AMATH 567: Problem Set 9

Jithin D. George, No. 1622555

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1. From the last exercise, we have,

$$f(k) = \pi \text{Res}_{z=k} f(z) \cot(\pi z),$$

provided $f(z)$ is analytic at $z = k$ ($k \in \mathbb{Z}$).

Now that f(z) may have a pole at z=0, this is valid for all $k \neq 0$.

And so,

$$\sum_{k=-\infty}^{\infty}{}' \frac{p(k)}{q(k)} = \pi \sum_{k=-\infty}^{\infty}{}' \text{Res}_{z=k} f(z) \cot(\pi z),$$

On a square contour, with corners at $(N + 1/2)(\pm 1 \pm i)$, we showed that

$$\lim_{N\to\infty} \left| \oint_{\Gamma_N} \frac{p(z)}{q(z)} \cot(\pi z) dz \right| = 0.$$

$$\sum \text{Res} f(z) \cot(\pi z) = 0$$

$$\sum_j \text{Res}_{z=z_j} f(z) \cot(\pi z) + \sum_{k=-\infty}^{\infty} \text{Res}_{z=k} f(z) \cot(\pi z) = 0$$

As the k=0 term is a pole of f(z), it can be included as one of the z_j to give

$$\sum_j \text{Res}_{z=z_j} f(z) \cot(\pi z) + \sum_{k=-\infty}^{\infty}{}' \text{Res}_{z=k} f(z) \cot(\pi z) = 0$$

$$\sum_{k=-\infty}^{\infty}{}' \text{Res}_{z=k} f(z) \cot(\pi z) = -\sum_j \text{Res}_{z=z_j} f(z) \cot(\pi z)$$

So,

$$\sum_{k=-\infty}^{\infty}{}' \frac{p(k)}{q(k)} = -\pi \sum_j \text{Res}_{z=z_j} f(z) \cot(\pi z)$$

2. (a)

$$\sum_{k=-\infty}^{\infty}{}' \frac{1}{k^2} = -\pi Res_{z=0} \frac{cot(\pi z)}{z^2}$$

This is a triple pole as $\cot(\pi z)$ is a simple pole at z=0 (last exercise).

$$cot(\pi z) = \frac{1}{\pi z} - \frac{\pi z}{3} - \frac{(\pi z)^3}{45} - \frac{2(\pi z)^5}{945} + \dots$$

$$zcot(\pi z) = \frac{1}{\pi} - \frac{\pi z^2}{3} - \frac{\pi^3 z^4}{45} - \frac{2\pi^5 z^6}{945} + \dots$$

$$\sum_{k=-\infty}^{\infty}{}' \frac{1}{k^2} = -\pi \lim_{z=0} \frac{1}{2!} \frac{d^2}{dz^2} \left(zcot(\pi z) \right) = \frac{\pi^2}{3}$$

(b)

$$\sum_{k=-\infty}^{\infty}{}' \frac{1}{k^3} = -\pi Res_{z=0} \frac{cot(\pi z)}{z^3}$$

This is a fourth order pole.

$$\sum_{k=-\infty}^{\infty}{}' \frac{1}{k^3} = -\pi \lim_{z=0} \frac{1}{3!} \frac{d^3}{dz^3} \left(zcot(\pi z) \right) = 0$$

(c)

$$\sum_{k=-\infty}^{\infty}{}' \frac{1}{k^4} = -\pi Res_{z=0} \frac{cot(\pi z)}{z^4}$$

This is a fifth order pole.

$$\sum_{k=-\infty}^{\infty}{}' \frac{1}{k^4} = -\pi \lim_{z=0} \frac{1}{4!} \frac{d^4}{dz^4} \left(zcot(\pi z) \right) = \frac{\pi^4}{45}$$

3.

$$\frac{\sin \pi z}{\pi z} = 1 - \frac{\pi^3 z^3}{\pi z 3!} + \ldots = 1 - \frac{\pi^2 z^2}{3!} + \ldots$$

$$\prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right) = \left(1 - z^2 - \frac{z^2}{2^2} - \frac{z^2}{3^2} - \frac{z^2}{4^2} + \ldots\right) = \left(1 - z^2 \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right) + \ldots\right)$$

$$\frac{\pi^2}{3!} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

4.

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{\sinh x} dx.$$

Sinh(x) has a zero at x =0 so the integral can't be defined. Sin(x) also has a zero there and the function ends up being have a removable singularity at 0. So, we use the principal value integral instead.

$$I_p = ⨍_{-\infty}^{\infty} \frac{\sin x}{\sinh x} dx$$

We use the contour in the figure below.

C_0 and C_π are two semi circles with centers at the singularities 0 and π and of radius ϵ as $\epsilon \to 0$ and thus in agreement with our principal value integral definition.

