Advanced Statistical Modelling: Ridge Regression

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Choosing the penalization parameter λ

The objective of this exercise is to implement Ridge Regression with two different approaches: $MSPE_{val}(\lambda)$ and $MSPE_{k-CV}(\lambda)$. In both cases we are going to take as input data the following:

- Matrix x and vector y corresponding to the training sample.
- Matrix x_{val} and vector y_{val} corresponding to the validation set.
- Vector *lambda.v* of candidate values for λ.

We are going to output for each element λ in *lambda.v* and the value of the $MSPE_{val}(\lambda)$ / $MSPE_{k-CV}(\lambda)$. Furthermore, we are going to plot these values against $log(1 + \lambda) - 1$.

Once we have build these two functions, we are going to use the prostate data used in class. We are going to choose a λ according to:

- Behaviour in the validation set (30 validations not included in the training sample)
- 5-fold, 10-fold cross-validation.
- Compare our results with those obtained with leave-one-out and generalized cross-validation.

Ridge regression based on $MSPE_{val}(\lambda)$

In order to choose the penalization parameter λ, we are going to write a function implementing the ridge regression penalization parameter λ choice based on the minimization of the $MSPE_{val}(\lambda)$.

```r
ridge_lambda_search <- function(x, y, x.val, y.val, lambda.v) {
  result <- data.frame(lambda=lambda.v, mspe=rep(0,length(lambda.v)), df=rep(0,length(lambda.v)))
  x.svd <- svd(x)
  for(i in 1:length(lambda.v)) {
    lambda <- lambda.v[i]
    d_inv <- diag(1/(x.svd$d*x.svd$d - lambda))
    xx_inv <- t( solve( t(x) %*% x + lambda*diag(1,ncol(x)) ))
    beta <- xx_inv %*% t(x) %*% y
    y.hat <- x.val %*% beta
    mspe <- sum((y.val - y.hat)^2) / length(y.val)
    df <- sum(x.svd$d^2 / (x.svd$d^2 +lambda))
    result$mspe[i] <- mspe
    result$df[i] <- df
  }
  plot(mspe~log(1+lambda), result, col=2)
  lambda.min <- result$lambda[which.min(result$mspe)]
  abline(v=log(1+lambda.min),col=2,lty=2)
  plot(mspe~df, result, col=3)
  df.min <- result$df[which.min(result$mspe)]
  abline(v=df.min,col=3,lty=2)
  return(result)
}
```

Ridge regression based on $MSPE_{k-CV}(\lambda)$

Now, we will write an R function implementing the ridge regression penalization parameterλ choice based on k-fold cross-validation $MSPE_{kCV}(\lambda)$

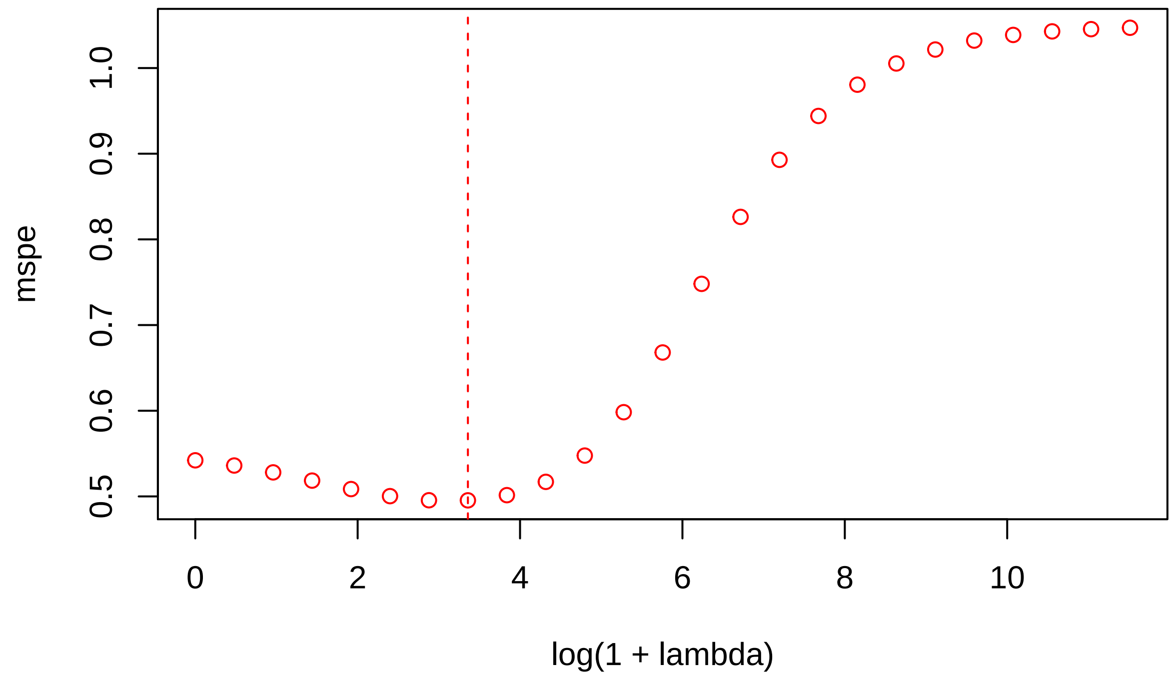
```
ridge_lambda_search_cv <- function(x, y, lambda.v, cv=10) {
  result <- data.frame(lambda=lambda.v, mspe=rep(0,length(lambda.v)), df=rep(0,length(lambda.v)))
  for(i in 1:length(lambda.v)) {
    lambda.v[i] <- lambda.v[i]
    folds <- createFolds(1:nrow(x), k = cv)
    result.cv <- data.frame(mspe=rep(0,cv), df=rep(0,cv))
    for(j in 1:cv) {
      fold <- folds[[j]]
      x.train <- x[-fold,]
      y.train <- y[-fold]
      x.val <- x[fold,]
      y.val <- y[fold]
      x.svd <- svd(x.train)
      d <- x.svd$d
      v <- x.svd$v
      d_inv <- diag(1/(d*d - lambda.v[i]))
      xx_inv <- t( solve( t(x.train) %*% x.train + lambda.v[i]*diag(1,ncol(x)) ))
      beta <- xx_inv %*% t(x.train) %*% y.train
      y.hat <- x.val %*% beta
      mspe <- sum((y.val - y.hat)^2) / length(y.val)
      df <- sum(d^2 / (d^2 +lambda.v[i]))
      result.cv$mspe[j] <- mspe
      result.cv$df[j] <- df
    }
    result$mspe[i] <- mean(result.cv$mspe)
    result$df[i] <- mean(result.cv$df)

  }
  plot(mspe~log(1+lambda), result, col=2)
  lambda.min <- result$lambda[which.min(result$mspe)]
  abline(v=log(1+lambda.min),col=2,lty=2)
  plot(mspe~df, result, col=3)
  df.min <- result$df[which.min(result$mspe)]
  abline(v=df.min,col=3,lty=2)

  return(result)

}
```

