AMATH 569: Problem Set 3

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1. We have
$$u_{tt} = u_{xx} + u_{yy}$$
We assume the solution is of the form
$$u = T(t)X(x)Y(y)$$
Plugging this in, we get
$$T''XY = TX''Y + TXY''$$
$$\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}$$
Since all the three terms contain terms independent of each other, each of the terms must equal a constant.
$$\frac{T''}{T} = k$$
$$\frac{X''}{X} = k_1$$
$$\frac{Y''}{Y} = k_2$$
$$k = k_1 + k_2$$
$$X'' - k_1 X = 0$$
From the boundary conditions, we have
$$X(0) = X(a) = 0$$
For a non-trivial solution, k_1 has to be negative.
$$k_1 = -\lambda^2$$
Then, for n = 1,2,...
$$X_n = sin(\frac{n\pi x}{a})$$
is a solution.

Similarly,
$$Y_m = sin(\frac{m\pi x}{b})$$

$$k = k_1 + k_2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$$

$$T_{n,m} = \alpha_{n,m} cos(\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}t) + \beta_{n,m} sin(\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}t)$$

$$u(x,y,t) = \sum_{n,m=1}^{\infty} \big(\alpha_{n,m} cos(\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}t) + \beta_{n,m} sin(\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}t)\big) sin(\frac{n\pi x}{a}) sin(\frac{n\pi y}{b})$$

$$u(x,y,0) = \phi(x,y)$$

$$\sum_{n,m=1}^{\infty} \alpha_{n,m} sin(\frac{n\pi x}{a}) sin(\frac{n\pi y}{b}) = \phi(x,y)$$

$$u_t(x,y,0) = \psi(x,y)$$

$$\sum_{n,m=1}^{\infty} \beta_{n,m}\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} sin(\frac{n\pi x}{a}) sin(\frac{n\pi y}{b}) = \psi(x,y)$$

Assuming ϕ and ψ have 2-D fourier series,

$$\beta_{n,m} = \frac{4}{ab}\int_0^a\int_0^b \phi(x,y) sin(\frac{n\pi x}{a}) sin(\frac{m\pi y}{b}) dydx$$

$$\alpha_{n,m} = \frac{4}{ab\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}}\int_0^a\int_0^b \psi(x,y) sin(\frac{n\pi x}{a}) sin(\frac{m\pi y}{b}) dydx$$

2. (a) We have

$$u_{tt} + ku_t = c^2 u_{xx}$$

We assume the solution is of the form

$$u = T(t)X(x)$$

Plugging this in, we get

$$T''X + kT'X = c^2 TX''$$

$$\frac{T''}{T} + k\frac{T'}{T} = c^2\frac{X''}{X}$$

$$\frac{X''}{X} = k_1$$

From the boundary conditions, we find

$$X_n = sin(\frac{n\pi x}{L})$$

is a solution.

We have

$$T'' + kT' = c^2 k_1 T$$

$$T'' + kT' - c^2\frac{n^2\pi^2}{L}T = 0$$

The solution to this ode (Mathematica) is

$$T = a_n e^{\frac{1}{2}t(\sqrt{\frac{4\pi^2c^2n^2+k^2L}{L}}-k)} + a_n e^{\frac{1}{2}t(-\sqrt{\frac{4\pi^2c^2n^2+k^2L}{L}}-k)}$$

$$u(x,t) = (a_n e^{\frac{1}{2}t(\sqrt{\frac{4\pi^2c^2n^2+k^2L}{L}}-k)} + b_n e^{\frac{1}{2}t(-\sqrt{\frac{4\pi^2c^2n^2+k^2L}{L}}-k)} sin(\frac{n\pi x}{L})$$

$$u(x,0) = f(x)$$

$$\sum_{n=1}^{\infty}(a_n + b_n)sin(\frac{n\pi x}{L}) = f(x)$$

$$u_t(x,0) = g(x)$$

$$\sum_{n=1}^{\infty}(a_n w_1 + b_n w_2)sin(\frac{n\pi x}{L}) = g(x)$$

where

$$w_1 = \sqrt{\frac{4\pi^2c^2n^2+k^2L}{L}} - k, w_2 = -\sqrt{\frac{4\pi^2c^2n^2+k^2L}{L}} - k$$

$$a_n + b_n = \frac{2}{L}\int_0^L f(x)sin(\frac{n\pi x}{L}) = f_n$$

$$a_n w_1 + b_n w_2 = \frac{2}{L}\int_0^L g(x)sin(\frac{n\pi x}{L}) = g_n$$

$$a_n = \frac{f_n w_2 - g_n}{w_2 - w_1}$$

$$b_n = \frac{f_n w_1 - g_n}{w_1 - w_2}$$

(b)

