Assignment 1.

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1. (a) From the conservation of mass, the total mass of water in the pipe should be conserved, i.e

$$\frac{d}{dt}\int_{container}(\rho V) = 0$$

where ρ is density and V is Volume. In incompressible flow, density is constant.

$$\rho\frac{d}{dt}\int_{container}(V) = \rho\int_{container}\frac{dV}{dt} = \rho\int_0^h A(h)v(h) = 0$$

where v(h) is the velocity at height h. Thus,

$$A(h)v(h) = A(0)v(0)$$

$$v(h) = \frac{A(0)v(0)}{A(h)} = -0.6\sqrt{2gh}\frac{A(0)}{A(h)}$$

(b)

$$\frac{dh}{dt} = -0.6A(0)\frac{\sqrt{2gh}}{A(h)}$$

$$h^{c-0.5}dh = -0.6A(0)\sqrt{2g}dt$$

$$\frac{h^{c+0.5}}{c+0.5} = -0.6A(0)\sqrt{2g}t + \frac{h_0^{c+0.5}}{c+0.5}$$

$$h^{c+0.5} = -0.6A(0)(c+0.5)\sqrt{2g}t + h_0^{c+0.5}$$

However, we would have $A(0) = 0^C$. So,

$$h^{c+0.5} = h_0^{c+0.5}$$

$$h = h_0$$

Thus, according to this model, the height stays the same. This is not physical when there is an outflow of water (violates conservation of mass).

2. (a) Using grams for Mass and minutes for time, we get

$$M = 5 - 5e^{-t}$$

If y is the mass in kg and τ is the time in minutes,

$$y = \frac{M}{1000}, \tau = \frac{t}{60}$$

So, we have

$$1000y = 5 - 5e^{-60\tau}$$
$$y = \frac{5}{1000} - \frac{5}{1000}e^{-60\tau}$$

(b) We start with

$$M = 5 - 5e^{-t}$$

where everything is a particular system, say SI. So, if we have $C_1 = 1kg^{-1}, C_2 = 1s^{-1}$.

Thus, we still have

$$C_1 M = 5 - 5e^{-C_2 t}$$

Suppose we change to another set of units. Then,

$$M = \alpha M', t = \beta t'$$

But we also will have

$$C_1 = \frac{1}{\alpha}C_1', C_2 = \frac{1}{\beta}C_2'$$

Thus, we'll end up with

$$C_1' M' = 5 - 5e^{-C_2' t'}$$

which is the same complete equation.

3. **Using dimensional analysis**

We know that the height $h : [L]$ depends on the pressure $P : [ML^{-1}T^{-2}]$, density of the oil $\rho : [ML^{-3}]$ and acceleration due to gravity $g : [LT^{-2}]$. There are 3 variables and 4 dimensions. So, from Buckingham Pi, there must be one dimensionless parameter π.

Equating height to an arbitary formula of the other terms, we have

$$\pi = h^-1 P^a g^b \rho^c A^d$$
$$1 = L^{-1} M^a L^{-a} T^{-2a} L^b T^{-2b} M^c L^{-3c}$$
$$1 = M^{a+c} L^{b-a-3c-1} T^{-2a-2b}$$

$$a + c = 0$$
$$b - a - 3c - 1 = 0$$
$$-2a - 2b = 0$$

we get a =1, b = -1 and c = -1. That implies

$$\pi = \frac{P}{\rho g h}$$

We know for some function f,

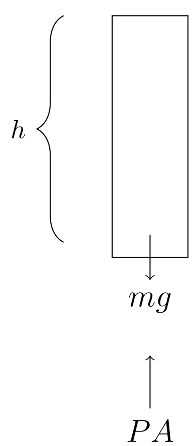
$$f(\pi) = 0$$

This function would have some root c. Thus,

$$\pi = c$$
$$\frac{P}{\rho g h} = c$$
$$h = c\frac{P}{\rho g}$$

Using a data point, we can find that c = 1.

