Assignment 2.

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1.
$$y'' + \epsilon y' - y = 1, y(0) = 0, y(1) = 1$$

Let
$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + O(\epsilon^3)$$
$$y_0'' + \epsilon y_1 + \epsilon^2 y_2 + \epsilon y_0' + \epsilon^2 y_1 + \epsilon^3 y_2 - y_0 - \epsilon y_1 - \epsilon^2 y_2 = 1$$

Equating the order 1 terms, we have

$$y_0'' - y_0 = 1, y_0(0) = 0, y_0(1) = 1$$

$$y_0 = ae^x + be^{-x} - 1$$

where
$$a = 1 - \frac{2-e}{\frac{1}{e} - e}, b = \frac{2-e}{\frac{1}{e} - e}$$

Equating the order ϵ terms, we have

$$y_1'' - y_0' - y_1 = 1, y_1(0) = 0, y_1(1) = 0$$

$$y_1'' - y_1 = 1 + ae^x - be^{-x}$$

$$y_1 = \frac{1}{2}axe^x - \frac{1}{2}bxe^{-x} + c_1 e^x + c_2 e^{-x} - 1$$

where
$$c_1 = \frac{1}{1+e}, c_2 = \frac{e^2 - e}{e^2 - 1}$$

Thus, the solution is

$$\begin{aligned} y &\sim y_0 + \epsilon y_1 \\ &\sim ae^x + be^{-x} - 1 + \epsilon \frac{1}{2}axe^x - \epsilon \frac{1}{2}bxe^{-x} + \epsilon c_1 e^x + \epsilon c_2 e^{-x} - \epsilon \end{aligned}$$

2. (a)
$$y'' - y + \epsilon y^3 = 0, y(0) = 0, y(1) = 1$$

Let
$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + O(\epsilon^3)$$

Equating the order 1 terms, we have

$$y_0'' - y_0 = 0, y_0(0) = 0, y_0(1) = 1$$

$$y_0 = \frac{e}{e^2-1}e^x - \frac{e}{e^2-1}e^{-x}$$

Equating the order ϵ terms, we have

$$y_1'' - y_1 + y_0^3 = 0, y_1(0) = 0, y_1(1) = 0$$

$$y_1'' - y_1 + \frac{e^3}{(e^2-1)^3}e^{3x} - \frac{e^3}{(e^2-1)^3}e^{-3x} - 3\frac{e^3}{(e^2-1)^3}e^x + 3\frac{e^3}{(e^2-1)^3}e^{-x} = 0$$

Using Wolfram, we find

$$y_1 = \frac{e}{8(e^2-1)^4}e^x - \frac{e}{8(e^2-1)^4}e^{-x} - \frac{e^3}{8(e^2-1)^3}(4xe^{-x} - 4xe^x + e^{-3x} - e^{3x})$$

$$\begin{aligned} y &\sim y_0 + \epsilon y_1 \\ &\sim \frac{e}{e^2-1}e^x - \frac{e}{e^2-1}e^{-x} + \epsilon(\frac{e}{8(e^2-1)^4}e^x - \frac{e}{8(e^2-1)^4}e^{-x} \\ &- \frac{e^3}{8(e^2-1)^3}(4xe^{-x} - 4xe^x + e^{-3x} - e^{3x})) \end{aligned}$$

(b)

$$y'' - y + y^3 = 0, y(0) = 0, y(1) = \epsilon$$

Let

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + O(\epsilon^3)$$

Equating the order 1 terms, we have

$$y_0'' - y_0 + y_0^3 = 0, y_0(0) = 0, y_0(1) = 0$$

We get the trivial solution.

$$y_0 = 0$$

Equating the order ϵ terms, we have

$$y_1'' - y_1 = 0, y_1(0) = 0, y_1(1) = 1$$

$$y_1 = \frac{e}{e^2-1}e^x - \frac{e}{e^2-1}e^{-x}$$

Equating the order ϵ^2 terms, we have

$$y_2'' - y_2 = 0, y_2(0) = 0, y_2(1) = 0$$

We get the trivial solution.

$$y_2 = 0$$

Equating the order ϵ^2 terms, we have

$$y_3'' - y_3 + y_1^3 = 0, y_3(0) = 0, y_3(1) = 0$$

$$y_3 = \frac{e}{8(e^2-1)^4}e^x - \frac{e}{8(e^2-1)^4}e^{-x} - \frac{e^3}{8(e^2-1)^3}(4xe^{-x} - 4xe^x + e^{-3x} - e^{3x})$$

$$\begin{aligned} y &= \epsilon y_1 + \epsilon^3 y_3 + O(\epsilon^4) \\ &\sim \frac{e}{e^2-1}e^x \epsilon y_1 - \frac{e}{e^2-1}e^{-x}\epsilon y_1 + \epsilon^3(\frac{e}{8(e^2-1)^4}e^x - \frac{e}{8(e^2-1)^4}e^{-x} \\ &- \frac{e^3}{8(e^2-1)^3}(4xe^{-x} - 4xe^x + e^{-3x} - e^{3x})) \end{aligned}$$

