AMath 574
Homework 2
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Problem #6.1 in the book

Solution:

$$\tilde{q}^n(x,t_n) = Q_i^n + \sigma_i^n(x - x_i)$$

The integral form is

$$\int_{C_i} q(x,t_{n+1})dx - \int_{C_i} q(x,t_n)dx = \int_{t_n}^{t_{n+1}} f(q(x_{i-\frac{1}{2}},t))dt - \int_{t_n}^{t_{n+1}} f(q(x_{i+\frac{1}{2}},t))dt$$

where C_i represents the ith cell.

$$Q_i^{n+1} = Q_i^n - \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} (f(q(x_{i+\frac{1}{2}},t)) - f(q(x_{i-\frac{1}{2}},t)))dt$$

6.13 looks more upwind than beam-warming so we take that into consideration.

$$\int_{t_n}^{t_{n+1}} f(q(x_{i-\frac{1}{2}},t)dt = \int_{t_n}^{t_{n+1}} uq(x_{i-\frac{1}{2}},t)dt$$

$$= \int_{t_n}^{t_{n+1}} uq(x_{i-\frac{1}{2}} - u(\Delta t), t_n)dt$$

$$= \int_{t_n}^{t_{n+1}} u(Q_{i-1}^n + \sigma_{i-1}^n(x_{i-\frac{1}{2}} - u(\Delta t) - x_{i-1}))dt$$

$$= \int_{t_n}^{t_{n+1}} u(Q_{i-1}^n + \sigma_{i-1}^n(\frac{1}{2}\Delta x - u(\Delta t)))dt$$

$$\int_{t_n}^{t_{n+1}} f(q(x_{i+\frac{1}{2}},t)dt = \int_{t_n}^{t_{n+1}} u(Q_i^n + \sigma_i^n(\frac{1}{2}\Delta x - u(\Delta t)))dt$$

$$Q_i^{n+1} = Q_i^n - \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} (u(Q_i^n - Q_{i-1}^n) - u(\frac{1}{2}\Delta x - u(\Delta t))(\sigma_i - \sigma_{i-1}))dt$$

$$= Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \frac{u}{\Delta x}(\frac{1}{2}\Delta x\Delta t - \frac{1}{2}u(\Delta t)^2)(\sigma_i - \sigma_{i-1})$$

$$= Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \frac{u\Delta t}{2\Delta x}(\Delta x - u(\Delta t))(\sigma_i - \sigma_{i-1})$$

Problem #6.2 in the book

Solution:

1.

$$
\begin{aligned}
TV(q(x)) &= TV(x=0) + TV(0<x<3) + TV(x=3) \\
&= |0-1| + \int_0^3 \pi cos(\pi x) + (2-0) \\
&= 3 + 12 \\
&= 15
\end{aligned}
$$

2.

$$
\begin{aligned}
TV(q(x)) &= TV(x=0) + TV(x=1) + TV(x=2) + TV(x=3) \\
&= 0 + 2 + 2 + 1 = 5
\end{aligned}
$$

Problem #6.5 in the book

Solution: When $u > 0$, we have the equation

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \frac{u\Delta t}{2\Delta x}(\Delta x - u(\Delta t))(\sigma_i - \sigma_{i-1})$$

It can be rewritten as

$$Q_i^{n+1} = Q_i^n - v(Q_i^n - Q_{i-1}^n) - \frac{1}{2}v(1-v)[\phi(\theta_{i+1/2}^n)(Q_{i+1}^n - Q_i^n) - \phi(\theta_{i-1/2}^n)(Q_i^n - Q_{i-1}^n)]$$

where

$$v = u\frac{dt}{dx}$$

Choosing

$$C_{i+1}^n = v - \frac{1}{2}v(1-v)(\frac{\phi(\theta_{i+1/2}^n)}{\theta_{i+1/2}^n} - \phi(\theta_{i-1/2}^n))$$

and

$$D_i^n = 0,$$

This equation satisfies Harten's theorem as long as if $0 \le C_{i-1} \le 1$, and this scheme is TVD. To be TVD, the flux limiter should satisfy

$$0 \le \phi(\theta) \le minmod(2, 2\theta)$$

For the minmod limiter,

$$\phi(\theta) = minmod(\theta, 1)$$

This satisfies the TVD condition and thus, the minmod scheme is TVD.

