AMath 574
Homework 4
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1. (Adapted from Exercise 13.7 in the book)

   Note: The Jupyter notebook `homeworks/hw4/Homework4-p-system.ipynb` will get you started on some parts of this problem (in particular parts (d), (e) and (g) are solved already). You can add to this notebook to produce the other plots you need and also adapt it for parts of Problem 2 below.

   Consider the p-system (described in Section 2.13),

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= 0, \end{aligned}$$

   where $p(v)$ is a given function of v. Note that $q = [v,\ u]^T$ for this system.

   (a) Compute the eigenvalues of the Jacobian matrix and show that the system is hyperbolic provided $p'(v) < 0$.

   Solution:

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p'(v)v_x &= 0, \end{aligned}$$

   So,

$$A = \begin{bmatrix} 0 & -1 \\ p'(v) & 0 \end{bmatrix}$$

   The eigenvalues are $\pm\sqrt{-p'(v)}$. So , for the system to be hyperbolic, $p'(v)$ has to be less than zero.

   (b) Use the Rankine-Hugoniot condition to show that a shock connecting $q = (v, u)$ to some fixed state $q^* = (v^*, u^*)$ must satisfy

$$u = u_* \pm \sqrt{-\left(\frac{p(v) - p(v_*)}{v - v_*}\right)}\,(v - v_*). \tag{1}$$

   Solution:

$$s = \frac{f(q1_*) - f(q1)}{q1_* - q1} = \frac{f(q2_*) - f(q2)}{q2_* - q2}$$

$$s = \frac{-(u_* - u)}{v_* - v} = \frac{p(v_*) - p(v)}{u_* - u}$$

$$
\begin{aligned}
(u_* - u)^2 &= -(p(v_*) - p(v))(v_* - v) \\
u_* &= u \pm \sqrt{-(p(v_*) - p(v))(v_* - v)} \\
u_* &= u \pm \sqrt{-\left(\frac{p(v) - p(v_*)}{v - v_*}\right)(v - v_*)}
\end{aligned}
$$

(c) What is the propagation speed for such a shock? How does this relate to the eigenvalues of the Jacobian matrix computed in part (a)?

Solution:

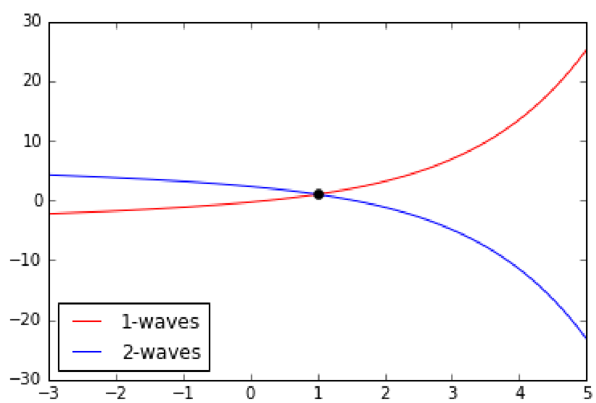
$$
s = \frac{-u_* + u}{v * - v} = \pm\sqrt{-\left(\frac{p(v) - p(v_*)}{v - v_*}\right)}
$$

As v gets very close to v_*, the shock speeds are the same as the eigenvalues.

(d) Plot the Hugoniot loci for the point $q_* = (1, 1)$ over the range $-3 \leq v \leq 5$ for the case $p(v) = -e^v$.

Note: This is not a realistic equation of state for a gas if v represents the specific volume $1/\rho$ as described in Section 2.13, since in that case $v > 0$ is required and $p(v) \to \infty$ as $v \to 0$. But mathematically this gives a fine hyperbolic system.

