Advanced Statistical Modelling: Ridge Regression

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Choosing the penalization parameter λ

The objective of this exercise is to implement Ridge Regression with two different approaches: $MSPE_{val}(\lambda)$ and $MSPE_{k-CV}(\lambda)$. In both cases we are going to take as input data the following:

- Matrix x and vector y corresponding to the training sample.
- Matrix x_{val} and vector y_{val} corresponding to the validation set.
- Vector *lambda.v* of candidate values for λ.

We are going to output for each element λ in *lambda.v* and the value of the $MSPE_{val}(\lambda)$ / $MSPE_{k-CV}(\lambda)$. Furthermore, we are going to plot these values against $log(1+\lambda)-1$.

Once we have build these two functions, we are going to use the prostate data used in class. We are going to choose a λ according to:

- Behaviour in the validation set (30 validations not included in the training sample)
- 5-fold, 10-fold cross-validation.
- Compare our results with those obtained with leave-one-out and generalized cross-validation.

Ridge regression based on $MSPE_{val}(\lambda)$

In order to choose the penalization parameter λ, we are going to write a function implementing the ridge regression penalization parameter λ choice based on the minimization of the $MSPE_{val}(\lambda)$.

```
ridge <- function(x, y, x.val, y.val, lambda.v) {
  result <- data.frame(lambda=lambda.v, mspe=rep(0,length(lambda.v)),
                      df=rep(0,length(lambda.v)))
  x.svd <- svd(x)
  for(i in 1:length(lambda.v)) {
    lambda <- lambda.v[i]
    d_inv <- diag(1/(x.svd$d*x.svd$d - lambda))
    xx_inv <- t( solve( t(x) %*% x + lambda*diag(1,ncol(x)) ))
    beta <- xx_inv %*% t(x) %*% y
    y.hat <- x.val %*% beta
    mspe <- sum((y.val - y.hat)^2) / length(y.val)
    df <- sum(x.svd$d^2 / (x.svd$d^2 +lambda))
    result$mspe[i] <- mspe
    result$df[i] <- df
  }
  plot(mspe~log(1+lambda), result, col=2)
  lambda.min <- result$lambda[which.min(result$mspe)]
  abline(v=log(1+lambda.min),col=2,lty=2)
  plot(mspe~df, result, col=3)
  df.min <- result$df[which.min(result$mspe)]
  abline(v=df.min,col=3,lty=2)
  return(result)
}
```

Ridge regression based on $MSPE_{k-CV}(\lambda)$

Now, we will write an R function implementing the ridge regression penalization parameterλ choice based on k-fold cross-validation $MSPE_{kCV}(\lambda)$

```
ridge_cv <- function(x, y, lambda.v, cv=10) {
  result <- data.frame(lambda=lambda.v, mspe=rep(0,length(lambda.v)),
                       df=rep(0,length(lambda.v)))
  for(i in 1:length(lambda.v)) {
    lambda.v[i] <- lambda.v[i]
    folds <- createFolds(1:nrow(x), k = cv)
    result.cv <- data.frame(mspe=rep(0,cv), df=rep(0,cv))
    for(j in 1:cv) {
      fold <- folds[[j]]
      x.train <- x[-fold,]
      y.train <- y[-fold]
      x.val <- x[fold,]
      y.val <- y[fold]
      x.svd <- svd(x.train)
      d <- x.svd$d
      v <- x.svd$v
      d_inv <- diag(1/(d*d - lambda.v[i]))
      xx_inv <- t( solve( t(x.train) %*% x.train + lambda.v[i]*diag(1,ncol(x)) ))
      beta <- xx_inv %*% t(x.train) %*% y.train
      y.hat <- x.val %*% beta
      mspe <- sum((y.val - y.hat)^2) / length(y.val)
      df <- sum(d^2 / (d^2 +lambda.v[i]))
      result.cv$mspe[j] <- mspe
      result.cv$df[j] <- df
    }
    result$mspe[i] <- mean(result.cv$mspe)
    result$df[i] <- mean(result.cv$df)

  }
  plot(mspe~log(1+lambda), result, col=2)
  lambda.min <- result$lambda[which.min(result$mspe)]
  abline(v=log(1+lambda.min),col=2,lty=2)
  plot(mspe~df, result, col=3)
  df.min <- result$df[which.min(result$mspe)]
  abline(v=df.min,col=3,lty=2)

  return(result)

}
```

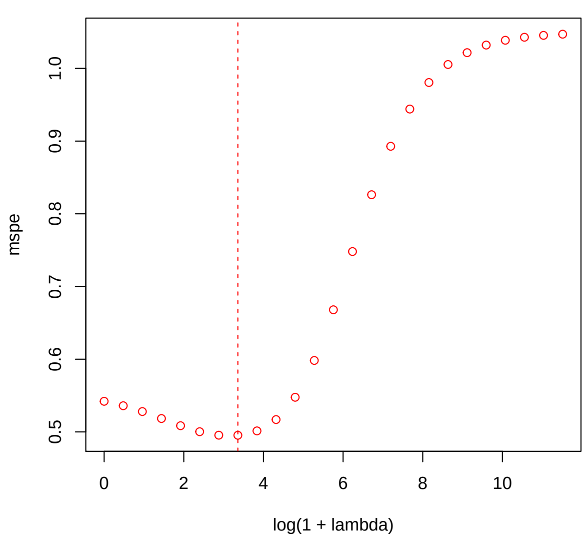
Comparation between penalization parameters of $MSPE_{val}(\lambda)$ and $MSPE_{k-CV}(\lambda)$

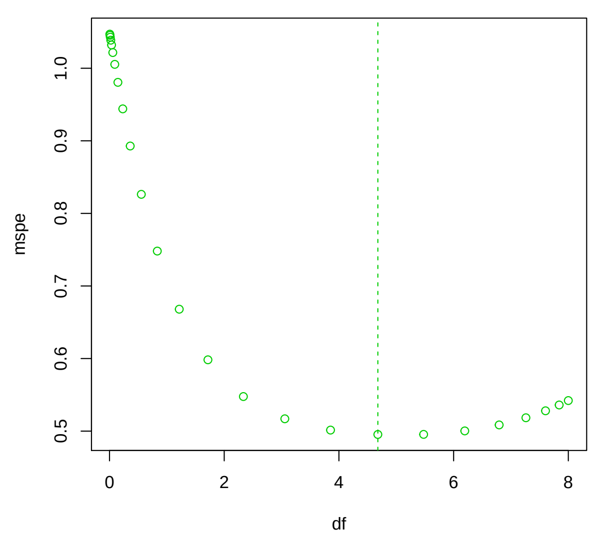
```
ridge_loocv_gcv <- function(x, y, lambda.v) {
  l <- length(lambda.v)
  result <- data.frame(lambda=lambda.v, loocv=rep(0,l), gcv=rep(0,l),df=rep(0,l))
  n <- nrow(x)
  x.svd <- svd(x)
  d <- x.svd$d
  v <- x.svd$v
```

```r
  u <- x.svd$u
  for(i in 1:l) {
    lambda <- lambda.v[i]
    d_inv <- diag(1/(d^2 - lambda))
    xx_inv <- t( solve( t(x) %*% x + lambda*diag(1,ncol(x)) ))
    beta <- (xx_inv %*% t(x)) %*% y
    y.hat <- x %*% beta
    df <- sum(d^2 / (d^2 +lambda))
    h <- x %*% xx_inv %*% t(x)
    result$loocv[i] <- sum( ( (y - y.hat)/(1 - diag(h)) )^2 ) / n
    result$gcv[i] <- sum( ( (y - y.hat)/(1 - df/n) )^2 ) / n
    result$df[i] <- df
  }
  plot(loocv~log(1+lambda), result, col=2)
  lambda.min <- result$lambda[which.min(result$loocv)]
  abline(v=log(1+lambda.min),col=2,lty=2)
  plot(loocv~df, result, col=3)
  df.min <- result$df[which.min(result$loocv)]
  abline(v=df.min,col=3,lty=2)
  plot(gcv~log(1+lambda), result, col=2)
  lambda.min <- result$lambda[which.min(result$gcv)]
  abline(v=log(1+lambda.min),col=2,lty=2)
  plot(gcv~df, result, col=3)
  df.min <- result$df[which.min(result$gcv)]
  abline(v=df.min,col=3,lty=2)
  return(result)
}
```

Validation

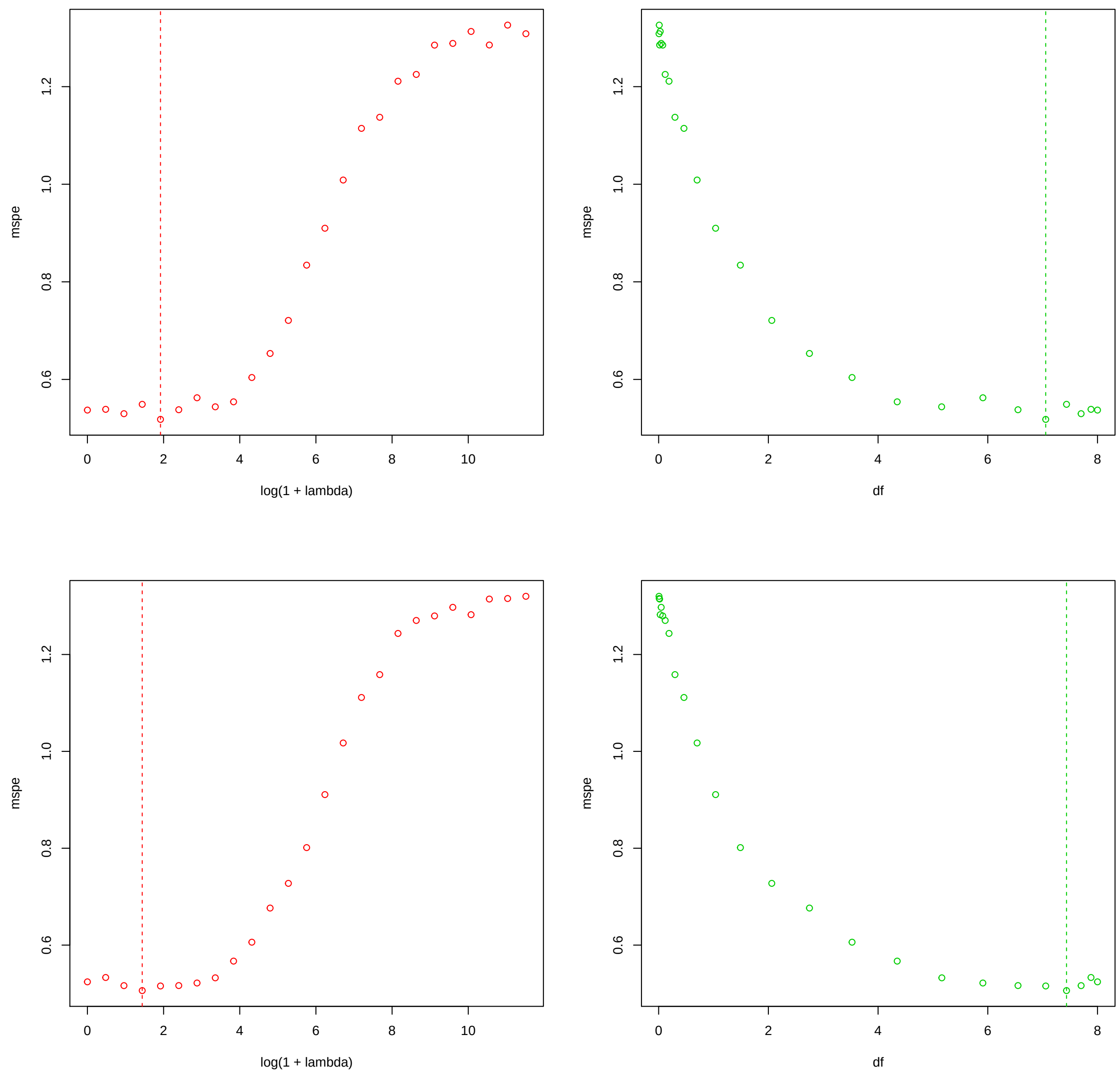
```r
op<-par(mfrow=c(1,2))
result.valid <- ridge(x, y, x.val, y.val, lambda.v)
```





