Assignment 1.

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1.

$$x^2 + (1 - \epsilon - \epsilon^2)x + \epsilon - 2e^{\epsilon^2} = 0$$

Let us assume

$$x = x_0 + x_1\epsilon^r + x_2\epsilon^{2r} + x_3\epsilon^{3r} + \dots$$

$$x^2 = x_0^2 + 2x_0x_1\epsilon^r + 2(x_0x_2 + x_1^2)\epsilon^{2r} + \dots$$

Plugging this into the algebraic equation,

$$x_0^2 + 2x_0x_1\epsilon^r + 2(x_0x_2 + x_1^2)\epsilon^{2r} + x_0 + x_1\epsilon^r + x_2\epsilon^{2r} - x_0\epsilon - x_1\epsilon^{1+r} - x_2\epsilon^{1+2r}$$
$$-x_0\epsilon^2 - x_1\epsilon^{2+r} - x_2\epsilon^{2+2r} + \epsilon - 2 - 2\epsilon^2 - \epsilon^4 + \dots = 0$$

Equating the zero order terms,

$$x_0^2 + x_0 - 2 = 0$$

$$x_0 = -2, 1$$

For dominant balance, we need

$$\epsilon^r \approx \epsilon$$

$$r = 1$$

Equating the zero order terms,

$$x_0^2 + x_0 - 2 = 0$$

$$x_0 = -2, 1$$

Equating the first order terms,

$$2x_0x_1 + x_1 - x_0 + 1 = 0$$

$$x_1 = 1, 0$$

Equating the second order terms,

$$2x_0x_2 + 2x_1^2 + x_2 - x_1 - x_0 - 2 = 0$$

$$x_2 = 0.5, 1$$

So, the solutions are

$$x = -2 + \epsilon + \frac{1}{2}\epsilon^2 + O(\epsilon^3)$$

and

$$x = 1 + \epsilon^2 + O(\epsilon^3)$$

2.

$$\epsilon x^3 - 3x + 1 = 0$$

Let us assume

$$x = \delta_0 x_0 + x_1\delta_1 + x_2\delta_2 + x_3\delta_3 + \dots$$

$$x^3 = \delta_0^3 x_0^3 + 3\delta_0^2\delta_1 x_0^2 x_1 + 3\delta_0^2\delta_2 x_0^2 x_2 + 3\delta_1^2\delta_0 x_1^2 x_0 + \dots$$

Trying the dominant terms,

$$\epsilon\delta_0^3 x_0^3 - 3\delta_0 x_0 + 1 = 0$$

Dominant balance gives two options

$$\delta_0 \sim 1, \delta_0 \sim \epsilon^{-\frac{1}{2}}$$

And

$$x_0 = \frac{1}{3}, x_0^3 - 3x_0 = 0 \rightarrow x_0 = \sqrt{3}, -\sqrt{3}$$

Equating the next order terms,

$$3\delta_0^2\delta_1 x_0^2 x_1 \epsilon - 3x_1\delta_1 = 0$$

$$x_0 = -2, 1$$

Let $x = r\epsilon^{-\frac{1}{2}}$. Then the main equation becomes

$$\frac{r^3}{\sqrt{\epsilon}} - 3\frac{r}{\sqrt{\epsilon}} + 1 = 0$$

$$r^3 - 3r + \sqrt{\epsilon} = 0$$

$$r = r_0 + r_1\epsilon^a + r_2\epsilon^{2a} + r_3\epsilon^{3a} + \dots$$

$$r^3 = r_0^3 + 3r_0^2 r_1\epsilon^a + 3(r_0 r_2 + r_1^2)\epsilon^{2a} + \dots$$

$$r_0^3 + 3r_0^2 r_1\epsilon^a + 3(r_0 r_2 + r_1^2)\epsilon^{2a} - 3r_0 - 3r_1\epsilon^a - 3r_2\epsilon^{2a} - 3r_3\epsilon^{3a} + \sqrt{\epsilon} = 0$$

