Computational techniques for problems in civil engineering: Finite Volumes

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Outline

Finite differences

The problem :

$$\frac{dx}{dt} = f(x, t)$$

The approximation:

$$\frac{dx(t)}{dt} = \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

Solution:

$$x(t + \Delta t) = x(t) + \Delta t \frac{dx}{dt} = x(t) + \Delta t f(t)$$

Projectile motion

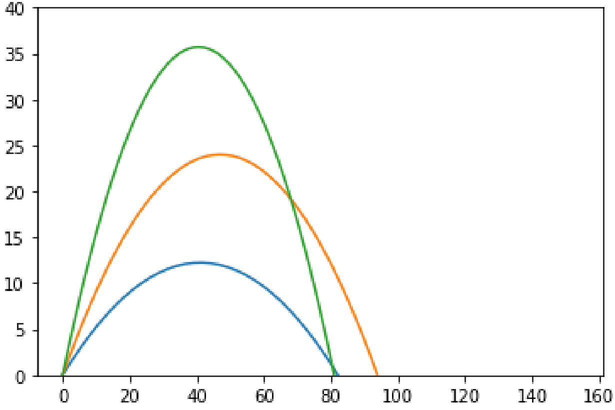
$$\frac{d^2x}{dt^2} = -g$$

$$\frac{d}{dt}\left[\begin{array}{c} x \\ v \end{array}\right] = \left[\begin{array}{c} v \\ -g \end{array}\right]$$

Projectile motion

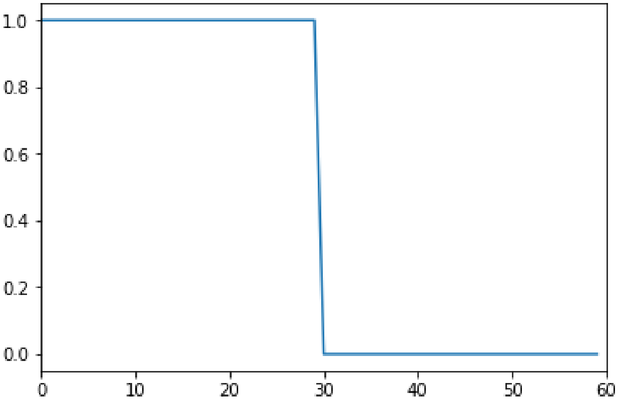
```
thetas= [np.pi/6,np.pi/4, np.pi/3];
a=np.array([[0,-9.81]]) #acceleration
x=np.zeros([60,2]) #initial position
u=30 #initial velocity magnitude

for theta in thetas:
    v=np.zeros([60,2])
    v[0,:]=[u*np.cos(theta),u*np.sin(theta)]
    h=0.1
    for i in range(1,60):
        x[i,:]=x[i-1,:] +h*v[i-1,:]
        v[i,:]=v[i-1,:] +h*a

    py.plot(x[:,0],x[:,1])
```

Projectile motion



The Riemann problem

The Riemann problem



$$f'(x) = \infty$$

Advection equation

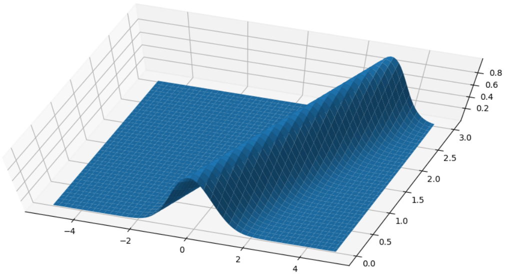
The advection equation:

$$\frac{\partial q}{\partial t} = c\frac{\partial q}{\partial x}$$

A solution is of the form $f(x+ct)$ satisfies this.

$$\frac{\partial f(x+ct)}{\partial t} = f'(x+ct)c, \frac{\partial f(x+ct)}{\partial x} = f'(x+ct)$$

Here, the function is $e^{-(x+t)^2}$



System of equations

$$\frac{\partial q}{dt} = A\frac{\partial q}{\partial x}$$

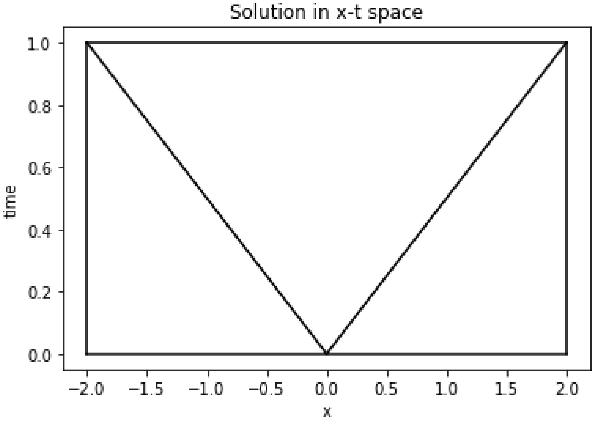
If A has real eigenvalues, the system is hyperbolic.

$$Av = \lambda v$$

Linear acoustics equation

$$\frac{\partial}{\partial t}\begin{bmatrix} p \\ u \end{bmatrix} + \begin{bmatrix} 0 & K \\ \frac{1}{\rho} & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} p \\ u \end{bmatrix} = 0$$

The eigenvalues are $\sqrt{\frac{K}{\rho}}$ and $-\sqrt{\frac{K}{\rho}}$



Linear acoustics

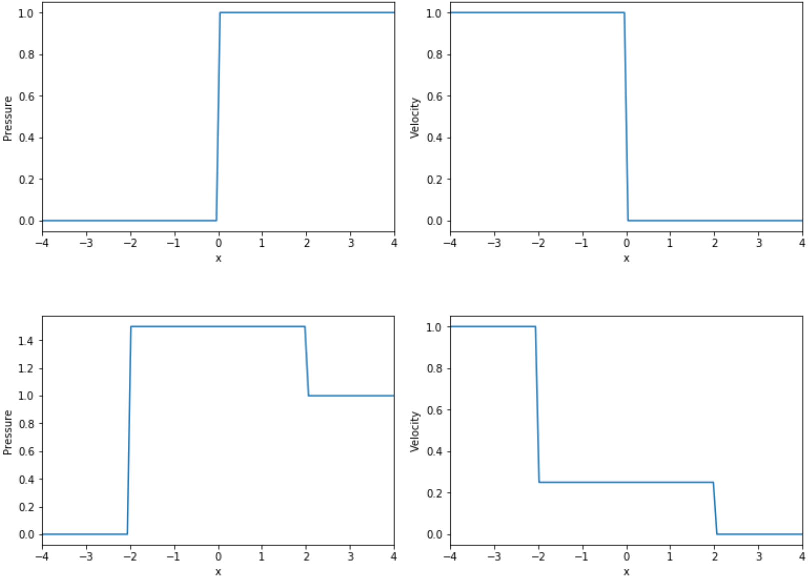
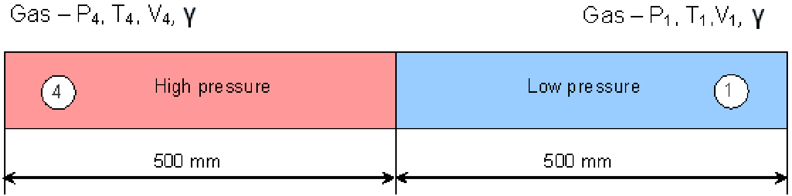


