AMATH 575: Problem Set 4

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1. (a) The first iteration has 20 blocks in 27 positions. After n iterations, this is $\frac{20}{27}^n$. As n goes to infinity, this becomes zero.

   (b) Let us look at epsilon as a series.

$$\epsilon_n = (\frac{1}{3})^n$$

   Here, in the first iteration, ϵ refers to the side length on one of the 27 cubes. We would need 20 blocks in the first iteration and 20^n in the following ones.

$$D = \lim_{n->\infty} \frac{ln(N)}{ln(\frac{1}{\epsilon}^n)} = \lim_{n->\infty} \frac{ln(20^n)}{ln(3^n)} = \frac{20}{3}$$

   (c) For a d-dimensional hypercube, it would be divided by into 3^d boxes with edge length $\frac{1}{3}$. Of these, one box is deleted at each d-1 face and one central box is deleted as well. There are 2d d-1 faces. Thus , the number of boxes is given by $3^d - 2d - 1$.

$$D = \lim_{n->\infty} \frac{ln(N)}{ln(\frac{1}{\epsilon}^n)} = \lim_{n->\infty} \frac{ln((3^d - 2d - 1)^n)}{ln(3^n)} = \frac{ln(3^d - 2d - 1)}{ln(3)}$$

   For 4-d, we get the dimension as 3.89

2. The standard map requires around 7 iterations to see good horseshoes. We get the following result.

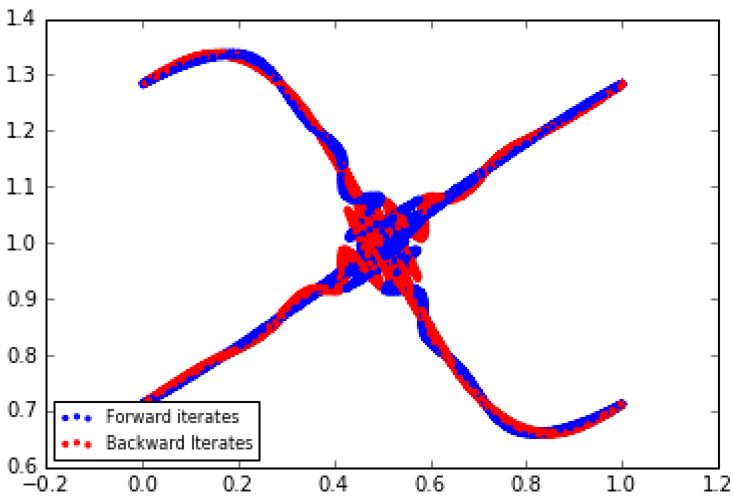

Figure 1: Forward and backward iterations with the standard map

Zooming in near the fixed point reveals horseshoes. The rectangle bounds one horseshoe quite well.

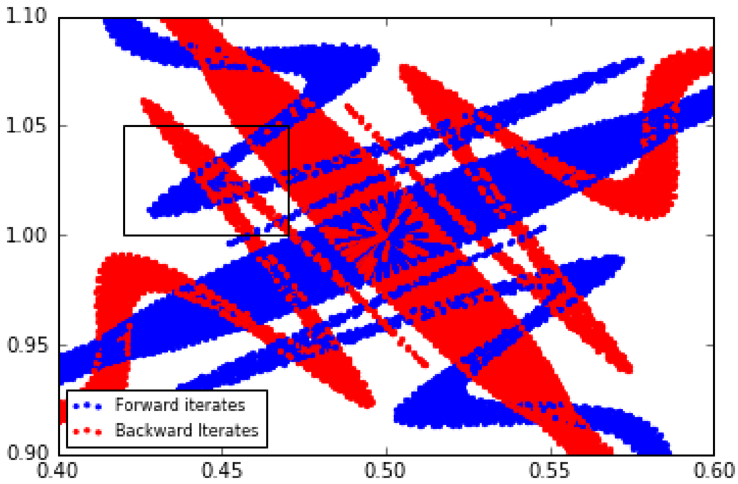

Figure 2: Horseshoes in the standard map

3. The following code generates the visualization of the horseshoe in the Henon map.

```
import numpy as np
import matplotlib.pyplot as plt
import decimal
decimal.getcontext().prec = 50

def hmap1(x):
        return [5-0.3*x[1]-x[0]**2,x[0]]
def hmap2(x):
        return [x[1],(5-x[0]-x[1]**2)/0.3]

def plotstuff():
        xspace = np.linspace(-3,3,300)
        yspace = np.linspace(-3,3,300)
        for x in xspace:
                for y in yspace:
                        [xnew,ynew]=hmap1([x,y])
                        [xnew2,ynew2]=hmap2([x,y])
                        plt.scatter(xnew,ynew,s=5,color='r')
                        plt.scatter(xnew2,ynew2,s=5,color='b')
plt.show

plotstuff()
```

