}_{dz}e^{-iwx}dx$$

$$= e^{-iwz}dz$$

$$P = \frac{1}{2\pi}\int e^{-iwz}dze^{iwx}e^{-|a-y||w|}dw$$

$$= \frac{1}{2\pi} \int e^{-iwz} dz e^{iwx} e^{-|a-y||w|} dw$$

$$P = \frac{1}{2\pi} \int e^{-iwz} dz e^{iwx} e^{-|a-y||w|} dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} e^{-iwz} dz e^{iwx} e^{|a-y|w} dw + \frac{1}{2\pi} \int_{0}^{\infty} e^{-iwz} dz e^{iwx} e^{-|a-y|w} dw$$

$$= \frac{dz}{2\pi} \left(\frac{1}{|a-y| + i(x-z)} + \frac{1}{|a-y| - i(x-z)} \right)$$

$$= \frac{dz}{\pi} \left(\frac{|a-y|}{|a-y|^2 + (x-z)^2} \right)$$

(c)

$$P = \frac{1}{2\pi} \int e^{-iwz} dz e^{iwx} e^{-|a-y||w|} dw$$

$$\hat{P} = e^{-iwz} dz e^{-|a-y||w|}$$

$$\hat{P}_t = 0$$

$$\hat{P}_{xx} = -w^2 \hat{P}$$

$$\hat{P}_{yy} = w^2 \hat{P}$$

$$\hat{P}_t + \hat{P}_{xx} + \hat{P}_{yy} = 0$$

Thus, this solves the PDE

$$P(t,x,y) = \frac{1}{2\pi} \int e^{-iwz} dz e^{iwx} e^{-|a-y||w|} dw$$

As t goes to T,

$$P(T,x,a) = \frac{1}{2\pi} \int e^{-iwz} dz e^{iwx} dw = \mathbb{I}_{dz}$$