Assignment 2.

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Due Monday, Jan. 22.

1. (Inverse matrix and Green's functions)

(a) Write out the 5×5 matrix A from (2.43) for the boundary value problem $u''(x) = f(x)$ with $u(0) = u(1) = 0$ for $h = 0.25$.

Solution:

$$A = \begin{bmatrix} 1. & 0. & 0. & 0. & 0. \\ 0.25 & -0.5 & 0.25 & 0. & 0. \\ 0. & 0.25 & -0.5 & 0.25 & 0. \\ 0. & 0. & 0.25 & -0.5 & 0.25 \\ 0. & 0. & 0. & 0. & 1. \end{bmatrix}$$

(b) Write out the 5×5 inverse matrix A^{-1} explicitly for this problem.

Solution:

$$A^{-1} = \begin{bmatrix} 1. & -0. & -0. & -0. & 0. \\ 0.75 & -0.046875 & -0.03125 & -0.015625 & 0.25 \\ 0.5 & -0.03125 & -0.0625 & -0.03125 & 0.5 \\ 0.25 & -0.015625 & -0.03125 & -0.046875 & 0.75 \\ 0. & 0. & 0. & 0. & 1. \end{bmatrix}$$

(c) If $f(x) = x$, determine the discrete approximation to the solution of the boundary value problem on this grid and sketch this solution and the five Green's functions whose sum gives this solution.

Solution:

$$u(x) = \int_0^1 \bar{x} G(x; \bar{x})\, d\bar{x} = 0.25hG_1 + 0.5hG_2 + 0.75hG_3$$

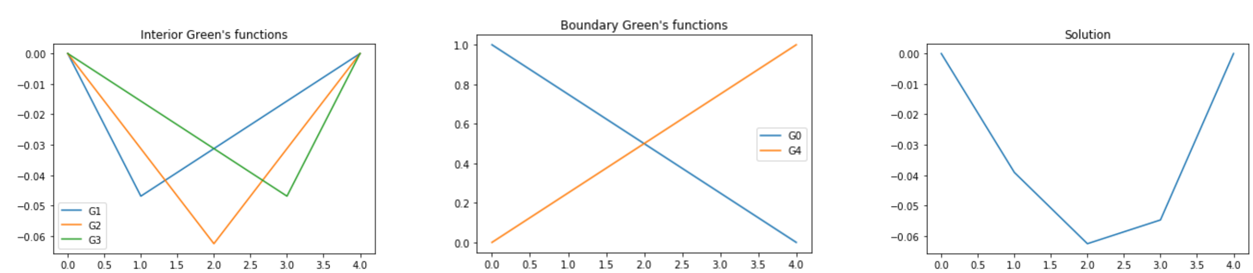

Figure 1: Interior Green's functions, Boundary Green's functions and Solution

2. (Another way of analyzing the error using Green's functions) The *composite trapezoid rule* for integration approximates the integral from a to b of a function g by dividing the interval into segments of length h and approximating the integral over each segment by the integral of the linear function that matches g at the endpoints of the segment. (For $g > 0$, this is the area of the trapezoid with height $g(x_j)$ at the left endpoint x_j and height $g(x_{j+1})$ at the right endpoint x_{j+1}.) Letting $h = (b-a)/(m+1)$ and $x_j = a + jh$, $j = 0, 1, \ldots, m, m+1$:

$$\int_a^b g(x)\,dx \approx h \sum_{j=0}^{m} \frac{g(x_j) + g(x_{j+1})}{2} = h\left[\frac{g(x_0)}{2} + \sum_{j=1}^{m} g(x_j) + \frac{g(x_{m+1})}{2}\right].$$

(a) Assuming that g is sufficiently smooth, show that the error in the composite trapezoid rule approximation to the integral is $O(h^2)$. [Hint: Show that the error on each subinterval is $O(h^3)$.]

Solution:

$$g(x) = g(x_i) + g'(x_i)(x - x_i) + \frac{1}{2}g''(x_i)(x-x_i)^2 + O((x-x_i)^3)$$

$$\begin{aligned}\int_{x_i}^{x_{i+1}} g(x)dx &= \int_{x_i}^{x_{i+1}} g(x_i)dx + \int_{x_i}^{x_{i+1}} g'(x_i)(x-x_i)dx + O((x_{i+1}-x_i)^3)dx \\ &= g(x_i)(x_{i+1}-x_i) + g'(x_i)(x_{i+1}-x_i)^2 + O((x_{i+1}-x_i)^3)\end{aligned}$$

$$\int_{x_i}^{x_{i+1}} g(x)dx = g(x_i)h + g'(x_i)\frac{h^2}{2} + O(h^3) \tag{1}$$