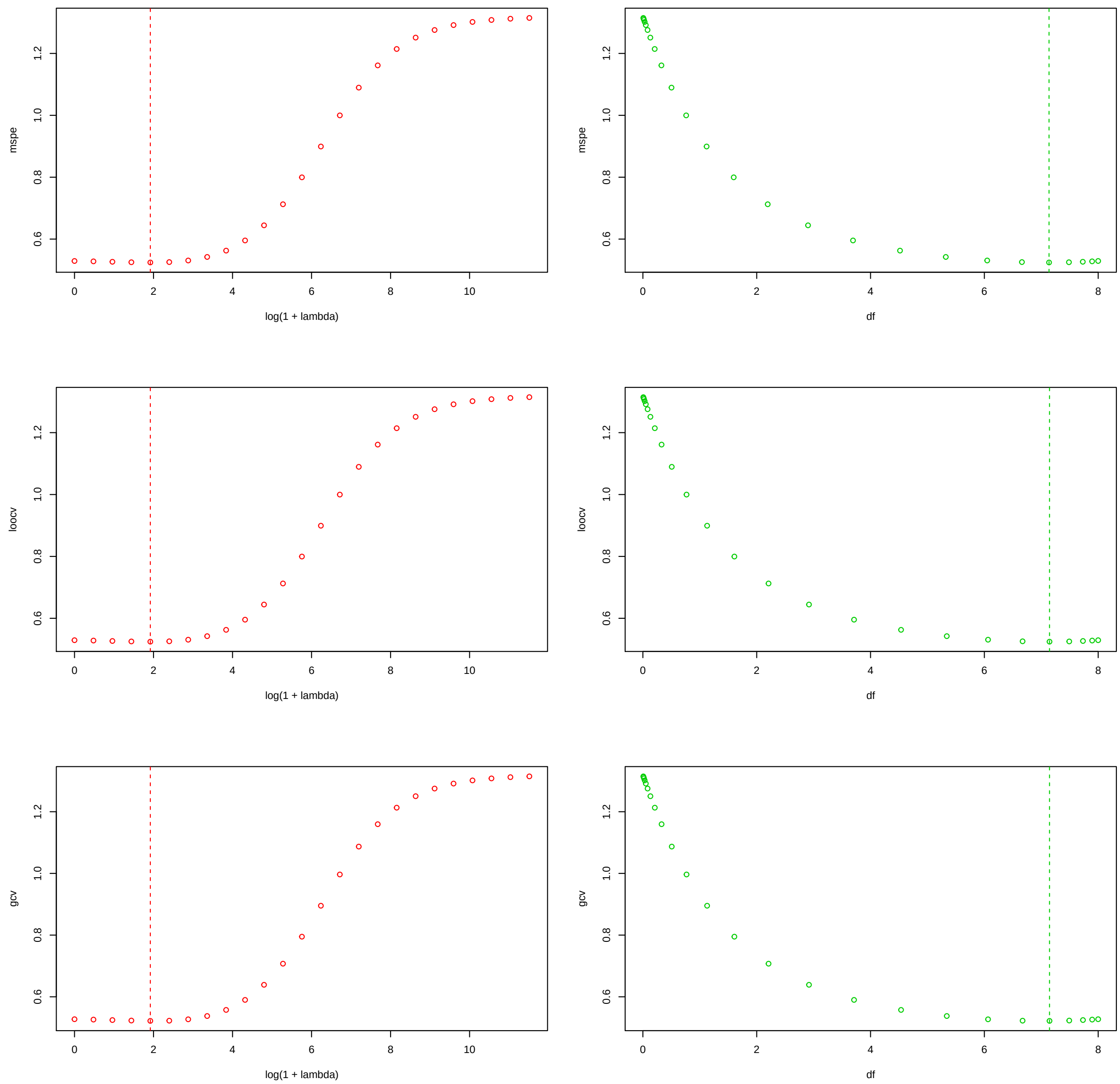
5-fold and 10-fold CV

```
x <- scale(data[,1:8], center = T, scale = T)
y <- scale(data[,9], center = T, scale = F)
op<-par(mfrow=c(2,2))
result.5.cv <- ridge_cv(x, y, cv = 10, lambda.v)
result.10.cv <- ridge_cv(x, y, cv = 10, lambda.v)
```



LOOCV and GCV

```r
op<-par(mfrow=c(3,2))
result.loocv <- ridge_cv(x, y, cv = nrow(x), lambda.v) #n-fold CV
result.loocv.gcv <- ridge_loocv_gcv(x, y, lambda.v)
```



We observe that the results of MSPE seem more stable in the LOOCV and GCV compared to 5-fold and 10-fold CV, as one would expect given the small size of the dataset. Furthermore, the validation set results are also relatively stable, although with a different curve than for the LOOCV/GCV.

Ridge regression for the Boston Housing data

The Boston Housing dataset is a classical dataset which containes the values of 506 suburbs of Boston corresponding to 1978. This dataset can be found in many places but we are going to use a version with some corrections that was provided to us, which additionally includes the UTM coordinates of the geographical centers of each neighborhood. Therefore, the variables are the following:

Variable	Description	Type
CRIM	per capita crime rate by town	Numeric
ZN	proportion of residential land zoned for lots over 25,000 sq.ft.	Numeric
INDUS	proportion of non-retail business acres per town	Numeric
CHAS	Charles River dummy variable (= 1 if tract bounds river; 0 otherwise)	Factor
NOX	nitric oxides concentration (parts per 10 million)	Numeric
RM	average number of rooms per dwelling	Numeric
AGE	proportion of owner-occupied units built prior to 1940	Numeric
DIS	weighted distances to five Boston employment centres	Numeric
RAD	index of accessibility to radial highways	Numeric
TAX	full-value property-tax rate per $10,000	Numeric
PTRATIO	pupil-teacher ratio by town	Numeric
B	1000(Bk - 0.63)^2 where Bk is the proportion of blacks by town	Numeric
LSTAT	% lower status of the population	Numeric
MEDV	Median value of owner-occupied homes in $1000's	Numeric

In this exercise we are going to use ridge regression on the Boston Housing dataset to fit the regression model where the response variable is $MEDV$ and the explanatory variables are the remaining 13 variables shown in the list. As we can see when loading the data, there are more variables than the ones listed ($TOWN$, $TOWNNO$, LON, LAT, $CMEDV$). We decided not to use them since the statement explicitly defines which to use.

```
load("boston.Rdata")
names(boston.c)
```

```
##  [1] "TOWN"    "TOWNNO"  "TRACT"   "LON"     "LAT"     "MEDV"    "CMEDV"
##  [8] "CRIM"    "ZN"      "INDUS"   "CHAS"    "NOX"     "RM"      "AGE"
## [15] "DIS"     "RAD"     "TAX"     "PTRATIO" "B"       "LSTAT"
```

```
dataset <- boston.c[,-c(1:5)] #"TOWN"    "TOWNNO"  "TRACT"   "LON"     "LAT"
dataset$CMEDV <- NULL
```