(c)

$$y'' - y + \epsilon y^3 = 0, y(0) = 0, y(1) = \epsilon$$

Let

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + O(\epsilon^3)$$

Equating the order 1 terms, we have

$$y_0 = 0$$

Equating the order ϵ terms, we have

$$y_1'' - y_1 + 0^3 = 0, y_1(0) = 0, y_1(1) = 1$$

$$y_1 = \frac{e}{e^2-1}e^x - \frac{e}{e^2-1}e^{-x}$$

Equating the order ϵ^2 terms, we have

$$y_2 = 0$$

Equating the order ϵ^3 terms, we have

$$y_3 = 0$$

Equating the order ϵ^4 terms, we have

$$y_4'' - y_4 + y_1^3 = 0, y_4(0) = 0, y_4(1) = 0$$

$$y_4 = \frac{e}{8(e^2-1)^4}e^x - \frac{e}{8(e^2-1)^4}e^{-x} - \frac{e^3}{8(e^2-1)^3}(4xe^{-x} - 4xe^x + e^{-3x} - e^{3x})$$

$$\begin{aligned} y &= \epsilon y_1 + \epsilon^4 y_4 + O(\epsilon^5) \\ &\sim \frac{e}{e^2-1}e^x \epsilon y_1 - \frac{e}{e^2-1}e^{-x}\epsilon y_1 + \epsilon^4(\frac{e}{8(e^2-1)^4}e^x - \frac{e}{8(e^2-1)^4}e^{-x} \\ &- \frac{e^3}{8(e^2-1)^3}(4xe^{-x} - 4xe^x + e^{-3x} - e^{3x})) \end{aligned}$$

3. (a)

$$x'' = -\frac{gR^2}{(x+R)^2} - \frac{k}{R+x}x'$$

Let

$$y = \frac{x}{\frac{v_0^2}{g}}, \tau = \frac{t}{\frac{v_0}{g}}$$

$$g\frac{d^2y}{d\tau^2} = -\frac{g}{(\frac{v_0^2 y}{gR}+1)^2} - \frac{\frac{kv_0}{R}}{1+\frac{v_0^2 y}{gR}}\frac{dy}{d\tau}$$

$$\frac{d^2y}{d\tau^2} = -\frac{1}{(\frac{v_0^2 y}{gR}+1)^2} - \frac{\frac{kv_0}{gR}}{1+\frac{v_0^2 y}{gR}}\frac{dy}{d\tau}$$

Replacing a few terms with dimensionless constants, we get the dimensionless equation

$$y'' = -\frac{1}{(\epsilon y+1)^2} - \frac{\alpha}{1+\epsilon y}y'$$

(b)

$$(\epsilon y+1)^2 y'' + \alpha(\epsilon y+1)y' + 1 = 0, y(0) = 0, y'(0) = 1$$

Let

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + O(\epsilon^3)$$

Equating the order 1 terms, we have

$$y_0'' + \alpha y_0' + 1 = 0, y_0(0) = 0, y_0'(0) = 1$$

$$y_0 = (-\frac{1}{\alpha} - \frac{1}{\alpha^2})e^{-\alpha\tau} - \frac{\tau}{\alpha} + \frac{1}{\alpha} + \frac{1}{\alpha^2}$$

Equating the order ϵ terms, we have

$$2y_0y_0'' + y_1'' + \alpha y_1' + \alpha y_0 y_0' + 1 = 0, y_1(0) = 0, y_1'(0) = 0$$

$$y_1'' + \alpha y_1' = -1 - 2y_0y_0'' - \alpha y_0 y_0'$$

Plugging this into mathematica, we get

$$y_1 = -\frac{e^{-2\alpha\tau}}{\tau\alpha^4} + (\frac{1}{\alpha^3} - \frac{1}{\alpha})\tau e^{-\alpha\tau} + (\frac{2}{\alpha^3} + \frac{1}{\alpha^2})\tau + (-\frac{1}{2\alpha^2})\tau^2$$
$$+ (\frac{1}{2\alpha^2} + \frac{1}{2\alpha})\tau^2 e^{-\alpha\tau} + \frac{1}{\alpha^3}\tau + (-\frac{1}{\alpha^3} - \frac{1}{2\alpha^2})e^{-2\tau\alpha} + ae^{-\tau\alpha} + b$$

where a and b can be found by plugging in boundary conditions.

$$y \sim y_0 + \epsilon y_1$$

(c) Ignoring the ϵ terms,

$$y'' + \alpha y' + 1 = 0, y(0) = 0, y'(0) = 1$$

Expanding in terms of α, the order 1 terms are

$$y_0'' + 1 = 0, y_0(0) = 0, y_0'(0) = 1$$

$$y_0 = t - \frac{1}{2}t^2$$

Expanding in terms of α, the order α terms are

$$y_1'' + y_0' = 0, y_1(0) = 0, y_1'(0) = 0$$

$$y_1 = -\frac{t^2}{2} + \frac{1}{3}t^3$$

$$y \sim t - \frac{1}{2}t^2 + \alpha(-\frac{t^2}{2} + \frac{1}{3}t^3)$$

Without the alpha terms, the roots occur at

$$t = 0, 2$$

With the alpha terms, the roots occur at

$$t = 0, \frac{2}{1+\alpha}$$

So, the air resistance decreases time of flight.

