# AMATH 569: Problem Set 5

Jithin D. George, No. 1622555

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1. (a)

$$Au + g = 0$$

We have to find $\phi_n$ and $\lambda_n$ such that

$$A\phi_n = \lambda_n \phi_n$$

Since A $= \partial_x^2$ and f(0)=0 and f(L)=0,

$$\phi_n = \sqrt{\frac{2}{L}} sin(\lambda_n x)$$

and

$$\lambda_n = \frac{n\pi}{L}$$

Since n $\in \mathbb{Z}$,

$$\lambda_0 = 0$$

For this $\lambda$,

$$\phi_0 = 0$$

So, $\lambda_0$ is not actually a solution and our equation has a unique solution.

$$\begin{aligned} u &= \sum_{n=-\infty, n\neq 0}^{\infty} \frac{< \phi_n, -g >}{\lambda_n} \sqrt{\frac{2}{L}} sin(\lambda_n x) \\ &= -\sum_{n=-\infty, n\neq 0}^{\infty} \frac{L}{n\pi} \sqrt{\frac{2}{L}} \sqrt{\frac{2}{L}} sin(\lambda_n x) \int_0^L g sin(\lambda_n x) dx \\ &= -\sum_{n=-\infty, n\neq 0}^{\infty} \frac{2}{n\pi} sin(\lambda_n x) \int_0^L g sin(\lambda_n x) dx \end{aligned}$$

Another way of looking at this is that if we take the second case of the Fredholm Alternative (infinitely many solutions). Then, we would have

$$u = -\sum_{n=-\infty, n\neq 0}^{\infty} \frac{2}{n\pi} sin(\lambda_n x) \int_0^L g sin(\lambda_n x) dx + c\phi_0$$

But, $\phi_0 = 0$ and we get back our unique solution.

(b)

$$Au + g = 0$$

We have to find $\phi_n$ and $\lambda_n$ such that

$$A\phi_n = \lambda_n \phi_n$$

Since A $= \partial_x^2$ and f'(0)=0 and f'(L)=0,

$$\phi_n = \sqrt{\frac{2}{L}} cos(\lambda_n x)$$

and

$$\lambda_n = \frac{n\pi}{L}$$

Since n $\in \mathbb{Z}$,

$$\lambda_0 = 0$$

For this $\lambda$,

$$\phi_0 = 1$$

We cannot ignore the eigenfunction here and our equation does not have a unique solution.

$$< \phi_0, g >= \int_0^L 1^* g dx = \int_0^L g dx = 0$$

Thus, by the Fredholm Alternative, the solution blows up at $\lambda_0$ =0 and there is no solution.

(c)

$$Au + g = 0$$

We have to find $\phi_n$ and $\lambda_n$ such that

$$A\phi_n = \lambda_n \phi_n$$

Since A $= \partial_x^2$ and f'(0)=0 and f'(L)=0,

$$\phi_n = \sqrt{\frac{2}{L}} cos(\lambda_n x)$$

and

$$\lambda_n = \frac{n\pi}{L}$$

Since n $\in \mathbb{Z}$,

$$\lambda_0 = 0$$

For this $\lambda$,

$$\phi_0 = 1$$

We cannot ignore the eigenfunction here and our equation does not have a unique solution.

$$< \phi_0, g >= \int_0^L 1^* g dx = \int_0^L g dx \neq 0$$

Thus, by the Fredholm Alternative, the solution has infinitely many solutions of the following form.

$$u = -\sum_{n=-\infty}^{\infty} \frac{2}{n\pi} cos(\lambda_n x) \int_0^L g cos(\lambda_n x) dx + c\phi_0$$

2.

$$(-\partial_t^2 + A)u = 0$$
$$A = \partial_x^2$$
$$u(0,x) = f(x), u_t(0,x) = g(x)$$

For A,

$$\phi_w = \frac{1}{\sqrt{2\pi}} e^{iwx}$$
$$\lambda_w = -w^2$$

Treating A as a constant and solving the equation as an ode, we get

$$u = cosh(t\sqrt{A})f + \frac{sinh(t\sqrt{A})g}{\sqrt{A}}$$

Treating cosh and sinh like the $\eta$ operator,

$$u = \int_R cosh(t\sqrt{\lambda_w}) < \phi_w, f > \phi_w dw + \int_R \frac{sinh(t\sqrt{\lambda_w})}{\sqrt{\lambda_w}} < \phi_w, g > \phi_w dw$$
$$= \frac{1}{\sqrt{2\pi}} \int_R cosh(iwt) < \phi_w, f > e^{iwx} dw + \frac{1}{\sqrt{2\pi}} \int_R \frac{sinh(iwt)}{iw} < \phi_w, g > e^{iwx} dw$$
$$= \frac{1}{\sqrt{2\pi}} \int_R cosh(iwt) < \phi_w, f > e^{iwx} dw + \frac{1}{\sqrt{2\pi}} \int_R \frac{sinh(iwt)}{iw} < \phi_w, g > e^{iwx} dw$$
$$= \frac{1}{\sqrt{2\pi}} \int_R \frac{e^{iwt} + e^{-iwt}}{2} < \phi_w, f > e^{iwx} dw + \frac{1}{\sqrt{2\pi}} \int_R \frac{sinh(iwt)}{iw} < \phi_w, g > e^{iwx} dw$$
$$= \frac{1}{2\pi} \int_R \frac{e^{iw(x+t)} + e^{iw(x-t)}}{2} < e^{iwx}, f > dw + \frac{1}{2\pi} \int_R \frac{sinh(iwt)}{iw} < e^{iwx}, g > e^{iwx} dw$$
$$= \frac{1}{2\pi} \int_R \frac{e^{iw(x+t)} + e^{iw(x-t)}}{2} \int_R e^{-iwx} f dx dw + \frac{1}{2\pi} \int_R \frac{e^{iw(x+t)} + e^{iw(x-t)}}{2iw} \int_R e^{-iwx} g dx dw$$
$$= \frac{1}{2\pi} \int_R \frac{e^{iw(x+t)} + e^{iw(x-t)}}{2} \hat{f} dw + \frac{1}{2\pi} \int_R \frac{e^{iw(x+t)} - e^{iw(x-t)}}{2iw} \hat{g} dw$$
$$= \frac{1}{2\pi} \int_R \frac{e^{iw(x+t)}\hat{f} + e^{iw(x-t)}\hat{f}}{2} dw + \frac{1}{2\pi} \int_R \frac{e^{iw(x+t)} - e^{iw(x-t)}}{2iw} \hat{g} dw$$
$$= \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2\pi} \int_R \frac{e^{iw(x+t)}\hat{g}}{2iw} dw - \frac{1}{2\pi} \int_R \frac{e^{iw(x-t)}\hat{g}}{2iw} dw$$

$$
\begin{aligned}
&= \frac{f(x+t)+f(x-t)}{2} + \int_{-\infty}^{x+t} \frac{g(z)}{2}dz - \int_{-\infty}^{x-t} \frac{g(z)}{2}dz \\
&= \frac{f(x+t)+f(x-t)}{2} + \int_{-\infty}^{x+t} \frac{g(z)}{2}dz + \int_{x-t}^{-\infty} \frac{g(z)}{2}dz \\
&= \frac{f(x+t)+f(x-t)}{2} + \int_{x-t}^{x+t} \frac{g(z)}{2}dz
\end{aligned}
$$

Inverse fourier transform tables were used in this problem.