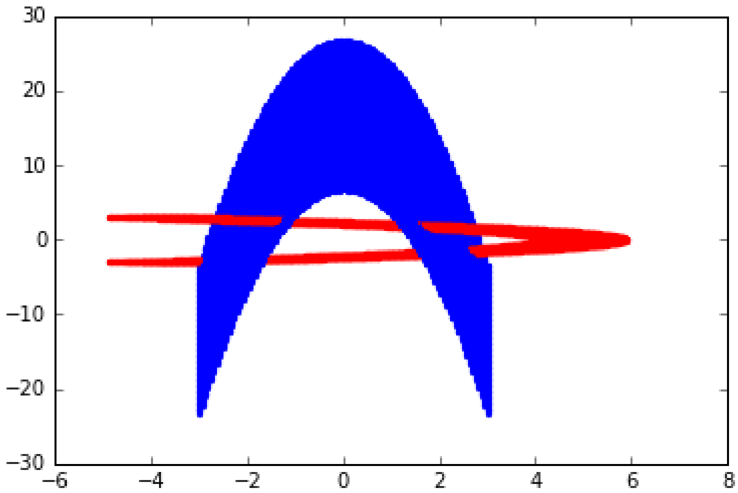

Figure 3: Horseshoe in the Henon Map

4. We have

$$y' = Jy + F_2^R(y) + F_3^R(y) + \ldots + F_{n+1}(y) + O(n+2)$$

We use

$$y = z + h_{n+1}(z)$$

$$z' + Dh_{n+1}(z)z' = Jz + Jh_{n+1}(z) + F_2^R(z + h_{n+1}(z)) + F_3^R(z + h_{n+1}(z)) + \ldots + F_{n+1}(y) + O(n+2)$$

$$[I + Dh_{n+1}(z)]z' = Jz + Jh_{n+1}(z) + F_2^R(z + h_{n+1}(z)) + \ldots + F_{n+1}(z + h_{n+1}(z)) + O(n+2)$$

$$z' = [I - Dh_{n+1}(z)](Jz + Jh_{n+1}(z) + F_2^R(z + h_{n+1}(z)) + \ldots + F_{n+1}(z + h_{n+1}(z)) + O(n+2))$$

$$= Jz + Jh_{n+1}(z) - JzDh_{n+1}(z) + F_2^R(z) + \ldots + F_{n+1}(z) + O(n+2))$$

$$= Jz + L^{n+1}(z) + F_2^R(z) + \ldots + F_{n+1}(z) + O(n+2))$$

The operator is given by

$$L^{n+1}(z) = Jh_{n+1}(z) - JDh_{n+1}(z)z$$

5.

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda 2 \end{bmatrix}$$

We need to see the outputs of the operator L on the span of H_2 where L is given by

$$Lh_2 = Jh_2 - Dh_2 J \begin{bmatrix} x \\ y \end{bmatrix}$$

Evaluating this would be much easier on a symbolic software. Instead of Mathematica, I use Sympy in Python. The code and output for the first case (x^2,0) are shown below.

```
from sympy import *
x= Symbol('x')
y= Symbol('y')
a= Symbol('a') #lambda1
b= Symbol('b') #lambda2
```

```
J= Matrix([[a,0], [0,b]])
H=Matrix([[x*2], [0]])
Dh=Matrix([[2*x,0], [0,0]])
L=J*H-Dh*J*Matrix([[x], [y]])
```

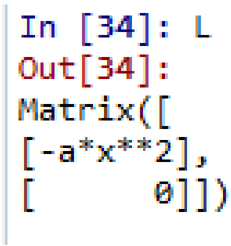

Figure 4: Output

$$L\begin{bmatrix} x^2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\lambda_1 x^2 \\ 0 \end{bmatrix}$$

$$L\begin{bmatrix} y^2 \\ 0 \end{bmatrix} = \begin{bmatrix} (\lambda_1 - 2\lambda_2)y^2 \\ 0 \end{bmatrix}$$

$$L\begin{bmatrix} xy \\ 0 \end{bmatrix} = \begin{bmatrix} -\lambda_2 xy \\ 0 \end{bmatrix}$$

$$L\begin{bmatrix} 0 \\ xy \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_1 xy \end{bmatrix}$$

$$L\begin{bmatrix} 0 \\ y^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_2 y^2 \end{bmatrix}$$

$$L\begin{bmatrix} 0 \\ x^2 \end{bmatrix} = \begin{bmatrix} 0 \\ (\lambda_2 - 2\lambda_1)x^2 \end{bmatrix}$$

