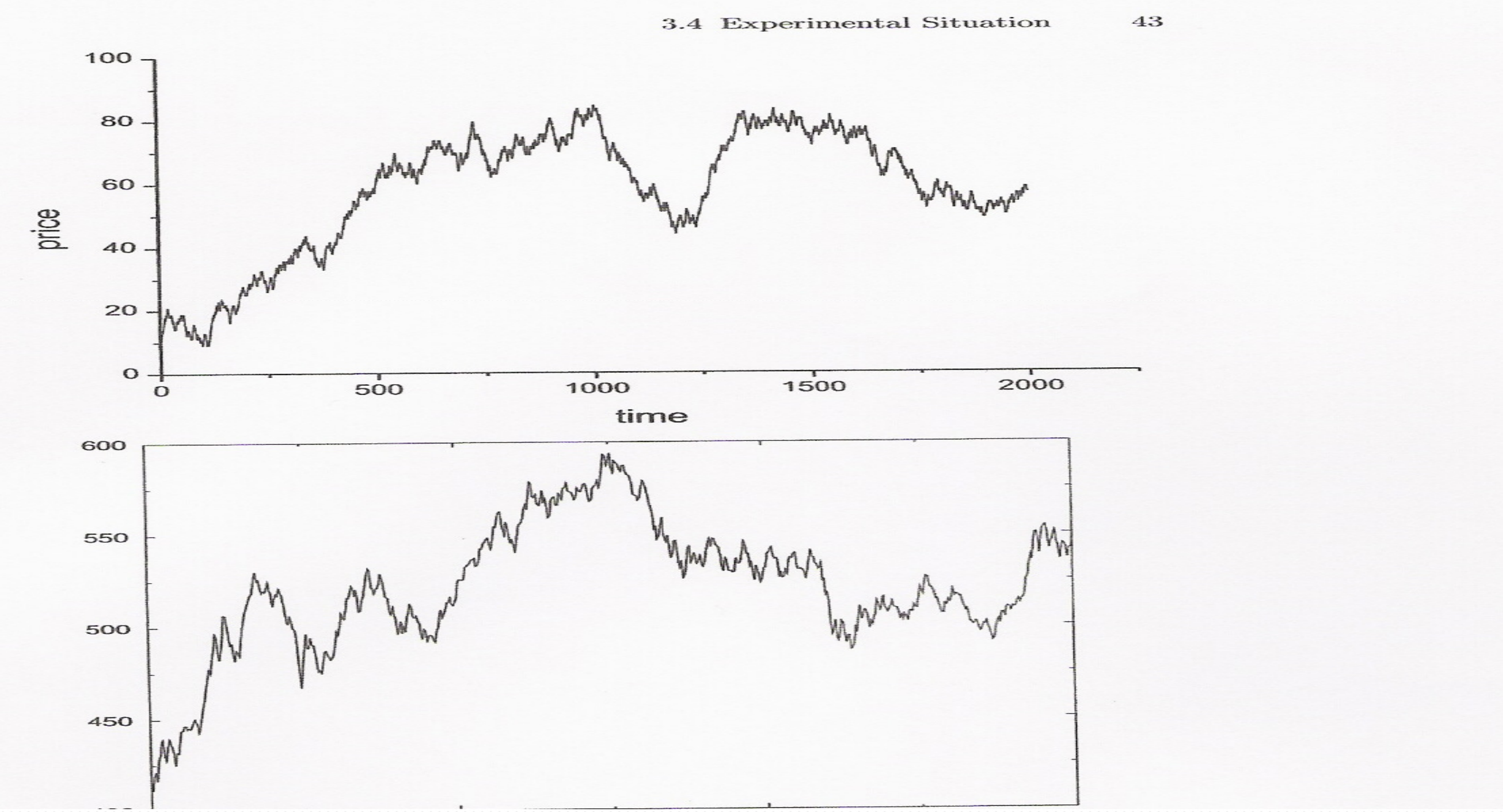
Introduction to Stochastic Calculus

Model or Market Data?



Markov Processes

- In a Markov process future movements in a variable depend only on where we are, not the history of how we got where we are
- We assume that stock prices follow Markov processes

Variances & Standard Deviations

- In Markov processes changes in successive periods of time are independent
- This means that variances are additive
- Standard deviations are not additive

A Wiener Process

- We consider a variable z whose value changes continuously
- The change in a small interval of time δt is δz
- The variable follows a Wiener process if

1. $\delta z = \varepsilon\sqrt{\delta t}$ where ε is a random drawing from $\phi(0,1)$
2. The values of δz for any 2 different (non-overlapping) periods of time are independent

Properties of a Wiener Process

- Mean of $[z(T) - z(0)]$ is 0
- Variance of $[z(T) - z(0)]$ is T
- Standard deviation of $[z(T) - z(0)]$ is $\sqrt{T}$

Taking Limits . . .

