AMath 574
Homework 3
Due by 11:00pm on February 9, 2017

For submission instructions, see:

`http://faculty.washington.edu/rjl/classes/am574w2017/homework3.html`

Problem #1

Consider the scalar conservation law with flux function $f(q) = q^3 - 4q^2 + 3q = q(q-1)(q-3)$.

(a) Show that the flux is convex as long as we restrict the data to fall within $-\infty < q < 4/3$ or within $4/3 < q < +\infty$.

Solution:

$$f''(q) = 6q - 8$$

This is a cubic function. So, the only point where the function is not convex(or concave) is when $f''(q) = 0$, i.e at q = 4/3. Thus, the flux is convex as long as $-\infty < q < 4/3$ or $4/3 < q < +\infty$.

(b) Determine the exact solution to the Riemann problem with data $q_\ell = 3, q_r = 2$.

Solution:

$$f'(q) = 3q^2 - 8q + 3$$

$$f'(q_l) = 27 - 24 + 3 = 6$$

$$f'(q_r) = 12 - 16 + 3 = -1$$

So, this is a shock.

The shock speed is given by

$$s = \frac{f(q_r) - f(q_l)}{q_r - q_l} = \frac{0+2}{-1} = -2$$

Thus,

$$q(x,t) = \begin{cases} 3 & \text{if } x/t < -2 \\ 2 & \text{if } x/t > -2 \end{cases}$$

(c) Determine the exact solution to the Riemann problem with data $q_\ell = 2, q_r = 3$. In this case the solution is a rarefaction wave, so determine the solution in the form of a similarity solution $q(x,t) = Q(x/t)$ and find an exact expression for the function $Q(\xi)$. At the trailing edge of the rarefaction wave there is a kink, a jump in the slope of the solution, from 0 to some non-zero value. What value (as a function of time)?

Solution:

$$f'(q_l) = -1$$

$$f'(q_r) = 6$$

So, this is a rarefaction.

$$q(x,t) = \begin{cases} 2 & \text{if } x/t < -1 \\ 3 & \text{if } x/t > 6 \end{cases}$$

For the middle state,

$$f'(q_m) = \frac{x}{t}$$

$$3q_m^2 - 8q_m + 3 = \frac{x}{t}$$

$$q_m = \frac{4}{3} + \sqrt{\frac{7}{9} + \frac{x}{3t}}$$

So, plugging in x = -t and x = 6t, we get 2 and 3. So, the function is continuous.

The kink is given by

$$q_x = \frac{1}{6}(\frac{7}{9} + \frac{x}{3t})^{-1/2}$$

(d) Re-do part (c) when $q_\ell = 4/3$, $q_r = 3$. Comment on why you expect the slope at the trailing edge of the rarefaction wave to be infinite in this case. Also plot the solution as a function of x at time $t = 1$.

Solution:

$$f'(q_l) = -\frac{7}{3}$$

$$f'(q_r) = 6$$

So, this is a rarefaction.

$$q(x,t) = \begin{cases} \frac{4}{3} & \text{if } x/t - \frac{7}{3} \\ 3 & \text{if } x/t > 6 \end{cases}$$

For the middle state,

$$f'(q_m) = \frac{x}{t}$$

$$3q_m^2 - 8q_m + 3 = \frac{x}{t}$$

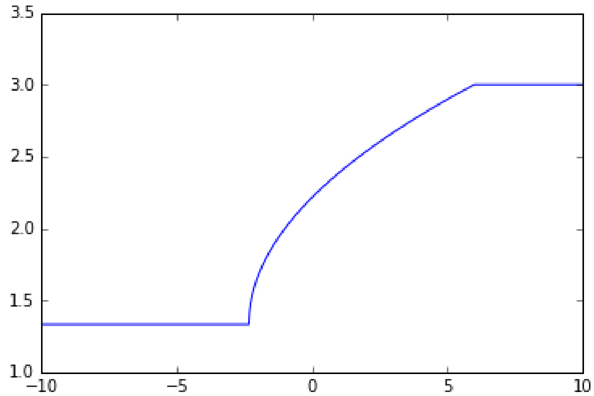
$$q_m = \frac{4}{3} + \sqrt{\frac{7}{9} + \frac{x}{3t}}$$

So, plugging in 3x = -7t and x = 6t, we get $\frac{4}{3}$ and 3. So, the function is continuous.