$$g(x) = g(x_{i+1}) + g'(x_{i+1})(x - x_{i+1}) + \frac{1}{2}g''(x_{i+1})(x-x_{i+1})^2 + O((x-x_i)^3)$$

$$g(x) = g(x_{i+1}) + g'(x_i)(x - x_{i+1}) + g''(x_i)h(x-x_{i+1}) + \frac{1}{2}g''(x_{i+1})(x-x_{i+1})^2 + O((x-x_i)^3)$$

$$\int_{x_i}^{x_{i+1}} g(x)dx = \int_{x_i}^{x_i} g(x_{i+1})dx + \int_{x_i}^{x_{i+1}} g'(x_{i+1})(x-x_{i+1}) + O((x_{i+1}-x_i)^3)$$

$$\int_{x_i}^{x_{i+1}} g(x)dx = g(x_{i+1})h - g'(x_i)\frac{h^2}{2} + O(h^3) \tag{2}$$

Adding (1) and (2) together,

$$\int_{x_i}^{x_{i+1}} g(x)dx = \frac{h}{2}(g(x_i) + g(x_{i+1})) + O(h^3)$$

(b) Recall that the true solution of the boundary value problem $u''(x) = f(x)$, $u(0) = u(1) = 0$ can be written as

$$u(x) = \int_0^1 f(\bar{x})G(x;\bar{x})\,d\bar{x}, \tag{3}$$

where $G(x;\bar{x})$ is the Green's function corresponding to $\bar{x}$. The finite difference approximation u_i to $u(x_i)$, using the centered finite difference scheme in (2.43), is

$$u_i = h\sum_{j=1}^{m} f(x_j)G(x_i;x_j), \quad i = 1,\ldots,m. \tag{4}$$

Show that formula (4) is the trapezoid rule approximation to the integral in (3) when $x = x_i$, and conclude from this that the error in the finite difference approximation is $O(h^2)$ at each node x_i. [Recall: The Green's function $G(x;x_j)$ has a *discontinuous* derivative at $x = x_j$. Why does this not degrade the accuracy of the composite trapezoid rule?]

Solution:

$$u(x) = \int_0^1 f(\bar{x})G(x;\bar{x})\,d\bar{x}$$

Setting x = x_i and using the Trapezoidal approximation,

$$u_i = h\frac{f(0)G(x_i;0)}{2} + h\sum_{j=1}^{m} f(x_j)G(x_i;x_j) + h\frac{f(1)G(x_i;1)}{2}$$

Since the Green's function is the solution to the ODE and since at the boundaries, u is 0, G(x,0)=G(x,1)=0. So, we get

$$u_i = h\sum_{j=1}^{m} f(x_j)G(x_i;x_j)$$

3. (Green's function with Neumann boundary conditions)

   (a) Determine the Green's functions for the two-point boundary value problem $u''(x) = f(x)$ on $0 < x < 1$ with a Neumann boundary condition at $x = 0$ and a Dirichlet condition at $x = 1$, i.e, find the function $G(x,\bar{x})$ solving

$$u''(x) = \delta(x-\bar{x}), \quad u'(0) = 0, \quad u(1) = 0$$

   and the functions $G_0(x)$ solving

$$u''(x) = 0, \quad u'(0) = 1, \quad u(1) = 0$$

   and $G_1(x)$ solving

$$u''(x) = 0, \quad u'(0) = 0, \quad u(1) = 1.$$

   Solution:

   We need

$$u'(0) = 0, u(1) = 0$$

and

$$u(\bar{x}+\epsilon) - u(\bar{x}-\epsilon) = 1$$

So, the Green's function satisying that would be given by

$$G(x,\bar{x}) = \left\{ \begin{array}{ll} \bar{x}-1, & \text{for } 0 \le x \le \bar{x} \\ x-1, & \text{for } \bar{x} \le x \le 1 \end{array} \right\}$$

Solving the odes for the boundaries, we have

$$G_0(x) = x - 1$$

$$G_1(x) = 1$$

(b) Using this as guidance, find the general formulas for the elements of the inverse of the matrix in equation (2.54). Write out the 5×5 matrices A and A^{-1} for the case $h = 0.25$.

Solution:

$$A = \begin{bmatrix} 3. & -4. & 1. & 0. & 0. \\ 0.25 & -0.5 & 0.25 & 0. & 0. \\ 0. & 0.25 & -0.5 & 0.25 & 0. \\ 0. & 0. & 0.25 & -0.5 & 0.25 \\ 0. & 0. & 0. & 0. & 1. \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1. & -0.75 & -0.5 & -0.25 & 1. \\ -0.75 & -0.75 & -0.5 & -0.25 & 1. \\ -0.5 & -0.5 & -0.5 & -0.25 & 1. \\ -0.25 & -0.25 & -0.25 & -0.25 & 1. \\ 0. & 0. & 0. & 0. & 1. \end{bmatrix}$$

4. (Solvability condition for Neumann problem) Determine the null space of the matrix A^T, where A is given in equation (2.58), and verify that the condition (2.62) must hold for the linear system to have solutions.

