Assignment 3.

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1. (a)

$$Lu = [p(x)u'(x)]' + q(x)u(x)$$

$$\begin{aligned}
\int_a^b vLudx &= \int_a^b v(x)([p(x)u'(x)]' + q(x)u(x))dx \\
&= [v(x)p(x)u'(x)]_a^b + \int_a^b -v'pu' + quvdx \\
&= [vpu' - v'pu]_a^b + \int_a^b v''pu + v'p'u + quvdx \\
&= [vpu' - v'pu]_a^b + u\int_a^b v''p + v'p' + qvdx
\end{aligned}$$

The adjoint equation is

$$L^*v = [p(x)v'(x)]' + q(x)v(x)$$

For the boundary conditions ,

$$v(b)p(b)u'(b) - v'(b)p(b)u(b) - v(a)p(a)u'(a) + v'(a)p(a)u(a) = 0$$

$$u(b)(-\frac{\alpha_1}{\alpha_2}v(b)p(b) - v'(b)p(b)) - u(a)(-\frac{\beta_1}{\beta_2}v(a)p(a) - v'(a)p(a)) = 0$$

Thus, the adjoint boundary conditions are

$$\alpha_1 v(b)p(b) + \alpha_2 v'(b)p(b) = 0$$

$$\beta_1 v(a)p(a) + \beta_2 v'(a)p(a) = 0$$

(b)

$$Lu = [p(x)u'(x)]' + q(x)u(x)$$

$$\begin{aligned}
\int_a^b vLudx &= \int_a^b v(x)([p(x)u'(x)]' + q(x)u(x))dx \\
&= [v(x)p(x)u'(x)]_a^b + \int_a^b -v'pu' + quvdx \\
&= [vpu' - v'pu]_a^b + \int_a^b v''pu + v'p'u + quvdx \\
&= [vpu' - v'pu]_a^b + u\int_a^b v''p + v'p' + qvdx
\end{aligned}$$

The adjoint equation is

$$L^*v = [p(x)v'(x)]' + q(x)v(x)$$

For the boundary conditions ,

$$v(b)p(b)u'(b) - v'(b)p(b)u(b) - v(a)p(a)u'(a) + v'(a)p(a)u(a) = 0$$

$$p(b)u'(b)(v(b) - v(a)) + u(a)(v'(a)p(a) - v'(b)p(b)) = 0$$

Thus, the adjoint boundary conditions are

$$v(b) - v(a) = 0$$

$$v'(a)p(a) - v'(b)p(b) = 0$$

(c)

$$Lu = u''(x) + [c(x) - x]u'(x) + c'(x)u(x)$$

$$\begin{aligned}
\int_a^b vLudx &= \int_a^b v(x)(u''(x) + [c(x) - x]u'(x) + c'(x)u(x))dx \\
&= [vu' + v(c-x)u]_a^b + \int_a^b -v'u' - u(v(c'-1) + v'(c-x)) + c'uvdx \\
&= [vu' + v(c-x)u - v'u]_a^b + \int_a^b v''u - u(v(c'-1) + v'(c-x)) + c'uvdx \\
&= [vu' + v(c-x)u - v'u]_a^b + u\int_a^b v'' - v(c'-1) + v'(c-x) + c'vdx \\
&= [vu' + v(c-x)u - v'u]_a^b + u\int_a^b v'' + v + v'(c-x)dx
\end{aligned}$$

The adjoint equation is

$$L^*v = v''(x) + (c(x) - x)v'(x) + v(x)$$

For the boundary conditions ,

$$v(1)u'(1) + v(1)(c(1)-1)u(1) - v'(1)u(1) - v(0)u'(0) - v(0)c(0)u(0) + v'(0)u(0) = 0$$

$$-v(1)u(0) - v'(1)u(1) + v(0)u(1) + v'(0)u(0) = 0$$

Thus, the adjoint boundary conditions are

$$v'(0) = v(1)$$

$$v'(1) = v(0)$$

2.

$$x(1-x)y'' - 2y' + 2y = 0$$

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n} = a_0 x^r + a_1 x^{r+1} + \dots$$

with

$$a_0 \neq 0$$

To find r, we plug $a_0 x^r$ into the ode.

$$r(r-1)x^{r-1} - r(r-1)x^r - 2rx^{r-1} + 2x^r = 0$$

Equating the lowest order terms,

$$r^2 - 3r = 0$$

$$r = 3, 0$$

So, the leading terms are 1 and x^3.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - 2\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

To find the reccurence relation,

$$\sum_{n=0}^{\infty} (n+r+1)(n+r) a_{n+1} x^{n+r} - \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - 2\sum_{n=0}^{\infty} (n+r+1) a_{n+1} x^{n+r} + 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$a_{n+1} = a_n \frac{(n+r-1)(n+r) - 2}{(n+r+1)(n+r) - 2(n+r+1)}$$

When r =0,

$$a_{n+1} = a_n \frac{n^2 - n - 2}{(n+1)(n-2)} = a_n$$

When r =3,

$$a_{n+1} = a_n \frac{(n+2)(n+3) - 2}{(n+4)(n+3) - 2(n+4)} = a_n \frac{n^2 + 5n + 4}{n^2 + 5n + 4} = a_n$$

So,

$$\begin{aligned} y &= c_1(1 + x + x^2 + x^3 + \dots) + c_2 x^3 (1 + x + x^2 + x^3 + \dots) \\ &= c_1 \frac{1}{1-x} + c_2 \frac{x^3}{1-x} \end{aligned}$$

3.

$$y'' - xy = 0$$

This is not singular. So,

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

So,

$$a_2 = 0$$

$$a_{n+2} = -\frac{1}{(n+2)(n+1)} a_{n-1}$$

$$y = a_0 \big(1 - \frac{x^3}{6} + \frac{x^6}{30} + \dots\big) + a_1 \big(x - \frac{x^4}{12} + \frac{x^7}{3*4*6*7} + \dots\big)$$