AMATH 561: Homework 2

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1. (a) X takes the values 1 and -1. So, the $\sigma(X)$ must contain the events decribed by those values in addition to the trivial σ-algebra.

$$\sigma(X) = \{\phi, \Omega, a \cup b, c \cup d\}$$

(b)

$$\mathbb{E}[Y|X](a) = \mathbb{E}[Y|X=1] = \frac{P(a)Y(a) + P(b)Y(b)}{P(x=1)} = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$$

$$\mathbb{E}[Y|X](b) = \mathbb{E}[Y|X=1] = \frac{P(a)Y(a) + P(b)Y(b)}{P(x=1)} = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$$

$$\mathbb{E}[Y|X](c) = \mathbb{E}[Y|X=-1] = \frac{P(c)Y(c) + P(d)Y(d)}{P(x=1)} = \frac{1}{2} - \frac{1}{2} = 0$$

$$\mathbb{E}[Y|X](d) = \mathbb{E}[Y|X=-1] = \frac{P(c)Y(c) + P(d)Y(d)}{P(x=1)} = \frac{1}{2} - \frac{1}{2} = 0$$

The partial averaging property.

For A = ϕ,

$$\mathbb{E}[\mathbb{I}_\phi \mathbb{E}[Y|X]] = \mathbb{E}[\mathbb{I}_\phi Y] = 0$$

For A = Ω,

$$\mathbb{E}[\mathbb{I}_\Omega \mathbb{E}[Y|X]] = P(a)\mathbb{E}[Y|X](a) + P(b)\mathbb{E}[Y|X](b) + 0 + 0 = -\frac{1}{18} - \frac{2}{18} = -\frac{1}{6}$$

$$\mathbb{E}[\mathbb{I}_\Omega Y] = P(a)Y(a) + P(b)Y(b) + P(c)Y(c) + P(d)Y(d) = -\frac{1}{6}$$

For A = a$\cup$b,

$$\mathbb{E}[\mathbb{I}_{a\cup b} \mathbb{E}[Y|X]] = P(a)\mathbb{E}[Y|X](a) + P(b)\mathbb{E}[Y|X](b) = -\frac{1}{18} - \frac{2}{18} = -\frac{1}{6}$$

$$\mathbb{E}[\mathbb{I}_{a\cup b} Y] = P(a)Y(a) + P(b)Y(b) = -\frac{1}{6}$$

For A = c$\cup$d,

$$\mathbb{E}[\mathbb{I}_{c\cup d} \mathbb{E}[Y|X]] = P(c)\mathbb{E}[Y|X](c) + P(d)\mathbb{E}[Y|X](d) = 0$$

$$\mathbb{E}[\mathbb{I}_{c\cup d} Y] = P(c)Y(c) + P(d)Y(d) = 0$$

(c) Z(a)= 2, Z(b)=Z(c)=0, Z(d)=-2

$$\mathbb{E}[Z|X](a) = \mathbb{E}[Z|X=1] = \frac{P(a)Z(a)+P(b)Z(b)}{P(x=1)} = \frac{2}{3}$$

$$\mathbb{E}[Z|X](b) = \mathbb{E}[Z|X=1] = \frac{P(a)Z(a)+P(b)Z(b)}{P(x=1)} = \frac{2}{3}$$

$$\mathbb{E}[Z|X](c) = \mathbb{E}[Z|X=-1] = \frac{P(c)Z(c)+P(d)Z(d)}{P(x=1)} = -1$$

$$\mathbb{E}[Z|X](d) = \mathbb{E}[Z|X=-1] = \frac{P(c)Z(c)+P(d)Z(d)}{P(x=1)} = -1$$

The partial averaging property.

For A = ϕ,

$$\mathbb{E}[\mathbb{I}_\phi \mathbb{E}[Z|X]] = \mathbb{E}[\mathbb{I}_\phi Z] = 0$$

For A = Ω,

$$\mathbb{E}[\mathbb{I}_\Omega \mathbb{E}[Z|X]] = P(a)\mathbb{E}[Z|X](a) + P(b)\mathbb{E}[Z|X](b) + P(c)\mathbb{E}[Y|X](c) + P(d)\mathbb{E}[Y|X](d) = -\frac{1}{6}$$

$$\mathbb{E}[\mathbb{I}_\Omega Z] = P(a)Z(a) + P(b)Z(b) + P(c)Z(c) + P(d)Z(d) == \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

For A = a$\cup$b,

$$\mathbb{E}[\mathbb{I}_\Omega \mathbb{E}[Z|X]] = P(a)\mathbb{E}[Z|X](a) + P(b)\mathbb{E}[Z|X](b) + P(c)\mathbb{E}[Y|X](c) + P(d)\mathbb{E}[Y|X](d) = \frac{1}{3}$$

$$\mathbb{E}[\mathbb{I}_\Omega Z] = P(a)Z(a) + P(b)Z(b) + P(c)Z(c) + P(d)Z(d) = \frac{1}{3}$$

