# AMATH 575: Problem Set 1

Jithin D. George, No. 1622555

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1.

$$\dot{x} = y$$
$$\dot{y} = -\delta y - \mu x - x^3$$

$\mu$ is 1. The fixed points are (0,0), (i,0) and (-i,0)

The jacobian is given by

$$\begin{bmatrix} 0 & 1 \\ -1-3x^2 & -\delta \end{bmatrix}$$

For (0,0), this is

$$\begin{bmatrix} 0 & 1 \\ -1 & -\delta \end{bmatrix}$$

$$\lambda_1 = \frac{1}{2}(-\sqrt{\delta^2 - 4} - \delta)$$
$$\lambda_2 = \frac{1}{2}(\sqrt{\delta^2 - 4} - \delta)$$

For the real part of both these eigenvalues to be negative, $\delta$ has to be positive.

So, for positive $\delta$, (0,0)is stable. For negative $\delta$, (0,0)is unstable. For $\delta = 0$, the eigenvalues are complex and we can't say anything about their stability.

For (i,0) and (-i,0), this is

$$\begin{bmatrix} 0 & 1 \\ 2 & -\delta \end{bmatrix}$$

$$\lambda_1 = \frac{1}{2}(-\sqrt{\delta^2 + 8} - \delta)$$
$$\lambda_2 = \frac{1}{2}(\sqrt{\delta^2 + 8} - \delta)$$

As long as $\delta$ is real, $\lambda_2$ will always have a positive real part. So, (i,0) and (-i,0) are always unstable fixed points.

2.

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = cos(4\theta)$$

The fixed points given the constraints on r and $\theta$ are $(0, \frac{\pi}{8})$,$(0, \frac{3\pi}{8})$, $(0, \frac{5\pi}{8})$, $(0, \frac{7\pi}{8})$, $(0, \frac{9\pi}{8})$,$(0, \frac{11\pi}{8})$, $(0, \frac{13\pi}{8})$, $(0, \frac{15\pi}{8})$,$(1, \frac{\pi}{8})$,$(1, \frac{3\pi}{8})$, $(1, \frac{5\pi}{8})$, $(1, \frac{7\pi}{8})$, $(1, \frac{9\pi}{8})$,$(1, \frac{11\pi}{8})$, $(1, \frac{13\pi}{8})$ and $(1, \frac{15\pi}{8})$. The Jacobian is

$$\begin{bmatrix} 1-3r^2 & 0 \\ 0 & -4sin(4\theta) \end{bmatrix}$$

When r= 0, the maximal eigenvalue is 1 and it is clearly unstable. When r= 1, it depends on the sign of $sin(4\theta)$

3.

$$x' = 10(-x + y)$$
$$y' = rx - y - xz$$
$$z' = -\frac{8}{3}z + xy$$

The fixed points are (0,0,0)$(\sqrt{\frac{8}{3}(r-1)}, \sqrt{\frac{8}{3}(r-1)}, r-1\ )$ and $(-\sqrt{\frac{8}{3}(r-1)}, -\sqrt{\frac{8}{3}(r-1)}, r-1\ )$

The jacobian is given by

$$\begin{bmatrix} -10 & 10 & 0 \\ r-z & -1 & -x \\ y & x & -8/3 \end{bmatrix}$$

For the first fixed point, this becomes

$$\begin{bmatrix} -10 & 10 & 0 \\ r & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix}$$

We can see the eigenvalues $-\frac{8}{3}$, $\frac{1}{2}(\sqrt{40r+81} - 11)$ and $\frac{1}{2}(-\sqrt{40r+81} - 11)$. So, (0,0,0)is an asymptotically stable fixed point if r <1 , unstable if r>1 and of unknown stability if r =1.

For the second fixed point, the jacobian becomes

$$\begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & -\sqrt{\frac{8}{3}(r-1)} \\ \sqrt{\frac{8}{3}(r-1)} & \sqrt{\frac{8}{3}(r-1)} & -8/3 \end{bmatrix}$$

$$\lambda^3 + \frac{41\lambda^2}{3} + \frac{80\lambda}{3} + \frac{8\lambda r}{3} + \frac{160(r-1)}{3} = 0$$

We get the same characteristic equation for the third fixed point as well.

For $(\sqrt{\frac{8}{3}(r-1)}, \sqrt{\frac{8}{3}(r-1)}, r-1\ )$ and $(-\sqrt{\frac{8}{3}(r-1)}, -\sqrt{\frac{8}{3}(r-1)}, r-1\ )$ to be stable, by the Routh-Hurwtiz table, we need

$$\frac{160(r-1)}{3} > 0, 41(80+80r) - 480(r-1) > 0$$

Thus, for those points to be stable, r has to between 1 and $\frac{470}{19}$.