Figure 2: Time = 1

Linear acoustics



Outline

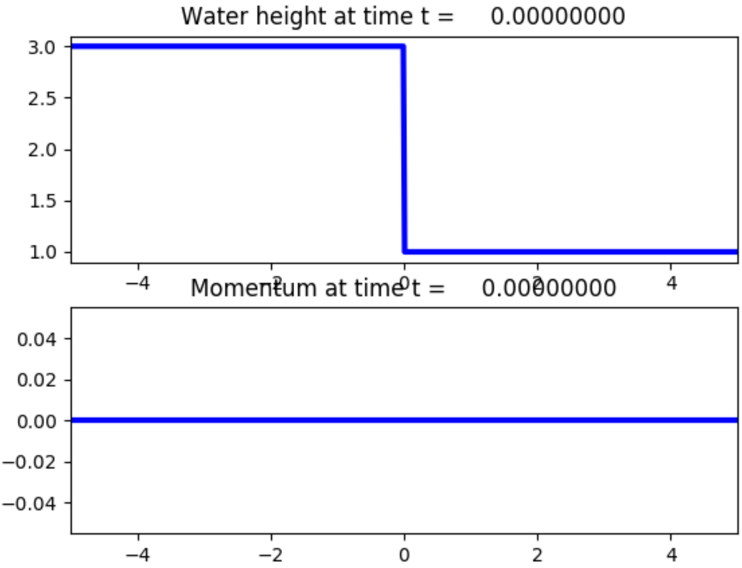
Shallow water equations

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2 + \frac{1}{2}gh^2)}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left[\begin{array}{c} h \\ hu \end{array}\right] + \frac{\partial}{\partial x}\left[\begin{array}{c} hu \\ hu^2 + \frac{1}{2}gh^2 \end{array}\right] = 0$$

Shallow water equations



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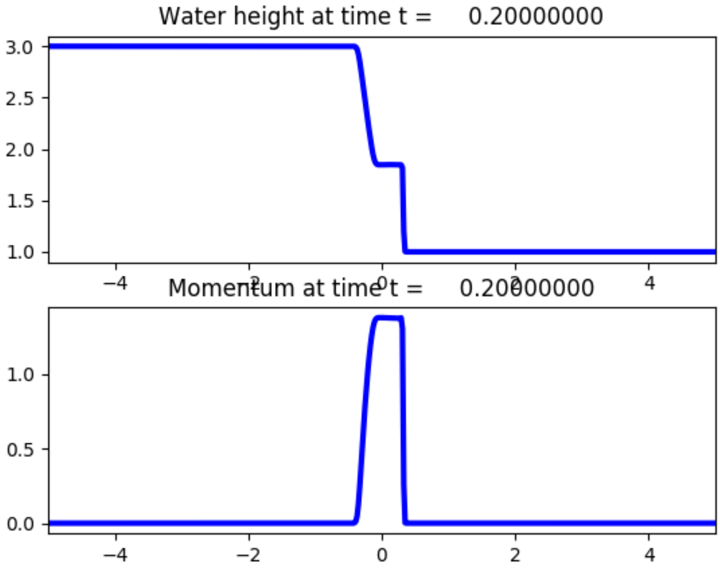
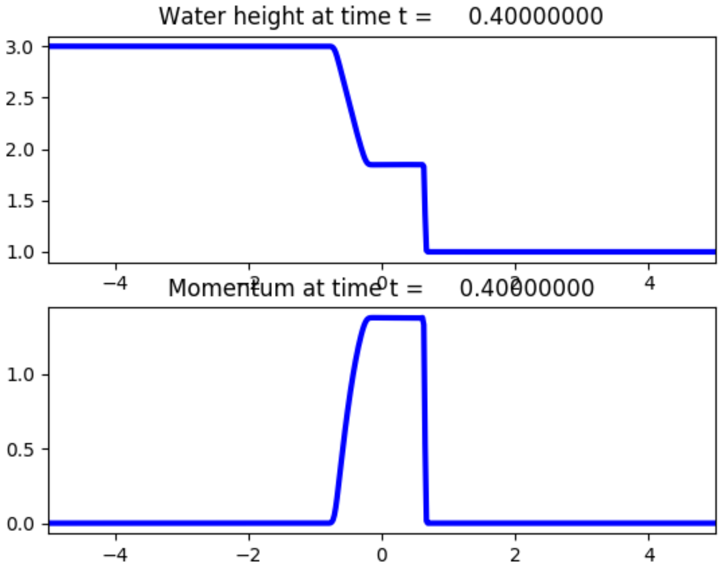


Shallow water equations



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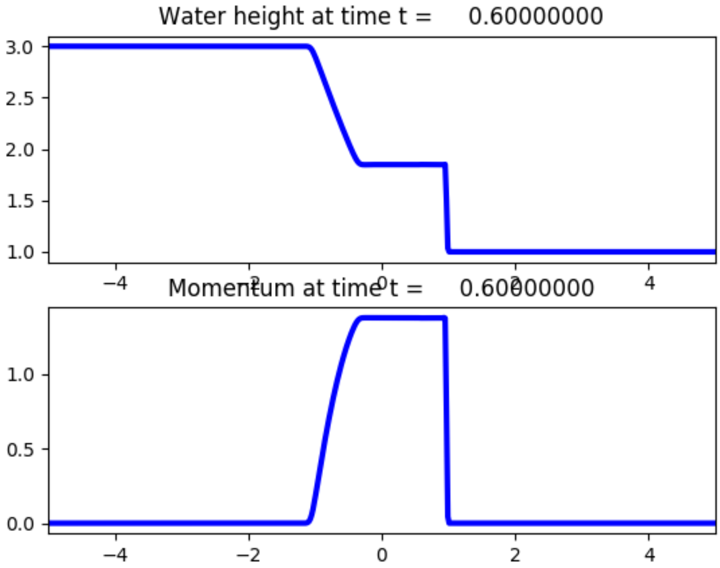

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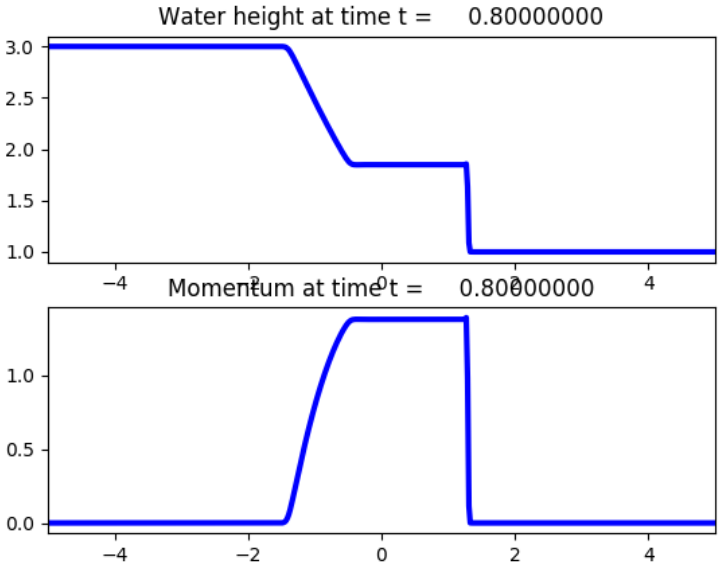

