Monte-Carlo Methods in Derivative Finance

American Style Options

American/Bermudan Options

- American versus Bermudan options
- Examples:
  - American Call/Put
  - Bermudan Swaptions
  - Callable Range Accrual Swaps
- Valuation of Bermudan Option on Binomial Trees

Monte Carlo Simulation and American Options

- **How to estimate the continuation value in each exercise point?**

- **Two main approaches:**
  - The least squares approach
  - The exercise boundary parameterization approach

Put Option

- Bermudan put option on stock
  payoff = max (K – S_T, 0) $payoff = \max(K - S_T, 0)$
- 8 simulation paths
- Initial stock price $S_0 = 1.00$
- Strike $K = 1.1$
- Maturity time $T = 3$

Least Square Monte-Carlo

- Determine Early exercise boundary by Polynomial Regression
- Proposed by Longstaff & Schwartz (2001)
- Widely used in Finance

Sampled Paths

Path	t=0	t=1	t=2	t=3
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

LSM: An Example Cont.

Path	t = 0	t = 1	t = 2	T = 3	Payoff T = 3
1	**1.00**	**1.09**	**1.08**	**1.34**	**0.00**
2	**1.00**	**1.16**	**1.26**	**1.54**	**0.00**
3	**1.00**	**1.22**	**1.07**	**1.03**	**0.07**
4	**1.00**	**0.93**	**0.97**	**0.92**	**0.18**
5	**1.00**	**1.11**	**1.56**	**1.52**	**0.00**
6	**1.00**	**0.76**	**0.77**	**0.90**	**0.20**
7	**1.00**	**0.92**	**0.84**	**1.01**	**0.09**
8	**1.00**	**0.88**	**1.22**	**1.34**	**0.00**

LSM : An Example Cont.

Path	Y	X
1	$0.00e^{-r}$	1.08
2	-	-
3	$0.07e^{-r}$	1.07
4	$0.18e^{-r}$	0.97
5	-	-
6	$0.20e^{-r}$	0.77
7	$0.09e^{-r}$	0.84
8	-	