3. We have

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

We assume the solution is of the form

$$u = v(r)w(\theta)$$

Pluggin this in,

$$v''w + \frac{v'w}{r} + \frac{vw''}{r^2} = 0$$
$$\frac{v''}{v} + \frac{v'}{rv} + \frac{w''}{r^2w} = 0$$
$$\frac{r^2v'' + rv'}{v} + \frac{w''}{w} = 0$$
$$\frac{w''}{w} = k$$

w(θ) has to be periodic since our boundary is on a disk. Thus, it cannot be an exponential or linear. This means k is negative.

$$w_n = a_n sin(\frac{n\pi\theta}{\pi}) + b_n cos(\frac{n\pi\theta}{\pi}) = a_n sin(n\theta) + b_n cos(n\theta)$$

The boundary condition only depends on θ as r is constant. Thus,

$$f = f(\theta)$$
$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)sin(\theta)d\theta$$
$$b_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)cos(\theta)d\theta$$

$$k = -\lambda^2 = -n^2$$
$$\frac{r^2v'' + rv'}{v} - n^2 = 0$$
$$r^2v'' + rv' - n^2v = 0$$

We assume a solution of the form r^k

$$r^kk(k-1) + kr^k - n^2r^k = 0$$
$$k = \pm n$$
$$v(r) = c_1r^n + c_2r^{-n}$$
$$v(1) = constant = \frac{f(\theta)}{w(\theta)}$$
$$u(r,\theta) = v(r)ww(\theta) = (c_1r^n + c_2r^{-n})(a_n sin(n\theta) + b_n cos(n\theta))$$

where $c_1 + c_2$ is constant.

4. Taking the Fourier transform (with respect to x)

$$\hat{u}_{tt} + c^2 w^2 \hat{u} = \hat{F}$$

We choose

$$\hat{v} = \hat{u} - \frac{\hat{F}}{c^2 w^2}$$

$$\hat{v}_{tt} + c^2 w^2 \hat{v} = 0$$

Although the solutions are sines and cosines, we'll keep it in exponential form to make the inverse fourier transform easier.

$$\hat{v} = ae^{icwt} + be^{-icwt}$$

$$\hat{u} = ae^{icwt} + be^{-icwt} + \frac{\hat{F}}{c^2 w^2}$$

a and b can be found using boundary conditions

$$a + b + \frac{\hat{F}}{c^2 w^2} = \hat{f}$$

$$aicw - bicw = \hat{g}$$

$$a = \frac{\hat{f}}{2} - \frac{\hat{F}}{2c^2 w^2} + \frac{\hat{g}}{2icw}$$

$$b = \frac{\hat{f}}{2} - \frac{\hat{F}}{2c^2 w^2} - \frac{\hat{g}}{2icw}$$

Plugging this in,

$$\hat{u} = ae^{icwt} + be^{-icwt} + \frac{\hat{F}}{c^2 w^2}$$

$$= \frac{e^{icwt} + e^{-icwt}}{2}\hat{f} + \frac{e^{icwt} - e^{-icwt}}{2icw}\hat{g} - \frac{e^{icwt} + e^{-icwt} - 2}{2c^2 w^2}\hat{F}$$

$$F^{-1}[e^{icwt}\hat{f}] = \int_{-\infty}^{\infty} e^{iw(ct+x)}\hat{f}dw = f(x+ct)$$

$$F^{-1}[\frac{e^{icwt} - e^{-icwt}}{2icw}\hat{g}] = \int_{-\infty}^{\infty} e^{iwx}\frac{e^{icwt} - e^{-icwt}}{2icw}\hat{g}$$

$$= \int_{-\infty}^{\infty} e^{iwx}\frac{e^{icwt} - e^{-icwt}}{2icw}\hat{g}dw$$

$$= \int_{-\infty}^{\infty}\int_{-t}^{t}(\frac{1}{2}e^{icws}ds)e^{iwx}\hat{g}dw$$

$$= \int_{-t}^{t}\int_{-\infty}^{\infty}\frac{1}{2}e^{icws}e^{iwx}\hat{g}dwds$$

$$= \frac{1}{2}\int_{-t}^{t} g(x+ct)ds$$

$$F^{-1}[\frac{e^{icwt}+e^{-icwt}-2}{2c^2w^2}\hat{F}] = \int_{-\infty}^{\infty} e^{iwx}\frac{e^{icwt}+e^{-icwt}-2}{2c^2w^2}\hat{F}dw$$

The c^2w^2 implies a double integral inside.

$$F^{-1}[\frac{e^{icwt}+e^{-icwt}-2}{2c^2w^2}\hat{F}] = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^t(\int_{-s}^{s} e^{icwr}drds)\hat{F}e^{iwx}dw$$

$$= \frac{1}{2}\int_0^t(\int_{-s}^{s}\int_{-\infty}^{\infty} e^{icwr}\hat{F}e^{iwx}dwdrds)$$

$$= \frac{1}{2}\int_0^t\int_{-s}^{s} F(x+cr)drds$$

$$u(t,x) = F^{-1}[\hat{u}] = \frac{1}{2}(f(x+ct)+f(x-ct)+\int_{-t}^{t} g(x+ct)ds - \int_0^t\int_{-s}^{s} F(x+cr)drds)$$