AMATH 568: Problem Set 6

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1. (a) Using Laplace's method, since the maximum of cos(t) is at t=0

$$\int_0^{\pi/4} e^{xcost} cos(nt)dt = \int_0^{\epsilon} e^{xcost} cos(0)dt$$

$$= \int_0^{\epsilon} e^{xcost} dt$$

Near 0, we can replace cos(t) by its taylor series.

$$= \int_0^{\epsilon} e^{x(1-t^2/2+\dots)} dt$$

$$= \int_0^{\epsilon} e^x e^{-xt^2/2} e^{xt^4/4!+\dots} dt$$

$$= \int_0^{\epsilon} e^x e^{-xt^2/2}(1 + (xt^4/4! - xt^6/6! + \dots) + (xt^4/4! - xt^6/6! + \dots)^2)\ dt$$

$$= e^x \int_0^{\epsilon} e^{-xt^2/2}(1 + (xt^4/4! - xt^6/6! + \dots) + (xt^4/4! - xt^6/6! + \dots)^2)\ dt$$

Using Laplace's method again,

$$= e^x \int_0^{\infty} e^{-xt^2/2}(1 + (xt^4/4! - xt^6/6! + \dots) + (xt^4/4! - xt^6/6! + \dots)^2)\ dt$$

$$= e^x \sqrt{\frac{\pi}{2x}} + e^x x \sqrt{\frac{9(4!)^5\pi}{16x^5}} + \dots$$

$$= e^x \sqrt{\frac{\pi}{2x}} + O(\frac{1}{\sqrt{x^3}})$$

(b) cosh(t) has a minimum at t =0 and it is an even function. Using Laplace's method,

$$\int_{-1}^{1} e^{-xcosht} dt = \int_{-\epsilon}^{\epsilon} e^{-xcosht} dt$$

$$= 2\int_0^{\epsilon} e^{-xcosht} dt$$

$$= 2\int_0^{\epsilon} e^{-x(1+t^2/2+\dots)} dt$$

$$= 2e^{-x} \int_0^{\epsilon} e^{-x(t^2/2+\dots)} dt$$

$$= 2e^{-x} \int_0^{\epsilon} e^{-x(t^2/2+\dots)} dt$$

$$= 2e^{-x}\int_0^\infty e^{-xt^2/2}(1 + (xt^4/4! + xt^6/6! + \ldots) + (xt^4/4! - xt^6/6! + \ldots)^2)\ dt$$

$$= 2e^{-x}\sqrt{\frac{\pi}{2x}} + 2e^{-x}x\sqrt{\frac{9(4!)^5\pi}{16x^5}} + \ldots$$

$$= 2e^{-x}\sqrt{\frac{\pi}{2x}} + O(\frac{1}{\sqrt{x^3}})$$

(c)

$$\int_0^1 t^x cos^n(\pi t)dt = \int_0^1 e^{xlog(t)}cos^n(\pi t)dt$$

The integral is valid only near t=1.

$$= \int_0^1 t^x(-1 + \frac{1}{2}\pi^2(t-1)^2 + \ldots)^n dt$$

Using Laplace's method,

$$= \int_{1-\epsilon}^1 t^x(-1 + \frac{1}{2}\pi^2(\epsilon)^2 + \ldots)^n dt$$

$$= \int_{1-\epsilon}^1 t^x(-1 + \frac{1}{2}\pi^2(\epsilon)^2 + \ldots)^n dt$$

As ϵ is determined by x,

$$= \int_0^1 t^x((-1)^n + \frac{1}{2}\pi^2(\frac{1}{x})^2 + \ldots)dt$$

$$= \Big[\frac{t^{x+1}(-1)^n}{x+1}\Big]_0^1 + \Big[\frac{t^{x+1}(-1)^n}{x+1}\Big]_0^1 \frac{1}{2}\pi^2(\frac{1}{x})^2 + \ldots$$

$$= \frac{(-1)^n}{x+1} + O(\frac{1}{x^3}) + \ldots$$

2. (a)

