Assignment 1.

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1. (a)

$$\Omega = \frac{N!}{N_h!(N-N_h)!}$$

$$\log \Omega = \log(N!) - \log(N_h!) - \log((N-N_h)!)$$

Considering N_h as a pertubation about $\frac{N}{2}$,

$$\begin{aligned}
\log \Omega =& \log(N!) - \log((\frac{N}{2}+\epsilon)!) - \log((\frac{N}{2}+\epsilon)!) \\
=& N\log N - N + \frac{1}{2}\log(2\pi N) \\
& - (\frac{N}{2}+\epsilon)\log(\frac{N}{2}+\epsilon) + (\frac{N}{2}+\epsilon) - \frac{1}{2}\log(2\pi(\frac{N}{2}+\epsilon)) \\
& - (\frac{N}{2}-\epsilon)\log(\frac{N}{2}-\epsilon) + (\frac{N}{2}-\epsilon) - \frac{1}{2}\log(2\pi(\frac{N}{2}-\epsilon)) \\
=& N\log N + \frac{1}{2}\log(2\pi N) \\
& - (\frac{N}{2}+\epsilon)\log((\frac{N}{2})(1+\frac{2\epsilon}{N})) - \frac{1}{2}\log((2\pi\frac{N}{2})(1+\frac{2\epsilon}{N})) \\
& - (\frac{N}{2}-\epsilon)\log((\frac{N}{2})(1-\frac{2\epsilon}{N})) - \frac{1}{2}\log((2\pi\frac{N}{2})(1-\frac{2\epsilon}{N})) \\
=& N\log N + \frac{1}{2}\log(2\pi N) - N\log(N/2) - \log(\pi N) \\
& - (\frac{N}{2}+\epsilon)\log(1+\frac{2\epsilon}{N}) - \frac{1}{2}\log(1+\frac{2\epsilon}{N}) \\
& - (\frac{N}{2}-\epsilon)\log(1-\frac{2\epsilon}{N}) - \frac{1}{2}\log(1-\frac{2\epsilon}{N}) \\
\approx& N\log N + \frac{1}{2}\log(2\pi N) - N\log(N/2) - \log(\pi N) \\
& - (\frac{N}{2}+\epsilon)(\frac{2\epsilon}{N} - \frac{2\epsilon^2}{N^2}) - \frac{1}{2}(\frac{2\epsilon}{N} - \frac{2\epsilon^2}{N^2}) \\
& - (\frac{N}{2}-\epsilon)(-\frac{2\epsilon}{N} - \frac{2\epsilon^2}{N^2}) - \frac{1}{2}(-\frac{2\epsilon}{N} - \frac{2\epsilon^2}{N^2}) \\
\approx& N\log N + \frac{1}{2}\log(2\pi N) - N\log(N/2) - \log(\pi N) - \frac{2\epsilon^2}{N} + \frac{2\epsilon^2}{N^2}
\end{aligned}$$

$$\Omega = \frac{2^{N+\frac{1}{2}}}{\sqrt{\pi N}} e^{-\frac{2\epsilon^2}{N}+\frac{2\epsilon^2}{N^2}}$$

(b) Maximum multiplicity is $\frac{2^{N+\frac{1}{2}}}{\sqrt{\pi N}}$. It is less than 2^N as $N \geq 0$.

