AMATH 567: Problem Set 8

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1. (a)

$$f(z) = \frac{z^2+1}{z^2-a^2} = \frac{z^2+1}{(z-a)(z+a)}$$

i. Since the singularities lie inside the unit circle as $a < 1$,

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = Res_{z=-a}f(z) + Res_{z=a}f(z)$$

$$= -\frac{a^2+1}{2a} + \frac{a^2+1}{2a} = 0$$

ii. Taking the residue at infinity,

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = -Res_{z=\infty}f(z) = Res_{t=0}(\frac{1}{t^2})f(\frac{1}{t})$$

$$= Res_{t=0}(\frac{1}{t^2})\frac{\frac{1}{t^2}+1}{\frac{1}{t^2}-a^2}$$

$$= Res_{t=0}(\frac{1}{t^2})\frac{t^2+1}{1-a^2t^2}$$

As it is a double pole at t=0,

$$= \lim_{t\to 0}\frac{d}{dt}\left(\frac{t^2+1}{1-a^2t^2}\right)$$

$$= \lim_{t\to 0}\left(\frac{2t(1-a^2t^2)+2a^2t(t^2+1)}{(1-a^2t^2)^2}\right)$$

$$= 0$$

(b)

$$f(z) = \frac{1}{z} + \frac{1}{z^3}$$

i. The singularity is at 0,

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = Res_{z=0}f(z) = 1$$

ii. Taking the residue at infinity,

$$\lim_{z\to\infty} f(z) = \frac{1}{\infty} + \frac{1}{\infty} = 0$$

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = Res_{z=\infty}f(z) = \lim_{z\to\infty} zf(z) = 1$$

(c)

$$f(z) = z^2 e^{-\frac{1}{z}} = z^2 - z + \frac{1}{2} - \frac{1}{3!z} + \frac{1}{4!z^2} + \dots$$

i. The singularity is at 0,

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = Res_{z=0}f(z) = -\frac{1}{3!}$$

ii. Taking the residue at infinity,

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = Res_{z=\infty}f(z) = Res_{t=0}(\frac{1}{t^2})f(\frac{1}{t})$$

$$= Res_{t=0}(\frac{1}{t^2})(\frac{1}{t^2} - \frac{1}{t} + \frac{1}{2} - \frac{t}{3!} + \frac{t^2}{4!} + \dots)$$

$$= Res_{t=0}(\frac{1}{t^4!} - \frac{1}{t^3} + \frac{1}{2t^2} - \frac{1}{3!t^2} + \frac{1}{4!} + \dots)$$

$$= -\frac{1}{3!}$$

2.

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \int_0^\infty \frac{dx}{(x+ia)(x-ia)(x+ib)(x-ib)}$$

Consider the contour of a semicircle of radius R with its diameter going from -R to R on the real line. Let C be the contour as R goes to infinity. The singularities inside C are at ib and ia. C_R is the arc going from 0 to 2π.

$$f(z) = \frac{1}{(z+ia)(z-ia)(z+ib)(z-ib)}$$

$$\int_C f(z)dz = 2\pi i(Res_{z=ib}f(z) + Res_{z=ia}f(z)$$

$$\int_{C_R} f(z)dz + \int_0^\infty f(x)dx + \int_{-\infty}^0 f(x)dx = 2\pi i(Res_{z=ib}f(z) + Res_{z=ia}f(z))$$

$$\underbrace{0}_{\text{Theorem 4.2.1}} + \int_0^\infty f(x)dx + \int_{-\infty}^0 f(x)dx = 2\pi i(\frac{1}{(i(b+a))(i(b-a))(2ib))} - \frac{1}{(i(b+a))(i(b-a))(2ia))})$$

$$\int_0^\infty f(x)dx + \int_{-\infty}^0 f(x)dx = 2\pi i(\frac{2i(a-b)}{(i(b+a))(i(a-b))(4ab))}) = \frac{4\pi}{((b+a))(4ab))}$$