Y: Payoff at time $T = 3$ discounted by r

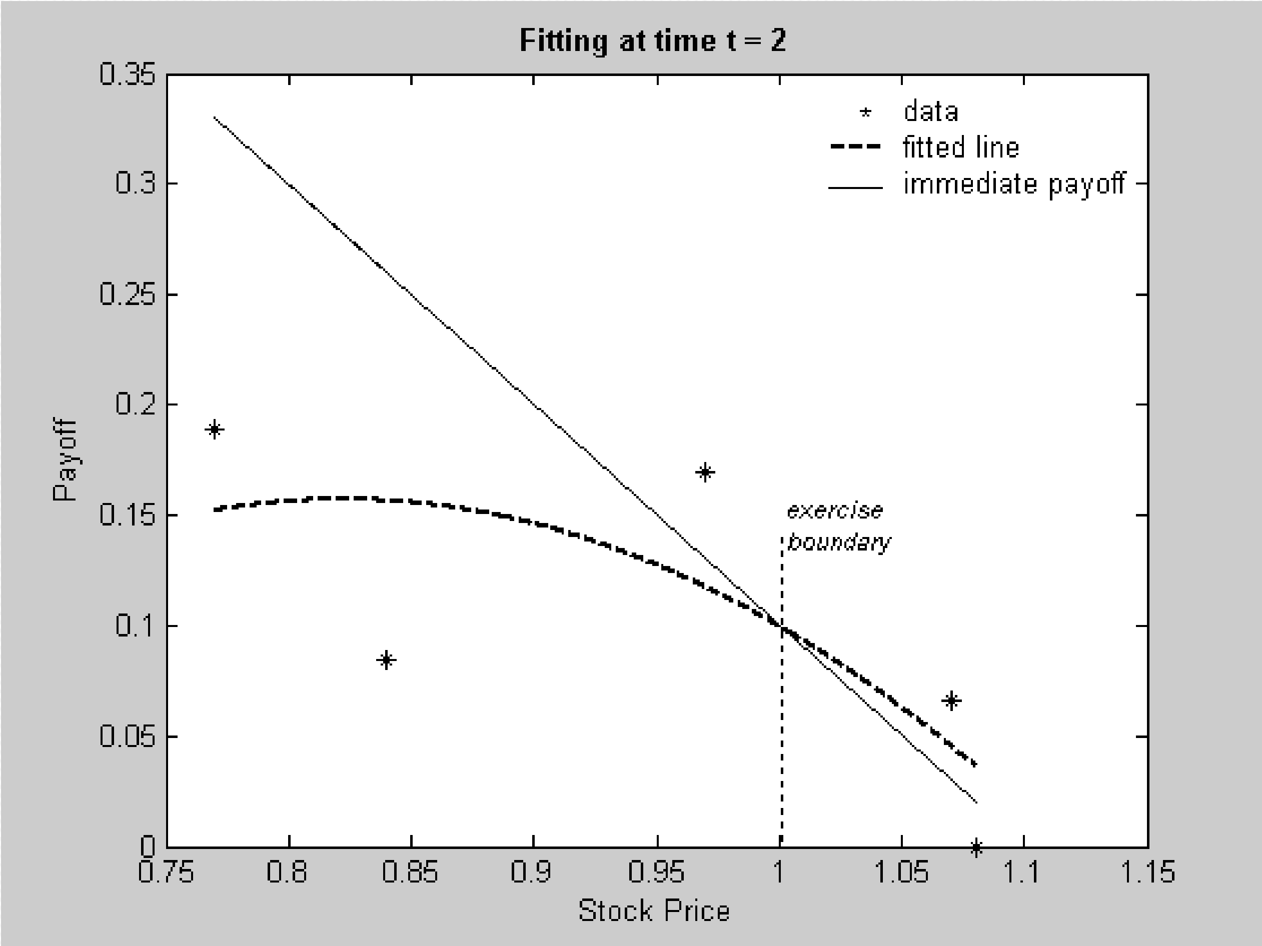
X: Stock price at time $t = 2$

Regression:

$$\boxed{E\{Y|X\} = c + \alpha X + \beta X^2}$$

Recursion for t=1 and t=0

Exercise or Continue?



What Controls Performance

- Order and type of Basis functions Standard Polynomials in Regression (Laquerre, Chebyshev or Hermite orthonormal polynomials)

- Choice of explanatory variables (e.g. Stock price)

American Put

	$S_0 = 36$ $\sigma = 0.2$ $T = 1$	$S_0 = 36$ $\sigma = 0.2$ $T = 2$	$S_0 = 36$ $\sigma = 0.4$ $T = 1$	$S_0 = 38$ $\sigma = 0.2$ $T = 1$
Closed Form European	3.844	3.763	6.711	2.852
Binomial Tree Approach	4.478	4.833	7.102	3.25
	Option price **(standard deviation)** ***difference with the binomial tree approach***			
$E(Y \mid S) = c + \alpha_1 S$	4.425 (0.021) *0.053*	4.756 (0.025) *0.077*	7.031 (0.024) *0.071*	3.205 (0.021) *0.045*
$E(Y \mid S) = c + \alpha_1 S + \alpha_2 S^2$	4.47 (0.021) *-0.008*	4.826 (0.021) *-0.007*	7.111 (0.020) *0.009*	3.259 (0.021) *-0.009*
$E(Y \mid S) = c + \alpha_1 S + \alpha_2 S^2 + \alpha_3 S^3$	4.478 (0.021) *0.000*	4.837 (0.02) *0.004*	7.126 (0.020) *0.024*	3.265 (0.01) *0.015*
$E(Y \mid S) = c + \alpha_1 L_0(S)$	4.445 (0.021) *0.033*	4.775 (0.029) *0.058*	7.076 (0.019) *0.026*	3.224 (0.014) *0.026*
$E(Y \mid S) = c + \alpha_1 L_0(S) + \alpha_2 L_1(S)$	4.467 (0.024) *0.011*	4.821 (0.025) *0.012*	7.108 (0.021) *-0.006*	3.251 (0.014) *-0.001*
$E(Y \mid S) = c + \alpha_1 L_0(S) + \alpha_2 L_1(S) + \alpha_2 L_2(S)$	4.474 (0.023) *-0.004*	4.829 (0.022) *-0.004*	7.116 (0.025) *0.014*	3.257 (0.011) *0.007*

Robustness Bermudan Put

- Results LSM in good agreement with Binomial Tree values for different parameter settings

- Convergence already with two basis functions (Quadratic or first two Laquere polynomials)

Asian Call

- *payoff (t)* = $\max(0, A(t,\tau) - K)$
- *Strike Price K = 100*
- *initial value A is variable and τ=0.25 years*
- *100 exercise points per year*
- *10,000 simulation paths and 20 trails*
- *Two-dimensional Regression (S and A)*
- *Comparison with Finite-Difference Results from Longstaff & Schwartz (2001)*

