AMATH 561: Homework 3

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1. We need the probability at each value of X to create the generating function. But the probability of the binomial distribution itself is a random variable with a uniform distribution over (0,1). Since it is a uniform distribution over (0,1), it has a probability of 1.

$$\begin{aligned} P(X=k) &= \int_0^1 P(X=K, U=r) P_U(r) dr \\ &= \int_0^1 \binom{n}{k} r^k (1-r)^{n-k} 1 dr \\ &= \int_0^1 \binom{n}{k} r^k (1-r)^{n-k} dr \end{aligned}$$

The generating function is given by

$$G(s) = \sum_k s^k P(X=k)$$

$$= \sum_k s^k \int_0^1 \binom{n}{k} r^k (1-r)^{n-k} dr$$

Assuming uniform convergrence (because these are probability distributions)

$$= \int_0^1 \sum_k \binom{n}{k} (sr)^k (1-r)^{n-k} dr$$

$$= \int_0^1 (sr + 1 - r)^n dr$$

$$= \frac{s^{n+1} - 1}{(n+1)(s-1)}$$

Luckily, this is the sum of a finite geometric series.

$$G(s) = \frac{1}{n+1}(1 + s + s^2 + \ldots + s^n)$$

Thus, P(X=k) for any value of k is $\frac{1}{n+1}$

2.

$$\begin{aligned} \mathbb{E}[Z_n Z_m] &= \mathbb{E}[\mathbb{E}[Z_n Z_m | Z_m]] \\ &= \mathbb{E}[Z_m \mathbb{E}[Z_n | Z_m]] \end{aligned}$$

We want to know what $\mathbb{E}[Z_n|Z_m]$ is

$$\mathbb{E}[Z_n|Z_m] = G'_{Z_n|Z_m}(1)$$
$$= G'_{Z_{n-1}|Z_m}(1)G'(1)$$
$$= G'_{Z_{n-1}|Z_m}(1)\mu$$
$$= G'_{Z_m|Z_m}(1)\mu^{n-m}$$
$$= \mathbb{E}Z_m|Z_m\mu^{n-m}$$
$$= Z_m\mu^{n-m}$$

Thus,

$$\mathbb{E}[Z_nZ_m] = \mathbb{E}[Z_mZ_m\mu^{n-m}$$
$$= \mu^{n-m}\mathbb{E}{Z_m}^2$$

$$\rho(Z_m, Z_n) = \frac{CoV(Z_m, Z_n)}{\sqrt{Var(Z_m)Var(Z_n)}}$$
$$= \frac{\mathbb{E}Z_mZ_n - \mathbb{E}Z_m\mathbb{E}Z_n}{\sqrt{Var(Z_m)Var(Z_n)}}$$

$$= \frac{\mu^{n-m}\mathbb{E}{Z_m}^2 - \mu^{n+m}}{\sqrt{Var(Z_m)Var(Z_n)}}$$
$$= \frac{\mu^{n-m}\mathbb{E}{Z_m}^2 - \mu^{n-m}\mu^{2m}}{\sqrt{Var(Z_m)Var(Z_n)}}$$
$$= \frac{\mu^{n-m}\mathbb{E}{Z_m}^2 - \mu^{n-m}(\mathbb{E}Z_m)^2}{\sqrt{Var(Z_m)Var(Z_n)}}$$
$$= \mu^{n-m}\frac{\sqrt{Var(Z_m)}}{\sqrt{Var(Z_n)}}$$

If $\mu = 1, Var(Z_k) = kVar(Z_1)$. Then,

$$\rho(Z_m, Z_n) = \mu^{n-m}\sqrt{\frac{m}{n}}$$

Otherwise,

$$Var(Z_k) = Var(Z_1)\frac{(\mu^k - 1)\mu^{k-1}}{\mu - 1}$$

Then,

$$\rho(Z_m, Z_n) = \mu^{n-m}\sqrt{\frac{(\mu^m - 1)\mu^{m-1}}{(\mu^n - 1)\mu^{n-1}}}$$

$$\rho(Z_m, Z_n) = \mu^{\frac{n-m}{2}}\sqrt{\frac{\mu^m - 1}{\mu^n - 1}}$$

3. Not required

4.

$$G_{Z_{n+1}}(s) = \mathbb{E}[s^{Z_{n+1}}]$$
$$= \mathbb{E}[s^{\sum_{i=1}^{Z_n} X_{n,i}+Y_n}]$$
$$= \mathbb{E}[s^{\sum_{i=1}^{Z_n} X_{n,i}} s^{Y_n}]$$

