# AMATH 561: Final Examination

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1.

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

$$Y_n = X_{S_n}$$

But, $S_n$ is a random variable. So, we need to incorporate its probability into the transition matrix. The probability of transitioning from state i to state j is given by

$$Q(i,j) = \sum_{k=0}^{\infty} prob(S_n = k) P^k(i,j)$$

Notice that it is $P^k(i,j)$ and not $(P(i,j))^k$. So, we can write the transition matrix as

$$Q = \sum_{k=0}^{\infty} prob(S_n = k) P^k$$

$$= \mathbb{E}[P^{S_n}]$$

$$= G_{S_n}(P)$$

where $G_{S_n}$ is the generating function for $S_n$.

$$P = \begin{bmatrix} 1 & -\frac{p}{q} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1-p-q \end{bmatrix} \begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ -\frac{q}{p+q} & \frac{q}{p+q} \end{bmatrix} = JAJ^{-1}$$

If a function has a power series representation which converges and a matrix P is diagonalisable to $JAJ^{-1}$,

$$f(P) = J^{-1} f(A) J \text{ (From wikipedia)}$$

Since $G_{S_n}$ is defined as a power series, we can use this property to find Q.

$$Q = G_{S_n}(P) = J^{-1} G_{S_n}(A) J$$

For geometric distribution,

$$G_{Geo} = \frac{sr}{1 - s(1-r)}$$

For $S_n$,

$$G_{S_n} = G_{Geo}^n = \left( \frac{sr}{1 - s(1-r)} \right)^n$$

$$G_{S_n}(A) = \begin{bmatrix} \left(\frac{r}{1-(1-r)}\right)^n & 0 \\ 0 & \left(\frac{(1-p-q)r}{1-(1-p-q)(1-r)}\right)^n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{(1-p-q)r}{1-(1-p-q)(1-r)}\right)^n \end{bmatrix}$$

(a) So,

$$Q = \begin{bmatrix} 1 & -\frac{p}{q} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{(1-p-q)r}{1-(1-p-q)(1-r)}\right)^n \end{bmatrix} \begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ -\frac{q}{p+q} & \frac{q}{p+q} \end{bmatrix}$$

(b) Thankfully, the eigenvalue 1 remains the same under $G_{S_n}$. So, the invariant distribution of Y is the same as that of X and given by the first row of 'J'.

$$\pi_Y = \pi_X = \left[\frac{q}{p+q}, \frac{p}{p+q}\right]$$

2.

$$\begin{aligned} g(X_i) &= \sum_j p(X_i, X_j) f(X_j) - f(X_i) \\ &= \mathbb{E}[f(X_{i+1})|X_i] - f(X_i) \\ M_n &= f(X_n) - \sum_{i=0}^{n-1} g(X_i) \end{aligned}$$

$$\mathbb{E}[M_n|\mathcal{F}_{n-1}] = \mathbb{E}[f(X_n)|\mathcal{F}_{n-1}] - \sum_{i=0}^{n-1} \mathbb{E}[g(X_i)|\mathcal{F}_{n-1}] \tag{1}$$

$$= \mathbb{E}[f(X_n)|\mathcal{F}_{n-1}] - \mathbb{E}[g(X_{n-1})|\mathcal{F}_{n-1}] - \sum_{i=0}^{n-2} \mathbb{E}[g(X_i)|\mathcal{F}_{n-1} \tag{2}$$

$$\mathbb{E}[g(X_{n-1})|\mathcal{F}_{n-1}] = \mathbb{E}[\mathbb{E}[f(X_n)|X_{n-1}]|\mathcal{F}_{n-1}] - \mathbb{E}[f(X_{n-1})|\mathcal{F}_{n-1}]$$