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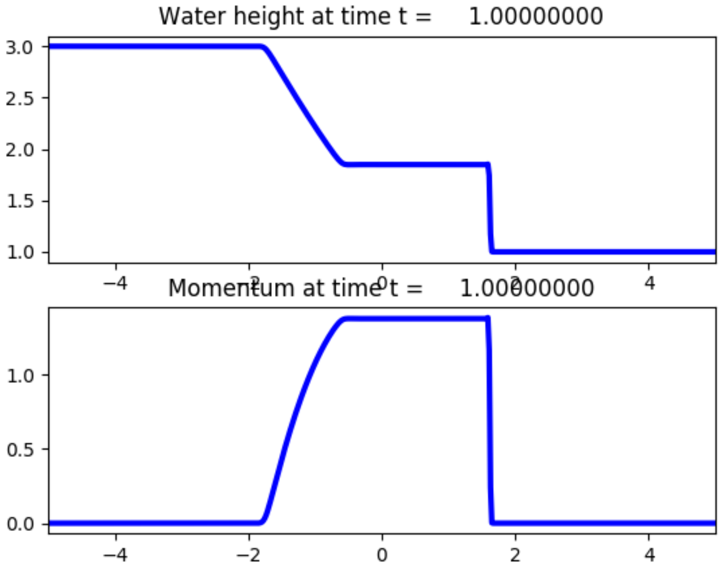

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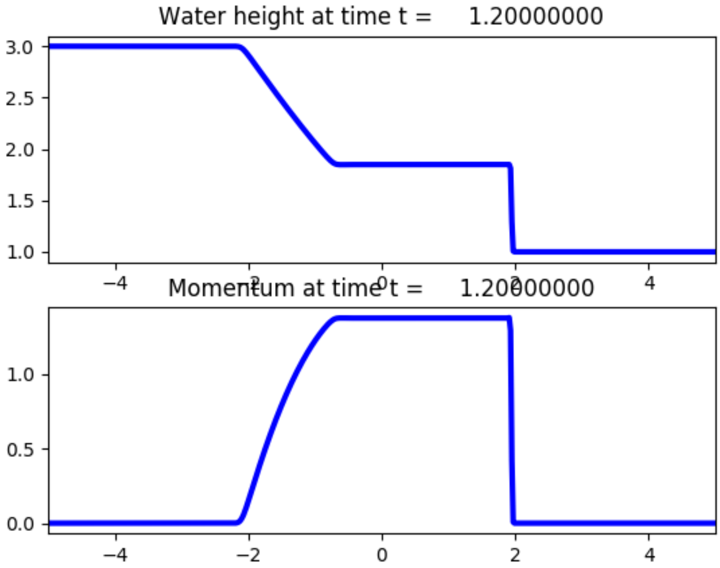

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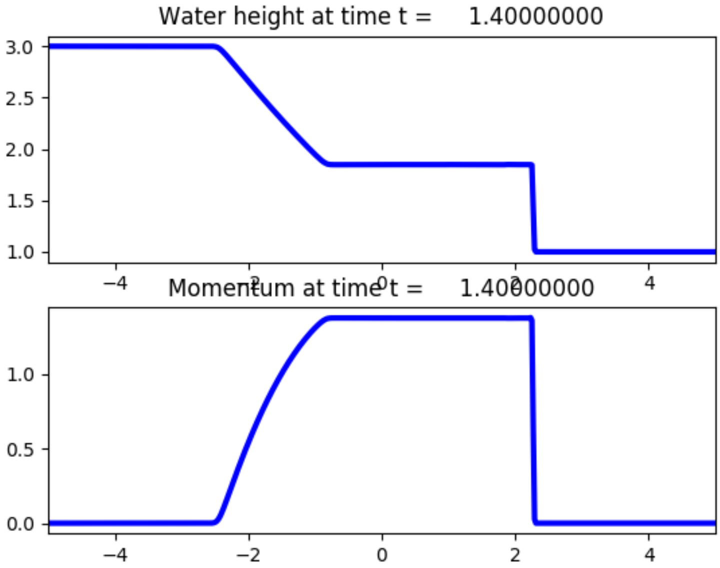

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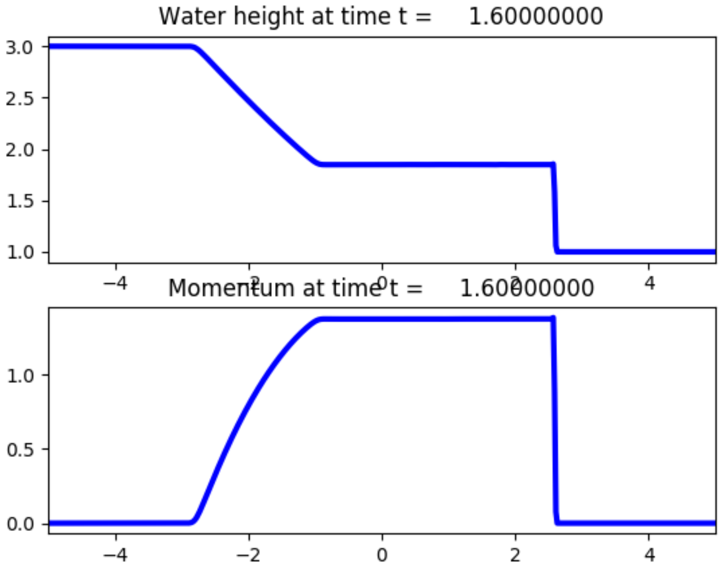

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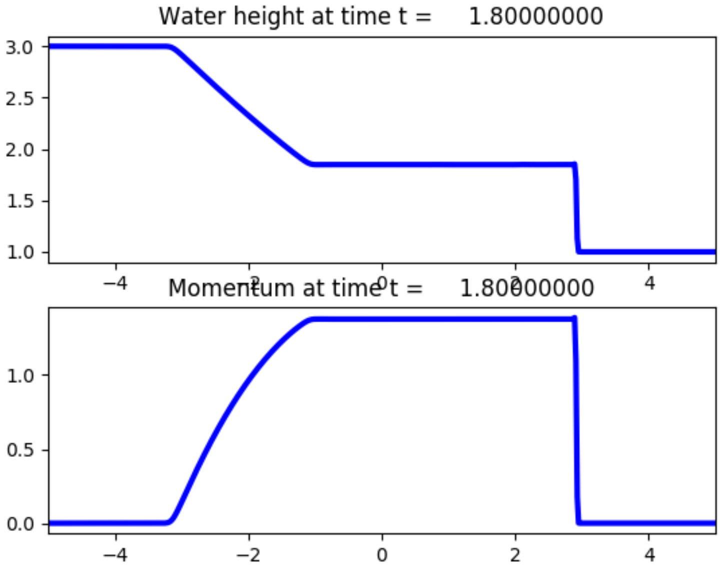