Since the Xs and Ys are independent,

$$= \mathbb{E}[s^{\sum_{i=1}^{Z_n} X_{n,i}}]\mathbb{E}[s^{Y_n}]$$

$$= \mathbb{E}[s^{\sum_{i=1}^{Z_n} X_{n,i}}]G_Y(s)$$

By definition, from Theorem 3.1.9, this is

$$= G_{Z_n}(G_X(s))G_Y(s)$$

$$G_{Z_{n+1}}(s) = G_{Z_n}(G_X(s))G_Y(s)$$
$$G_{Z_1}(s) = G_{Z_0}(G_X(s))G_Y(s)$$
$$= \mathbb{E}[s^{\sum_{i=1}^{1} X_{n,i}}]G_Y(s)$$
$$= G_X(s)G_Y(s)$$
$$G_{Z_2}(s) = G_{Z_1}(G_X(s))G_Y(s)$$
$$= G_X(G_X(s))G_Y(G_X(s))G_Y(s)$$
$$= (G_X(s))^2 G_Y(G_X(s))G_Y(s)$$

5.

$$\phi_{X^2}(t) = \mathbb{E}e^{itX^2}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{itx^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{itx^2-\frac{(x-\mu)^2}{2\sigma^2}}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(1-2\sigma^2 it)x^2-2\mu x+\mu^2}{2\sigma^2}}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(1-2\sigma^2 it)}{2\sigma^2}(x^2-\frac{2\mu x}{1-2\sigma^2 it}+\frac{\mu^2}{1-2\sigma^2 it})}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(1-2\sigma^2 it)}{2\sigma^2}((x-\frac{\mu}{1-2\sigma^2 it})^2-\frac{\mu^2}{(1-2\sigma^2 it)^2}+\frac{\mu^2}{1-2\sigma^2 it})}$$
$$= e^{\frac{-it}{(1-2\sigma^2 it)}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(1-2\sigma^2 it)}{2\sigma^2}(x-\frac{2\mu}{1-it})^2}$$
$$= e^{\frac{-it}{(1-2\sigma^2 it)}} \frac{1}{\sqrt{1-2\sigma^2 it}}$$
$$\phi_{X^2}(t) = \frac{1}{\sqrt{1-2\sigma^2 it}} e^{\frac{-it}{(1-2\sigma^2 it)}}$$

In this case, we assume it is possible to analytically continue the complex function.

6. (a)

$$F_{X_n}(x) = (x - \frac{sin(2n\pi x)}{2n\pi})\mathbb{I}_{0\leq x\leq 1} + \mathbb{I}_{x>1}$$

This is 0 when x $\leq$ 0 and 1 when $x > 0$. Also, when x is in between 0 and 1, this is a strictly increasing function. Thus, this is a distribution function.

It is smooth between 0 and 1. So, we can differentiate it

$$f(x) = \begin{cases} 1 - cos(2n\pi x) & \text{if } 0 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$\lim_{n\to\infty} F(x) = x$$

This is the uniform distribution function.

$$\lim_{n\to\infty} f(x) = 1 - \lim_{n\to\infty} cos(2n\pi x)$$

This limit doesn't exist. It implies that in that limit, the cos term is wildly oscillating and is not the constant uniform density function.

7. When k = 1, this process is a geomtric distribution where you get tails N-1 times and then heads. When k = 2, this is like one geometric distribution after another one and thus getting two heads. These two geometric distributions are independent of each other.

Thus, for any k, the process is a sequence of k geometric distribution. So, the generating function is $G_{Geo}(s)^k$, where $G_{Geo}(s)$ is the generating function of the geometric distribution. Since the characteristic function is a generating function too, we can say the characteristic function of this process is

$$\phi_N(t) = (\phi_{geo}(t))^k$$
$$= (e^{it}p + e^{2it}qp + e^{3it}q^2p + \ldots)^k$$
$$= (\frac{e^{it}p}{1 - e^{it}(1-p)})^k$$
$$\phi_{2Np}(t) = \left(\frac{e^{2itp}p}{1 - e^{itp}(1-p)}\right)^k$$
$$\lim_{p\to 0} \phi_{2Np}(t) = \left(\frac{1}{1 - it}\right)^k$$

Characteristic function for a gamma distribution is given by

$$\phi(t) = \int_0^\infty \frac{1}{\Gamma(s)}\lambda^s x^{s-1} e^{-\lambda x + itx} dx$$

Using a result from complex analysis, we know

$$= \left(\frac{\lambda}{\lambda - it}\right)^s$$

For $\lambda = 1$ and s=k, these characteristic function converges to that of a gamma distribution as p goes to zero.

Thus, by the corollary of the Continuity Theorem, the distribution function for the process converges to a Gamma distribution.