4. (a)

$$\psi_0'' - V_0(x)\psi_0 = -E_0\psi_0$$

(b)

$$e^{\phi}\phi'' + e^{\phi}\phi'^2 - [V_0(x) + \epsilon V_1(x)]e^{\phi} = -Ee^{\phi}$$
$$\phi'' + \phi'^2 - [V_0(x) + \epsilon V_1(x)] = -E$$

(c)

$$\phi \sim \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2$$
$$\phi^{'2} \sim \phi_0^{'2} + 2\epsilon\phi_1^{'}\phi_0^{'} + \epsilon^2(2\phi_2^{'}\phi_0^{'} + \phi_1^{'2})$$
$$\phi_1'' + 2\phi_1'\phi_0' = V_1 - E_1$$

$$(\phi_1' e^{2\int \phi_0' dx})' = V_1 e^{2\int \phi_0' dx} - E_1 e^{2\int \phi_0' dx}$$

$$(\phi_1'\psi_0^2)' = V_1\psi_0^2 - E_1\psi_0^2 \quad (1)$$

Integrating from $-\infty$ to ∞, the left side evaluates to zero at the limits.

$$0 = \int_{-\infty}^{\infty} V_1(x)\psi_0^2 - E_1 \int_{-\infty}^{\infty} \psi_0^2$$

$$E_1 = \int_{-\infty}^{\infty} V_1(x)\psi_0^2$$

$$\phi_2'' + 2\phi_2^{'}\phi_0^{'} + \phi_1^{'2} = -E_2$$
$$\phi_2'' + 2\phi_2^{'}\phi_0^{'} = -\phi_1^{'2} - E_2$$

Similarly, we get

$$E_2 = -\int_{-\infty}^{\infty} \phi_1^{'2}\psi_0^2$$

We can find ϕ' by integrating equation 1 from $-\infty$ to x.

$$\phi_1'(x)\psi_0^2(x) = \int_{-\infty}^{x} V_1(x)\psi_0^2(x) - \int_{-\infty}^{x} E_1\psi_0^2(x)$$

(d) We know that

$$\int_{-\infty}^{\infty} \psi_0^2(x)dx = 1$$

So, we would think that ψ_0^2 is a Gaussian and $\psi_0 = e^{-ax}$. But that doesn't satisfy the boundary condition $\psi_0(-\infty) = 0$. So, we choose e^{-ax^2} which satisfies everything. So,

$$\phi_0 = -ax^2$$

Plugging into

$$\phi'' + \phi'^2 - \lambda^2 x^2 = -E_0$$
$$-2a + 4a^2x^2 - \lambda^2x^2 = -E_0$$
$$a = \frac{\lambda}{2}$$
$$E_0 = 2a = \lambda$$
$$\psi_0 = e^{-\frac{\lambda}{2}x^2}$$

The normalized ψ_0^2

$$\psi_0^2 = \frac{\sqrt{\lambda}}{\sqrt{\pi}}e^{-\lambda x^2}$$

$$\begin{aligned} E_1 &= \int_{-\infty}^{\infty} V_1(x)e^{-\lambda x^2} \\ &= \int_{-\infty}^{\infty} \alpha x e^{-\gamma x^2}e^{-\lambda x^2}dx \\ &= 0 \text{ (odd function)} \end{aligned}$$

$$\phi_1'(x)\psi_0^2(x) = \int_{-\infty}^{x} V_1(x)\psi_0^2(x) - \int_{x}^{\infty} E_1\psi_0^2(x)$$

$$\phi_1'(x)e^{-\lambda x^2} = \int_{-\infty}^{x} \frac{1}{4}\alpha x e^{-(\gamma+\lambda)x^2} dx = \frac{\alpha e^{-(\gamma+\lambda)x^2}}{2(\gamma+\lambda)}$$

$$\phi_1'(x) = \frac{\alpha e^{-\gamma x^2}}{2(\gamma+\lambda)}$$

$$\begin{aligned} E_2 &= -\int_{-\infty}^{\infty} \phi_1'^2 \psi_0^2 dx \\ &= -\int_{-\infty}^{\infty} \frac{1}{4}\left(\frac{\alpha}{\gamma+\lambda}\right)^2 \frac{\sqrt{\lambda}}{\sqrt{\pi}} e^{-(\lambda+2\gamma)x^2} dx \\ &= -\frac{1}{4}\left(\frac{\alpha}{\gamma+\lambda}\right)^2 \sqrt{\frac{\lambda}{\lambda+2\gamma}} \end{aligned}$$

$$E \sim \lambda - \frac{1}{4}\left(\frac{\alpha\epsilon}{\gamma+\lambda}\right)^2 \sqrt{\frac{\lambda}{\lambda+2\gamma}}$$