AMATH 535: Homework Problem 4.3

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Solution:

$$N_{t+1} = \frac{R_0^\beta N_t}{[1+((R_0-1)/K)N_t]^\beta}$$

At equilibrium, $N_{t+1} = N_t$

We see that $N_t = 0$ is a solution.

$$N_t = \frac{R_0^\beta N_t}{[1+((R_0-1)/K)N_t]^\beta}$$

$$[1+((R_0-1)/K)N_t]^\beta = R_0^\beta$$

$$((R_0-1)/K)N_t = R_0 - 1$$

$$N_t^* = K$$

Now, to find the stability, we need the slope.

$$f'(N_t) = \frac{R_0^\beta + [(R_0-1)/K]N_t R_0^\beta - \beta((R_0-1)/K)N_t R_0^\beta}{[1+((R_0-1)/K)N_t]^{\beta+1}}$$

$$= \frac{R_0^\beta + (1-\beta)((R_0-1)/K)N_t R_0^\beta}{[1+((R_0-1)/K)N_t]^{\beta+1}}$$

When N_t^* =0,

$$f'(N_t) = R_0^\beta$$

Since $f'(N_t) < 1$ to be stable, the fixed point at 0 is stable for $R_0 < 1$.

For N_t^* =K,

$$f'(N_t) = \frac{R_0^\beta + (1-\beta)(R_0-1)R_0^\beta}{R_0^{\beta+1}}$$

$$= \frac{1+(1-\beta)(R_0-1)}{R_0}$$

$$= 1 - \beta + \frac{\beta}{R_0}$$

For this to be stable,

$$|1 - \beta + \frac{\beta}{R_0}| < 1$$

$$1 - \beta + \frac{\beta}{R_0} < 1 \text{ and } 1 - \beta + \frac{\beta}{R_0} > -1$$

$$\beta > \frac{\beta}{R_0} \text{ and } \beta - \frac{\beta}{R_0} < 2$$

Since both β and R_0 are known to be greater than 0,

$$R_0 > 1 \text{ and } \beta < \frac{2}{(1 - \frac{1}{R_0})}$$

$$f'(N_t) = 1 - \beta\big(1 - \frac{1}{R_0}\big)$$

For a monotonic damping,

$$\beta\big(1 - \frac{1}{R_0}\big) < 1)$$

$$\beta < \frac{1}{(1 - \frac{1}{R_0})}$$

For an oscillatory damping,

$$1 < \beta\big(1 - \frac{1}{R_0}\big) < 2)$$

$$\frac{1}{(1 - \frac{1}{R_0})} < \beta < \frac{2}{(1 - \frac{1}{R_0})}$$

Here are my plots and python equivalents since my talents in sketching are not so great.

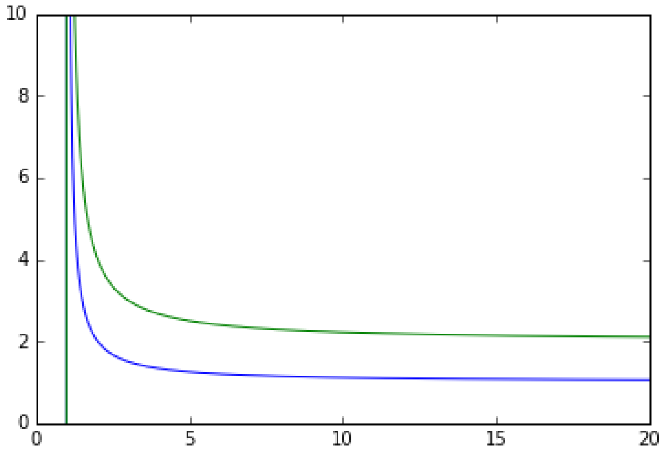

Figure 1: Python plots and hand-drawn curves