Solution:

$$A = \begin{pmatrix} -h & h & & & & \\ 1 & -2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & -2 & 1 \\ & & & & h & -h \end{pmatrix}$$

$$A^T = \begin{pmatrix} -h & 1 & & & & \\ h & -2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & -2 & h \\ & & & & 1 & -h \end{pmatrix}$$

The nullspace of this matrix is the vector [1,h,h,h,h....,h,1] because we can see multiplying them yields 0. So, a solution only exists if the f-vector is orthogonal to the nullspace. Setting the dot-product equal to 0, we obtain the following criteria

$$\sigma_0 + (h/2)f(x_0) + hf(x_1) + \ldots + hf(x_{m-1}) + hf(x_m) - \sigma_1 + \frac{h}{2}f(x_{m+1}) = 0$$

which is the same as 2.62.

5. (Symmetric tridiagonal matrices)

   (a) Consider the **Second approach** described on p. 31 for dealing with a Neumann boundary condition. If we use this technique to approximate the solution to the boundary value problem $u''(x) = f(x)$, $0 \le x \le 1$, $u'(0) = \sigma$, $u(1) = \beta$, then the resulting linear system $A\mathbf{u} = \mathbf{f}$ has the following form:

$$\frac{1}{h^2}\begin{pmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \\ u_m \end{pmatrix} = \begin{pmatrix} \sigma + (h/2)f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \beta/h^2 \end{pmatrix}.$$

   Show that the above matrix is similar to a symmetric tridiagonal matrix via a *diagonal* similarity transformation; that is, there is a diagonal matrix D such that DAD^{-1} is symmetric.

   Solution: Experimenting with a smaller version,

$$\begin{pmatrix} d_0 & & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_2 \end{pmatrix} \begin{pmatrix} -h & h & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{d_0} & & 0 \\ 0 & \frac{1}{d_1} & 0 \\ 0 & 0 & \frac{1}{d_2} \end{pmatrix} = \begin{pmatrix} -h & h\frac{d_0}{d_1} & 0 \\ \frac{d_1}{d_0} & -2 & \frac{d_1}{d_2} \\ 0 & \frac{d_2}{d_1} & -2 \end{pmatrix}$$

   So, for the resulting matrix to be symmetric, we choose the diagonal vector given by $[1 \ \sqrt{h} \ \sqrt{h} \ \sqrt{h} \ \sqrt{h}]$

$$\begin{pmatrix} 1 & & & & \\ & \sqrt{h} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \sqrt{h} \end{pmatrix} \begin{pmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & \frac{1}{\sqrt{h}} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \frac{1}{\sqrt{h}} \end{pmatrix}$$

$$
= \begin{pmatrix} -h & \sqrt{h} & & & \\ \sqrt{h} & -2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix}
$$

This is a symmetric matrix.

(b) Consider the **Third approach** described on pp. 31-32 for dealing with a Neumann boundary condition. [**Note:** If you have an older edition of the text, there is a typo in the matrix (2.57) on p. 32. There should be a row above what is written there that has entries $\frac{3}{2}h$, $-2h$, and $\frac{1}{2}h$ in columns 1 through 3 and 0's elsewhere. I believe this was corrected in newer editions.] Show that if we use that first equation (given at the bottom of p. 31) to eliminate u_0 and we also eliminate u_{m+1} from the equations by setting it equal to β and modifying the right-hand side vector accordingly, then we obtain an m by m linear system $A\mathbf{u} = \mathbf{f}$, where A is similar to a symmetric tridiagonal matrix via a diagonal similarity transformation.

Solution:

$$
\frac{1}{h^2} \begin{pmatrix} \frac{3}{2}h & -2h & \frac{1}{2}h & & & \\ 1 & -2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \\ u_m \end{pmatrix} = \begin{pmatrix} \sigma \\ f(x_1) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \beta/h^2 \end{pmatrix}.
$$

$$
\frac{1}{h^2} \begin{pmatrix} 3 & -4 & 1 & & \\ 3 & -6 & 3 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \\ u_m \end{pmatrix} = \begin{pmatrix} 2\frac{\sigma}{h} \\ 3f(x_1) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \beta/h^2 \end{pmatrix}.
$$

Eliminating the boundary element,

$$
\frac{1}{h^2} \begin{pmatrix} -1 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_{m-1} \\ u_m \end{pmatrix} = \begin{pmatrix} \frac{3}{2}f(x_1) - \frac{\sigma}{h} \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \beta/h^2 \end{pmatrix}.
$$

This is a symmetric matrix.