Assignment 4.

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1.

$$I_p(x) = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n+p}}{n!(n+p)!}$$

$$\begin{aligned}
\frac{d}{dx}[x^p I_p(\alpha x)] &= \frac{d}{dx}\sum_{n=0}^{\infty} \frac{x^p\left(\frac{\alpha x}{2}\right)^{2n+p}}{n!(n+p)!} \\
&= \sum_{n=0}^{\infty} \frac{(2n+2p)x^{2n+2p-1}\alpha^{2n+p}}{2^{2n+p}n!(n+p)!} \\
&= \sum_{n=0}^{\infty} \frac{x^{2n+2p-1}\alpha^{2n+p}}{2^{2n+p-1}n!(n+p-1)!} \\
&= \sum_{n=0}^{\infty} \frac{x^2\left(\frac{x}{2}\right)^{2n+p-1}\alpha^{2n+p}}{n!(n+p)!} \\
&= \sum_{n=0}^{\infty} \frac{\alpha x^2\left(\frac{\alpha x}{2}\right)^{2n+p-1}}{n!(n+p)!} \\
&= \alpha x^p I_{p-1}(\alpha x)
\end{aligned}$$

2.

$$xy'' + (1+4x^2)y' + x(5+4x^2)y = 0$$
$$x^2y'' + x(1+4x^2)y' + (5x^2+4x^4)y = 0$$

This is of the form

$$x^2y'' + x(a+2bx^2)y' + (c + dx^{2s} - b(1-a-r)x^r + b^2x^{2r})y = 0$$

where a=1,b=2,r=2,c=0,s=1 and d=1 So,

$$y(x) = e^{-x^2}Z(\eta)$$

$$\eta = x$$

Z is a Bessel equation of order p where

$$p = 0$$

$$y(x) = e^{-x^2}J_0(x) + e^{-x^2}Y_0(x)$$

3.
$$x^4y'' + \alpha^2 y = 0$$

This is of the form
$$x^2y'' + dx^{2s}y = 0$$
where d = 1, s =-1 and everything else is 0 So,
$$y(x) = \sqrt{x}Z(\eta)$$
$$\eta = -\frac{\alpha}{x}$$
Z is a Bessel equation of order p where
$$p = \frac{1}{2}$$

So,
$$\begin{aligned} y(x) &= \sqrt{x}J_{\frac{1}{2}}(\eta) + \sqrt{x}J_{-\frac{1}{2}}(\eta) \\ &= a_1\sqrt{x}\frac{1}{\sqrt{x}}sin(\eta) + a_2\sqrt{x}\frac{1}{\sqrt{x}}cos(\eta) \\ &= c_1\sin(\frac{\alpha}{x}) + c_2\cos(\frac{\alpha}{x}) \end{aligned}$$

4. Let's look at the homogeneous equation.
$$xy'' - y' + 4x^3y = 0$$
$$x^2y'' - xy' + 4x^4y = 0$$

This is of the form
$$x^2y'' + x(a + 2bx^2)y' + (c + dx^{2s} - b(1 - a - r)x^r + b^2x^{2r})y = 0$$
where a=-1,b=0,s=2 and d=2 So,
$$y(x) = xZ(\eta)$$
$$\eta = x^2$$
Z is a Bessel equation of order p where
$$p = \frac{1}{2}$$

So,
$$\begin{aligned} y(x) &= xJ_{\frac{1}{2}}(\eta) + xJ_{-\frac{1}{2}}(\eta) \\ &= c_1\sin(x^2) + c_2\cos(x^2) \end{aligned}$$

With the initial conditions, we find that the solution is

$$y = c_1 sin(x^2)$$

Clearly, this isn't trivial. So, we check the solvability condition.

$$\int_0^{\pi^2} x^4 sin(x^2) dx \neq 0$$

Thus, there is no solution.