AMATH 562: Homework 2

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1.
$$\begin{aligned}\mathbb{E}[W(t)^2 - t|\mathbb{F}_s] &= \mathbb{E}[W(t)^2 - W(s)^2 - t + s|\mathbb{F}_s] + \mathbb{E}[W(s)^2 - s|\mathbb{F}_s] \\ &= \mathbb{E}[W(t)^2 - W(s)^2 - t + s|\mathbb{F}_s] + W(s)^2 - s\end{aligned}$$

For $W(t)^2 - t$, we need
$$\mathbb{E}[W(t)^2 - W(s)^2 - t + s|\mathbb{F}_s = 0$$

or
$$\mathbb{E}[W(t)^2 - W(s)^2|\mathbb{F}_s] = t - s$$

$$\begin{aligned}\mathbb{E}[W(t)^2 - W(s)^2|\mathbb{F}_s] &= \mathbb{E}[(W(t) - W(s))(W(t) + W(s))|\mathbb{F}_s] \\ &= \mathbb{E}[(W(t) - W(s))W(t)|\mathbb{F}_s] + \mathbb{E}[(W(t) - W(s))W(s)|\mathbb{F}_s] \\ &= \mathbb{E}[(W(t) - W(s))W(t)|\mathbb{F}_s] + \mathbb{E}W(s)\mathbb{E}[W(t) - W(s)|\mathbb{F}_s] \\ &= \mathbb{E}[(W(t) - W(s))(W(t) - W(s) + W(s))|\mathbb{F}_s] + 0 \\ &= \mathbb{E}[(W(t) - W(s))^2|\mathbb{F}_s] + \mathbb{E}[(W(t) - W(s))W(s)|\mathbb{F}_s] \\ &= \mathbb{E}[(W(t) - W(s))^2|\mathbb{F}_s] \\ &= t - s\end{aligned}$$

2.
$$\begin{aligned}\phi_{W(N)}(u) &= \mathbb{E}[e^{iuW(N)}] \\ &= \mathbb{E}[\mathbb{E}[e^{iuW(N)}]|N] \\ &= \mathbb{E}[e^{-\frac{1}{2}Nu^2}] \\ &= \sum_0^\infty e^{-\frac{1}{2}ku^2} e^{-\lambda}\frac{\lambda^k}{k!} \\ &= e^{-\lambda}\sum_0^\infty \frac{(\lambda e^{-\frac{1}{2}u^2})^k}{k!} \\ &= e^{-\lambda}e^{\lambda e^{-\frac{u^2}{2}}} \\ &= e^{\lambda e^{-\frac{u^2}{2}} - \lambda}\end{aligned}$$

3.

$$V_T(1,W) = \lim_{||\pi||\to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|$$

$$V_T(3,W) = \lim_{||\pi||\to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3$$

We know

$$V_T(2,W) = T$$

$$T = \lim_{||\pi||\to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2$$

$$T \leq \lim_{||\pi||\to 0} max(|W(t_{j+1}) - W(t_j)|) \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|$$

$$T \leq \lim_{||\pi||\to 0} max(|W(t_{j+1}) - W(t_j)|) \lim_{||\pi||\to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|$$

$$T \leq \lim_{||\pi||\to 0} max(|W(t_{j+1}) - W(t_j)|) V_T(1,W)$$

Since $\lim_{||\pi||\to 0} max(|W(t_{j+1}) - W(t_j)|) = 0$, $V_T(1,W)$ has to be infinity.

$$V_T(3,W) = \lim_{||\pi||\to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3$$

$$V_T(3,W) \leq \lim_{||\pi||\to 0} max(|W(t_{j+1}) - W(t_j)|) \lim_{||\pi||\to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2$$

$$V_T(3,W) \leq \lim_{||\pi||\to 0} max(|W(t_{j+1}) - W(t_j)|) T$$

$$V_T(3,W) \leq 0$$

Since the variations are non-negative, $V_T(3,W) = 0$.