$$\begin{aligned} \mathbb{E}[\mathbb{E}[f(X_n)|X_{n-1}]|\mathcal{F}_{n-1}] &= \mathbb{E}[\mathbb{E}[f(X_n)|\mathcal{F}_{n-1}]|\mathcal{F}_{n-1}] \text{ (Since X is markov)} \\ &= \mathbb{E}[f(X_n)|\mathcal{F}_{n-1}] \text{ (Iterated Conditiong)} \end{aligned}$$

$$\mathbb{E}[f(X_{n-1})|\mathcal{F}_{n-1}] = f(X_{n-1})$$

Thus,

$$\mathbb{E}[g(X_{n-1})|\mathcal{F}_{n-1}] = \mathbb{E}[f(X_n)|\mathcal{F}_{n-1}] - f(X_{n-1})$$

Plugging this into (2),

$$\mathbb{E}[M_n|\mathcal{F}_{n-1}] = f(X_{n-1}) - \sum_{i=0}^{n-2} \mathbb{E}[g(X_i)|\mathcal{F}_{n-1}]$$

$$\sum_{i=0}^{n-2} \mathbb{E}[g(X_i)|\mathcal{F}_{n-1}] = \mathbb{E}[\sum_{i=0}^{n-2} \mathbb{E}[f(X_{i+1})|X_i]\mathcal{F}_{n-1}] - \mathbb{E}[\sum_{i=0}^{n-2} f(X_i)|\mathcal{F}_{n-1}]$$

Since $X_0, X_1, .....X_{i-1} \in \mathcal{F}_{n-1}$,

$$\sum_{i=0}^{n-2} \mathbb{E}[g(X_i)|\mathcal{F}_{n-1}] = \sum_{i=0}^{n-2} \mathbb{E}[f(X_{i+1})|X_i] - \sum_{i=0}^{n-2} f(X_i)$$
$$= \sum_{i=0}^{n-2} g(X_i)$$

Thus,

$$\mathbb{E}[M_n|\mathcal{F}_{n-1}] = f(X_{n-1}) - \sum_{i=0}^{n-2} g(X_i)$$
$$= M_{n-1}$$

Similarly, we can show

$$\mathbb{E}[M_{n+m}|\mathcal{F}_n] = \mathbb{E}[\mathbb{E}[M_{n+m}|\mathcal{F}_{n+m-1}]|\mathcal{F}_n]$$
$$= \mathbb{E}[M_{n+m-1}|\mathcal{F}_n]$$
$$= ......................$$
$$= \mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$$

So, the process $M_n$ is a martingale with respect to this filtration.

3. (a) Let $A_j$ represent the number of accidents in the jth year

$$\mathbb{E}[A_2|A_1 = n] = \mathbb{E}[\mathbb{E}[A_2|type = i]|A_1 = n] \text{ (Iterated Conditioning)}$$

We know

$$\mathbb{E}[\mathbb{E}[A_2|type = i]] = \sum_i Prob(type = i)\lambda_i$$

Thus,

$$\mathbb{E}[\mathbb{E}[A_2|type = i]|A_1 = n] = \sum_i Prob(type = i|A_1 = n)\lambda_i$$

$$Prob(type = i|A_1 = n) = \frac{Prob(A_1 = n|type = i)Prob(type = i)}{Prob(A_1 = n)} \text{ (Bayes Rule)}$$
$$= \frac{e^{-\lambda_i}\frac{\lambda_i^n}{n!}p_i}{\sum_{j=1}^k e^{-\lambda_j}\frac{\lambda_j^n}{n!}p_j}$$

Thus,

$$\mathbb{E}[A_2|A_1 = n] = \frac{\sum_i e^{-\lambda_i}\frac{\lambda_i^n}{n!}p_i\lambda_i}{\sum_{j=1}^k e^{-\lambda_j}\frac{\lambda_j^n}{n!}p_j}$$