- What does an expression involving dz and dt mean?
- It should be interpreted as meaning that the corresponding expression involving δz and δt is true in the limit as δt tends to zero
- In this respect, stochastic calculus is analogous to ordinary calculus

Generalized Wiener Processes

$$\delta x = a\,\delta t + b\,\varepsilon\sqrt{\delta t}$$

- Mean change in x in time T is aT
- Variance of change in x in time T is b^2T
- Standard deviation of change in x in time T is

$$b\sqrt{T}$$

Examples

- A stock price starts at 40 and has a probability distribution of $\phi(40,10)$ at the end of the year
- If we assume the stochastic process is Markov with no drift then the process is

$$dS = 10dz$$

- If the stock price were expected to grow by $8 on average during the year, so that the year-end distribution is $\phi(48,10)$, the process is

$$dS = 8dt + 10dz$$

Ito Process

- In an Ito process the drift rate and the variance rate are functions of time

$$dx=a(x,t)dt+b(x,t)dz$$

- The discrete time equivalent

$$\delta x = a(x,t)\delta t + b(x,t)\varepsilon\sqrt{\delta t}$$

is only true in the limit as δt tends to zero

Why a Generalized Wiener Process is not Appropriate for Stocks

- For a stock price we can conjecture that its expected percentage change in a short period of time remains constant, not its expected absolute change in a short period of time
- We can also conjecture that our uncertainty as to the size of future stock price movements is proportional to the level of the stock price

An Ito Process for Stock Prices

$$dS = \mu S\,dt + \sigma S\,dz$$

where μ is the expected return σ is the volatility.

The discrete time equivalent is

$$\delta S = \mu S \delta t + \sigma S \varepsilon \sqrt{\delta t}$$

Ito's Lemma

- If we know the stochastic process followed by x, Ito's lemma tells us the stochastic process followed by some function $G(x, t)$
- Since a derivative security is a function of the price of the underlying and time, Ito's lemma plays an important part in the analysis of derivative securities

Taylor Series Expansion

A Taylor's series expansion of $G(x, t)$ gives

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + ½\frac{\partial^2 G}{\partial x^2}\delta x^2$$

$$+ \frac{\partial^2 G}{\partial x \partial t}\delta x\ \delta t + ½\frac{\partial^2 G}{\partial t^2}\delta t^2 + \ldots$$

Ignoring Terms of Higher Order Than δt

In ordinary calculus we have

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t$$

In stochastic calculus this becomes

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + ½\frac{\partial^2 G}{\partial x^2}\delta x^2$$

because δx has a component which is of order $\sqrt{\delta t}$

Substituting for δx

Suppose

$$dx = a(x,t)dt + b(x,t)dz$$

so that

$$\delta x = a\,\delta t + b\,\varepsilon\sqrt{\delta t}$$

Then ignoring terms of higher order than δt

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + ½\frac{\partial^2 G}{\partial x^2}b^2\varepsilon^2\delta t$$

The $\varepsilon^2 \Delta t$ Term

Since $\varepsilon \approx \phi(0,1)\ E(\varepsilon) = 0$

$$E(\varepsilon^2) - [E(\varepsilon)]^2 = 1$$

$$E(\varepsilon^2) = 1$$

It follows that $E(\varepsilon^2 \delta t) = \delta t$

The variance of δt is proportion al to δt^2 and can be ignored. Hence

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2 \delta t$$

Taking Limits

Taking limits $$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + ½ \frac{\partial^2 G}{\partial x^2} b^2 dt$$

Substituting $$dx = a\ dt + b\ dz$$

We obtain $$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b\ dz$$

This is Ito's Lemma

Application of Ito's Lemma to a Stock Price Process

The stock price process is

$$dS = \mu S\, dt + \sigma S\, dz$$

For a function G of S and t

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S\, dz$$

Examples

The forward price of a stock for a contract maturing at time T

$$G = S\, e^{r(T-t)}$$

$$dG = (\mu - r)G\, dt + \sigma G\, dz$$

Examples

$$G = \ln S$$

$$dG = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma \, dz$$