Dynamic Inventory Rationing among Multiple Classes with Stochastic Demands and Backordering

Article *in* European Journal of Operational Research · January 2015

DOI: 10.1016/j.ejor.2015.01.022

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Closed-Form Expressions for Dynamic Inventory Rationing among Multiple Classes with Stochastic Demands and Backordering

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Abstract

This paper considers dynamic inventory rationing for systems with multiple demand classes, stationary stochastic demands, and backordering. In the literature, dynamic programming is often applied to problems of same type. However, due to the curse of dimensionality, computation remains a critical challenge for dynamic programming. In this paper, an innovative two-step approach is proposed based on an idea similar to the certainty equivalence principle. The deterministic inventory rationing problem, whose future demand is set to be the expectation of the stochastic demand process, is first studied. The important properties obtained from solving the problem with the KKT conditions are then used to come up with effective dynamic rationing policies for stochastic demands, which yields closed-form expressions for dynamic rationing thresholds. The expressions are easy to calculate and are applicable to any number of demand classes. It is shown by numerical results that the expressions are close to and provide a lower bound for the optimal dynamic thresholds. They also shed light on important managerial insights, for example, the relation between different parameters and the rationing thresholds.

Keywords: Dynamic inventory rationing, multiple demand classes, backordering, closed-form expressions, KKT conditions

1. Introduction

Customers of a product or service often have different penalty costs of shortage or service level requirements. To reduce cost or improve service, firms usually classify customers into several classes with different shortage costs and provide a different service level to each class based on certain inventory rationing policy. Applications of inventory rationing arise frequently in practice.

For instance, Dekker et al. (1998) considered a spare part used by different machines in a large petrochemical plant. The breakdown of the machines brings different losses to the firm. Hence, when the inventory of the spare part is low, the system may reject demands from less important machines with lower loss of breakdown to reserve stock for potential future demands from more important machines. Analogously, for a consumable part which has different importance to the US Navy and Army, Deshpande et al. (2003) assigned demands with different service level requirements.

Due to the importance of inventory rationing, the problem has been well addressed since the 1960s. Topkis (1968) showed that the optimal inventory rationing policy is a dynamic threshold one for discrete-time periodic review systems with zero lead time. Under such a policy, for any time in the planning horizon, there is a threshold of the on-hand inventory for each demand class such that the demand of a certain class is satisfied immediately if and only if the on-hand inventory is above the threshold. Moreover, under certain conditions, the thresholds are nondecreasing with respect to the remaining time. For two demand classes, Evans (1968) and Kaplan (1969) presented similar results.

Since then, progress has been made in different perspectives. For example, Nahmias and Demmy (1981); Cohen et al. (1988); Dekker et al. (1998); Moon and Kang (1998) analyzed service levels under static rationing policies with given ordering. Melchiors et al. (2000) introduced an approach to evaluate the cost associated with a static threshold policy for an (R, Q)-inventory model with two demand classes in a lost sales environment. For an (R, Q)-system with two demand classes and backorders, Deshpande et al. (2003) developed an approach to optimize the static thresholds and the parameters of ordering policy. Arslan et al. (2007) studied a similar problem using a model which allows for any number of demand classes. Mollering and Thonemann (2008) considered a periodic review model with two demand classes and found the optimal static thresholds under the optimal backordering clearing mechanism. Fadıloğlu and Bulut (2010b) proposed a method using embedded Markov chain to analyze an $(S-1, S)$-inventory system with two demand classes, Poisson demand, and backordering.

In most of the above-mentioned papers, the focuses are on static threshold policies, where the thresholds are assumed to be invariant over time. Dynamic inventory rationing models started to be looked into more recently. Assuming an (R, Q)-inventory policy and using a lost sales model with Poisson demand and multiple demand classes, Melchiors (2003) introduced a dynamic threshold policy where different threshold levels are allowed for different time slots. Fadıloğlu and Bulut (2010a) studied a dynamic rationing policy using the information of outstanding orders for an (R, Q)-inventory system with two demand classes and Poisson demand, complemented with simulation to investigate the benefit of the dynamic rationing policy for both lost sales and backordering models.

