AMATH 561: Homework 1

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1. Setting A as B and as ϕ, we find that $B, \phi \in \mathcal{G}$.

   If $A_1, A_2, A_3, \ldots \in \mathcal{F}$, then $\cup_i A_i \in \mathcal{F}$.

   Since $\cup_i A_i \in \mathcal{F}$.,
   $$\cup_i (A_i \cap B) = (\cup_i A_i) \cap B \text{ exists in } \mathcal{G}.$$

   For any A $\in \mathcal{F}$, there exist its complement A^c.

   So, the following exist.
   $$Y_1 = A \cap B, Y_2 = A^c \cap B$$
   The relative complement of Y_1 with respect to B is given by
   $$B \setminus Y_1 = B \setminus (A \cap B) = (B \setminus B) \cup (B \setminus A) = \phi \cup (B \setminus A) = B \setminus A = B \cap A^c = Y_2$$
   So, if Y is an element in G, then its complement exists as well in G.

   Thus, $\mathcal{F}$ is a σ -algebra on B

2. Since $\mathcal{F}$ and $\mathcal{G}$ are σ-algebra,
   $$\phi \in \mathcal{F}, \phi \in \mathcal{G}$$
   So,
   $$\phi \in \mathcal{F} \cap \mathcal{G}$$
   Also, if A is in $\mathcal{F} \cap \mathcal{G}$, then
   $$A \in \mathcal{F} \text{ and } A \in \mathcal{G}$$
   and
   $$A^c \in \mathcal{F} \text{ and } A^c \in \mathcal{G}$$
   Thus,
   $$A^c \in \mathcal{F} \cap \mathcal{G}$$
   Finally, if
   $$A_1, A_2, A_3, A_4, \ldots \in \mathcal{F} \cap \mathcal{G}$$
   then for countable unions,
   $$\cup_i A_i \in \mathcal{F} \cap \mathcal{G}$$
   since
   $$A_1, A_2, A_3, A_4, \ldots, \cup_i A_i \in \mathcal{F} \text{ and } A_1, A_2, A_3, A_4, \ldots, \cup_i A_i \in \mathcal{G}$$
   Thus, $\mathcal{F} \cap \mathcal{G}$ is a σ-algebra.

3. (a) Here, Ω is the set of all possible outcomes and is given by

$$\Omega = \{TTT, HHH, HTT, HHT, HTH, THH, THT, TTH\}$$

The trivial σ -algebra is given by

$$\mathcal{F}_{/} = \{\phi, \Omega\}$$

with the probability measure maps them to 0,1 A slightly more resolved σ -algebra is given by

$$\mathcal{F}_1 = \{\phi, \Omega, HXX, TXX\}$$

where HXX represents the event where the firstcoin shows a Head while TXX is its complement where the first coin shows a tail.

If p is the probability of getting a Head, the probability measure maps the σ -algebra to {0,1,p,1-p}

(b)

$$\Omega = \{BlueBlue, RedRed, BlueRed, RedBlue\}$$

A slightly resolved σ -algebra is given by

$$\mathcal{F} = \{\phi, \Omega, \text{ ‘Two balls of same color ‘}, \text{ ‘Two balls of different color ‘}\}$$

The probability measure maps this σ -algebra to {0,1,$\frac{1}{3}$, $\frac{1}{3}$}

(c) Here, Ω is infinite and given by

$$\Omega = \{H, TH, TTH, TTTH, TTTTH, TTTTTH \ldots\}$$

A σ -algebra is given by

$$\mathcal{F} = \{\phi, \Omega, \text{ ‘Events where you get Head within 3 tries‘}, \text{ ‘Events where you don’t get Head within 3}$$

If p is the probability of getting a Head is p,the probability measure maps this σ -algebra to {0,1,$p + (1-p)p + (1-p)^2 p$,1-($p + (1-p)p + (1-p)^2 p$)}

4. Since g is a strictly increasing function, it is a one-one function. Since it is continuous, it is onto. Thus, g is invertible.