4.

$$Z = e^{\sigma X - (\sigma\mu t + \sigma^2 t/2)} = e^{\sigma\mu t + \sigma W - (\sigma\mu t + \sigma^2 t/2)} = e^{\sigma W - \sigma^2 t/2}$$

We need to show

$$\mathbb{E}[e^{\sigma W_t - \sigma^2 t/2} | \mathbb{F}_s] = e^{\sigma W_s - \sigma^2 s/2}$$

$$\mathbb{E}[e^{\sigma W_t - \sigma W_s - \sigma^2 t/2 + \sigma^2 s/2} e^{\sigma W_s - \sigma^2 s/2} | \mathbb{F}_s] = e^{\sigma W_s - \sigma^2 s/2}$$

$$\mathbb{E}[e^{\sigma W_t - \sigma W_s - \sigma^2 t/2 + \sigma^2 s/2} | \mathbb{F}_s] e^{\sigma W_s - \sigma^2 s/2} = e^{\sigma W_s - \sigma^2 s/2}$$

$$\mathbb{E}[e^{\sigma W_t - \sigma W_s - \sigma^2 t/2 + \sigma^2 s/2} | \mathbb{F}_s] = 1$$

$$\mathbb{E}[e^{\sigma W_t - \sigma W_s} | \mathbb{F}_s] e^{-\sigma^2 t/2 + \sigma^2 s/2} = 1$$

$$\mathbb{E}[e^{\sigma W_t - \sigma W_s} | \mathbb{F}_s] = e^{\sigma^2 (t-s)/2}$$

$$\mathbb{E}[e^{\sigma W_t - \sigma W_s} | \mathbb{F}_s] = \mathbb{E}[e^{\sigma N(0, t-s)}]$$

$$= \mathbb{E}[\sigma N(0,t-s)] + \mathbb{E}[\frac{\sigma^2 N(0,t-s)^2}{2}] + \mathbb{E}[\frac{\sigma^3 N(0,t-s)^3}{3!}] + \mathbb{E}[\frac{\sigma^4 N(0,t-s)^4}{4!}] + \dots$$

$$= \frac{\sigma^2(t-s)}{2} + \frac{3\sigma^4(t-s)^2}{4!} + \frac{15\sigma^6(t-s)^3}{6!} + \frac{105\sigma^4(t-s)^4}{8!} + \dots$$

$$= \frac{\sigma^2(t-s)}{2} + \frac{\sigma^4(t-s)^2}{2^2 * 2!} + \frac{\sigma^6(t-s)^3}{2^3 * 3!} + \frac{\sigma^4(t-s)^4}{2^4 4!} + \dots$$

$$= e^{\sigma^2(t-s)/2}$$

Thus, Z is a martingale.

$$Z = e^{\sigma W_t - \sigma^2 t/2}$$

By assumption, Z^m the stopping process is a martingale. So,

$$1 = Z_0^m = \mathbb{E}Z_t^m = \mathbb{E}e^{\sigma W_{t\wedge\tau_m} - \sigma^2 t\wedge\tau_m/2}$$

$$1 = \lim_{t\to\infty} \mathbb{E}e^{\sigma W_{t\wedge\tau_m} - \sigma^2 t\wedge\tau_m/2}$$

$$= \mathbb{E}\lim_{t\to\infty} e^{\sigma W_{t\wedge\tau_m} - \sigma^2 t\wedge\tau_m/2}$$

$$= \mathbb{E}e^{\sigma W_{\tau_m} - \sigma^2 \tau_m/2}$$

$$= \mathbb{E}e^{\sigma(m-\mu\tau_m) - \sigma^2 \tau_m/2}$$

$$= \mathbb{E}e^{\sigma(m-\mu\tau_m) - \sigma^2 \tau_m/2}$$

So,

$$e^{\sigma m}\mathbb{E}e^{-\sigma\mu\tau_m - \sigma^2\tau_m/2} = 1$$

$$\mathbb{E}e^{-(\sigma\mu+\sigma^2/2)\tau_m} = e^{-\sigma m}$$

We can set

$$\sigma = \frac{-2\mu + \sqrt{4\mu^2 + 8\alpha}}{2}$$

$$\alpha = \sigma\mu + \sigma^2/2$$

Then,

$$\mathbb{E}e^{-\alpha\tau_m} = e^{-\frac{(-2\mu+\sqrt{4\mu^2+8\alpha})}{2}m}$$