For a system with two demand classes, Poisson demands, and backordering, Teunter and Haneveld (2008) considered dynamic inventory rationing over a single period and developed a set of formulas to determine the optimal dynamic thresholds, which are in general not applicable to more than two demand classes. Chew et al. (2011) generalized the problem to multiple demand classes by using a one-dimensional dynamic-programming model, which is embedded into the multiple period systems with positive lead time, to eliminate the curse of dimensionality. Bounds on the optimal costs are established and the costs of the proposed dynamic policies are verified to be close to those of the optimal policies. Hung et al. (2012) extended the results to general demand processes and proposed a method to sequentially obtain the dynamic thresholds and the parameters of the ordering policies.

Besides inventory systems, another common application field of rationing is production. Ha (1997a,b, 2000); de Véricourt et al. (2002); Gayon et al. (2009) proved the optimality of static threshold policies in a variety of make-to-stock production systems. By allowing multiple production channels, Bulut and Fadiloğlu (2011) showed that the rationing thresholds and the production base stock levels are state-dependent. For a make-to-stock system with advance demand information and two demand classes, Iravani et al. (2007) illustrated that the optimal production and rationing policies are threshold-type policies. More recently, Piplani and Liu (2014) extended the concept of rationing to a make-to-order production system.

The studies of inventory rationing are mainly based on dynamic programming. The thresholds obtained from these models hence convey few insights about the quantitative relations with respect to the input parameters such as the demand rates, the penalty costs, and the remaining time. It is also difficult to evaluate the effect on the cost when a parameter cannot be estimated accurately. Moreover, the dynamic-programming approach usually encounters computational difficulties because of the curse of dimensionality.

Aimed to fill the existing gaps by deriving closed-form expressions for the dynamic thresholds, this paper proposes an innovative approach to dynamic inventory rationing based on an idea similar to the certainty equivalence principle (CEP) developed by Simon (1956) and Theil (1957) for linear systems with quadratic cost. CEP tells that for certain systems with stochastic variables for the future process, e.g., linear systems with quadratic costs, the optimal decision at any given time is equal to the optimal decision of a deterministic system whose input parameters are set to the expectation of the random variables in the stochastic counterpart.

For a system where this property does not hold, intuitively, such an approach may still obtain effective solutions. In particular, the idea of CEP has been applied to inventory models yielding closed-to-optimal solutions. For an (R, Q)-inventory system, Zheng (1992) showed that the cost incurred by using the economic order quantity (EOQ) solution is at most 1.125 times of the optimal cost, while numerical experiments indicate that the performance of the EOQ solution is significantly better than the worst-case bound. Furthermore, if the expected values are the only available information about the stochastic inputs at the time of decision making, there are few choices except for building a deterministic model using the expectations.

Based on this idea, in this paper, a method is developed to address the dynamic inventory rationing problem with stochastic demands. First, at any decision time, constant demand rates are used to replace stationary stochastic demand processes and a deterministic model is considered for inventory rationing. In this way, the original decision problems with stochastic demands are transformed into a sequence of decision problems with deterministic demands, one at every customer arrival. Each deterministic decision problem is then solved using the KKT conditions. From the solution to the deterministic decision problem, the dynamic inventory rationing thresholds are obtained for the stochastic model, which are characterized by closed-form expressions.

The method proposed in this paper yields near-optimal closed-form expressions for the dynamic thresholds, from which important managerial insights are obtained. For deterministic demands, the closed-form expressions are optimal. The new method distinguishes from those based on dynamic programming in different perspectives. For example, the seminal work of Topkis (1968) focuses on discrete-time systems, assuming independent demands across time periods and requiring complete demand distributions. It suffers from the curse of dimensionality and lacks of managerial insights as no relation can be obtained among thresholds and parameters. In contrast, the new method is applicable to a wide range of demand processes without restrictions on the number of demand classes, whereas most existing models, especially those with backordering, consider only two demand classes due to the complexity associated with the dynamic inventory rationing problems. It can also be implemented in both discrete-time and continuous-time inventory systems.