Equating the zero order terms,

$$r_0^3 - 3r_0 = 0$$

$$r_0 = 0, -\sqrt{3}, \sqrt{3}$$

Equating the next order terms,

$$a = \frac{1}{2}$$

$$3r_0^2 r_1 - 3r_1 + 1 = 0$$

$$r_1 = \frac{1}{3}, \frac{1}{6}, \frac{1}{6}$$

Equating the next order terms,

$$3(r_0 r_2 + r_1^2) - 3r_2 = 0$$

$$r_2 = \frac{1}{9}, ...$$

So, the solutions in r are

$$r = \frac{1}{3}\epsilon^{\frac{1}{2}} + \frac{1}{9}\epsilon + O(\epsilon^3)$$

$$r = -\sqrt{3} + \frac{1}{6}\epsilon^{\frac{1}{2}} + +O(\epsilon)$$

$$r = \sqrt{3} + \frac{1}{6}\epsilon^{\frac{1}{2}} + +O(\epsilon^)$$

and thus,

$$x = \frac{1}{3} + \frac{1}{9}\epsilon^{\frac{1}{2}} + O(\epsilon)$$

$$x = -\sqrt{3}\epsilon^{-\frac{1}{2}} + \frac{1}{6} + O(\epsilon^{\frac{1}{2}})$$

$$x = \sqrt{3}\epsilon^{-\frac{1}{2}} + \frac{1}{6} + O(\epsilon^{\frac{1}{2}})$$

3.

$$\epsilon^2 x^3 - x + \epsilon = 0$$

Trying dominant balance,

$$\epsilon^2 \delta_0^3 x_0^3 - x_0 \delta_0 + \epsilon = 0$$

There are two options.

$$\delta_0 \sim \epsilon, \delta_0 \sim \epsilon^{-1}$$

The second option is the more 'dominant' of the two. So, we can rescale the original equation by it.

Let

$$x = r\epsilon^{-1}$$

$$r^3 - r + \epsilon^2 = 0$$

$$r = r_0 + r_1\epsilon^a + r_2\epsilon^{2a} + r_3\epsilon^{3a} + \dots$$

$$r^3 = r_0^3 + 3r_0^2 r_1 \epsilon^a + 3(r_0 r_2 + r_1^2)\epsilon^{2a} + \dots$$

$$r_0^3 + 3r_0^2 r_1\epsilon^a + 3(r_0 r_2 + r_1^2)\epsilon^{2a} - r_0 - r_1\epsilon^a - r_2\epsilon^{2a} - r_3\epsilon^{3a} + \epsilon^2 + \dots = 0$$

Equating the zero order terms,

$$r_0^3 - r_0 = 0$$

$$r_0 = 0, -1, 1$$

Equating the next order terms,

$$a = 2$$

$$3r_0^2 r_1 - r_1 + 1 = 0$$

$$r_1 = 1, -\frac{1}{2}, -\frac{1}{2}$$

Equating the next order terms,

$$3(r_0 r_2 + r_1^2) - r_2 = 0$$

$$r_2 = 1, ...$$

So, the solutions in r are

$$r = \epsilon^2 + \epsilon^4 + O(\epsilon^6)$$

$$r = -1 - \frac{1}{2}\epsilon^2 + +O(\epsilon^4)$$

$$r = 1 - \frac{1}{2}\epsilon^2 + +O(\epsilon^4)$$

and thus,

$$x = \epsilon + \epsilon^3 + O(\epsilon^5)$$

$$x = -\frac{1}{\epsilon} - \frac{1}{2}\epsilon + +O(\epsilon^3)$$

$$x = \frac{1}{\epsilon} - \frac{1}{2}\epsilon + +O(\epsilon^3)$$

4.

$$x^2 + \sqrt{1 + \epsilon x} = e^{\frac{1}{2+\epsilon}}$$

This is not singular. So,

$$x = x_0 + x_1\epsilon^r + x_2\epsilon^{2r} + x_3\epsilon^{3r} + \dots$$

$$x_0^2 + 2x_0x_1\epsilon^r + \sqrt{1 + \epsilon x_0 + x_1\epsilon^{r+1} + \dots} = e^{\frac{1}{2+\epsilon}} + \dots$$

$$x_0^2 + 2x_0x_1\epsilon^r + 1 + \frac{1}{2}\epsilon x_0 + \frac{1}{2}x_1\epsilon^{r+1} = e^{\frac{1}{2+\epsilon}} = \sqrt{e} - \sqrt{e}\frac{\epsilon}{4}$$

