AMATH 562: Homework 6

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1. (a) Poisson process is the special case of compound poisson process. Here, the jump can only be of size one.

$$v(U) = \mathbb{E}N(1,U) = \mathbb{E}\sum_{0\le s\le 1} \mathbb{1}_{\Delta P\in U}$$

$$= \mathbb{E}\sum_{1}^{P_1} \mathbb{1}_{1\in U}$$

$$= \mathbb{E}\mathbb{E}[\sum_{1}^{P_1} \mathbb{1}_{1\in U}|P_1]$$

$$= \mathbb{E}[\sum_{1}^{P_1} \mathbb{E}[\mathbb{1}_{1\in U}|P_1]]$$

$$= \mathbb{E}[P_1\mathbb{E}[\mathbb{1}_{1\in U}|P_1]]$$

$$= \mathbb{E}[P_1\delta_1(U)]$$

$$= \delta_1(U)\mathbb{E}[P_1]$$

$$= \delta_1(U)\lambda$$

(b)

$$dX_t = dP_t$$

If X_t =x,

$$X_T = P_T - P_t + x$$

$$u(t,x) = \mathbb{E}[\phi(P_T - P_t + x)|X_t = x]$$

$$= \sum_n \phi(n+x)\mathbb{P}(P_T - P_t = n)$$

$$= \sum_n \phi(n+x)\mathbb{P}(P_{T-t} = n)$$

$$= \sum_n \phi(n+x)\frac{(\lambda(T-t))^n}{n!}e^{-\lambda(T-t)}$$

$$\mathbf{A}(t)\phi(x) = \lim_{s\downarrow t}\frac{P(s,t)\phi(x) - \phi(x)}{s-t}$$

$$= \lim_{s\downarrow t}\frac{\mathbb{E}[\phi(X_s)|X_t = x] - \phi(x)}{s-t}$$

As t goes to s, there can either be a jump of size one or no jump.

$$\mathbf{A}(t)\phi(x) = \lim_{s\downarrow t} \frac{P(s,t)\phi(x) - \phi(x)}{s-t}$$

$$= \lim_{s\downarrow t} \frac{\mathbb{E}[\phi(X_s)|X_t = x] - \phi(x)}{s-t}$$

$$= \lim_{dt\to 0} \frac{\phi(x+1)(\lambda dt e^{-\lambda dt}) + \phi(x)(e^{-\lambda dt}) - \phi(x)}{dt}$$

$$= \lim_{dt\to 0} \frac{\phi(x+1)(\lambda dt(1-\lambda dt+\ldots)) + \phi(x)(1-\lambda dt+\ldots) - \phi(x)}{dt}$$

$$= \phi(x+1)\lambda - \lambda\phi(x)$$

$$\partial_t u(t,x) = \sum_n \lambda\phi(n+x)\frac{(\lambda(T-t))^n}{n!}e^{-(T-t)} + \lambda\phi(n+x+1)\frac{(\lambda(T-t))^n}{n!}e^{-(T-t)}$$

$$= \lambda u(t,x) - \lambda u(t,x+1)$$

$$= -\mathbb{A}(t,x)u(t,x)$$

Thus, it satisfies the KBE.

2. (a)

$$Z_t = \frac{X_t}{Y_t}$$

This looks like a great place to use Ito's multi-dimensional formula.

$$f_{X_t} = \frac{1}{Y_t}, f_{Y_t} = -\frac{X_t}{Y_t^2}$$

$$f_{XX} = 0, f_{YY} = \frac{2X_t}{Y_t^3}, f_{XY} = f_{YX} = -\frac{1}{Y_t^2}$$

$$f(X_{t-}+(e^{\gamma_t(z)}-1)X_{t-}, Y_{t-}+(e^{g_t(z)}-1)Y_{t-}) = \frac{X_{t-}+(e^{\gamma_t(z)}-1)X_{t-}}{Y_{t-}+(e^{g_t(z)}-1)Y_{t-}} = Z_{t-}e^{\gamma_t(z)-g_t(z)}$$

$$\begin{aligned} dZ_t = (\mu_t\frac{X_t}{Y_t} - b_t\frac{X_t}{Y_t} - 2\frac{1}{2}\sigma_t a_t\frac{X_t}{Y_t} + a_t^2\frac{X_t}{Y_t})dt \\ +(\sigma_t\frac{X_t}{Y_t} - a_t\frac{X_t}{Y_t})dW_t \\ +\int_R Z_{t-}e^{\gamma_t(z)-g_t(z)}\widetilde{N}(dt,dz) \\ +\int_R (Z_{t-}e^{\gamma_t(z)-g_t(z)} - (e^{\gamma_t(z)}-1)Z_{t-} + (e^{g_t(z)}-1)Z_{t-})v(dz)dt \end{aligned}$$

