AMATH 535: Homework Problem 3.6

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1.

$$P_n(t+\Delta t) = P_{n-1}(t)[(n-1)\Delta t\beta + I\Delta t] + P_{n+1}(t)\mu(n+1)\Delta t$$
$$+P_n(t)[1-((\beta+\mu)n+I)\Delta t]$$

$$\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = P_{n-1}(t)[(n-1)\beta + I] + P_{n+1}(t)\mu(n+1)$$
$$-P_n(t)((\beta+\mu)n+I)$$

As Δt goes to 0, this becomes,

$$\frac{dP_n(t)}{dt} = P_{n-1}(t)[(n-1)\beta + I] + P_{n+1}(t)\mu(n+1)$$
$$-P_n(t)((\beta+\mu)n+I)$$

2.

$$F(x,t) = \sum_{n=0}^{\infty} P_n x^n$$

$$\frac{\partial F}{\partial t} = \sum_{n=0}^{\infty} \dot{P}_n x^n$$

$$= \sum_{n=0}^{\infty}(P_{n-1}(t)[(n-1)\beta + I] + P_{n+1}(t)\mu(n+1) - P_n(t)((\beta+\mu)n+I))x^n$$

$$= \beta x^2 \sum_{n=0}^{\infty} P_{n-1}(t)(n-1)x^{n-2} + \mu \sum_{n=0}^{\infty} P_{n+1}(t)(n+1)x^n - (\beta+\mu)x\sum_{n=0}^{\infty} P_n(t)(n)x^{n-1}$$

$$+Ix\sum_{n=0}^{\infty} P_{n-1}(t)(n-1)x^{n-1} - I\sum_{n=0}^{\infty} P_n(t)x^n$$

$$\frac{\partial F}{\partial t} = (\beta x^2 - (\beta+\mu)x + \mu)\frac{\partial F}{\partial x} + F(Ix - I)$$

$$\frac{\partial F}{\partial t} = (x-1)(\beta x - \mu)\frac{\partial F}{\partial x} + FI(x-1)$$

Initial conditions :

$$F(x,0) = x^{n_0}$$

3. At equilibrium,

$$\frac{\partial F}{\partial t} = 0$$

$$(x-1)(\beta x - \mu)\frac{\partial F}{\partial x} + FI(x-1) = 0$$
$$\frac{dF}{dx} = -\frac{I}{\beta x - \mu}F$$
$$\frac{dF}{F} = -\frac{I}{\beta x - \mu}dx$$

Integrating, we get

$$ln(F) = -\frac{I}{\beta}ln(x - \frac{\mu}{\beta}) + C$$
$$F = \frac{A}{(x - \frac{\mu}{\beta})^{\frac{I}{\beta}}}$$

All the probabilities must sum to 1. That imposes the following constraint.

$$F(1,t) = 1$$
$$\frac{A}{(1 - \frac{\mu}{\beta})^{\frac{I}{\beta}}} = 1$$
$$A = (1 - \frac{\mu}{\beta})^{\frac{I}{\beta}}$$

So, the equilibrium P.G.F is

$$F^* = (\frac{1 - \frac{\mu}{\beta}}{x - \frac{\mu}{\beta}})^{\frac{I}{\beta}}$$

4. The expected population size at equilibrium is given by

$$E^*[N(t)] = \frac{dF}{dx}\Big|_{x=1} = -\frac{I}{\beta}(\frac{1 - \frac{\mu}{\beta}}{1 - \frac{\mu}{\beta}})^{\frac{I}{\beta}}\frac{1}{1 - \frac{\mu}{\beta}} = \frac{I}{\mu - \beta}$$

5. We can see

$$\frac{d^2F}{dx^2}\Big|_{x=1} = -\frac{I}{\beta}(-\frac{I}{\beta} - 1)(\frac{1 - \frac{\mu}{\beta}}{1 - \frac{\mu}{\beta}})^{\frac{I}{\beta}}\frac{1}{(1 - \frac{\mu}{\beta})^2} = \frac{I^2 + I\beta}{(\mu - \beta)^2}$$

The variance at equilibrium is given by

$$Var^*[N(t)] = \frac{d^2F}{dx^2} + \frac{dF}{dx} - (\frac{dF}{dx})^2\Big|_{x=1} = \frac{I^2 + I\beta + I\mu - I\beta - I^2}{(\mu - \beta)^2} = \frac{I\mu}{(\mu - \beta)^2}$$

6. The probability of the population being extinct at equilibrium is given by

$$p_0* = F*(0) = (\frac{\beta - \mu}{\mu})^{\frac{I}{\beta}}$$