AMATH 569: Final Exam

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1. (a)

$$(-\partial_t + A)\Gamma(t,x,y) = 0$$
$$\Gamma(0,x,y) = \delta_y(x)$$

Taking the fourier transform of the pde,

$$-\int_R dxe^{-iwx}\partial_t\Gamma(t,x,y) - \int_R dxe^{-iwx}x\partial_x\Gamma(t,x,y) + \frac{1}{2}\int_R dxe^{-iwx}\partial_x^2\Gamma(t,x,y) = 0$$

$$-\partial_t\hat{\Gamma} - i\frac{d(iw\hat{\Gamma})}{dw} - \frac{w^2}{2}\hat{\Gamma} = 0$$
$$-\partial_t\hat{\Gamma} + \frac{d(w\hat{\Gamma})}{dw} - \frac{w^2}{2}\hat{\Gamma} = 0$$
$$-\partial_t\hat{\Gamma} + \hat{\Gamma} + w\hat{\Gamma}_w - \frac{w^2}{2}\hat{\Gamma} = 0$$

Taking the fourier transform of the initial condition,

$$\int_R dxe^{-iwx}\Gamma(0,x,y) = \int_R dxe^{-iwx}\delta_y(x) = e^{-iwy}$$

$$\textbf{PDE :}\ \partial_t\hat{\Gamma} - \hat{\Gamma} - w\hat{\Gamma}_w + \frac{w^2}{2}\hat{\Gamma} = 0$$
$$\textbf{IC :}\ \Gamma(0,w,y) = e^{-iwy}$$

(b)

$$\partial_t\hat{\Gamma} - w\hat{\Gamma}_w = \left(1 - \frac{w^2}{2}\right)\hat{\Gamma}$$

We use method of characteristics to solve this pde

$$\frac{dw}{dt} = -w$$

$$w = w_0 e^{-t}$$
$$\frac{d\hat{\Gamma}}{dt} = \left(1 - \frac{w^2}{2}\right)\hat{\Gamma}$$

$$\frac{d\hat{\Gamma}}{dw}\frac{dw}{dt} = (1 - \frac{w^2}{2})\hat{\Gamma}$$

$$\frac{d\hat{\Gamma}}{dw} - w = (1 - \frac{w^2}{2})\hat{\Gamma}$$

$$\frac{d\hat{\Gamma}}{dw} = (\frac{w}{2} - \frac{1}{w})\hat{\Gamma}$$

$$\frac{d\hat{\Gamma}}{\hat{\Gamma}} = (\frac{w}{2} - \frac{1}{w})dw$$

Integrating,

$$ln\hat{\Gamma} = w^2 - ln(w) + C$$

$$\hat{\Gamma} = A\frac{e^{w^2}}{w}$$

At t=0,

$$\hat{\Gamma}(0, w_0) = A\frac{e^{w_0^2}}{w_0}$$

$$e^{-iw_0 y} = A\frac{e^{w_0^2}}{w_0}$$

$$A = w_0 e^{-iw_0 y - w_0^2}$$

$$A = we^t e^{-iwe^t y - w^2 e^{2t}}$$

$$\hat{\Gamma} = we^t e^{-iwe^t y - w^2 e^{2t}}\frac{e^{w^2}}{w}$$

$$\boxed{\hat{\Gamma} = e^t e^{-iwe^t y - w^2 e^{2t} + w^2}}$$

(c)

$$\hat{\Gamma} = e^t e^{-iwe^t y - w^2 e^{2t} + w^2}$$

$$\Gamma = \frac{1}{2\pi}\int_R \hat{\Gamma} e^{iwx} dw$$

$$= \frac{1}{2\pi}\int_R e^t e^{-iwe^t y - w^2 e^{2t} + w^2} e^{iwx} dw$$

$$\Gamma(t, x, y) = \frac{e^t}{2\pi}\int_R e^{w(ix - ie^t y) - (e^{2t} - 1)w^2} dw$$

$$\boxed{\Gamma(t, x, y) = \frac{e^t}{\sqrt{4\pi(e^{2t} - 1)}} exp(\frac{(e^t y - x)^2}{4(e^{2t} - 1)})}$$

2. (a)

$$A = \frac{1}{2}\partial_x^2 - \lambda$$

Let us assume the eigenfunction are of the form e^{iwx}. For the boundary conditions, f(0)= f(π)=0, we have

$$w = n$$

So,

$$\phi_n = e^{inx}$$

$$A\phi_n = \lambda_n \phi_n$$

$$(-\frac{n^2}{2} - \lambda)\phi_n = \lambda_n \phi_n$$

$$\lambda_n = -\frac{n^2}{2} - \lambda$$

When λ is $-\frac{n^2}{2}$, ϕ_n is not neccesarily zero. So, at this value, there is no unique solution.

$$\boxed{\lambda = -\frac{n^2}{2}}$$

(b) For at least one solution, the solution must not blow up if the eigenvalue is zero. When eigenvalue is zero,

$$\lambda = -\frac{n^2}{2}$$

$$n = \sqrt{-2\lambda}$$

$$\phi = e^{i\sqrt{-2\lambda}x}$$

We need

$$< \phi, g >= 0$$

$$\boxed{\int_R dxe^{-i\sqrt{-2\lambda}x} g = 0}$$

(c) Any solution of the following form would satisfy the pde.

$$u = \sum_{n=-\infty, n\neq\sqrt{-2\lambda}}^{\infty} \lambda_n e^{inx} + ce^{i\sqrt{-2\lambda}x}$$

where c can be any number.

$$\boxed{u = \sum_{n=-\infty, n\neq\sqrt{-2\lambda}}^{\infty} (-\frac{n^2}{2} - \lambda)e^{inx} + ce^{i\sqrt{-2\lambda}x}}$$

3. (a)

$$A = \frac{1}{2} e^{x^2} \partial_x e^{-x^2} \partial_x = x\partial_x + \frac{1}{2}\partial_x^2$$

Naturally, we see that Γ from (1) is the Green's function for this equation.

$$\boxed{u(t,x) = \int_R dy \Gamma(t,x,y) f(x) + \int_0^t ds \Gamma(t-s,x,y) g(s,x)}$$

(b) If A is expressed in the following form,

$$\frac{1}{m(x)} \partial_x \frac{1}{s(x)} \partial_x$$

then it is formally self-adjoint in L(, m)

$$A = \frac{1}{2} e^{x^2} \partial_x e^{-x^2} \partial_x$$

Looking at this, we can get the following expression for m.

$$\boxed{m(x) = \frac{c}{2} e^{-x^2}}$$

(c) Γ solves this pde

$$(-\partial_t + A)u = 0$$

Treating A like a constant and solving it like and ode, we have

$$\Gamma = e^{At}\delta_y(x)$$

$$= \sum_n e^{\lambda_n t} < \phi_n(x), \delta_y(x) > \phi_n$$

$$= \sum_n e^{\lambda_n t} < \phi_n(x), \delta_y(x) > \phi_n$$

$$\boxed{\Gamma = \sum_n e^{\lambda_n t} \phi_n(y) \phi_n}$$