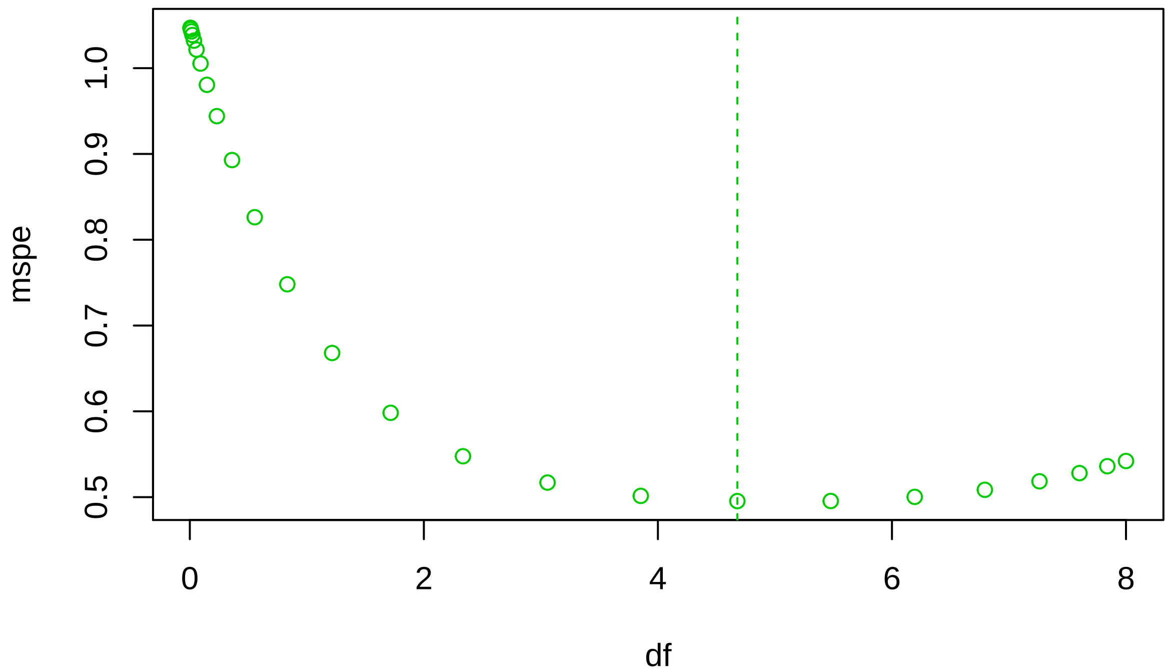
Comparation between penalization parameters of $MSPE_{val}(\lambda)$ and $MSPE_{k-CV}(\lambda)$

```
ridge_lambda_search_loocv_gcv <- function(x, y, lambda.v) {
  l <- length(lambda.v)
  result <- data.frame(lambda=lambda.v, loocv=rep(0,l), gcv=rep(0,l),df=rep(0,l))
  n <- nrow(x)
  x.svd <- svd(x)
  d <- x.svd$d
  v <- x.svd$v
  u <- x.svd$u
```

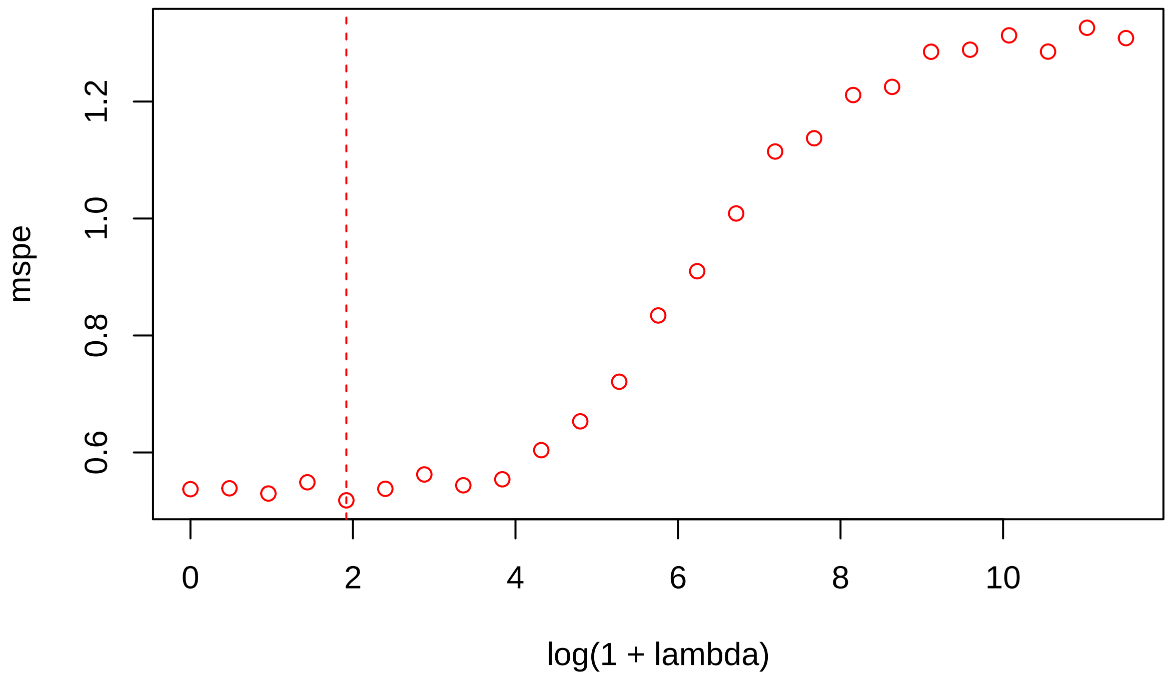
```r
  for(i in 1:l) {
    lambda <- lambda.v[i]
    d_inv <- diag(1/(d^2 - lambda))
    xx_inv <- t( solve( t(x) %*% x + lambda*diag(1,ncol(x)) ))
    beta <- (xx_inv %*% t(x)) %*% y
    y.hat <- x %*% beta
    df <- sum(d^2 / (d^2 +lambda))
    h <- x %*% xx_inv %*% t(x)
    result$loocv[i] <- sum( ( (y - y.hat)/(1 - diag(h)) )^2 ) / n
    result$gcv[i] <- sum( ( (y - y.hat)/(1 - df/n) )^2 ) / n
    result$df[i] <- df
  }
  plot(loocv~log(1+lambda), result, col=2)
  lambda.min <- result$lambda[which.min(result$loocv)]
  abline(v=log(1+lambda.min),col=2,lty=2)
  plot(loocv~df, result, col=3)
  df.min <- result$df[which.min(result$loocv)]
  abline(v=df.min,col=3,lty=2)
  plot(gcv~log(1+lambda), result, col=2)
  lambda.min <- result$lambda[which.min(result$gcv)]
  abline(v=log(1+lambda.min),col=2,lty=2)
  plot(gcv~df, result, col=3)
  df.min <- result$df[which.min(result$gcv)]
  abline(v=df.min,col=3,lty=2)
  return(result)
}
```