Beside from eliminating variables, we divided the dataset between train and test. To do so, we have decided to do 3/4 training, 1/4 test.

```
set.seed(42)
trainIndex <- createDataPartition(dataset$MEDV, p = 0.75, list = F)
train <- dataset[trainIndex,]
test <- dataset[-trainIndex,]
```

```
train
```

```
##     MEDV     CRIM    ZN INDUS CHAS    NOX    RM   AGE     DIS RAD TAX
## 1   24.0  0.00632  18.0  2.31    0 0.5380 6.575  65.2  4.0900   1 296
## 2   21.6  0.02731   0.0  7.07    0 0.4690 6.421  78.9  4.9671   2 242
## 3   34.7  0.02729   0.0  7.07    0 0.4690 7.185  61.1  4.9671   2 242
## 4   33.4  0.03237   0.0  2.18    0 0.4580 6.998  45.8  6.0622   3 222
## 5   36.2  0.06905   0.0  2.18    0 0.4580 7.147  54.2  6.0622   3 222
## 6   28.7  0.02985   0.0  2.18    0 0.4580 6.430  58.7  6.0622   3 222
```

```
## 8   27.1  0.14455  12.5  7.87    0 0.5240 6.172  96.1  5.9505   5 311
## 9   16.5  0.21124  12.5  7.87    0 0.5240 5.631 100.0  6.0821   5 311
## 10  18.9  0.17004  12.5  7.87    0 0.5240 6.004  85.9  6.5921   5 311
## 12  18.9  0.11747  12.5  7.87    0 0.5240 6.009  82.9  6.2267   5 311
## 16  19.9  0.62739   0.0  8.14    0 0.5380 5.834  56.5  4.4986   4 307
## 17  23.1  1.05393   0.0  8.14    0 0.5380 5.935  29.3  4.4986   4 307
## 19  20.2  0.80271   0.0  8.14    0 0.5380 5.456  36.6  3.7965   4 307
## 21  13.6  1.25179   0.0  8.14    0 0.5380 5.570  98.1  3.7979   4 307
## 23  15.2  1.23247   0.0  8.14    0 0.5380 6.142  91.7  3.9769   4 307
## 27  16.6  0.67191   0.0  8.14    0 0.5380 5.813  90.3  4.6820   4 307
## 28  14.8  0.95577   0.0  8.14    0 0.5380 6.047  88.8  4.4534   4 307
## 29  18.4  0.77299   0.0  8.14    0 0.5380 6.495  94.4  4.4547   4 307
## 30  21.0  1.00245   0.0  8.14    0 0.5380 6.674  87.3  4.2390   4 307
## 31  12.7  1.13081   0.0  8.14    0 0.5380 5.713  94.1  4.2330   4 307
## 32  14.5  1.35472   0.0  8.14    0 0.5380 6.072 100.0  4.1750   4 307
## 33  13.2  1.38799   0.0  8.14    0 0.5380 5.950  82.0  3.9900   4 307
## 34  13.1  1.15172   0.0  8.14    0 0.5380 5.701  95.0  3.7872   4 307
## 35  13.5  1.61282   0.0  8.14    0 0.5380 6.096  96.9  3.7598   4 307
## 36  18.9  0.06417   0.0  5.96    0 0.4990 5.933  68.2  3.3603   5 279
## 37  20.0  0.09744   0.0  5.96    0 0.4990 5.841  61.4  3.3779   5 279
## 38  21.0  0.08014   0.0  5.96    0 0.4990 5.850  41.5  3.9342   5 279
## 39  24.7  0.17505   0.0  5.96    0 0.4990 5.966  30.2  3.8473   5 279
## 41  34.9  0.03359  75.0  2.95    0 0.4280 7.024  15.8  5.4011   3 252
## 42  26.6  0.12744   0.0  6.91    0 0.4480 6.770   2.9  5.7209   3 233
## 44  24.7  0.15936   0.0  6.91    0 0.4480 6.211   6.5  5.7209   3 233
## 48  16.6  0.22927   0.0  6.91    0 0.4480 6.030  85.5  5.6894   3 233
## 51  19.7  0.08873  21.0  5.64    0 0.4390 5.963  45.7  6.8147   4 243
## 53  25.0  0.05360  21.0  5.64    0 0.4390 6.511  21.1  6.8147   4 243
## 56  35.4  0.01311  90.0  1.22    0 0.4030 7.249  21.9  8.6966   5 226
## 57  24.7  0.02055  85.0  0.74    0 0.4100 6.383  35.7  9.1876   2 313
## 58  31.6  0.01432 100.0  1.32    0 0.4110 6.816  40.5  8.3248   5 256
## 59  23.3  0.15445  25.0  5.13    0 0.4530 6.145  29.2  7.8148   8 284
## 60  19.6  0.10328  25.0  5.13    0 0.4530 5.927  47.2  6.9320   8 284
## 61  18.7  0.14932  25.0  5.13    0 0.4530 5.741  66.2  7.2254   8 284
## 62  16.0  0.17171  25.0  5.13    0 0.4530 5.966  93.4  6.8185   8 284
## 63  22.2  0.11027  25.0  5.13    0 0.4530 6.456  67.8  7.2255   8 284
## 64  25.0  0.12650  25.0  5.13    0 0.4530 6.762  43.4  7.9809   8 284
## 66  23.5  0.03584  80.0  3.37    0 0.3980 6.290  17.8  6.6115   4 337
## 67  19.4  0.04379  80.0  3.37    0 0.3980 5.787  31.1  6.6115   4 337
## 68  22.0  0.05789  12.5  6.07    0 0.4090 5.878  21.4  6.4980   4 345
## 69  17.4  0.13554  12.5  6.07    0 0.4090 5.594  36.8  6.4980   4 345
## 72  21.7  0.15876   0.0 10.81    0 0.4130 5.961  17.5  5.2873   4 305
## 73  22.8  0.09164   0.0 10.81    0 0.4130 6.065   7.8  5.2873   4 305
## 74  23.4  0.19539   0.0 10.81    0 0.4130 6.245   6.2  5.2873   4 305
## 75  24.1  0.07896   0.0 12.83    0 0.4370 6.273   6.0  4.2515   5 398
## 77  20.0  0.10153   0.0 12.83    0 0.4370 6.279  74.5  4.0522   5 398
## 79  21.2  0.05646   0.0 12.83    0 0.4370 6.232  53.7  5.0141   5 398
## 80  20.3  0.08387   0.0 12.83    0 0.4370 5.874  36.6  4.5026   5 398
## 81  28.0  0.04113  25.0  4.86    0 0.4260 6.727  33.5  5.4007   4 281
## 82  23.9  0.04462  25.0  4.86    0 0.4260 6.619  70.4  5.4007   4 281
## 83  24.8  0.03659  25.0  4.86    0 0.4260 6.302  32.2  5.4007   4 281
## 84  22.9  0.03551  25.0  4.86    0 0.4260 6.167  46.7  5.4007   4 281
## 86  26.6  0.05735   0.0  4.49    0 0.4490 6.630  56.1  4.4377   3 247
## 88  22.2  0.07151   0.0  4.49    0 0.4490 6.121  56.8  3.7476   3 247
```

```
## 89  23.6  0.05660   0.0  3.41    0 0.4890 7.007  86.3  3.4217    2 270
## 91  22.6  0.04684   0.0  3.41    0 0.4890 6.417  66.1  3.0923    2 270
## 92  22.0  0.03932   0.0  3.41    0 0.4890 6.405  73.9  3.0921    2 270
## 94  25.0  0.02875  28.0 15.04    0 0.4640 6.211  28.9  3.6659    4 270
## 95  20.6  0.04294  28.0 15.04    0 0.4640 6.249  77.3  3.6150    4 270
## 98  38.7  0.12083   0.0  2.89    0 0.4450 8.069  76.0  3.4952    2 276
## 99  43.8  0.08187   0.0  2.89    0 0.4450 7.820  36.9  3.4952    2 276
## 100 33.2  0.06860   0.0  2.89    0 0.4450 7.416  62.5  3.4952    2 276
## 102 26.5  0.11432   0.0  8.56    0 0.5200 6.781  71.3  2.8561    5 384
## 104 19.3  0.21161   0.0  8.56    0 0.5200 6.137  87.4  2.7147    5 384
## 107 19.5  0.17120   0.0  8.56    0 0.5200 5.836  91.9  2.2110    5 384
## 108 20.4  0.13117   0.0  8.56    0 0.5200 6.127  85.2  2.1224    5 384
## 109 19.8  0.12802   0.0  8.56    0 0.5200 6.474  97.1  2.4329    5 384
## 110 19.4  0.26363   0.0  8.56    0 0.5200 6.229  91.2  2.5451    5 384
## 111 21.7  0.10793   0.0  8.56    0 0.5200 6.195  54.4  2.7778    5 384
## 112 22.8  0.10084   0.0 10.01    0 0.5470 6.715  81.6  2.6775    6 432
## 113 18.8  0.12329   0.0 10.01    0 0.5470 5.913  92.9  2.3534    6 432
## 114 18.7  0.22212   0.0 10.01    0 0.5470 6.092  95.4  2.5480    6 432
## 115 18.5  0.14231   0.0 10.01    0 0.5470 6.254  84.2  2.2565    6 432
## 116 18.3  0.17134   0.0 10.01    0 0.5470 5.928  88.2  2.4631    6 432
## 118 19.2  0.15098   0.0 10.01    0 0.5470 6.021  82.6  2.7474    6 432
## 119 20.4  0.13058   0.0 10.01    0 0.5470 5.872  73.1  2.4775    6 432
## 120 19.3  0.14476   0.0 10.01    0 0.5470 5.731  65.2  2.7592    6 432
## 121 22.0  0.06899   0.0 25.65    0 0.5810 5.870  69.7  2.2577    2 188
## 122 20.3  0.07165   0.0 25.65    0 0.5810 6.004  84.1  2.1974    2 188
## 123 20.5  0.09299   0.0 25.65    0 0.5810 5.961  92.9  2.0869    2 188
## 124 17.3  0.15038   0.0 25.65    0 0.5810 5.856  97.0  1.9444    2 188
## 125 18.8  0.09849   0.0 25.65    0 0.5810 5.879  95.8  2.0063    2 188
## 126 21.4  0.16902   0.0 25.65    0 0.5810 5.986  88.4  1.9929    2 188
## 128 16.2  0.25915   0.0 21.89    0 0.6240 5.693  96.0  1.7883    4 437
## 131 19.2  0.34006   0.0 21.89    0 0.6240 6.458  98.9  2.1185    4 437
## 132 19.6  1.19294   0.0 21.89    0 0.6240 6.326  97.7  2.2710    4 437
## 133 23.0  0.59005   0.0 21.89    0 0.6240 6.372  97.9  2.3274    4 437
## 134 18.4  0.32982   0.0 21.89    0 0.6240 5.822  95.4  2.4699    4 437
## 136 18.1  0.55778   0.0 21.89    0 0.6240 6.335  98.2  2.1107    4 437
## 137 17.4  0.32264   0.0 21.89    0 0.6240 5.942  93.5  1.9669    4 437
## 140 17.8  0.54452   0.0 21.89    0 0.6240 6.151  97.9  1.6687    4 437
## 142 14.4  1.62864   0.0 21.89    0 0.6240 5.019 100.0  1.4394    4 437
## 143 13.4  3.32105   0.0 19.58    1 0.8710 5.403 100.0  1.3216    5 403
## 144 15.6  4.09740   0.0 19.58    0 0.8710 5.468 100.0  1.4118    5 403
## 145 11.8  2.77974   0.0 19.58    0 0.8710 4.903  97.8  1.3459    5 403
## 146 13.8  2.37934   0.0 19.58    0 0.8710 6.130 100.0  1.4191    5 403
## 147 15.6  2.15505   0.0 19.58    0 0.8710 5.628 100.0  1.5166    5 403
## 148 14.6  2.36862   0.0 19.58    0 0.8710 4.926  95.7  1.4608    5 403
## 149 17.8  2.33099   0.0 19.58    0 0.8710 5.186  93.8  1.5296    5 403
## 150 15.4  2.73397   0.0 19.58    0 0.8710 5.597  94.9  1.5257    5 403
## 152 19.6  1.49632   0.0 19.58    0 0.8710 5.404 100.0  1.5916    5 403
## 153 15.3  1.12658   0.0 19.58    1 0.8710 5.012  88.0  1.6102    5 403
## 156 15.6  3.53501   0.0 19.58    1 0.8710 6.152  82.6  1.7455    5 403
## 157 13.1  2.44668   0.0 19.58    0 0.8710 5.272  94.0  1.7364    5 403
## 158 41.3  1.22358   0.0 19.58    0 0.6050 6.943  97.4  1.8773    5 403
## 159 24.3  1.34284   0.0 19.58    0 0.6050 6.066 100.0  1.7573    5 403
## 162 50.0  1.46336   0.0 19.58    0 0.6050 7.489  90.8  1.9709    5 403
## 163 50.0  1.83377   0.0 19.58    1 0.6050 7.802  98.2  2.0407    5 403
```

