Finite-Difference Techniques for Financial Derivatives - part I

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Scope of the lecture

Why one should use Finite Difference

- For small dimensions, Finite Difference may be faster than Monte Carlo (even with variance reduction techniques).
- It can handle early exercise options and complex boundaries and barriers.
- Finite Differences are ideally for computing some Greeks
- Disadvantage: Method is not feasible for high dimensional problems ≥ 4

Derivation - I

- Assume Geometric Brownian motion is driving the underlying stock dynamics:

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{1}$$

- We want to determine the price of an option: $V(S,t)$.
- Apply Itô calculus to (1):

$$dV_t = \left(\mu S_t \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW_t. \tag{2}$$

Derivation - II

- Use an arbitrage argument:
- set up a portfolio with value Π by selling one derivative and buy $\frac{\partial V}{\partial S}$ shares:

$$\Pi_t = -V_t + \frac{\partial V}{\partial S} S_t. \tag{3}$$

- We assume that there is a risk-free rate r such that:

$$d\Pi_t = -dV_t + \frac{\partial V}{\partial S} dS_t. \tag{4}$$

- Now we can substitute equations (1) and (2) into (4).

Derivation - III

- Result:

$$d\Pi_t = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt. \tag{5}$$

- Together with $d\Pi_t = r\Pi_t dt$, we get:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV. \tag{6}$$

The Standard Black-Scholes Equation

The PDE is linear-hyperbolic

$$\begin{cases} \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV. \\ \qquad\qquad + \\ \text{Initial \& Boundary conditions.} \end{cases}$$

- V: Asset value
- S: Underlying value
- r: Risk-free (or arbitrage free) interest rate
- σ: Volatility (standard deviation) of price (movement)

Risk Terminology for Black-Scholes

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV.$$

- Theta (Θ) is a time-rate term
- Delta (Δ) is the convective term
- Gamma (Γ) is the diffusion term

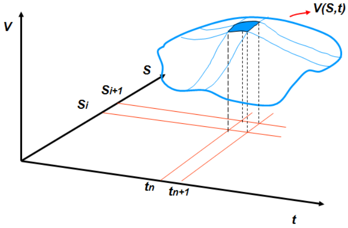
So in risk parameters, the PDE becomes:

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV.$$

Finite Difference Procedure

We are looking for a surface in 3 dimensions

- Divide interval $[0, T]$ into N equal sized subintervals equidistantly
- Divide interval $[0, S]$ into M equal sized subintervals, also equidistantly



Boundary Conditions - I

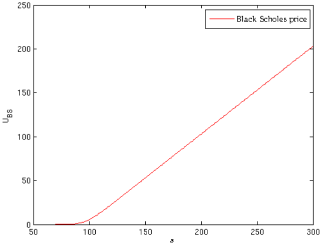
- Finite number of grid points imply Boundary Conditions (BC)
- BCs are deduced by looking at the nature of derivatives
- For example a European Call:

$$\begin{aligned} V(S,t) =& N(d_1)S - N(d_2)Ke^{-r(T-t)}, \\ d_{1,2} =& \frac{\ln \frac{S}{K} + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

- Note that the limiting behavior should be model independent!

Boundary Conditions-II

- At $S = 0$ we have: $V(0, \tau) = 0$
- $\lim_{S\to\infty} V(S, \tau) = S - Ke^{-r(T-\tau)} \approx S$
- Also possible to take the derivative: $\lim_{S\to\infty} \frac{\partial V(S,\tau)}{\partial S} = 1$



Black-Scholes PDE

- Boundary Value Problem (BVP):

$$\begin{cases} \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV, \\ V(0,t) = 0, \\ V(S_{\text{max}}, t) = S_{\text{max}}, \\ V(S,0) = \phi(S). \end{cases}$$

- Now we have a Initial Boundary Value Problem (IBVP)
- Problem: Don't know the option value at $t = 0$, what now?

Payoff Function of European Call-Option

- We do know the option price at expiry!
- At expiry the final payoff of a European call option equals

$$\max(0, S_T - K).$$

- Time can be reversed (similar as in binomial tree method): $\tau := T - t$
- Final payoff becomes initial payoff
- In case of European Call-option:

$$\phi(S) = \max(0, S_{\tau=0} - K).$$

Transformation of Black-Scholes PDE

- PDE can be transformed to constant coefficient PDE
- Introduce:

$$X = \ln S.$$

- The Black-Scholes PDE then turns into:

$$\frac{\partial V}{\partial \tau} = \left(r - \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial X} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial X^2} - rV.$$

