Monte-Carlo Methods in Derivative Finance

Basics
Numerical Integration of Stochastic Differential Equations

Risk-Neutral Valuation of a Call Option

$$c = e^{-rT}\mathrm{E}[MAX(S_T - K, 0)]$$

1. The expected return from the stock price is the risk-free rate
2. Calculate the expected payoff from the option
3. Discount at the risk-free rate

Calculation of Expectation

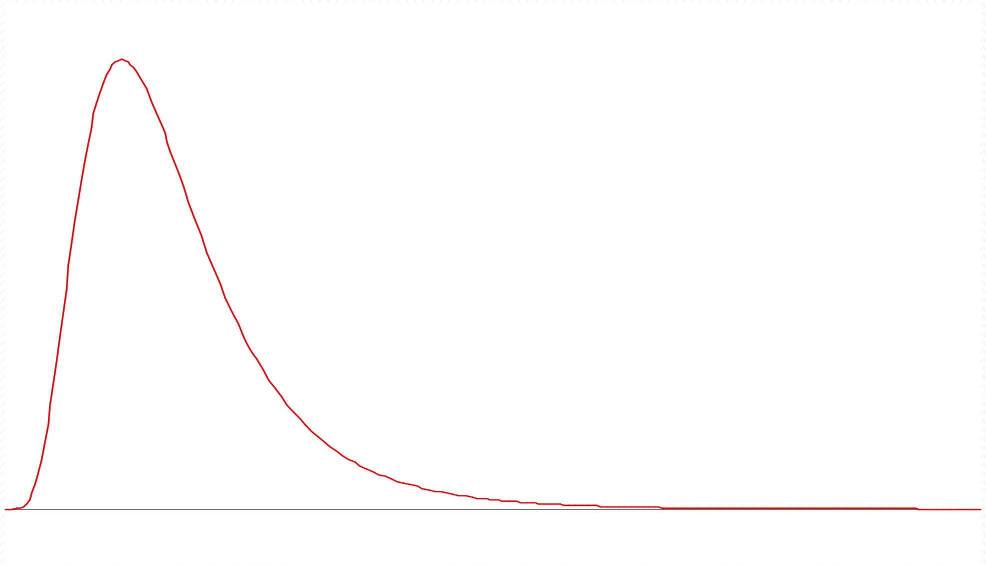
- Expectation of the payoff in risk-neutral measure

$$\mathrm{E}[MAX(S_T - K, 0)] = \int_0^\infty MAX(S_T - K, 0) f(S_T) dS_T$$

- Note that the probability density function of the stock price follows a lognormal distribution

$$\ln S_T - \ln S_0 \approx \phi\left[\left(r - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

The Lognormal Distribution



$$E(S_T) = S_0\, e^{rT}$$

$$\operatorname{var}(S_T) = S_0^{\,2} e^{2rT} (e^{\sigma^2 T} - 1)$$

Monte Carlo Simulation

When used to value European stock options, this involves the following steps:

1. Simulate 1 path for the stock price in a risk neutral world
2. Calculate the payoff from the stock option
3. Repeat steps 1 and 2 many times to get many sample payoff
4. Calculate mean payoff
5. Discount mean payoff at risk free rate to get an estimate of the value of the option

Properties of Monte Carlo Estimate

$$\mathrm{E}[MAX(S_T - K, 0)] \approx \tfrac{1}{N}(\sum_{i=1}^{N} MAX(S_i - K, 0) = \Theta$$

- MC estimate converges to true value *(Law of Large Numbers)*
- MC estimate is asymptotically normally distributed *(Central Limit Theorem)*
- For large N, standard deviation of MC estimate is given by:

$$\frac{\sigma(payoff)}{\sqrt{N}}$$

Monte Carlo Simulations: Features

- MC can deal relatively easy with options with complex payoffs
- Path dependent options
- Supports variety of Stochastic Processes
- Extremely useful for high-dimensional problems

Path-Dependent Options

- **Barrier Options: A down-and-out call has a payoff of zero if the asset crosses some predefined barrier $B < S_0$ at some time in $[0,T]$ and otherwise the payoff becomes $MAX(S_T-K,0)$**

- **Asian Options: An Asian option has a payoff of a call option where the underlying rate is the average of the asset price over a time-window**

- **These options depends on the asset dynamics**

Monte Carlo Simulations: Shortcomings

- Numerical Simulation of SDE can be tricky
- MC cannot easily deal with American-style options
- MC is slow for low dimensional problems (Use Variance Reduction Techniques)
- Unstable estimates for the Greeks when discontinuous payoffs are considered

