AMATH 562: Homework 1

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January 12, 2018

1. **6.2**

   Let us define
   $$A := \{s \in S : \lim_{n\to\infty} |X_n(s) - X(s)| = 0\}$$
   $$A := \{s \in S : \lim_{n\to\infty} |X_n(s) - s| = 0\}$$
   For all s $\in (0, 1)$,
   $$\lim_{n\to\infty} X_n(s) = s$$
   Even
   $$\lim_{n\to\infty} X_n(1) = 1$$
   But,
   $$\lim_{n\to\infty} X_n(0) = 0$$
   Thus,
   $$A = S - \{0\}$$
   Since the probability distribution is uniform and so, the probability at any particular point is zero,
   $$P(A) = 1$$
   Thus, the sequence X_n converges almost surely to X under this probability measure.

2. **6.3**

   We know there exists A and B such that
   $$A := \{s \in S : \lim_{n\to\infty} |X_n(s) - X(s)| = 0\}$$
   $$B := \{s \in S : \lim_{n\to\infty} |Y_n(s) - Y(s)| = 0\}$$
   and P(A)=P(B)=1
   $$\lim_{n\to\infty} |X_n(s) - X(s) + Y_n(s) - Y(s)| \leq \lim_{n\to\infty} |X_n(s) - X(s)| + |Y_n(s) - Y(s)|$$
   So,
   $$C := \{s \in S : \lim_{n\to\infty} |X_n(s) + Y_n(s) - X(s) - Y(s)| = 0\} \supseteq A \cap B$$
   $$P(C) \geq P(A \cap B) = P(A) + P(B) - P(A \cup B) = 1$$
   Thus, the sequence $X_n + Y_n$ converges almost surely to X+Y under this probability measure.

3. **6.6**

   (a) For convergence in probability to 0, we need

   $$\lim_{n\to\infty} P(|X_n| > \epsilon) = 0$$

   $$\lim_{n\to\infty} P(|X_n| > \epsilon) = \lim_{n\to\infty} P(|X_n| = 1) = \lim_{n\to\infty} p_n$$

   Thus by definition, for this problem, convergence in probability occurs if and only if p_n go to zero.

   (b) Since this converges in probability, we can get almost sure convergence by Theorem 6.6.2, if

   $$\sum_n^\infty P(|X_n| > \epsilon) < \infty$$

   $$\sum_n^\infty P(|X_n| > \epsilon) = \sum_n^\infty P(|X_n| = 1) = \sum_n^\infty p_n < \infty$$

   Now, we have to show if X_n converges to 0 almost surely, then $\sum_n^\infty p_n < \infty$.

   We try to prove the contra-positive. If $\sum_n^\infty p_n = \infty$, then X_n does not converge to 0. Since the events are independent, by Borel Cantelli,

   $$P(|X_n| > \epsilon, i.o) = 1$$

   So,

   $$P(\limsup_{n\to\infty} |X_n| > \epsilon) = 0$$

   which is the same as

   $$P(\liminf_{n\to\infty} |X_n| < \epsilon) = 1$$

   which is the definition of almost sure convergence (From Wikipedia).

4. **6.7**

   For $\limsup_{n\to\infty} \frac{logX_n}{logn} = C$, we need

   $$P\left(\frac{logX_n}{logn} < C + \epsilon, a.b.f.m\right) = 1$$

   $$P\left(\frac{logX_n}{logn} < C - \epsilon, i.o\right) = 1$$

   Let $Y_n = \frac{logX_n}{logn}$

   $$P(Y_n > y) = P(\frac{logX_n}{logn} > y) = P(X_n > n^y) = \frac{1}{n^{5y}}$$

   If y> $\frac{1}{5}$

   $$\sum_n^\infty P(Y_n > y) < \infty$$

Thus, by Borel-Cantelli, for y$> \frac{1}{5}$,

$$P(Y_n > y, i.o) = 0$$

So,

$$P(Y_n \leq y, a.b.f.m) = 1$$

$$P(Y_n < \frac{1}{5} + \epsilon, a.b.f.m) = 1 \tag{1}$$

If y$\leq \frac{1}{5}$

$$\sum_{n}^{\infty} P(Y_n \leq y) = \infty$$

Thus, by Borel-Cantelli, for y$\leq \frac{1}{5}$,

$$P(Y_n > y, i.o) = 1$$

$$P(Y_n > \frac{1}{5} - \epsilon, i.o) = 1 \tag{2}$$

From [1] and [2], we can say

$$\limsup_{n\to\infty} \frac{logX_n}{logn} = C = \frac{1}{5}$$