AMATH 567: Problem Set 7

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11/16/16

1. (a)

$$\wp(z) = \frac{1}{z^2} + \sum_{j,k=-\infty}^{\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right)$$

Here, the term where both j and k are zero is removed.

$$\wp(z + M\omega_1 + N\omega_2) = \frac{1}{(z + M\omega_1 + N\omega_2)^2} + \sum_{j,k=-\infty}^{\infty} \left(\frac{1}{(z + (M-j)\omega_1 + (N-k)\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right)$$

$$=$$

$$\wp(z + M\omega_1) = \frac{1}{(z + M\omega_1)^2} + \sum_{j,k=-\infty}^{\infty} \left(\frac{1}{(z + (M-j)\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right)$$

$$= \frac{1}{(z + M\omega_1)^2} + \sum_{j,k=-\infty}^{\infty} \left(\frac{1}{(z + (M-j)\omega_1 - k\omega_2)^2} - \frac{1}{(z - j\omega_1 - k\omega_2)^2} + \frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right)$$

$$= \frac{1}{(z + M\omega_1)^2} +$$

$$\underbrace{\sum_{j,k=-\infty}^{\infty} \left(\frac{1}{(z + (M-j)\omega_1 - k\omega_2)^2} - \frac{1}{(z - j\omega_1 - k\omega_2)^2} \right)}_{\text{This is a telescoping series and hence not divergent}} + \underbrace{\sum_{j,k=-\infty}^{\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right)}_{\text{This is the same type of sequence as before (so not divergent)}}$$

$$= \frac{1}{(z + M\omega_1)^2} +$$

$$\underbrace{\sum_{k=0, j-\infty, j\neq 0}^{\infty} \left(\frac{1}{(z + (M-j)\omega_1 - k\omega_2)^2} - \frac{1}{(z - j\omega_1 - k\omega_2)^2} \right)}_{\text{if } k \neq 0 \text{, every term is cancelled by the Mth term after it}} + \sum_{j,k=-\infty}^{\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right)$$

$$= \frac{1}{(z + M\omega_1)^2} + \sum_{j-\infty, j\neq 0}^{\infty} \left(\frac{1}{(z + (M-j)\omega_1)^2} - \frac{1}{(z - j\omega_1)^2} \right) + \wp(z) - \frac{1}{z^2}$$

$$= \frac{1}{(z + M\omega_1)^2} + \left(\frac{1}{(z)^2} - \frac{1}{(z + M\omega_1)^2} \right) + \wp(z) - \frac{1}{z^2}$$

$$= \wp(z)$$

$$\wp(z + Mw_1) = \wp(z)$$

Similarly, we can show

$$\wp(z + Nw_2) = \wp(z)$$

Thus,

$$\wp(z + M\omega_1 + N\omega_2) = \wp(z + Mw_1) = \wp(z)$$

(b)

$$\wp(z) = \frac{1}{z^2} + \sum_{j,k=-\infty}^{\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right)$$

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \frac{1}{(z-\omega_1)^2} + \frac{1}{(z+\omega_1)^2} + \frac{1}{(z-\omega_2)^2} + \frac{1}{(z+\omega_2)^2} + \frac{2}{(\omega_1)^2} + \frac{2}{(\omega_2)^2} \\
&+ \sum_{j,k=1}^{\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) + \sum_{j=1,k=-1}^{j=\infty,k=-\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) \\
&+ \sum_{j=-1,k=1}^{j=-\infty,k=\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) + \sum_{j,k=-1}^{-\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) \\
&= \frac{1}{z^2} + \frac{1}{(z-\omega_1)^2} + \frac{1}{(z+\omega_1)^2} + \frac{1}{(z-\omega_2)^2} + \frac{1}{(z+\omega_2)^2} + \frac{2}{(\omega_1)^2} + \frac{2}{(\omega_2)^2} \\
&+ \sum_{j,k=1}^{\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) + \sum_{j,k=1}^{\infty} \left(\frac{1}{(z - j\omega_1 + k\omega_2)^2} - \frac{1}{(j\omega_1 - k\omega_2)^2} \right) \\
&+ \sum_{j,k=1}^{\infty} \left(\frac{1}{(z + j\omega_1 - k\omega_2)^2} - \frac{1}{(-j\omega_1 + k\omega_2)^2} \right) + \sum_{j,k=1}^{\infty} \left(\frac{1}{(z + j\omega_1 + k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right)
\end{aligned}$$