The remainder of this paper is organized as follows. In Section 2, following the description of the dynamic inventory rationing problem, the deterministic rationing problem is first formulated and solved, under the assumption that the future demand is known. The insights from the deterministic decision problem is then employed to derive closed-form expressions for dynamic thresholds, the accuracy of which is numerically demonstrated in Section 3. Section 4 summarizes the results and concludes the paper. Proofs of theorems are given in the appendix.

2. Model and Result

Similar to Topkis (1968) and Teunter and Haneveld (2008), this paper focuses on a single-period inventory system, which contains $K(\geq 2)$ customer classes differing in penalty cost of shortage. Let T_P be the length of the period. The time points in the period are labeled backward. That is, the beginning of the interval is labeled as time T_P whereas the end as time 0. The amount of the on-hand inventory is monitored constantly.

The demands from any class $k, k \in \{1, \ldots, K\}$ follow a stationary stochastic process with an expected rate d_k. That is, the expected number of demands in any unit time interval is d_k. When a demand from class k arrives, the system decides whether to satisfy it immediately or not. If it is not fulfilled, the demand is backordered and a penalty cost π_k, $k \in \{1, \ldots, K\}$ per unit per unit time is incurred. The on-hand inventory also bears a holding cost h per unit per unit time. At the end of the period, the system reorders the inventory with zero lead time and the outstanding backorders are fulfilled. Note that the assumption of zero lead time does not limit the applicability of this model. As Teunter and Haneveld (2008) pointed out, there are indeed cases of zero lead time in practice. Following an approach similar to the one outlined in Chew et al. (2011), the assumption can be relaxed as well.

Without loss of generality, assume that $\pi_i > \pi_j$ for $i < j$. As the penalty costs of different classes are different, it is natural to backlog certain demands from lower priority classes with lower penalty costs to reserve stock for future demands from higher priority classes. One type of rationing policies is to set a threshold for each class at any time. The demands from a class are satisfied at a particular time if and only if the on-hand inventory is above the corresponding threshold. Otherwise, they are backordered. The threshold-type policies have been proved to be optimal under various conditions, e.g., in Topkis (1968). In this paper, it is aimed to find well-performing thresholds with closed-form expressions.

2.1. Inventory Rationing for Deterministic Demands

First consider a deterministic rationing problem, under the assumption that the future demands from any class k, $k \in \{1, \ldots, K\}$, are constant with rate d_k. That is, the stochastic future demands are replaced with deterministic processes with the same expected arrival rates. Given the on-hand inventory at any time point, the optimal decision rule at that point is characterized by closed-form expressions.

2.1.1. Model Formulation

Let an arbitrary time $T \in [0, T_P]$ denote the remaining time to the end of the period and let s be the amount of the on-hand inventory at T. The following structural property holds for the optimal rationing policy.

Proposition 1. *Given the on-hand inventory s at time T, for any demand class k, $k \in \{1, \ldots, K\}$, there exists a time $t_k \in [0, T]$ such that it is optimal to satisfy the demands from class k at any time $t \geq t_k$ and to backorder the demands at any time $t < t_k$. Moreover, $t_i \leq t_j$ for any $i < j$ and the on-hand inventory at time t_1 must be zero if $t_1 > 0$.*

Proof. See the appendix. □

Proposition 1 shows that the time interval to backorder demands from class k is $[0, t_k)$, which implies that t_k is the threshold in time for deciding whether to satisfy demands from class k or not. Such a rationing policy is referred as the Time Threshold Policy (TTP), which is distinguished from the threshold policies in the literature, where the threshold levels are applied to the on-hand inventory. Note that t_k, the backordering time for demand class k, also determines the fill rate of class k given the inventory s at time T, i.e., $FR_k(s) = 1 - \frac{t_k}{T}$, which is an important measure of the service level provided by the inventory system.

For models with deterministic demands, an equivalent static on-hand inventory threshold policy can be obtained from a given TTP. That is, given the time threshold t_k and the initial stock s, the TTP is equivalent to an inventory threshold policy with constant on-hand inventory rationing thresholds. For stochastic demand models, the TTP is different from the classical threshold policy whose optimality has been proved in Topkis (1968).