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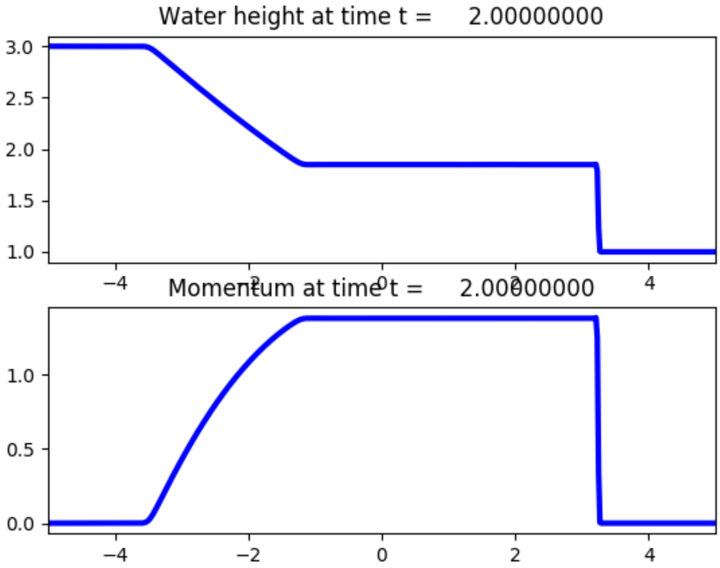
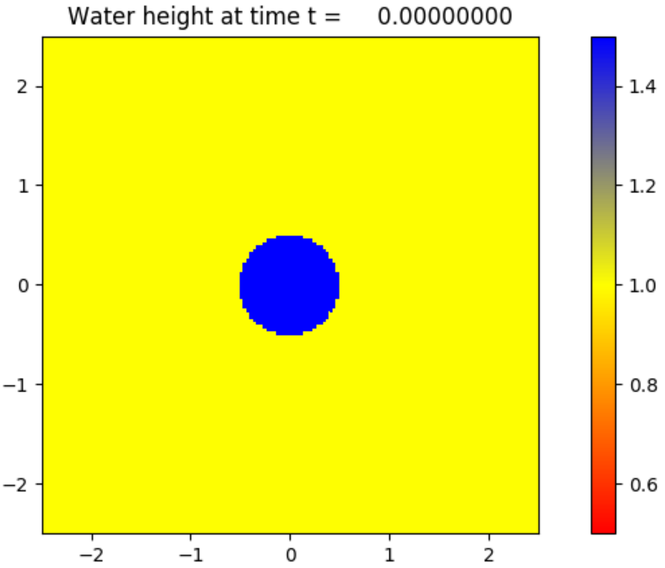


2D Shallow water equations

$$\frac{\partial}{\partial t}\left[\begin{array}{c} h \\ hu \\ hv \end{array}\right] + \frac{\partial}{\partial x}\left[\begin{array}{c} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{array}\right] + \frac{\partial}{\partial y}\left[\begin{array}{c} 0 \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{array}\right] = 0$$

2D Shallow water equations



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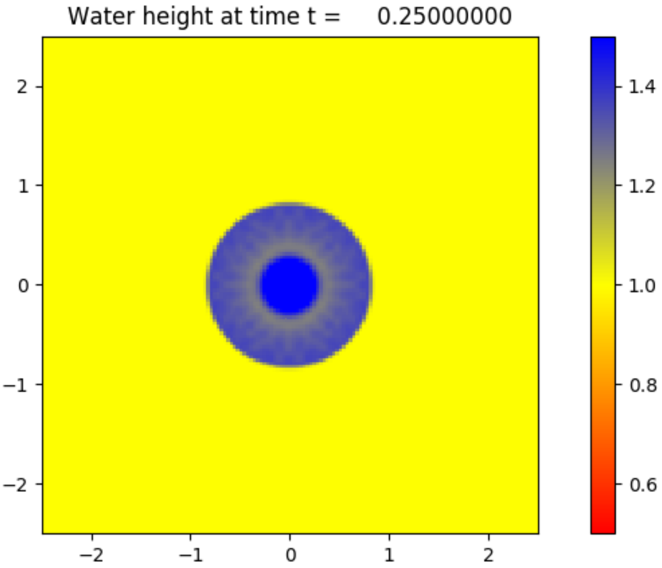

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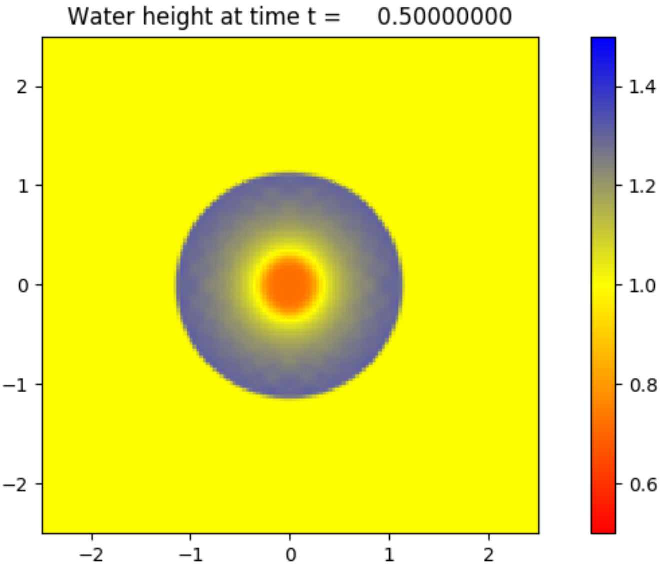

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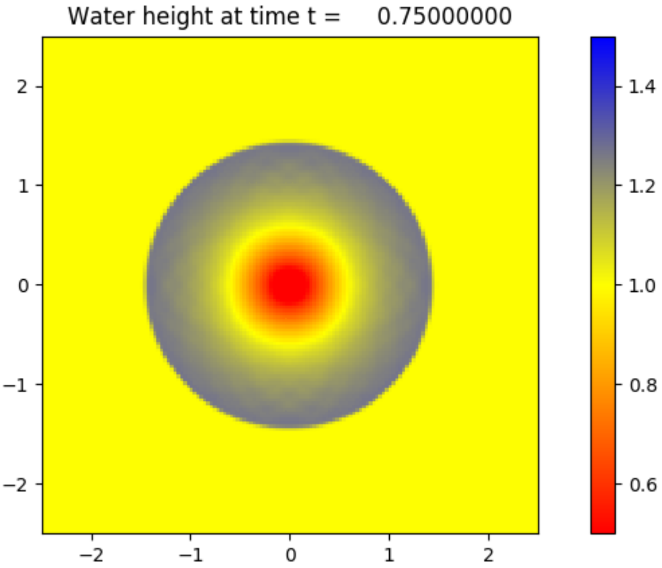

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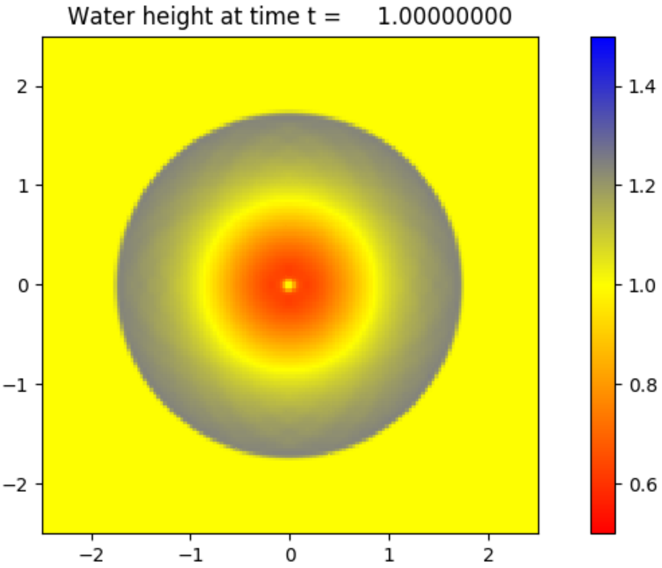