Simulation of SDE: Euler Scheme

- Geometric Brownian Motion

$$dS = rS\ dt + \sigma S\ dz$$

- Simulate a path by choosing time steps of length δt and using the discrete version

$$\tilde{S}(t+\delta t) = \tilde{S}(t) + r\tilde{S}(t)\delta t + \sigma\tilde{S}(t)\,\varepsilon\sqrt{\delta t}$$

where ε is a random sample from φ(0,1)

Sampling from Normal Distribution

- One simple way to obtain a sample
  from $\phi(0,1)$ is to generate 12 random numbers between 0.0 & 1.0, take the sum, and subtract 6.0
- Use e.g. Inverse Transform Methods *(Course 'Distributed Stochastic Simulations')*

A More Accurate Approach

Use $d \ln S = \left(r - \sigma^2/2\right)dt + \sigma\, dz$

The multi-step discrete version is

$$\ln \tilde{S}(t + \delta t) - \ln \tilde{S}(t) = \left(r - \sigma^2/2\right)\delta t + \sigma\varepsilon\sqrt{\delta t}$$

Of course it can be simulated in a single-step

$$\tilde{S}(T) = S(0)\, e^{\left(r - \sigma^2/2\right)T + \sigma\varepsilon\sqrt{T}}$$

Convergence of the Numerical Discretisation

Strong Convergence

- **Important when trajectory itself is important**
- **Path-dependent options**

$$E[|S(T)-\tilde{S}_{\delta}(T)|]\leq c_s\delta^{\gamma}$$

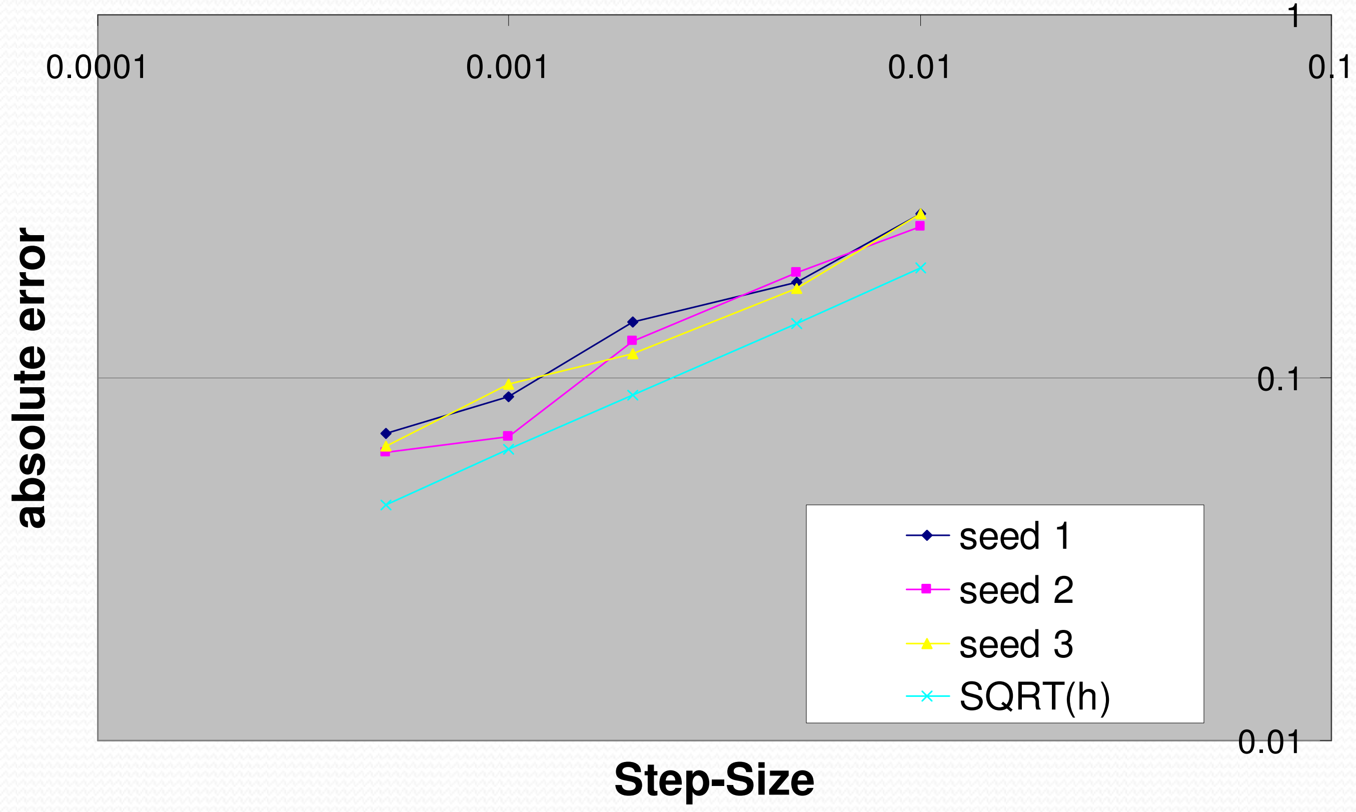
Convergence of the Numerical Discretisation