When $u < 0$, problem 6.7 (done later) ends with the TVD condition,

$$|\phi(\theta_{i+1/2}^n) - \frac{\phi(\theta_{i-1/2}^n)}{\theta_{i-1/2}^n}| \le 2$$

from which the previous TVD condition is obtained and again, the minmod limiter satisfies that.

Problem #6.6 in the book

Solution: For $u > 0$,

$$\begin{aligned} F^n_{i-1/2} &= uQ^n_{i-1} + \frac{1}{2}u(1 - \frac{u\Delta t}{\Delta x})\delta^n_{i-1/2} \\ &= uQ^n_{i-1} + \frac{1}{2}u(1 - \frac{u\Delta t}{\Delta x})(Q^n_i - Q^n_{i-1}) \\ F^n_{i+1/2} &= uQ^n_i + \frac{1}{2}u(1 - \frac{u\Delta t}{\Delta x})(Q^n_{i+1} - Q^n_i) \\ Q^{n+1}_i &= Q^n_i - \frac{\Delta t}{\Delta x}(F_{i+1/2} - F_{i-1/2}) \end{aligned}$$

$$\begin{aligned} Q^{n+1}_i = Q^n_i &- \frac{u\Delta t}{2\Delta x}(2Q^n_i - 2Q^n_{i-1} + Q^n_{i+1} - Q^n_i - Q^n_i + Q^n_{i-1}) \\ &+ \frac{(u\Delta t)^2}{2\Delta x^2}(Q^n_{i+1} - 2Q^n_i + Q^n_{i-1}) \\ = Q^n_i &- \frac{u\Delta t}{2\Delta x}(Q^n_{i+1} - Q^n_{i-1}) + \frac{(u\Delta t)^2}{2\Delta x^2}(Q^n_{i+1} - 2Q^n_i + Q^n_{i-1}) \end{aligned}$$

which is the Lax-Wendroff scheme.

For $u < 0$, Taking u as the magnitude of the velocity

$$\begin{aligned} F^n_{i-1/2} &= -uQ^n_i + \frac{1}{2}u(1 - \frac{u\Delta t}{\Delta x})\delta^n_{i-1/2} \\ &= -uQ^n_i + \frac{1}{2}u(1 - \frac{u\Delta t}{\Delta x})(Q^n_i - Q^n_{i-1}) \\ F^n_{i+1/2} &= -uQ^n_{i+1} + \frac{1}{2}u(1 - \frac{u\Delta t}{\Delta x})(Q^n_{i+1} - Q^n_i) \\ Q^{n+1}_i &= Q^n_i - \frac{\Delta t}{\Delta x}(F_{i+1/2} - F_{i-1/2}) \end{aligned}$$

$$\begin{aligned} Q^{n+1}_i = Q^n_i &- \frac{u\Delta t}{2\Delta x}(2Q^n_i - 2Q^n_{i+1} + Q^n_{i+1} - Q^n_i - Q^n_i + Q^n_{i-1}) \\ &+ \frac{(u\Delta t)^2}{2\Delta x^2}(Q^n_{i+1} - 2Q^n_i + Q^n_{i-1}) \\ = Q^n_i &+ \frac{u\Delta t}{2\Delta x}(Q^n_{i+1} - Q^n_{i-1}) + \frac{(u\Delta t)^2}{2\Delta x^2}(Q^n_{i+1} - 2Q^n_i + Q^n_{i-1}) \end{aligned}$$

Taking u as the velocity vector back again, we have

$$Q^{n+1}_i = Q^n_i - \frac{u\Delta t}{2\Delta x}(Q^n_{i+1} - Q^n_{i-1}) + \frac{(u\Delta t)^2}{2\Delta x^2}(Q^n_{i+1} - 2Q^n_i + Q^n_{i-1})$$

which is the Lax-Wendroff scheme.