Solution:



(e) Determine the 2-shock solution to the Riemann problem for the p-system with $p(v) = -e^v$ and data

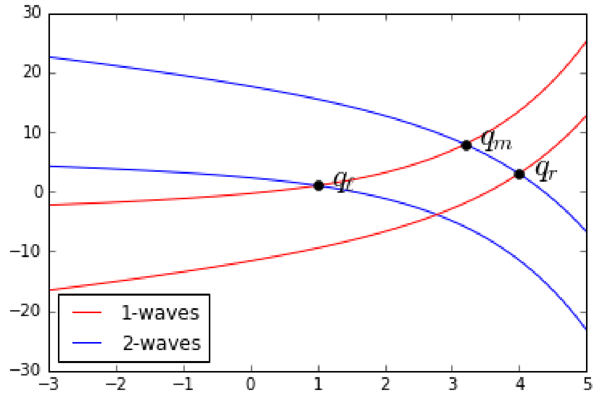
$$
q_\ell = (1, 1), \qquad q_r = (4, 3).
$$

Do this in two ways:

i. Plot the relevant Hugoniot loci and estimate where they intersect.

ii. Set up and solve the proper scalar nonlinear equation for v_m, using `scipy.optimize.fsolve`.

Solution:



Using fsolve, we get

$$q_m = \begin{bmatrix} 3.19779 \\ 7.91551 \end{bmatrix}$$

(f) Does the Riemann solution found in the previous part satisfy the Lax Entropy Condition? Sketch the structure of the solution in the x-t plane showing also some sample 1-characteristics and 2-characteristics.

Solution:

$$v_l < v_m < v_r$$

$$\lambda_1 = -e^{v/2}$$

Thus, for λ_1,

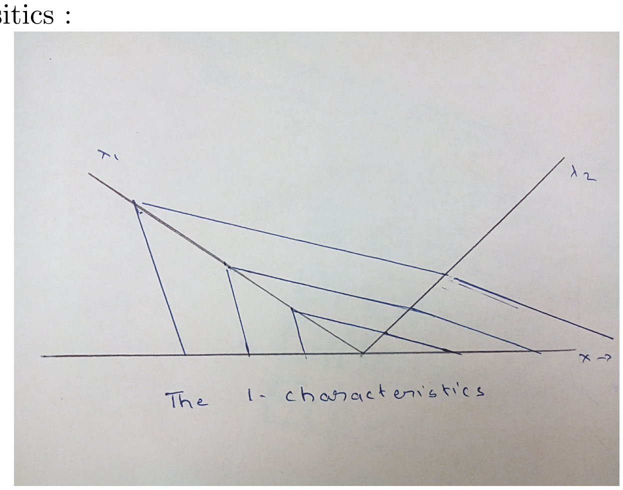
$$\lambda_l > \lambda_m$$

This satisfies the condition for a 1-shock. However, for $\lambda_2 = e^{v/2}$,

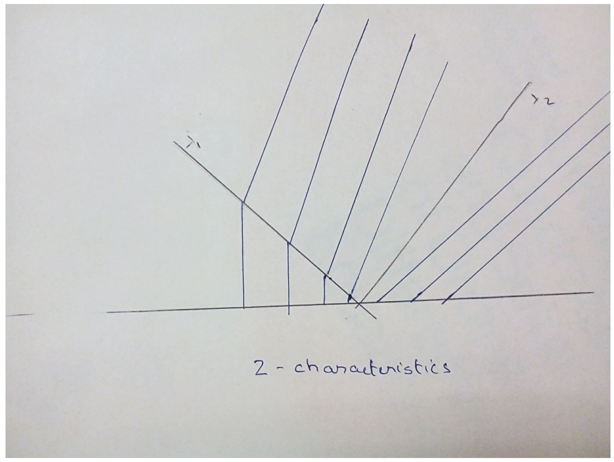
$$\lambda_m < \lambda_r$$

Thus, it is a 2-rarefaction. So, it does not satisfy the Lax Entropy condition.

The 1-charactersitics :



The 2-characteristics :



(g) For the given left state $q_\ell = (1, 1)$, in what region of the phase plane must the right state q_r lie in order for the 2-shock Riemann solution to satisfy the Lax Entropy Condition? (This is already done in the notebook, you'll have to do something similar in 2(f) below.)