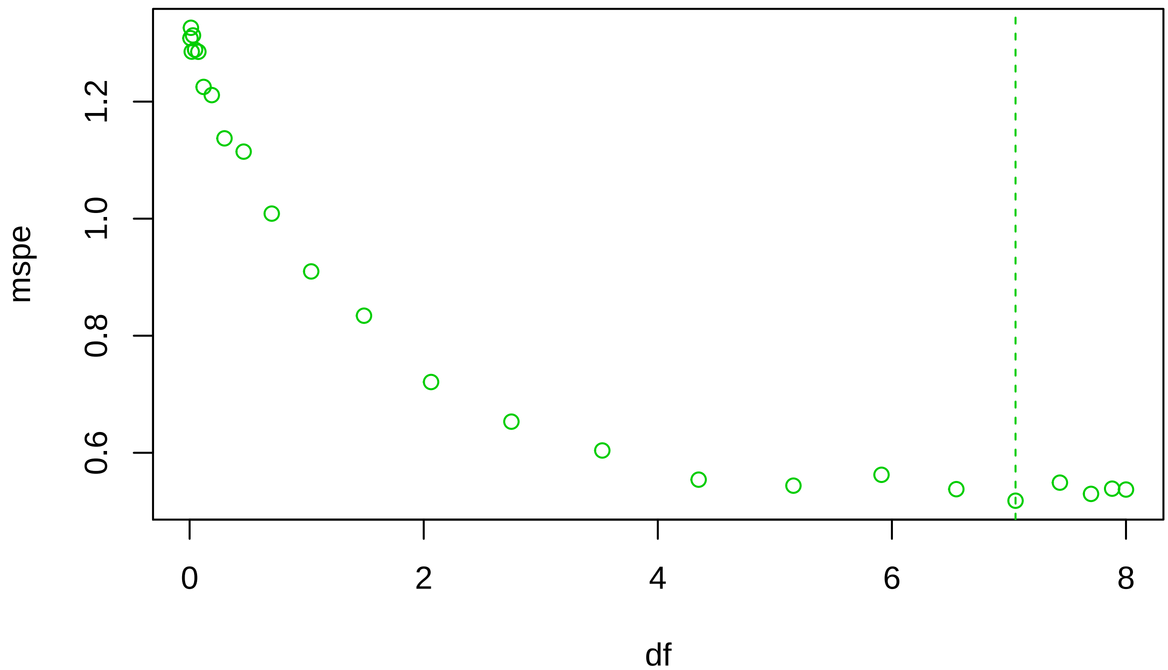
```r
prostate <- fread(file="prostate_data.txt", header=TRUE)
prostate$train <- as.logical(prostate$train)
prostate$No <- NULL
prostate$train <- as.logical(prostate$train)
prostate <- as.data.frame(prostate)
data <- prostate[,1:9]
lambda.max <- 1e5
n.lambdas <- 25
lambda.v <- exp(seq(0,log(lambda.max+1),length=n.lambdas))-1
validation.ind <- prostate$train
validation <- data[!validation.ind,]
training <- data[validation.ind,]

x <- scale(training[,1:8], center = T, scale = T)
y <- scale(training[,9], center = T, scale = F)

x.val <- scale(validation[,1:8], center = T, scale = T)
y.val <- scale(validation[,9], center = T, scale = F)
result.valid <- ridge_lambda_search(x, y, x.val, y.val, lambda.v)
```


mspe
0.5
0.6
0.7
0.8
0.9
1.0
0
2
4
6
8
10
log(1 + lambda)

