Assignment 1.

Jithin D. George

Due Oct 8

1.

$$x^2y'' + 7xy' + 13y = 0$$

This is an equidimensional equation. Plugging in x^m

$$m(m-1) + 7m + 13 = 0$$

$$m^2 + 6m + 13 = 0$$

$$m = -3 + 2i, -3 - 2i$$

So, the solution is

$$y = c_1x^{-3}cos(2x) + c_2x^{-3}sin(2x)$$

2.

$$y'' - 3y' + 2y = xe^x + xe^{-x}$$

The homogeneous solution is $c_1e^x + c_2e^{2x}$ The particular solution should have the format

$$ae^x + be^{-x} + cxe^x + dxe^{-x}$$

But e^x is part of the homogeneous solution. So, we replace it by x^2e^x since xe^x is already part of our guess. Our new ansatz is

$$ax^2e^x + be^{-x} + cxe^x + dxe^{-x}$$

Plugging it into the solution, we have

$$\begin{aligned} ax^2e^x + 4axe^x + 2ae^x + be^{-x} + 2ce^x - 2de^{-x} + cxe^x + dxe^{-x} \\ -3cxe^x - 3ce^x - 3ax^2e^x - 6axe^x + 3be^{-x} + 3dxe^{-x} - 3de^{-x} \\ +2ax^2e^x + 2be^{-x} + 2cxe^x + 2dxe^{-x} \\ = xe^x + xe^{-x} \end{aligned}$$

$$2a + 2c - 3c = 0 \to c = 2a$$

$$b - 2d - 3d + 3b + 2b = 0 \to 6b = 5d$$

$$4a + c - 3c - 6a + 2c = 1 \to a = -\frac{1}{2}$$

$$d + 3d + 2d = 1 \to d = \frac{1}{6}$$

Thus, the general solution is

$$y = c_1e^x + c_2e^{2x} - \frac{1}{2}x^2e^x + \frac{5}{6}e^{-x} - xe^x + \frac{1}{6}xe^{-x}$$

3.

$$y'' + y = (e^x + 1)\sin x$$

The homogeneous solution is

$$y = c_1 \sin x + c_2 \cos x$$

Since sin and cos appear in the homogeneous solution, our ansatz for the particular solution is

$$y_p = ax\sin x + bx\cos x + ce^x \sin x + de^x \cos x$$

$$y_p' = a\sin x + ax\cos x + b\cos x - bx\sin x + ce^x \sin x + ce^x \cos x + de^x \cos x - de^x \sin x$$

$$y_p'' = 2a\cos x - ax\sin x - 2b\sin x - bx\cos x + 2ce^x \cos x - 2de^x \sin x$$

Plugging it into the ode, we get

$$a = 0, b = -\frac{1}{2}, c = \frac{1}{5}, d = -\frac{2}{5}$$

So, the general solution is

$$y = c_1 \sin x + c_2 \cos x - \frac{1}{2}x\cos x + \frac{1}{5}e^x \sin x - \frac{2}{5}e^x \cos x$$

4.

$$y''' - 2y'' + y' = 1 + xe^x$$

This is an initial value problem with smooth coefficients and the first coefficient is 1. So, the solution exists everywhere and is unique. Solving the homogeneous equation with the guess e^{mx},

$$m^3 - 2m^2 + m = 0$$

$$m = 0, 1, 1$$

Thus, the homogeneous solution is

$$y = c_1 + c_2 e^x + c_3 xe^x$$

The particular solution will have the form

$$y_p = ax + bx^2 e^x + cx^3 e^x$$

$$y_p' = a + (3c + b)x^2 e^x + 2bxe^x + cx^3 e^x$$

$$y_p'' = (6c + b)x^2 e^x + (6c + 4b)xe^x + 2be^x + cx^3 e^x$$

$$y_p''' = (9c + b)x^2 e^x + (18c + 6b)xe^x + (6c + 6b)be^x + cx^3 e^x$$

Plugging it into the ode, we have

$$a = 1, c = \frac{1}{6}, b = -\frac{1}{2}$$

The general solution is

$$y = c_1 + c_2 e^x + c_3 x e^x + x - \frac{1}{2}x^2 e^x + \frac{1}{6}x^3 e^x$$

Using the initial conditions,

$$c_1 + c_2 = 0$$
$$c_2 + c_3 + 1 = 0$$
$$c_2 + 2c_3 - 1 = 1$$

$$y = 4 - 4e^x + 3xe^x + x - \frac{1}{2}x^2 e^x + \frac{1}{6}x^3 e^x$$

5.

$$xy'' - (x+1)y' + y = x^2 e^{2x}, y(1) = 0, y'(1) = e$$

This is an initial value problem starting at 1. The solution exists near 1 but there is a singularity at x reaches 0.

$$y_1 = x + 1$$
$$y_2 = (x+1)u = xu + u$$
$$y_2' = xu' + u' + u$$
$$y_2'' = u' + xu'' + u'' + u' = 2u' + (x+1)u''$$