(b)

$$p(A_2 = m|A_1 = n) = \frac{p(A_2 = m \cap A_1 = n)}{p(A_1 = n)}$$
$$= \frac{\sum_i p(A_2 = m \cap A_1 = n|type = i)p(type = i)}{\sum_i p(A_1 = n|type = i)p(type = i)}$$
$$= \frac{\sum_i e^{-2\lambda_i}\frac{\lambda_i^{n+m}}{n!m!}p_i}{\sum_i e^{-\lambda_i}\frac{\lambda_i^n}{n!}p_i}$$

4. The inter-arrival times of machines arriving from the serviceman is determined by an exponential distribution with parameter $\mu$. Thus, the number of machines arriving is given by a Poisson distribution with parameter $\mu$. The inter-'departure' times of **a** machine going to the serviceman is determined by an exponential distribution with mean $\frac{1}{\lambda}$. Thus, for n machines, the mean time would be (probability of 1 of n machines failing) * (expected failing time of one machine) $= \frac{1}{n\lambda}$. So, the number of machines leaving to the serviceman is given by a Poisson distribution with parameter $n\lambda$.

   Then, we can construct the following generator for the number of machines running by thinking of it like a birth-death process.

$$\begin{bmatrix} -\mu & \mu & 0 & 0 \\ \lambda & -(\mu+\lambda) & \mu & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & (M-1)\lambda & -(\mu+(M-1)\lambda) & \mu \\ 0 & 0 & M\lambda & -M\lambda \end{bmatrix}$$

   Let $\pi$ be

$$[\pi_0, \pi_1, \pi_2, \pi_3, \ldots]$$

$$-\mu\pi_0 + \lambda\pi_1 = 0$$

$$\pi_1 = \frac{\mu}{\lambda}\pi_0$$

$$\mu\pi_0 - (\mu+\lambda)\pi_1 + n\lambda\pi_2 = 0$$

$$\pi_2 = \frac{\mu}{2\lambda}\pi_1$$

$$\vdots$$

$$\pi_n = \frac{\mu}{n\lambda}\pi_{n-1}$$

   Thus,

$$\pi_n = \frac{1}{n!}\left(\frac{\mu}{\lambda}\right)^n \pi_0$$

   To normalize,

$$\sum_{n=0}^{M} \frac{1}{n!}\left(\frac{\mu}{\lambda}\right)^n \pi_0 = 1$$

$$\pi_0 \sum_{n=0}^{M} \frac{1}{n!}\left(\frac{\mu}{\lambda}\right)^n = 1$$

$$\pi_0 = \frac{1}{\sum_{n=0}^{M} \frac{1}{n!}\left(\frac{\mu}{\lambda}\right)^n}$$

   (a) Expected number of machinces not in use =

$$= \sum_{k=0}^{M}(M-k)\pi(k)$$

$$= \sum_{k=0}^{M} M\pi(k) - \sum_{k=0}^{M} k\pi(k)$$

$$= M \sum_{k=0}^{M} \pi(k) - \sum_{k=0}^{M} k\pi(k)$$

$$= M - \sum_{k=0}^{M} \frac{k}{k!} \left(\frac{\mu}{\lambda}\right)^k \pi_0$$

$$= M - \frac{\sum_{k=0}^{M} \frac{k}{k!} \left(\frac{\mu}{\lambda}\right)^k}{\sum_{k=0}^{M} \frac{1}{k!} \left(\frac{\mu}{\lambda}\right)^k}$$

(b) Probability that a given machine ($M_x$) is in use = (Probability of n machines in use * Probability that $M_x$ is one of them) for all n

$$= \sum_{n=0}^{M} \frac{n}{M} \pi(n)$$

$$= \frac{1}{M} \sum_{n=0}^{M} n\pi(n)$$

$$= \frac{\sum_{k=0}^{M} \frac{k}{k!} \left(\frac{\mu}{\lambda}\right)^k}{M \sum_{k=0}^{M} \frac{1}{k!} \left(\frac{\mu}{\lambda}\right)^k}$$