4.

$$\theta_{n+1} = \theta_n + I_n - \frac{K}{2\pi} sin 2\pi\theta_n mod 1$$
$$I_{n+1} = I_n - \frac{K}{2\pi} sin 2\pi\theta_n$$

The expression we get after taking th determinant to find the eigenvalues is The mod makes theta periodic between 0 and 1. The jacobian is given by

$$\begin{bmatrix} 1 - Kcos2\pi\theta & 1 \\ -Kcos2\pi\theta & 1 \end{bmatrix}$$

If K=0, the fixed points are at ($\theta$,n) where n is any integer and $\theta$ lies in [0,1). The jacobian becomes

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

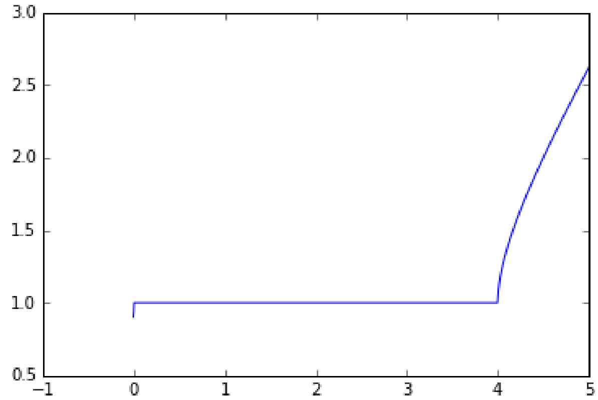
The eigenvalues are 1. So, we need higher order terms for stability. Otherwise, the fixed points are at (0,0) and (0,$\frac{1}{2}$). The fixed point at (0,1) does not exist because of the mod.

$$\begin{bmatrix} 1 - K & 1 \\ -K & 1 \end{bmatrix}$$

The eigenvalues are given by

$$\lambda = \frac{1}{2}(2 - k \pm \sqrt{k^2 - 4k})$$

We plot the bigger eigenvalue.




We can see that, for K > 4, the absolute value of the maximal eigenvalue is greater than 1 and both the fixed points are unstable (saddle). For K in between 0 and 4, the eigenvalues are 1 and we would need the higher order terms to find the stability.

5. **Flow**:

$$\dot{x} = x^2 + y^2$$
$$\dot{y} = x^2 + y^2$$

At the fixed point (0,0), the jacobian is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The eigenvalues are (0,0) indicating a center. However, for any value of x and y, $\dot{x}$ and $\dot{y}$ >0. So, it is clearly non-linearly unstable.

**Map**:

$$x_{n+1} = x_n + x_n^2 + y_n^2$$
$$y_{n+1} = y_n + x_n^2 + y_n^2$$

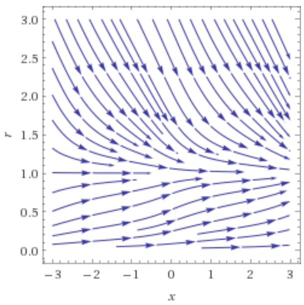
At the fixed point (0,0), the jacobian is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The eigenvalues are (1,1) indicating a center. However, for any value of x and y, $x_{n+1} > x_n$ and $y_{n+1} > y_n$. So, it is clearly non-linearly unstable.

6.

$$\dot{\theta} = 1 + sin^2\theta + (1-r)^2$$
$$\dot{r} = r(1-r)$$




If we define the distance between two points $(r_1, \theta_1)$and $(r_2, \theta_2)$

$$a^2 = (r_1 - r_2)^2 + (\theta_1 - \theta_2)^2$$

We get

$$\frac{da^2}{dt} = (r_1 - r_2)^2 + (\theta_1 - \theta_2)^2$$
$$= 2(r_1 - r_2)(r_1' - r_2') + 2(\theta_1 - \theta_2)(\theta_1' - \theta_2')$$
$$= 2(r_1 - r_2)(r_1 - r_1^2 - r_2 + r_2^2) + 2(\theta_1 - \theta_2)(sin^2(\theta_1) - sin^2(\theta_2) + r_1^2 - r_2^2 - 2r_1 + 2r_2)$$
$$= 2(r_1 - r_2)^2(1 - r_1 - r_2) + 2(\theta_1 - \theta_2)((sin(\theta_1) - sin(\theta_2))(sin(\theta_1) + sin(\theta_2)) + (r_1 - r_2)(1 - r_1 - r_2))$$

Using $|sin(\theta_1) - sin(\theta_2)| \leq |(\theta_1) - (\theta_2)|$

$$\leq 2(r_1 - r_2)^2(1 - r_1 - r_2) + 2(\theta_1 - \theta_2)((\theta_1 - \theta_2)(sin(\theta_1) + sin(\theta_2)) + (r_1 - r_2)(1 - r_1 - r_2))$$

Using $|(\theta_1) - (\theta_2)| \leq a$ and $|(r_1) - (r_2)| \leq a$

$$\leq 2a^2(1 - r_1 - r_2) + 2a(a(sin(\theta_1) + sin(\theta_2)) + a(1 - r_1 - r_2))$$
$$\leq 2a^2(sin(\theta_1) + sin(\theta_2) + 2(1 - (r_1 + r_2)))$$

Since $|r_1| \leq max(1, r_0)$ because trajectories seem to go to r=1,

$$\leq 2a^2(4 - (r_{01} + r_{02}))$$
$$\frac{da^2}{dt} \leq Ka^2$$

Since the distance is bounded in this fashion, we have a bound on the distance. So given two points initially separated by a $\delta$, we can find a bound on the $\epsilon$, the distance between them for the rest of time. Thus, this system is orbitally stable.