Solution: To satisfy the Lax Entropy condition,

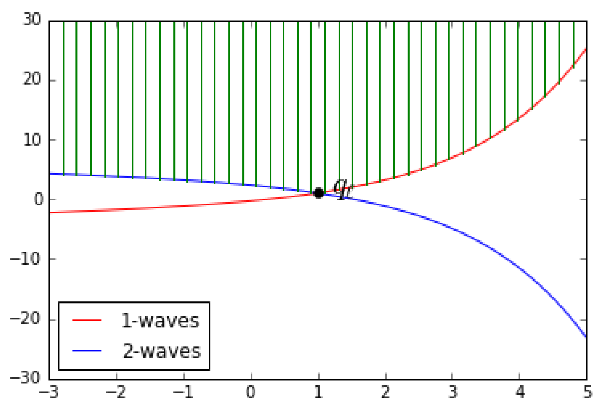
$$\lambda_l^1 > \lambda_m^1$$

$$\lambda_m^2 < \lambda_r^2$$

For this to happen,

$$v_m > max(v_l, v_r)$$

Given q_l = (1,1), q_r has to lie such that the intersection of the 2-wave through q_r with the 1-wave through q_l lies to the right of both q_r and q_l. The region is shaded in the figure below.



2. Consider the p-system of Problem 1 with $p(v) = -e^v$.

   (a) Follow the procedure of 13.8.1 to show that along any integral curve of r^1 the relation

   $$u = u_* - 2\left(e^{v_*/2} - e^{v/2}\right)$$

   must hold, where (v_*, u_*) is a particular point on the integral curve. Conclude that

   $$w^1(q) = u - 2e^{v/2}$$

   is a 1-Riemann invariant for this system.

   Solution: Along any integral curve of r^1,

   $$q' = \alpha r^1$$

   $$r^1 = \begin{bmatrix} 1 \\ \sqrt{-p'(v)} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{v/2} \end{bmatrix}$$

   Taking alpha =1,

   $$v'(\xi) = 1$$
   $$v = \xi$$
   $$u'(\xi) = e^{\xi/2}$$
   $$u(\xi) = 2e^{\xi/2} + c$$

   Since $u(v_*) = u_*$,

   $$u(v) = u_*(v_*) + 2e^{v/2} - 2e^{v_*/2}$$

   $$u(v) - 2e^{v/2} = u_*(v_*) - 2e^{v_*/2}$$

   Thus,

   $$w^1(q) = u - 2e^{v/2}$$

   is the 1-Riemann invariant for this system

   (b) Follow the procedure of Section 13.8.5 to show that through a centered rarefaction wave

   $$\tilde{u}(\xi) = A - 2\xi,$$

   where

   $$A = u_l - 2e^{v_l/2} = u_r - 2e^{v_r/2},$$

   and determine the form of $\tilde{v}(\xi)$.

   Solution:

   $$\lambda_1 = -\sqrt{-p'(v)}$$

   $$r^1 = \begin{bmatrix} 1 \\ \sqrt{-p'(v)} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{v/2} \end{bmatrix}$$

$$\nabla\lambda_1 = \begin{bmatrix} \frac{d}{dv}(-\sqrt{-p'(v)}) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{d}{dv}(-e^{v/2}) \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}e^{v/2} \\ 0 \end{bmatrix}$$

$$q' = \frac{r^1}{\nabla\lambda_1.r^1} = \begin{bmatrix} -2e^{-v/2} \\ -2 \end{bmatrix}$$

$$u' = -2$$

$$u = -2\xi + A$$

$$A = u + 2\xi = u_l - 2\sqrt{-p'(v_l)} = u_r - 2\sqrt{-p'(v_r)}$$

Thus,

$$A = u + 2\xi = u_l - 2e^{v_l}/2 = u_r - 2e^{v_r}/2$$

$$v' = -2e^{-v/2}$$

$$v'e^{v/2} = -2$$

$$2e^{v/2} = -2\xi + 2B$$

$$e^{v/2} = -\xi + B$$

$$v(\xi) = 2log(B - \xi)$$

where

$$B = e^{v_l/2} - e^{v_l/2} = 0$$

$$v(\xi) = 2log(-\xi)$$

We can get the same result by using the 1-Riemann invariant.

(c) Show that this field is genuinely nonlinear for all q.

