AMATH 568: Problem Set 2

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1.

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{bmatrix}$$

$$A^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}$$

The diagonal elements of e^{At} are of the form

$$1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \frac{\lambda^3 t^3}{3!} + \ldots = e^{\lambda t}$$

The first off-diagonal elements of e^{At} are of the form

$$t + \frac{2\lambda t^2}{2!} + \frac{3\lambda^2 t^3}{3!} + \ldots = f$$

$$1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \frac{\lambda^3 t^3}{3!} + \ldots = \frac{f}{t}$$

$$e^{\lambda t} = \frac{f}{t}$$

$$f = te^{\lambda t}$$

The second off-diagonal elements of e^{At} are of the form

$$\frac{t^2}{2!} + \frac{3\lambda t^3}{3!} + \frac{6\lambda^2 t^4}{4!} + \ldots = g$$

$$\frac{t^2}{} + \frac{\lambda t^3}{1} + \frac{\lambda^2 t^4}{2!} + \ldots = 2g$$

$$1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \frac{\lambda^3 t^3}{3!} + \ldots = \frac{2g}{t^2}$$

$$e^{\lambda t} = \frac{2g}{t^2}$$

$$g = \frac{t^2 e^{\lambda t}}{2}$$

$$e^{At} = \begin{bmatrix} \lambda e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

2. (a) We know that,

$$\frac{d}{dt} det[\Phi(t)] = det[\Phi(t)]Tr(B)$$

where

$$\Phi B = A\Phi$$

$$\frac{d}{dt} ln(det[\Phi(t)]) = \frac{1}{det[\Phi(t)]}\frac{d}{dt} det\Phi$$

$$= \frac{det[\Phi(t)]Tr(B)}{det[\Phi(t)]}$$
$$= Tr(B)$$
$$= Tr(A(t))$$

(b)

$$\Phi'(t+T) = A(t+T)\Phi(t+T)$$
$$= A(t)\Phi(t+T)$$

So, both $\Phi(t)$ and $\Phi(t+T)$ solve $x' = Ax$. They are both fundamental matrices. Since they solve a group of linearly independent equations,

$$\Phi(t+T) = \Phi(t)C$$

where C is a constant matrix. Since $\Phi(0)$ is non-singular, $\Phi(t)$ is too and C is non-singular as well. So, we can find a B such that

$$C = e^{BT}$$

Let

$$\Phi(t) = e^{Bt}Q(t)$$
$$Q(t) = \Phi(t)e^{-B(t)}$$
$$Q(t+T) = \Phi(t+T)e^{-B(t+T)}$$
$$= \Phi(t)e^{BT}e^{-B(t+T)}$$
$$= \Phi(t)e^{Bt} = Q(t)$$

So, Q(t) is periodic. And we have,

$$\Phi(t) = e^{Bt}Q(t)$$

3. (a) The fixed points are at (0,0), (0,1) , (2,0) and (0.5,1.5)

(b) The Jacobian is given by

$$J*(x,y) = \begin{bmatrix} 2-2x-y & -x \\ \frac{y^2}{(1+x)^2} & 1-\frac{2y}{1+x} \end{bmatrix}$$

At (0,0),

$$J*(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

The eigenvalues are 2 and 1. Since both are positive, this point is unstable. The eigenvectors are (1,0) and (0,1)

At (0,1),

$$J*(0,1) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

The eigenvalues are -1 and 1. Since one value has a positive real part, this point is unstable. The eigenvectors are (0,1) and (0.89442719, 0.4472136).

At (2,0),

$$J*(2,0) = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}$$

The eigenvalues are -2 and 1. Since one value has a positive real part, this point is unstable. The eigenvectors are (1,0) and (-0.5547002, 0.83205029).

At (0.5,1.5),

$$J*(0.5,1.5), = \begin{bmatrix} -0.5 & -0.5 \\ 1 & -1 \end{bmatrix}$$

The eigenvalues are -0.75 +0.6614i and -0.75-0.6614i. Since both values have a negative real part, this point is stable. The eigenvectors are (0.2041 + 0.5401i,0.8165) and (0.2041 - 0.5401i, 0.8165).

(c) Done in part(b)

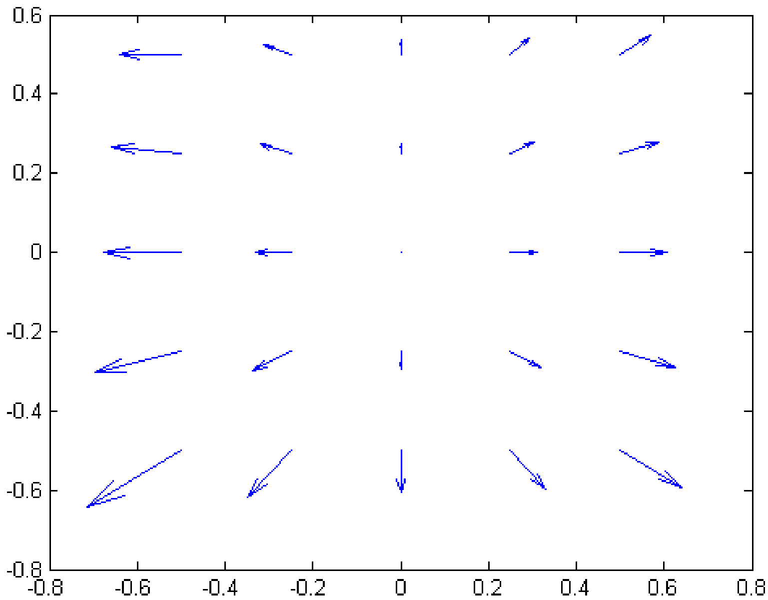

Figure 1: Phase portrait in vicinity of (0,0)

(d) The eigenvectors (1,0) and (0,1) are orthogonal and is well-represented by the stretching in perpendicular directions.