$$\begin{aligned} dZ_t = (\mu_t - b_t - \sigma_t a_t + a_t^2)Z_t dt + (\sigma_t - a_t)Z_t dW_t \\ +\int_R Z_{t-}e^{\gamma_t(z)-g_t(z)}\widetilde{N}(dt,dz) \\ +\int_R (e^{\gamma_t(z)-g_t(z)} + (e^{g_t(z)} - e^{\gamma_t(z)}))v(dz)Z_{t-}dt \end{aligned}$$

(b) For Z_t to be a martingale, it should have no drift Thus

$$\mu_t = b_t + \sigma_t a_t + a_t^2 - \frac{\int_R (e^{\gamma_t(z) - g_t(z)} + (e^{g_t(z)} - e^{\gamma_t(z)}))v(dz)Z_{t-}}{Z_t}$$

3. (a)

$$dX_t = \kappa(\theta - X_t)dt + d\eta_t$$
$$Y_t = X_t - \theta$$
$$dY_t = -\kappa Y_t dt + d\eta_t$$
$$f = e^{kt}Y_t$$
$$df = ke^t Y_t + e^{kt}dY_t$$
$$df = ke^t Y_t - \kappa e^{kt} Y_t + e^{kt} d\eta_t$$
$$df = e^{kt} d\eta_t$$
$$f_t = f_0 + \int_0^t e^{ks} d\eta_s$$
$$Y_t = e^{-kt} f_0 + \int_0^t e^{-k(t-s)} d\eta_s$$
$$Y_t = e^{-kt} Y_0 + \int_0^t e^{-k(t-s)} d\eta_s$$
$$X_t = \theta + e^{-kt}(X_0 - \theta) + \int_0^t e^{-k(t-s)} d\eta_s$$
$$X_t = \theta + e^{-kt}(X_0 - \theta) + \int_0^t e^{-k(t-s)} \sigma dW_s + + \int_0^t \int_R e^{-k(t-s)} z \widetilde{N}(ds, dz)$$

(b)

$$m(t) = \mathbb{E}[X_t] = \theta + e^{-kt}(X_0 - \theta)$$

Here, we assume the X_0 is a constant and a random variable.

$$\mathbb{E}[(X_s - m(s))(X_t - m(t))] = \mathbb{E}[(X_s - m(s))(X_t - m(t))]$$

$$Z_s = e^{ks}(X_s - m(s)) = \int_0^t e^{kr} \sigma dW_r + + \int_0^t \int_R e^{kr} z \widetilde{N}(dr, dz)$$

So, this process has no drift and we can do something similar to Ito Isometry.

$$f(Z_s, Z_t) = Z_s Z_t$$

If s $<$ t,I(t) = g(I(s)) and vice versa. Thus we can use Ito's multidimensional formula.

$$\partial_{Z_s} f(Z_s, Z_t) = Z_t$$

$$\partial_{Z_s} \partial_{Z_s} f(Z_s, Z_t) = 0$$
$$\partial_{Z_t} \partial_{Z_s} f(Z_s, Z_t) = 1$$
$$f(Z_{s-} + \gamma_s, Z_{t-} + \gamma_t) - f(Z_{s-}, Z_{t-}) - \gamma . \nabla f(Z_s, Z_t)$$

$$= Z_{s-}Z_{t-} + \gamma^2 + \gamma Z_{t-} + \gamma Z_{s-} - Z_{s-}Z_{t-} - \gamma Z_{t-} + -\gamma Z_{s-}$$
$$= \gamma^2 = e^{2kr} z^2$$

$$\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{eff}\sigma_{eff}^T(i,j)\partial_{Z_i}\partial_{Z_j} f(Z_s, Z_t) = \sigma_{eff}^2 = e^{2kr}\sigma^2$$