Table 1: Comparison between the optimal and approximate policies

Case	d_1	d_2	d_3	π_1	π_2	π_3	T_P	$CD_{\max}(\%)$	$c_2(T_P)$, $\tilde{c}_2(T_P)$	$c_3(T_P)$, $\tilde{c}_3(T_P)$
1	300	300	300	27	9	3	0.08	0.78	16, 15.4	36, 35.0
2	–	–	–	10	–	–	–	0.07	2, 2.2	30, 29.7
3	–	–	–	18	–	–	–	0.30	11, 11.4	34, 33.3
4	–	–	–	36	–	–	–	1.45	18, 17.5	38, 35.8
5	–	–	–	45	–	–	–	2.26	20, 18.8	39, 36.3
6	–	–	–	63	–	–	–	4.24	22, 20.3	40, 36.9
7	–	–	–	90	–	–	–	7.70	24, 21.4	41, 37.4
8	300	300	300	27	9	2	–	1.17	16, 15.4	40, 38.2
9	–	–	–	–	–	4	–	0.60	16, 15.4	33, 31.7
10	–	–	–	–	–	6	–	0.45	16, 15.4	26, 25.2
11	–	–	–	–	–	8	–	0.40	16, 15.4	19, 18.7
12	100	300	300	27	9	3	–	1.50	6, 5.1	22, 21.3
13	200	–	–	–	–	–	–	1.04	11, 10.3	29, 28.1
14	400	–	–	–	–	–	–	0.61	21, 20.6	43, 41.8
15	500	–	–	–	–	–	–	0.49	26, 25.7	50, 48.7
16	300	300	100	27	9	3	–	0.63	16, 15.4	36, 35.0
17	–	–	200	–	–	–	–	0.72	16, 15.4	36, 35.0
18	–	–	400	–	–	–	–	0.80	16, 15.4	36, 35.0
19	–	–	500	–	–	–	–	0.81	16, 15.4	36, 35.0
20	–	–	700	–	–	–	–	0.79	16, 15.4	36, 35.0
21	–	–	900	–	–	–	–	0.76	16, 15.4	36, 35.0
22	100	100	100	27	9	3	–	3.00	6, 5.14	13, 11.7
23	200	200	200	–	–	–	–	1.34	11, 10.3	25, 23.3
24	400	400	400	–	–	–	–	0.51	21, 20.6	48, 46.6
25	500	500	500	–	–	–	–	0.36	26, 25.7	60, 58.3
26	300	300	300	27	9	3	0.04	1.91	8, 7.7	19, 17.5
27	–	–	–	–	–	–	0.12	0.43	24, 23.1	54, 52.5
28	–	–	–	–	–	–	0.14	0.34	28, 27.0	63, 61.2

explains why $CD_{\max}$ is not sensitive to the demand rate of class 3.

Cases 26-28 demonstrate the effect of the period length on the performance of the closed-form expressions. Note that the coefficients of variation of the demands decreases with the period length T_P. That is, the demand processes are closer to the deterministic ones for longer periods. Therefore, as the length of the period T_P increases, the maximal difference $CD_{\max}$ decreases and the performance of the closed-form expressions improves.

To summarize, the thresholds obtained from the closed-form expressions provide close approximations for the optimal ones, which indicates that the expressions capture the essential characteristics of the dynamic rationing thresholds. This approximate approach performs better for systems with higher demand rates, smaller ratios of two adjacent penalty costs, and longer periods. The most important factor that affects the cost difference is the ratio of penalty costs. In the above numerical study, the penalty cost ratio $\frac{\pi_1}{\pi_2}$ can be as large as 10, which cover a variety of practical situations, e..g, those considered in Deshpande et al. (2003). Furthermore, it is a common practice to use different selling prices of the product as the penalty costs for different classes. The resulting ratio between adjacent penalty costs is usually less than 5. Certainly, there exist situations where the penalty cost ratio can be large, e.g., greater than 10. In this case, the system needs to find more accurate rationing thresholds and those from the closed-form expressions can serve as a lower bound.

