Estimating Sensitivities

The following notes develop sophisticated methods for estimating the derivatives of derivative prices commonly referred to as "Greeks", as they were discussed in class. These methods, when applicable, produce unbiased estimates by using information about the simulated stochastic process to replace numerical differentiation with exact integration.

Pathwise Method

The *pathwise method* differentiates each simulated outcome with respect to the parameter of interest.

$$\begin{aligned}
S(\theta) &= (S_0(\theta), ..., S_T(\theta)) \\
&= \text{path of underlying at parameter } \theta \\
f(S(\theta)) &= \text{discounted option payoff} \\
C = E[f(S(\theta))] &= \text{option price}
\end{aligned}$$

For smooth payoffs:

$$\begin{aligned}
\frac{dC}{d\theta} &= \frac{d}{d\theta} E[f[S(\theta)]] \\
&= E\left[\frac{df(S(\theta)}{d\theta}\right]
\end{aligned}$$

We use the payoff derivative $df(S)/d\theta$ to estimate the price derivative. Following, there are some examples of how $df(S)/d\theta$ can be computed.

Black-Scholes Delta by the Pathwise Method

Price:

$$C = E\left[e^{-rT} \max(S_T - K, 0)\right]$$

Simulation:

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}, \qquad Z \sim N(0,1)$$

We want to estimate

$$\Delta = \frac{dC}{dS_0} = E\left[e^{-rT}\frac{d\max(S_T-K,0)}{dS_0}\right]$$

Thus, we need to make sense of

$$\frac{d\max(S_T-K,0)}{dS_0}$$

$$\begin{aligned}
\frac{d\max(S_T-K,0)}{dS_0} &= \frac{d\max(S_T-K,0)}{dS_T}\frac{dS_T}{dS_0} \\
&= \mathbf{1}_{\{S_T>K\}}\frac{dS_T}{dS_0} \\
&= \mathbf{1}_{\{S_T>K\}}\frac{S_T}{S_0},
\end{aligned}$$

since for a geometric Brownian Motion

$$\frac{dS_T}{dS_0} = \frac{S_T}{S_0}$$

This follows from

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}$$

Hence,

$$\hat{\Delta} = e^{-rT}\mathbf{1}_{\{S_T>K\}}\frac{S_T}{S_0}$$

The expected value of this expression is indeed the Black-Scholes delta, so the estimator is unbiased

$$\Delta = E\left[\hat{\Delta}\right]$$

Note that the relation

$$\frac{dS_T}{dS_0} = \frac{S_T}{S_0}$$

holds more widely than GBM and is often useful as an approximation.

Estimating Vega by the Pathwise Method

$$C = E\left[e^{-rT}\max(S_T-K,0)\right]$$

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}$$

We want to estimate

$$\nu = \frac{dC}{d\sigma} = E\left[e^{-rT}\frac{d\max(S_T-K,0)}{d\sigma}\right]$$

$$\begin{aligned}
\frac{d\max(S_T-K,0)}{d\sigma} &= \frac{d\max(S_T-K,0)}{dS_T}\frac{dS_T}{d\sigma} \\
&= \mathbf{1}_{\{S_T>K\}}\frac{dS_T}{d\sigma} \\
&= \mathbf{1}_{\{S_T>K\}}S_T(-\sigma T+\sqrt{T}Z)
\end{aligned}$$

Hence,

$$\hat{\nu} = e^{-rT}\mathbf{1}_{\{S_T>K\}}S_T(-\sigma T+\sqrt{T}Z)$$

The expected value of this estimator is the Black-Scholes vega, so this estimator is unbiased

$$\nu = E\left[\hat{\nu}\right].$$

Path-Dependent Examples

As in the previous example, the underlying asset is modelled by geometric Brownian motion, but now let the payoff be path dependent. The Asian option is written on the average

$$\bar{S} = \frac{1}{m}\sum_{i=1}^{m} S_i$$

Payoff

$$\max\left(\bar{S}-K,0\right)$$

Pathwise delta estimate:

$$e^{-rT}\frac{\bar{S}}{S_0}\mathbf{1}_{\{\bar{S}>K\}}$$

Since

$$\frac{d\bar{S}}{dS_0} = \frac{1}{m}\frac{d}{dS_0}\sum_{i=1}^{m} S_i = \frac{1}{m}\sum_{i=1}^{m}\frac{S_i}{S_0} = \frac{\bar{S}}{S_0}.$$