So, we can write

$$F_Y(x) = P(Y \leq x) = P(g(X) \leq x) = P(X \leq g^{-1}(x)) = F_X(g^{-1}(x))$$

By using chain rule, we can get

$$f_Y(x) = F'_Y(x) = F'_X(g^{-1}(x)) g^{-1'}(x)$$

5. (a)

$$F_Y(x) = P(Y \leq x) = P(X^2 \leq x) = P(-\sqrt{x} \leq X \leq \sqrt{x}) = F_X(\sqrt{x}) - F_X(-\sqrt{x})$$

Here, x should be greater than or equal to 0. So,

$$F_Y = \begin{cases} 0 & \text{if } x < 0 \\ F_X(\sqrt{x}) - F_X(-\sqrt{x}) & \text{if } x \geq 0 \end{cases}$$

(b)

$$F_Y(x) = P(Y \leq x) = P(\sqrt{|X|} \leq x) = P(|X| \leq x^2) = P(-x^2 \leq X \leq x^2) = F_X(x^2) - F_X(-x^2)$$

Since usually the square root function is assumed to be its positive value and not multivalued, So,

$$F_Y = \begin{cases} 0 & \text{if } x < 0 \\ F_X(x^2) - F_X(-x^2) & \text{if } x \geq 0 \end{cases}$$

(c)

$$F_Y(x) = P(Y \leq x) = P(sin(X) \leq x)$$

This is only valid for $-1 \leq x \leq 1$. Within this region, for $sin(X) \leq x$, the random variable X can take any value in R depending on x. Also, for a particular x, the corresponding X such that $sin(X) \leq x$ is a collection of infinitely many sets since sin(x) is periodic.

We define

$$\eta = \cup_k \{-\pi + 2k\pi \leq X \leq arcsin(x) + 2k\pi\} \forall k \in \mathcal{Z}$$

Then,

$$F_Y = \begin{cases} 0 & \text{if } x < -1 \\ \\ \int_\eta dx F_X'(x) & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

(d)

$$F_Y(x) = P(Y \leq x) = P(F_X(X) \leq x)$$

Although F_X is non-decreasing, it may not be continuous and increasing. So, it may not be invertible. But, for any x, the random variable X that satisfies $F_X(X) \leq x$) by an open or closed set like in the graph below (since it's non-decreasing and right-continuous). Thus, everything is defined on a Borel σ-algebra and F_X(x) is still a measurable function.

Unlike the previous case, here the set $\{X : F_X(X) \leq x\}$ is one single set (since F_X is non-decreasing). So, we can find F_Y by evaluating F_X at K such that $K = sup\{X : F_X(X) \leq x\}$.

$$F_Y = \begin{cases} 0 & \text{if } x < 0 \\ \\ F_X(sup\{X : F_X(X) \leq x\}) & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

6. (a) We need $\mathbb{E}$Z =1. By definition,

$$\mathbb{E}Z = \int_R dx Z f(x) = \int_R dx \frac{g(x)}{f(x)} f(x)$$

Since f > 0 always, we don't need to worry about singularities in the integral.

$$\mathbb{E}Z = \int_R dx Z f(x) = \int_R dx g(x) = \int_{-\infty}^{\infty} g(x) dx$$

That is the distribution function corresponding to g evaluated at infinity, which should be 1. Thus,

$$\mathbb{E}Z = 1$$

Also, since g $\geq$ 0 because it is a density function and f > 0, Z $\geq$ 0.

Thus, Z fulfills the requirements to be a Radon-Nikodym derivative.

(b) To find the density of X, we start with the distribution of X (under the new probability measure).

$$\widetilde{\mathcal{P}}(X \leq b) = \mathbb{E}Z1_{\{X \leq b\}} = \int_{\{X \leq b\}} dx Z f(x) = \int_{\{X \leq b\}} dx g(x) = \int_{-\infty}^{b} g(x) dx$$

Looking at this, we can see that the density function of X under the new measure is g(x).