AMath 574
Homework 1
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Problem #2.7 in the book

Solution:

$$v_t - u_x = 0$$
$$u_t - p(v)_x = 0$$

This is the same as

$$u_t - p'(v)v_x = 0$$

$$\begin{bmatrix} v \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & 1 \\ -p'(v) & 0 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}_x = 0$$

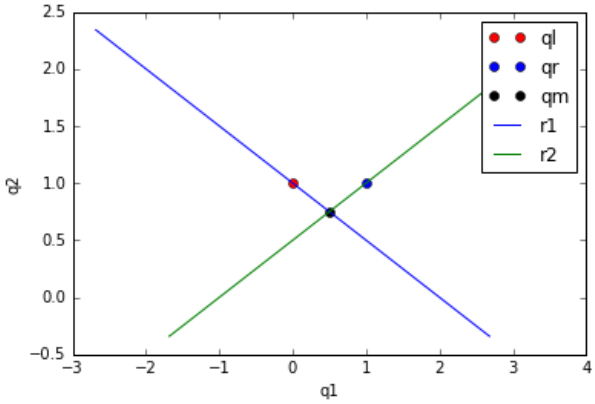
$$\lambda^2 = -p'(v)$$

For the eigenvalues to be real, p'(v) has to be negative. Thus, for the matrix to be diagonalizable and the system of equations to be hyperbolic, p'(v) must be less than 0.

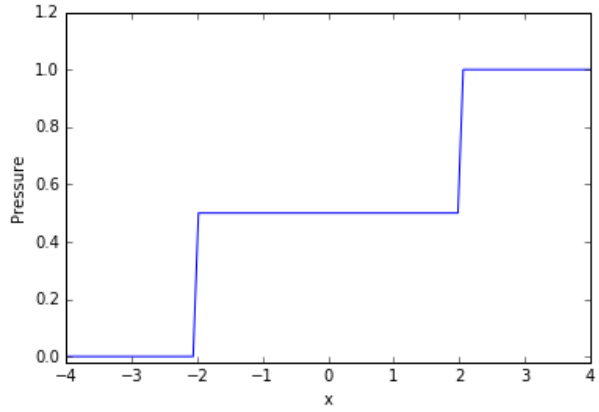
Problem #3.1 in the book You might want to do Problem 3.2 first.

Solution:

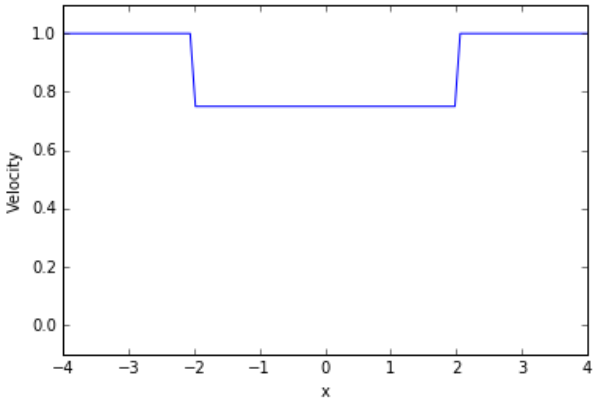
(a) • The phase plot of q



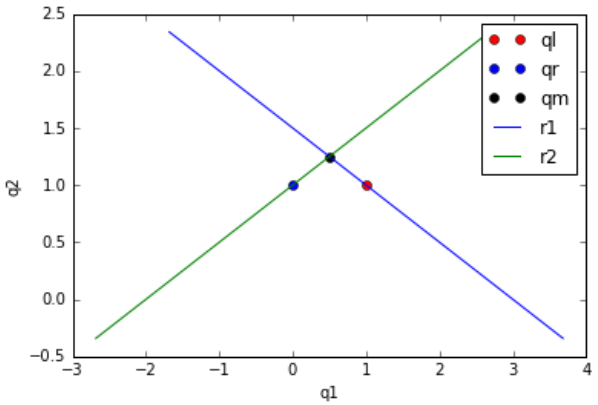
- Pressure at time =1



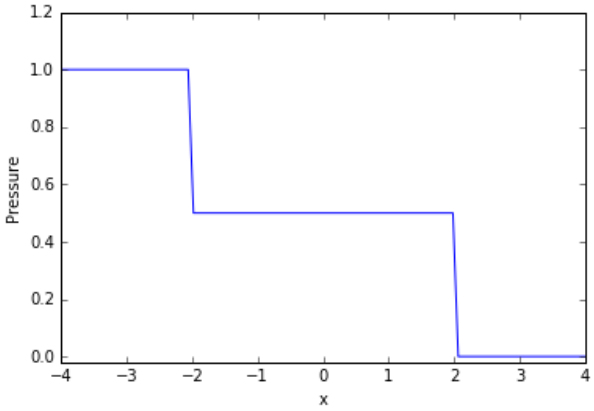
- Velocity at time =1



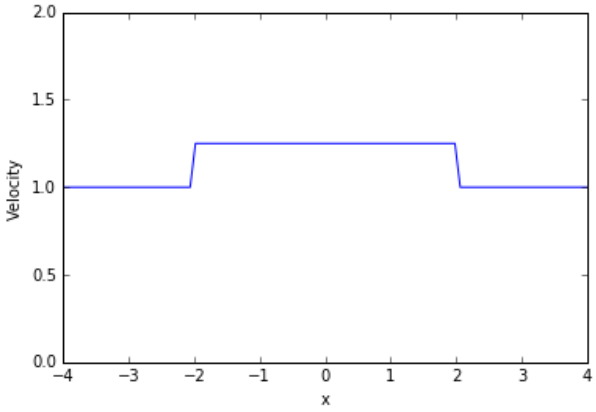
(b) • The phase plot of q



- Pressure at time =1

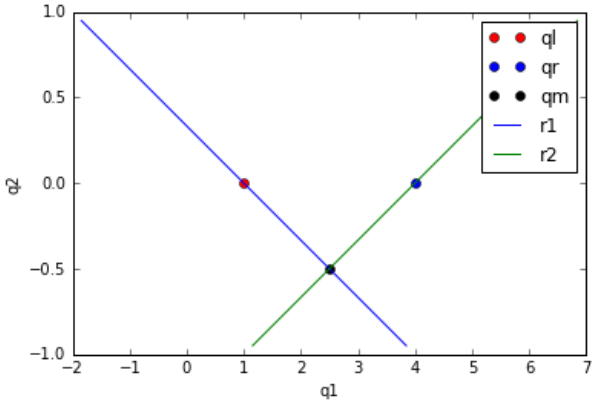


- Velocity at time =1

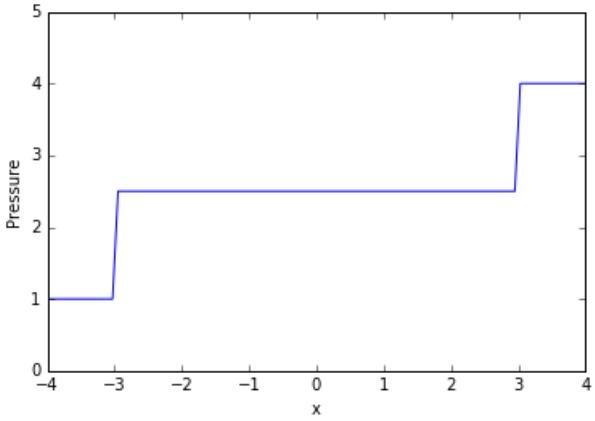


(c)

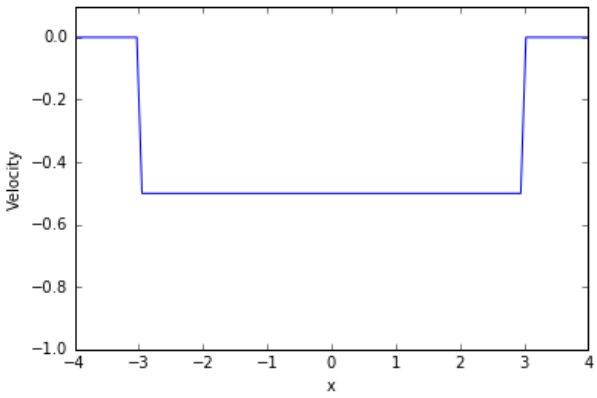
- The phase plot of q



- Pressure at time =1

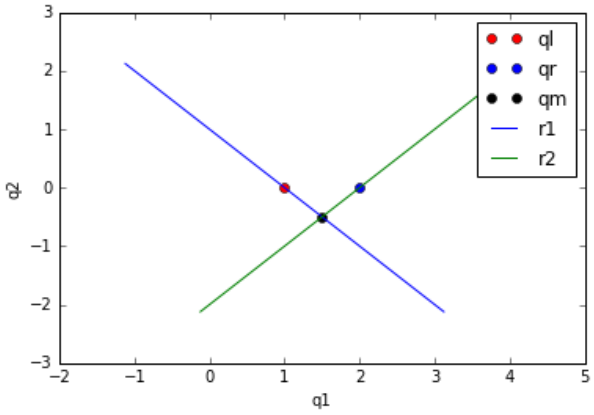


- Velocity at time =1

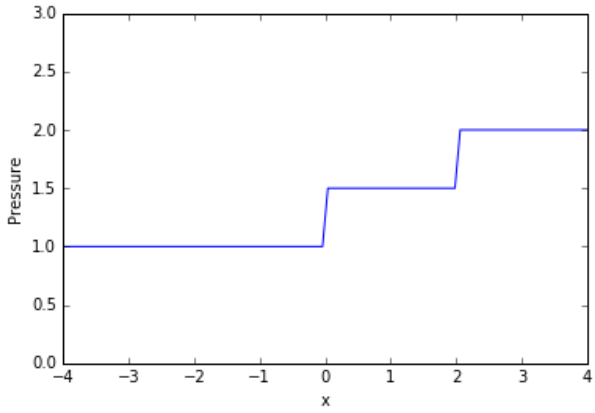


(d)