From here, it is clear that

$$r = 1$$

$$x_0 = \sqrt{\sqrt{e} - 1}, -\sqrt{\sqrt{e} - 1}$$

$$x_1 = -\frac{\sqrt{e}}{4\sqrt{\sqrt{e} - 1}} - \frac{1}{4}, \frac{\sqrt{e}}{4\sqrt{\sqrt{e} - 1}} - \frac{1}{4}$$

So, the solutions are

$$x = \sqrt{\sqrt{e}-1} - \frac{\sqrt{e}}{4\sqrt{\sqrt{e}-1}}\epsilon - \frac{1}{4}\epsilon + o(\epsilon)$$

and

$$x = -\sqrt{\sqrt{e}-1} + \frac{\sqrt{e}}{4\sqrt{\sqrt{e}-1}}\epsilon - \frac{1}{4}\epsilon + o(\epsilon)$$

5.

$$x^2 + \epsilon\sqrt{2+x} = cos(\epsilon)$$

$$x = x_0 + x_1\epsilon^r + x_2\epsilon^{2r} + x_3\epsilon^{3r} + \dots$$

$$x_0^2 + 2x_0x_1\epsilon^r + \epsilon\sqrt{2 + x_0 + x_1\epsilon^r + \dots} = 1 - \frac{\epsilon^2}{2} + \dots$$

$$x_0^2 + 2x_0x_1\epsilon^r + \epsilon\sqrt{2+x_0} + \frac{x_1\epsilon^{r+1}}{2\sqrt{2+x_0}} + \dots = 1 - \frac{\epsilon^2}{2} + \dots$$

From here, it is clear that

$$r = 1$$

$$x_0 = 1, -1$$

$$x_1 = -\frac{3}{2}, \frac{1}{2}$$

Thus, the solutions are

$$x = 1 - \frac{3}{2}\epsilon + o(\epsilon)$$

$$x = -1 + \frac{1}{2}\epsilon + o(\epsilon)$$

6.

$$\epsilon e^{x^2} = 1 + \frac{\epsilon}{1+x^2}$$

$$\epsilon e^{x^2} + x^2\epsilon e^{x^2} = 1 + x^2 + \epsilon$$

Using the dominant terms,

$$\epsilon e^{x_0^2\delta_0^2} + x_0^2\delta_0^2\epsilon e^{x_0^2\delta_0^2} = 1 + x_0^2\delta_0^2 + \epsilon$$

The dominant balance seems to be

$$\delta_0^2\epsilon e^{\delta_0^2} \sim \delta_0^2$$

$$e^{\delta_0^2} \sim \frac{1}{\epsilon}$$

$$\delta_0 \sim \sqrt{\log \frac{1}{\epsilon}}$$

Let

$$x = r\sqrt{\log \frac{1}{\epsilon}}$$

Plugging this into the original equation, we get

$$\epsilon \left(\frac{1}{\epsilon}\right)^{r^2} = 1 + \frac{\epsilon}{1 - r^2 \log \epsilon}$$

Since the last term on the right goes to zero, we can ignore it to get a leading order approximation.

$$\epsilon \left(\frac{1}{\epsilon}\right)^{r^2} \sim 1$$

$$r^2 \sim 1$$

$$r \sim 1, -1$$

Let

$$r^2 \sim 1 + k$$

Now, for the tricky part

$$\left(\frac{1}{\epsilon}\right)^{r^2} = \frac{1}{\epsilon} + \frac{1}{1 - r^2 \log \epsilon}$$

$$\left(\frac{1}{\epsilon}\right)\left(\frac{1}{\epsilon}\right)^{k} = \frac{1}{\epsilon}\left(1 + \frac{\epsilon}{1 - r^2 \log \epsilon}\right)$$

$$\left(\frac{1}{\epsilon}\right)^{k} = 1 + \frac{\epsilon}{1 - r^2 \log \epsilon}$$

It seems appropriate to take a log on both sides because the second term on the right is pretty small.

