AMATH 569: Problem Set 4

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1. (a) We assume u is of the form $Ae^{ikx-iwt}$

$$u_t = -u_{xxxx}$$

$$-iw = -(ik)^4$$

$$w = -ik^4$$

$$\int_{-\infty}^{\infty} |u(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u(k)}|^2 dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |A(k)|^2 |e^{2(ikx-k^4t)}| dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |A(k)|^2 e^{-2k^4t} dk$$

This decays to zero for all k as time goes to infinity. So, the pde is well-posed.

(b) We assume u is of the form $Ae^{ikx-iwt}$

$$u_t = iu_x$$

$$-iw = i(ik)$$

$$w = -ik$$

$$\int_{-\infty}^{\infty} |u(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u(k)}|^2 dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |A(k)|^2 |e^{2(ikx-kt)}| dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |A(k)|^2 e^{-2kt} dk$$

This blows up for negative k as time goes to infinity. So, the pde is ill-posed.

(c) We assume u is of the form $Ae^{ik_xx+ik_yy-iwt}$

$$u_t = u_{xx} - u_{yy}$$

$$-iw = -k_x^2 + k_y^2$$

$$w = -ik_x^2 + ik_y^2$$

$$\int_{-\infty}^{\infty} |u(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |u(\hat{k})|^2 dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |A(k)|^2 |e^{2(ik_x x + ik_y y - iwt)}| dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |A(k)|^2 |e^{2(ik_x x + ik_y y - (k_x^2 - k_y^2)t)}| dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |A(k)|^2 e^{-2(k_x^2 - k_y^2)t} dk$$

When $k_x^2 < k_y^2$, this blows up as time goes to infinity and the pde is ill-posed.

(d) We assume u is of the form $Ae^{ik_x x + ik_y y - iwt}$

$$u_t = u_{xx} - u_y$$

$$-iw = -k_x^2 + ik_y$$

$$w = -ik_x^2 - k_y$$

$$\int_{-\infty}^{\infty} |u(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |u(\hat{k})|^2 dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |A(k)|^2 |e^{2(ik_x x + ik_y y - ik_y t - k_x^2 t)}| dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |A(k)|^2 e^{-2k_x^2 t} dk$$

This decays to zero for all k as time goes to infinity. So, the pde is well-posed.

2.

$$u_t = L(u)$$

L is of the form $\sum_{p=0}^{n} c_n \partial_{x_p}$

$$u_t = \sum_{p=0}^{n} c_p \partial_{x_p} u$$

$$u(x,0) = f(x)$$

We assume u is of the form $Ae^{ikx - iwt}$. We get the following dispersion relationship.

$$-iw = \sum_{p=0}^{n} c_p i^p k^p$$

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(k) e^{ikx + \sum_{p=0}^{n} c_p i^p k^p t} dk$$

At t=0,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} A(k) e^{ikx} dk = f(x)$$

$$A(k) = \int_{-\infty}^{\infty} e^{-iky} f(y)dy$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k)e^{ikx+\sum_{p=0}^{n} c_p i^p k^p t}dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+\sum_{p=0}^{n} c_p i^p k^p t}\int_{-\infty}^{\infty} e^{-iky} f(y)dydk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+\sum_{p=0}^{n} c_p i^p k^p t}\int_{-\infty}^{\infty} e^{-iky} f(y)dydk$$
$$= \int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-iky+\sum_{p=0}^{n} c_p i^p k^p t}dkf(y)dy$$

The Green's function is given by

$$G(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-iky+\sum_{p=0}^{n} c_p i^p k^p t}dk$$

$$G(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\sum_{p=0}^{n} c_p i^p k^p t} cos(k(x-y))dk$$

3. We solve the homogeneous case first.

$$u_t = \sigma u_{xx}$$

We assume the solution is of the form

$$u = T(t)X(x)$$

Plugging this in, we get

$$T'X = \sigma T X''$$
$$\frac{T''}{\sigma T} = \frac{X''}{X} = -\lambda^2$$
$$X'' + \lambda^2 X = 0$$

$$X = asin(\lambda X) + bcos(\lambda X)$$
$$X(0) = 0 \text{ and } X'(L) = 0$$

So, b=0 and the typical solution is of the form

$$\lambda_n = \frac{(2n-1)\pi}{2L}$$

$$x_n = sin(\lambda_n x)$$

$$T' = -\sigma \lambda_n^2 T$$

$$T_n = a_n e^{-\sigma\lambda_n^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\sigma\lambda_n^2 t} sin(\lambda_n x)$$

$$u(x,0) = \sum_{n=1}^{\infty} a_n sin(\lambda_n x) = 0$$

So, all a_n are zeros. So, we have a trivial solution. However, the non-homogeneous case will have a particular solution arising from the forcing term. Let us assume it is of the following form.

$$u_p(x,t) = \sum_{n=1}^{\infty} d_n(t) sin(\lambda_n x)$$

We assume F has a fourier representation.

$$F = \sum_{n=1}^{\infty} F_n(t) sin(\lambda_n x)$$

Here, λ_n is the same as before and the cosine terms dissappear to be in tune with the boundary conditions. Furthermore,

$$F_n(0) = 0$$

Plugging these together, we have

$$\sum_{n=1}^{\infty} d_n'(t) sin(\lambda_n x) = -\sigma\lambda^2 \sum_{n=1}^{\infty} d_n(t) sin(\lambda_n x) + \sum_{n=1}^{\infty} F_n(t) sin(\lambda_n x)$$

$$d_n' + \sigma\lambda^2 d_n = F_n$$

$$(e^{\sigma\lambda_n^2 t} d_n)' = e^{\sigma\lambda_n^2 t} F_n(t)$$

$$e^{\sigma\lambda_n^2 t} d_n = \int_0^t e^{\sigma\lambda_n^2 \tau} F_n(\tau) d\tau$$

$$d_n = e^{-\sigma\lambda_n^2 t} \int_0^t e^{\sigma\lambda_n^2 \tau} F_n(\tau) d\tau$$

$$d_n = \int_0^t e^{-\sigma\lambda^2 (t-\tau)} F_n(\tau) d\tau$$

And the solution becomes

$$u(x,t) = \sum_{n=1}^{\infty} sin(\lambda_n x) \int_0^t e^{-\sigma\lambda^2 (t-\tau)} F_n(\tau) d\tau$$

We do a verification by seeing that the boundary conditions are satisfied with this particular solution.