AMATH 575: Problem Set 3

Jithin D. George, No. 1622555

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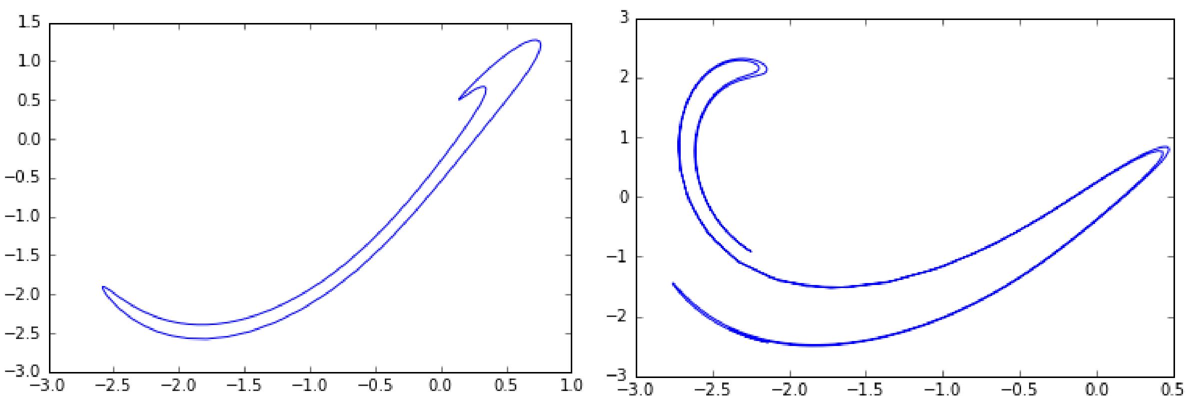
1. This nifty code aids us in plotting the Goodwin equation in time.

```
import numpy as np
from scipy import integrate
import matplotlib as mpl
from mpl_toolkits.mplot3d import Axes3D
import matplotlib.pyplot as plt
def Goodwin(X,t):
        return X[1], -0.75*((X[0]**2-1)/(X[0]**2+1))*X[1]+\\
        0.5*X[0]-0.5*X[0]**3 +14*np.sin(t)
fig = plt.figure()
ax = fig.gca(projection='3d')
circa =np.linspace(0,2*np.pi,1000)
index=0
plotspace1= np.zeros((1000,2))
for i in circa:
  a_t = np.arange(0, 4*np.pi, 0.01)
  asol = integrate.odeint(Goodwin, [np.cos(i),np.sin(i)], a_t)
  plotspace1[index,:]=asol[-1,:]
  index +=1
  ax.plot(a_t,asol[:,0],asol[:,1])
plt.show
```

The first two iterates are plotted below.



(a) The first iteration

(b) The second iteration

Figure 1: The first two iterations

This code generates a 3-dimensional plot in (y,y',t) phase.

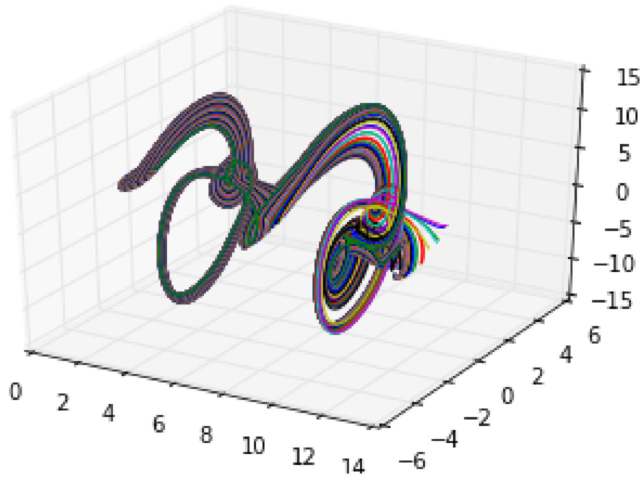


Figure 2: The 3-d plot with time as the third axis

2.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^T J A = J$$

$$\begin{bmatrix} a^T & c^T \\ b^T & d^T \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

$$\begin{bmatrix} -c^T & a^T \\ -d^T & b^T \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

$$\begin{bmatrix} a^T c - c^T a & a^T d - c^T b \\ b^T c - d^T a & b^T d - d^T b \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

(a) Thus, we see

$$a^T c - c^T a = 0, b^T d - d^T b = 0$$

For this to work, $a^T c$ and $b^T d$ have to be symmetric.