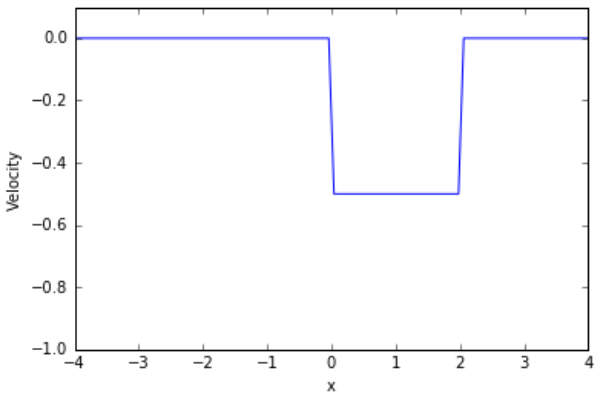
- The phase plot of q



- Pressure at time =1

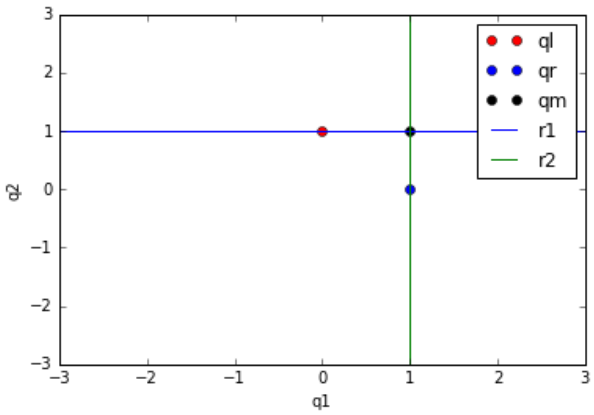


- Velocity at time =1

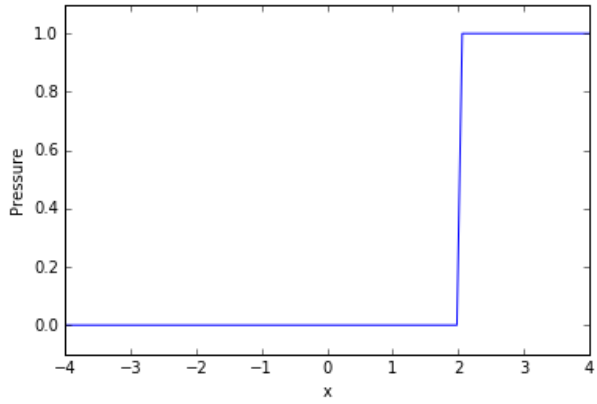


(e)

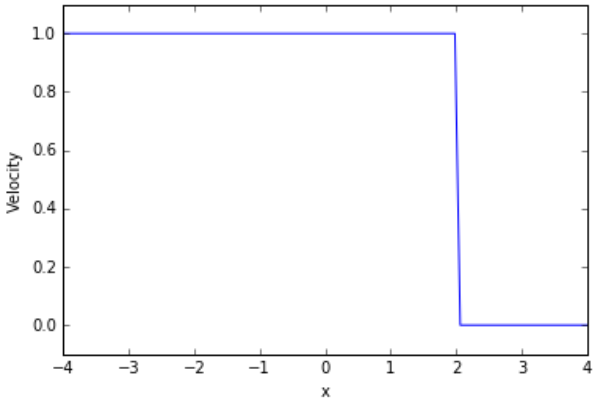
- The phase plot of q



- Pressure at time =1

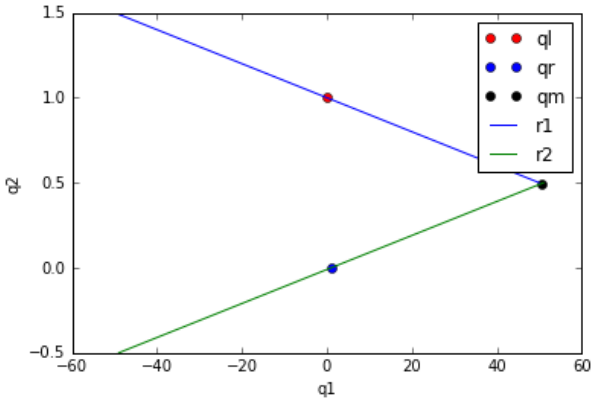


- Velocity at time =1

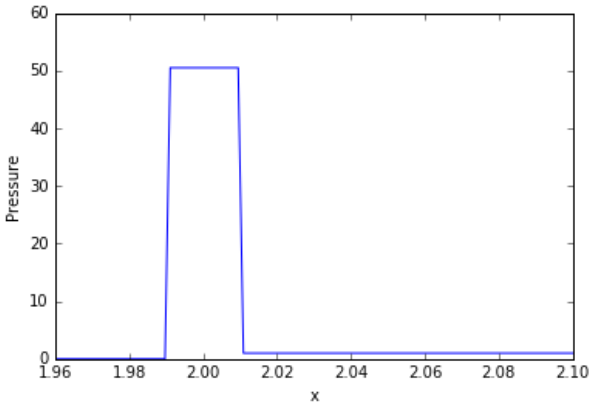


(f)