Using force balance on a column of water



Fromthe force balance, we have

$$PA = mg$$
$$PA = \rho V g$$
$$PA = \rho A h g$$
$$h = \frac{P}{\rho g}$$

4. (a) Here, position is positive downwards in the direction of gravity.

$$x'' = g + k|x'|^2$$

Note that k is not the spring constant , but the spring constant divided by mass.

(b)

$$v' = g + k|v|^2$$

(c) The solution to the above ode is

$$v = \sqrt{\frac{g}{k}} \tan(c_1\sqrt{gk} + \sqrt{gk}t)$$

Since our initial velocity is 0,

$$v = \sqrt{\frac{g}{k}} \tan(\sqrt{gk}t)$$

Thus, the position is given by

$$x = \frac{1}{k} \log(|\sec(\sqrt{gk}t)|)$$

This blows up when $\sqrt{gk}t = \frac{\pi}{2}$. In the plots, we see that it is point of divergence. So,

$$t_c = \frac{\pi}{2\sqrt{gk}}$$

(d) Plots

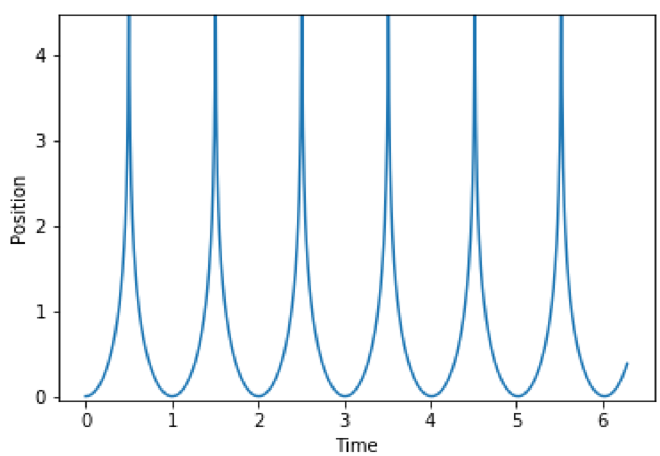


Figure 1: Position versus time

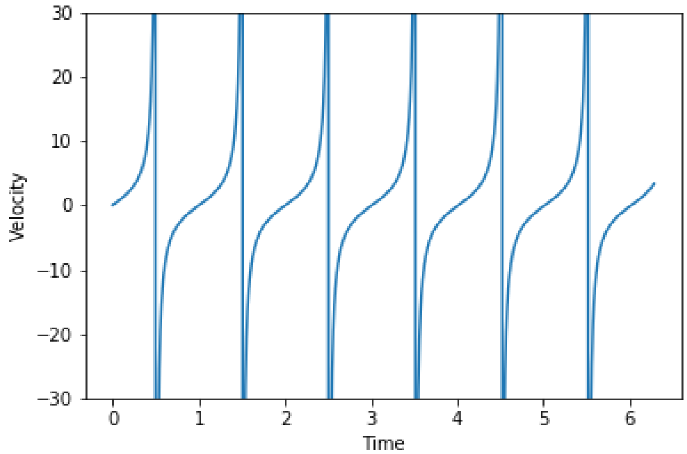


Figure 2: Velocity versus time

Of course, the analytical solution is only valid as a model until $t_c = \frac{\pi}{2\sqrt{gk}}$. It doesn't make any physical sense for a ball to go to infinity and return without the presence of any attractive force.

(e) In that case, the position would be given by

$$x(t) = \frac{g}{k^2}e^{kt} - \frac{g}{k}$$

This does not blow up anywhere and hence there is no time of divergence. This is a bad model because you should expect things to go very far away in finite time if drag and gravity acted in the same direction.

5. Let the distance between the centers of the dice be x. Thus, the dynamics of the dice are given by

$$x_1'' = \frac{Gm}{x^2}$$
$$x_2'' = -\frac{Gm}{x^2}$$

$$x'' = (x_2 - x_1)'' = -\frac{2Gm}{x^2}$$
$$2x'x'' = -\frac{4Gmx'}{x^2}$$
$$(x')^2 = C + \frac{4Gm}{x}$$

Since the initial velocity is zero,

$$C = -\frac{4Gm}{10^{-1}}$$

All units are SI to match with the G.

$$x' = -\sqrt{-\frac{4Gm}{10^{-2}} + \frac{4Gm}{x}} = -2\sqrt{Gm}\sqrt{-10 + \frac{1}{x}}$$

Here, the negative square root is taken because gravity is an attactive force and velocity is in the opposite direction of displacement.