$$\int_{-\infty}^\infty e^{-x(t-u)^2}g(t)dt = \int_{-\infty}^u e^{-x(t-u)^2}g(t)dt + \int_u^\infty e^{-x(t-u)^2}g(t)dt$$

$$= \int_{-\infty}^u e^{-x(t-u)^2}g(t)dt + \int_u^\infty e^{-x(t-u)^2}g(t)dt$$

$$= \int_{-\infty}^u e^{-x(t-u)^2}g(t)dt + \int_u^\infty e^{-x(t-u)^2}g(t)dt$$

$$= -\int_0^\infty e^{-xs}g(u-\sqrt{s})\frac{ds}{2\sqrt{s}} + \int_0^\infty e^{-xs}g(u+\sqrt{s})\frac{ds}{2\sqrt{s}}$$

$$= \int_0^\infty e^{-xs}\frac{g(u+\sqrt{s}) - g(u-\sqrt{s})ds}{2\sqrt{s}}$$

$$g(u+\sqrt{s}) = g(u) + \sqrt{s}g'(u) + \sqrt{s}g''(u) + \ldots$$

$$g(u-\sqrt{s}) = g(u) - \sqrt{s}g'(u) + \sqrt{s}g''(u) + \ldots$$

$$g(u+\sqrt{s}) - g(u-\sqrt{s}) = 2\sqrt{s^3}g'(u) + 2sg'''(u) + \ldots$$

$$= \int_0^\infty e^{-xs}(g'(u) + sg'''(u) + \ldots)$$

$$= \int_0^\infty e^{-xs}\sum_{k=0}^\infty g^{2k+1}(u)s^k)$$

$$= \sum_{k=0}^{\infty} \frac{g^{2k+1}(u)}{2k+1} \int_0^{\infty} e^{-xs} s^k$$

Using v=xs,

$$= \sum_{k=0}^{\infty} \frac{g^{2k+1}(u)}{2k+1} \int_0^{\infty} e^{-v} (\frac{v}{x})^k$$

$$= \sum_{k=0}^{\infty} \frac{g^{2k+1}(u)}{2k+1} \frac{\Gamma(k+1)}{x^{k+1}}$$

(b)

$$\int_{-\infty}^{\infty} e^{-x(t-u)^2} \sum_0^{\infty} \frac{g^k(u)}{k!} (t-u)^k dt = \int_{-\infty}^{\infty} e^{-xs^2} \sum_0^{\infty} \frac{g^k(u)}{k!} (s)^k ds$$

$$= \sum_0^{\infty} \frac{g^k(u)}{k!} \int_{-\infty}^{\infty} e^{-xs^2} (s)^k ds$$

Since the odd moments are odd functions,

$$= \sum_0^{\infty} \frac{g^{2k}(u)}{2k!} \int_{-\infty}^{\infty} e^{-xs^2} (s)^{2k} ds$$

$$= \sum_0^{\infty} \frac{g^{2k}(u)}{2k!} \frac{(2k-1)!!}{x^k 2^k} \sqrt{\frac{\pi}{x}}$$

Using an identity from online,

$$= \sum_0^{\infty} \frac{g^{2k}(u)}{2k!} \frac{\Gamma(k+1/2)}{x^{k+1/2}}$$

3.

$$\int_0^{\infty} e^{-xt^3} g(t) dt = \int_0^{\infty} e^{-xt^3} (g(0) + g'(0)t + \ldots) dt$$

$$= \sum_{k=0}^{\infty} \frac{g^k(0)}{k!} \int_0^{\infty} e^{-xt^3} t^k dt$$

Using a substitution,

$$= \sum_{k=0}^{\infty} \frac{1}{3} \frac{g^k(0)}{k!} \int_0^{\infty} e^{-xv} v^{k/3} v^{-2/3} dv$$

$$= \frac{1}{3} \sum_{k=0}^{\infty} \frac{g^k(0)}{k!} \int_0^{\infty} e^{-xv} v^{(k-2)/3} dv$$