$$df(Z_s, Z_t) = e^{2kr}\sigma^2 dr + \int_{\mathbb{R}} e^{2kr} z^2 v(dz)dr + \int_{\mathbb{R}} (\dots)\widetilde{N}(dt, dz) + (\dots)dW_t$$
$$d(Z_s Z_t) = e^{2kr}\sigma^2 dr + \int_{\mathbb{R}} e^{2kr} z^2 v(dz)dr + \int_{\mathbb{R}} (\dots)\widetilde{N}(dt, dz) + (\dots)dW_t$$
$$\mathbb{E}[Z_s Z_t] = \mathbb{E}\int_0^T e^{2kr}\sigma^2 dr + \int_0^T \int_{\mathbb{R}} e^{2kr} z^2 v(dz)dr$$
$$= \frac{(e^{2kT}-1)\sigma^2}{2k} + \frac{(e^{2kT}-1)}{2k}\int_{\mathbb{R}} z^2 v(dz)$$
$$\mathbb{E}[Z_s Z_t] = \mathbb{E}[e^{k(t+s)}(X_s - m(s))(X_t - m(t))]$$
$$c(t,s) = \mathbb{E}[(X_s - m(s))(X_t - m(t))] = \mathbb{E}[e^{-k(t+s)}Z_s Z_t] = e^{-k(t+s)}\mathbb{E}[Z_s Z_t]$$
$$= \frac{e^{-k(t+s)}(e^{2kT}-1)\sigma^2}{2k} + \frac{e^{-k(t+s)}(e^{2kT}-1)}{2k}\int_{\mathbb{R}} z^2 v(dz)$$

4.

$$dX_t = \mu dt + \sigma dW_t + \int_{\mathbb{R}} \gamma \widetilde{N}(t, dz)$$
$$d\phi(X_t) = (\mu_t\phi'(X_t) + \frac{1}{2}\sigma^2\phi''(X_t))dt + \int_{\mathbb{R}} (\phi(X_{t-} + \gamma_t(z)) - \phi(X_{t-}) - \gamma_t(z)\phi'(X_t))v(dz)dt$$
$$+ \int_{\mathbb{R}} (\dots)\widetilde{N}(dt, dz) + (\dots)dW_t$$
$$\phi(X_s) = \phi(x) + \int_t^s (\mu_r\phi'(X_r) + \frac{1}{2}\sigma^2\phi''(X_r))dr + \int_t^s \int_{\mathbb{R}} (\phi(X_{r-} + \gamma_t(z)) - \phi(X_{r-}) - \gamma_r(z)\phi'(X_r))v(dz)dr$$
$$+ \int_t^s \int_{\mathbb{R}} (\dots)\widetilde{N}(dr, dz) + \int_t^s (\dots)dW_r$$
$$\mathbb{E}[\phi(X_s)|X_t = x] = \phi(x) + \int_t^s \mathbb{E}[(\mu_r\phi'(X_r) + \frac{1}{2}\sigma^2\phi''(X_r))dr|X_t = x]$$
$$+ \int_t^s \mathbb{E}[\int_{\mathbb{R}} (\phi(X_{r-} + \gamma_t(z)) - \phi(X_{r-}) - \gamma_r(z)\phi'(X_r))v(dz)dr|X_t = x]$$
$$+0 + 0$$
$$\mathbb{E}[\phi(X_s)|X_t = x] - \phi(x) = \int_t^s \mathbb{E}[(\mu_r\phi'(X_r) + \frac{1}{2}\sigma^2\phi''(X_r))dr|X_t = x]$$
$$+ \int_t^s \mathbb{E}[\int_{\mathbb{R}} (\phi(X_{r-} + \gamma_t(z)) - \phi(X_{r-}) - \gamma_r(z)\phi'(X_r))v(dz)dr|X_t = x]$$
$$\lim_{s\downarrow t} \frac{1}{s-t}\Big(\mathbb{E}[\phi(X_s)|X_t = x] - \phi(x)\Big) = \lim_{s\downarrow t}\frac{1}{s-t}\int_t^s \mathbb{E}[(\mu_r\phi'(X_r) + \frac{1}{2}\sigma^2\phi''(X_r))dr|X_t = x]$$

$$+\lim_{s\downarrow t}\frac{1}{s-t}\int_t^s \mathbb{E}[\int_{\mathbb{R}}(\phi(X_{r^-}+\gamma_r(z))-\phi(X_{r^-})-\gamma_r(z)\phi'(X_r))v(dz)dr|X_t=x]$$

Thus,

$$\mathbf{A}(t)\phi(x)=$$

$$=\mathbb{E}[(\mu_r\phi'(X_t)+\frac{1}{2}\sigma^2\phi''(X_t))|X_t=x]$$

$$+\mathbb{E}[\int_{\mathbb{R}}(\phi(X_{t^-}+\gamma_t(z))-\phi(X_{t^-})-\gamma_t(z)\phi'(X_t))v(dz)|X_t=x]$$

$$=(\mu_t\phi'(x)+\frac{1}{2}\sigma^2\phi''(x))+\int_{\mathbb{R}}(\phi(x+\gamma_t(z))-\phi(x)-\gamma_t(z)\phi'(x))v(dz)$$