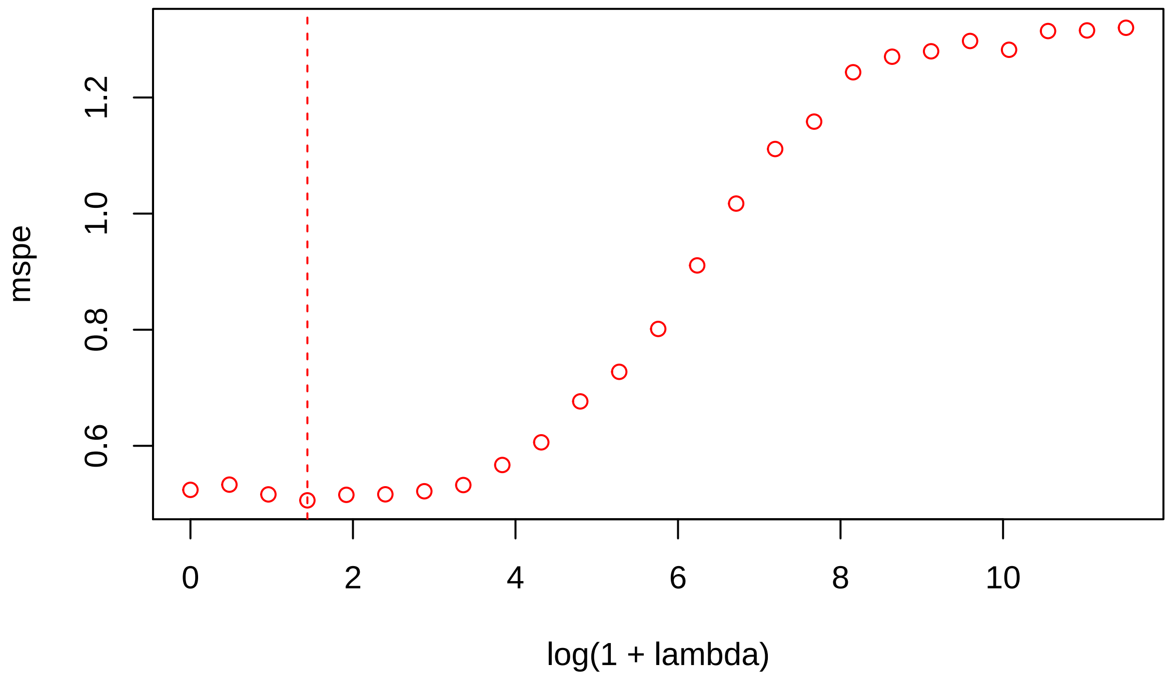


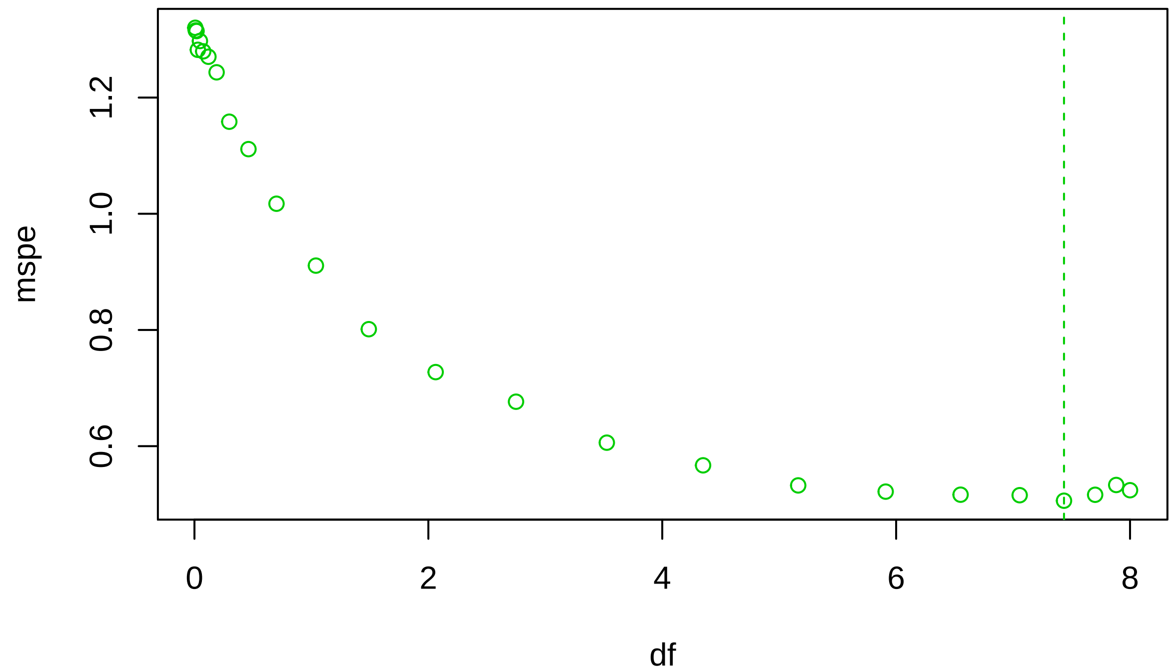

```r
# 5-fold and 10-fold CV
# With CV
x <- scale(data[,1:8], center = T, scale = T)
y <- scale(data[,9], center = T, scale = F)
result.5.cv <- ridge_lambda_search_cv(x, y, cv = 10, lambda.v)
```


mspe
1.2
1.0
0.8
0.6
0
2
4
6
8
10
log(1 + lambda)

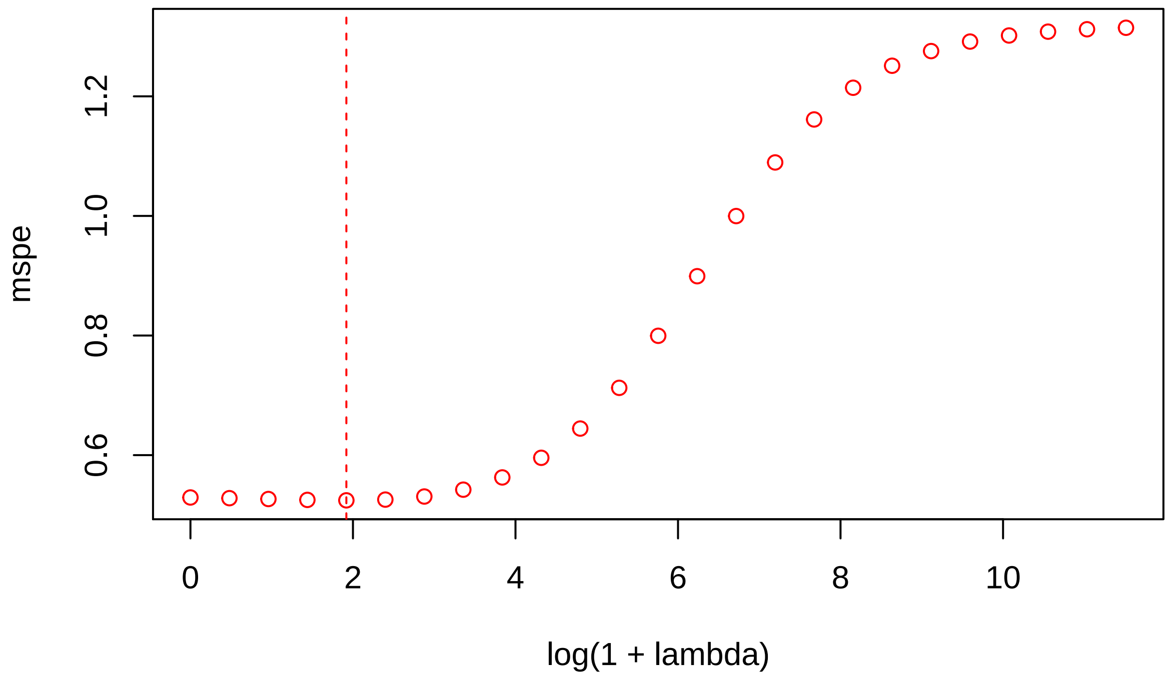



```
result.10.cv <- ridge_lambda_search_cv(x, y, cv = 10, lambda.v)
```

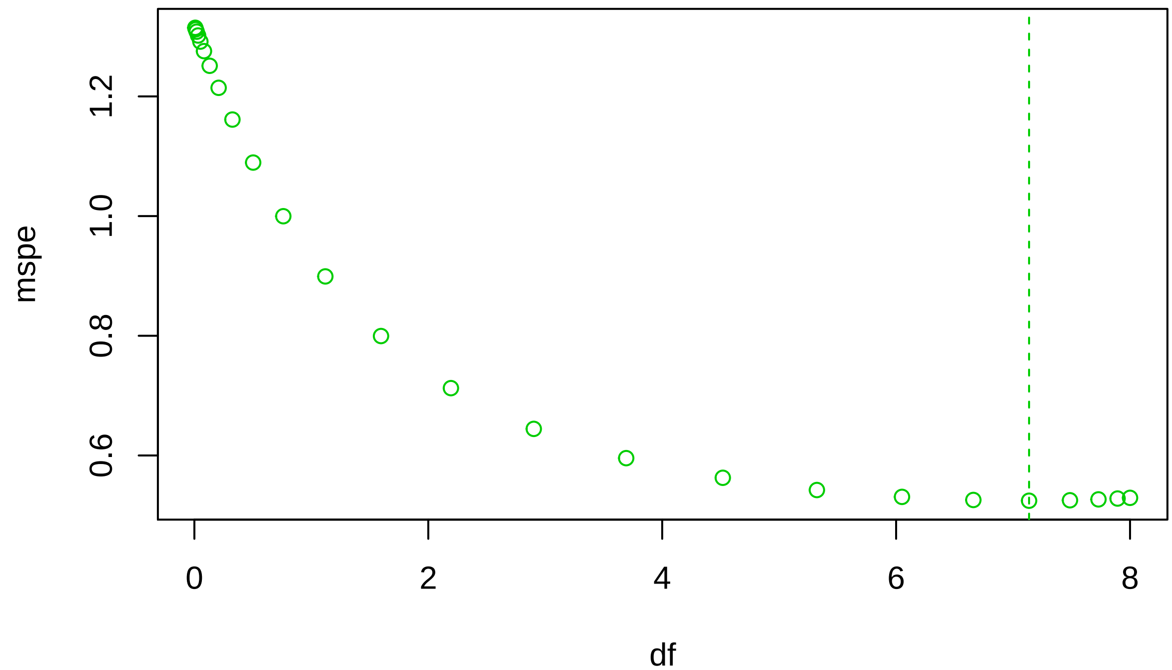

mspe
0.6
0.8
1.0
1.2
0
2
4
6
8
10
log(1 + lambda)