Robustness and Accuracy of Asian Call: Standard Polynomials in S and A

Basis-function	S=110 A=100	S=120 A=110	S=90 A=110
$c+\alpha_1 S+\alpha_2 S^2$	**13.24** ***2.48***	**22.53** ***2.92***	**3.37** ***0.77***
$c+\alpha_1 S+\alpha_2 S^2+\alpha_3 S^3$	**13.21** ***2.51***	**22.50** ***2.95***	**3.36** ***0.78***
$c+\alpha_1 A+\alpha_2 A^2$	**14.56** ***1.16***	**23.55** ***1.90***	**3.93** ***0.21***
$c+\alpha_1 A+\alpha_2 A^2+\alpha_3 A^3$	**14.56** ***1.16***	**23.46** ***1.99***	**3.93** ***0.21***

Robustness and Accuracy of Asian Call: Standard Polynomials in S and A

Basis-function	S=110 A=100	S=12 A=110	S=90 A=110
$c+\alpha_1 S+\alpha_2 S^2+\alpha_3 A+\alpha_4 A^2$	**15.57** ***0.15***	**25.35** ***0.10***	**4.13** ***0.01***
$c+\alpha_1 S+\alpha_2 S^2+\alpha_3 A+\alpha_4 A^2+\alpha_5 SA$	**15.60** ***0.12***	**25.38** ***0.07***	**4.14** ***0.00***
$c+\alpha_1 S+\alpha_2 S^2+\alpha_3 A+\alpha_4 A^2+\alpha_5 SA+\alpha_6 S^2 A^2$	**15.63** ***0.09***	**25.39** ***0.06***	**4.16** ***-0.02***
$c+\alpha_1 S+\alpha_2 S^2+\alpha_3 A+\alpha_4 A^2+\alpha_5 SA+\alpha_6 S^2 A^2+\alpha_7 S^3 A^3$	**15.63** ***0.09***	**25.39** ***0.06***	**4.16** ***-0.02***

Convergence Asian Options

- Similar convergence behaviour if Laguere polynomials are used
- Two explanatory variables are required!
- Method is robust, however, options with path-dependent characteristics require a higher number of basis functions

The Early Exercise Boundary Parametrization Approach

- We assume that the early exercise boundary can be parameterized in some way
- We carry out a first Monte Carlo simulation and work back from the end calculating the optimal parameter values
- We then discard the paths from the first Monte Carlo simulation and carry out a new Monte Carlo simulation using the early exercise boundary defined by the parameter values.

Sampled Paths

Path	t=0	t=1	t=2	t=3
1	1.00	1.09	1.08	1.34
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Application to Example

- We parameterize the early exercise boundary by specifying a critical asset price, S^*, below which the option is exercised.
- At t=1 the optimal S^* for the eight paths is 0.88. At t=2 the optimal S^* is 0.84
- In practice we would use many more paths to calculate the S^*

Lower Bound versus Upper Bound Estimates

- Exercise decision estimated by using both methods is suboptimal and therefore the estimated value is lower then the true value (lower bound estimate)

- Algorithm based on Duality principle can be used to estimate upper bounds (See section 8.7 of Glasserman's book: Monte Carlo Methods in Financial Engineering)

Monte-Carlo Methods in Derivative Finance

Estimation of Greeks

Greeks in Derivative Finance

- What are Greeks?
- Why are Greeks important in finance?
  - Valuation and Hedging
  - Quantification of Portfolio Risk Exposure
- Typically the following Greeks are of interest: Delta, Vega and Gamma

Euler Scheme: Bump and Revalue

- $V(S)$ – option price at time T
- $\delta = \partial V/\partial S$
- Use the Euler formula to approximate δ:

$$\delta = \frac{V(S+\varepsilon) - V(S)}{\varepsilon}$$

- We choose ε as small as possible but not too close to machine precision
- Then we run Monte Carlo at two points:

$$V(S+\varepsilon), \quad V(S)$$

Results based on Random Seed

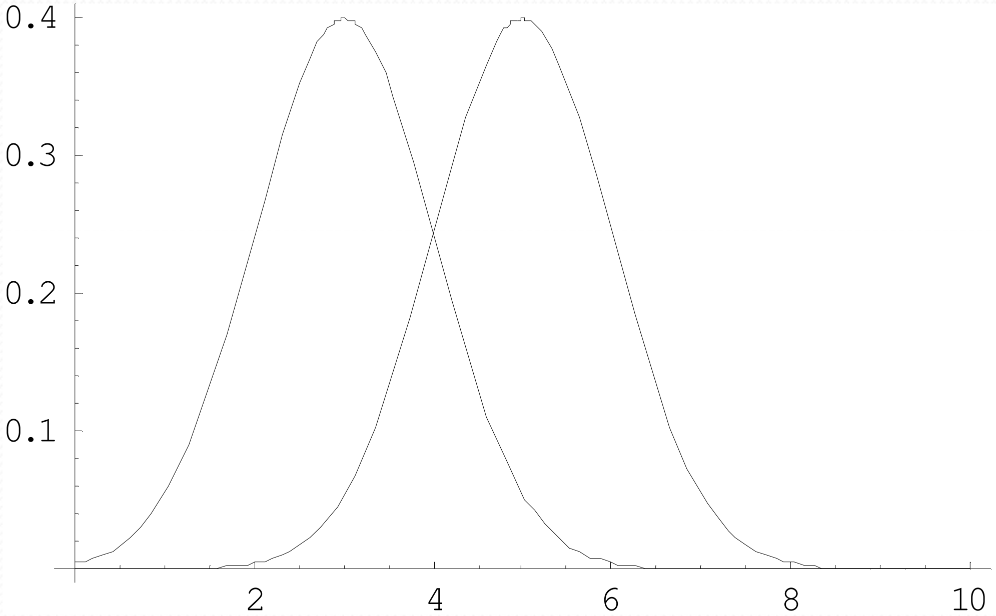
- r = 6%, σ =20%, S=100, K=99, T=1
- Result is unstable
  - Increasing iterations does not improve accuracy
  - Variance increases for smaller ε
- How can we reduce variance?

Size	$\varepsilon = 0.01$	$\varepsilon = 0.02$	$\varepsilon = 0.5$
10^4	0.484%	0.005%	0.494%
10^5	0.008%	0.026%	0.599%
10^6	53.444%	8.537%	2.726%
10^7	26.532%	21.801%	0.488%

Relative error is shown with analytical value as reference

Controlling Variance

$$Var(\delta) = \frac{1}{\varepsilon^2}\left(Var(V(S+\varepsilon)) + Var(V(S)) - 2Cov(V(S+\varepsilon), V(S))\right)$$



We can increase covariance by using the same random seed!

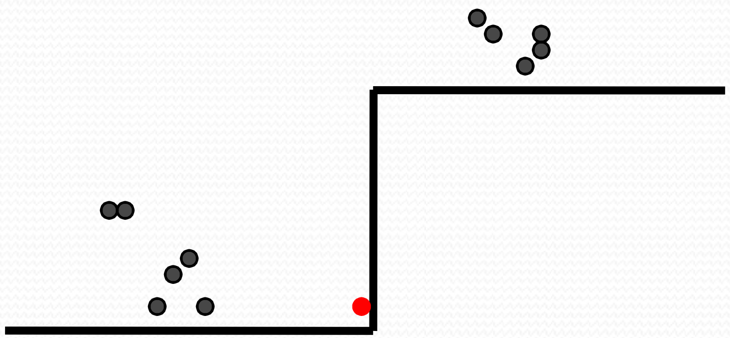
Results based on Same Seed

- r = 6%, σ =20%, S=100, K=99, T=1
- As expected a clear improvement of the results

Size	$\varepsilon = 0.01$	$\varepsilon = 0.02$	$\varepsilon = 0.5$
10^4	0.102%	0.087%	0.572%
10^5	0.121%	0.132%	0.745%
10^6	0.047%	0.061%	0.696%
10^7	0.003%	0.016%	0.653%

Relative error is shown with analytical value as reference

Stability Problems for Digital Options



$$\text{Digital Payoff} = I_{S_T > K}$$

If shift-size is **very small** and with **insufficient number of paths** the value corresponding to the bumped-scenario might change drastically due to one path crossing the discontinuous boundary.

Note that for this effect will be magnified in the estimate because we are dividing with the shift-size.

Results Digital Call

Results not stable and not nearly as good as with a European option

Size	$\varepsilon = 0.01$	$\varepsilon = 0.02$	$\varepsilon = 0.5$
10^4	3.49%	9.45%	6.08%
10^5	3.49%	0.90%	1.22%
10^6	5.30%	1.94%	5.18%
10^7	15.89%	15.28%	0.92%

Relative error is shown with analytical value as reference

Pathwise method for Delta with zero interest-rates

$$\Delta = \frac{\partial E[f(S_T)]}{\partial S_0} = E[\frac{\partial f(S_T)}{\partial S_0}]$$

$$\frac{\partial f(S_T)}{\partial S_0} = \frac{\partial f(S_T)}{\partial S_T}\frac{\partial S_T}{\partial S_0}$$

Minimal Condition for interchange:

Payoff should be smooth and differentiable **almost everywhere**

Likelihood ratio method

- θ : current stock price
- S_T : stock price at time T
- $f(S_T)$: payoff at time T
- $g(S_T,\theta)$: p.d.f. of S_T with initial stock price θ
- $V(\theta) = \mathrm{E}[f(S_T)]$

$$V(\theta) = \int f(S_T)\, g(S_T, \theta)\, dS_T$$

Likelihood ratio method

$$\frac{dV}{d\theta} \stackrel{?}{=} \int f(S_T)\frac{\partial g}{\partial \theta} dS_T$$

$$\square \int f(S_T)\dot{g}\, dS_T$$

$$= \int f(S_T)\frac{\dot{g}}{g} g\, dS_T$$

$$= E\left[f \frac{\dot{g}}{g} \right]$$

Likelihood ratio method

Thus we've found an unbiased estimator for delta which is based on a much smoother function – the p.d.f. (or the likelihood) of a certain payoff

Delta is estimated by the following expression

$$e^{-rT} f\left(S_T\right) \frac{\dot{g}\left(S_T, \theta\right)}{g\left(S_T, \theta\right)}$$

Likelihood ratio method

Example: delta of a digital

$$\ln S_T \approx \Phi\left[\ln\theta + \left(\mu - \sigma^2/2\right)T, \sigma\sqrt{T}\right]$$

$$g(x,\theta) = \frac{1}{x\sigma\sqrt{T}}\Phi\left(\zeta(x,\theta)\right)$$

$$\zeta(x,\theta) = \frac{\log(x/\theta) - \left(r - \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

$$S_T = \theta e^{\left(e-\sigma^2/2\right)T + \sigma\sqrt{T}Z}$$

$$e^{-rT} f(S_T)\frac{Z}{\sigma\theta\sqrt{T}} \xrightarrow{\text{Digital}} e^{-rT} I\{S_T > K\}\frac{Z}{\sigma\theta\sqrt{T}}$$

Likelihood method: Results

Good, stable results (r = 6%, σ =20%, S=100, K=99, T=1)

sample size	absolute error	relative error
10^4	0.006115	0.907%
10^5	0.001838	0.273%
10^6	0.000027	0.004%
10^7	0.000068	0.010%

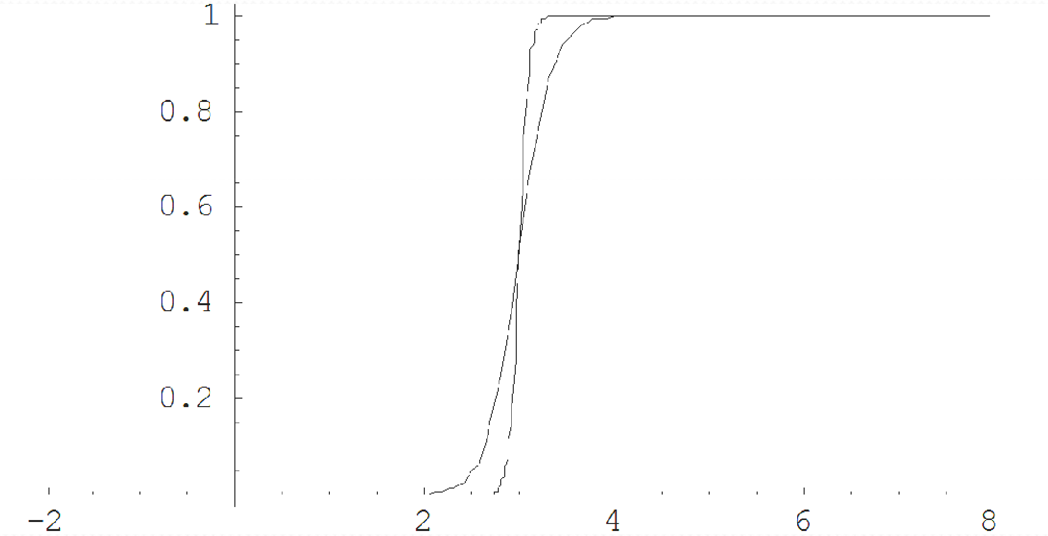
Table 7: Results for a European option

sample size	absolute error	relative error
10^4	0.000311	1.708%
10^5	0.000052	0.287%
10^6	0.000005	0.027%
10^7	0.000015	0.082%

Table 8: Results for a Digital option

Direct Smoothing Methods

- Smooth the discontinuous payoff function (Use e.g. CDF of Normal Distribution as smoothing function



- Bias should be minimized by suitable smoothing parameters