For A = c$\cup$d,

$$\mathbb{E}[\mathbb{I}_\Omega \mathbb{E}[Z|X]] = P(c)\mathbb{E}[Y|X](c) + P(d)\mathbb{E}[Y|X](d) = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$\mathbb{E}[\mathbb{I}_\Omega Z] = P(c)Z(c) + P(d)Z(d) = -\frac{1}{2}$$

2.

$$\mathbb{V}(Y-X) = \mathbb{V}(Y - \mathbb{E}[Y|G] + \mathbb{E}[Y|G] - X)$$
$$= \mathbb{V}(Y - \mathbb{E}[Y|G]) + \mathbb{V}(X - \mathbb{E}[Y|G]) - 2Cov\mathbb{V}(Y - \mathbb{E}[Y|G], X - \mathbb{E}[Y|G])$$

Intuitively, the Cov$\mathbb{V}$ should be zero, since X-$\mathbb{E}[Y|G]$ is in G and Y-$\mathbb{E}[Y|G]$ is orthogonal to it.

$$Cov\mathbb{V}(Y - \mathbb{E}[Y|G], X - \mathbb{E}[Y|G]) = \mathbb{E}(Y - \mathbb{E}[Y|G])(X - \mathbb{E}[Y|G]) - \mathbb{E}(Y - \mathbb{E}[Y|G])\mathbb{E}(X - \mathbb{E}[Y|G])$$
$$= \mathbb{E}(YX) - \mathbb{E}[\mathbb{E}[Y|G]X] - \mathbb{E}[Y\mathbb{E}[Y|G]] + \mathbb{E}[\mathbb{E}[Y|G]\mathbb{E}[Y|G]] - \mathbb{E}Y\mathbb{E}X$$
$$-(\mathbb{E}[\mathbb{E}[Y|G]])^2 + \mathbb{E}X\mathbb{E}[\mathbb{E}[Y|G]] + \mathbb{E}Y\mathbb{E}[\mathbb{E}[Y|G]]$$
$$= \mathbb{E}(YX) - \mathbb{E}[X\mathbb{E}[Y|G]] - \mathbb{E}[Y\mathbb{E}[Y|G]|G] + \mathbb{E}[Y|G]\mathbb{E}[Y|G] - \mathbb{E}Y\mathbb{E}X$$
$$-(\mathbb{E}Y)^2 + \mathbb{E}X\mathbb{E}Y + (\mathbb{E}Y)^2$$
$$= \mathbb{E}(YX) - \mathbb{E}[XY] - \mathbb{E}[Y|G]\mathbb{E}[Y|G] + \mathbb{E}[Y|G]\mathbb{E}[Y|G] = 0$$

Thus,

$$\mathbb{V}(Y-X) = \mathbb{V}(Y - \mathbb{E}[Y|G]) + \mathbb{V}(X - \mathbb{E}[Y|G]) \geq \mathbb{V}(Y - \mathbb{E}[Y|G])$$

3.
$$\Omega = \{a, b, c, d\}$$
$$X = \{1, 2, 3, 4\}$$

$$f(X) = \begin{cases} 1 & \text{if } x > 2 \\ \\ 0 & \text{if } x \leq 2 \end{cases}$$

$$\sigma(f(X)) = \{\phi, \Omega, a \cup b, c \cup d\}$$

This is clearly, strictly smaller than $\sigma(X)$ since it doesn't have terms like $a \cup c$

If g is a constant function, $\sigma(g(X)) = \{\phi, \Omega\}$ since the random variable produces the same value for every event (apart from the null event).

4.
$$X_n = \mathbb{E}[X|F_n]$$
$$\mathbb{E}[X_n|F_s] = \mathbb{E}[\mathbb{E}[X|F_n]|F_s] = \mathbb{E}[X|F_s] = X_s$$

Thus, this is a martingale.

5. For convenience, let
$$q = 1 - p$$
$$Z_{n+1} = \left(\frac{q}{p}\right)^{2S_{n+1}-(n+1)} = \left(\frac{q}{p}\right)^{2S_n+2X_{n+1}-n-1} = Z_n\left(\frac{q}{p}\right)^{2X_{n+1}-1}$$
$$\mathbb{E}[Z_{n+1}|F_n] = \mathbb{E}[Z_n\left(\frac{q}{p}\right)^{2X_{n+1}-1}|F_n]$$

Since Z_n is known from F_n,

$$\mathbb{E}[Z_{n+1}|F_n] = \mathbb{E}[Z_n\left(\frac{q}{p}\right)^{2X_{n+1}-1}|F_n] = Z_n\mathbb{E}[\left(\frac{q}{p}\right)^{2X_{n+1}-1}|F_n]$$

$$= Z_n(q\left(\frac{p}{q}\right) + p\left(\frac{q}{p}\right))$$
$$= Z_n(p + q) = Z_n$$

Similarly, we can show

$$\mathbb{E}[Z_{n+m}|F_n] = \mathbb{E}[\mathbb{E}[Z_{n+m}|F_{n+m-1}]|F_n]$$
$$= \mathbb{E}[Z_{n+m-1}|F_n]$$
$$= \ldots\ldots\ldots\ldots\ldots\ldots$$
$$= \mathbb{E}[Z_{n+1}|F_n] = Z_n$$

Thus, Z_n is a martingale with respect to this filtration.