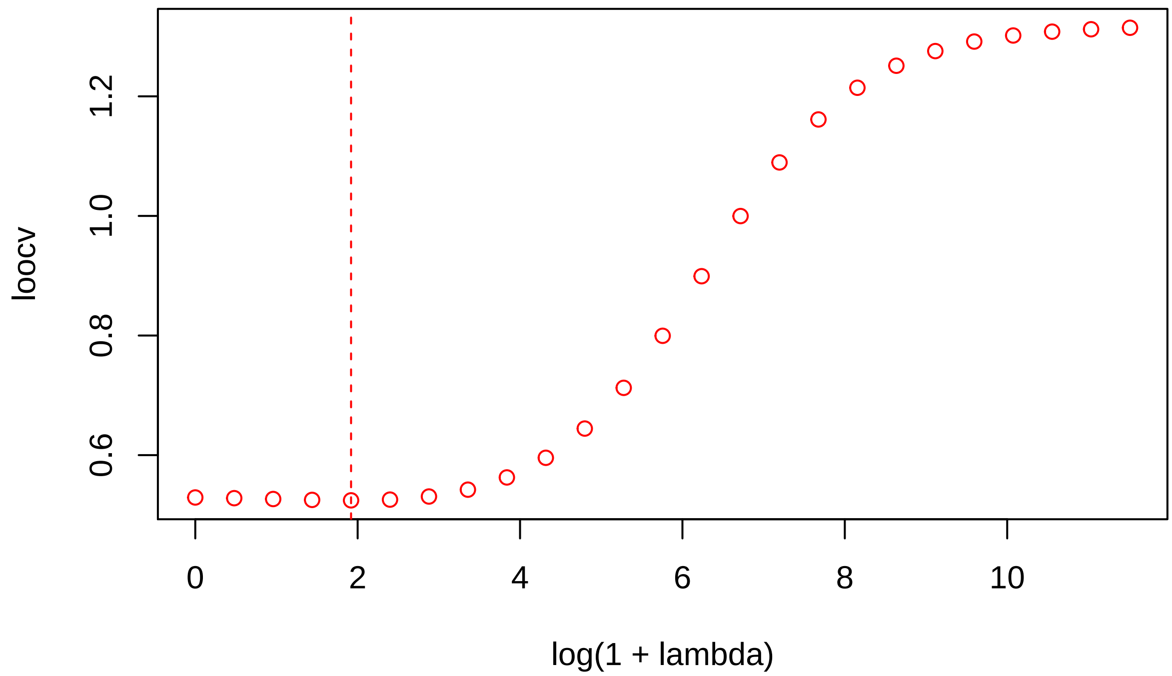

```
# LOOCV and GCV estimates
result.loocv <- ridge_lambda_search_cv(x, y, cv = nrow(x), lambda.v)
```

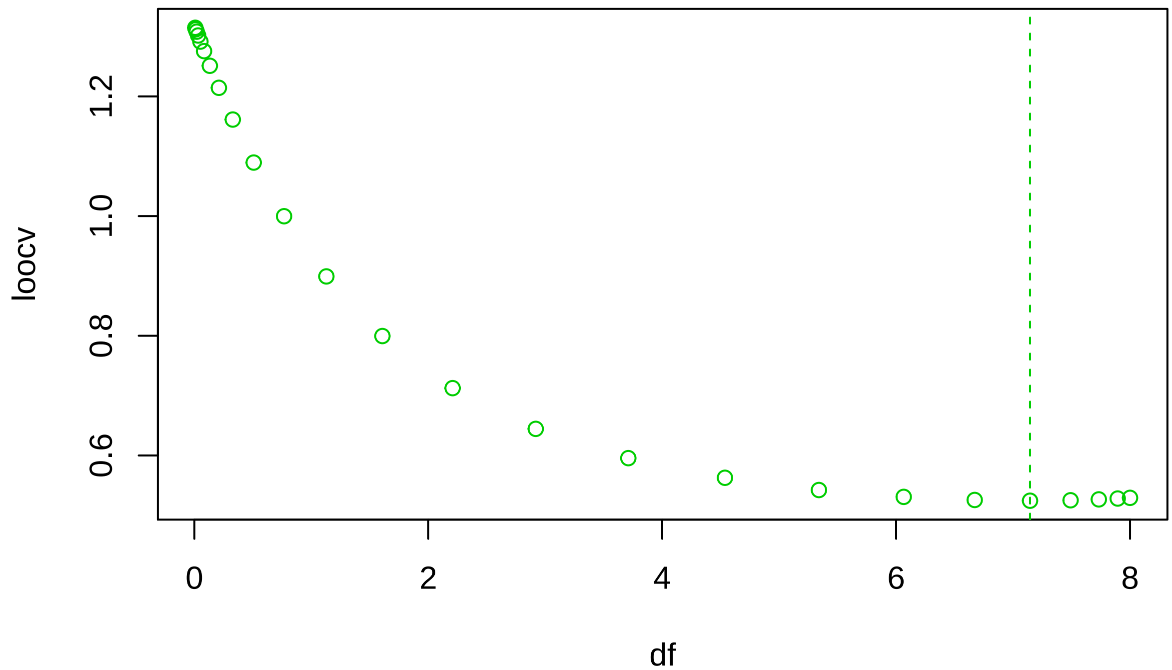


mspe
1.2
1.0
0.8
0.6
0
2
4
6
8
10
log(1 + lambda)

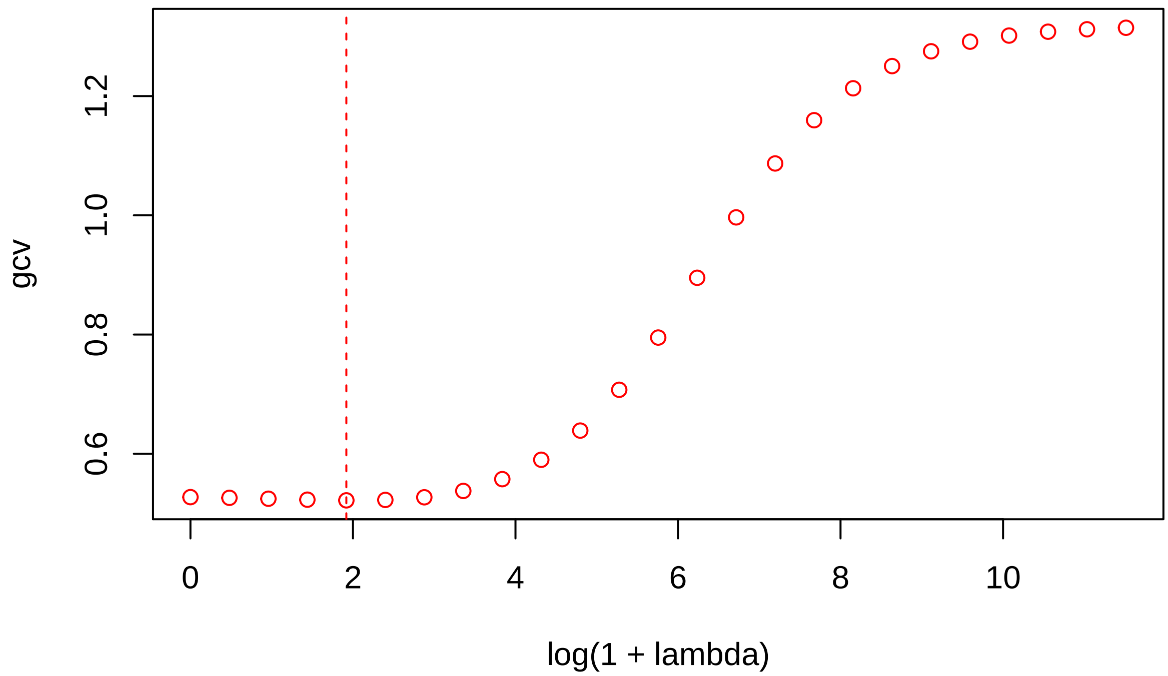



```r
result.loocv.gcv <- ridge_lambda_search_loocv_gcv(x, y, lambda.v)
```

