Assignment 2.

Jithin D. George

Due Oct 23

1. (a) If $N' = 0$, then

$$N = \frac{p}{Gn + f}$$

$$\begin{aligned} n' &= \frac{Gnp}{Gn + f} - kn \\ &= n\left(\frac{Gp}{Gn + f} - k\right) \end{aligned}$$

(b) It's easy to see that $n^* = 0$ is an equilibirum. Consider a perturbation ϵ

$$\begin{aligned} \epsilon' &= \epsilon\left(\frac{Gp}{G\epsilon + f} - k\right) \\ &\sim \epsilon\left(\frac{Gp}{f} - k\right) \end{aligned}$$

So, for the perturbation to decay to zero, we need

$$p < \frac{kf}{G} = p_c$$

Thus, if $p > p_c$, the equilibrium at zero is unstable.

(c) At p_c, the equilibrium changes from a stable one to an unstable one. The other equilibrium is unaffected at p_c. So, the closest known bifurcation to it is transcritical.

(d) Because N relaxes much more rapidly than n, the two processes take place at very different timescales. Effectively, what happens is something of the form

$$\epsilon N' = f(n, N)$$

$$N' = \frac{1}{\epsilon} f(n, N) \sim -\frac{k}{\epsilon}$$

This approximation is made because with such a huge derivative, everything says the same while N attains equilibirum quickly.

Rewriting our main equations,

$$n' = GnN - kn$$

$$N' = -f(\frac{GnN}{f} + N - \frac{p}{f})$$

If f is very large say $\frac{1}{\epsilon}$,

$$N' = -\frac{1}{\epsilon}N$$

This attains equilibirum very quickly.
Thus, we need $f >> G, p, k$

2. (a)

$$P = DE$$

$$\lambda + 1 - D - \lambda EP = 0$$

$$\lambda + 1 - D - \lambda E^2 D = 0$$

$$D = \frac{\lambda + 1}{\lambda E^2 + 1}$$

$$P = \frac{\lambda + 1}{\lambda E^2 + 1}E$$

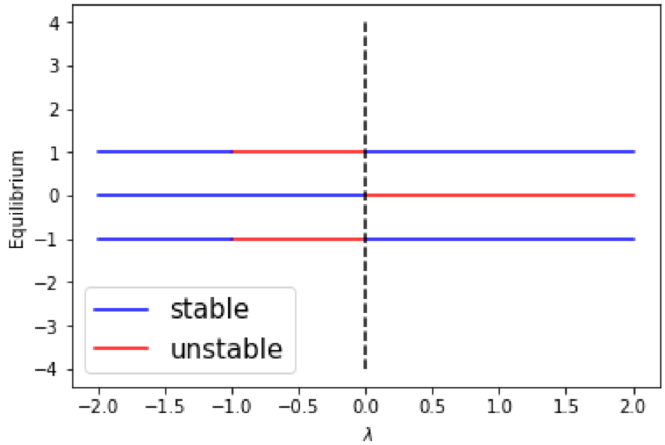
$$\begin{aligned} E' &= \kappa E\left(\frac{\lambda + 1}{\lambda E^2 + 1} - 1\right) \\ &= \kappa E\left(\frac{\lambda(1 - E^2)}{\lambda E^2 + 1}\right) \end{aligned}$$

(b) The fixed points are 0,1 and -1 if $\lambda \neq 0$. Else, any value of E is a fixed point.

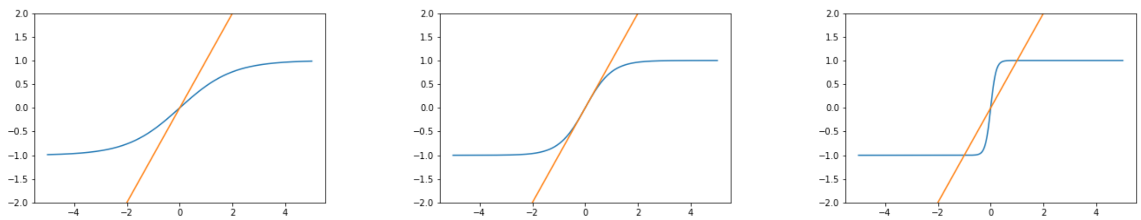
(c)

$$E'' = -\kappa\lambda\left(\frac{\lambda E^4 + (\lambda + 3)E^2 - 1}{(\lambda E^2 + 1)^2}\right)$$

If $\lambda < 0$, 0 is stable and viceversa. If $-1 < \lambda < 0$, -1,1 are unstable and viceversa. When $\lambda = 0$, all equilibria are neither unstable nor stable. A small perturbation will not grow or decay but it will stay within a distance. Orbital stability ? Hence, equilibria at $\lambda = 0$ are characterized only by a dashed line.