(b) We also see that

$$a^T d - c^T b = I$$

(c) An attempt at the solution

$$det(A) = det(ad - bc) = det(a^T d - c^T b) = det(I) = 1$$

I was unable to show the middle step: det(ad - bc)=det($a^T d - c^T b$) .

3. Assume there are two symplectic transformations. One between (q_1, p_1) and (q_2, p_2) and another between (q_2, p_2) and (q_3, p_3). Our goal is to prove the transformation between (q_1, p_1) and (q_3, p_3) is symplectic. The Jacobian A of that transformation is given by

$$A = \begin{bmatrix} \frac{\partial q_3}{\partial q_1} & \frac{\partial q_3}{\partial p_1} \\ \frac{\partial p_3}{\partial q_1} & \frac{\partial p_3}{\partial p_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial q_3}{\partial q_2}\frac{\partial q_2}{\partial q_1} + \frac{\partial q_3}{\partial p_2}\frac{\partial p_2}{\partial q_1} & \frac{\partial q_3}{\partial q_2}\frac{\partial q_2}{\partial p_1} + \frac{\partial q_3}{\partial p_2}\frac{\partial p_2}{\partial p_1} \\ \frac{\partial p_3}{\partial q_2}\frac{\partial q_2}{\partial q_1} + \frac{\partial p_3}{\partial p_2}\frac{\partial p_2}{\partial q_1} & \frac{\partial p_3}{\partial q_2}\frac{\partial q_2}{\partial p_1} + \frac{\partial p_3}{\partial p_2}\frac{\partial p_2}{\partial p_1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial q_3}{\partial q_2} & \frac{\partial q_3}{\partial p_2} \\ \frac{\partial p_3}{\partial q_2} & \frac{\partial p_3}{\partial p_2} \end{bmatrix} \begin{bmatrix} \frac{\partial q_2}{\partial q_1} & \frac{\partial q_2}{\partial p_1} \\ \frac{\partial p_2}{\partial q_1} & \frac{\partial p_2}{\partial p_1} \end{bmatrix}$$
$$= A_2 A_1$$

Where A_1 and A_2 are the Jacobians for the first two transformations.

$$A^T J A = (A_2 A_1)^T J (A_2 A_1) = A_1^T A_2^T J A_2 A_1 = A_1^T J A_1 = J$$

Thus, the composition of two symplectic transformations is sympletic.

4. (a) The Jacobian of a Hamiltonian is a infinitesimal symplectic matrix. So, if the jacobian has an eigenvalue λ, then $-\lambda$ is also an eigenvalue. Thus, there would be no fixed point with only negative eigenvalues. Thus, an asymptotic fixed point is not possible.

   (b) If the jacobian of a Hamiltonian vector field has an eigenvalue λ, then $\bar{\lambda}$ is also an eigenvalue. Also, if 0 is an eigenvalue, it has even multiplicity. Thus, because of this, there is no way to obtain a center manifold of odd dimensions.

5. If the trace of the Jacobian of a system is zero, it is volume preserving. Thus, the following matrix B is volume-preserving since it has no diagonal elements.

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B^T + JB = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ -2 & 0 & 1 & 0 \end{bmatrix}$$

So, this is definitely not a Hamiltonian vector field. Thus, the following vector field is volume preserving and not Hamiltonian.

$$x_1' = x_4$$
$$x_2' = x_1$$
$$x_3' = x_4$$
$$x_4' = x_3$$

6. (a)

$$q_1' = p_1$$
$$q_2' = -p_2$$
$$p_1' = \lambda q_1 - 3\phi^2 q_1 - q_2$$
$$p_2' = -\lambda q_2 + \phi^2 q_2 - q_1$$

(b) The Jacobian B is given by

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \lambda - 3\phi^(x) & -1 & 0 & 0 \\ -1 & \phi^(x) - \lambda & 0 & 0 \end{bmatrix}$$

$$B^T J + JB = \begin{bmatrix} 0 & 0 & \lambda - 3\phi^2(x) & -1 \\ 0 & 0 & -1 & \phi^2(x) - \lambda \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \lambda - 3\phi^(x) & -1 & 0 & 0 \\ -1 & \phi^(x) - \lambda & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -(\lambda - 3\phi^2(x)) & 1 & 0 & 0 \\ 1 & \lambda - \phi^2(x) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} (\lambda - 3\phi^2(x)) & -1 & 0 & 0 \\ -1 & -\lambda + \phi^2(x) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= 0$$

Thus,$B^T J + JB = 0$ seems to hold.