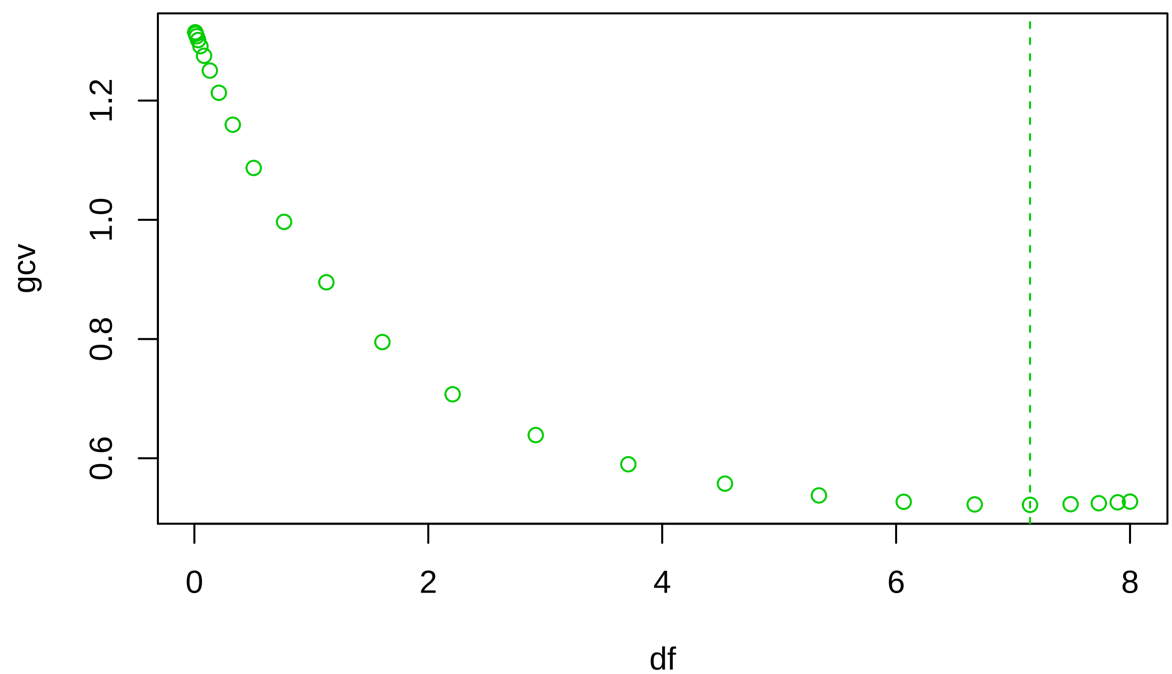

loocv
0.6
0.8
1.0
1.2
0
2
4
6
8
10
log(1 + lambda)



1.2
1.0
0.8
0.6
loocv
0
2
4
6
8
df



gcv
0.6
0.8
1.0
1.2
0
2
4
6
8
10
log(1 + lambda)



We observe that the results of MSPE seem more stable in the LOOCV and GCV compared to 5-fold and 10-fold CV, as one would expect given the small size of the dataset. Furthermore, the validation set results are also relatively stable, although with a different curve than for the LOOCV/GCV.

Ridge regression for the Boston Housing data

The Boston Housing dataset is a classical dataset which containes the values of 506 suburbs of Boston corresponding to 1978. This dataset can be found in many places but we are going to use a version with some corrections that was provided to us, which additionally includes the UTM coordinates of the geographical centers of each neighborhood. Therefore, the variables are the following:

Variable	Description	Type
CRIM	per capita crime rate by town	Numeric
ZN	proportion of residential land zoned for lots over 25,000 sq.ft.	Numeric
INDUS	proportion of non-retail business acres per town	Numeric
CHAS	Charles River dummy variable (= 1 if tract bounds river; 0 otherwise)	Factor
NOX	nitric oxides concentration (parts per 10 million)	Numeric
RM	average number of rooms per dwelling	Numeric
AGE	proportion of owner-occupied units built prior to 1940	Numeric
DIS	weighted distances to five Boston employment centres	Numeric
RAD	index of accessibility to radial highways	Numeric
TAX	full-value property-tax rate per $10,000	Numeric
PTRATIO	pupil-teacher ratio by town	Numeric
B	1000(Bk - 0.63)^2 where Bk is the proportion of blacks by town	Numeric
LSTAT	% lower status of the population	Numeric

Variable	Description	Type
MEDV	Median value of owner-occupied homes in $1000's	Numeric

In this exercise we are going to use ridge regression on the Boston Housing dataset to fit the regression model where the response variable is $MEDV$ and the explanatory variables are the remaining 13 variables shown in the list. As we can see when loading the data, there are more variables than the ones listed ($TOWN$, $TOWNNO$, LON, LAT, $CMEDV$). We decided not to use them since the statement explicitly defines which to use.