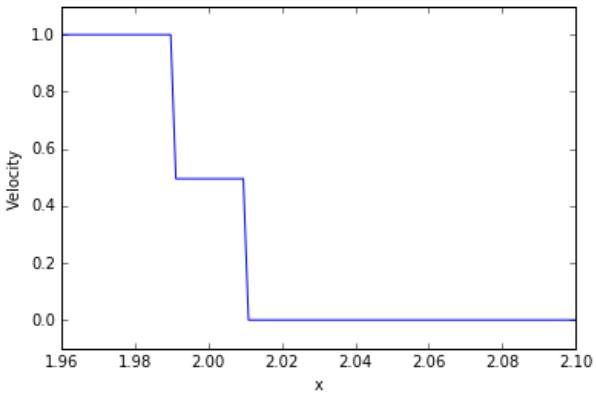
- The phase plot of q



- Pressure at time =1



- Velocity at time =1



Problem #3.2 in the book

You can use Matlab for this one, but I suggest you try writing the program in Python. A Jupyter notebook will be provided to help you get started.

Note that the module `numpy.linalg` contains an `eig` function similar to Matlab.

Solution:

We start with the initial conditions.

```
import numpy as np
import pylab as pl
A = np.array([[2.,2],[0,-3]])
ql = np.array([1,1])
qr = np.array([0,3])
```

Make some useful functions

```
#Get Eigen-values/vectors
def Reisolve(A,ql,qr):
        lam,R = np.linalg.eig(A)
        T= qr-ql
        Rinv = np.linalg.inv(R)
        alpha =np.dot(Rinv,T)
        return alpha,lam ,R

#Phaseplots
def phaseplot(ql,qr,R,qm):
        k1= -3
        k2= 3
        r1 = R[:,0]
        r2 = R[:,1]
        u1 = ql+k1*r1
        u2 = ql + k2*r1
        u = np.vstack((u1,u2))
        v1 = qr+k1*r2
        v2 = qr + k2*r2
```

```
        v = np.vstack((v1,v2))
        pl.plot(ql[0],ql[1], 'ro') #ql
        pl.plot(qr[0],qr[1], 'bo') #qr
        pl.plot(qm[0],qm[1], 'ko') #qm
        pl.plot(u[:,0],u[:,1]) #First eigenvector
        pl.plot(v[:,0],v[:,1]) #Second Eigenvector
        pl.xlabel("q1")
        pl.ylabel("q2")

#Solve for various x at particular time
def qsolve(xcol,t,lamda,R,alpha):
        i=0
        q=np.zeros((2,len(xcol)))
        for x in xcol:
                if x/t <lamda[0]: #Naturally, there is a bug at t=0
                        q[:,i]=ql
                elif x/t <lamda[1]:
                        q[:,i] = ql + alpha[0]*R[:,0]
                else:
                        q[:,i]=qr
                i+=1
        return q

#Solve for middle states
def qmsolve(R,alpha,ql):
        i=0
        qm=np.zeros((2,1))
        qm[:,i] = ql + alpha[0]*R[:,0]
        return qm
```

Solve and plot.

```
al, lamda, R1= Reisolve(A,ql,qr)
ind = np.argsort(lamda)
lamda=lamda[ind]
R= R1[:,ind]
al= al[ind]
x=np.linspace(-10,10,100)
t=1
qm=qmsolve(x,t,lamda,R,al)
pl.figure(3)
phaseplot(ql,qr,R,qm)
pl.figure(1)
pl.plot(x,q[0,:])
pl.xlabel("x")
pl.ylabel("q1")

pl.figure(2)
pl.plot(x,q[1,:])
pl.xlabel("x")
pl.ylabel("q2")
```

Problem #3.3 in the book

Following the sort of thing done in script `problem_3_5.py` might be useful if you want to insert a figure in your solution, or you can draw with another programming language, or sketch the solution by hand and scan.

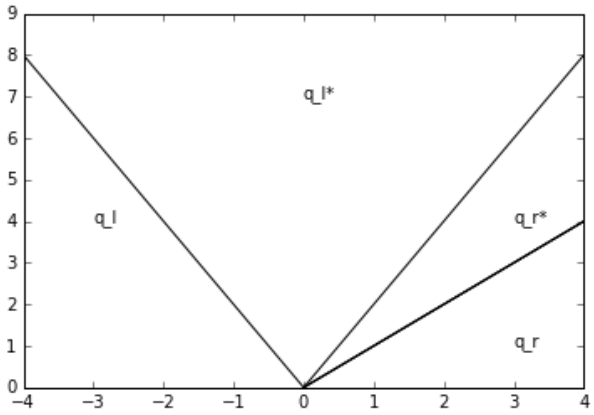
Solution:

The code to plot the figure is given below.

```
from pylab import *

clf()  # clear figure

plot([-4,4],[0,0],'k') # x-axis  'k' means black line
plot([-4,-4],[0,6],'k') # t-axis
plot([4,4],[0,6],'k') # right boundary

def plot_rightgoing(x1,t1,s):
"""
plot right-going wave of slope s starting at (x1,t1) to right boundary
"""
        t2 = t1 + s*(4-x1)
        plot([x1,4], [t1,t2], 'k')

def plot_leftgoing(x1,t1,s):
"""
plot left-going wave of slope s starting at (x1,t1) to left boundary
"""
        t2 = t1 + s*(-4-x1)
        plot([x1,-4], [t1,t2], 'k')

plot_rightgoing(0,0,1)
plot_rightgoing(0,0,1)
def Reisolve(A,ql,qr):
        lam,R = linalg.eig(A)
        T= qr-ql
        Rinv = linalg.inv(R)
        alpha =dot(Rinv,T)
        return alpha,lam ,R
def qsolve(xcol,t,lamda,R,alpha):
        i=0
        q=np.zeros((2,len(xcol)))
        for x in xcol:
        if x/t <lamda[0]: #Naturally, there is a bug at t=0
                q[:,i]=ql
        elif x/t <lamda[1]:
                q[:,i] = ql + alpha[0]*R[:,0]
        else:
                q[:,i]=qr
        i+=1
        return q
A = np.array([[0,0,4],[0,1,0],[1,0,0]])
```

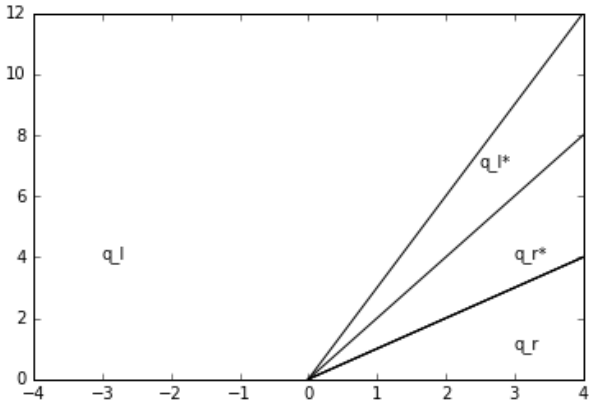
```
ql = np.array([1,2,0])
qr = np.array([1,5,1])
al, lamda, R1= Reisolve(A,ql,qr)
ind = np.argsort(lamda)
lamda=lamda[ind]
R= R1[:,ind]

#Plotting the regions
for l in lamda:
        if l>0:
                plot_rightgoing(0,0,l)
        else:
                plot_leftgoing(0,0,l)
text(-3, 4, 'q_l')
text(0, 7, 'q_l*')
text(3.0, 4, 'q_r*')
text(3, 1, 'q_r')
```

(a)



$$q_l* = \begin{bmatrix} 0 \\ 2 \\ 0.5 \end{bmatrix}, q_r* = \begin{bmatrix} 0 \\ 5 \\ 0.5 \end{bmatrix}$$

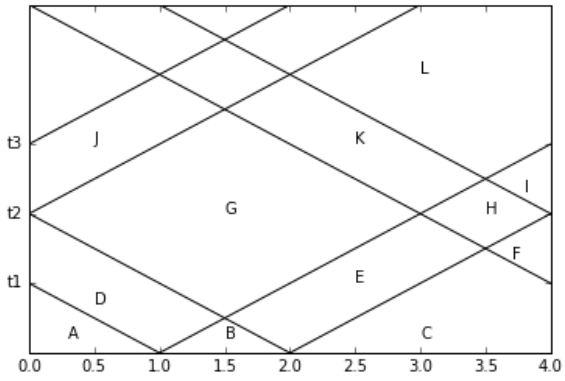
(b)



$$q_l* = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, q_r* = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Problem #3.5 in the book

The script `problem_3_5.py` was used to generate this figure:



To solve this problem, determine the states $A,\ B,\ \ldots,\ M$ and also the times $t_1,\ t_2,\ t_3$. The times can be written in terms of the parameters ρ_0 and K_0, which were not stated in the problem.

