Assignment 4.

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1.

$$I_p(x) = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n+p}}{n!(n+p)!}$$

$$\begin{aligned}
\frac{d}{dx}[x^p I_p(\alpha x)] &= \frac{d}{dx}\sum_{n=0}^{\infty} \frac{x^p\left(\frac{\alpha x}{2}\right)^{2n+p}}{n!(n+p)!} \\
&= \sum_{n=0}^{\infty} \frac{(2n+2p)x^{2n+2p-1}\alpha^{2n+p}}{2^{2n+p}n!(n+p)!} \\
&= \sum_{n=0}^{\infty} \frac{x^{2n+2p-1}\alpha^{2n+p}}{2^{2n+p-1}n!(n+p-1)!} \\
&= \sum_{n=0}^{\infty} \frac{x^2\left(\frac{x}{2}\right)^{2n+p-1}\alpha^{2n+p}}{n!(n+p)!} \\
&= \sum_{n=0}^{\infty} \frac{\alpha x^2\left(\frac{\alpha x}{2}\right)^{2n+p-1}}{n!(n+p)!} \\
&= \alpha x^p I_{p-1}(\alpha x)
\end{aligned}$$

2.

$$xy'' + (1+4x^2)y' + x(5+4x^2)y = 0$$
$$x^2y'' + x(1+4x^2)y' + (5x^2+4x^4)y = 0$$

This is of the form

$$x^2y'' + x(a+2bx^2)y' + (c + dx^{2s} - b(1-a-r)x^r + b^2x^{2r})y = 0$$

where a=1,b=2,r=2,c=0,s=1 and d=1 So,

$$y(x) = e^{-x^2}Z(\eta)$$

$$\eta = x$$

Z is a Bessel equation of order p where

$$p = \frac{1}{2}$$

$$
\begin{aligned}
y(x) &= e^{-x^2} J_{\frac{1}{2}}(x) + e^{-x^2} J_{-\frac{1}{2}}(x) \\
&= c_1 \frac{e^{-x^2}}{\sqrt{x}} \sin(x) + c_2 \frac{e^{-x^2}}{\sqrt{x}} \cos(x)
\end{aligned}
$$

3.

$$x^4 y'' + y = 0$$

This is of the form

$$x^2 y'' + dx^{2s} y = 0$$

where d = 1, s =-1 and everything else is 0 So,

$$y(x) = \sqrt{x} Z(\eta)$$

$$\eta = -\frac{1}{x}$$

Z is a Bessel equation of order p where

$$p = \frac{1}{2}$$

So,

$$
\begin{aligned}
y(x) &= \sqrt{x} J_{\frac{1}{2}}(\eta) + \sqrt{x} J_{-\frac{1}{2}}(\eta) \\
&= a_1 \sqrt{x} \frac{1}{\sqrt{x}} sin(\eta) + a_2 \sqrt{x} \frac{1}{\sqrt{x}} cos(\eta) \\
&= c_1 \sin(\frac{1}{x}) + c_2 \cos(\frac{1}{x})
\end{aligned}
$$

4. Let's look at the homogeneous equation.

$$xy'' - y' + 4x^3 y = 0$$

$$x^2 y'' - xy' + 4x^4 y = 0$$

This is of the form

$$x^2 y'' + x(a + 2bx^2) y' + (c + dx^{2s} - b(1 - a - r)x^r + b^2 x^{2r}) y = 0$$

where a=-1,b=0,s=2 and d=2 So,

$$y(x) = xZ(\eta)$$

$$\eta = x^2$$

Z is a Bessel equation of order p where

$$p = \frac{1}{2}$$

So,

$$\begin{aligned} y(x) &= xJ_{\frac{1}{2}}(\eta) + xJ_{-\frac{1}{2}}(\eta) \\ &= c_1 \sin(x^2) + c_2 \cos(x^2) \end{aligned}$$

With the initial conditions, we find that the solution is

$$y = c_1 sin(x^2)$$

Clearly, this isn't trivial. So, we check the solvability condition.

$$\int_0^{\pi^2} x^4 sin(x^2) dx \neq 0$$

Thus, there is no solution.