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## 165 22.7  2.24236   0.0 19.58    0 0.6050 5.854  91.8  2.4220    5 403
## 166 25.0  2.92400   0.0 19.58    0 0.6050 6.101  93.0  2.2834    5 403
## 167 50.0  2.01019   0.0 19.58    0 0.6050 7.929  96.2  2.0459    5 403
## 168 23.8  1.80028   0.0 19.58    0 0.6050 5.877  79.2  2.4259    5 403
## 169 23.8  2.30040   0.0 19.58    0 0.6050 6.319  96.1  2.1000    5 403
## 170 22.3  2.44953   0.0 19.58    0 0.6050 6.402  95.2  2.2625    5 403
## 171 17.4  1.20742   0.0 19.58    0 0.6050 5.875  94.6  2.4259    5 403
## 172 19.1  2.31390   0.0 19.58    0 0.6050 5.880  97.3  2.3887    5 403
## 173 23.1  0.13914   0.0  4.05    0 0.5100 5.572  88.5  2.5961    5 296
## 174 23.6  0.09178   0.0  4.05    0 0.5100 6.416  84.1  2.6463    5 296
## 175 22.6  0.08447   0.0  4.05    0 0.5100 5.859  68.7  2.7019    5 296
## 176 29.4  0.06664   0.0  4.05    0 0.5100 6.546  33.1  3.1323    5 296
## 178 24.6  0.05425   0.0  4.05    0 0.5100 6.315  73.4  3.3175    5 296
## 179 29.9  0.06642   0.0  4.05    0 0.5100 6.860  74.4  2.9153    5 296
## 180 37.2  0.05780   0.0  2.46    0 0.4880 6.980  58.4  2.8290    3 193
## 181 39.8  0.06588   0.0  2.46    0 0.4880 7.765  83.3  2.7410    3 193
## 182 36.2  0.06888   0.0  2.46    0 0.4880 6.144  62.2  2.5979    3 193
## 184 32.5  0.10008   0.0  2.46    0 0.4880 6.563  95.6  2.8470    3 193
## 186 29.6  0.06047   0.0  2.46    0 0.4880 6.153  68.8  3.2797    3 193
## 187 50.0  0.05602   0.0  2.46    0 0.4880 7.831  53.6  3.1992    3 193
## 188 32.0  0.07875  45.0  3.44    0 0.4370 6.782  41.1  3.7886    5 398
## 189 29.8  0.12579  45.0  3.44    0 0.4370 6.556  29.1  4.5667    5 398
## 190 34.9  0.08370  45.0  3.44    0 0.4370 7.185  38.9  4.5667    5 398
## 191 37.0  0.09068  45.0  3.44    0 0.4370 6.951  21.5  6.4798    5 398
## 192 30.5  0.06911  45.0  3.44    0 0.4370 6.739  30.8  6.4798    5 398
## 193 36.4  0.08664  45.0  3.44    0 0.4370 7.178  26.3  6.4798    5 398
## 195 29.1  0.01439  60.0  2.93    0 0.4010 6.604  18.8  6.2196    1 265
## 196 50.0  0.01381  80.0  0.46    0 0.4220 7.875  32.0  5.6484    4 255
## 197 33.3  0.04011  80.0  1.52    0 0.4040 7.287  34.1  7.3090    2 329
## 198 30.3  0.04666  80.0  1.52    0 0.4040 7.107  36.6  7.3090    2 329
## 199 34.6  0.03768  80.0  1.52    0 0.4040 7.274  38.3  7.3090    2 329
## 200 34.9  0.03150  95.0  1.47    0 0.4030 6.975  15.3  7.6534    3 402
## 202 24.1  0.03445  82.5  2.03    0 0.4150 6.162  38.4  6.2700    2 348
## 203 42.3  0.02177  82.5  2.03    0 0.4150 7.610  15.7  6.2700    2 348
## 204 48.5  0.03510  95.0  2.68    0 0.4161 7.853  33.2  5.1180    4 224
## 205 50.0  0.02009  95.0  2.68    0 0.4161 8.034  31.9  5.1180    4 224
## 206 22.6  0.13642   0.0 10.59    0 0.4890 5.891  22.3  3.9454    4 277
## 208 22.5  0.25199   0.0 10.59    0 0.4890 5.783  72.7  4.3549    4 277
## 209 24.4  0.13587   0.0 10.59    1 0.4890 6.064  59.1  4.2392    4 277
## 212 19.3  0.37578   0.0 10.59    1 0.4890 5.404  88.6  3.6650    4 277
## 213 22.4  0.21719   0.0 10.59    1 0.4890 5.807  53.8  3.6526    4 277
## 214 28.1  0.14052   0.0 10.59    0 0.4890 6.375  32.3  3.9454    4 277
## 216 25.0  0.19802   0.0 10.59    0 0.4890 6.182  42.4  3.9454    4 277
## 218 28.7  0.07013   0.0 13.89    0 0.5500 6.642  85.1  3.4211    5 276
## 219 21.5  0.11069   0.0 13.89    1 0.5500 5.951  93.8  2.8893    5 276
## 220 23.0  0.11425   0.0 13.89    1 0.5500 6.373  92.4  3.3633    5 276
## 221 26.7  0.35809   0.0  6.20    1 0.5070 6.951  88.5  2.8617    8 307
## 223 27.5  0.62356   0.0  6.20    1 0.5070 6.879  77.7  3.2721    8 307
## 225 44.8  0.31533   0.0  6.20    0 0.5040 8.266  78.3  2.8944    8 307
## 226 50.0  0.52693   0.0  6.20    0 0.5040 8.725  83.0  2.8944    8 307
## 227 37.6  0.38214   0.0  6.20    0 0.5040 8.040  86.5  3.2157    8 307
## 228 31.6  0.41238   0.0  6.20    0 0.5040 7.163  79.9  3.2157    8 307
## 230 31.5  0.44178   0.0  6.20    0 0.5040 6.552  21.4  3.3751    8 307
## 231 24.3  0.53700   0.0  6.20    0 0.5040 5.981  68.1  3.6715    8 307
```

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## 234 48.3  0.33147   0.0  6.20    0 0.5070 8.247  70.4  3.6519   8 307
## 235 29.0  0.44791   0.0  6.20    1 0.5070 6.726  66.5  3.6519   8 307
## 236 24.0  0.33045   0.0  6.20    0 0.5070 6.086  61.5  3.6519   8 307
## 237 25.1  0.52058   0.0  6.20    1 0.5070 6.631  76.5  4.1480   8 307
## 239 23.7  0.08244  30.0  4.93    0 0.4280 6.481  18.5  6.1899   6 300
## 240 23.3  0.09252  30.0  4.93    0 0.4280 6.606  42.2  6.1899   6 300
## 242 20.1  0.10612  30.0  4.93    0 0.4280 6.095  65.1  6.3361   6 300
## 243 22.2  0.10290  30.0  4.93    0 0.4280 6.358  52.9  7.0355   6 300
## 244 23.7  0.12757  30.0  4.93    0 0.4280 6.393   7.8  7.0355   6 300
## 246 18.5  0.19133  22.0  5.86    0 0.4310 5.605  70.2  7.9549   7 330
## 247 24.3  0.33983  22.0  5.86    0 0.4310 6.108  34.9  8.0555   7 330
## 248 20.5  0.19657  22.0  5.86    0 0.4310 6.226  79.2  8.0555   7 330
## 249 24.5  0.16439  22.0  5.86    0 0.4310 6.433  49.1  7.8265   7 330
## 250 26.2  0.19073  22.0  5.86    0 0.4310 6.718  17.5  7.8265   7 330
## 251 24.4  0.14030  22.0  5.86    0 0.4310 6.487  13.0  7.3967   7 330
## 252 24.8  0.21409  22.0  5.86    0 0.4310 6.438   8.9  7.3967   7 330
## 254 42.8  0.36894  22.0  5.86    0 0.4310 8.259   8.4  8.9067   7 330
## 255 21.9  0.04819  80.0  3.64    0 0.3920 6.108  32.0  9.2203   1 315
## 256 20.9  0.03548  80.0  3.64    0 0.3920 5.876  19.1  9.2203   1 315
## 258 50.0  0.61154  20.0  3.97    0 0.6470 8.704  86.9  1.8010   5 264
## 260 30.1  0.65665  20.0  3.97    0 0.6470 6.842 100.0  2.0107   5 264
## 262 43.1  0.53412  20.0  3.97    0 0.6470 7.520  89.4  2.1398   5 264
## 264 31.0  0.82526  20.0  3.97    0 0.6470 7.327  94.5  2.0788   5 264
## 265 36.5  0.55007  20.0  3.97    0 0.6470 7.206  91.6  1.9301   5 264
## 266 22.8  0.76162  20.0  3.97    0 0.6470 5.560  62.8  1.9865   5 264
## 267 30.7  0.78570  20.0  3.97    0 0.6470 7.014  84.6  2.1329   5 264
## 268 50.0  0.57834  20.0  3.97    0 0.5750 8.297  67.0  2.4216   5 264
## 269 43.5  0.54050  20.0  3.97    0 0.5750 7.470  52.6  2.8720   5 264
## 270 20.7  0.09065  20.0  6.96    1 0.4640 5.920  61.5  3.9175   3 223
## 271 21.1  0.29916  20.0  6.96    0 0.4640 5.856  42.1  4.4290   3 223
## 273 24.4  0.11460  20.0  6.96    0 0.4640 6.538  58.7  3.9175   3 223
## 274 35.2  0.22188  20.0  6.96    1 0.4640 7.691  51.8  4.3665   3 223
## 275 32.4  0.05644  40.0  6.41    1 0.4470 6.758  32.9  4.0776   4 254
## 276 32.0  0.09604  40.0  6.41    0 0.4470 6.854  42.8  4.2673   4 254
## 277 33.2  0.10469  40.0  6.41    1 0.4470 7.267  49.0  4.7872   4 254
## 279 29.1  0.07978  40.0  6.41    0 0.4470 6.482  32.1  4.1403   4 254
## 280 35.1  0.21038  20.0  3.33    0 0.4429 6.812  32.2  4.1007   5 216
## 281 45.4  0.03578  20.0  3.33    0 0.4429 7.820  64.5  4.6947   5 216
## 282 35.4  0.03705  20.0  3.33    0 0.4429 6.968  37.2  5.2447   5 216
## 283 46.0  0.06129  20.0  3.33    1 0.4429 7.645  49.7  5.2119   5 216
## 285 32.2  0.00906  90.0  2.97    0 0.4000 7.088  20.8  7.3073   1 285
## 286 22.0  0.01096  55.0  2.25    0 0.3890 6.453  31.9  7.3073   1 300
## 287 20.1  0.01965  80.0  1.76    0 0.3850 6.230  31.5  9.0892   1 241
## 288 23.2  0.03871  52.5  5.32    0 0.4050 6.209  31.3  7.3172   6 293
## 289 22.3  0.04590  52.5  5.32    0 0.4050 6.315  45.6  7.3172   6 293
## 290 24.8  0.04297  52.5  5.32    0 0.4050 6.565  22.9  7.3172   6 293
## 291 28.5  0.03502  80.0  4.95    0 0.4110 6.861  27.9  5.1167   4 245
## 292 37.3  0.07886  80.0  4.95    0 0.4110 7.148  27.7  5.1167   4 245
## 294 23.9  0.08265   0.0 13.92    0 0.4370 6.127  18.4  5.5027   4 289
## 296 28.6  0.12932   0.0 13.92    0 0.4370 6.678  31.1  5.9604   4 289
## 297 27.1  0.05372   0.0 13.92    0 0.4370 6.549  51.0  5.9604   4 289
## 299 22.5  0.06466  70.0  2.24    0 0.4000 6.345  20.1  7.8278   5 358
## 301 24.8  0.04417  70.0  2.24    0 0.4000 6.871  47.4  7.8278   5 358
## 303 26.4  0.09266  34.0  6.09    0 0.4330 6.495  18.4  5.4917   7 329
```

