AMATH 569: Problem Set 1

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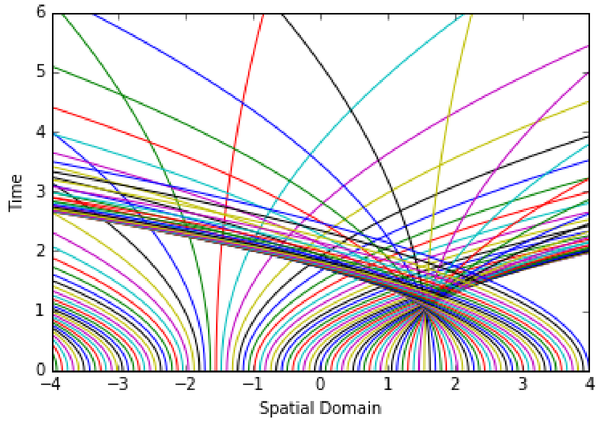
April 10, 2017

1. We have
$$u_t + au_x + bu_y = f$$
We know
$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u_x\frac{dx}{dt} + u_y\frac{dy}{dt}$$
Comparing the above two equations, we get, along the curves satisfying
$$\frac{dx}{dt} = a$$
$$\frac{dy}{dt} = b$$
we have
$$\frac{du}{dt} = f$$

2.
$$u_t + tuu_x = 0$$
Since this is homogeneous, along the characteristics, u = $u_0(x_0)$
$$\frac{dx}{dt} = tu$$
Integrating,
$$x = \frac{1}{2}t^2u_0 + x_0 \tag{1}$$
$$x_0 = x - \frac{1}{2}t^2u_0$$
Since u is constant along it,
$$u(x,t) = u(x_0,0) = cos(x - \frac{1}{2}t^2u)$$
To find breaking time, we differentiate (1) w.t.r t ,
$$0 = tu + \frac{1}{2}t^2u_{x_0}x_{0t} + x_{0t}$$
$$x_{0t} = \frac{-ut}{\frac{1}{2}t^2u_{x_0} + 1}$$
At breaking time, this is infinite. So,
$$t* = \sqrt{\frac{-2}{u_{x_0}}} = \sqrt{\frac{-2}{cos(x_0)}}$$
The minimal realistic (positive) breaking time is t* = $\sqrt{2}$

We can see the characteristics crossing somewhere close to 1.



3.

$$u_t + xu_x = 0$$

$$\frac{dx}{dt} = x$$

Integrating,

$$x = x_0 e^t \tag{2}$$

$$x_0 = xe^{-t}$$

Since this is non-homogeneous, along the characteristics,

$$\frac{du}{dt} = 1$$

$$u(x,t) = t + e^{x_0}$$

$$u(x,t) = t + e^{xe^{-t}}$$

To find breaking time, we differentiate (2) w.t.r t ,

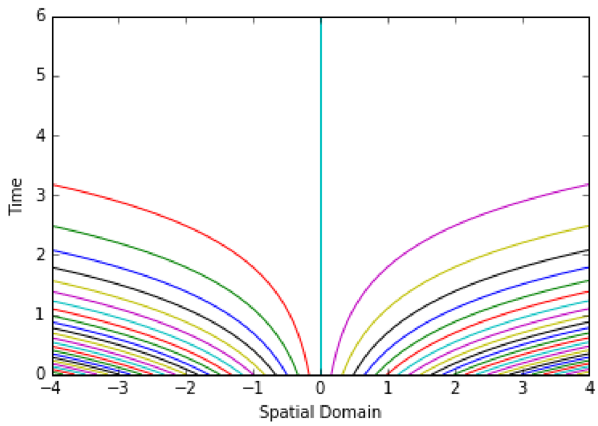
$$0 = x_0 e^t + x_{0t} e^t$$

$$x_{0t} = -x_0$$

This does not blow up. Differentiating (2) w.t.r x , we get

$$x_{0x} = e^{-t}$$

Since this does not go to infinity either, we can say that the characteristics do not cross. This is further confirmed by the plots below.



