The Black-Scholes Model

The Concepts Underlying Black-Scholes

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
- This leads to the Black-Scholes differential equation

Derivation of the Black-Scholes Differential Equation

$$\delta S = \mu S\, \delta t + \sigma S\, \delta z$$

$$\delta f = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + ½ \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \delta t + \frac{\partial f}{\partial S} \sigma S\, \delta z$$

We set up a portfolio consisting of

$$-1: \text{ derivative}$$

$$+\frac{\partial f}{\partial S}: \text{ shares}$$

Derivation of the Black-Scholes Differential Equation continued

The value of the portfolio Π is given by

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change in its value in time δt is given by

$$\delta\Pi = -\delta f + \frac{\partial f}{\partial S} \delta S$$

The Derivation of the Black-Scholes Differential Equation continued

The return on the portfolio must be the risk - free rate. Hence

$$\delta\Pi = r\,\Pi\,\delta t$$

We substitute for δf and δS in these equations to get the Black - Scholes differential equation :

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

The Differential Equation

- Any security whose price is dependent on the stock price satisfies the differential equation
- The particular security being valued is determined by the boundary conditions of the differential equation
- Do you recognize this PDE?

Risk-Neutral Valuation

- The variable μ does not appear in the Black-Scholes equation
- The equation is independent of all variables affected by risk preference
- The solution to the differential equation is therefore the same in a risk-free world as it is in the real world
- This leads to the principle of risk-neutral valuation

Applying Risk-Neutral Valuation: Stochastic Simulation

1. Assume that the expected return from the stock price is the risk-free rate
2. Calculate the expected payoff from the option
3. Discount at the risk-free rate

Topic of Part II of this Course

Applying Risk-Neutral Valuation: Numerical Methods

1. Determine Boundary Conditions for Black-Scholes PDE
2. Discretize Black-Scholes PDE
3. Solve System of Equations Iteratively

Topic of Part III of this Course

Example: Pricing Forward Contracts using PDE

- Any security whose price is dependent on the stock price satisfies the differential equation
- In a forward contract the boundary condition is $f = S - K$ when $t = T$
- The solution to the equation is

$$f = S - K e^{-r(T-t)}$$

- At $t = 0$ the value of the forward contract is

$$f = S - K e^{-rT}$$

Example: Pricing Forward Contracts using Expectation in Risk-Neutral Measure

Valuation Formula

$$f = e^{-rT} E[(S_T - K)] = S_0 - Ke^{rT}$$

Recall From Binomial Tree Method

$$E[S_T] = pS_u + (1-p)S_d$$

$$E[S_T] = \tfrac{e^{rT}-d}{u-d} uS_o + \tfrac{u-e^{rT}}{u-d} dS_o$$

$$E[S_T] = e^{rT} S_0$$

The Stock Price Assumption

- Consider a stock whose price is *S*
- In a short period of time of length δt, the return on the stock is normally distributed:

$$\frac{\delta S}{S} \approx \phi\left(\mu \delta t, \sigma \sqrt{\delta t}\right)$$

where μ is expected return and σ is volatility

The Lognormal Property

- It follows from this assumption that

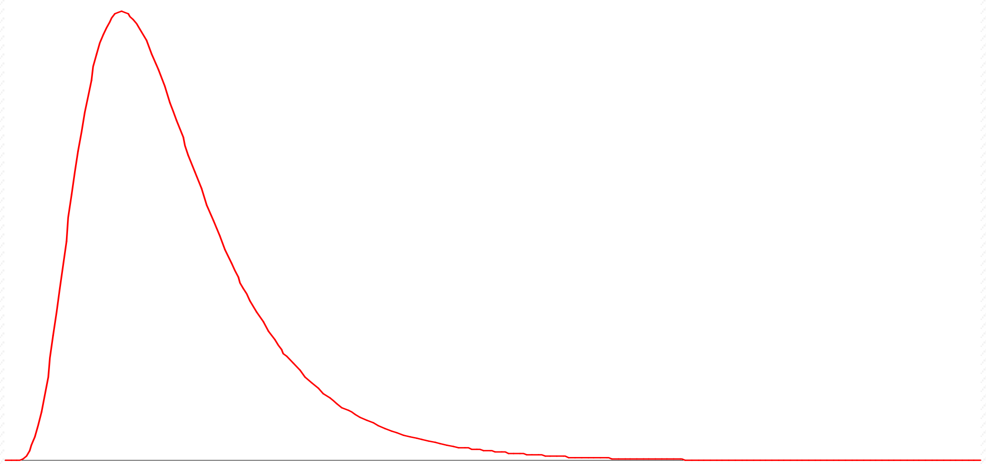
$$\ln S_T - \ln S_0 \approx \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

