Assignment 3.

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Due Monday, Jan. 29.

1. (nonlinear pendulum)

   (a) Write a program to solve the boundary value problem for the nonlinear pendulum as discussed in the text. See if you can find yet another solution for the boundary conditions illustrated in Figures 2.4 and 2.5.

   Solution:

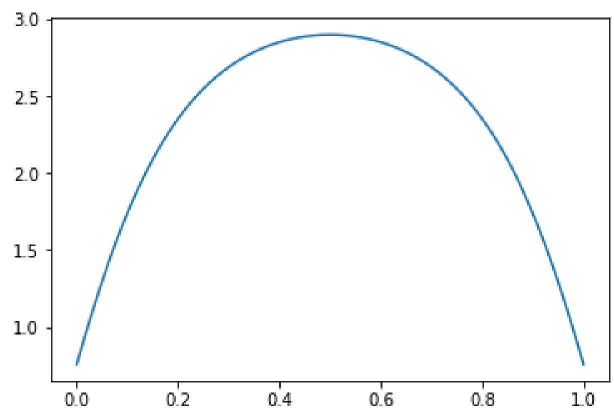


Figure 1: Initial condition of 0.7cos(t)+20sin(t)

```
def tridiag(a, b, c, k1=-1, k2=0, k3=1):
    return np.diag(a, k1) + np.diag(b, k2) + np.diag(c, k3)

def mat(thetavec):
    N = len(thetavec)
    h =2*np.pi/N
    a1 = np.append([0.7],thetavec[:-1])
    a2 = np.append(thetavec[1:],[0.7])
    b = -2*thetavec+ a1+a2 + (h**2)* np.sin(thetavec)
    return b


def jacobian(thetavec):
    N = len(thetavec)
    h =2*np.pi/N
    a = np.ones(N-1)
    b = -2  + (h**2)* np.cos(thetavec)
    return tridiag(a,b,a)
```

```
N=200
x = np.linspace(0,1,N)
thetavec = 0.7*np.cos(x)+2*np.sin(x)
for i in range(40):
    b = -mat(thetavec)
    A = jacobian(thetavec)
    delta = np.linalg.solve(A,b)
    thetavec += delta

plt.plot(x,thetavec)
```

(b) Find a numerical solution to this BVP with the same general behavior as seen in Figure 2.5 for the case of a longer time interval, say, $T = 20$, again with $\alpha = \beta = 0.7$. Try larger values of T. What does $\max_i \theta_i$ approach as T is increased? Note that for large T this solution exhibits "boundary layers".

Solution:

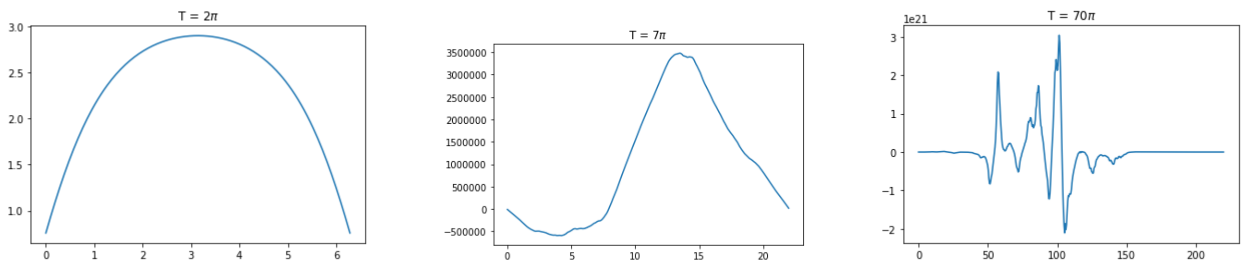


Figure 2: Solutions at t = 2π,7π and 70π

As T increases, maximum value of theta goes to infinity.

2. (Gerschgorin's theorem and stability) Consider the boundary value problem

$$-u_{xx} + (1 + x^2)u = f, \quad 0 \le x \le 1,$$

$$u(0) = 0, \quad u(1) = 0.$$

On a uniform grid with spacing $h = 1/(m+1)$, the following set of difference equations has local truncation error $O(h^2)$:

$$\frac{2u_i - u_{i+1} - u_{i-1}}{h^2} + (1 + x_i^2)u_i = f(x_i), \quad i = 1, \ldots, m.$$

(a) Use Gerschgorin's theorem to determine upper and lower bounds on the eigenvalues of the coefficient matrix for this set of difference equations.

Solution:

The global error is related to the local error by

$$Ae = \tau$$

Where A is

$$\frac{1}{h^2}\begin{pmatrix} 2+(1+x_0^2)h^2 & -1 & & & & \\ -1 & 2+(1+x_i^2)h^2 & -1 & & & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2+(1+x_i^2)h^2 & -1 \\ & & & & -1 & 2+(1+x_n^2)h^2 \end{pmatrix}$$

By Gerschgorin's theorem,

$$|\lambda - (2 + (1 + x_i^2)h^2)| \leq 2$$

$$-2 \leq \lambda - (2 + (1 + x_i^2)h^2) \leq 2$$
$$(1 + x_i^2)h^2 \leq \lambda \leq 4 + (1 + x_i^2)h^2$$
$$h^2 \leq \lambda \leq 4 + 2h^2$$
$$1 \leq \frac{\lambda}{h^2} \leq \frac{4}{h^2} + 2$$

Thus, the eigenvalues of the coefficient matrix lie in 1 and $\frac{4}{h^2} + 2$

(b) Show that the L_2-norm of the *global error* is of the same order as the local truncation error.