7. The general solution is spanned by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -11 \begin{bmatrix} -10 \\ 1 \\ 0 \end{bmatrix} - \frac{8}{3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The dimension of the stable manifold is 2 while that of the center manifold is 1. The stable manifold is in the form

$$y = ax + bz + cx^2 + dz^2 + exz + O(x^3)$$

$$y' = (a + 2cx + ez)x' + (b + 2dz + ex)z'$$

$$x - y - xz = (a + 2cx + ez)(-10x + 10y) + (b + 2dz + ex)(-\frac{8}{3}z + xy)$$

$$-30a^2x - 3abx^2 - 30abz - 90acx^2 - 6adx^2z - 30adz^2 - 3aex^3 - 60aexz + 27ax - 3b^2xz - 3bcx^3 - 60bcxz$$
$$-9bdxz^2 - 6bex^2z - 30bez^2 + 5bz - 60c^2x^3 - 6cdx^3z - 60cdxz^2 - 3cex^4 - 90cex^2z + 57cx^2$$
$$-6d^2xz^3 - 9dex^2z^2 - 30dez^3 + 13dz^2 - 3e^2x^3z - 30e^2xz^2 + 35exz - 3xz + 3x = 0$$

Equating coefficients, we have

$$a = 1, -\frac{1}{10}$$
$$b = 0, c = 0, d = 0$$
$$e = \frac{3}{41}, -\frac{3}{25}$$

The stable manifold is given by the appropriate coefficients

$$y = -\frac{1}{10}x - \frac{3}{25}xz$$

For the center manifold, we express

$$y = h_1(x) = ax + bx^2$$
$$z = h_2(x) = cx + dx^2$$

$$y' = (a + 2bx)x'$$
$$z' = (c + 2dx)x'$$

$$x - y - xz = (a + 2bx)(-10x + 10y)$$
$$x - (ax + bx^2) - x(cx + dx^2) = (a + 2bx)(-10x + 10(ax + bx^2))$$
$$-\frac{8}{3}z + xy = (c + 2dx)(-10x + 10y)$$
$$-\frac{8}{3}(cx + dx^2) + x(ax + bx^2) = (c + 2dx)(-10x + 10(ax + bx^2))$$

Equating coefficients, we have

$$a = 1, -\frac{1}{10}$$
$$b = 0, c = 0,$$

$$d = \frac{3}{500}, \frac{3}{8}$$

The center manifold is given by the appropriate coefficients

$$y = x$$

$$z = \frac{3}{8}x^2 = \frac{3}{8}y^2$$

The unstable manifold is the empty set.

8. At the saddle points, the dimension of the stable manifold is 1 and that of the unstable manifold is 1 (and center 0). The manifold would be of the form

$$\theta_{n+1} = aI_{n+1} + bI_{n+1}^2 + \dots$$

$$\theta_n + I_n - K\theta_n = a(I_n - K\theta_n) + b(I_n - K\theta_n)^2 + \dots$$

$$(a - Ka + 1)I_n + b(1 - K)I_n^2 = a(I_n - K(aI_n + bI_n^2)) + b(I_n - K(aI_n + bI_n^2))^2 + \dots$$

$$a = \frac{\sqrt{K} - \sqrt{K-4}}{2\sqrt{K}}, \frac{\sqrt{K} + \sqrt{K-4}}{2\sqrt{K}}$$

$$b = 0$$

Thus, the stable manifold would be

$$\theta_n = \frac{\sqrt{K} - \sqrt{K-4}}{2\sqrt{K}} I_n$$

The unstable manifold would be

$$\theta_n = \frac{\sqrt{K} + \sqrt{K-4}}{2\sqrt{K}} I_n$$

9. (a)

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{-2x^2 + 2xy^2}{-x + y^2} = 2x$$

$$y = x^2 + c$$

Thus, $y = x^2$ is an invariant manifold because it is a solution curve to the systems of 2 odes.

(b)

$$\dot{x} = -x + y^2$$

Plugging in y = $x^2$,

$$\dot{x} = -x + x^4$$

The solution to this is

$$x(t) = \frac{1}{(e^{c_1 + 3t} + 1)}^{1/3}$$

As t goes from $-\infty$ to $\infty$, (x,y) goes from (0,0) to (1,1). Thus , y = $x^2$ is the trajectory connecting (0,0) and (1,1)

(c) At (0,0), we get that the eigenvectors give us the following span.

$$\begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus, we see that the center manifold should be tangent to the y-axis. However, $y = x^2$ is tangent to the x-axis which means it is the stable manifold.