Solution:

$$\nabla\lambda_1.r^1 = -\frac{1}{2}e^{v/2}$$

This is never equal to zero for all finite values of q.

(d) Determine the 2-Riemann invariants and the form of a 2-rarefaction.

Solution: Along any integral curve of $r2$,

$$q' = \alpha r^2$$

$$r^2 = \begin{bmatrix} 1 \\ -\sqrt{-p'(v)} \end{bmatrix} = \begin{bmatrix} 1 \\ -e^{v/2} \end{bmatrix}$$

Taking alpha =1,

$$v'(\xi) = 1$$

$$v = \xi$$

$$u'(\xi) = -e^{\xi/2}$$

$$u(\xi) = -2e^{\xi/2} + c$$

Since $u(v_*) = u_*$,

$$u(v) = u_*(v_*) - 2e^{v/2} + 2e^{v_*/2}$$

$$u(v) + 2e^{v/2} = u_*(v_*) + 2e^{v_*/2}$$

Thus,

$$w^2(q) = u + 2e^{v/2}$$

is the 2-Riemann invariant for this system

For the 2-rarefaction,

$$\lambda_2 = \sqrt{-p'(v)}$$

$$r^2 = \begin{bmatrix} 1 \\ -\sqrt{-p'(v)} \end{bmatrix} = \begin{bmatrix} 1 \\ -e^{v/2} \end{bmatrix}$$

$$\nabla\lambda_2 = \begin{bmatrix} \frac{d}{dv}(\sqrt{-p'(v)}) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{d}{dv}(e^{v/2}) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{v/2} \\ 0 \end{bmatrix}$$

$$q' = \frac{r^2}{\nabla\lambda_2.r^2} = \begin{bmatrix} 2e^{-v/2} \\ -2 \end{bmatrix}$$

$$u' = -2$$

$$u = -2\xi + A$$

$$A = u + 2\xi = u_l - 2\sqrt{-p'(v_l)} = u_r - 2\sqrt{-p'(v_r)}$$

Thus,

$$A = u + 2\xi = u_l - 2e^{v_l}/2 = u_r - 2e^{v_r}/2$$

$$v' = 2e^{-v/2}$$

$$v'e^{v/2} = -2$$

$$2e^{v/2} = 2\xi + 2B$$

$$e^{v/2} = \xi + B$$

$$v(\xi) = 2log(B + \xi)$$

where

$$B = e^{v_l/2} - e^{v_l/2} = 0$$

$$v(\xi) = 2log(\xi)$$

(e) Suppose arbitrary states q_ℓ and q_r are specified and we wish to construct a Riemann solution consisting of two "rarefaction waves" (which might not be physically realizable). Determine the point $q_m = (v_m, u_m)$ where the two relevant integral curves intersect.

Solution:

$$u_m(v) = u_l(v_l) + 2e^{v_m/2} - 2e^{v_l/2}$$

$$u_m(v) = u_r(v_r) - 2e^{v_m/2} + 2e^{v_r/2}$$

$$4e^{v_m/2} = u_r(v_r) - u_l(v_l) + 2e^{v_r/2} + 2e^{v_l/2}$$

$$v_m = 2log(\frac{u_r(v_r) - u_l(v_l) + 2e^{v_r/2} + 2e^{v_l/2}}{4})$$

u_m can be then found using the first equation.

(f) What conditions must be satisfied on q_ℓ and q_r for this to be the physically correct solution to the Riemann problem?

In particular, for the left state $q_\ell = (1, 1)$, shade the region of phase space where q_r must lie in order to have the Riemann solution consist of two rarefaction waves.