- This equation is solved numerically via a discrete variant

Taylor Expansion Techniques - I

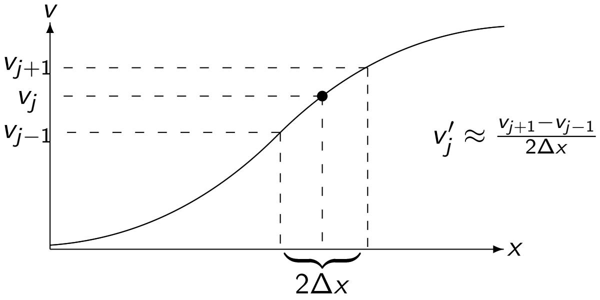
- For example the 1-d expansion of the option value in the underlying value
- Let $v_j^n = V(X_j, \tau_n)$, $X_{j+1} - X_j = \Delta x$ and let $\left(\frac{\partial V}{\partial x}\right)_j^n = (v_j^n)'$, then two directions:

$$\begin{array}{r}
v_{j+1}^n = v_j^n + \Delta x_j (v_j^n)' + \dfrac{\Delta x_j^2}{2}(v_j^n)'' + \ldots, \\
v_{j-1}^n = v_j^n - \Delta x_j (v_j^n)' + \dfrac{\Delta x_j^2}{2}(v_j^n)'' - \ldots -, \\
\hline
v_{j+1}^n - v_{j-1}^n = 2\Delta x_j (v_j^n)' + \mathcal{O}(\Delta x^3),
\end{array}$$

$$\Leftrightarrow (v_j^n)' = \frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} + \mathcal{O}(\Delta x^2).$$

Taylor Expansion Techniques-II

Visually the approximation looks as follows:



Approximations of Other Derivatives

- Forward approximation of first derivative with respect to time:

$$\left(\frac{\partial V}{\partial \tau}\right)_j^n \approx \frac{v_j^{n+1} - v_j^n}{\Delta \tau},$$

- central approximation of first derivative with respect to space:

$$\left(\frac{\partial V}{\partial x}\right)_j^n \approx \frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x},$$

- central approximation of second derivative with respect to space:

$$\left(\frac{\partial^2 V}{\partial x^2}\right)_j^n \approx \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta x^2}.$$

Matrix Representation - I

- We have expressions for local derivatives in linear combinations of neighboring points
- If we take a vector $\vec{v}$ which contains all our grid points, we have:

$$\frac{\partial V}{\partial X} \approx A\vec{v}, \tag{7}$$

$$\begin{pmatrix} v'_1 \\ v'_2 \\ \vdots \\ \vdots \\ \vdots \\ v'_M \end{pmatrix} \approx \frac{1}{2\Delta x} \begin{pmatrix} k_1 & k_2 & 0 & & & \\ 1 & 0 & -1 & & & \\ 0 & 1 & 0 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & -1 \\ & & & 0 & k_3 & k_4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ \vdots \\ \vdots \\ v_M \end{pmatrix}$$

- What do we do at the boundaries: k_1, k_2, k_3 and k_4?

Boundary Conditions-III

- Information now needs to be stored in FD framework
- In the example at $S = 0$, we know the option price $\Rightarrow$ Dirichlet boundary condition
- At $S = S_{\max}$ we know the approximated option value *and* the derivative $\Rightarrow$ Neumann boundary condition
- So in these limiting cases we have:

$$V(0,\tau) = 0 \text{ and}$$
$$\lim_{x \to x_{\max}} V(x,\tau) = e^x.$$

Boundary Conditions-IV

- From Dirichlet bounday condition, at $S = 0$, $v(0, \tau) = 0$. So, $v'_1 = 0$.
- From Von Neumann bounday condition, at $S = S_{\text{max}}$, $V(x_{\text{max}}, \tau) = e^{x_{\text{max}}}$. Therefore, $v'_M = e^{x_{\text{max}}}$
- In matrix notation, this can be represented as:

$$\begin{pmatrix} v'_1 \\ v'_2 \\ \vdots \\ \vdots \\ v'_M \end{pmatrix} \approx \frac{1}{2\Delta x} \begin{pmatrix} 0 & 0 & 0 & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ \vdots \\ v_M \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ e^{X_{\text{max}}} \end{pmatrix}$$