Since the contour encloses no singularity, by Cauchy's theorem,

$$\left(\int_{-R}^{-\epsilon} + \int_{C_0} + \int_{\epsilon}^{R} + \int_{R}^{R+\pi i} + \int_{R+\pi i}^{\epsilon+\pi i} + \int_{C_\pi} + \int_{-\epsilon+\pi i}^{-R+\pi i} + \int_{-R+\pi i}^{R}\right) f(z)dz = 0$$

Taking the curve C_3,

$$z = R + i\pi t$$

where t goes from 0 to 1.

$$\Big|\frac{\sin z}{\sinh z}\Big| = \Big|\frac{(e^{iz} - e^{-iz})}{i(e^z - e^{-z})}\Big|$$

$$= \Big|\frac{(e^{iR}e^{-\pi t} - e^{-iR}e^{\pi t})}{i(e^R e^{i\pi t} - e^{-R}e^{-i\pi t})}\Big|$$

$$\leq \Big|\frac{|e^{iR}||e^{-\pi t}| + |e^{-iR}||e^{\pi t}|}{|e^R||e^{i\pi t}| - |e^{-R}||e^{-i\pi t}|}\Big|$$

$$\leq |(e^{i-1})^R|\Big|\frac{|e^{-\pi t}| + |e^{-2iR}||e^{\pi t}|}{|e^{i\pi t}| - |e^{-2R}||e^{-i\pi t}|}\Big|$$

$$\leq (e^{-R})\Big|\frac{|e^{-\pi t}| + |e^{-2iR}||e^{\pi t}|}{|e^{i\pi t}| - |e^{-2R}||e^{-i\pi t}|}\Big|$$

As R $\to \infty$, this value goes to zero.

A similar case can be made for C_6.

Taking the curve C_4,

$$z = t + 2\pi i$$

where t goes from ϵ to R.

$$\int_{\epsilon+\pi i}^{R+\pi i} \frac{\sin z}{\sinh z} dz = \int_{\epsilon+\pi i}^{R+\pi i} \frac{\sin(t+\pi i)}{\sinh(t+\pi i)} dz$$

$$= \int_{\epsilon+\pi i}^{R+\pi i} \frac{\sin(t)cos(\pi i) + sin(\pi i)cos(t)}{\sinh(t)cosh(\pi i) + sinh(\pi i)cosh(t)} dz$$

$$= -\int_{\epsilon}^{R} \frac{\sin(z)cos(\pi i)}{\sinh(z)} dz$$

Similarly, on C_6, we can show,

$$\int_{-\epsilon+\pi i}^{-R+\pi i} \frac{\sin z}{\sinh z} dz = -cos(\pi i) \int_{-\epsilon}^{-R} \frac{\sin(z)}{\sinh(z)} dz$$

So, we have now,

$$\left(\int_{-R}^{-\epsilon} + \int_{C_0} + \int_{\epsilon}^{R} + cos(\pi i) \int_{\epsilon}^{R} + \int_{C_\pi} + cos(\pi i) \int_{-R}^{-\epsilon}\right) f(z)dz = 0$$

$$(1 + cos(\pi i))\left(\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R}\right) f(z)dz = -\left(\int_{C_0} + \int_{C_\pi}\right) f(z)dz$$

As R goes to ∞,

$$(1 + cos(\pi i))I_p = -\left(\int_{C_0} + \int_{C_\pi}\right) f(z)dz$$

At 0, the function has a removable singularity. So to find the integral over C_0 using Theorem 4.3.1(b)(Baby Limit theorem), we need to change it into a form which has simple poles.

$$\int_{C_0} f(z)dz = -\int_{C_0} \frac{e^{iz} - e^{-iz}}{2isinh(z)}dz = -\int_{C_0} \frac{e^{iz}}{2isinh(z)}dz + \int_{C_0} \frac{e^{-iz}}{2isinh(z)}dz$$

The minus sign comes from the integral being in the clockwise direction. Both the terms have a simple pole at z =0 since

$$\frac{1}{sinh(z)} = \frac{1}{z} - \frac{z}{6} + \dots$$

$$\int_{C_0} f(z)dz = \pi i\frac{1}{2i} - \pi i\frac{1}{2i} = 0$$

At πi, the function has a simple pole and as

$$\frac{1}{sinh(z-\pi i)} = -\frac{1}{z-\pi i} + \frac{z-\pi i}{6} + \dots$$

We can use Theorem 4.3.1(b) directly to obtain

$$\int_{C_\pi} f(z)dz = -(-\pi isin(\pi i)) = \pi isin(\pi i)$$

Substituting that in

$$(1 + cos(\pi i))I_p = -\left(\int_{C_0} + \int_{C_\pi}\right) f(z)dz$$

,

$$I_p = -\pi i\frac{sin(\pi i)}{(1 + cos(\pi i))} = -\pi itan(\frac{\pi i}{2}) = \pi tanh(\frac{\pi}{2})$$