Problem #6.7 in the book

Solution: Since $u < 0$,

$$\theta^n_{i-1/2} = \frac{Q^n_{i+1} - Q^n_i}{Q^n_i - Q^n_{i-1}}$$

$$Q^{n+1}_i = Q^n_i - C^n_{i-1}(Q^n_i - Q^n_{i-1}) + D^n_i(Q^n_{i+1} - Q^n_i)$$
$$= Q^n_i + (-v + \frac{1}{2}v(1+v)(\phi(\theta^n_{i+1/2}) - \frac{\phi(\theta^n_{i-1/2})}{\theta^n_{i-1/2}}))(Q^n_{i+1} - Q^n_i)$$
$$= Q^n_i - v(Q^n_{i+1} - Q^n_i) + \frac{1}{2}v(1+v)[\phi(\theta^n_{i+1/2})(Q^n_{i+1} - Q^n_i) - \frac{\phi(\theta^n_{i-1/2})}{\theta^n_{i-1/2}}(Q^n_{i+1} - Q^n_i)]$$
$$= Q^n_i - v(Q^n_{i+1} - Q^n_i) + \frac{1}{2}v(1+v)[\phi(\theta^n_{i+1/2})(Q^n_{i+1} - Q^n_i) - \phi(\theta^n_{i-1/2})(Q^n_i - Q^n_{i-1})]$$

This is 6.41.

$$D = -v + \frac{1}{2}v(1+v)(\phi(\theta^n_{i+1/2}) - \frac{\phi(\theta^n_{i-1/2})}{\theta^n_{i-1/2}})$$

If

$$|\phi(\theta^n_{i+1/2}) - \frac{\phi(\theta^n_{i-1/2})}{\theta^n_{i-1/2}}| \leq 2$$

then,

$$|D_{max}| = |-v + v(1+v)| or |-v - v(1+v)|$$
$$= |v^2| or |-2v - v^2|$$

The first term has a maximum of 1 and the second term has a maximum of 0.25 because the velocity is negative and $-1 < v < 0$. Thus, the conditions for Theorem 6.1 are satisfied and 6.41 is TVD.

Problem #7.1 in the book

Solution:

$$F^n_{1/2} = \frac{1}{2}u(Q^n_0 + Q^n_1) - \frac{1}{2}\frac{\Delta t}{\Delta x}u^2(Q^n_1 - Q^n_0)$$

Taking unit Courant number,

$$F^n_{1/2} = uQ^n_0 = ug_0(t_n + \frac{\Delta x}{2u}) = ug_0(t_n + \frac{\Delta t}{2})$$

which is 7.6 exactly.

Problem #7.2 in the book

Solution:

1. For the acoustic equation,

$$q_m = q_l + \alpha^1 r^1 = \frac{1}{2}\begin{bmatrix} (p_l + p_r) - Z_0(u_r - u_l) \\ (u_l + u_r) - (p_r - p_l)/Z_0 \end{bmatrix} = \begin{bmatrix} p_1 - Z_0 u_1 \\ 0 \end{bmatrix}$$

   This middle state condition agrees with the condition of zero velocity at the boundary.

2.

$$q_m = q_l + \alpha^1 r^1 = \frac{1}{2}\begin{bmatrix} (p_l + p_r) - Z_0(u_r - u_l) \\ (u_l + u_r) - (p_r - p_l)/Z_0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2p_1 - Z_0(2u_1 - 2U(t_n)) \\ 2\epsilon sin(wt) \end{bmatrix}$$

$$= \begin{bmatrix} p_1 - Z_0(u_1 - U(t_n)) \\ \epsilon sin(wt) \end{bmatrix}$$

   Thus, the middle state has an oscillating velocity.

Additional Problem:

Show that the flux-limiter method (6.40),(6.41) can be written as a *wave limiter* method as:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\bar{u}^+ \mathcal{W}_{i-1/2} + \bar{u}^- \mathcal{W}_{i+1/2}) - \frac{\Delta t}{\Delta x}(\widetilde{F}_{i+1/2} - \widetilde{F}_{i-1/2}),$$

where $\mathcal{W}_{i-1/2} = Q_i^n - Q_{i-1}^n$ and the "correction flux" is

$$\widetilde{F}_{i-1/2} = \frac{1}{2}|\bar{u}|\left(1 - \frac{\Delta t}{\Delta x}|\bar{u}|\right)\widetilde{\mathcal{W}}_{i-1/2},$$

with the limited waves $\widetilde{\mathcal{W}}$ defined by

$$\widetilde{\mathcal{W}}_{i-1/2} = \phi(\theta_{i-1/2})\mathcal{W}_{i-1/2}.$$

The ratio $\theta_{i-1/2}$ is defined in (6.35) and the function ϕ might be one of limiters from (6.39).