Total time till collision is given by

$$\begin{aligned}
T &= \int_{0.1}^{0} dt \\
&= \int_{0.1}^{0} \frac{dx}{v} \\
&= -\frac{1}{2\sqrt{Gm}} \int_{0.1}^{0} \frac{dx}{\sqrt{-10 + \frac{1}{x}}} \\
&\approx -\frac{1}{2\sqrt{Gm}} \int_{0.1}^{0.001} \frac{dx}{\sqrt{-10 + \frac{1}{x}}} \\
&\approx \frac{0.0497}{2\sqrt{Gm}} \\
&\approx 9621s
\end{aligned}$$

6. (a) Assume $l_1 > l_2$. The push of the first spring should balance the pull of the second spring so as to have equilibrium. Thus, the equilibrium position x_0 is in between the l_1 and l_2.

$$k_1(l_1 - x_0) = k_2(x_0 - l_2)$$
$$x_0 = \frac{k_1 l_1 + k_2 l_2}{k_1 + k_2}$$

(b) If the mass is given a small perturbation Δx about x_0, the position of the mass would be given by

$$x = x_0 + \Delta x$$

Its motion can be estimated from the forces acting on it.

$$mx'' = k_1(l_1 - x) - k_2(x - l_2)$$

$$mx'' = -k_1(x - l_1) - k_2(x - l_2)$$

$$\begin{aligned}
mx'' &= -k_1(x - l_1) - k_2(x - l_2) \\
&= -(k_1 + k_2)x + k_1 l_1 + k_2 l_2
\end{aligned}$$

Writing the position as relative to the equilibrium,

$$\begin{aligned} m(x_0 + \Delta x)'' &= -(k_1 + k_2)(x_0 + \Delta x) + k_1 l_1 + k_2 l_2 \\ &= -(k_1 + k_2)\Delta x \end{aligned}$$

$$\Delta x'' = -\frac{k_1 + k_2}{m}\Delta x$$

Thus, the mass follows an S.H.M about x_0 .

(c) The solution to $y'' = -\frac{k_1+k_2}{m}y$ is $c_1 \sin(\sqrt{\frac{k_1+k_2}{m}}t) + c_2 \cos(\sqrt{\frac{k_1+k_2}{m}}t)$. So, the period of oscillation is $\frac{2\pi\sqrt{m}}{\sqrt{k_1+k_2}}$

(d) Since the dynamics are governed by

$$\Delta x'' = -\frac{k_1 + k_2}{m}\Delta x$$

which can be written as

$$x'' = -\frac{k_1 + k_2}{m}(x - x_0)$$

we can see that a system equivalent to this must have the spring constant $\frac{k_1+k_2}{m}$ and an unstretched length x_0.

7. For the spings to be at rest, the spring should be at their unstretched lengths. The massless point with be at l_1 and the mass would be at l_1+l_2 which would the equilibrium positions.

Let x_2 and x_1 be the perturbations about the equilibrium positions of the mass and the unstretched lengths. The force acting on the mass is

$$k_2(x_2 - x_1)$$

Since the spring is massless, this is the force on the massless point from the right side. So,

$$\begin{aligned} k_2(x_2 - x_1) &= k_1 x_1 \\ x_1 &= \frac{k_2}{k_1 + k_2}x_2 \end{aligned}$$

Thus, the dynamics become

$$mx_2'' = -k_2(x_2 - \frac{k_2}{k_1 + k_2}x_2) = -\frac{k_1 k_2}{k_1 + k_2}x_2$$

This are the dynamics of an S.H.M with period $\frac{2\pi\sqrt{m(k_1+k_2)}}{\sqrt{k_1 k_2}}$.

The spring constant of an equivalent system with one spring would be $\frac{k_1 k_2}{k_1+k_2}$ and the unstretched length would be $l_1 + l_2$.