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## 304 33.1  0.10000  34.0  6.09    0 0.4330 6.982  17.7  5.4917   7 329
## 305 36.1  0.05515  33.0  2.18    0 0.4720 7.236  41.1  4.0220   7 222
## 306 28.4  0.05479  33.0  2.18    0 0.4720 6.616  58.1  3.3700   7 222
## 307 33.4  0.07503  33.0  2.18    0 0.4720 7.420  71.9  3.0992   7 222
## 308 28.2  0.04932  33.0  2.18    0 0.4720 6.849  70.3  3.1827   7 222
## 309 22.8  0.49298   0.0  9.90    0 0.5440 6.635  82.5  3.3175   4 304
## 311 16.1  2.63548   0.0  9.90    0 0.5440 4.973  37.8  2.5194   4 304
## 313 19.4  0.26169   0.0  9.90    0 0.5440 6.023  90.4  2.8340   4 304
## 314 21.6  0.26938   0.0  9.90    0 0.5440 6.266  82.8  3.2628   4 304
## 316 16.2  0.25356   0.0  9.90    0 0.5440 5.705  77.7  3.9450   4 304
## 317 17.8  0.31827   0.0  9.90    0 0.5440 5.914  83.2  3.9986   4 304
## 318 19.8  0.24522   0.0  9.90    0 0.5440 5.782  71.7  4.0317   4 304
## 319 23.1  0.40202   0.0  9.90    0 0.5440 6.382  67.2  3.5325   4 304
## 320 21.0  0.47547   0.0  9.90    0 0.5440 6.113  58.8  4.0019   4 304
## 321 23.8  0.16760   0.0  7.38    0 0.4930 6.426  52.3  4.5404   5 287
## 322 23.1  0.18159   0.0  7.38    0 0.4930 6.376  54.3  4.5404   5 287
## 323 20.4  0.35114   0.0  7.38    0 0.4930 6.041  49.9  4.7211   5 287
## 324 18.5  0.28392   0.0  7.38    0 0.4930 5.708  74.3  4.7211   5 287
## 327 23.0  0.30347   0.0  7.38    0 0.4930 6.312  28.9  5.4159   5 287
## 328 22.2  0.24103   0.0  7.38    0 0.4930 6.083  43.7  5.4159   5 287
## 330 22.6  0.06724   0.0  3.24    0 0.4600 6.333  17.2  5.2146   4 430
## 331 19.8  0.04544   0.0  3.24    0 0.4600 6.144  32.2  5.8736   4 430
## 332 17.1  0.05023  35.0  6.06    0 0.4379 5.706  28.4  6.6407   1 304
## 333 19.4  0.03466  35.0  6.06    0 0.4379 6.031  23.3  6.6407   1 304
## 334 22.2  0.05083   0.0  5.19    0 0.5150 6.316  38.1  6.4584   5 224
## 335 20.7  0.03738   0.0  5.19    0 0.5150 6.310  38.5  6.4584   5 224
## 336 21.1  0.03961   0.0  5.19    0 0.5150 6.037  34.5  5.9853   5 224
## 337 19.5  0.03427   0.0  5.19    0 0.5150 5.869  46.3  5.2311   5 224
## 338 18.5  0.03041   0.0  5.19    0 0.5150 5.895  59.6  5.6150   5 224
## 340 19.0  0.05497   0.0  5.19    0 0.5150 5.985  45.4  4.8122   5 224
## 341 18.7  0.06151   0.0  5.19    0 0.5150 5.968  58.5  4.8122   5 224
## 343 16.5  0.02498   0.0  1.89    0 0.5180 6.540  59.7  6.2669   1 422
## 345 31.2  0.03049  55.0  3.78    0 0.4840 6.874  28.1  6.4654   5 370
## 346 17.5  0.03113   0.0  4.39    0 0.4420 6.014  48.5  8.0136   3 352
## 347 17.2  0.06162   0.0  4.39    0 0.4420 5.898  52.3  8.0136   3 352
## 349 24.5  0.01501  80.0  2.01    0 0.4350 6.635  29.7  8.3440   4 280
## 350 26.6  0.02899  40.0  1.25    0 0.4290 6.939  34.5  8.7921   1 335
## 351 22.9  0.06211  40.0  1.25    0 0.4290 6.490  44.4  8.7921   1 335
## 352 24.1  0.07950  60.0  1.69    0 0.4110 6.579  35.9 10.7103   4 411
## 353 18.6  0.07244  60.0  1.69    0 0.4110 5.884  18.5 10.7103   4 411
## 355 18.2  0.04301  80.0  1.91    0 0.4130 5.663  21.9 10.5857   4 334
## 356 20.6  0.10659  80.0  1.91    0 0.4130 5.936  19.5 10.5857   4 334
## 358 21.7  3.84970   0.0 18.10    1 0.7700 6.395  91.0  2.5052  24 666
## 359 22.7  5.20177   0.0 18.10    1 0.7700 6.127  83.4  2.7227  24 666
## 360 22.6  4.26131   0.0 18.10    0 0.7700 6.112  81.3  2.5091  24 666
## 361 25.0  4.54192   0.0 18.10    0 0.7700 6.398  88.0  2.5182  24 666
## 363 20.8  3.67822   0.0 18.10    0 0.7700 5.362  96.2  2.1036  24 666
## 365 21.9  3.47428   0.0 18.10    1 0.7180 8.780  82.9  1.9047  24 666
## 366 27.5  4.55587   0.0 18.10    0 0.7180 3.561  87.9  1.6132  24 666
## 367 21.9  3.69695   0.0 18.10    0 0.7180 4.963  91.4  1.7523  24 666
## 369 50.0  4.89822   0.0 18.10    0 0.6310 4.970 100.0  1.3325  24 666
## 370 50.0  5.66998   0.0 18.10    1 0.6310 6.683  96.8  1.3567  24 666
## 373 50.0  8.26725   0.0 18.10    1 0.6680 5.875  89.6  1.1296  24 666
## 374 13.8 11.10810   0.0 18.10    0 0.6680 4.906 100.0  1.1742  24 666
```

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## 375 13.8 18.49820   0.0 18.10    0 0.6680 4.138 100.0  1.1370  24 666
## 376 15.0 19.60910   0.0 18.10    0 0.6710 7.313  97.9  1.3163  24 666
## 379 13.1 23.64820   0.0 18.10    0 0.6710 6.380  96.2  1.3861  24 666
## 380 10.2 17.86670   0.0 18.10    0 0.6710 6.223 100.0  1.3861  24 666
## 382 10.9 15.87440   0.0 18.10    0 0.6710 6.545  99.1  1.5192  24 666
## 383 11.3  9.18702   0.0 18.10    0 0.7000 5.536 100.0  1.5804  24 666
## 384 12.3  7.99248   0.0 18.10    0 0.7000 5.520 100.0  1.5331  24 666
## 385  8.8 20.08490   0.0 18.10    0 0.7000 4.368  91.2  1.4395  24 666
## 386  7.2 16.81180   0.0 18.10    0 0.7000 5.277  98.1  1.4261  24 666
## 387 10.5 24.39380   0.0 18.10    0 0.7000 4.652 100.0  1.4672  24 666
## 388  7.4 22.59710   0.0 18.10    0 0.7000 5.000  89.5  1.5184  24 666
## 389 10.2 14.33370   0.0 18.10    0 0.7000 4.880 100.0  1.5895  24 666
## 390 11.5  8.15174   0.0 18.10    0 0.7000 5.390  98.9  1.7281  24 666
## 392 23.2  5.29305   0.0 18.10    0 0.7000 6.051  82.5  2.1678  24 666
## 394 13.8  8.64476   0.0 18.10    0 0.6930 6.193  92.6  1.7912  24 666
## 396 13.1  8.71675   0.0 18.10    0 0.6930 6.471  98.8  1.7257  24 666
## 397 12.5  5.87205   0.0 18.10    0 0.6930 6.405  96.0  1.6768  24 666
## 398  8.5  7.67202   0.0 18.10    0 0.6930 5.747  98.9  1.6334  24 666
## 399  5.0 38.35180   0.0 18.10    0 0.6930 5.453 100.0  1.4896  24 666
## 400  6.3  9.91655   0.0 18.10    0 0.6930 5.852  77.8  1.5004  24 666
## 401  5.6 25.04610   0.0 18.10    0 0.6930 5.987 100.0  1.5888  24 666
## 402  7.2 14.23620   0.0 18.10    0 0.6930 6.343 100.0  1.5741  24 666
## 404  8.3 24.80170   0.0 18.10    0 0.6930 5.349  96.0  1.7028  24 666
## 405  8.5 41.52920   0.0 18.10    0 0.6930 5.531  85.4  1.6074  24 666
## 407 11.9 20.71620   0.0 18.10    0 0.6590 4.138 100.0  1.1781  24 666
## 408 27.9 11.95110   0.0 18.10    0 0.6590 5.608 100.0  1.2852  24 666
## 409 17.2  7.40389   0.0 18.10    0 0.5970 5.617  97.9  1.4547  24 666
## 410 27.5 14.43830   0.0 18.10    0 0.5970 6.852 100.0  1.4655  24 666
## 411 15.0 51.13580   0.0 18.10    0 0.5970 5.757 100.0  1.4130  24 666
## 412 17.2 14.05070   0.0 18.10    0 0.5970 6.657 100.0  1.5275  24 666
## 417  7.5 10.83420   0.0 18.10    0 0.6790 6.782  90.8  1.8195  24 666
## 418 10.4 25.94060   0.0 18.10    0 0.6790 5.304  89.1  1.6475  24 666
## 419  8.8 73.53410   0.0 18.10    0 0.6790 5.957 100.0  1.8026  24 666
## 421 16.7 11.08740   0.0 18.10    0 0.7180 6.411 100.0  1.8589  24 666
## 422 14.2  7.02259   0.0 18.10    0 0.7180 6.006  95.3  1.8746  24 666
## 423 20.8 12.04820   0.0 18.10    0 0.6140 5.648  87.6  1.9512  24 666
## 425 11.7  8.79212   0.0 18.10    0 0.5840 5.565  70.6  2.0635  24 666
## 426  8.3 15.86030   0.0 18.10    0 0.6790 5.896  95.4  1.9096  24 666
## 427 10.2 12.24720   0.0 18.10    0 0.5840 5.837  59.7  1.9976  24 666
## 428 10.9 37.66190   0.0 18.10    0 0.6790 6.202  78.7  1.8629  24 666
## 429 11.0  7.36711   0.0 18.10    0 0.6790 6.193  78.1  1.9356  24 666
## 430  9.5  9.33889   0.0 18.10    0 0.6790 6.380  95.6  1.9682  24 666
## 431 14.5  8.49213   0.0 18.10    0 0.5840 6.348  86.1  2.0527  24 666
## 432 14.1 10.06230   0.0 18.10    0 0.5840 6.833  94.3  2.0882  24 666
## 433 16.1  6.44405   0.0 18.10    0 0.5840 6.425  74.8  2.2004  24 666
## 434 14.3  5.58107   0.0 18.10    0 0.7130 6.436  87.9  2.3158  24 666
## 435 11.7 13.91340   0.0 18.10    0 0.7130 6.208  95.0  2.2222  24 666
## 436 13.4 11.16040   0.0 18.10    0 0.7400 6.629  94.6  2.1247  24 666
## 437  9.6 14.42080   0.0 18.10    0 0.7400 6.461  93.3  2.0026  24 666
## 438  8.7 15.17720   0.0 18.10    0 0.7400 6.152 100.0  1.9142  24 666
## 439  8.4 13.67810   0.0 18.10    0 0.7400 5.935  87.9  1.8206  24 666
## 440 12.8  9.39063   0.0 18.10    0 0.7400 5.627  93.9  1.8172  24 666
## 441 10.5 22.05110   0.0 18.10    0 0.7400 5.818  92.4  1.8662  24 666
## 442 17.1  9.72418   0.0 18.10    0 0.7400 6.406  97.2  2.0651  24 666
```