Weak Convergence

- Pointwise approximation of *S(T)* is not real aim but proxy its moment(s)
- European option valuations

$$\left| E[g(S(T))] - E[g(\tilde{S}_{\delta}(T))] \right| \leq c_w \delta^{\beta}$$

Strong Convergence of Euler Scheme (γ=0.5)



Spurious Paths in Euler Scheme

Recall Euler Scheme

$$\tilde{S}(t+\delta t)=\tilde{S}(t)+r\tilde{S}(t)\delta t+\sigma\tilde{S}(t)\,\varepsilon\sqrt{\delta t}$$

$\tilde{S}(t+\delta t)$ will become negative if

$$\varepsilon<-\frac{1+r\delta t}{\sigma\sqrt{\delta t}}$$

Milstein Scheme

$$\mathrm{dS} = \mathrm{a(S, t)dt} + \mathrm{b(S, t)dW}$$

Since error is dominated by diffusion term
we should improve its discretisation by adding a
correction term

$$\frac{1}{2} b \frac{\partial b(t,S)}{\partial S}\left(\varepsilon^2 - 1\right)\delta t$$

$$S(t+\delta t) = S(t)\left(1 + (r - \tfrac{1}{2}\sigma^2)\delta t + \sigma\varepsilon\sqrt{\delta t} + \tfrac{1}{2}\sigma^2\varepsilon^2\delta t\right)$$

Milstein Scheme

- Strong convergence of order h **(γ=1)**
- However, difficult to extend to multiple dimensions

Transformation of stochastic variables

$$\mathrm{dv} = \mathrm{a(t, v)dt} + \mathrm{b(t, v)dW}$$

Consider $\mathrm{u} = \mathrm{F(v)}$

$$\mathrm{du} = \left(\frac{\partial \mathrm{F}}{\partial \mathrm{v}} a(t,v) + \frac{1}{2} \frac{\partial^2 \mathrm{F}}{\partial \mathrm{v}^2} b^2(t,v) \right) dt + \frac{\partial \mathrm{F}}{\partial \mathrm{v}} b(t,v) dW$$

Choose F such that $\dfrac{\partial \mathrm{F}}{\partial \mathrm{v}} b(t,v)$ is constant and

this diffusion term becomes simple

Euler on Transformed SDE

$$\frac{\partial \mathrm{F}}{\partial \mathrm{v}} = \frac{1}{b(t,v)}$$

$$\frac{\partial^2 \mathrm{F}}{\partial \mathrm{v}^2} = -\frac{1}{b^2(t,v)} \frac{\partial b(t,v)}{\partial v}$$

$$du = \left(\frac{a(t,v)}{b(t,v)} - \frac{1}{2} b^2(t,v) \frac{\partial b(t,v)}{\partial v} \right) dt + dW$$

Apply Euler Method on Transformed SDE

Example: GBM Process

Geometric Brownian Motion

$$dS = rSdt + \sigma SdW$$

$$\frac{\partial F}{\partial v} \propto \frac{1}{S}$$

$$F = \ln(S)$$

Example: CIR Process

$$\text{Mean - Reverting Square - Root Process}$$

$$\mathrm{dv} = \mathrm{a}(\theta - \mathrm{v})\mathrm{dt} + \lambda\sqrt{v}\,\mathrm{dW}$$

$$\frac{\partial \mathrm{F}}{\partial \mathrm{v}} \propto \frac{1}{\lambda\sqrt{v}}$$

$$F = \sqrt{v}$$

Monte-Carlo Methods in Derivative Finance

Multi-factor Models
Variance Reduction Techniques

Multi-Asset Options

- When a derivative depends on several underlying variables we can simulate paths for each of them in a risk-neutral world to calculate the option value

- Consider Spread Option with payoff:

$$MAX\,(S_1(T) - S_2(T) - K, 0)$$

- What are the risk factors of this option?

Multi-Asset Price Dynamics

$$\delta \ln(S_i) = ..\ \delta t + \sigma_i \varepsilon_i \sqrt{\delta t} \quad \text{for } i = 1, 2$$

ε_1 and ε_2 Gaussian variables with correlation ρ

$$\rho = \frac{E[\ln(S_1)\ln(S_2)]}{\sigma_1 \sigma_2}$$

What are the moments of the distribution of the log of both assets?