Due to the fact that there is no analytical formula for the price of an Asian option, this unbiased estimator has genuine practical value. Given that $\bar{S}$ would be simulated anyway in estimating the price of the option, this estimator requires minimal additional effort.

Multi-Asset Examples

Spread option

Payoff:

$$e^{-rT}\max\left(\left(S_T^1 - S_T^2\right) - K, 0\right)$$

S^1-Delta:

$$e^{-rT}\frac{S_T^1}{S_0^1}\mathbf{1}_{\{S_T^1 - S_T^2 > K\}}$$

S^2-Delta:

$$-e^{-rT}\frac{S_T^2}{S_0^2}\mathbf{1}_{\{S_T^1 - S_T^2 > K\}}$$

Max option

Payoff:

$$e^{-rT}\max\left(\max\left(S_T^1, ..., S_T^k\right) - K, 0\right)$$

S^i-Delta:

$$e^{-rT}\frac{S_T^i}{S_0^i}\mathbf{1}_{\{S_T^i > K, S_T^i > S_T^j, j\neq i\}}$$

More generally

$$\begin{aligned}
f\left(S_T^1, ..., S_T^d\right) &= \text{portfolio value.} \\
\frac{\partial f}{\partial S_T^i}\frac{\partial S_T^i}{\partial S_0^i} &= \text{pathwise delta,} \qquad i = 1, ..., k.
\end{aligned}$$

Stochastic Volatility

The expressions derived in the exampled apply more generally. Consider an underlying asset described by an SDE

$$\begin{aligned}
\frac{dS_t}{S_t} &= r_t dt + \sigma_t dW_t^{(1)} \\
d\sigma_t &= \alpha(\sigma_t)dt + b(\sigma_t)dW_t^{(2)}
\end{aligned}$$

Then

$$S_T = S_0 \exp\left(\int_0^T (r_t - \frac{1}{2}\sigma_t^2)dt + \int_0^T \sigma_t dW_t^{(1)}\right),$$

Standard Call: Delta estimate for $E\left[e^{-rT}\max(S_T - K, 0)\right]$ is

$$e^{-rT}\frac{S_T}{S_0}\mathbf{1}_{\{S_T>K\}}$$

Asian Option: Delta estimate for $\max\left(\bar{S} - K, 0\right)$ is

$$e^{-rT}\frac{\bar{S}}{S_0}\mathbf{1}_{\{\bar{S}>K\}}.$$

Likelihood Ratio Method

The *likelihood ratio method* differentiates a probability density rather than a path. The method provides an alternative approach to derivative estimation requiring no smoothness at all in the discounted payoff. The epxected discounted payoff is expressed as an integral:

$$C(S_0) = E\left[f(S_T)\right] = \int f(x) g\left(x, S_0\right) dx$$

Dependence on S_0 is now in the density.

Likelihood Ratio Method Derivative Estimates

Price

$$C = E\left[f(S_T)\right] = \int_0^\infty f(x) g(x;\theta) dx$$

θ a parameter of the density rather than an outcome. Thus,

$$\begin{aligned}
\frac{dC}{d\theta} &= \int_0^\infty f(x) \frac{g(x;\theta)}{\partial\theta} dx \\
&= \int_0^\infty f(x) \frac{\partial g(x;\theta)}{\partial\theta} \frac{g(x;\theta)}{g(x;\theta)} dx \\
&= \int_0^\infty f(x) \frac{\partial \log(g(x))}{\partial\theta} g(x;\theta) dx \\
&= E\left[f(S_T) \frac{\partial \log(g(S_T))}{\partial\theta}\right]
\end{aligned}$$

This is the same for all payoffs f.