Beside from eliminating variables, we divided the dataset between train and test. To do so, we have decided to do 3/4 training, 1/4 test.

```
train
```

```
##      MEDV    CRIM    ZN INDUS CHAS    NOX    RM   AGE     DIS RAD TAX
## 1    24.0 0.00632  18.0  2.31    0 0.5380 6.575  65.2  4.0900   1 296
## 2    21.6 0.02731   0.0  7.07    0 0.4690 6.421  78.9  4.9671   2 242
## 3    34.7 0.02729   0.0  7.07    0 0.4690 7.185  61.1  4.9671   2 242
## 4    33.4 0.03237   0.0  2.18    0 0.4580 6.998  45.8  6.0622   3 222
## 5    36.2 0.06905   0.0  2.18    0 0.4580 7.147  54.2  6.0622   3 222
## 6    28.7 0.02985   0.0  2.18    0 0.4580 6.430  58.7  6.0622   3 222
## 8    27.1 0.14455  12.5  7.87    0 0.5240 6.172  96.1  5.9505   5 311
## 9    16.5 0.21124  12.5  7.87    0 0.5240 5.631 100.0  6.0821   5 311
## 10   18.9 0.17004  12.5  7.87    0 0.5240 6.004  85.9  6.5921   5 311
## 12   18.9 0.11747  12.5  7.87    0 0.5240 6.009  82.9  6.2267   5 311
## 16   19.9 0.62739   0.0  8.14    0 0.5380 5.834  56.5  4.4986   4 307
## 17   23.1 1.05393   0.0  8.14    0 0.5380 5.935  29.3  4.4986   4 307
## 19   20.2 0.80271   0.0  8.14    0 0.5380 5.456  36.6  3.7965   4 307
## 21   13.6 1.25179   0.0  8.14    0 0.5380 5.570  98.1  3.7979   4 307
## 23   15.2 1.23247   0.0  8.14    0 0.5380 6.142  91.7  3.9769   4 307
## 27   16.6 0.67191   0.0  8.14    0 0.5380 5.813  90.3  4.6820   4 307
## 28   14.8 0.95577   0.0  8.14    0 0.5380 6.047  88.8  4.4534   4 307
## 29   18.4 0.77299   0.0  8.14    0 0.5380 6.495  94.4  4.4547   4 307
## 30   21.0 1.00245   0.0  8.14    0 0.5380 6.674  87.3  4.2390   4 307
## 31   12.7 1.13081   0.0  8.14    0 0.5380 5.713  94.1  4.2330   4 307
## 32   14.5 1.35472   0.0  8.14    0 0.5380 6.072 100.0  4.1750   4 307
## 33   13.2 1.38799   0.0  8.14    0 0.5380 5.950  82.0  3.9900   4 307
## 34   13.1 1.15172   0.0  8.14    0 0.5380 5.701  95.0  3.7872   4 307
## 35   13.5 1.61282   0.0  8.14    0 0.5380 6.096  96.9  3.7598   4 307
## 36   18.9 0.06417   0.0  5.96    0 0.4990 5.933  68.2  3.3603   5 279
## 37   20.0 0.09744   0.0  5.96    0 0.4990 5.841  61.4  3.3779   5 279
## 38   21.0 0.08014   0.0  5.96    0 0.4990 5.850  41.5  3.9342   5 279
## 39   24.7 0.17505   0.0  5.96    0 0.4990 5.966  30.2  3.8473   5 279
## 41   34.9 0.03359  75.0  2.95    0 0.4280 7.024  15.8  5.4011   3 252
## 42   26.6 0.12744   0.0  6.91    0 0.4480 6.770   2.9  5.7209   3 233
## 44   24.7 0.15936   0.0  6.91    0 0.4480 6.211   6.5  5.7209   3 233
## 48   16.6 0.22927   0.0  6.91    0 0.4480 6.030  85.5  5.6894   3 233
## 51   19.7 0.08873  21.0  5.64    0 0.4390 5.963  45.7  6.8147   4 243
## 53   25.0 0.05360  21.0  5.64    0 0.4390 6.511  21.1  6.8147   4 243
## 56   35.4 0.01311  90.0  1.22    0 0.4030 7.249  21.9  8.6966   5 226
## 57   24.7 0.02055  85.0  0.74    0 0.4100 6.383  35.7  9.1876   2 313
## 58   31.6 0.01432 100.0  1.32    0 0.4110 6.816  40.5  8.3248   5 256
## 59   23.3 0.15445  25.0  5.13    0 0.4530 6.145  29.2  7.8148   8 284
## 60   19.6 0.10328  25.0  5.13    0 0.4530 5.927  47.2  6.9320   8 284
```

```
## 61  18.7  0.14932  25.0  5.13    0 0.4530 5.741  66.2  7.2254   8 284
## 62  16.0  0.17171  25.0  5.13    0 0.4530 5.966  93.4  6.8185   8 284
## 63  22.2  0.11027  25.0  5.13    0 0.4530 6.456  67.8  7.2255   8 284
## 64  25.0  0.12650  25.0  5.13    0 0.4530 6.762  43.4  7.9809   8 284
## 66  23.5  0.03584  80.0  3.37    0 0.3980 6.290  17.8  6.6115   4 337
## 67  19.4  0.04379  80.0  3.37    0 0.3980 5.787  31.1  6.6115   4 337
## 68  22.0  0.05789  12.5  6.07    0 0.4090 5.878  21.4  6.4980   4 345
## 69  17.4  0.13554  12.5  6.07    0 0.4090 5.594  36.8  6.4980   4 345
## 72  21.7  0.15876   0.0 10.81    0 0.4130 5.961  17.5  5.2873   4 305
## 73  22.8  0.09164   0.0 10.81    0 0.4130 6.065   7.8  5.2873   4 305
## 74  23.4  0.19539   0.0 10.81    0 0.4130 6.245   6.2  5.2873   4 305
## 75  24.1  0.07896   0.0 12.83    0 0.4370 6.273   6.0  4.2515   5 398
## 77  20.0  0.10153   0.0 12.83    0 0.4370 6.279  74.5  4.0522   5 398
## 79  21.2  0.05646   0.0 12.83    0 0.4370 6.232  53.7  5.0141   5 398
## 80  20.3  0.08387   0.0 12.83    0 0.4370 5.874  36.6  4.5026   5 398
## 81  28.0  0.04113  25.0  4.86    0 0.4260 6.727  33.5  5.4007   4 281
## 82  23.9  0.04462  25.0  4.86    0 0.4260 6.619  70.4  5.4007   4 281
## 83  24.8  0.03659  25.0  4.86    0 0.4260 6.302  32.2  5.4007   4 281
## 84  22.9  0.03551  25.0  4.86    0 0.4260 6.167  46.7  5.4007   4 281
## 86  26.6  0.05735   0.0  4.49    0 0.4490 6.630  56.1  4.4377   3 247
## 88  22.2  0.07151   0.0  4.49    0 0.4490 6.121  56.8  3.7476   3 247
## 89  23.6  0.05660   0.0  3.41    0 0.4890 7.007  86.3  3.4217   2 270
## 91  22.6  0.04684   0.0  3.41    0 0.4890 6.417  66.1  3.0923   2 270
## 92  22.0  0.03932   0.0  3.41    0 0.4890 6.405  73.9  3.0921   2 270
## 94  25.0  0.02875  28.0 15.04    0 0.4640 6.211  28.9  3.6659   4 270
## 95  20.6  0.04294  28.0 15.04    0 0.4640 6.249  77.3  3.6150   4 270
## 98  38.7  0.12083   0.0  2.89    0 0.4450 8.069  76.0  3.4952   2 276
## 99  43.8  0.08187   0.0  2.89    0 0.4450 7.820  36.9  3.4952   2 276
## 100 33.2  0.06860   0.0  2.89    0 0.4450 7.416  62.5  3.4952   2 276
## 102 26.5  0.11432   0.0  8.56    0 0.5200 6.781  71.3  2.8561   5 384
## 104 19.3  0.21161   0.0  8.56    0 0.5200 6.137  87.4  2.7147   5 384
## 107 19.5  0.17120   0.0  8.56    0 0.5200 5.836  91.9  2.2110   5 384
## 108 20.4  0.13117   0.0  8.56    0 0.5200 6.127  85.2  2.1224   5 384
## 109 19.8  0.12802   0.0  8.56    0 0.5200 6.474  97.1  2.4329   5 384
## 110 19.4  0.26363   0.0  8.56    0 0.5200 6.229  91.2  2.5451   5 384
## 111 21.7  0.10793   0.0  8.56    0 0.5200 6.195  54.4  2.7778   5 384
## 112 22.8  0.10084   0.0 10.01    0 0.5470 6.715  81.6  2.6775   6 432
## 113 18.8  0.12329   0.0 10.01    0 0.5470 5.913  92.9  2.3534   6 432
## 114 18.7  0.22212   0.0 10.01    0 0.5470 6.092  95.4  2.5480   6 432
## 115 18.5  0.14231   0.0 10.01    0 0.5470 6.254  84.2  2.2565   6 432
## 116 18.3  0.17134   0.0 10.01    0 0.5470 5.928  88.2  2.4631   6 432
## 118 19.2  0.15098   0.0 10.01    0 0.5470 6.021  82.6  2.7474   6 432
## 119 20.4  0.13058   0.0 10.01    0 0.5470 5.872  73.1  2.4775   6 432
## 120 19.3  0.14476   0.0 10.01    0 0.5470 5.731  65.2  2.7592   6 432
## 121 22.0  0.06899   0.0 25.65    0 0.5810 5.870  69.7  2.2577   2 188
## 122 20.3  0.07165   0.0 25.65    0 0.5810 6.004  84.1  2.1974   2 188
## 123 20.5  0.09299   0.0 25.65    0 0.5810 5.961  92.9  2.0869   2 188
## 124 17.3  0.15038   0.0 25.65    0 0.5810 5.856  97.0  1.9444   2 188
## 125 18.8  0.09849   0.0 25.65    0 0.5810 5.879  95.8  2.0063   2 188
## 126 21.4  0.16902   0.0 25.65    0 0.5810 5.986  88.4  1.9929   2 188
## 128 16.2  0.25915   0.0 21.89    0 0.6240 5.693  96.0  1.7883   4 437
## 131 19.2  0.34006   0.0 21.89    0 0.6240 6.458  98.9  2.1185   4 437
## 132 19.6  1.19294   0.0 21.89    0 0.6240 6.326  97.7  2.2710   4 437
## 133 23.0  0.59005   0.0 21.89    0 0.6240 6.372  97.9  2.3274   4 437
```

