AMATH 561: Homework 5

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1. Since the inter-treatment time for patients is exponential with parameter μ. The number of patients treated would be a Poisson process with parameter μ. Thus, we get the following generator

$$\begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ \mu & -(\mu+\lambda) & \lambda & 0 & \dots \\ 0 & \mu & -(\mu+\lambda) & \lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

To find the invariant distribution,

$$\pi G = 0$$

Let π be

$$[x_0, x_1, x_2, x_3, \dots]$$

$$-\lambda x_0 + \mu x_1 = 0$$

$$x_1 = \frac{\lambda}{\mu} x_0$$

$$\lambda x_{n-1} - (\mu+\lambda)x_n + \mu x_{n+1} = 0$$

$$x_2 = \frac{\lambda}{\mu} x_1$$

Thus,

$$x_n = \left(\frac{\lambda}{\mu}\right)^n x_0$$

We can only normalize and get an invariant distribution if the x_ns are finite. So, λ has to be less than μ

To normalize,

$$\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n x_0 = 1$$

$$x_0 = \frac{\mu - \lambda}{\mu}$$

Expected waiting time = (probability of n patients being present x Expected time of waiting for n patients and treating yourself) for all n =

$$= \sum_{n}^{\infty} x_0 \left(\frac{\lambda}{\mu}\right)^n \frac{(n+1)}{\mu}$$

$$= \frac{x_0 \mu}{(\mu-\lambda)^2} = \frac{1}{\mu - \lambda}$$

2. Y is the discrete time process and T_n is the nth sampled time. X is the continous markov chain.

$$\begin{aligned} P[Y_{n+1}|Y_n, Y_{n-1}, Y_{n-2}, ...] &= P[X_{T_{n+1}}|X_{T_n}, X_{T_{n-1}}, X_{T_{n-2}}, ...] \\ &= P[X_{T_{n+1}}|X_{T_n}] \text{ (X is markov)} \\ &= P[Y_{n+1}|Y_n] \end{aligned}$$

Thus, Y is Markov and a discrete time Markov chain with a transition matrix P.

If X is homogeneous with fixed semi-group $\mathbb{P}$ and the times are i.i.d,

$$P[X_{T_{n+1}}|X_{T_n}] = P[X_{T_1}|X_{T_0}]$$

$$P[Y_{n+1}|Y_n] = P[Y_1|Y_0]$$

Thus, Y is homogeneous with transition matrix P. This P corresponds to $\mathbb{P}_\tau$ where τ is any of $\tau_1, \tau_2, ...$ We know

$$\pi_x \mathbb{P}_t = \pi_x$$

Setting t = τ,

$$\pi_x \mathbb{P}_\tau = \pi_x$$

$$\pi_x P = \pi_x$$

Thus, π_x is the invariant distribtuion corresponding to P and thus the invariant distribution for Y.

Alternate solution : If X was reversible, we could use detail balance and use the approach in the Metropolis-Hastings algorithm to show that the invariant distribution of Y is the same as that of X. If π is the invariant distribution for X, there exists some sort of detail balance. Let there be two states X_1 and X_2.

$$p(X_1|X_2)\pi(X_2) = p(X_2|X_1)\pi(X_1)$$

$$\begin{aligned} p(Y_1|Y_2) &= p(X_{1_{t_n}}|X_{2_{t_{n-1}}})p(t_n|t_{n-1}) \\ p(Y_2|Y_1) &= p(X_{2_{t_n}}|X_{1_{t_{n-1}}})p(t_n|t_{n-1}) \\ \frac{p(Y_1|Y_2)}{p(Y_2|Y_1)} &= \frac{p(X_1|X_2)}{p(X_2|x_1)} = \frac{\pi(X_1)}{\pi(X_2)} \\ p(Y_1|X_2)\pi(X_2) &= p(Y_2|Y_1)\pi(X_1) \end{aligned}$$

Thus, we get the same invariant distribution for Y.

3.

$$\begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ \mu & -(\mu+\lambda) & \lambda & 0 & \dots \\ 0 & 2\mu & -(2\mu+\lambda) & \lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

$$\frac{dP(i,j)}{dt} = p(i,j-1)g(j-1,j) + p(i,j)g(j,j) + p(i,j+1)g(j+1,j)$$

$$= \lambda p(i,j-1) - \lambda p(i,j) - j\mu p(i,j) + (j+1)\mu p(i,j+1)$$

Multiplying with s^j, we obtain the generating function conditioned that x_0 is j,

$$\frac{\partial G}{\partial t} = \lambda s G - \lambda G - \mu s \frac{\partial G}{\partial s} + \mu \frac{\partial G}{\partial s}$$

$$G(s,0) = s^j$$

Using Mathematica,

$$G(s,t) = (se^{-\mu t} + 1 - e^{-\mu t})^j e^{\frac{\lambda(s-1)(1-e^{-\mu t})}{\mu}}$$

$$G(s,\infty) = e^{\frac{\lambda}{\mu}(s-1)}$$

Thus, from the generating function, we get that the limiting distribution is Poisson with parameter $\frac{\lambda}{\mu}$