4.

$$u_t + uu_x = 0$$

Since this is homogeneous, along the characteristics, u $= u_0(x_0)$

$$\frac{dx}{dt} = u$$

Integrating,

$$x = tu + x_0 \tag{3}$$

$$x_0 = x - tu$$

Since u is constant along it,

$$u(x,t) = u(x_0, 0) = u(x - tu, 0)$$

Differentiating (3) w.t.r x,

$$x_{0x} = \frac{1}{u_{x_0}t + 1}$$

For this to be infinite,

$$t = \frac{1}{u_{x_0}}$$

(a)

$$u(x,0) = -x$$

$$u(x,t) = u(x_0,0) = u(x - tu, 0) = tu - x$$

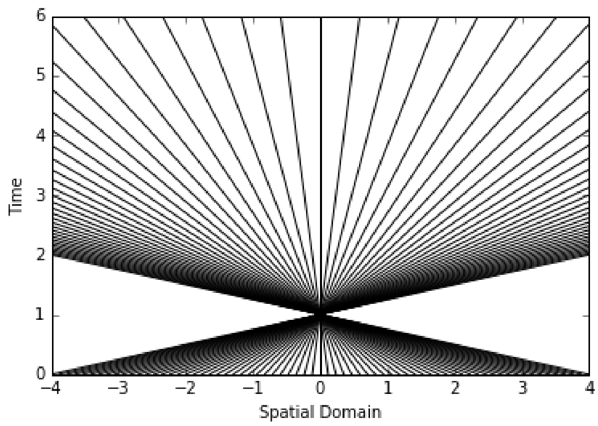
$$u(x,t) == \frac{-x}{1-t}$$

$$u(x,0) = -x$$

$$u_{x_0} = -1$$

$$t* = 1$$

Thus, we have the same realistic breaking time for all characteristics as shown below.



(b)

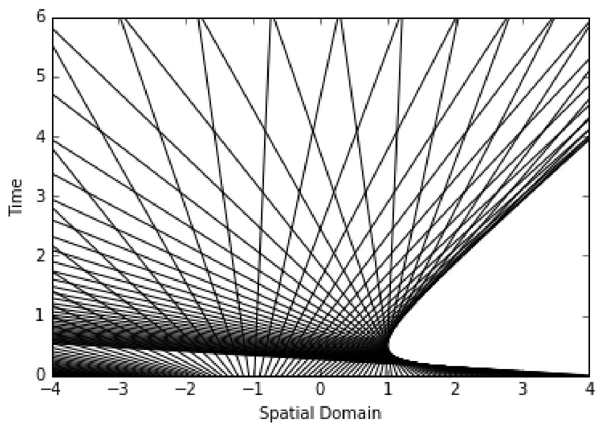
$$u(x,0) = 1 - x^2$$
$$u(x,t) = u(x_0,0) = u(x - tu, 0) = 1 - (x - tu)^2 = 1 - x^2 - t^2u^2 + 2tux$$
$$t^2u^2 + (1-2t)u + x^2 - 1 = 0$$
$$u(x,t) = \frac{2t - 1 \pm \sqrt{(1-2t)^2 - 4(x^2-1)t^2}}{2t^2} = \frac{2t - 1 \pm \sqrt{(1 - 4t - 4x^2t^2 + 8t^2}}{2t^2}$$
$$u_{x_0} = -2x$$

$$t* = \frac{1}{2x}$$

The minimal realistic breaking time goes to 0 as x goes to infinity.

$$t* = 0$$

This means the solution is invalid for all time. This agrees for our multivalued result for u. (One of them has to be non-physical)



(c)

$$u(x,0) = sin(x)$$
$$u(x,t) = u(x_0,0) = u(x - tu, 0) = sin(x - tu)$$
$$u(x,0) = sin(x)$$

$$u_{x_0} = cos(x)$$

$$t* = \frac{-1}{cos(x)}$$

The minimal realistic breaking time is 1 when cos(x) is -1.

$$t* = 1$$