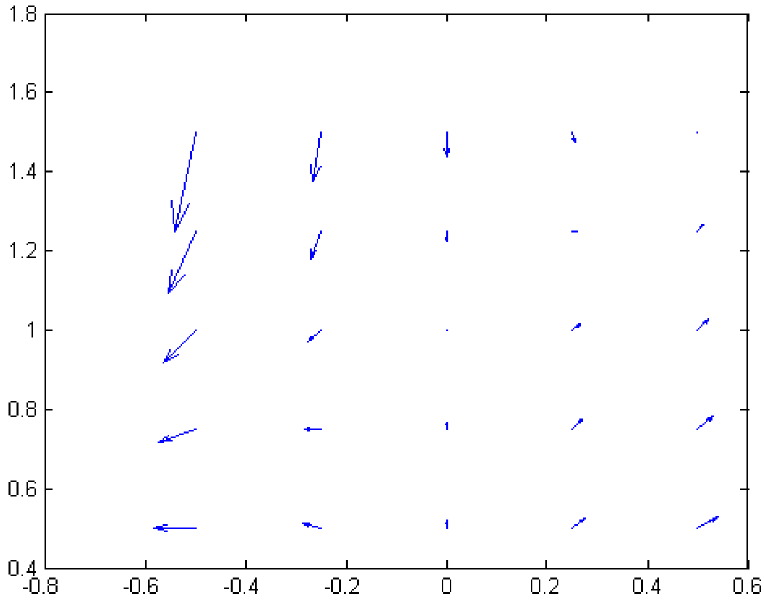

Figure 2: Phase portrait in vicinity of (0,1)

The eigenvectors (0,1) corresponds to stretching in the vertical direction.

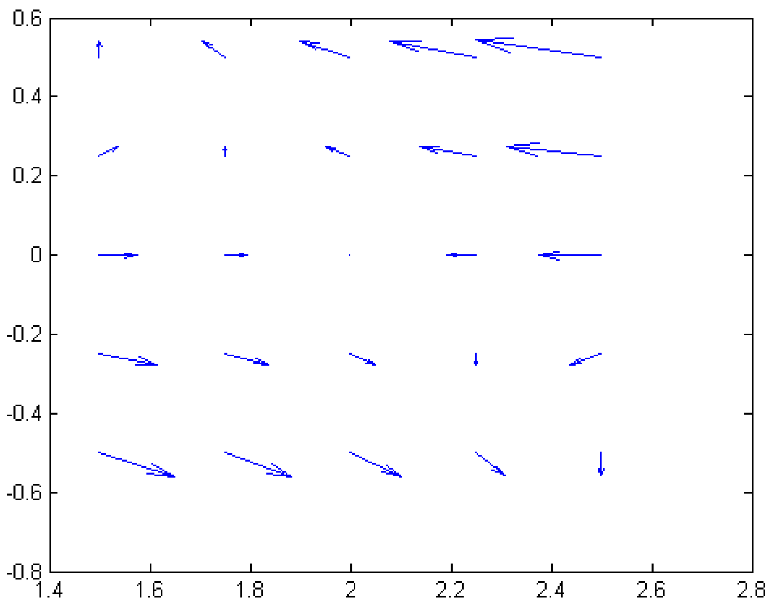

Figure 3: Phase portrait in vicinity of (2,0)

The eigenvectors (1,0) represents stretching in the horizontal direction but since the eigenvalue is negative, it is inward compression.

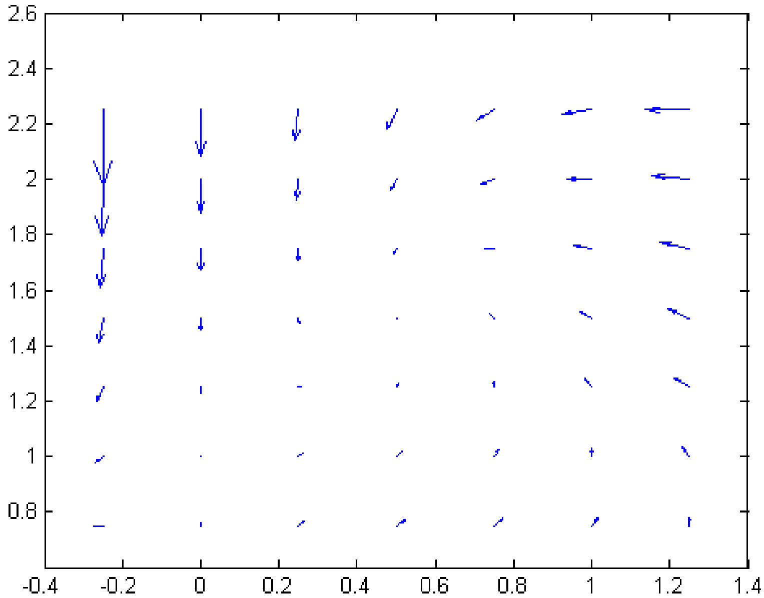

Figure 4: Phase portrait in vicinity of (0.5,1.5)

The complex eigenvectors correspond to the spiral. The plots were obtained in Matlabd using the following code

```
clear all; close all; clc;
a=0.5;
b=1.5;
[x,y]=meshgrid(-0.5+a:0.25:a+0.5,-0.5+b:0.25:0.5+b);
xdot=x.*(2-x)-x.*y;
ydot= y.*(1-(y./(1+x)));
quiver(x,y,xdot,ydot);
```