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## 443 18.4  5.66637   0.0 18.10    0 0.7400 6.219 100.0  2.0048  24 666
## 444 15.4  9.96654   0.0 18.10    0 0.7400 6.485 100.0  1.9784  24 666
## 446 11.8 10.67180   0.0 18.10    0 0.7400 6.459  94.8  1.9879  24 666
## 447 14.9  6.28807   0.0 18.10    0 0.7400 6.341  96.4  2.0720  24 666
## 449 14.1  9.32909   0.0 18.10    0 0.7130 6.185  98.7  2.2616  24 666
## 450 13.0  7.52601   0.0 18.10    0 0.7130 6.417  98.3  2.1850  24 666
## 451 13.4  6.71772   0.0 18.10    0 0.7130 6.749  92.6  2.3236  24 666
## 452 15.2  5.44114   0.0 18.10    0 0.7130 6.655  98.2  2.3552  24 666
## 454 17.8  8.24809   0.0 18.10    0 0.7130 7.393  99.3  2.4527  24 666
## 455 14.9  9.51363   0.0 18.10    0 0.7130 6.728  94.1  2.4961  24 666
## 457 12.7  4.66883   0.0 18.10    0 0.7130 5.976  87.9  2.5806  24 666
## 458 13.5  8.20058   0.0 18.10    0 0.7130 5.936  80.3  2.7792  24 666
## 459 14.9  7.75223   0.0 18.10    0 0.7130 6.301  83.7  2.7831  24 666
## 461 16.4  4.81213   0.0 18.10    0 0.7130 6.701  90.0  2.5975  24 666
## 462 17.7  3.69311   0.0 18.10    0 0.7130 6.376  88.4  2.5671  24 666
## 463 19.5  6.65492   0.0 18.10    0 0.7130 6.317  83.0  2.7344  24 666
## 466 19.9  3.16360   0.0 18.10    0 0.6550 5.759  48.2  3.0665  24 666
## 467 19.0  3.77498   0.0 18.10    0 0.6550 5.952  84.7  2.8715  24 666
## 468 19.1  4.42228   0.0 18.10    0 0.5840 6.003  94.5  2.5403  24 666
## 469 19.1 15.57570   0.0 18.10    0 0.5800 5.926  71.0  2.9084  24 666
## 470 20.1 13.07510   0.0 18.10    0 0.5800 5.713  56.7  2.8237  24 666
## 471 19.9  4.34879   0.0 18.10    0 0.5800 6.167  84.0  3.0334  24 666
## 472 19.6  4.03841   0.0 18.10    0 0.5320 6.229  90.7  3.0993  24 666
## 473 23.2  3.56868   0.0 18.10    0 0.5800 6.437  75.0  2.8965  24 666
## 474 29.8  4.64689   0.0 18.10    0 0.6140 6.980  67.6  2.5329  24 666
## 475 13.8  8.05579   0.0 18.10    0 0.5840 5.427  95.4  2.4298  24 666
## 476 13.3  6.39312   0.0 18.10    0 0.5840 6.162  97.4  2.2060  24 666
## 477 16.7  4.87141   0.0 18.10    0 0.6140 6.484  93.6  2.3053  24 666
## 478 12.0 15.02340   0.0 18.10    0 0.6140 5.304  97.3  2.1007  24 666
## 479 14.6 10.23300   0.0 18.10    0 0.6140 6.185  96.7  2.1705  24 666
## 480 21.4 14.33370   0.0 18.10    0 0.6140 6.229  88.0  1.9512  24 666
## 483 25.0  5.73116   0.0 18.10    0 0.5320 7.061  77.0  3.4106  24 666
## 484 21.8  2.81838   0.0 18.10    0 0.5320 5.762  40.3  4.0983  24 666
## 485 20.6  2.37857   0.0 18.10    0 0.5830 5.871  41.9  3.7240  24 666
## 486 21.2  3.67367   0.0 18.10    0 0.5830 6.312  51.9  3.9917  24 666
## 488 20.6  4.83567   0.0 18.10    0 0.5830 5.905  53.2  3.1523  24 666
## 490  7.0  0.18337   0.0 27.74    0 0.6090 5.414  98.3  1.7554   4 711
## 492 13.6  0.10574   0.0 27.74    0 0.6090 5.983  98.8  1.8681   4 711
## 493 20.1  0.11132   0.0 27.74    0 0.6090 5.983  83.5  2.1099   4 711
## 494 21.8  0.17331   0.0  9.69    0 0.5850 5.707  54.0  2.3817   6 391
## 495 24.5  0.27957   0.0  9.69    0 0.5850 5.926  42.6  2.3817   6 391
## 496 23.1  0.17899   0.0  9.69    0 0.5850 5.670  28.8  2.7986   6 391
## 497 19.7  0.28960   0.0  9.69    0 0.5850 5.390  72.9  2.7986   6 391
## 498 18.3  0.26838   0.0  9.69    0 0.5850 5.794  70.6  2.8927   6 391
## 499 21.2  0.23912   0.0  9.69    0 0.5850 6.019  65.3  2.4091   6 391
## 500 17.5  0.17783   0.0  9.69    0 0.5850 5.569  73.5  2.3999   6 391
## 501 16.8  0.22438   0.0  9.69    0 0.5850 6.027  79.7  2.4982   6 391
## 502 22.4  0.06263   0.0 11.93    0 0.5730 6.593  69.1  2.4786   1 273
## 503 20.6  0.04527   0.0 11.93    0 0.5730 6.120  76.7  2.2875   1 273
## 505 22.0  0.10959   0.0 11.93    0 0.5730 6.794  89.3  2.3889   1 273
## 506 11.9  0.04741   0.0 11.93    0 0.5730 6.030  80.8  2.5050   1 273
##     PTRATIO      B LSTAT
## 1      15.3 396.90  4.98
## 2      17.8 396.90  9.14
```

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## 3       17.8 392.83  4.03
## 4       18.7 394.63  2.94
## 5       18.7 396.90  5.33
## 6       18.7 394.12  5.21
## 8       15.2 396.90 19.15
## 9       15.2 386.63 29.93
## 10      15.2 386.71 17.10
## 12      15.2 396.90 13.27
## 16      21.0 395.62  8.47
## 17      21.0 386.85  6.58
## 19      21.0 288.99 11.69
## 21      21.0 376.57 21.02
## 23      21.0 396.90 18.72
## 27      21.0 376.88 14.81
## 28      21.0 306.38 17.28
## 29      21.0 387.94 12.80
## 30      21.0 380.23 11.98
## 31      21.0 360.17 22.60
## 32      21.0 376.73 13.04
## 33      21.0 232.60 27.71
## 34      21.0 358.77 18.35
## 35      21.0 248.31 20.34
## 36      19.2 396.90  9.68
## 37      19.2 377.56 11.41
## 38      19.2 396.90  8.77
## 39      19.2 393.43 10.13
## 41      18.3 395.62  1.98
## 42      17.9 385.41  4.84
## 44      17.9 394.46  7.44
## 48      17.9 392.74 18.80
## 51      16.8 395.56 13.45
## 53      16.8 396.90  5.28
## 56      17.9 395.93  4.81
## 57      17.3 396.90  5.77
## 58      15.1 392.90  3.95
## 59      19.7 390.68  6.86
## 60      19.7 396.90  9.22
## 61      19.7 395.11 13.15
## 62      19.7 378.08 14.44
## 63      19.7 396.90  6.73
## 64      19.7 395.58  9.50
## 66      16.1 396.90  4.67
## 67      16.1 396.90 10.24
## 68      18.9 396.21  8.10
## 69      18.9 396.90 13.09
## 72      19.2 376.94  9.88
## 73      19.2 390.91  5.52
## 74      19.2 377.17  7.54
## 75      18.7 394.92  6.78
## 77      18.7 373.66 11.97
## 79      18.7 386.40 12.34
## 80      18.7 396.06  9.10
## 81      19.0 396.90  5.29
## 82      19.0 395.63  7.22
```