$$\begin{aligned}
\wp(-z) &= \frac{1}{z^2} + \frac{1}{(-z-\omega_1)^2} + \frac{1}{(-z+\omega_1)^2} + \frac{1}{(-z-\omega_2)^2} + \frac{1}{(-z+\omega_2)^2} + \frac{2}{(\omega_1)^2} \frac{2}{(\omega_2)^2} \\
&+ \sum_{j,k=1}^{\infty} \left(\frac{1}{(-z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) + \sum_{j,k=1}^{\infty} \left(\frac{1}{(-z - j\omega_1 + k\omega_2)^2} - \frac{1}{(j\omega_1 - k\omega_2)^2} \right) \\
&+ \sum_{j,k=1}^{\infty} \left(\frac{1}{(-z + j\omega_1 - k\omega_2)^2} - \frac{1}{(-j\omega_1 + k\omega_2)^2} \right) + \sum_{j,k=1}^{\infty} \left(\frac{1}{(-z + j\omega_1 + k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) \\
&= \frac{1}{z^2} + \frac{1}{(z+\omega_1)^2} + \frac{1}{(z-\omega_1)^2} + \frac{1}{(z+\omega_2)^2} + \frac{1}{(z-\omega_2)^2} + \frac{2}{(\omega_1)^2} + \frac{2}{(\omega_2)^2} \\
&+ \sum_{j,k=1}^{\infty} \left(\frac{1}{(z + j\omega_1 + k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) + \sum_{j,k=1}^{\infty} \left(\frac{1}{(z + j\omega_1 - k\omega_2)^2} - \frac{1}{(-j\omega_1 + k\omega_2)^2} \right) \\
&+ \sum_{j,k=1}^{\infty} \left(\frac{1}{(z - j\omega_1 + k\omega_2)^2} - \frac{1}{(j\omega_1 - k\omega_2)^2} \right) + \sum_{j,k=1}^{\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) \\
&= \wp(z)
\end{aligned}$$

(c) $\wp(z)$ has a singularity at the origin. But

$$f(z) = \wp(z) - \frac{1}{z^2}$$

has no singularity around the origin and we can obtain a taylor series around the origin.

$$f'(z) = -2 \sum_{j,k=-\infty}^{\infty} \frac{1}{(z - j\omega_1 - k\omega_2)^3}$$

$$f'(0) = -2 \sum_{j,k=-\infty}^{\infty} \frac{1}{(-j\omega_1 - k\omega_2)^3}$$

$$= -2\left(\frac{1}{(\omega_1)^3} + \frac{1}{(-\omega_1)^3} + \frac{1}{(\omega_2)^3} + \frac{1}{(-\omega_2)^3}\right)$$

$$-2\left(\sum_{j,k=1}^{\infty} \frac{1}{(j\omega_1 + k\omega_2)^3} + \sum_{j,k=1}^{\infty} \frac{1}{(-j\omega_1 + k\omega_2)^3} + \sum_{j,k=1}^{\infty} \frac{1}{(j\omega_1 - k\omega_2)^3} + \sum_{j,k=1}^{\infty} \frac{1}{(-j\omega_1 - k\omega_2)^3}\right)$$

$$= 0$$

Similarly, $f'''(z) = 0$ and $f^{2n+1}(z) = 0$ where n is a non-negative integer.

$$f^{2n}(z) = (2n+1)! \sum_{j,k=-\infty}^{\infty} \frac{1}{(j\omega_1 + k\omega_2)^{2n+2}}, f(0) = 0$$