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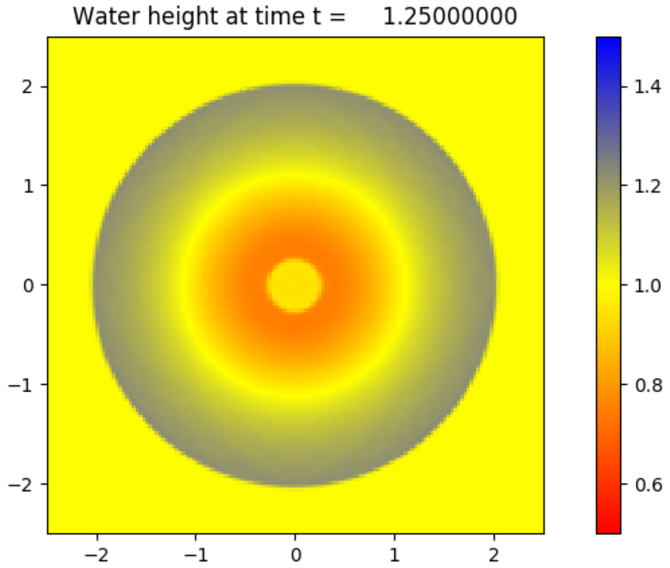

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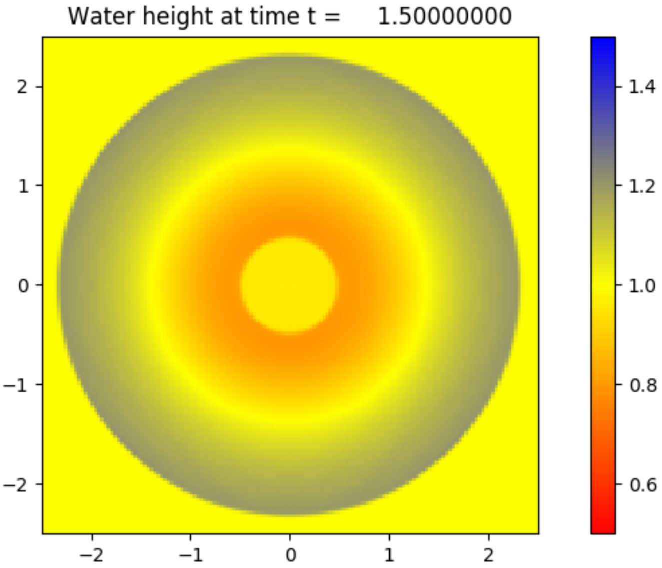

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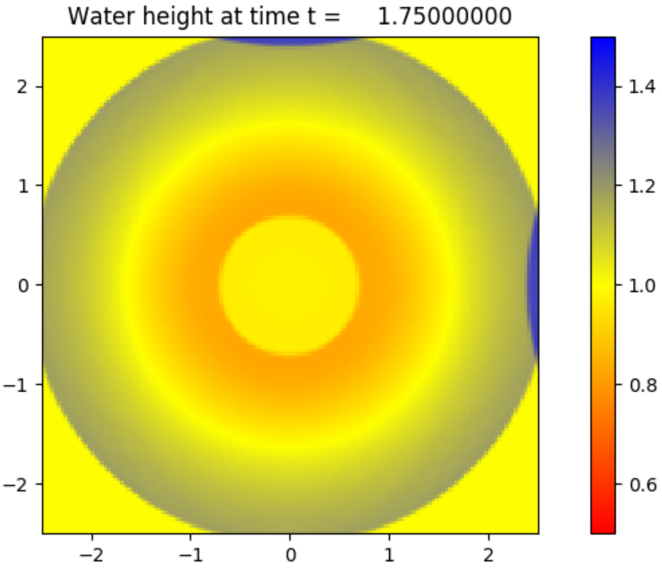

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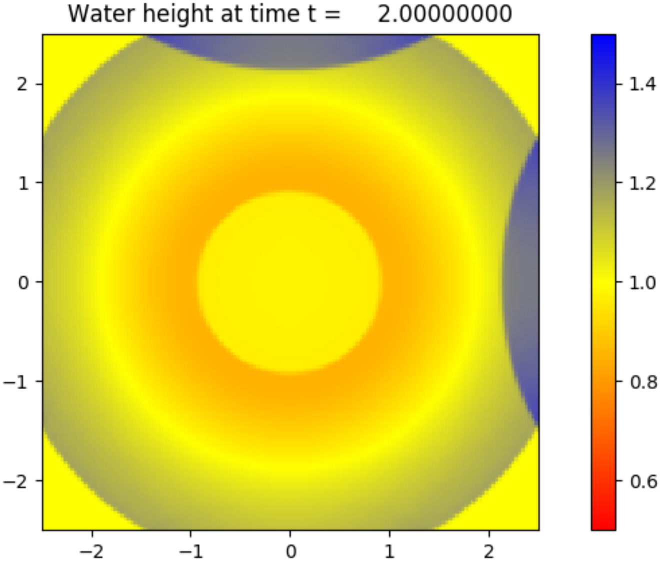

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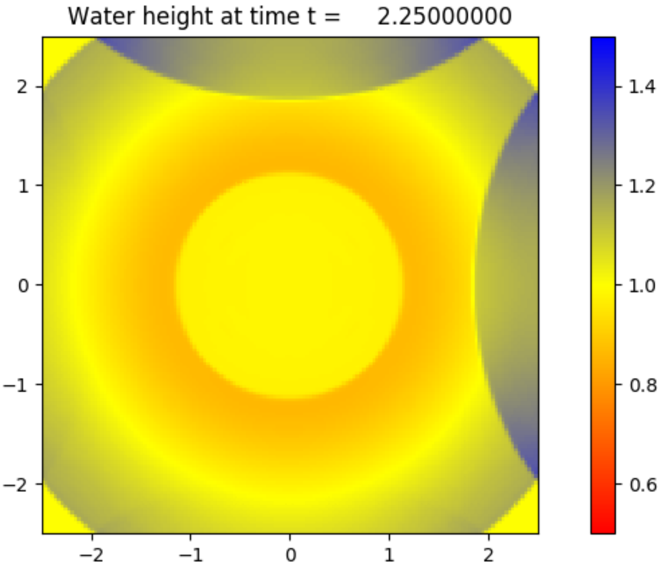