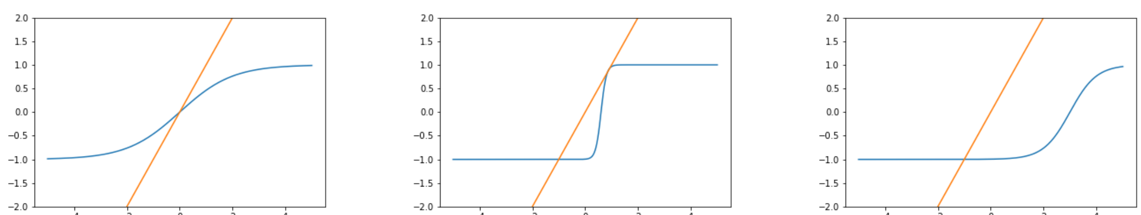


3. (a) We get a variety of solution depending on h and $\frac{Jn}{T}$.



(a) $\frac{Jn}{T} = 0.5$ (b) $\frac{Jn}{T} = 1$ (c) $\frac{Jn}{T} = 5$

Figure 1: When h = 0



(a) h=3, $\frac{Jn}{T} = 5$ (b) h = -3, $\frac{Jn}{T} = 5$ (c) h = -3, $\frac{Jn}{T} = 1$

Figure 2: When $h \neq 0$

(b) When $h = 0$,

$$m = \tanh(\frac{Jnm}{T})$$

Looking at Figure 1, we see that a single equilibrium translates to 3 fixed points after a particular value of $\frac{Jnm}{T}$. This value is when the slope of the line $y = m$ is equal to $y = tanh(\frac{Jnm}{T})$ at $m = 0$.

That happens when

$$\frac{Jn}{T} = 1$$
$$T_c = Jn$$

4. (a)

$$x' + y' + z' = 0$$
$$(x + y + z)' = 0$$
$$x + y + z = N$$

(b)

$$x' = -kx\frac{z'}{l}$$
$$\log(x) = -k\frac{z}{l} + C$$
$$x = x_0 e^{-\frac{kz}{l}}$$

(c)

$$z' = ly$$
$$z' = l(N - z - x_0 e^{-\frac{kz}{l}})$$

(d) Let

$$t = \frac{1}{kx_0}\tau, N = ax_0, z = u\frac{l}{k}$$
$$\frac{lk}{kx_0}\frac{du}{d\tau} = l(ax_0 - \frac{l}{k}u - e^{-u})$$
$$\frac{du}{d\tau} = a - bu - e^{-u}$$

(e)

$$a = \frac{N}{x_0} \geq 1$$
$$b = \frac{l}{kx_0} > 0$$

(f) Since $a \geq 1$ and $b > 0$, there are either two fixed points or 1.

This is determined by the intersection of $a - bu$ with e^{-u}

If $a > 1$, there are definitely two fixed points. The smaller one is unstable and the larger one is stable.

If $a = 1$, there might be only one fixed point. If $a - bu$ is the tangent to e^{-u} at u =0, i.e b=1 , there is only one fixed point. This is a saddle point (unstable). Otherwise, if $b < 1$, there are two fixed point and the one at $u = 0$ is unstable while the other is stable.

Since u cannot be negative, the number of physically relevant fixed points is usually different from this

(g) Since $u' = \frac{k}{l}z'$, they have the same maximum peaks. Also, at the maximum value of z,

$$z'' = ly' = 0$$

Thus, y has a fixed point ends up being a maximum.

(h)

$$u(0) = 0$$

$$u'' = -b + e^{-u}$$

So,

$$u''(0) = 1 - b > 0$$

Thus, u keeps on increasing. As u increases, e^{-u} decreases until u'' has a fixed point. That's the peak value of u'. Because after that, u'' is negative and u' starts to decrease.

(i) If $b > 1$, u'' will always be negative. Since $u'(0)$is negative, u keeps on decreasing. Hence, the peak occurs at $t = 0$.