$$H = \int_0^1 < f(tx), Jx > dt$$

$$= \int_0^1 \begin{bmatrix} tp_1 \\ -tp_2 \\ \lambda t q_1 - 3\phi^2(x) t q_1 - t q_2 \\ -\lambda t q_2 + \phi^2(x) t q_2 - t q_1 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ -q_1 \\ -q_2 \end{bmatrix} dt$$

$$= \int_0^1 t(p_1^2 - p_2^2 - \lambda(q_1^2 + q_2^2) + 3\phi^2(x) q_1^2 + \phi^2(x) q_2^2 - 2q_1 q_2) dt$$

$$H = \frac{1}{2}(p_1^2 - p_2^2 - \lambda(q_1^2 + q_2^2) + 3q_1^2\phi^2(x) + q_2^2\phi^2(x) + 2q_1 q_2)$$

$$\frac{\partial H}{\partial p_1} = p_1 = q_1'$$

$$\frac{\partial H}{\partial p_2} = -p_2 = q_2'$$

$$\frac{\partial H}{\partial q_1} = -\lambda q_1 + 3q_1\phi^2(x) + q_1 = -p_1'$$

$$\frac{\partial H}{\partial q_2} = -\lambda q_2 + 3q_2\phi^2(x) + q_1 = -p_2'$$

Thus, there exists a Hamiltonian too.

So, the Homotopy theorem seems to hold in this case.

(c) Using Mathematica,

$$\frac{dH}{dx} = p_1p_1' - p_2p_2' - \lambda(q_1q_1' + q_2q_2') + 3q_1q_1'\phi^2(x) + 3q_1^2\phi(x)\phi(x)'$$

$$+q_2q_2'\phi^2(x) + q_2^2\phi(x)\phi(x)' + 2q_1'q_2 + 2q_2'q_1$$
$$= 3q_1^2\phi(x)\phi(x)' + q_2^2\phi(x)\phi(x)'$$

Thus, since it is non-autonomous, the Hamiltonian is not a first integral.

(d)

$$q_1' = q_2$$
$$q_2' = \lambda q_1 - 3\phi^2 q_1 - p_1$$
$$p_1' = p2$$
$$p_2' = \lambda p_1 - \phi^2 p_1 + q_1$$

Here, the Jacobian is given by

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \lambda - 3\phi^2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & \lambda - \phi^2 & 0 \end{bmatrix}$$

$$B^T J + JB \text{ is not zero}$$

This is not zero (shown in the last Mathematica attachment). So, it is not infinitely symplectic.

$$H = \int_0^1 < f(tx), Jx > dt$$

$$= \int_0^1 \begin{bmatrix} tq_2 \\ t(\lambda q_1 - 3\phi^2 q_1 - p_1) \\ tp_2 \\ t(\lambda p_1 - \phi^2 p_1 + q_1) \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ -q_1 \\ -q_2 \end{bmatrix} dt$$

$$= (1 - \lambda + \phi^2(x))p_1q_2 + \lambda q_1p_2 - 3\phi^2 q_1p_2 - p_1p_2 - p_2q_1 - q_1q_2$$

$$\frac{\partial H}{\partial p_1} \text{ is not } q_1'$$

Thus, the Homotopy operator does not yield a Hamiltonian.

7. All these are done in the Mathematica attachment.

   (a) Here, we split things up to real and imaginary parts and we show that $B^TJ + JB = 0$

   (b) Here, we show that the two transformation are equivalent using the Expand function and equating them.

   (c) Here, we equate the derivative of the Hamiltonian and make sure that it matches with q_1', p_1' etc.

8.

$$x' = \alpha x - \beta xy$$

$$y' = \delta xy - \gamma y$$

The conserved quantity V for the Lotka-Volterra model is given by

$$V = -\delta x + \gamma ln(x) - \beta y + \alpha ln(y)$$

Let's see if it's a Hamiltonian under the coordinates (q,p)–(ln(x),ln(y)).

$$H = -\delta e^q + \gamma q - \beta e^p + \alpha p$$

$$q' = \frac{\partial H}{\partial p} = -\beta e^p + \alpha$$

$$\frac{1}{x} x' = -\beta y + \alpha$$

$$x' = \alpha x - \beta xy$$

$$q' = -\frac{\partial H}{\partial q} = \delta e^q - \gamma$$

$$\frac{1}{y} y' = \delta x - \gamma$$

$$y' = \delta xy - \gamma y$$

Thus, we can recover the original system and our conserved quantity is a Hamiltonian