The slope is given by

$$q_x = \frac{1}{6}(\frac{7}{9} + \frac{x}{3t})^{-1/2}$$

At 3x = -t, this becomes infinite. This is easily seen in the solution at time =1 plotted below.



(e) Consider this same conservation law with initial data

$$q(x,0) = \begin{cases} 4 & \text{if } x < 0, \\ 3 & \text{if } 0 < x < 1, \\ 2 & \text{if } x > 1 \end{cases}$$

The solution consists of two shocks that merge at some time — determine the time when they merge, and the new shock speed.

Solution:

$$s_1 = \frac{f(q_r) - f(q_l)}{q_r - q_l} = \frac{0 - 12}{3 - 4} = 12$$

$$s_2 = \frac{f(q_r) - f(q_l)}{q_r - q_l} = \frac{-2 - 0}{2 - 3} = 2$$

There is a right going shock from x=0 and a right going shock from x=1 traveling slower. Let us suppose they merge at time t.

$$0 + s_1 t = 1 + s_2 t$$

$$12t = 1 + 2t$$

$$t = \frac{1}{10}$$

The new shock speed is

$$s = \frac{f(2) - f(4)}{2 - 4} = \frac{-2 - 12}{-2} = 7$$

Problem #2.

Some sample code in `$AM574/homeworks/hw3/burgers` should help get you started with this problem. I will also provide a video.

Modify the code provided to solve the conservation law from Problem #1 above, assuming that the initial data satisfies $q(x,0) \geq 4/3$ for all x.

Test it out on $-2 \leq x \leq 2$ with initial data

$$q(x,0) = \begin{cases} 4/3 & \text{if } -2 \leq x < 0, \\ 3 & \text{if } 0 \leq x \leq 2, \end{cases}$$

and *periodic boundary conditions*. Solve the problem up to time $t = 0.25$.

Modify the plotting specified in the provided `setplot.py` file so that you plot the true solution to this problem along with the approximate solution.

Experiment with this code and comment on what you observe. You might want to try:

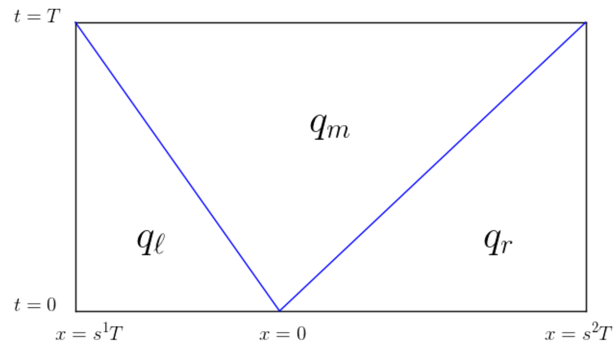
- Changing the grid resolutions, e.g. using 100, 200, or more grid cells.
- Use the first order accurate method (specify `clawdata.order = 1`) in `setrun.py` and compare to the high resolution method with `clawdata.order = 2` and `clawdata.limiter = ['mc']`.
- Compare results with and without the entropy fix.

What to turn in:

- Create a new directory `cubic` that has the necessary files, in particular `Makefile, setrun.py, setplot.py, rp1_cubic.f90, qinit.f90, setprob.f90`. Note that `rp1_cubic.f90` will be a modified version of `rp1_burgers.f90`.
- Turn in a tar file of this directory, set up for the case where the high-resolution method is used with the entropy fix and 100 grid cells.
- Provide plots of a few other key results to illustrate your discussion.
- One way to write up your observations would be to modify the Jupyter notebook `burgers.ipynb` to make a new notebook `cubic.ipynb` that contains some discussion and examples illustrating your results.

Problem #3.

