Assignment 4.

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Due Monday, Feb. 5.

1. (finite elements) Use the Galerkin finite element method with continuous piecewise linear basis functions to solve the problem

$$-\frac{d}{dx}\left((1+x^2)\frac{du}{dx}\right) = f(x), \quad 0 \le x \le 1,$$

$$u(0) = 0, \quad u(1) = 0.$$

(a) Derive the matrix equation that you will need to solve for this problem.

Solution:

$$\hat{u} = \sum_{j=1}^{n-1} c_j \psi_j$$

$$\begin{pmatrix} a_{ii} & a_{ii+1} & & & \\ a_{ii-1} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & a_{ii-1} & a_{ii} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} \langle f, \psi_1 \rangle \\ \vdots \\ \vdots \\ \langle f, \psi_{n-1} \rangle \end{pmatrix}$$

$$\begin{aligned} a_{ii} &= \int_0^1 (1+x^2)(\psi_i')^2 dx \\ &= \int_{x_{i-1}}^{x_i} (1+x^2)\frac{1}{(x_i - x_{i-1})^2} dx + \int_{x_i}^{x_{i+1}} (1+x^2)\frac{1}{(x_{i+1} - x_i)^2} dx \\ &= \frac{1}{x_i - x_{i-1}} + \frac{x_i^3 - x_{i-1}^3}{3(x_i - x_{i-1})^2} + \frac{1}{x_{i+1} - x_i} + \frac{x_{i+1}^3 - x_i^3}{3(x_{i+1} - x_i)^2} \end{aligned}$$

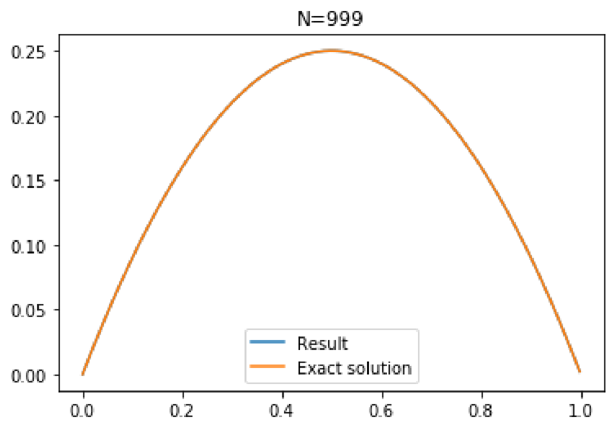
$$\begin{aligned} a_{ii-1} &= \int_0^1 (1+x^2)\psi_i \psi_{i-1} dx \\ &= -\int_{x_{i-1}}^{x_i} (1+x^2)\frac{1}{(x_i - x_{i-1})^2} dx \\ &= -\frac{1}{x_i - x_{i-1}} - \frac{x_i^3 - x_{i-1}^3}{3(x_i - x_{i-1})^2} \end{aligned}$$

$$\begin{aligned} a_{ii+1} &= \int_0^1 (1+x^2)\psi_i \psi_{i+1} dx \\ &= -\frac{1}{x_{i+1} - x_i} - \frac{x_{i+1}^3 - x_i^3}{3(x_{i+1} - x_i)^2} \end{aligned}$$

$$\langle f, \psi_i \rangle = \int_0^1 f\psi_i dx = \int_{x_{i-1}}^{x_i} f\psi_i dx + \int_{x_i}^{x_{i+1}} f\psi_i dx$$

(b) Write a code to solve this set of equations. You can test your code on a problem where you know the solution by choosing a function $u(x)$ that satisfies the boundary conditions and determining what $f(x)$ must be in order for $u(x)$ to satisfy the differential equation. Try $u(x) = x(1-x)$. Then $f(x) = 2(3x^2 - x + 1)$.