For example,

$$A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad , B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad , C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \ldots$$

Solution:

From the initial conditions, we have

$$A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad , B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad , C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The eigenvalues are $\sqrt{K_0/\rho_0}$ and - $\sqrt{K_0/\rho_0}$.

$$t_1 = \frac{\Delta x}{-\lambda} = \sqrt{\frac{\rho_0}{K_0}}$$

$$t_2 = 2\sqrt{\frac{\rho_0}{K_0}}$$

$$t_3 = 3\sqrt{\frac{\rho_0}{K_0}}$$

The eigenvector matrix and it's inverse are

$$R = \begin{bmatrix} -Z_0 & Z_0 \\ 1 & 1 \end{bmatrix}, \quad , R^{-1} = \frac{1}{2Z_0} \begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix} \quad ,$$

where

$$Z_0 = \sqrt{K_0 \rho_0}$$

D:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = R^{-1}(B-A) = \frac{1}{2Z_0}\begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2Z_0}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$W_1 = a_1 R_1 = \frac{1}{2Z_0}\begin{bmatrix} Z_0 \\ -1 \end{bmatrix}$$

$$W_2 = a_2 R_2 = \frac{1}{2Z_0}\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + a_1 R_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2Z_0} \end{bmatrix}$$

E:

$$\begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = R^{-1}(C-B) = \frac{1}{2Z_0}\begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \frac{1}{2Z_0}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_1 R_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2Z_0} \end{bmatrix}$$

$$W_3 = a_3 R_1 = \frac{1}{2Z_0}\begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}$$

$$W_4 = a_4 R_2 = \frac{1}{2Z_0}\begin{bmatrix} -Z_0 \\ -1 \end{bmatrix}$$

F: At the boundary, the left going wave W_1 reflects and we have the new wave W_1'

$$W_1' = -W_1 = \frac{1}{2Z_0}\begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}$$

$$F = D - W_1' = D + W_1 = \begin{bmatrix} 1 \\ \frac{-1}{Z_0} \end{bmatrix}$$

G:

$$G = D + W_1' = D - W_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

H:

$$H = E + W_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

I:

$$I = F + W_3 = \frac{1}{2Z_0}\begin{bmatrix} Z_0 \\ -1 \end{bmatrix}$$

J:

$$J = G + W_4' = G - W_4 = \frac{1}{2Z_0}\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

K:

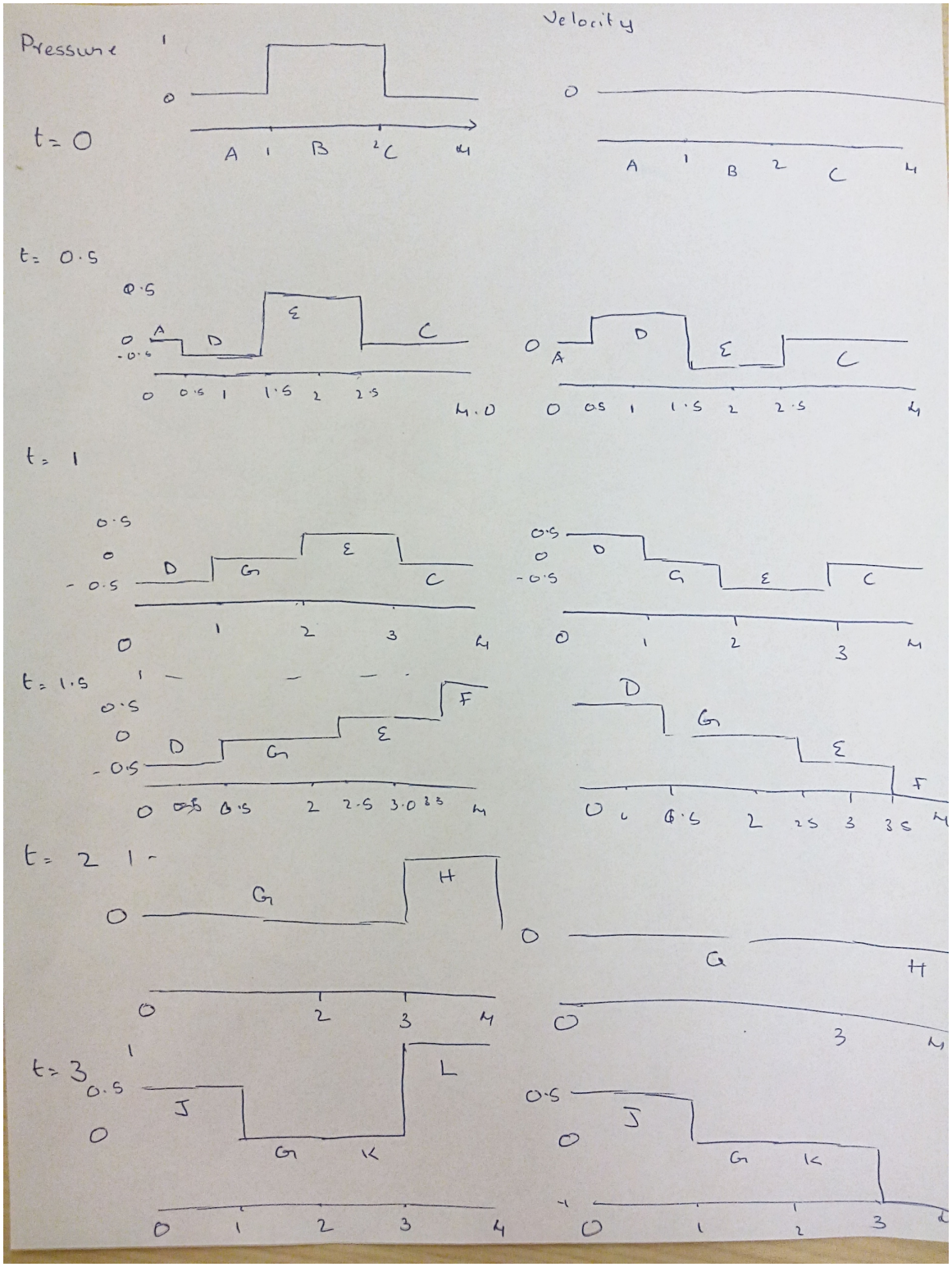
$$K = I - W_3' = I + W_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