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{f^{2n}}{2n!} z^{2n}$$

$$\wp(z) = \frac{1}{z^2} + \alpha_0 + \alpha_2 z^2 + \alpha_4 z^4 ...$$

$$\alpha_0 = 0$$

$$\alpha_2 = 3 \sum_{j,k=-\infty}^{\infty} \frac{1}{(j\omega_1 + k\omega_2)^4}$$

$$\alpha_4 = 5 \sum_{j,k=-\infty}^{\infty} \frac{1}{(j\omega_1 + k\omega_2)^6}$$

$$\wp'(z) = \frac{-2}{z^3} + \sum_{n=1}^{\infty} 2n \frac{f^{2n}}{2n!} z^{2n-1}$$

$$\wp'(z) = -\frac{2}{z^3} + \beta_1 z + \beta_3 z^3 + \dots$$

$$\beta_1 = 6 \sum_{j,k=-\infty}^{\infty} \frac{1}{(j\omega_1 + k\omega_2)^4}$$

$$\beta_2 = 20 \sum_{j,k=-\infty}^{\infty} \frac{1}{(j\omega_1 + k\omega_2)^6}$$

(d)

$$f(z) = (\wp')^2 - a\wp^3 - b\wp^2 - c\wp$$

f(z) is bounded within the rectangle with vertices $(0,0), (w_1, 0), (0, w_2)$ and (w_1, w_2). If we choose a,b and c such that the singularities cancel out, since $\wp(z)$ and its derivatives are periodic, f(z) will be bounded everywhere in addition to being entire. Thus by Louiville's theorem, f(z) is a constant. So, once we cancel out the singularities, we don't need to worry about the higher order terms because since the function is a constant,they disappear.

$$(\wp')^2 = a\wp^3 + b\wp^2 + c\wp + d,$$

$$(-\frac{2}{z^3} + \beta_1 z + \beta_3 z^3 + \dots)^2 = a(\frac{1}{z^2} + \alpha_2 z^2..)^3 + b(\frac{1}{z^2} + \alpha_2 z^2..)^2 + c(\frac{1}{z^2} + \alpha_2 z^2..) + d$$

$$(\frac{4}{z^6} - \frac{4\beta_1}{z^2} - 4\beta_3 + \beta_1^2 z^2 + 2\beta_1\beta_3 z^4 + \beta_3^2 z^6 ...)$$
$$= a(\alpha_4^3 z^{12} + 3\alpha_4^2\alpha_2 z^{10} + 3\alpha_4^2 z^6 + 3\alpha_4\alpha_2^2 z^8 + 6\alpha_4\alpha_2 z^4 + 3\alpha_4 + \alpha_2^3 z^6 + 3\alpha_2^2 z^2 + \frac{3\alpha_2}{z^2} + \frac{1}{z^6}..)$$
$$+b(\alpha_4^2 z^6 + 2\alpha_4\alpha_2 z^5 + 2\alpha_4 z + \alpha_2^2 z^4 + 2\alpha_2 + \frac{1}{z^4}..) + c(\frac{1}{z^2} + \alpha_2 z^2..) + d$$

Equating the coefficients of z^{-6},

$$a = 4$$

Since the first term on the right side and the left side does not have a z^{-4} term.

$$b = 0$$

Equating the coefficients of z^{-2},

$$c = -4\beta_1 - a3\alpha_2 = -4\beta_1 - 12\alpha_2$$

Equating constants,

$$d = -4\beta_3 - a3\alpha_4 = -4\beta_3 - 12\alpha_4$$

2. (a) Substituting U(x) into the KdV equation

$$U_t = 6UU_x + U_{xxx}$$

$$0 = 6UU_x + U_{xxx}$$

On integration,

$$C = 3U^2 + U_{xx}$$

Multiplying with U_x,

$$CU_x = 3U^2 U_x + U_{xx}U_x$$

Integrating,

$$k + CU = U^3 + \frac{U_x^2}{2}$$

$$U_x^2 = -2U^3 + CU + k$$

(b) Substituting $U = U_0\wp(x)$ (taking $x_0 = 0$),

$$U_0^2\wp_x^2 = -2U_0^3\wp^3 + CU_0\wp + k$$

Substituting $(\wp')^2 = a\wp^3 + b\wp^2 + c\wp + d$,

$$U_0^2(a\wp^3 + b\wp^2 + c\wp + d) = -2U_0^3\wp^3 + CU_0\wp + k$$

Equating coefficients of $\wp^3$,

$$a = -2U_0$$

$$U_0 = -\frac{a}{2} = -2$$

3. (a)