4. Kolmogorov Forward Equation

$$\frac{dp(i,j)}{dt} = p(i,j-1)g(j-1,j) + p(i,j)g(j,j)$$

$$= \lambda p(i,j-1) - \lambda p(i,j)$$

$$\frac{\partial G}{\partial t} = \lambda(t) s G - \lambda(t) G$$

$$G = Ce^{\int_0^t \lambda(t)(s-1)dt}$$

At t=0,

$$G = s^{N_0} = s^0 = 1$$

Thus,

$$G = e^{\int_0^t \lambda(t)(s-1)dt}$$

$$G = e^{(s-1)\int_0^t \lambda(t)dt}$$

This is a Poisson process with λ as $\int_0^t \lambda(t)dt$

$$p_t(0,0) = e^{-\int_0^t \lambda(t)dt}$$

Kolmogorov Backward Equation

$$\frac{dp(i,j)}{dt} = g(i,i+1)p(i+1,j) + g(i,i)p(i,j)$$

Since this is a Poisson process

$$p(i+1,j) = p(i,j-1)$$

$$\frac{dp(i,j)}{dt} = g(i,i+1)p(i,j-1) + g(i,i)p(i,j)$$

$$\frac{dp(i,j)}{dt} = \lambda p(i,j-1) - \lambda p(i,j)$$

$$\frac{\partial G}{\partial t} = \lambda(t) s G - \lambda(t) G$$

which is the same pde as before. Again, we see this is a Poisson process with λ as $\int_0^t \lambda(t)dt$

When

$$\lambda(t) = \frac{c}{1+t}$$

$$G = e^{s-1}(1+t)^c$$

$$p(\tau_1 > t) = p_t(0,0) = (1+t)^{-c}$$

$$p(\tau_1 < t) = 1 - p(\tau_1 > t) = 1 - (1+t)^{-c}$$

$$p(\tau_1 = t) = c(1+t)^{-c-1}$$

$$\mathbb{E}(\tau_1) = \int_0^\infty tc(1+t)^{-c-1}dt$$

$$= \left.\frac{t}{(1+t)^c}\right|_0^\infty + \int_0^\infty \frac{1}{(1+t)^c}dt$$

The first term is only finite for c > 1.

Thus, $\mathbb{E}(\tau_1) < \infty$ only when c > 1.

5.

$$G_{N_t}(s) = \mathbb{E}[s^{N_t}]$$

$$= \mathbb{E}[\mathbb{E}[s^{N_t}|\lambda]]$$

$$= p\mathbb{E}[s^{N_t}|\lambda_1] + (1-p)p\mathbb{E}[s^{N_t}|\lambda_2]$$

$$= pe^{\lambda_1 t(s-1)} + (1-p)e^{\lambda_2 t(s-1)}$$

The mean is given by

$$\left.\frac{\partial G(s)}{\partial s}\right|_{s=1} = p\lambda_1 t + (1-p)\lambda_2 t$$

The variance is given by

$$\left.\frac{\partial^2 G(s)}{\partial^2 s}\right|_{s=1} - \left(\left.\frac{\partial G(s)}{\partial s}\right|_{s=1}\right)^2 + \left.\frac{\partial G(s)}{\partial s}\right|_{s=1}$$

$$= p\lambda_1^2 t^2 + (1-p)\lambda_2^2 t^2 - p^2\lambda_1^2 t^2 - (1-p)^2\lambda_2^2 t^2 - 2p(1-p)\lambda_1\lambda_2 t^2 + p\lambda_1 t + (1-p)\lambda_2 t$$

6.

$$\begin{bmatrix} -\mu & \mu & 0 & 0 \\ \lambda & -(\mu+\lambda) & \mu & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \lambda & -(\mu+\lambda) & \mu \\ 0 & 0 & \lambda & -\lambda \end{bmatrix}$$

Let π be

$$[\pi_0, \pi_1, \pi_2, \pi_3, \ldots]$$

$$-\mu\pi_0 + \lambda\pi_1 = 0$$

$$\pi_1 = \frac{\mu}{\lambda}\pi_0$$

$$\mu\pi_0 - (\mu+\lambda)\pi_1 + \lambda\pi_2 = 0$$

$$\pi_2 = \frac{\mu}{\lambda}\pi_1$$

$$\vdots$$

$$\pi_n = \frac{\mu}{\lambda}\pi_{n-1}$$

Thus,

$$\pi_n = \left(\frac{\mu}{\lambda}\right)^n \pi_0$$

We can only normalize and get an invariant distribution if the π_ns are finite. So, λ has to be greater than μ

To normalize,

$$\sum_{n=0}^{\infty} \left(\frac{\mu}{\lambda}\right)^n \pi_0 = 1$$

$$\pi_0 = \frac{\lambda - \mu}{\lambda}$$

(a) Expected number of machinces not in use =

$$= \sum_{k=0}^{M} (M-k)\pi(k)$$

$$= \sum_{k=0}^{M} M\pi(k) - \sum_{k=0}^{M} k\pi(k)$$

$$= M - \sum_{k=0}^{M} k\pi(k)$$

$$= M - \sum_{k=0}^{M} k\left(\frac{\mu}{\lambda}\right)^k \pi_0$$

The term on the right is an arithmetic geometric sequence.

$$= M - \left(\lambda\frac{M(\frac{\mu}{\lambda})^{m+1}}{\lambda-\mu} + \lambda^2\frac{\frac{\mu}{\lambda} - (\frac{\mu}{\lambda})^M}{(\lambda-\mu)^2}\right)\frac{\lambda-\mu}{\lambda}$$

$$= M - M\left(\frac{\mu}{\lambda}\right)^{m+1} - \frac{\mu - \frac{\mu^M}{\lambda^{M-1}}}{(\lambda-\mu)}$$

(b) Probability that a given machine (M_x) is in use = (Probability of n machines in use * Probability that M_x is one of them) for all n

$$= \sum_{n=0}^{M} \frac{n}{M}\pi(n)$$

$$= \frac{1}{M}\sum_{n=0}^{M} n\pi(n)$$

$$= \left(\frac{\mu}{\lambda}\right)^{m+1} + \frac{\mu - \frac{\mu^M}{\lambda^{M-1}}}{M(\lambda-\mu)}$$

(c)