4. Conclusion

This paper studies dynamic inventory rationing for systems with multiple demand classes, stationary stochastic demand processes, and backordering. An innovative method is proposed based on a similar idea to the certainty equivalence principle, which yields closed-form expressions for dynamic rationing thresholds. The expressions are easy to compute and are applicable to any number of demand classes. They also provide important managerial insights.

Under the assumption of Poisson demands, numerical results show that the closed-form expressions capture the essential characteristics of optimal thresholds. The thresholds calculated using the closed-form expressions are reasonably close to the optimal ones. These approximations are more accurate and perform better for systems with smaller ratios between two adjacent penalty costs, larger demand rates, and longer periods.

Since the certainty equivalence principle has no restrictions on the demand processes and the distribution of random variables, the method may be applied to a wide range of demand processes, with continuous or discrete demands, independent or dependent demands for different classes, and stationary or nonstationary demands. It is hence interesting to investigate the accuracy of approximate expressions for demand processes other than the Poisson process presented in this paper. The method is expected to be equally effective. The closed-form expressions may also be adjusted to improve their performance for some special cases, e.g., in the case that the ratios between adjacent penalty costs are extremely large. Finally, it will be useful to develop closed-form expressions for the lost sale problems, which could be applied in airline seat rationing.

Acknowledgments

The authors gratefully thank two anonymous referees for carefully reviewing an earlier version of this paper and for providing valuable comments and suggestions, which helped to improve the paper further.

yields

$$\frac{dTC_n^*(s)}{ds}|_{s=S_{n-1}} = hT - (\pi_n + h)T = -\pi_n T.$$

By (A.3), it holds that

$$\begin{aligned}
\lim_{s\to S_{n-1}^-}\frac{dTC_{n-1}^*(s)}{ds} &= hT + 2\frac{\sum_{k=1}^{n-1} d_k T - S_{n-1}}{\sum_{k=1}^{n-1}\rho_{(n-1)k}d_k}\cdot\frac{-1}{\sum_{k=1}^{n-1}\rho_{(n-1)k}d_k}\cdot\sum_{k=1}^{n-1}\frac{(\pi_k+h)\rho_{(n-1)k}^2 d_k}{2}\\
&= hT + 2\frac{\sum_{k=1}^{n-1}\rho_{nk}d_k T}{\sum_{k=1}^{n-1}\rho_{(n-1)k}d_k}\cdot\frac{-1}{\sum_{k=1}^{n-1}\rho_{(n-1)k}d_k}\cdot\sum_{k=1}^{n-1}\frac{\rho_{(n-1)k}d_k}{2}(\pi_{n-1}+h)\\
&= hT + \frac{\sum_{k=1}^{n-1}\rho_{nk}d_k T}{\sum_{k=1}^{n-1}\rho_{nk}d_k}\cdot\frac{(\pi_n+h)\cdot(-1)}{(\pi_{n-1}+h)}(\pi_{n-1}+h)\\
&= -\pi_n T\\
&= \frac{dTC_n^*(s)}{ds}|_{s=S_{n-1}}.
\end{aligned}$$

Similarly, it can be verified that $TC^*(s)$ is differentiable at $s = S_{K-1}$ and $\frac{dTC_K^*(s)}{ds}|_{s=S_{K-1}} = -\pi_K T$.

Finally, for the differentiability of $TC^*(s)$ at $s = S_K$, clearly, by (A.3),

$$\frac{dTC_{K+1}^*(s)}{ds}|_{s=S_K} = hT$$

and

$$\begin{aligned}
\lim_{s\to S_K^-}\frac{dTC_K^*(s)}{ds} &= hT + 2\frac{\sum_{k=1}^{K} d_k T - S_K}{\sum_{k=1}^{K}\rho_{Kk}d_k}\cdot\frac{-1}{\sum_{k=1}^{K}\rho_{Kk}d_k}\cdot\sum_{k=1}^{K}\frac{(\pi_k+h)\rho_{Kk}^2 d_k}{2}\\
&= hT\\
&= \frac{dTC_{K+1}^*(s)}{ds}|_{s=S_K}.
\end{aligned}$$

This completes the proof. □

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