$$-k = \log\left(1 + \frac{\epsilon}{1 - r^2 \log \epsilon}\right) \sim \frac{\epsilon}{1 - r^2 \log \epsilon} \sim \frac{\epsilon}{1 - \log \epsilon}$$

$$k = \frac{\epsilon}{\log \epsilon - 1} + o\left(\frac{\epsilon}{1 - \log \epsilon}\right)$$

So, bn

$$r^2 \sim 1 + \frac{\epsilon}{\log \epsilon - 1} + o\left(\frac{\epsilon}{1 - \log \epsilon}\right)$$

$$r \sim 1 + \frac{\epsilon}{2\log\epsilon - 2}, -1 + \frac{\epsilon}{2 - 2\log\epsilon}$$

So,

$$x \sim \sqrt{\log\frac{1}{\epsilon}} + \sqrt{\log\frac{1}{\epsilon}}\frac{\epsilon}{2\log\epsilon - 2}$$

and

$$x \sim -\sqrt{\log\frac{1}{\epsilon}} + \sqrt{\log\frac{1}{\epsilon}}\frac{\epsilon}{2 - 2\log\epsilon}$$

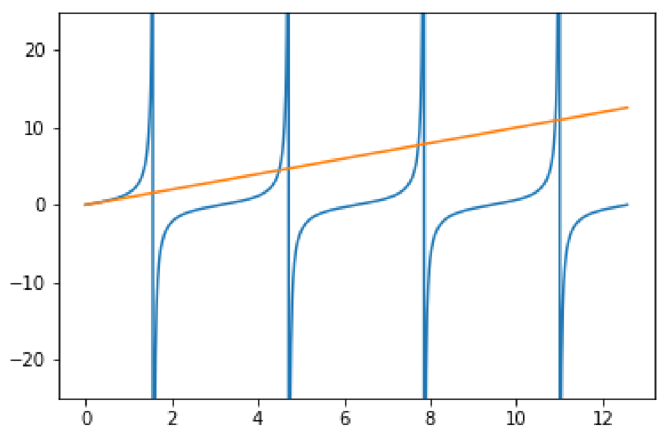

Figure 1: tan(x) vs x

7. (a) The plots cross at infinitely many points. So, there are infinitely many solutions.

   (b)

$$\tan\lambda = \lambda$$

With usual techniques, we would get λ near zero. But, we want solutions when λ is very large. This is only possible when $\lambda \sim \frac{(2n+1)\pi}{2}$ where n is a very large integer.

We are going to assume

$$\lambda \sim \frac{\lambda_0}{\epsilon^\alpha} + \lambda_1\epsilon^\beta$$

This is slightly different from the guess in the book.

Now, in this ansatz, as ϵ goes to zero, the second term disappears and the first term dominates.

$$\frac{\lambda_0}{\epsilon^\alpha} \approx \frac{(2n+1)\pi}{2}$$

$$\tan\left(\frac{\lambda_0}{\epsilon^\alpha} + \lambda_1\epsilon^\beta\right) = \frac{\lambda_0}{\epsilon^\alpha} + \lambda_1\epsilon^\beta$$

$$\tan\left(\frac{(2n+1)\pi}{2} + \lambda_1\epsilon^\beta\right) = \frac{\lambda_0}{\epsilon^\alpha} + \lambda_1\epsilon^\beta$$

$$\cot\left(-\lambda_1\epsilon^\beta\right) = \frac{\lambda_0}{\epsilon^\alpha} + \lambda_1\epsilon^\beta$$

$$-\frac{1}{\lambda_1\epsilon^\beta} + \frac{\lambda_1\epsilon^\beta}{3} + \ldots = \frac{\lambda_0}{\epsilon^\alpha} + \lambda_1\epsilon^\beta$$

Doing dominant balance,

$$-\frac{1}{\lambda_1\epsilon^\beta} \sim \frac{\lambda_0}{\epsilon^\alpha}$$

$$\alpha = \beta$$

$$-\frac{1}{\lambda_1} = \lambda_0$$

Let α be 1.

$$\frac{\lambda_0}{\epsilon} \approx \frac{(2n+1)\pi}{2} \sim \pi n$$

So,

$$n \sim \frac{1}{\epsilon}$$

and

$$\lambda_0 = \pi$$

Thus,

$$\lambda = \frac{\pi}{\epsilon} - \frac{1}{\pi}\epsilon + o(\epsilon)$$