$$-ln(\Gamma) = ln(z) + \gamma z + \sum_{n=1}^{\infty}\left(ln(\frac{n+z}{n}) - \frac{z}{n}\right)$$

$$ln(\Gamma) = -ln(z) - \gamma z - \sum_{n=1}^{\infty}\left(ln(\frac{n+z}{n}) - \frac{z}{n}\right)$$

Differentiating,

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{n=1}^{\infty}\left(\frac{1}{n}\frac{n}{n+z} - \frac{1}{n}\right)$$

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)$$

(b)

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)$$

$$\begin{aligned}
\frac{\Gamma'(z+1)}{\Gamma(z+1)} &= -\frac{1}{z+1} - \gamma - \sum_{n=1}^{\infty}\left(\frac{1}{z+1+n} - \frac{1}{n}\right) \\
&= -\frac{1}{z+1} - \gamma - \sum_{1+n=2}^{\infty}\left(\frac{1}{z+1+n} - \frac{1}{n}\right) \\
&= -\frac{1}{z+1} - \gamma - \sum_{j=2}^{\infty}\left(\frac{1}{z+j} - \frac{1}{j-1}\right) \\
&= -\gamma - \sum_{j=1}^{\infty}\left(\frac{1}{z+j} - \frac{1}{j}\right) \\
&= -\gamma - \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right) \\
&= \frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z}
\end{aligned}$$

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{1}{z} = 0$$

Integrating,

$$\ln(\Gamma(z+1)) - \ln(\Gamma(z)) - ln(z) = b$$

$$\frac{\Gamma(z+1)}{\Gamma(z)} = Cz$$

$$\Gamma(z+1) = Cz\Gamma(z)$$

(c)

$$\lim_{z\to 0} z\Gamma(z) = \lim_{z\to 0}\frac{1}{e^{\gamma z}\prod_{n=1}^{\infty}(1+\frac{z}{n})e^{\frac{-z}{n}}} = 1$$

$$\lim_{z\to 0}\Gamma(z+1) = \lim_{z\to 0} Cz\Gamma(z)$$

$$\Gamma(1) = C$$

(d) Choosing C such that

$$\Gamma(1) = 1$$

$$\frac{1}{e^{\gamma}\prod_{n=1}^{\infty}(1+\frac{1}{n})e^{\frac{-1}{n}}} = 1$$

$$e^{-\gamma} = \prod_{n=1}^{\infty}(1+\frac{1}{n})e^{\frac{-1}{n}}$$

(e)

$$\prod_{n=1}^{\infty}(1+\frac{1}{n})e^{\frac{-1}{n}} = \lim_{N\to\infty}\prod_{n=1}^{N}(1+\frac{1}{n})e^{\frac{-1}{n}} = \lim_{N\to\infty}\frac{2}{1}\frac{3}{2}\frac{4}{3}..\frac{N+1}{N}e^{-S_N} = \lim_{N\to\infty}(N+1)e^{-S_N}$$

where

$$S_N = 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{N}$$

$$e^{-\gamma} = \prod_{n=1}^{\infty}(1+\frac{1}{n})e^{\frac{-1}{n}}$$

$$\gamma = -ln(\lim_{N\to\infty}(N+1)e^{-S_N})$$

Taking principal branch of ln and since it is continuous,

$$\gamma = \lim_{N\to\infty} -ln((N+1)e^{-S_N}) = \lim_{N\to\infty}\sum_{k=1}^{\infty}\frac{1}{k} - ln(N+1)$$