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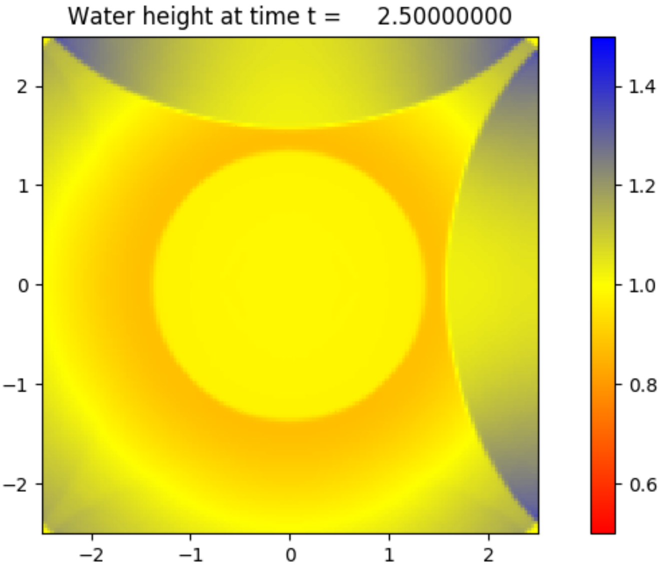
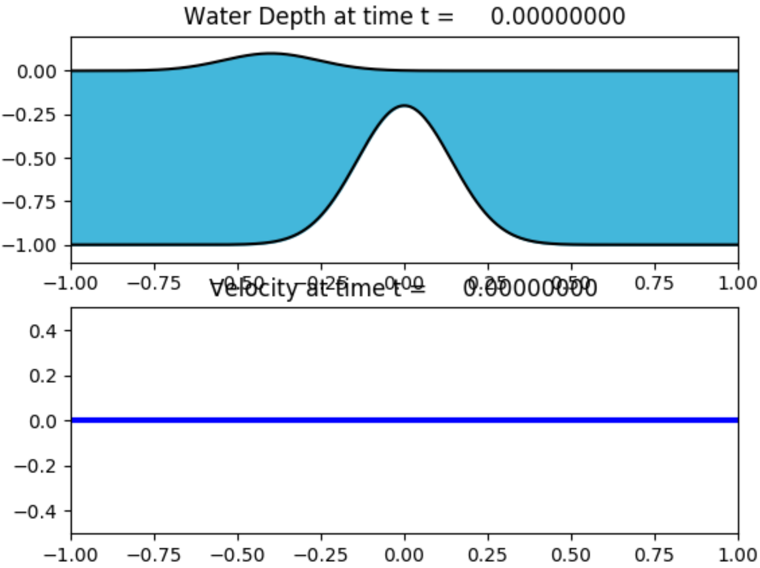


Shallow water equations with bathymetry

$$\frac{\partial}{\partial t}\begin{bmatrix} h \\ hu \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -ghb_x \end{bmatrix}$$

Shallow water equations with bathymetry



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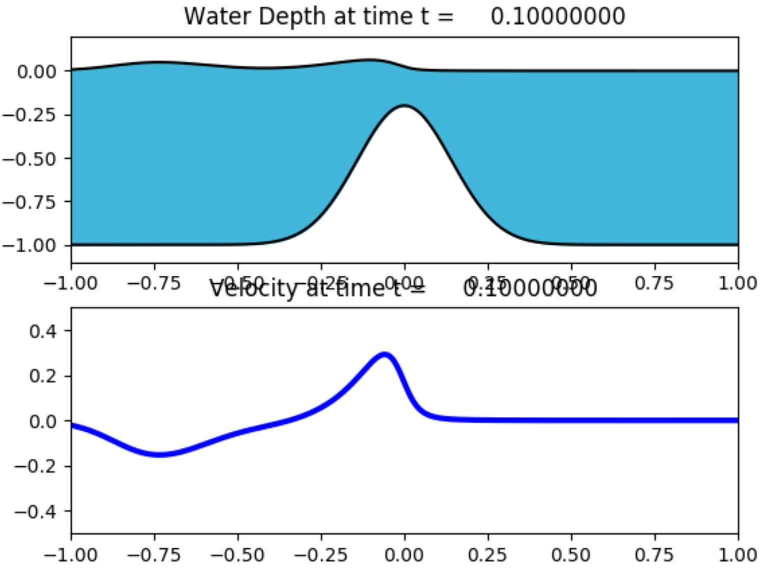

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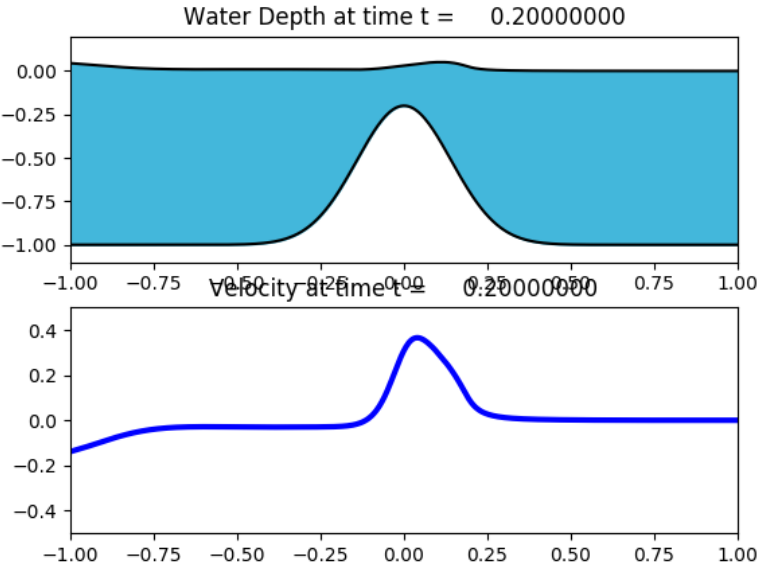

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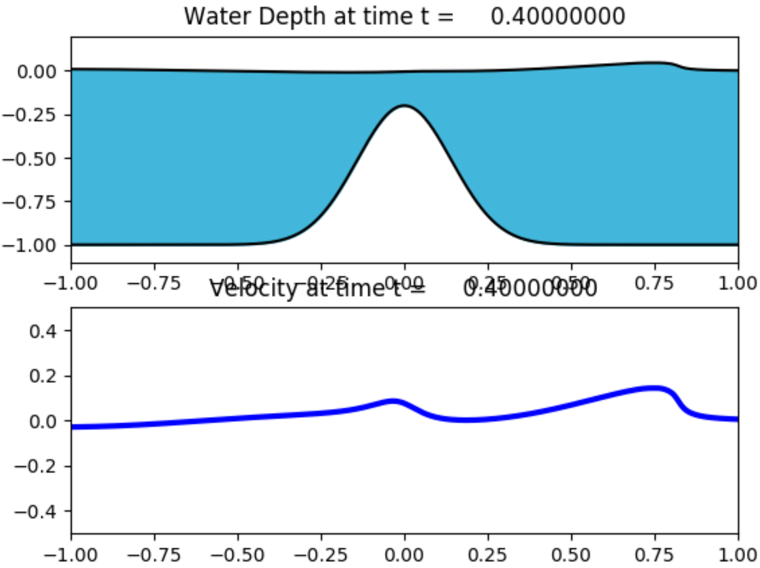