Suppose the solution to a Riemann problem for some system of conservation laws $q_t + f(q)_x = 0$ consists of exactly 2 waves with wave speeds $s^1 < 0 < s^2$. Given Riemann data q_ℓ and q_r, let q_m be the resulting state between the two waves. Then we can integrate over the space-time region shown below in order to find a simple expression for q_m in terms of q_ℓ, q_r, s^1, and s^2 (in a similar manner to how integrating over the region shown in Figure 11.7 gives the Rankine-Hugoniot condition).



(a) Use this approach to compute the formula for q_m.

Solution:

$$\int_X q(x, t_{n+1})dx - \int_X q(x, t_n)dx = \int_{t_n}^{t_{n+1}} f(q_l)dt - \int_{t_n}^{t_{n+1}} f(q_r)dt$$

$$\int_{s^1T}^{s^2T} q_m dx - \int_{s^1T}^{0} q_l dx - \int_{0}^{s^2T} q_r dx = \int_0^T f(q_l)dt - \int_0^T f(q_r)dt$$

$$(s^2 - s^1)Tq_m = -s^1Tq_l + s^2q_rT + T(f(q_l) - f(q_r))$$

$$q_m = \frac{-s^1q_l + s^2q_r + f(q_l) - f(q_r)}{s^2 - s^1}$$

(b) Define two waves by $\mathcal{W}^1 = q_m - q_\ell$ and $\mathcal{W}^2 = q_r - q_m$. Show that

$$s^1\mathcal{W}^1 + s^2\mathcal{W}^2 = f(q_r) - f(q_\ell).$$

Solution:

$$
\begin{aligned}
(s^2 - s^1)q_m &= -s^1 q_l + s^2 q_r + (f(q_l) - f(q_r)) \\
s^1(q_l - q_m) + s^2(q_m - q_r) &= f(q_l) - f(q_r) \\
q_m &= \frac{-s^1 q_l + s^2 q_r + f(q_l) - f(q_r)}{s^2 - s^1}
\end{aligned}
$$

(c) Apply the formula from (a) to the case of constant coefficient linear acoustics with $s^1 = -c$ and $s^2 = +c$ and show that the resulting q_m agrees with what was found in (3.32) in the book based on the eigenvectors of the coefficient matrix.

Solution:

$$
\begin{aligned}
q_m &= \frac{-s^1 q_l + s^2 q_r + f(q_l) - f(q_r)}{s^2 - s^1} \\
f(q) &= \begin{bmatrix} K_0 u \\ \frac{P}{\rho_0} \end{bmatrix} \\
q_m(1) &= \frac{-cP_l + cP_r + K_0(u_l - u_r)}{2c} \\
&= \frac{1}{2}(-P_l + P_r + Z_0(u_l - u_r)) \\
q_m(2) &= \frac{-cu_l + cu_r + \frac{1}{\rho_0}(P_l - P_r)}{2c} \\
&= \frac{1}{2}(-u_l + u_r + \frac{1}{Z_0}(P_l - P_r)) \\
q_m &= \frac{1}{2}\begin{bmatrix} P_r - P_l - Z_0(u_r - u_l) \\ u_r - u_l - \frac{P_r - P_l}{Z_0} \end{bmatrix}
\end{aligned}
$$

This agrees with 3.32.

Note: For any system of m equations we could choose any values $s^1 < s^2$ and use the formula you found to define a state q_m and hence define two waves $\mathcal{W}^1 = q_m - q_\ell$ and $\mathcal{W}^2 = q_r - q_m$. These waves could then be used to obtain a conservative method (which follows from (b), as we will see later). Limiters and high-resolution correction terms can also be based on this waves.

This won't be the exact Riemann solution except for special cases like the acoustics equation (or any linear system of two equations where s^1 and s^2 are chosen to be the two eigenvalues). But it defines an *approximate Riemann solver* that is very cheap to compute and sometimes works well enough. This is called the *HLL solver* after the original work on this idea by Harten, Lax, and van Leer. This is discussed in Section 15.3.7 along with some extensions. (You'll find the solution to part (a) there too.)