Plugging this into the ode, we have

$$2xu' + x(x+1)u'' - (x+1)^2 u' - (x+1)u' + (x+1)u = x^2 e^{2x}$$

$$2xu' + x(x+1)u'' - (x+1)^2 u' = x^2 e^{2x}$$

$$u'' - \left(\frac{x+1}{x} - \frac{2}{x+1}\right)u' = \frac{xe^{2x}}{x+1}$$

$$e^{-\left(\frac{x+1}{x} - \frac{2}{x+1}\right)} u'' - e^{-\left(\frac{x+1}{x} - \frac{2}{x+1}\right)} \left(\frac{x+1}{x} - \frac{2}{x+1}\right)u' = e^{-\left(\frac{x+1}{x} - \frac{2}{x+1}\right)} \frac{xe^{2x}}{x+1}$$

$$\left(e^{\int -\left(\frac{x+1}{x} - \frac{2}{x+1}\right)dx} u'\right)' = e^{\int -\left(\frac{x+1}{x} - \frac{2}{x+1}\right)dx} \frac{xe^{2x}}{x+1}$$

$$e^{-x}\frac{(x+1)^2}{x}u' = \int e^{-x}\frac{(x+1)^2}{x}\frac{xe^{2x}}{x+1}dx = \int (x+1)e^x dx$$

$$e^{-x}\frac{(x+1)^2}{x}u' = xe^x + c_1$$

$$u' = \frac{x^2 e^{2x}}{(x+1)^2} + c_1 \frac{xe^x}{(x+1)^2}$$

$$u' = \frac{e^{2x}(x-1)}{2(x+1)} + c_1 \frac{e^x}{x+1} + c_2$$

$$y(x) = \frac{1}{2}e^{2x}(x-1) + c_1 e^x + c_2(x+1)$$

Plugging in initial conditions,

$$c_1 e + 2c_2 = 0$$

$$\frac{e^2}{2} + c_2 + c_1 e = e$$

$$c_1 = 2 - e, c_2 = \frac{e^2}{2} - e$$

$$y(x) = \frac{1}{2}e^{2x}(x-1) + (2-e)e^x + (\frac{e^2}{2} - e)(x+1)$$

6.

$$(x-1)y'' - xy' + y = (x-1)^2$$

$$y = xu$$

$$y' = xu' + u$$

$$y'' = xu'' + 2u'$$

Putting this into the homogeneous ode,

$$x(x-1)u'' + 2(x-1)u' - x^2u' - xu + xu = 0$$

$$x(x-1)u'' = (x^2 - 2x + 2)u'$$

$$u' = c_1 \frac{e^x(1-x)}{x^2}$$

$$u = -c_1 \frac{e^x}{x} + c_2$$

$$y = -c_1 e^x + c_2 x$$

From the initial conditions, we get $c_1 = 0, c_2 = 0$ This is a trivial solution for the homogeneous problem. So, the inhomogeneous problem has a unique solution.

Plugging the derivatives into the inhomogeneous ode, we get

$$x(x-1)u'' - (x^2 - 2x + 2)u' = (x-1)^2$$

$$u'' - \frac{(x^2 - 2x + 2)}{x(x-1)}u' = 1 - \frac{1}{x}$$

$$(\frac{e^{-x}x^2u'}{1-x})' = -e^{-x}x$$

$$
\begin{aligned}
\frac{e^{-x}x^2u'}{1-x} &= e^{-x}x + e^{-x} + c_1 \\
u' &= \frac{1}{x^2} - 1 + c_1\frac{e^{-x}(1-x)}{x} \\
u &= -\frac{1}{x} - x + c_1\frac{e^x}{x} + c_2 \\
y &= -1 - x^2 + c_1 e^x + c_2
\end{aligned}
$$

Plugging in the initial conditions, we get

$$y = -1 - x^2 + \frac{1}{4\sqrt{e}-1}e^x + \frac{4\sqrt{e}-5}{4\sqrt{e}-1}$$

7.

$$4y'' + y = x, y(\pi) = y(-\pi) = 0$$

The homogeneous solution is

$$y = c_1\cos(\frac{x}{2})$$

in addition to the trivial solution. This means that the inhomogeneous equation has either no solution or infinitely many solutions.

Checking the solvability condition,

$$\int_{-\pi}^{\pi} xc_1\cos(\frac{x}{2}) = 0$$

Thus, the BVP is solvable and has infinitely many solutions. x is a particular solution. So, the general solution is

$$y = c_1\cos(\frac{x}{2}) + c_2\sin(\frac{x}{2}) + x$$

From the BCs, we get $c_2 = -\pi$.

$$y = c_1\cos(\frac{x}{2}) - \pi\sin(\frac{x}{2}) + x$$

8. This is an initial value problem where the coefficients of the derivatives are smooth. So, the solution exists and is unique.

The general solution is

$$y = -\cos x + c_1 x + c_2$$

From the initial conditions, we can find that

$$c_2 = 1, c_1 = 0$$

Thus, the solution is

$$y = -\cos x + 1$$

9.
$$9y'' + y = xe^{-x^2}, y(-3\pi) = y(3\pi) = 0$$

The homogeneous solution is
$$y = c_1 \sin(\frac{x}{3})$$
in addition to the trivial solution. This means that the inhomogeneous equation has either no solution or infinitely many solutions. Checking the solvability condition,
$$\int_{-3\pi}^{3\pi} xe^{-x^2} c_1 \sin(\frac{x}{3}) dx = 18c_1 \int_0^{\pi} ue^{-9u^2} \sin(u) du > 0$$

Hence, there is no solution for the inhomogeneous ode.