or

$$\ln S_T \approx \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

- Since the logarithm of S_T is normal, S_T is lognormally distributed

The Lognormal Distribution



$$E(S_T) = S_0\, e^{\mu T}$$

$$\text{var}(S_T) = S_0^{\ 2} e^{2\mu T} (e^{\sigma^2 T} - 1)$$

The Black-Scholes Formulas

$$c = S_0\, N(d_1) - K\, e^{-rT} N(d_2)$$

$$p = K\, e^{-rT}\; N(-d_2) - S_0\, N(-d_1)$$

$$\text{where } \; d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / K) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Input Black-Scholes Formula

Two types of input parameters are required for the Black-Scholes Formula:

- **Option contract parameters**
- **Market parameters such as Volatility**

The Volatility

- The volatility of an asset is the standard deviation of the continuously compounded rate of return in 1 year
- As an approximation it is the standard deviation of the percentage change in the asset price in 1 year

Estimating Volatility from Historical Data

- Take observations $S_0, S_1, \ldots, S_n$ at intervals of τ years
- Calculate the continuously compounded return in each interval as:

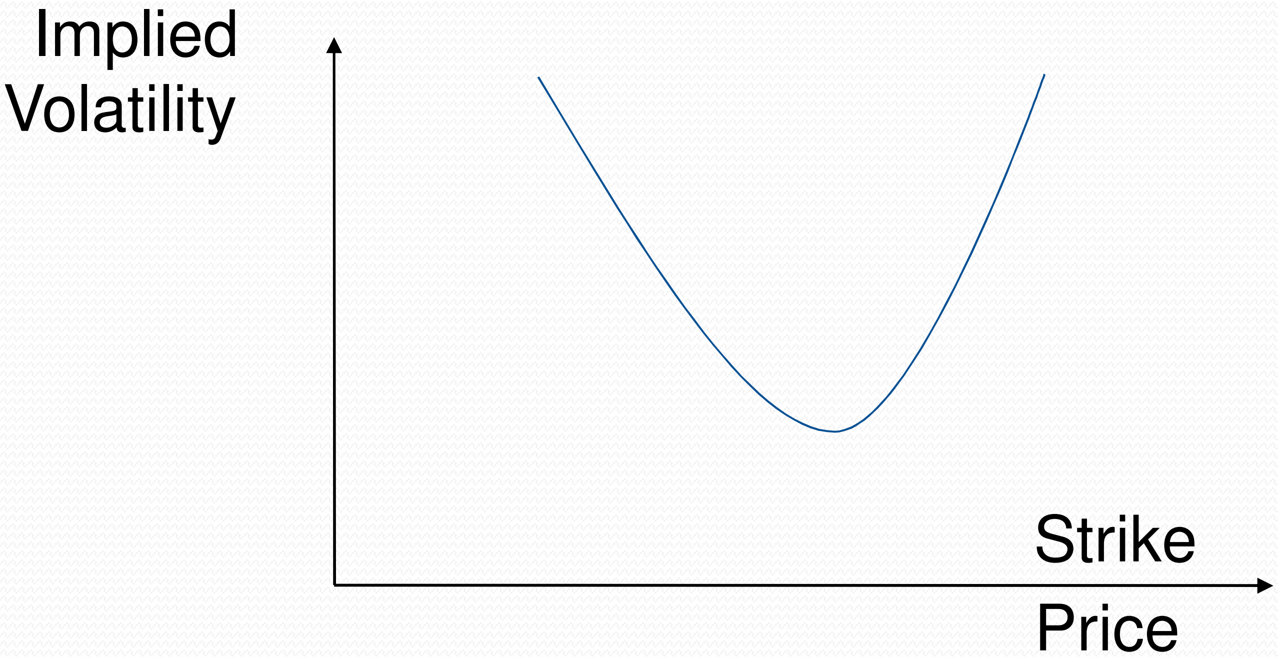
$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

- Calculate the standard deviation, s , of the u_i′s
- The historical volatility estimate is: $\hat{\sigma} = \frac{s}{\sqrt{\tau}}$

Implied Volatility

- The implied volatility of an option is the volatility for which the Black-Scholes price equals the market price
- The is a one-to-one correspondence between prices and implied volatilities
- Traders and brokers often quote implied volatilities rather than dollar prices

The Volatility Smile for Foreign Currency Options



Implied Distribution for Foreign Currency Options

- Both tails are heavier than the lognormal distribution

- It is also “more peaked than the lognormal distribution