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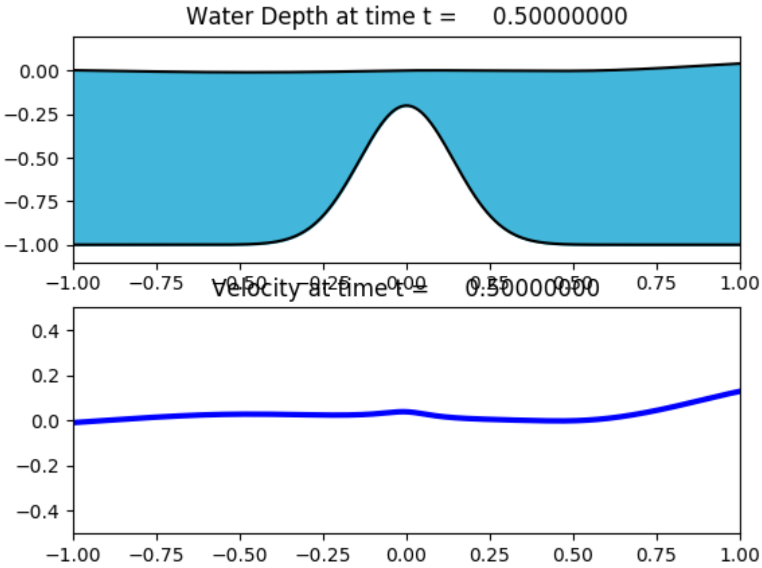

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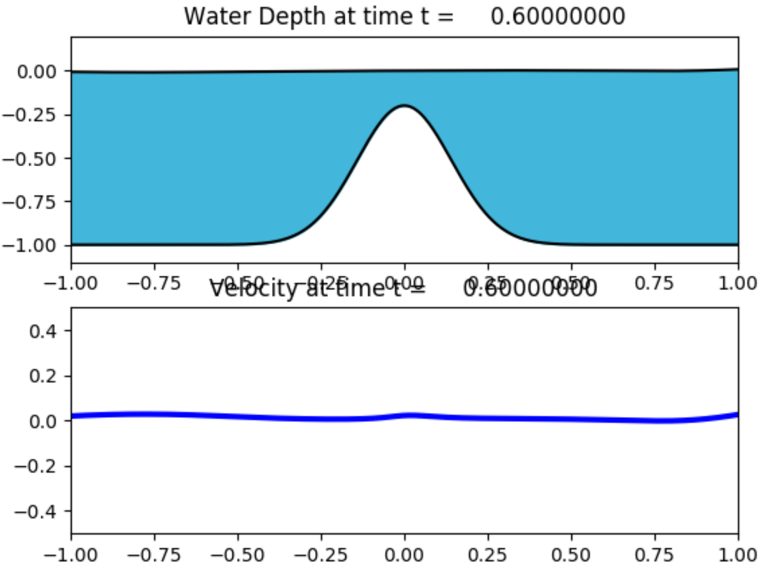

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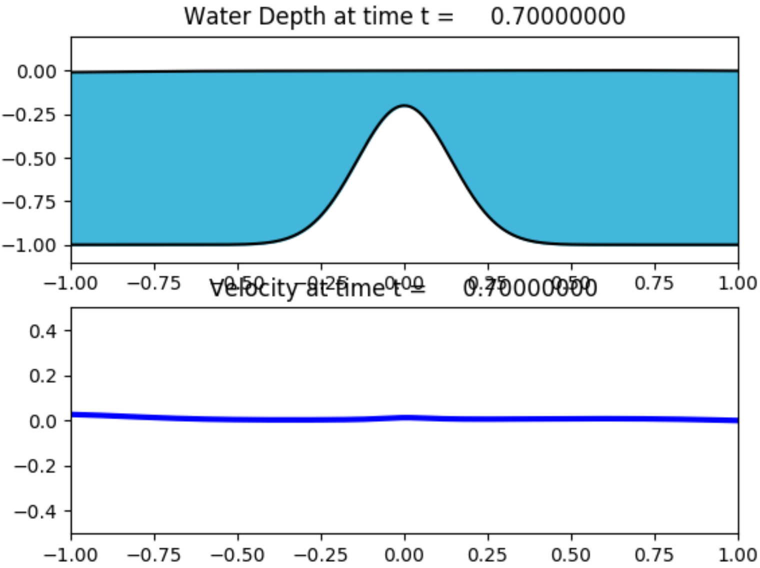
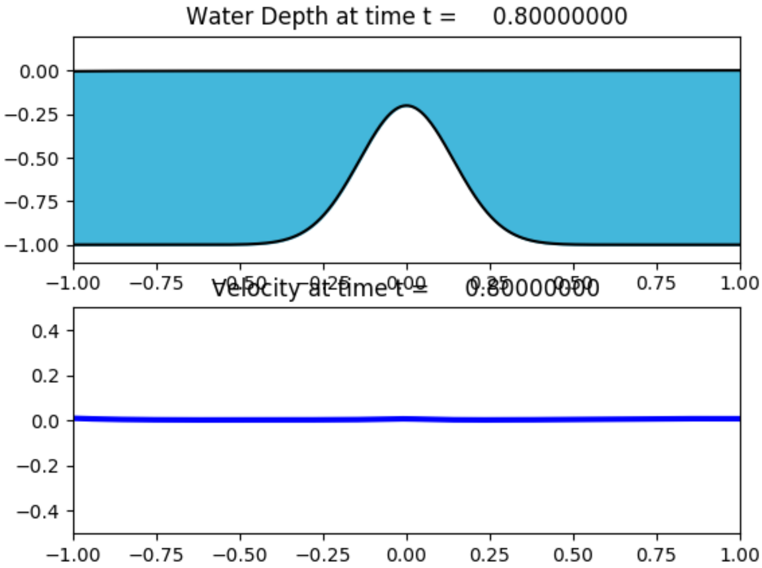


Shallow water equations with bathymetry



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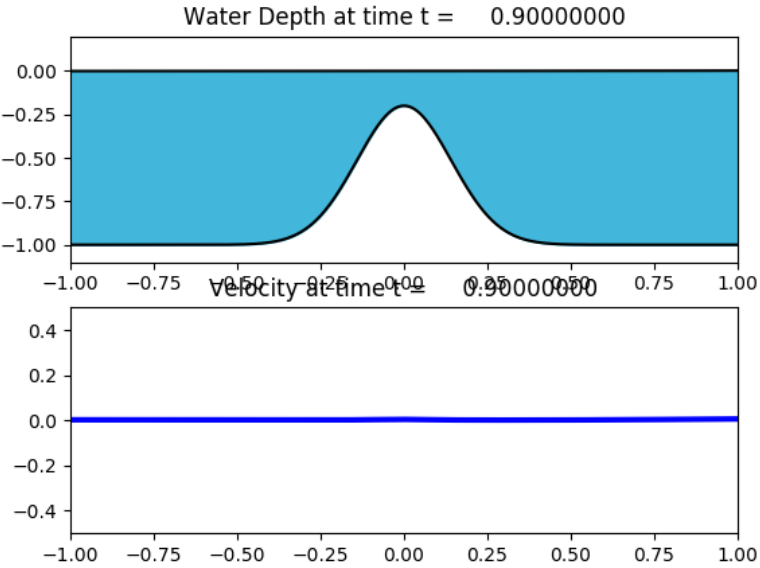
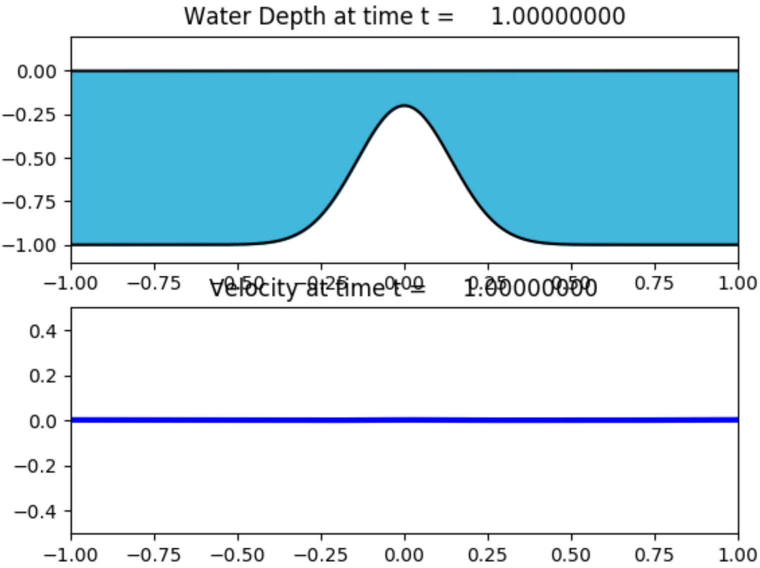