Black-Scholes Delta by the Likelihood Ratio Method

$$C = E\left[e^{-rT} \max(S_T - K, 0)\right]$$

The log-normal density of S_T is

$$g(x) = \frac{1}{x\sigma\sqrt{2\pi T}} \exp\left[-\frac{1}{2}\left(\frac{\log(x/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2\right],$$

or

$$g(x) = \frac{1}{x\sigma\sqrt{T}} \phi\left(d(x)\right),$$

where

$$d(x) = \frac{\log(x/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

and

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

the standard normal density. Note that

$$\phi'(x) = -x\phi(x).$$

Then

$$\begin{aligned}
\frac{\partial \log(g(x))}{\partial S_0} &= \frac{\partial \log(g(x))}{\partial g(x)} \cdot \frac{\partial g(x)}{\partial S_0} = \frac{1}{g(x)} \cdot \frac{\partial g(x)}{\partial S_0} = \frac{1}{g(x)} \cdot \frac{\partial}{\partial S_0}\left[\frac{1}{x\sigma\sqrt{T}} \phi\left(d(x)\right)\right] \\
&= \frac{1}{g(x)} \cdot \frac{1}{x\sigma\sqrt{T}} \phi'\left(d(x) \frac{\partial d(x)}{\partial S_0}\right) = \frac{1}{g(x)} \cdot \frac{1}{x\sigma\sqrt{T}}\left(-d(x)\phi\left(d(x)\right) \frac{\partial d(x)}{\partial S_0}\right) \\
&= -d(x) \frac{\partial d(x)}{\partial S_0} = \frac{\log(x/S_0) - (r - \frac{1}{2}\sigma^2)T}{S_0 \sigma^2 T},
\end{aligned}$$

and thus

$$\frac{\partial \log(g(S_T))}{\partial S_0} = \frac{\log(S_T/S_0) - (r - \frac{1}{2}\sigma^2)T}{S_0\sigma^2 T} = \frac{Z}{S_0\sigma\sqrt{T}}$$

$$\Delta = \frac{dC}{dS_0} = E\left[e^{-rT}\max(S_T - K, 0)\frac{Z}{S_0\sigma\sqrt{T}}\right].$$

For digital option, just replace $e^{-rT}\max(S_T - K, 0)$ with $e^{-rT}\mathbf{1}_{\{S_T \geq K\}}$.

Likelihood Ratio Method for Options on Multiple Assets

Option on $(S_T^1, ..., S_T^k)$,

$$dS_t^i = rS_t^i dt + \sigma_i S_t^i dW_t^i, \qquad i = 1, ..., k,$$

and

$$E\left[dW_t^i \cdot dW_t^j\right] = \rho_{ij},$$

e.g., basket option

$$\text{discounted payoff} = e^{-rT}\max(\alpha_1 S_T^1 + ... + \alpha_k S_T^k - K, 0)$$

Covariance matrix $\Sigma_{ij} = \sigma_i\sigma_j\rho_{ij}$.

$$S_T^i = S_0^i \exp\left((r - \frac{1}{2}\sigma_i^2)T + \sqrt{T}X^i\right)$$

where $X \sim N(0, \Sigma)$.

Density of $(S_T^1, ..., S_T^k)$ multivariate lognormal $g(x_1, ..., x_k)$ and

$$\frac{\partial \log g}{\partial S_0^i} = \left(X\Sigma^{-1}\right)_i \cdot \frac{1}{S_0^i\sqrt{T}},$$

Likelihood Ratio Method estimate on the ith delta

$$\text{discounted option payoff} \times \left[\left(X\Sigma^{-1}\right)_i \cdot \frac{1}{S_0^i\sqrt{T}}\right].$$

References

[1] Glasserman, Paul. *Monte Carlo methods in financial engineering.* Vol. 53. Springer Science & Business Media, 2013.