Solution:

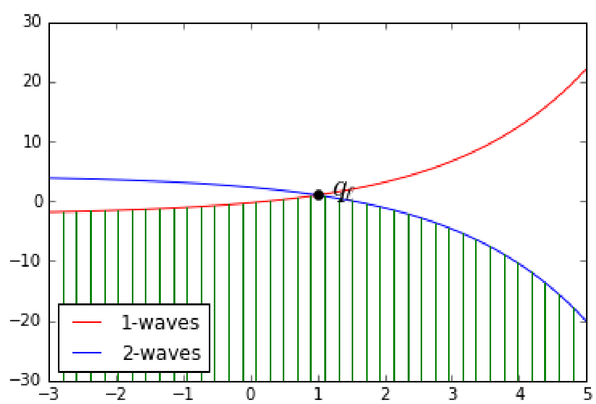
For the solution to be valid, the terms inside the log function must be positive. So,

$$u_r(v_r) - u_l(v_l) + 2e^{v_r/2} + 2e^{v_l/2} \geq 0$$

For a 2-rarefaction solution, $\lambda_r^1 > \lambda_m^1$ and $\lambda_m^2 > \lambda_l^2$.

Since $\lambda^1 = -e^{v/2}$ and $\lambda^2 = e^{v/2}$, this means v_m should lie to the left of both v_l and v_r.

The following figure shows the possible states of q_r for which the 2-wave through q_r intersects the 1-wave through q_l such that v_m lies to the left of both of them.

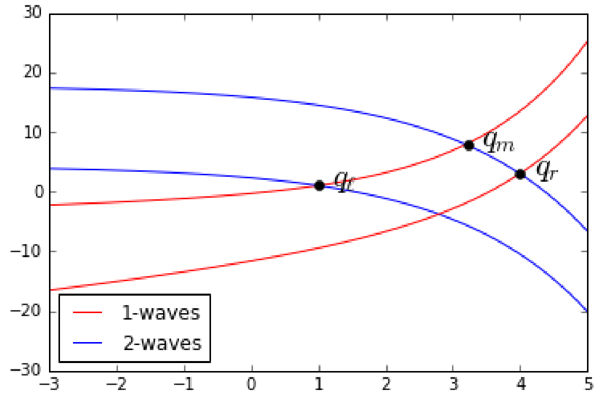


(g) Determine the correct Riemann solution (consisting of one shock and one rarefaction wave) for the problem with

$$q_\ell = (1, 1), \qquad q_r = (4, 3).$$

Solution:

This is obtained by finding the intersection of the 1-Hugoniot curve through q_l and the 2-integral curve through q_r.



Using fsolve, we can find

$$q_m = \begin{bmatrix} 3.19466 \\ 7.89844 \end{bmatrix}$$

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3. For the general p-system of Problem 1, determine the condition on the function $p(v)$ that must be satisfied in order for both fields to be genuinely nonlinear for all q.

Solution:

$$r^1 = \begin{bmatrix} 1 \\ \sqrt{-p'(v)} \end{bmatrix}$$

$$r^2 = \begin{bmatrix} 1 \\ -\sqrt{-p'(v)} \end{bmatrix}$$

$$\nabla\lambda_1 = \begin{bmatrix} \frac{p''(v)}{2\sqrt{-p'(v)}} \\ 0 \end{bmatrix}$$

$$\nabla\lambda_2 = \begin{bmatrix} -\frac{p''(v)}{2\sqrt{-p'(v)}} \\ 0 \end{bmatrix}$$

Thus, for genuine non-linearity ,

$$\nabla\lambda_p.r^p \neq 0$$

That means

$$\frac{p''(v)}{2\sqrt{-p'(v)}} \neq 0$$

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4. Again consider the p-system. For given states q_ℓ and q_r, define the matrix $\hat{A}(q_\ell, q_r)$ as

$$\hat{A} = \begin{bmatrix} 0 & -1 \\ \frac{p(v_r)-p(v_\ell)}{v_r - v_\ell} & 0 \end{bmatrix}.$$

(a) Show that this matrix satisfies the condition

$$\hat{A}(q_r - q_\ell) = f(q_r) - f(q_\ell).$$

corresponding to equation (15.18) in the book. This means we can solve the linear Riemann problem $q_t + \hat{A}q_x = 0$ with left and right states q_ℓ and q_r to obtain an approximate Riemann solution that has nice properties as described in Section 15.3.2. Since (15.18) is satisfied, this is called a "Roe solver".