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## 83      19.0 396.90  6.72
## 84      19.0 390.64  7.51
## 86      18.5 392.30  6.53
## 88      18.5 395.15  8.44
## 89      17.8 396.90  5.50
## 91      17.8 392.18  8.81
## 92      17.8 393.55  8.20
## 94      18.2 396.33  6.21
## 95      18.2 396.90 10.59
## 98      18.0 396.90  4.21
## 99      18.0 393.53  3.57
## 100     18.0 396.90  6.19
## 102     20.9 395.58  7.67
## 104     20.9 394.47 13.44
## 107     20.9 395.67 18.66
## 108     20.9 387.69 14.09
## 109     20.9 395.24 12.27
## 110     20.9 391.23 15.55
## 111     20.9 393.49 13.00
## 112     17.8 395.59 10.16
## 113     17.8 394.95 16.21
## 114     17.8 396.90 17.09
## 115     17.8 388.74 10.45
## 116     17.8 344.91 15.76
## 118     17.8 394.51 10.30
## 119     17.8 338.63 15.37
## 120     17.8 391.50 13.61
## 121     19.1 389.15 14.37
## 122     19.1 377.67 14.27
## 123     19.1 378.09 17.93
## 124     19.1 370.31 25.41
## 125     19.1 379.38 17.58
## 126     19.1 385.02 14.81
## 128     21.2 392.11 17.19
## 131     21.2 395.04 12.60
## 132     21.2 396.90 12.26
## 133     21.2 385.76 11.12
## 134     21.2 388.69 15.03
## 136     21.2 394.67 16.96
## 137     21.2 378.25 16.90
## 140     21.2 396.90 18.46
## 142     21.2 396.90 34.41
## 143     14.7 396.90 26.82
## 144     14.7 396.90 26.42
## 145     14.7 396.90 29.29
## 146     14.7 172.91 27.80
## 147     14.7 169.27 16.65
## 148     14.7 391.71 29.53
## 149     14.7 356.99 28.32
## 150     14.7 351.85 21.45
## 152     14.7 341.60 13.28
## 153     14.7 343.28 12.12
## 156     14.7  88.01 15.02
## 157     14.7  88.63 16.14
```

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## 158    14.7 363.43  4.59
## 159    14.7 353.89  6.43
## 162    14.7 374.43  1.73
## 163    14.7 389.61  1.92
## 165    14.7 395.11 11.64
## 166    14.7 240.16  9.81
## 167    14.7 369.30  3.70
## 168    14.7 227.61 12.14
## 169    14.7 297.09 11.10
## 170    14.7 330.04 11.32
## 171    14.7 292.29 14.43
## 172    14.7 348.13 12.03
## 173    16.6 396.90 14.69
## 174    16.6 395.50  9.04
## 175    16.6 393.23  9.64
## 176    16.6 390.96  5.33
## 178    16.6 395.60  6.29
## 179    16.6 391.27  6.92
## 180    17.8 396.90  5.04
## 181    17.8 395.56  7.56
## 182    17.8 396.90  9.45
## 184    17.8 396.90  5.68
## 186    17.8 387.11 13.15
## 187    17.8 392.63  4.45
## 188    15.2 393.87  6.68
## 189    15.2 382.84  4.56
## 190    15.2 396.90  5.39
## 191    15.2 377.68  5.10
## 192    15.2 389.71  4.69
## 193    15.2 390.49  2.87
## 195    15.6 376.70  4.38
## 196    14.4 394.23  2.97
## 197    12.6 396.90  4.08
## 198    12.6 354.31  8.61
## 199    12.6 392.20  6.62
## 200    17.0 396.90  4.56
## 202    14.7 393.77  7.43
## 203    14.7 395.38  3.11
## 204    14.7 392.78  3.81
## 205    14.7 390.55  2.88
## 206    18.6 396.90 10.87
## 208    18.6 389.43 18.06
## 209    18.6 381.32 14.66
## 212    18.6 395.24 23.98
## 213    18.6 390.94 16.03
## 214    18.6 385.81  9.38
## 216    18.6 393.63  9.47
## 218    16.4 392.78  9.69
## 219    16.4 396.90 17.92
## 220    16.4 393.74 10.50
## 221    17.4 391.70  9.71
## 223    17.4 390.39  9.93
## 225    17.4 385.05  4.14
## 226    17.4 382.00  4.63
```

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## 227    17.4 387.38  3.13
## 228    17.4 372.08  6.36
## 230    17.4 380.34  3.76
## 231    17.4 378.35 11.65
## 234    17.4 378.95  3.95
## 235    17.4 360.20  8.05
## 236    17.4 376.75 10.88
## 237    17.4 388.45  9.54
## 239    16.6 379.41  6.36
## 240    16.6 383.78  7.37
## 242    16.6 394.62 12.40
## 243    16.6 372.75 11.22
## 244    16.6 374.71  5.19
## 246    19.1 389.13 18.46
## 247    19.1 390.18  9.16
## 248    19.1 376.14 10.15
## 249    19.1 374.71  9.52
## 250    19.1 393.74  6.56
## 251    19.1 396.28  5.90
## 252    19.1 377.07  3.59
## 254    19.1 396.90  3.54
## 255    16.4 392.89  6.57
## 256    16.4 395.18  9.25
## 258    13.0 389.70  5.12
## 260    13.0 391.93  6.90
## 262    13.0 388.37  7.26
## 264    13.0 393.42 11.25
## 265    13.0 387.89  8.10
## 266    13.0 392.40 10.45
## 267    13.0 384.07 14.79
## 268    13.0 384.54  7.44
## 269    13.0 390.30  3.16
## 270    18.6 391.34 13.65
## 271    18.6 388.65 13.00
## 273    18.6 394.96  7.73
## 274    18.6 390.77  6.58
## 275    17.6 396.90  3.53
## 276    17.6 396.90  2.98
## 277    17.6 389.25  6.05
## 279    17.6 396.90  7.19
## 280    14.9 396.90  4.85
## 281    14.9 387.31  3.76
## 282    14.9 392.23  4.59
## 283    14.9 377.07  3.01
## 285    15.3 394.72  7.85
## 286    15.3 394.72  8.23
## 287    18.2 341.60 12.93
## 288    16.6 396.90  7.14
## 289    16.6 396.90  7.60
## 290    16.6 371.72  9.51
## 291    19.2 396.90  3.33
## 292    19.2 396.90  3.56
## 294    16.0 396.90  8.58
## 296    16.0 396.90  6.27
```

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## 297     16.0 392.85  7.39
## 299     14.8 368.24  4.97
## 301     14.8 390.86  6.07
## 303     16.1 383.61  8.67
## 304     16.1 390.43  4.86
## 305     18.4 393.68  6.93
## 306     18.4 393.36  8.93
## 307     18.4 396.90  6.47
## 308     18.4 396.90  7.53
## 309     18.4 396.90  4.54
## 311     18.4 350.45 12.64
## 313     18.4 396.30 11.72
## 314     18.4 393.39  7.90
## 316     18.4 396.42 11.50
## 317     18.4 390.70 18.33
## 318     18.4 396.90 15.94
## 319     18.4 395.21 10.36
## 320     18.4 396.23 12.73
## 321     19.6 396.90  7.20
## 322     19.6 396.90  6.87
## 323     19.6 396.90  7.70
## 324     19.6 391.13 11.74
## 327     19.6 396.90  6.15
## 328     19.6 396.90 12.79
## 330     16.9 375.21  7.34
## 331     16.9 368.57  9.09
## 332     16.9 394.02 12.43
## 333     16.9 362.25  7.83
## 334     20.2 389.71  5.68
## 335     20.2 389.40  6.75
## 336     20.2 396.90  8.01
## 337     20.2 396.90  9.80
## 338     20.2 394.81 10.56
## 340     20.2 396.90  9.74
## 341     20.2 396.90  9.29
## 343     15.9 389.96  8.65
## 345     17.6 387.97  4.61
## 346     18.8 385.64 10.53
## 347     18.8 364.61 12.67
## 349     17.0 390.94  5.99
## 350     19.7 389.85  5.89
## 351     19.7 396.90  5.98
## 352     18.3 370.78  5.49
## 353     18.3 392.33  7.79
## 355     22.0 382.80  8.05
## 356     22.0 376.04  5.57
## 358     20.2 391.34 13.27
## 359     20.2 395.43 11.48
## 360     20.2 390.74 12.67
## 361     20.2 374.56  7.79
## 363     20.2 380.79 10.19
## 365     20.2 354.55  5.29
## 366     20.2 354.70  7.12
## 367     20.2 316.03 14.00
```