Solution:

$$e = A^{-1}\tau$$

$$||e||_2 = ||A^{-1}\tau||_2 \leq ||A^{-1}||_2||\tau||_2 \leq \lambda(A^{-1})_{max}||\tau||_2$$

Since

$$1 \leq \lambda(A) \leq \frac{4}{h^2} + 2$$

$$\frac{h^2}{4 + 2h^2} \leq \lambda(A^{-1}) \leq 1$$

Thus,

$$||e||_2 \leq ||\tau||_2$$

3. (Richardson extrapolation)

   (a) Use your code from problem 6 in assignment 1, or download the code from the course web page to do the following exercise. Run the code with $h = .1$ (10 subintervals) and with $h = .05$ (20 subintervals) and apply Richardson extrapolation to obtain more accurate solution values on the coarser grid. Record the L_2-norm or the ∞-norm of the error in the approximation obtained with each h value and in that obtained with extrapolation.

   Solution:

   Using the infinity norm.

h	Error
0.1	0.00222556
0.05	0.0005504

   Using Richardson's extrapolation, we get the error = 7.9865541829916443e-06 which is less than our previous errors.

   (b) Suppose you assume that the coarse grid approximation is piecewise linear, so that the approximation at the midpoint of each subinterval is the average of the values at the two endpoints. Can one use Richardson extrapolation with the fine grid approximation and these interpolated values on the coarse grid to obtain a more accurate approximation at these points? Explain why or why not?

   Solution:

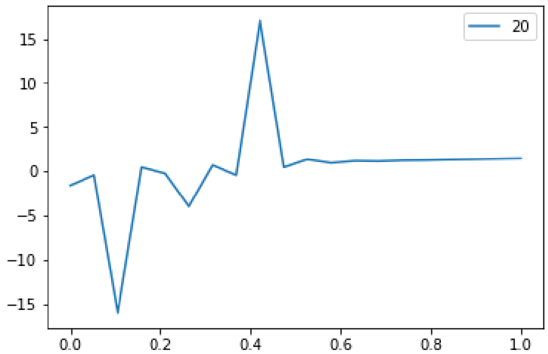
```
Y1 = prob_6([rho,f,u],x,1/10)
Y2 = prob_6([rho,f,u],x,1/20)

def interplot(Y):
    j=[Y[0]]
    for i in range(1,len(Y)):
        j.append(0.5*(Y[i]+Y[i-1]))
        j.append(Y[i])
    return j
Ynew = interplot(Y1)
rich2 =( 4*Y2-Ynew)/3
eee=U-rich2
error+=[np.linalg.norm(eee,np.inf)]
```

   The error obtained using Richardson's extrapolation in this way is 0.00083991977137813927. This is worse than the error when we used h = 0.05. The reason is that the solution is $(1-x)^2$ which is quadratic. So, using a piecewise linear interpolation would create errors.

4. Write down the Jacobian matrix associated with Example 2.2 and the nonlinear difference equations (2.106) on p. 49. Write a code to solve these difference equations when $a = 0$, $b = 1$, $\alpha = -1$, $\beta = 1.5$, and $\epsilon = 0.01$. Use an initial guess of the sort suggested in the text. Try, say, $h = 1/20$, $h = 1/40$, $h = 1/80$, and $h = 1/160$, and turn in a plot of your results.

Solution:



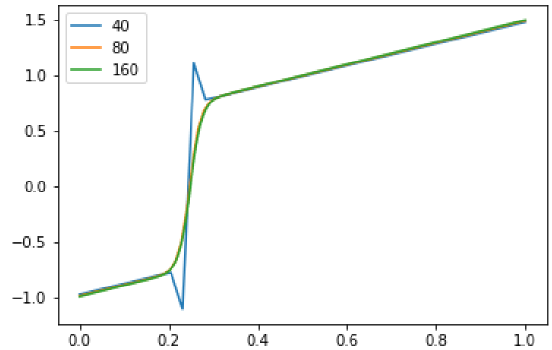


Figure 3: Boundary layers at N = 20 and N= 40, 80 and 160

```
def tridiag(a, b, c, k1=-1, k2=0, k3=1):
    return np.diag(a, k1) + np.diag(b, k2) + np.diag(c, k3)

def mat2(thetavec,h,e=0.01):
    a1 = np.append([-1],thetavec[:-1])
    a2 = np.append(thetavec[1:],[1.5])
    b = e*(-2*thetavec+ a1+a2) + thetavec*(0.5*h*(a2-a1)-h**2)
    return b

def jacobian2(thetavec,h,e=0.01):
    a1 = np.append([-1],thetavec[:-1])
    a2 = np.append(thetavec[1:],[1.5])
    b = -2*e  + (0.5*h*(a2-a1)-h**2)
    return tridiag(e-0.5*h*thetavec[1:],b,e+0.5*h*thetavec
       [:-1])
N=60
h =1/N
e=0.01
w_0 =1.5
x_0 = 0.25
x= np.linspace(0,1,N)

thetavec = x - x_0 +w_0*np.tanh(w_0*(x-x_0)/(2*e))
for i in range(100):
    b = -mat2(thetavec,h,e)
    A = jacobian2(thetavec, h,e)
    delta = np.linalg.solve(A,b)
    thetavec += delta
plt.plot(x,thetavec)
```