(c) Width is given by $2\epsilon^*$ where

$$\frac{2\epsilon^{2*}}{N} - \frac{2\epsilon^{2*}}{N^2} = 1$$

$$\epsilon^* = \frac{N}{\sqrt{2N+2}}$$

Thus, the width is $\frac{2N}{\sqrt{2N+2}}$

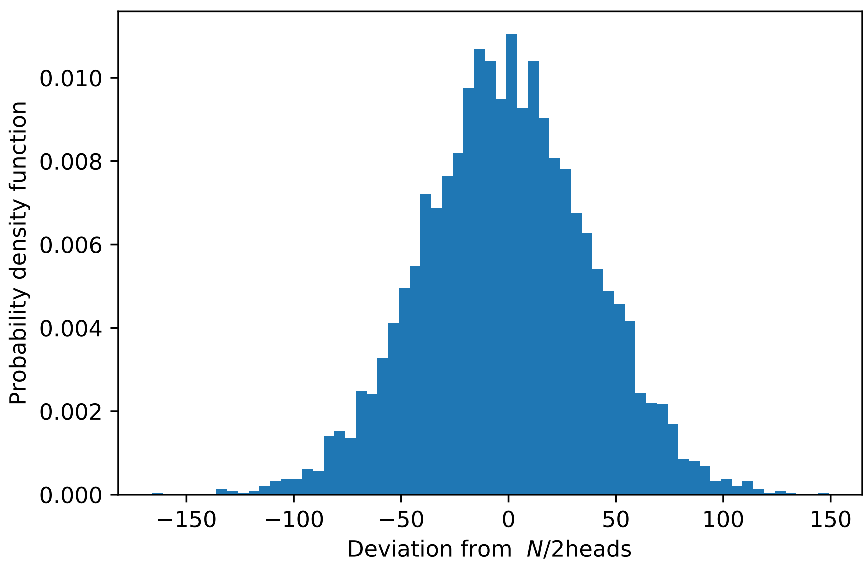

Figure 1: The histogram

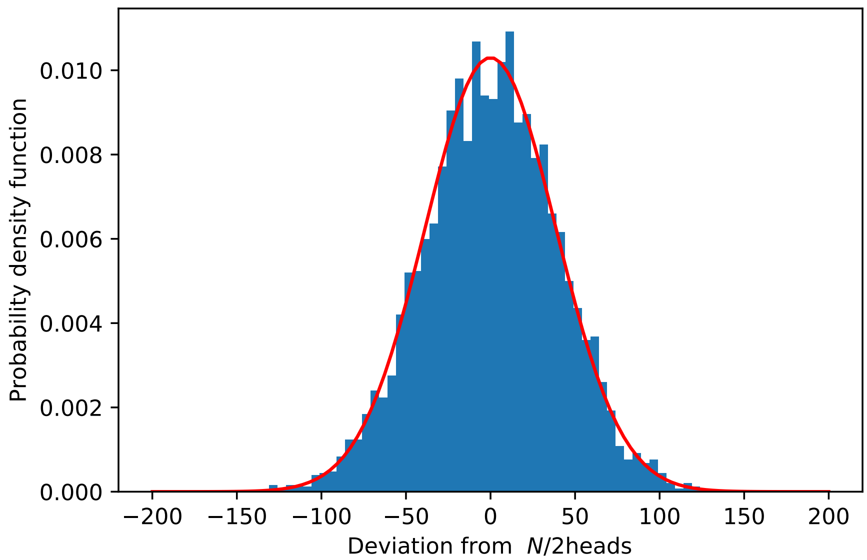

Figure 2: The histogram with the probability density function

2. (a)
   (b) Done in part(a).
   (c) Done in part(a).
   (d) It would be a probability distribution which would be accurate near $\frac{N}{2}$.
   (e) Looking at the Normalization description from the Matlab help page, I think dividing the histogram by the bin width would give us the probability density function.
   (f) This code allows us to obtain the width of distribution analytically.

```
# September 2017 -- Coin toss
import numpy as np
import matplotlib.pyplot as plt
N_trials = 5000;
Sample_sizes = [2000,3000,6000];
# Note that you can run just one or several Sample_sizes.
# For example, replace the line above with Sample_sizes =
    [1000, 2000]
# to just have the two Sample_sizes: 1000 and 2000.
width_experimental =[]
width_analytical =[]
for k in Sample_sizes:
        N_samples = k
        Total_heads = []
        for i in range(N_trials):
```

```
                        # Generate [1 x N_samples] vector of uniformly distributed random
                        # integers from 0 or 1.
                        rs = np.random.randint(2,size = N_samples);
                        # Calculate the total number of heads.
                        Total_heads.append(sum(rs));

        BW = 5; # Bin width
        data = np.array(Total_heads)-N_samples/2
        histdata = plt.hist(data,normed=True, bins=np.arange(min(data), max(data) + int(BW), int(BW)))
        x = np.linspace(-200, 200,100) # returns a row vector of 100 evenly spaced points between -200 and 200
        POmega = (1/np.sqrt(np.pi*N_samples/2))*np.exp(-2*(x**2)/N_samples+ -2*(x**2)/N_samples**2);
        plt.plot (x, POmega, 'r-')# 'LineWidth', 2);

        plt.xlabel('Deviation from '+' $N/2$'+'heads');
        plt.ylabel('Probability density function') # Label probability density function


        plt.xlim([-200, 200])



        '''Finding the width'''
        Max_value = max(histdata[0])
        delta =Max_value/20

        elements = np.where(  np.abs(histdata[0]-Max_value/np.exp(1)) <delta)[0]

        width_experimental.append(max(elements[1:]-elements[:-1])*BW) # This is the list of the widths found experimentally
        width_analytical.append(2*N_samples/np.sqrt(2*N_samples+2))

plt.clf()
plt.plot(width_analytical,width_experimental,'r*')
plt.xlabel('Analytical width');
plt.ylabel('Experimental Width')
for xy in zip(width_analytical, width_experimental):
```

```
                                                # <--
                plt.annotate('(%s, %s)' % xy, xy=xy,
                   textcoords='data') # <--


plt.savefig('widths.pdf')
plt.show()
```

The end result is the following plot

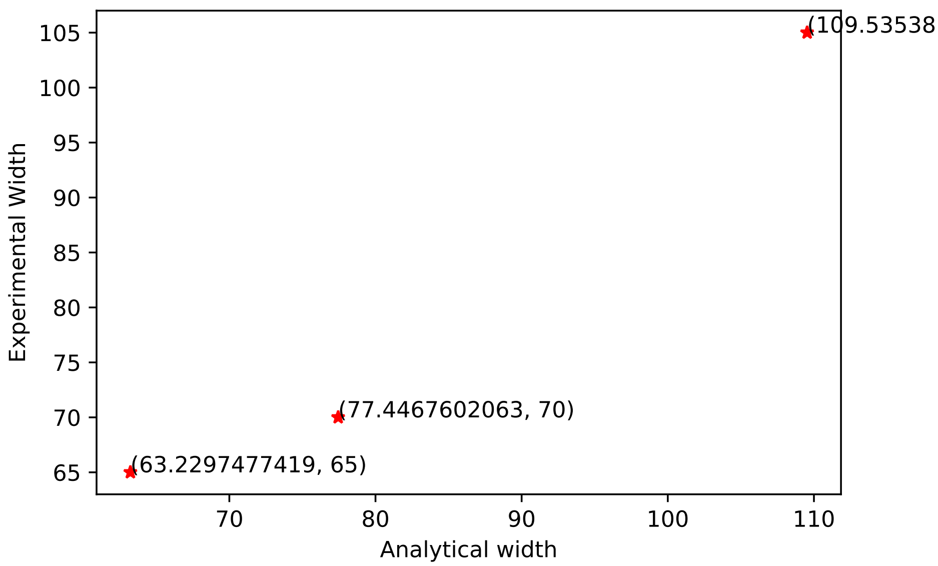

Figure 3: The widths for 3 distributions