Applying Watson's Lemma,

$$= \frac{1}{3} \sum_{k=0}^{\infty} \frac{g^k(0)\Gamma(k/3+1/3)}{k! x^{k/3+1/3}}$$

$$= \frac{g(0)\Gamma(k/3+1/3)}{3k! x^{1/3}} + O(\frac{1}{x^{2/3}})$$

4.

$$\int_0^1 dy \int_0^y \exp(\frac{\phi(y)-\phi(x)}{\epsilon})dx$$

$$= \int_0^1 dy \exp(\frac{\phi(y)}{\epsilon}) \underbrace{\int_0^y \exp(\frac{-\phi(x)}{\epsilon})dx}_{g(y)}$$

$$= \int_0^1 dy \exp(\frac{\phi(y)}{\epsilon}) g(y)$$

Using Laplace's method and the maxima at x_2,

$$= \int_{x_2-\epsilon}^{x_2+\epsilon} dy \exp(\frac{\phi(x_2)+\phi''(x_2)(x-x_2)^2+\dots}{\epsilon})g(x_2)$$

$$= \int_{x_2-\epsilon}^{x_2+\epsilon} e^{\phi(x_2)} e^{\phi''(x_2)(x-x_2)^2} \exp(\frac{\phi'''(x_2)(x-x_2)^3+\phi''''(x_2)(x-x_2)^4+\dots}{\epsilon})g(x_2)dy$$

$$= \int_{x_2-\epsilon}^{x_2+\epsilon} e^{\frac{\phi(x_2)}{\epsilon}} e^{\frac{\phi''(x_2)(x-x_2)^2}{\epsilon}} [1+(\frac{\phi'''(x_2)(x-x_2)^3+\phi''''(x_2)(x-x_2)^4+\dots}{\epsilon})$$
$$+(\frac{\phi'''(x_2)(x-x_2)^3+\phi''''(x_2)(x-x_2)^4+\dots}{\epsilon})^2+\dots]g(x_2)dy$$

$$= \int_{x_2-\epsilon}^{x_2+\epsilon} e^{\frac{\phi(x_2)}{\epsilon}} e^{\frac{\phi''(x_2)(x-x_2)^2}{\epsilon}} [1+(\frac{\phi'''(x_2)(x-x_2)^3+\phi''''(x_2)(x-x_2)^4+\dots}{\epsilon})$$
$$+(\frac{\phi'''(x_2)(x-x_2)^3+\phi''''(x_2)(x-x_2)^4+\dots}{\epsilon})^2+\dots]g(x_2)dy$$

$$= e^{\frac{\phi(x_2)}{\epsilon}} g(x_2) \int_{x_2-\epsilon}^{x_2+\epsilon} e^{\frac{\phi''(x_2)(x-x_2)^2}{\epsilon}} [1+(\frac{\phi''''(x_2)(x-x_2)^4}{\epsilon})+\dots]dy$$

$$= e^{\frac{\phi(x_2)}{\epsilon}} g(x_2) \int_{-\infty}^{\infty} e^{\frac{\phi''(x_2)(x-x_2)^2}{\epsilon}} [1+(\frac{\phi''''(x_2)(x-x_2)^4}{\epsilon})+\dots]dy$$

$$= e^{\frac{\phi(x_2)}{\epsilon}} g(x_2)\sqrt{\frac{2\pi\epsilon}{\phi''(x_2)}} + O(\epsilon^{3/2})$$

$$g(x_2) = \int_0^{x_2} \exp(\frac{-\phi(x)}{\epsilon})dx$$

This uses the minimum at x_1 which is before x_2 and like before, we can find

$$g(x_2) = e^{\frac{-\phi(x_1)}{\epsilon}} \sqrt{\frac{2\pi\epsilon}{\phi''(x_1)}}$$

Thus,

$$\int_0^1 dy \int_0^y \exp(\frac{\phi(y)-\phi(x)}{\epsilon})dx = e^{\frac{\phi(x_2)-\phi(x_1)}{\epsilon}} \frac{2\pi\epsilon}{\sqrt{\phi''(x_1)\phi''(x_2)}}$$