AMATH 562: Homework 3

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1.

$$d(W_T^4) = 4W_T^3 dW_T + 6W_T^2 dt$$

$$W_T^4 = \int_0^T 4W_T^3 dW_T + \int_0^T 6W_T^2 dt$$

$$E[W_T^4] = E[\int_0^T 4W_T^3 dW_T] + E[\int_0^T 6W_T^2 dt]$$

The first term is an Ito integral. So, it is a martingale. So, its expectation is the value at t=0 which is zero.

$$E[W_T^4] = E[\int_0^T 6W_T^2 dt]$$

We can use Fubini's theorem to take the expectation inside the integral.

$$E[W_T^4] = \int_0^T 6E[W_T^2]dt$$

$$E[W_T^4] = \int_0^T 6tdt$$

$$E[W_T^4] = 3T^2$$

$$E[W_T^6] = \int_0^T E[6W_T^5 dW_T] + \int_0^T 15E[W_T^4]dt$$

$$E[W_T^6] = \int_0^T 45t^2 dt$$

$$E[W_T^6] = 15T^3$$

2.

$$F_t = e^{-\alpha W_t + \frac{1}{2}\alpha^2 t}$$

$$dF_t = \frac{1}{2}\alpha^2 F_t dt - \alpha F_t dW_t + \frac{1}{2}\alpha^2 F_t dt$$

$$dF_t = \alpha^2 F_t dt - \alpha F_t dW_t$$

$$d(Y_tF_t) = Y_t dF_t + F_t dY_t + d[Y,F]_t$$
$$d[Y,F]_t = -\alpha^2 Y_t F_t dt$$
$$d(Y_tF_t) = Y_t\alpha^2 F_t dt - \alpha Y_t F_t dW_t + F_t r dt + F_t \alpha Y_t dW_t + -\alpha^2 Y_t F_t dt$$
$$d(Y_tF_t) = F_t r dt$$

$$Y_tF_t - Y_0F_0 = \int_0^t F_s r ds$$
$$Y_tF_t = Y_0 + \int_0^t F_s r ds$$
$$Y_t = Y_0 e^{\alpha W_t - \frac{1}{2}\alpha^2 t} + e^{\alpha W_t - \frac{1}{2}\alpha^2 t}\int_0^t e^{-\alpha W_s + \frac{1}{2}\alpha^2 s} r ds$$
$$Y_t = Y_0 e^{\alpha W_t - \frac{1}{2}\alpha^2 t} + \int_0^t e^{\alpha(W_t - W_s) - \frac{1}{2}\alpha^2(t-s)} r ds$$

3.
$$d(\Pi_t) = X_t d(\Delta_t) + \Delta_t d(X_t) + d[X,\Delta]_t$$
$$d(\Delta_t) = d(\frac{\partial f}{\partial x})$$
$$d(\frac{\partial f}{\partial x}) = \frac{\partial (df)}{\partial x}$$
$$df = (\partial_t + \frac{1}{2}\sigma^2 X^2 \partial_x^2) f dt + \partial_x f dX_t$$
$$d(\Delta_t) = d(\frac{\partial f}{\partial x}) = \frac{\partial (df)}{\partial x} = (\partial_x\partial_t + \frac{1}{2}\sigma^2 X_t^2 \partial_x^3) f dt + \partial_x^2 f dX_t$$
$$= (\partial_x\partial_t + \frac{1}{2}\sigma^2 X_t^2 \partial_x^3) f dt + \sigma X_t \partial_x^2 f dW_t$$
$$d[X,\Delta]_t = \partial_x^2 f d[X,X]_t = \sigma^2 X_t^2 \partial_x^2 f dt$$

We know that
$$(\partial_t + \frac{1}{2}\sigma^2 X^2 \partial_x^2) f = 0$$
$$\partial_x(\partial_t + \frac{1}{2}\sigma^2 X^2 \partial_x^2) f = 0$$
$$(\partial_x\partial_t + \sigma^2 X \partial_x^2 + \frac{1}{2}\sigma^2 X^2 \partial_x^3) f = 0$$
$$(\partial_x\partial_t + \frac{1}{2}\sigma^2 X^2 \partial_x^3) f = -\sigma^2 X \partial_x^2 f$$

Plugging this into $d(\Delta_t)$
$$d(\Delta_t) = -\sigma^2 X_t \partial_x^2 f dt + \sigma X_t \partial_x^2 f dW_t$$
$$X_t d(\Delta_t) = -\sigma^2 X_t^2 \partial_x^2 f dt + \sigma X_t^2 \partial_x^2 f dW_t$$

$$d(\Pi_t) = -\sigma^2 X_t^2 \partial_x^2 f dt + \sigma X_t^2 \partial_x^2 f dW_t + \Delta_t d(X_t) + d[X,\Delta]_t$$

$$= -\sigma^2 X_t^2 \partial_x^2 f dt + \sigma X_t^2 \partial_x^2 f dW_t + \sigma X_t f_x dW_t + \sigma^2 X_t^2 \partial_x^2 f dt$$
$$= \sigma X_t^2 \partial_x^2 f dW_t + \sigma X_t f_x dW_t$$

Integrating this from 0 to T,

$$\Pi_T - \Pi_0 = \int_0^T \sigma X_t^2 \partial_x^2 f dW_t + \sigma X_t f_x dW_t$$

Since $X_0 = 0$, Π_0 =0.

$$\Pi_T = \int_0^T (\sigma X_t^2 \partial_x^2 f + \sigma X_t f_x) dW_t$$

Since f and X_t are $\mathcal{F}$-adapted, the right hand side is an Ito integral which is a martingale. Thus, Π is a martingale with filtration $\mathcal{F}$.

4.

$$df = (\partial_t + \frac{1}{2}\sigma^2 \partial_x^2) f dt + \partial_x f dX_t$$

$$df = (\partial_t + \frac{1}{2}\sigma^2 \partial_x^2) f dt + \partial_x f \mu dt + \partial_x f \sigma dW_t$$

Integrating this from 0 to T,

$$f(T, X_T) - f(0, X_0) = \int_0^T (\partial_t + \frac{1}{2}\sigma^2 \partial_x^2) f dt + \partial_x f \mu dt + \partial_x f \sigma dW_t$$

$$f(T, X_T) - f(0, X_0) - \int_0^T (\partial_t + \frac{1}{2}\sigma^2 \partial_x^2) f dt + \partial_x f \mu dt = \int_0^T \partial_x f \sigma dW_t$$

$$M_T^f = \int_0^T \partial_x f \sigma dW_t$$

Since f and σ are $\mathcal{F}$-adapted, the right hand side is an Ito integral which is a martingale. Thus, M^f is a martingale with filtration $\mathcal{F}$.