L:

$$L = I + W_4' = I - W_4 = \begin{bmatrix} 1 \\ \frac{-1}{Z_0} \end{bmatrix}$$

M:

$$M = J + W_2' = J - W_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Problem #3.5A Solve #3.5 with *periodic* boundary conditions instead of reflecting walls. Sketch the solution in the x–t plane up to at least time t_3 (as in #3.5 the time the right-going wave from $x_0 = 1$ hits the right boundary) and indicate the state in each section. You might

want to modify the script `problem_3_5.py` to make the plot.

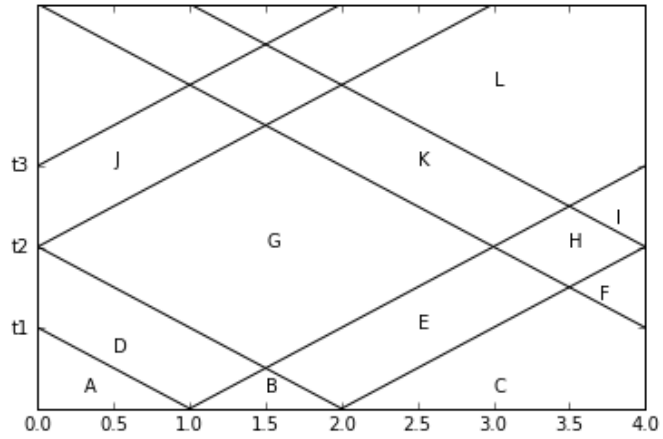
Solution:

With periodic boundary conditions, we get

$$W_2(0,t) = W_1(4,t)$$

$$W_1(4,t) = W_2(0,t)$$

And this following image is obtained.



A,B,C,D and E remain the same.

$$A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad , B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad , C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The eigenvalues are $\sqrt{K_0/\rho_0}$ and - $\sqrt{K_0/\rho_0}$.

$$t_1 = \frac{\Delta x}{-\lambda} = \sqrt{\frac{\rho_0}{K_0}}$$

$$t_2 = 2\sqrt{\frac{\rho_0}{K_0}}$$

$$t_3 = 3\sqrt{\frac{\rho_0}{K_0}}$$

The eigenvector matrix and it's inverse are

$$R = \begin{bmatrix} -Z_0 & Z_0 \\ 1 & 1 \end{bmatrix}, \quad , R^{-1} = \frac{1}{2Z_0} \begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix} \quad ,$$

where

$$Z_0 = \sqrt{K_0 \rho_0}$$

D:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = R^{-1}(B - A) = \frac{1}{2Z_0} \begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2Z_0} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$W_1 = a_1 R_1 = \frac{1}{2Z_0} \begin{bmatrix} Z_0 \\ -1 \end{bmatrix}$$

$$W_2 = a_2 R_2 = \frac{1}{2Z_0} \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + a_1 R_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2Z_0} \end{bmatrix}$$

E:

$$\begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = R^{-1}(C - B) = \frac{1}{2Z_0} \begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \frac{1}{2Z_0} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_1 R_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2Z_0} \end{bmatrix}$$

$$W_3 = a_3 R_1 = \frac{1}{2Z_0} \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}$$

$$W_4 = a_4 R_2 = \frac{1}{2Z_0} \begin{bmatrix} -Z_0 \\ -1 \end{bmatrix}$$

F:

$$F = C + W_1 = \frac{1}{2Z_0} \begin{bmatrix} Z_0 \\ -1 \end{bmatrix}$$

G:

$$G = D + W_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

H:

$$H = E + W_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Problem #4.1 in the book

Solution:

$$\lambda_1 = u_0 - c_0$$

$$\lambda_2 = u_0 + c_0$$

- If the flow is subsonic, there is one positive eigenvalue and one negative one.

$$A^+ = R\lambda_2 R^{-1} = \begin{bmatrix} -Z_0 & Z_0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & (u_0 + c_0) \end{bmatrix} \frac{1}{2Z_0} \begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix} = \frac{1}{2Z_0} \begin{bmatrix} Z_0\lambda_2 & Z_0^2\lambda_2 \\ \lambda_2 & Z_0\lambda_2 \end{bmatrix}$$

$$A^- = R\lambda_1 R^{-1} = \begin{bmatrix} -Z_0 & Z_0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (u_0 - c_0) & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{2Z_0} \begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix} = \frac{1}{2Z_0} \begin{bmatrix} Z_0\lambda_1 & Z_0^2\lambda_1 \\ \lambda_1 & Z_0\lambda_1 \end{bmatrix}$$