(j) $b = 1$ implies that

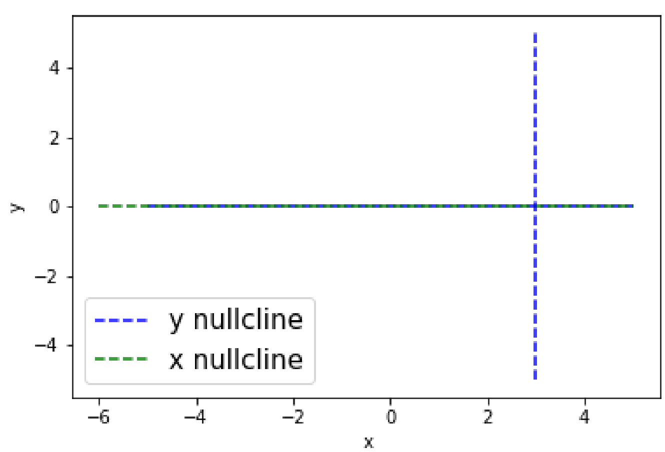
$$kx_0 = l$$

This means that a constant number of people are sick.

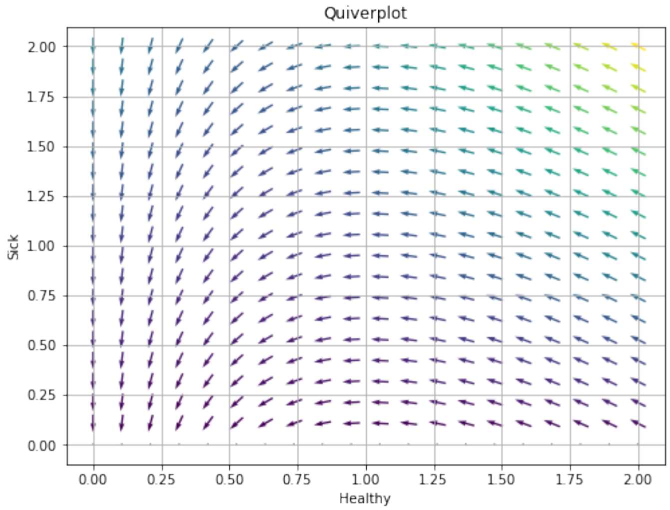
(k) AIDS is not an epidemic so the rate at which the people fall sick would be different. Also, the rate of death might not be constant.

5. (a) For $x' = 0$ and $y' = 0$, we need $y = 0$ Thus, the fixed points are the x axis

(b) These are the nullclines.

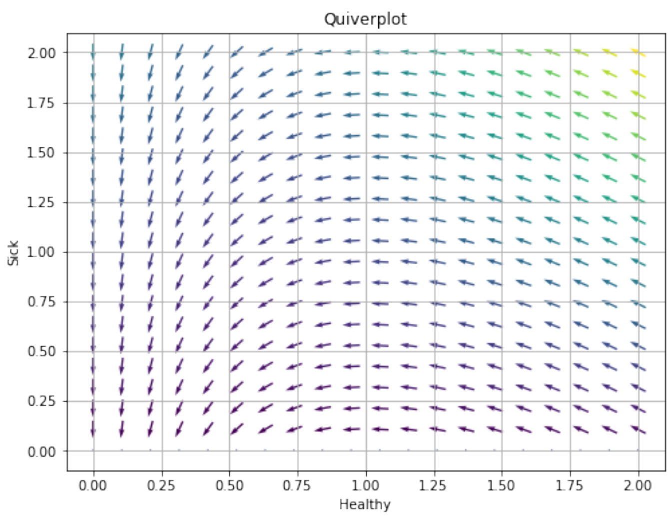


This is the vector field.



(c)

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{kxy - ly}{-kxy}$$
$$\frac{dy}{dx} = -1 + \frac{l}{kx}$$
$$y + x - \frac{l}{k}\log x = c$$



(d) It looks as though y goes to 0 and x goes to some point in between (0,1).

(e) Epidemic occurs for $x > \frac{l}{k}$

6. (a)

$$\frac{du}{d\theta} = v$$
$$\frac{dv}{d\theta} = \alpha + \epsilon u^2 - u$$

(b)

$$u = \frac{1 \pm \sqrt{1 - 4\alpha\epsilon}}{2\epsilon}, v = 0$$

(c) The eigenvalues of the Jacobian are given by

$$\lambda^2 = \pm\sqrt{1 - 4\alpha\epsilon}$$

Thus, the equilibrium $\frac{1-\sqrt{1-4\alpha\epsilon}}{2\epsilon}$ is a center. It's a nonlinear center too because this is a hamiltonian system with conserved quantities like energy and angular momentum.

(d) The following equilibrium is a center.

$$u = \frac{1 - \sqrt{1 - 4\alpha\epsilon}}{2\epsilon}$$

$$\frac{1}{r} = \frac{1 - \sqrt{1 - 4\alpha\epsilon}}{2\epsilon}$$

$$r = \frac{2\epsilon}{1 - \sqrt{1 - 4\alpha\epsilon}}$$

That's a circular orbit