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## 369    20.2 375.52  3.26
## 370    20.2 375.33  3.73
## 373    20.2 347.88  8.88
## 374    20.2 396.90 34.77
## 375    20.2 396.90 37.97
## 376    20.2 396.90 13.44
## 379    20.2 396.90 23.69
## 380    20.2 393.74 21.78
## 382    20.2 396.90 21.08
## 383    20.2 396.90 23.60
## 384    20.2 396.90 24.56
## 385    20.2 285.83 30.63
## 386    20.2 396.90 30.81
## 387    20.2 396.90 28.28
## 388    20.2 396.90 31.99
## 389    20.2 372.92 30.62
## 390    20.2 396.90 20.85
## 392    20.2 378.38 18.76
## 394    20.2 396.90 15.17
## 396    20.2 391.98 17.12
## 397    20.2 396.90 19.37
## 398    20.2 393.10 19.92
## 399    20.2 396.90 30.59
## 400    20.2 338.16 29.97
## 401    20.2 396.90 26.77
## 402    20.2 396.90 20.32
## 404    20.2 396.90 19.77
## 405    20.2 329.46 27.38
## 407    20.2 370.22 23.34
## 408    20.2 332.09 12.13
## 409    20.2 314.64 26.40
## 410    20.2 179.36 19.78
## 411    20.2   2.60 10.11
## 412    20.2  35.05 21.22
## 417    20.2  21.57 25.79
## 418    20.2 127.36 26.64
## 419    20.2  16.45 20.62
## 421    20.2 318.75 15.02
## 422    20.2 319.98 15.70
## 423    20.2 291.55 14.10
## 425    20.2   3.65 17.16
## 426    20.2   7.68 24.39
## 427    20.2  24.65 15.69
## 428    20.2  18.82 14.52
## 429    20.2  96.73 21.52
## 430    20.2  60.72 24.08
## 431    20.2  83.45 17.64
## 432    20.2  81.33 19.69
## 433    20.2  97.95 12.03
## 434    20.2 100.19 16.22
## 435    20.2 100.63 15.17
## 436    20.2 109.85 23.27
## 437    20.2  27.49 18.05
## 438    20.2   9.32 26.45
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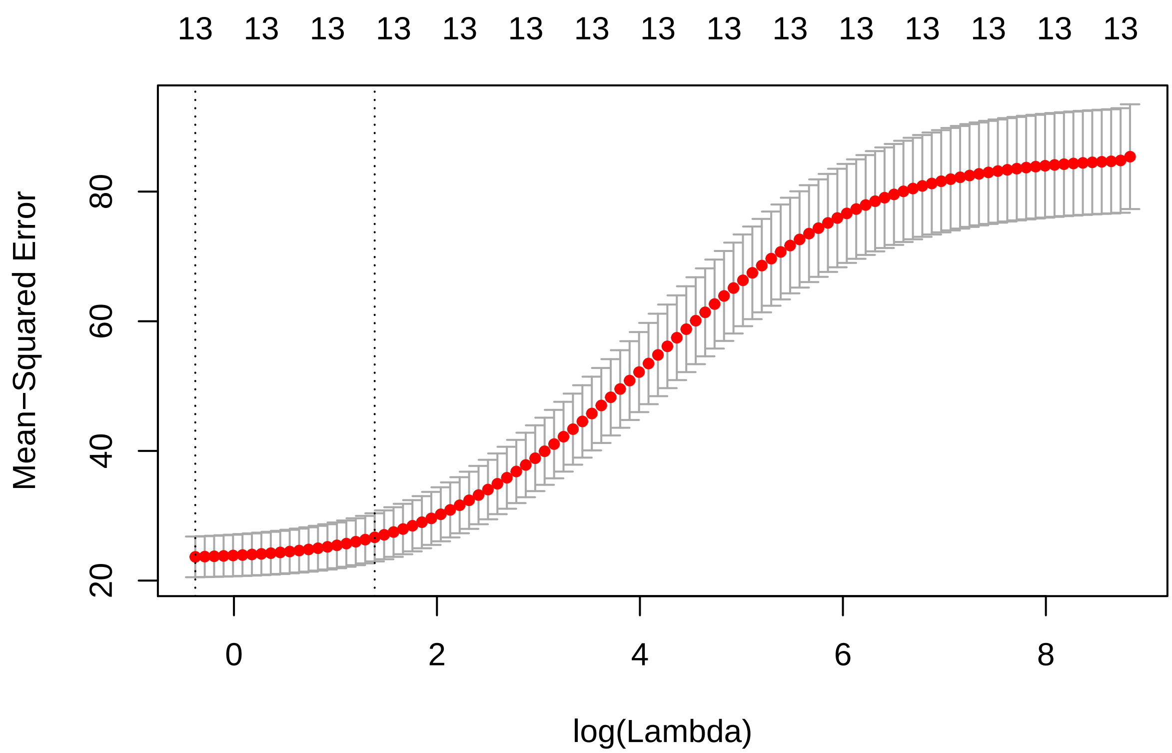
## 439    20.2  68.95 34.02
## 440    20.2 396.90 22.88
## 441    20.2 391.45 22.11
## 442    20.2 385.96 19.52
## 443    20.2 395.69 16.59
## 444    20.2 386.73 18.85
## 446    20.2  43.06 23.98
## 447    20.2 318.01 17.79
## 449    20.2 396.90 18.13
## 450    20.2 304.21 19.31
## 451    20.2   0.32 17.44
## 452    20.2 355.29 17.73
## 454    20.2 375.87 16.74
## 455    20.2   6.68 18.71
## 457    20.2  10.48 19.01
## 458    20.2   3.50 16.94
## 459    20.2 272.21 16.23
## 461    20.2 255.23 16.42
## 462    20.2 391.43 14.65
## 463    20.2 396.90 13.99
## 466    20.2 334.40 14.13
## 467    20.2  22.01 17.15
## 468    20.2 331.29 21.32
## 469    20.2 368.74 18.13
## 470    20.2 396.90 14.76
## 471    20.2 396.90 16.29
## 472    20.2 395.33 12.87
## 473    20.2 393.37 14.36
## 474    20.2 374.68 11.66
## 475    20.2 352.58 18.14
## 476    20.2 302.76 24.10
## 477    20.2 396.21 18.68
## 478    20.2 349.48 24.91
## 479    20.2 379.70 18.03
## 480    20.2 383.32 13.11
## 483    20.2 395.28  7.01
## 484    20.2 392.92 10.42
## 485    20.2 370.73 13.34
## 486    20.2 388.62 10.58
## 488    20.2 388.22 11.45
## 490    20.1 344.05 23.97
## 492    20.1 390.11 18.07
## 493    20.1 396.90 13.35
## 494    19.2 396.90 12.01
## 495    19.2 396.90 13.59
## 496    19.2 393.29 17.60
## 497    19.2 396.90 21.14
## 498    19.2 396.90 14.10
## 499    19.2 396.90 12.92
## 500    19.2 395.77 15.10
## 501    19.2 396.90 14.33
## 502    21.0 391.99  9.67
## 503    21.0 396.90  9.08
## 505    21.0 393.45  6.48
```

```
## 506    21.0 396.90  7.88
```

```
x <- model.matrix(MEDV~., train)[,-1]
y = train$MEDV
y.test = y[-trainIndex]

#perform cross-validation to choose tuning parameter lambda
cv.out <- cv.glmnet(x, y, alpha = 0)
plot(cv.out)
```




```
bestlambda <- cv.out$lambda.min #lambda that results in lowest cross validation error
grid <- 10^seq(10,-2,length = 100) #from 10^10 to 10^-2
ridge.mod <- glmnet(x,y,alpha= 0, lambda = grid, thresh = 1e-12)
#make predictions for lambda = bestlambda
ridge.pred <- predict(ridge.mod, s=bestlambda, newx = x[-trainIndex,])
```

```
library(plotmo)

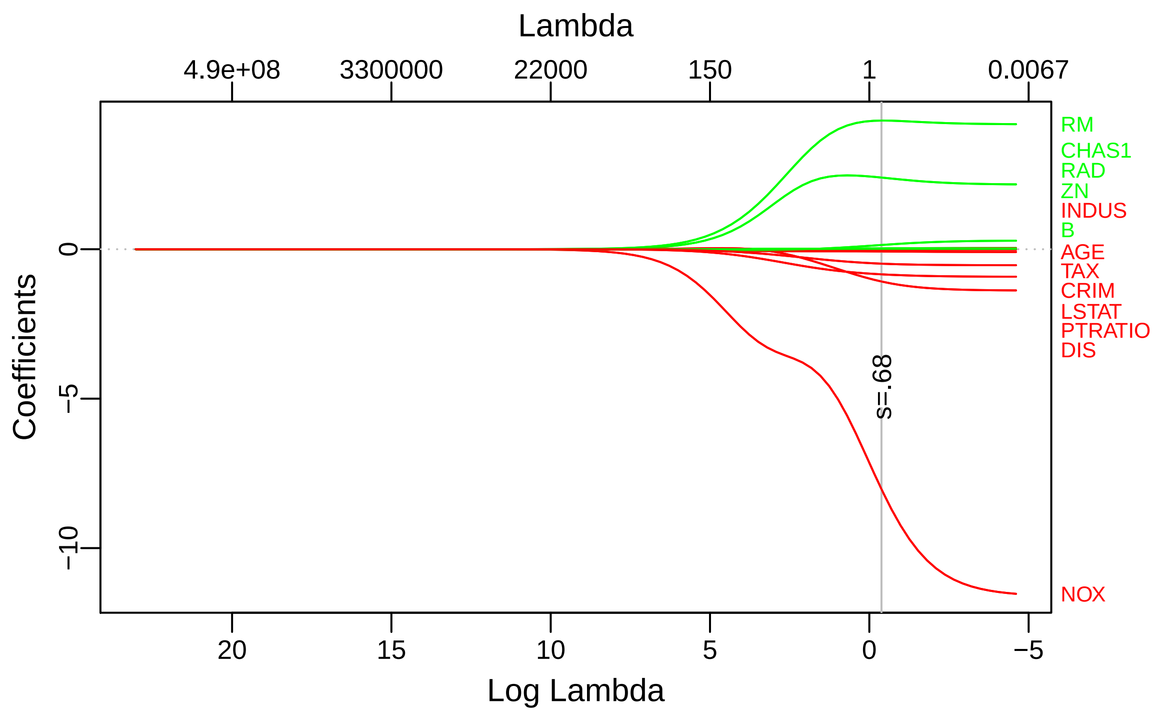
glmcoef<-coef(ridge.mod,bestlambda )
coef.increase<-dimnames(glmcoef[glmcoef[,1]>0,0])[[1]]
coef.decrease<-dimnames(glmcoef[glmcoef[,1]<0,0])[[1]]

#get ordered list of variables as they appear at smallest lambda
allnames<-names(coef(ridge.mod)[,
            ncol(coef(ridge.mod))][order(coef(ridge.mod)[,
            ncol(coef(ridge.mod))],decreasing=TRUE)])
```

```
#remove intercept
allnames<-setdiff(allnames,allnames[grep("Intercept",allnames)])

#assign colors
cols<-rep("gray",length(allnames))
cols[allnames %in% coef.increase]<-"green"      # higher medv is good
cols[allnames %in% coef.decrease]<-"red"        # lower medv is not

plot_glmnet(ridge.mod,label=TRUE,s=bestlambda,col=cols)
```



To select the best model, we now use 10x10-CV using the lambda that best minimised the error in cross-validation, which is lambda.ridge.

```
trainContrl <- trainControl (method="repeatedcv", number=10, repeats=10)
model <- train(MEDV ~ ., data = train, trControl=trainContrl, method='glmnet',tuneGrid=expand.grid(alpha
model.ridge.FINAL <- glmnet(x, y, alpha = 0, lambda = bestlambda)



normalized <- (length(train$MEDV)-1)*var(train$MEDV)
NMSE.ridge.train.error <- crossprod(predict (model) - train$MEDV) / normalized
normTest <- (length(test$MEDV)-1)*var(test$MEDV)
sse_raw <- test$MEDV - model.ridge.FINAL$a0- data.matrix(test[,-dim(train)[2]]) %*% model.ridge.FINAL$be
sse <- crossprod (as.matrix(sse_raw))

NMSE.ridge.test.error <- sse/normTest
```

Table 2: Model Errors Summary

regression_method	train_MSE	test_MSE
ridge regression	0.255	4101.969