Solution:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\bar{u}^+ \mathcal{W}_{i-1/2} + \bar{u}^- \mathcal{W}_{i+1/2}) - \frac{\Delta t}{\Delta x}(\widetilde{F}_{i+1/2} - \widetilde{F}_{i-1/2})$$

$$= Q_i^n - \frac{\Delta t}{\Delta x}(\bar{u}^+(Q_i^n - Q_{i-1}^n) + \bar{u}^-(Q_{i+1}^n - Q_i^n))$$

$$-\frac{\Delta t}{\Delta x}\frac{1}{2}|\bar{u}|\left(1-\frac{\Delta t}{\Delta x}|\bar{u}|\right)(\phi(\theta_{i+1/2})\mathcal{W}_{i+1/2}-\phi(\theta_{i-1/2})\mathcal{W}_{i-1/2})$$

Taking v as $\frac{\Delta t}{\Delta x}\frac{1}{2}|\bar{u}|\left(1-\frac{\Delta t}{\Delta x}|\bar{u}|\right)$, we have

$$Q_i^{n+1}=Q_i^n-(v^+(Q_i^n-Q_{i-1}^n)+v^-(Q_{i+1}^n-Q_i^n))$$

$$-\frac{1}{2}|v|\left(1-|v|\right)(\phi(\theta_{i+1/2})(Q_{i+1}^n-Q_i^n)-\phi(\theta_{i-1/2})(Q_i^n-Q_{i-1}^n))$$

When $u>0$,

$$Q_i^{n+1}=Q_i^n-v(Q_i^n-Q_{i-1}^n)-\frac{1}{2}v\left(1-v\right)(\phi(\theta_{i+1/2})(Q_{i+1}^n-Q_i^n)-\phi(\theta_{i-1/2})(Q_i^n-Q_{i-1}^n))$$

which is 6.40

When $u<0$,

$$Q_i^{n+1}=Q_i^n-v(Q_i^n-Q_{i-1}^n)-\frac{1}{2}|v|\left(1-|v|\right)(\phi(\theta_{i+1/2})(Q_{i+1}^n-Q_i^n)-\phi(\theta_{i-1/2})(Q_i^n-Q_{i-1}^n))$$

$$=Q_i^n-v(Q_i^n-Q_{i-1}^n)+\frac{1}{2}v\left(1+v\right)(\phi(\theta_{i+1/2})(Q_{i+1}^n-Q_i^n)-\phi(\theta_{i-1/2})(Q_i^n-Q_{i-1}^n))$$

which is 6.41

Programming problem

Modify the IPython notebook `$AM574/notebooks/Advection_Examples2.ipynb` to create a new notebook `Advection_Limiters.ipynb` that illustrates your solutions to the following:

a. Implement the wave limiter methods for advection, as described in the previous problem. Note that it's impossible to use half-integer indices, so you might want to declare arrays such as `Ftilde` in which `Ftilde[i]` holds the correction flux $\widetilde{F}_{i-1/2}$. (This is the convention used in Clawpack — the index i often refers to information at the left edge of the cell $x_{i-1/2}$.)

   Copy the `upwind` function definition to a new cell in the notebook and modify it to create a new function `wave_limiter` that has one additional argument `phi` in the calling sequence, so that a limiter function $\phi(\theta)$ can be passed in. The function ϕ might be one of those listed in (6.39a,b) in the book.

   For example:

```
def phi_minmod(theta):
    return(max(0., min(theta,1)))
```

   would define the minmax limiter, and then

```
figs = wave_limiter(x,tfinal,nsteps,u,qtrue,nplot,phi_minmod)
```

   should solve the problem using the minmod wave-limiter method and return a list of figures to animate.

b. Test your function by writing a notebook that produces animations up to time $t = 10$ that correspond to Figures 6.2 and 6.3 from the book. This requires trying out several different limiter functions with 2 sets of initial conditions.

Submit your notebook `Advection_Limiters.ipynb`.

Note: Before submitting your notebook, clear all the output (From the "Cell" menu select "All Output" and then "Clear"), and then save the notebook. Otherwise png figures are stored in the notebook file and it may be very large.

Installing Clawpack. You don't need to turn in anything for this, but you should make sure you have Clawpack installed and working for future assignments.

I posted two videos at `http://faculty.washington.edu/rjl/classes/am574w2017/lectures.html` with tips on installing and getting started with running the code and changing the inputs.

You might want to also clone the `apps` repository, see `http://www.clawpack.org/apps.html`. The directory `apps/fvmbook/chap6` has the Clawpack code that creates Figures 6.2 and 6.3 (newly updated for Version 5.4.0). You might want to use the plots you can create in this directory for comparison.