Solution:




```
import numpy as np
import matplotlib.pyplot as plt

def f(x):
    return 6*x**2-2*x+2
def xf(x):
    return 6*x**3-2*x**2+2*x
def intf(x):
    return 2*x**3-x**2+2*x
def intxf(x):
    return (6/4)*x**4-(2/3)*x**3+x**2
def fmaker(x,i):
    return (x[i+1]/(x[i+1]-x[i]))*(intf(x[i+1])-intf(x[i]))
       - (x[i-1]/(x[i]-x[i-1]))*(intf(x[i])-intf(x[i-1]))
       +(1/(x[i]-x[i-1]))*(intxf(x[i])-intxf(x[i-1])) -(1/(x
       [i+1]-x[i]))*(intxf(x[i+1])-intxf(x[i]))
def aij(x,i):
    return 1/(x[i]-x[i-1]) +(x[i]**3-x[i-1]**3)/(3*(x[i]-x[i
       -1])**2)
def aii(x,i):
    return aij(x,i)+aij(x,i+1)
def tridiag(a, b, c, k1=-1, k2=0, k3=1):
    return np.diag(a, k1) + np.diag(b, k2) + np.diag(c, k3)
N=1000
x=np.array([])
```

```python
for i in range(N):
    x= np.append(x,(i/(N-1))**2)
#x=np.linspace(0,1,N)
fe = np.array([])
aiivec = np.array([])
aijvec = np.array([])
for i in range(1,len(x)-1):
    fe=np.append(fe,fmaker(x,i))
    aiivec =np.append(aiivec,aii(x,i))
    aijvec =np.append(aijvec,-aij(x,i))
mat2=tridiag(aijvec[1:],aiivec,aijvec[1:])
z=np.zeros(N-2)
k = np.linalg.solve(mat2,fe)
y=x[1:-1]
plt.plot(y,k, label = "Result")
exact=y*(1-y)
plt.plot(y,exact, label = "Exact solution")
plt.legend()
print(np.linalg.norm(k-exact, np.inf))
plt.title("N="+str(N))
```

(c) Try several different values for the mesh size h. Based on your results, what would you say is the order of accuracy of the Galerkin method with continuous piecewise linear basis functions?

Solution:

The infinity norm is used here.

h	Error
0.01	3.2112324338e-06
0.001	3.1537675276e-08

The order seems to be O(h^2)

(d) Now try a nonuniform mesh spacing, say, $x_i = (i/(m+1))^2$, $i = 0, 1, \ldots, m+1$. Do you see the same order of accuracy, if h is defined as the maximum mesh spacing, $\max_i(x_{i+1} - x_i)$?

Solution:

The infinity norm is used here.

h	Error
0.02	1.23156638885e-05
0.002	1.20969674222e-07

Again, the order seems to be O(h^2).

(e) Suppose the boundary conditions were $u(0) = a$, $u(1) = b$. Show how you would represent the approximate solution $\hat{u}(x)$ as a linear combination of hat functions and how the matrix equation in part (a) would change.

Solution:

$$\hat{u} = \sum_{j=0}^{n} c_j \psi_j$$

$$\psi_0(x) = \left\{ \begin{array}{ll} \frac{x_1 - x}{x_1}, & \text{for } 0 \le x \le x_1 \\ 0, & \text{elsewhere} \end{array} \right\}$$

$$\psi_n(x) = \left\{ \begin{array}{ll} \frac{1 - x_{n-1}}{x_n - x_{n-1}}, & \text{for } x_{n-1} \le x \le x_n \\ 0, & \text{elsewhere} \end{array} \right\}$$

$$\begin{pmatrix} 1 & & & & & \\ a_{ii-1} & a_{ii} & a_{ii+1} & & & \\ & a_{ii-1} & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{ii-1} & a_{ii} & a_{ii+1} \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ c_{n-1} \\ c_n \end{pmatrix} = \begin{pmatrix} a \\ \langle f, \psi_1 \rangle \\ \vdots \\ \vdots \\ \langle f, \psi_{n-1} \rangle \\ b \end{pmatrix}$$

2. (spectral methods, `chebfun`) Download the package `chebfun` from `www.chebfun.org`. This package works with functions that are represented (to machine precision) as sums of Chebyshev polynomials. It can solve 2-point boundary value problems using spectral methods. Use `chebfun` to solve the same problem as in the previous exercise and check the L_2-norm and the ∞-norm of the error.

   Solution:

```
L = chebop(0, 1);
L.op = @(x,u)- diff((1+x.^2).*diff(u,1),1) ;
L.lbc = 0; L.rbc = 0;
x = chebfun('x', [0, 1]);
f = 2*(3*x.^2-x+1);
u = L\f;
LW = 'linewidth'; lw = 0.6;
plot(u, 'm', LW, lw)
w=x.*(1-x);

norm(w-u,Inf)
norm(w-u,2)
```

   L_2 norm : 2.3919e-15

   ∞ norm : 3.2752e-15

   Spectral methods seem to give near machine precision results.