$$\int_0^\infty f(x)dx + \int_{-\infty}^0 f(x)dx = \int_0^\infty f(x)dx + \int_\infty^0 \frac{e^{\pi i}}{(t^2+a^2)(t^2+b^2)}dt$$

$$= \int_0^\infty f(x)dx - \int_\infty^0 \frac{1}{(t^2+a^2)(t^2+b^2)}dt$$

$$= \int_0^\infty f(x)dx + \int_0^\infty \frac{1}{(t^2+a^2)(t^2+b^2)}dt$$

$$= 2\int_0^\infty f(x)dx$$

$$\int_0^\infty f(x)dx = \frac{\pi}{2((b+a))(ab))}$$

When a=b,

$$f(z) = \frac{1}{(z+ia)^2(z-ia)^2}$$

$$2\int_0^\infty f(x)dx = 2\pi i(Res_{z=ia}f(z))$$

Since there is a double pole at ia,

$$2\int_0^\infty f(x)dx = 2\pi i \lim_{z\to ia}\frac{d}{dx}\frac{1}{(z+ia)^2}$$

$$\int_0^\infty f(x)dx = \pi i \lim_{z\to ia}\frac{-2}{(z+ia)^3} = \frac{\pi}{4a^3}$$

3.

$$\int_{-\infty}^\infty \frac{cos(kx)dx}{(x^2+a^2)(x^2+b^2)} = \int_{-\infty}^\infty \frac{cos(kx)dx}{(x+ia)(x-ia)(x+ib)(x-ib)}$$

Consider the contour of a semicircle of radius R with its diameter going from -R to R on the real line. Let C be the contour as R goes to infinity. The singularities inside C are at ib and ia. C_R is the arc going from 0 to 2π.

$$p(z) = \frac{cos(kx)dx}{(x+ia)(x-ia)(x+ib)(x-ib)}$$

$$p(z) = Re(f(z))$$

$$f(z) = \frac{e^{ikz}}{(z+ia)(z-ia)(z+ib)(z-ib)}$$

$$\int_{C_R} f(z)dz + \int_{-\infty}^\infty f(x)dx = 2\pi i(Res_{z=ia}f(z) + Res_{z=ib}f(z)$$

$$\underbrace{0}_{\text{Jordan's Lemma}} + \int_{-\infty}^\infty f(x)dx = 2\pi i(Res_{z=ia}f(z) + Res_{z=ib}f(z))$$

$$\int_{-\infty}^\infty f(x)dx = 2\pi i(\frac{e^{-ka}}{(2ia)i(a+b)i(a-b)} - \frac{e^{-kb}}{i(b+a)i(a-b)(2ib)})$$

$$= \pi(-\frac{e^{-ka}}{(a)(a+b)(a-b)} + \frac{e^{-kb}}{(b+a)(a-b)(b)})$$

$$= \pi(\frac{ae^{-kb} - be^{-ka}}{(b+a)(a-b)(ab)})$$

$$\int_{-\infty}^\infty p(x)dx = Re\left(\int_{-\infty}^\infty f(x)dx\right) = \pi(\frac{ae^{-kb} - be^{-ka}}{(b+a)(a-b)(ab)})$$

When a=b, there is a double pole at ia,

$$\int_{-\infty}^\infty f(x)dx = 2\pi i(Res_{z=ia}\frac{e^{ikz}}{(z+ia)^2(z-ia)^2})$$

$$= 2\pi i \lim_{z\to ia}\frac{d}{dx}\frac{e^{ikz}}{(z+ia)^2}$$

$$= 2\pi i \lim_{z\to ia}\frac{-2e^{ikz} + ike^{ikz}(z+ia)}{(z+ia)^3}$$

$$= 2\pi i\frac{-2e^{-ka} - 2kae^{-ka}}{(-8ia^3)}$$

$$= \pi\frac{e^{-ka} + kae^{-ka}}{(2a^3)}$$