If $\lambda_2 - 2\lambda_1 = 0$, the normal form is given by

$$x' = \lambda_1 x + O(3)$$

$$y' = \lambda_2 y + a_1 x^2 + O(3)$$

or if $\lambda_1 - 2\lambda_2 = 0$, the normal form is given by

$$x' = \lambda_1 x + a_1 y^2 + O(3)$$

$$y' = \lambda_2 y + O(3)$$

Otherwise, the normal form is

$$x' = \lambda_1 x + O(3)$$

$$y' = \lambda_2 y + O(3)$$

6.

$$L\begin{bmatrix} x^3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2\lambda_1 x^3 \\ 0 \end{bmatrix}$$

$$L\begin{bmatrix} y^3 \\ 0 \end{bmatrix} = \begin{bmatrix} (\lambda_1 - 3\lambda_2)y^2 \\ 0 \end{bmatrix}$$

$$L\begin{bmatrix} x^2y \\ 0 \end{bmatrix} = \begin{bmatrix} -(\lambda_1+\lambda_2)x^2y \\ 0 \end{bmatrix}$$

$$L\begin{bmatrix} xy^2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2\lambda_2 xy^2 \\ 0 \end{bmatrix}$$

$$L\begin{bmatrix} 0 \\ x^2y \end{bmatrix} = \begin{bmatrix} 0 \\ -2\lambda_1 x^2y \end{bmatrix}$$

$$L\begin{bmatrix} 0 \\ xy^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -(\lambda_1+\lambda_2)xy^2 \end{bmatrix}$$

$$L\begin{bmatrix} 0 \\ y^3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2\lambda_2 y^3 \end{bmatrix}$$

$$L\begin{bmatrix} 0 \\ x^3 \end{bmatrix} = \begin{bmatrix} 0 \\ (\lambda_2-3\lambda_1)x^3 \end{bmatrix}$$

If $\lambda_2 - 3\lambda_1 = 0$, the normal form is given by

$$x' = \lambda_1 x + O(4)$$

$$y' = \lambda_2 y + a_1 x^3 + O(4)$$

or if $\lambda_1 - 3\lambda_2 = 0$, the normal form is given by

$$x' = \lambda_1 x + a_1 y^3 + O(4)$$

$$y' = \lambda_2 y + O(4)$$

or if $\lambda_1 + \lambda_2 = 0$, the normal form is given by

$$x' = \lambda_1 x + a_1 x^2 y + O(4)$$

$$y' = \lambda_2 y + a_2 xy^2 + O(4)$$

Otherwise, the normal form has only the terms from second order and in the best case scenario becomes

$$x' = \lambda_1 x + O(4)$$
$$y' = \lambda_2 y + O(4)$$

7. Takens-Bogdanov has the following J.

$$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$L\begin{bmatrix} x^3 \\ 0 \end{bmatrix} = \begin{bmatrix} -3x^2y \\ 0 \end{bmatrix}$$

$$L\begin{bmatrix} y^3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$L\begin{bmatrix} x^2y \\ 0 \end{bmatrix} = \begin{bmatrix} -2xy^2 \\ 0 \end{bmatrix}$$

$$L\begin{bmatrix} xy^2 \\ 0 \end{bmatrix} = \begin{bmatrix} -y^3 \\ 0 \end{bmatrix}$$

$$L\begin{bmatrix} 0 \\ x^2y \end{bmatrix} = \begin{bmatrix} x^2y \\ -2xy^2 \end{bmatrix}$$

$$L\begin{bmatrix} 0 \\ xy^2 \end{bmatrix} = \begin{bmatrix} xy^2 \\ -y^3 \end{bmatrix}$$

$$L\begin{bmatrix} 0 \\ y^3 \end{bmatrix} = \begin{bmatrix} y^3 \\ 0 \end{bmatrix}$$

$$L\begin{bmatrix} 0 \\ x^3 \end{bmatrix} = \begin{bmatrix} x^3 \\ -3x^2y \end{bmatrix}$$

We see that L cannot span $\begin{bmatrix} 0 \\ x^3 \end{bmatrix}$ and one of $\begin{bmatrix} x^3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -3x^2y \end{bmatrix}$ Thus,we get the normal form to be

$$x' = y + a_1x^2 + a_2x^3 + O(4)$$
$$y' = a_2x^2 + a_4x^3 + a_5y^3 + O(4)$$