If Q_i and Q_{i-1} are given, let

$$\Delta Q_{i-1/2} = Q_i - Q_{i-1}$$

$$W^1_{i-1/2} = \frac{1}{\lambda_1} A^- \Delta Q_{i-1/2}$$

$$W^2_{i-1/2} = \frac{1}{\lambda_2} A^+ \Delta Q_{i-1/2}$$

- If $u_0 > c_0$, both eigenvalues are positive,

$$A^+ = R\lambda R^{-1} = A$$

$$A^- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- If $u_0 < -c_0$,

$$A^+ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^- = R\lambda R^{-1} = A$$

For the last two cases,

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = R^{-1} \Delta Q_{i-1/2}$$

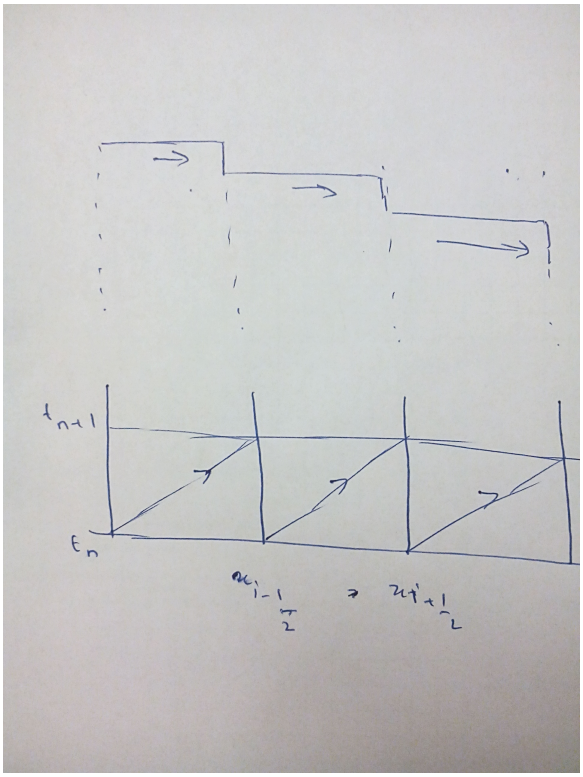
$$W^1_{i-1/2} = a_1 R_1$$

$$W^2_{i-1/2} = a_2 R_2$$

Problem #4.2 in the book

Solution:

(a)



Ex. 2

b) $$Q_i^{n+1} = \frac{1}{2}(Q_{i-1}^n + Q_{i+1}^n) - \frac{\Delta t}{2\Delta x}(u Q_{i+1}^n - u Q_{i-1}^n)$$

$$= \frac{1}{2}(Q_{i-1}^n + Q_{i+1}^n) - \frac{1}{2}(Q_{i+1}^n - Q_{i-1}^n)$$

$$= Q_{i-1}^n$$

∴ The Lax-Friedrich satisfies it.

Lax-Wendroff

$$Q_{i-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(Q_{i-1}^n + Q_i^n) - \frac{\Delta t}{2\Delta x}(u Q_i^n - u Q_{i-1}^n)$$

$$= Q_{i-1}^n$$

$$\therefore Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n)$$

$$= Q_i^n - \frac{\Delta t}{\Delta x}(u Q_i^n - u Q_{i-1}^n)$$

$$= Q_{i-1}^n$$

∴ LW also satisfies it

c) $q_{i-\frac{1}{2}} = \frac{1}{2}\begin{bmatrix} p_i + p_{i-1} \\ -(p_i - p_{i-1}) z_0 \end{bmatrix}$

Solving the Riemann Problem

$$F^n_{i-\frac{1}{2}} = f\left(q_{i-\frac{1}{2}}\right)$$

$$= \frac{c}{2}\begin{bmatrix} p_i + p_{i-1} \\ -(p_i - p_{i-1}) z_0 \end{bmatrix}$$

$$Q_i^{n+1} = Q_i^n - \frac{c\Delta t}{2\Delta x}\begin{bmatrix} p_{i+1} + p_i - p_i - p_{i-1} \\ -(p_{i+1} - p_{i-1}) z_0 + (p_i - p_{i-1}) z_0 \end{bmatrix}$$

$$\begin{bmatrix} p_i^{n+1} \\ u_i^{n+1} \end{bmatrix} = \begin{bmatrix} p_i^n \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} p_{i+1} - p_{i-1} \\ -(p_{i+1} + p_{i-1}) z_0 + 2 p_i z_0 \end{bmatrix}$$

$$p_i^{n+1} = p_i^n - \frac{1}{2}(p_{i+1} - p_{i-1})$$

$$u_i^{n+1} = 0 - \frac{z_0}{2}(2p_i - (p_{i+1} + p_{i-1}))$$

d) Possibly