AMATH 561: Homework 4

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1. Not required.
2. (a) In a random walk, X becomes X+1 with probability p and X-1 with probability 1-p.

   In this process, X becomes X+2 with probability p^2, X-2 with probability $(1-p)^2$ and stays at X with probability 2p(1-p).

   The transition matrix will looks like

$$\begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & (1-p)^2 & 0 & 2p(1-p) & 0 & p^2 & 0 & 0 \\ 0 & 0 & (1-p)^2 & 0 & 2p(1-p) & 0 & p^2 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

   (b) If the initial generation is 0, the future generations are 0 as well.

   If the generating function for a single individual is G(s), the generating function for an individual after two generations is G(G(s)) or $G_2(s)$

   Thus, we can obtain the transition probabilities by differentiating it.

$$P(i,j) = \frac{1}{j!}\frac{d^j}{ds^j}(G_2(s))^i|_{s=0}$$

   The transition matrix will looks like

$$\begin{bmatrix} 1 & 0 & 0 & \dots \\ p(1,0) & p(1,1) & p(1,2) & \dots \\ p(2,0) & p(2,1) & p(2,2) & \dots \\ \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

3. not required
4.

$$P(\tau_j < \tau_i | X_0 = i) = P(\tau_i < \tau_j | X_0 = j) = p$$

Probability that j is visited once $= P(X_0 = i)P(\tau_j < \tau_i|X_0 = i)P(\tau_i < \tau_j|X_0 = j) = p^2 P(X_0 = i)$

$$\text{Probability that j is visited once given you start at i} = p^2$$

$$\text{Probability that j is visited n times given you start at i} =$$
$$= p^2(\text{Probability that it transitions between j and j n-1 times}) = p^2 P(\tau_j < \tau_i | X_0 = j)^{n-1} = (1-p)^{n-1}p^2$$

$$\text{Expected number of visits} = \sum_{n=1}^{\infty} n(1-p)^{n-1}p^2 = \sum_{n=0}^{\infty}(n+1)(1-p)^n p^2$$

This is an arithmetic geometric progression. So, the infinite sume is given by

$$\text{Expected number of visits} = p^2\left(\frac{1}{1-(1-p)} + \frac{1-p}{(1-(1-p))^2}\right) = p^2\left(\frac{1}{p} + \frac{1-p}{p^2}\right) = 1$$

5. not required
6. Written at the end
7. not required
8. Since $\sum_i P(i,j) = 1$, we can replace all the μs to get the following transition matrix.

$$\begin{bmatrix} 1-\lambda_0 & \lambda_0 & 0 & 0 & \dots \\ 1-\lambda_1 & 0 & \lambda_1 & 0 & \dots \\ 0 & 1-\lambda_2 & 0 & \lambda_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \lambda_{n-1} \\ 0 & 0 & 0 & 1-\lambda_n & \lambda_n \end{bmatrix}$$

For equilibrium, we need

$$\pi P = P$$

Let π be

$$[x_0, x_1, x_2, x_3, \dots]$$

$$(1-\lambda_0)X_0 + (1-\lambda_1)x_1 = x_0$$

$$x_1 = \frac{\lambda_0}{(1-\lambda_1)}x_0$$

$$(\lambda_0)x_0 + (1-\lambda_2)x_2 = x_1$$

$$(1-\lambda_1)x_1 + (1-\lambda_2)x_2 = x_1$$

$$(1-\lambda_2)x_2 = \lambda_1 x_1$$

$$x_2 = \frac{\lambda_1}{(1-\lambda_2)}x_1$$

Similarly, for j upto n-1,

$$x_j = \frac{\lambda_{j-1}}{(1-\lambda_j)}x_{j-1}$$

For n,

$$\lambda_{n-1}x_{n-1} = \lambda_n x_n$$

$$x_n = \frac{\lambda_{n-1}}{1-\lambda_n} x_{n-1}$$

For reversibility, we need

$$\pi(i)p(i,j) = \pi(j)p(j,i)$$

Since this MC has finite space and the πs should sum to 1, we can safely say all the πs are finite. For diagonal elements,

$$\pi(i)p(i,i) = \pi(i)p(i,i)$$

We only need to worry about off-diagonal elements of the form j=i+1. (because all other probabilities are zero). For i =0,

$$\pi(0)p(0,1) = x_0\lambda_0 = \frac{\lambda_0}{(1-\lambda_1)} x_0(1-\lambda_1) = \pi(1)p(1,0)$$

For i upto n-2,

$$\pi(i)p(i,i+1) = x_i\lambda_i = \frac{\lambda_i}{(1-\lambda_{i+1})} x_0(1-\lambda_{i+1}) = \pi(i+1)p(i+1,i)$$

For i = n-1,

$$\pi(n-1)p(n-1,n) = x_{n-1}\lambda_{n-1} = \frac{\lambda_{n-1}}{(1-\lambda_n)} x_0(1-\lambda_n) = \pi(n)p(n,n-1)$$

Thus, the system is reversible in equilibrium.