Correlated Normal Samples

Obtain independent normal samples

x_1 and x_2 and set

$$\varepsilon_1 = x_1$$

$$\varepsilon_2 = \rho x_1 + x_2\sqrt{1-\rho^2}$$

A procedure known a Cholesky's decomposition can be used when samples are required from more than two normal variables

Alternative Approach

- Assume both assets follow a bi-variate lognormal distribution. Sample this in MC and calculate the expected value

- Joint-density is not necessarily a bi-variate lognormal distribution

- Use Copulas for other dependence structures. A copula is a function that generates a joint distribution from two marginal distribution functions

Variance Reduction Techniques

Recall that the standard error of the MC estimate is given by

$$\frac{\sigma(payoff)}{\sqrt{N}}$$

Accuracy can be improved by reducing the variance of the sampling

Main approaches used for option valuation:

- Antithetic variable technique
- Control variate technique
- Importance sampling
- Moment matching
- Using quasi-random sequences

Antithetic variable technique

$$\ln(S^i{}_T) = \ln(S^i{}_0) + rT + \sigma\varepsilon^i\sqrt{T}$$

ε^i follows $N(0,1)$

$$V = \frac{V^+ + V^-}{2}$$

$V^+ =$ MC estimate based on ε^i

$V^- =$ MC estimate based on -ε^i

$$Var(V) = \frac{1}{4}Var\left(V^+\right) + \frac{1}{4}Var\left(V^-\right) + \frac{1}{2}Cov\left(V^+, V^-\right)$$

Variance Reduction due to **negative** correlation

Control Variate Technique

- Goal is to value derivative A using information of simpler derivative B
- Note that Derivative A and B are closely related

$$\tilde{C}_A = \text{Control variate estimate of derivative A}$$

$$C_B = \text{Accurate value of derivative B}$$

$$\hat{C}_B = \text{MC estimate of derivative B}$$

$$\hat{C}_A = \text{MC estimate of derivative A}$$

$$\tilde{C}_A = \hat{C}_A - \beta(\hat{C}_B - C_B)$$

Control Variate Technique

The control variate estimate is unbiased because (Note C_A is the true value)

$$E[\tilde{C}_A] = E[\hat{C}_A - \beta(\hat{C}_B - C_B)] = E[\hat{C}_A] = C_A$$

Standard Error of Control Variate Estimate: $\sigma_A^2 + \beta^2\sigma_B^2 - 2\rho\beta\sigma_A\sigma_B$

Variance Reduction if $\beta^2\sigma_B^2 - 2\rho\beta\sigma_A\sigma_B < 0 \Rightarrow \rho > \dfrac{\beta\sigma_B}{2\sigma_A}$

The optimal coefficient which minimizes the variance: $\beta^* = \dfrac{\sigma_A}{\sigma_B}\rho$

Control Variate Technique

Ratio of the Var of optimally controlled estimator to that of uncontrolled

$$\frac{\mathrm{Var}[\tilde{C}_A]}{\mathrm{Var}[\hat{C}_A]} = 1 - \rho^2$$

Remarks:

• Effectiveness is determined by the strength of the correlation between A and B

• The reduction factor increases very sharply as |ρ| approaches 1, and, accordingly, it drops off quickly as |ρ| decreases away from 1.

American Put Case

- John Hull and Allen White, *The use of the Control Variate Technique in Option Pricing, J. Financial and Quantitative Analysis (1988)*
  - Considered American Put with European Put (BS) as control variate
  - Reported efficiency gains ranging from 1 to 100 depending on option parameters
- Challenge is to find a **good** control variate for which analytical value is known or an accurate numerical estimate can be calculated efficiently.

Control Variate Technique: Asian Call Case

Call option on arithmetic average $S_A = \frac{1}{n}\sum_{i=1}^{n} S(t_i)$ requires simulation

Use call option on geometric average $S_G = \left(\prod_{i=1}^{n} S(t_i)\right)^{1/n}$ which can be priced in closed form

Very strong correlation between Asian based on arithmetic average and geometric average (0.99)

Control Variate Technique: Hedges as Controls

V is replicated through a delta - hedging strategy

$$V(T) = V(0) + \int_0^T \sum_{j=1}^{d} \frac{\partial V(t)}{\partial S_j} dS_j(t)$$

V(T) should be highly correlated with $\sum_{\mathrm{i}=1}^{\mathrm{m}} \sum_{j=1}^{d} \frac{\partial V(t_i)}{\partial S_j} \left(S_j(t_i) - S_j(t_{i-1}) \right)$

In general $\frac{\partial V(t_i)}{\partial S_j}$ is not known, however in practice approximations are used