Solution:

$$\hat{A}(q_r - q_l) = \begin{bmatrix} 0 & -1 \\ \frac{p(v_r)-p(v_\ell)}{v_r - v_\ell} & 0 \end{bmatrix} \begin{bmatrix} v_r - v_l \\ u_r - u_l \end{bmatrix} = \begin{bmatrix} -u_r + u_l \\ p(v_r) - p(v_\ell) \end{bmatrix} = f(q_r) - f(q_\ell)$$

(b) Let

$$c = \sqrt{\frac{p(v_r) - p(v_\ell)}{v_r - v_\ell}}$$

Show that the eigenvalues and eigenvectors of $\hat{A}$ are:

$$\Lambda = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 1 \\ c & -c \end{bmatrix}$$

and compute the inverse R^{-1}. The waves in the approximate Riemann solution are then $\mathcal{W}^1 = \alpha^1 r^1$ and $\mathcal{W}^2 = \alpha^2 r^2$ where $\alpha = R^{-1}(q_r - q_\ell)$. You will need to use this in the next problem.

Solution:

$$\lambda^2 = \frac{-(p(v_r) - p(v_\ell))}{v_r - v_\ell}$$

$$\lambda = \pm\sqrt{\frac{-(p(v_r) - p(v_\ell))}{v_r - v_\ell}}$$

Thus, if

$$c = \sqrt{\frac{-(p(v_r) - p(v_\ell))}{v_r - v_\ell}}$$

$$\Lambda = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 1 \\ c & -c \end{bmatrix}$$

and

$$R^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -\frac{1}{c} \\ 1 & \frac{1}{c} \end{bmatrix}$$

5. The sample code in `$AM574/homeworks/hw4/swe_collide` solves the shallow water equations for a case similar to what is shown in Figure 13.19, illustrating that when two shocks collide in a nonlinear system the resulting interact gives waves in both families.

   Using this code as a starting point, implement the Roe solver for the p-system in a modified version of the shallow water Riemann solver. Note that it will be much simpler since you don't need to worry about an entropy fix.

   Clean up the code and document it so that it doesn't contain extraneous things left over from the shallow water equations. Create a new directory `psystem` containing this code.

   Test your code for initial data consisting of the Riemann problem with a single jump, with

$$q_\ell = (1, 1), \qquad q_r = (4, 3)$$

   as studied above.

   Solve on the domain $-5 \le x \le 5$ for $0 \le t \le 0.5$, using 1000 grid cells.

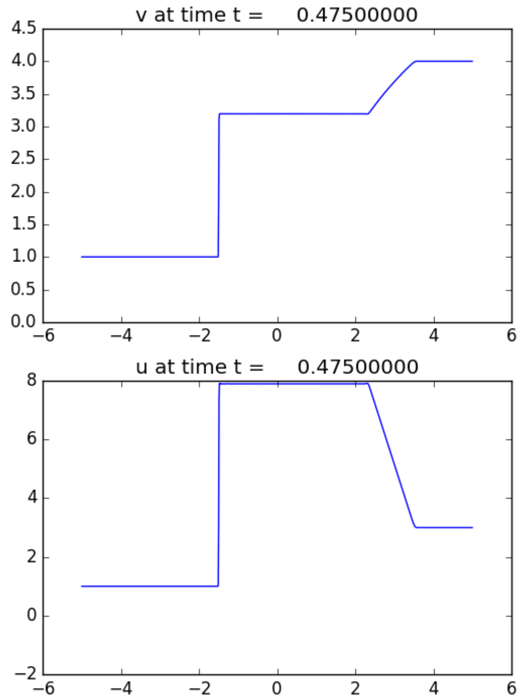
   By examining the output files in the `_output` directory, confirm that the intermediate state observed in the numerical solution agrees to at least 2 or 3 significant digits with the exact intermediate state q_m you found in Problem 2(g). Note that even though an approximate Riemann solver is used in the numerical method, it should converge to the exact solution of the Riemann problem as the grid is refined, when viewed at some fixed time.

Solution:

We get the following middle state

$$q_m = \begin{bmatrix} 3.1948 \\ 7.897 \end{bmatrix}$$

which is in the order of our previous calculations.


v at time t = 0.47500000
u at time t = 0.47500000