Boundary Conditions-V

- If we want to use the Neumann BC, we still need the second derivative!
- idea: Use information about the first derivative to approximate the second:

$$\frac{v_{M+1}-v_{M-1}}{2\Delta x} = e^{X_{\max}} \Rightarrow v_{M+1} = 2\Delta x e^{X_{\max}} + v_{M-1},$$
$$\frac{v_{M+1}-2v_M+v_{M-1}}{\Delta x^2} = \frac{2\Delta x e^{X_{\max}} - 2v_M + 2v_{M-1}}{\Delta x^2}.$$

$$\begin{pmatrix} v_1'' \\ v_2'' \\ \vdots \\ \vdots \\ v_M'' \end{pmatrix} \approx \frac{1}{\Delta x^2}\begin{pmatrix} 0 & 0 & 0 & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ \vdots \\ v_M \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \frac{2}{\Delta x} e^{X_{\max}} \end{pmatrix}$$

Matrix Representation-II

- Now all spatial derivatives are approximated by a matrix vector multiplication
- Stored in one matrix A one has:

$$\frac{\partial \vec{v}}{\partial \tau} = A\vec{v} + \vec{k}.$$

- Now we approximated the spatial derivatives and incorporated the boundary conditions
- Only the time dimension needs to be tackled

The Euler Forward scheme

Substituting the former gives the FTCS (Forward in Time, Central in Space)

$$\frac{v_j^{n+1} - v_j^n}{\Delta\tau} = \left(r - \frac{\sigma^2}{2}\right)\frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} + \frac{\sigma^2}{2}\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta x^2} - rv_j^n,$$

or in matrix notation:

$$\frac{1}{\Delta\tau}\left(\vec{v}^{n+1} - \vec{v}^n\right) = A\vec{v}^n + \vec{k},$$
$$\Rightarrow \vec{v}^{n+1} = \vec{v}^n + \Delta\tau\left(A\vec{v}^n + \vec{k}\right).$$

The Euler Backward scheme

Another option is to use the Euler Backward scheme:

$$\begin{aligned} \frac{1}{\Delta\tau}\left(\vec{v}^{n+1} - \vec{v}^{n}\right) &= A\vec{v}^{n+1} + \vec{k}, \\ \Rightarrow (I - \Delta\tau A)\,\vec{v}^{n+1} &= \vec{v}^{n} + \Delta\tau\vec{k}. \end{aligned}$$

- if you want a very fast method, which would you prefer?

Time Discretizations

- Euler forward: $\vec{v}^{n+1} = \vec{v}^{n} + \Delta\tau \left(A\vec{v}^{n} + \vec{k} \right)$
  1. Solution can be obtained explicitly
  2. Severe stability condition on $\Delta\tau$
- Euler backward: $\vec{v}^{n+1} = \vec{v}^{n} + \Delta\tau \left(A\vec{v}^{n+1} + \vec{k} \right)$
  1. Unconditionally stable, also for large values of $\Delta\tau$
  2. Solution can only be obtained implicitly, so matrix inversion is needed
- θ-scheme uses a weighted average of implicit and explicit:

$$\frac{1}{\Delta\tau}\left(v^{n+1} - v^{n}\right) = \theta \left(Av^{n} + \vec{k}^{n} \right) + (1-\theta)\left(Av^{n+1} + \vec{k} \right).$$

- This θ-scheme can be second order and stable!

Lax-Equivalence Theorem

Theorem

A finite difference approximation converges (towards the solution of the PDE) if and only if

- *The scheme is consistent (for $d\tau \to 0$ and $dx \to 0$ the difference scheme agrees with the original differential equation)*
- *The difference scheme is stable*

- So we can show convergence by showing:
  1. Consistency (Taylor Expansions)
  2. Stability (von Neumann Stability Analysis)

Consistency

- Approximations of derivatives will converge to actual derivative value.
- Order of this convergence for every derivative can be derived from the Taylor Expansions:

$$\frac{\partial}{\partial s} v_j = \frac{v_{j+1} - v_{j-1}}{2\Delta s} + \mathcal{O}(\Delta s^2).$$

Similar technique for second derivative:

$$\frac{\partial^2}{\partial s^2} v_j = \frac{v_{j+1} - 2v_j + v_{j-1}}{\Delta s^2} + \mathcal{O}(\Delta s^2).$$

Analysing Order of Convergence - I

- Order of convergence $\Rightarrow$ substitute Taylor Expansions
- Consider the θ-scheme for the convection equation:

$$
\begin{aligned}
\frac{\partial v}{\partial t} &= a\frac{\partial v}{\partial x}, \\
\frac{v_j^{n+1} - v_j^n}{\Delta t} &= \frac{v_j^n + \Delta t\,(v_t)_j^n + \frac{\Delta t^2}{2!}\,(v_{tt})_j^n + \mathcal{O}(\Delta t^3) - v_j^n}{\Delta t}, \\
&= (v_t)_j^n + \frac{\Delta t}{2!}\,(v_{tt})_j^n + \mathcal{O}(\Delta t^2), \\
\frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} &= (v_x)_j^n + \frac{\Delta x^2}{3!}\,(v_{xxx})_j^n + \mathcal{O}(\Delta x^3), \\
\frac{v_{j+1}^{n+1} - v_{j-1}^{n+1}}{2\Delta x} &= (v_x)_j^n + \Delta t\,(v_{xt})_j^n + \mathcal{O}(\Delta x^2, \Delta t^2).
\end{aligned}
$$