Shallow water equations with bathymetry



How to actually implement Finite Volumes

- PDE:

$$\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = 0$$

- Conservation form:

$$\frac{\partial}{\partial t} \int q(x,t)dx = f(q(x_{left},t)) - f(q(x_{right},t))$$

- We use cell averages instead of pointwise values.

$$Q = \int q(x,t)dx$$

How to actually implement Finite Volumes

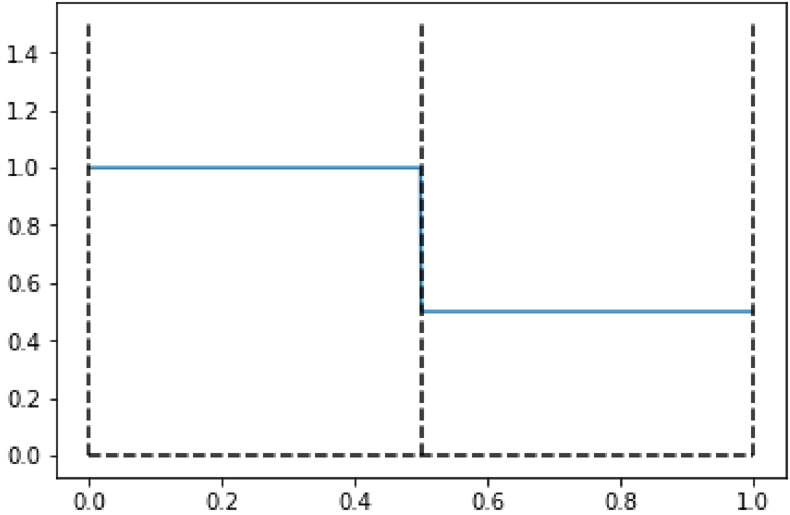


Figure 3: Start values

How to actually implement Finite Volumes

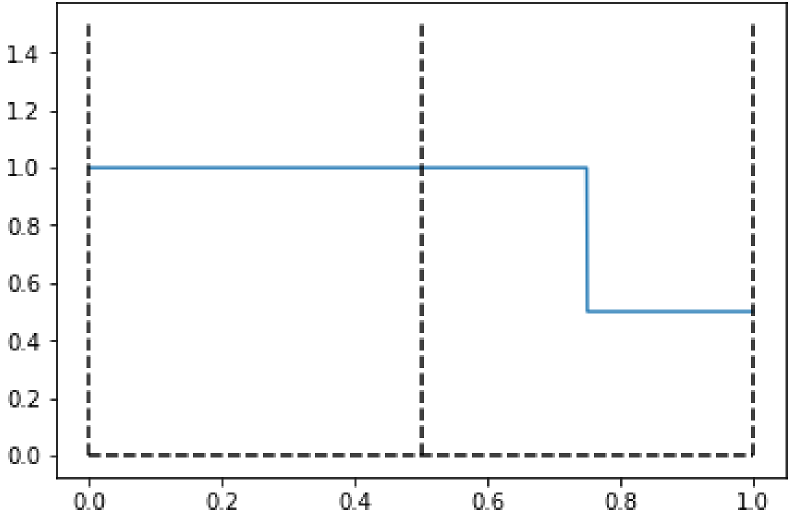


Figure 4: Estimate flow (flux) in timestep

How to actually implement Finite Volumes

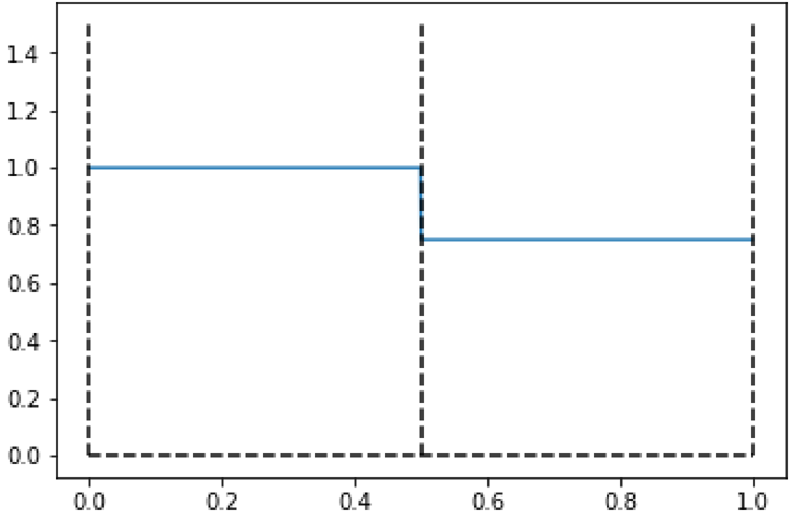
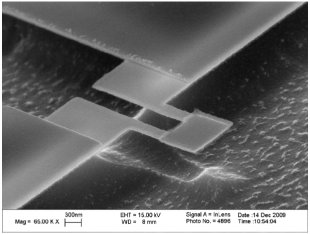


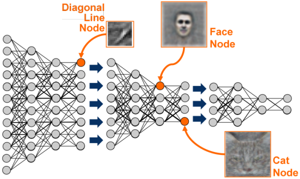
Figure 5: Update values

Outline

Areas of active research in engineering and mathematics



(a) MEMS and Nanoscale devices



(b) Deep Learning

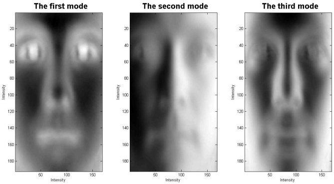


Figure 7: Singular Value Decomposition(SVD/PCA)

Resources

- General
  - www.udacity.com
  - www.coursera.com
- Scientific Computing
  - http://courses.washington.edu/am301/
  - http://faculty.washington.edu/kutz/page5/page23/
  - Spectral Methods in Matlab, Lloyd N. Trefethen
- Finite Volumes
  - Finite Volume Methods for Hyperbolic Problems, R.J. Leveque
  - Clawpack