Analysing Order of Convergence-II

- Since θ-scheme is a weighted average of implicit and explicit schemes:

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = a\left((1-\theta)\frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} + \theta\frac{v_{j+1}^{n+1} - v_{j-1}^{n+1}}{2\Delta x}\right).$$

- Substituting terms from Taylor series expansion in last slide, we have:

$$(v_t)_j^n + \frac{\Delta t}{2!}(v_{tt})_j^n + \mathcal{O}(\Delta t^2) = a[(1-\theta)((v_x)_j^n + \frac{\Delta x^2}{3!}(v_{xxx})_j^n$$
$$+\mathcal{O}(\Delta x^3)) + \theta((v_x)_j^n + \Delta t\,(v_{xt})_j^n + \mathcal{O}(\Delta x^2, \Delta t^2))]$$

- Ignoring Δx and Δt terms with order 2 and higher, we obtain:

$$(v_t)_j^n + \frac{\Delta t}{2}(v_{tt})_j^n = a[(1-\theta)(v_x)_j^n + \theta((v_x)_j^n + \Delta t\,(v_{xt})_j^n)]$$
$$+\mathcal{O}(\Delta x^2, \Delta t^2)$$

Analysing Order of Convergence-III

- Rearranging terms and ignoring sub- and superscripts n, j for v_t, v_{tt}, v_x, v_{xt}, we obtain:

$$v_t = av_x - \Delta t(\frac{1}{2}v_{tt} - a\theta v_{xt}) + \mathcal{O}(\Delta x^2, \Delta t^2).$$

- What is the order of $\frac{1}{2}v_{tt} - a\theta v_{xt}$?
- For $\theta = \frac{1}{2}$, $\frac{1}{2}v_{tt} - a\theta v_{xt} = \frac{1}{2}(v_{tt} - av_{xt})$. From the last equation on the previous slide:

$$v_t = av_x + \mathcal{O}(\Delta t, \Delta x^2) \Leftrightarrow$$
$$\frac{\partial}{\partial t}(v_t - av_x) = v_{tt} - av_{xt} = \mathcal{O}(\Delta t, \Delta x^2).$$

- Note that differentiating the error term $(\mathcal{O}(\Delta t, \Delta x^2))$ will not change the order of error.

von Neumann Stability Analysis - I

- Can we use $\theta = \frac{1}{2}$?
- Idea is to limit the growth of the error:

$$\begin{aligned} v_j^n &= v(j\Delta s, n\Delta\tau) + \epsilon_j^n \\ v_j^n &: \text{ Computed Solution from FD Scheme} \\ v(j\Delta s, n\Delta\tau) &: \text{ Exact Solution} \\ \epsilon_j^n &: \text{ Error at time level } n \text{ mesh point } j \end{aligned}$$

- The idea is to bound the error ϵ_j^n as one advances in time

von Neumann Stability Analysis-II

- Now introduce:

$$\begin{aligned}\epsilon_j^n = e^{\omega n \Delta t} e^{jki\Delta s},\\ i = \sqrt{-1},\\ k \in [0, 2\pi].\end{aligned}$$

- All methods should advance in time to the desired solution, so look at growth of the exponential by substituting $v_j^n = e^{n\Delta t} e^{jki\Delta s}$ in

$$\begin{aligned}\frac{v_j^{n+1} - v_j^n}{\Delta \tau} = \left(r - \frac{\sigma^2}{2}\right) \frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x},\\ + \frac{\sigma^2}{2} \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta x^2} - r v_j^n.\end{aligned}$$

von Neumann stability analysis-III

- Now we can construct a recursive formula for the amplification factor $\rho = \frac{e^{\omega(n+1)\Delta t}}{e^{\omega n \Delta t}} = e^{\omega \Delta t}$
- Now we want to make sure that $|\rho|$ doesn't explode
- For example the θ-scheme for the convection equation:

$$\frac{\partial v}{\partial t} = a\frac{\partial v}{\partial x},$$

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = a\left((1-\theta)\frac{v_{j+1}^n - v_{j-1}^n}{\Delta x} + \theta\frac{v_{j+1}^{n+1} - v_{j-1}^{n+1}}{\Delta x}\right).$$