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## 134 18.4  0.32982   0.0 21.89    0 0.6240 5.822  95.4  2.4699   4 437
## 136 18.1  0.55778   0.0 21.89    0 0.6240 6.335  98.2  2.1107   4 437
## 137 17.4  0.32264   0.0 21.89    0 0.6240 5.942  93.5  1.9669   4 437
## 140 17.8  0.54452   0.0 21.89    0 0.6240 6.151  97.9  1.6687   4 437
## 142 14.4  1.62864   0.0 21.89    0 0.6240 5.019 100.0  1.4394   4 437
## 143 13.4  3.32105   0.0 19.58    1 0.8710 5.403 100.0  1.3216   5 403
## 144 15.6  4.09740   0.0 19.58    0 0.8710 5.468 100.0  1.4118   5 403
## 145 11.8  2.77974   0.0 19.58    0 0.8710 4.903  97.8  1.3459   5 403
## 146 13.8  2.37934   0.0 19.58    0 0.8710 6.130 100.0  1.4191   5 403
## 147 15.6  2.15505   0.0 19.58    0 0.8710 5.628 100.0  1.5166   5 403
## 148 14.6  2.36862   0.0 19.58    0 0.8710 4.926  95.7  1.4608   5 403
## 149 17.8  2.33099   0.0 19.58    0 0.8710 5.186  93.8  1.5296   5 403
## 150 15.4  2.73397   0.0 19.58    0 0.8710 5.597  94.9  1.5257   5 403
## 152 19.6  1.49632   0.0 19.58    0 0.8710 5.404 100.0  1.5916   5 403
## 153 15.3  1.12658   0.0 19.58    1 0.8710 5.012  88.0  1.6102   5 403
## 156 15.6  3.53501   0.0 19.58    1 0.8710 6.152  82.6  1.7455   5 403
## 157 13.1  2.44668   0.0 19.58    0 0.8710 5.272  94.0  1.7364   5 403
## 158 41.3  1.22358   0.0 19.58    0 0.6050 6.943  97.4  1.8773   5 403
## 159 24.3  1.34284   0.0 19.58    0 0.6050 6.066 100.0  1.7573   5 403
## 162 50.0  1.46336   0.0 19.58    0 0.6050 7.489  90.8  1.9709   5 403
## 163 50.0  1.83377   0.0 19.58    1 0.6050 7.802  98.2  2.0407   5 403
## 165 22.7  2.24236   0.0 19.58    0 0.6050 5.854  91.8  2.4220   5 403
## 166 25.0  2.92400   0.0 19.58    0 0.6050 6.101  93.0  2.2834   5 403
## 167 50.0  2.01019   0.0 19.58    0 0.6050 7.929  96.2  2.0459   5 403
## 168 23.8  1.80028   0.0 19.58    0 0.6050 5.877  79.2  2.4259   5 403
## 169 23.8  2.30040   0.0 19.58    0 0.6050 6.319  96.1  2.1000   5 403
## 170 22.3  2.44953   0.0 19.58    0 0.6050 6.402  95.2  2.2625   5 403
## 171 17.4  1.20742   0.0 19.58    0 0.6050 5.875  94.6  2.4259   5 403
## 172 19.1  2.31390   0.0 19.58    0 0.6050 5.880  97.3  2.3887   5 403
## 173 23.1  0.13914   0.0  4.05    0 0.5100 5.572  88.5  2.5961   5 296
## 174 23.6  0.09178   0.0  4.05    0 0.5100 6.416  84.1  2.6463   5 296
## 175 22.6  0.08447   0.0  4.05    0 0.5100 5.859  68.7  2.7019   5 296
## 176 29.4  0.06664   0.0  4.05    0 0.5100 6.546  33.1  3.1323   5 296
## 178 24.6  0.05425   0.0  4.05    0 0.5100 6.315  73.4  3.3175   5 296
## 179 29.9  0.06642   0.0  4.05    0 0.5100 6.860  74.4  2.9153   5 296
## 180 37.2  0.05780   0.0  2.46    0 0.4880 6.980  58.4  2.8290   3 193
## 181 39.8  0.06588   0.0  2.46    0 0.4880 7.765  83.3  2.7410   3 193
## 182 36.2  0.06888   0.0  2.46    0 0.4880 6.144  62.2  2.5979   3 193
## 184 32.5  0.10008   0.0  2.46    0 0.4880 6.563  95.6  2.8470   3 193
## 186 29.6  0.06047   0.0  2.46    0 0.4880 6.153  68.8  3.2797   3 193
## 187 50.0  0.05602   0.0  2.46    0 0.4880 7.831  53.6  3.1992   3 193
## 188 32.0  0.07875  45.0  3.44    0 0.4370 6.782  41.1  3.7886   5 398
## 189 29.8  0.12579  45.0  3.44    0 0.4370 6.556  29.1  4.5667   5 398
## 190 34.9  0.08370  45.0  3.44    0 0.4370 7.185  38.9  4.5667   5 398
## 191 37.0  0.09068  45.0  3.44    0 0.4370 6.951  21.5  6.4798   5 398
## 192 30.5  0.06911  45.0  3.44    0 0.4370 6.739  30.8  6.4798   5 398
## 193 36.4  0.08664  45.0  3.44    0 0.4370 7.178  26.3  6.4798   5 398
## 195 29.1  0.01439  60.0  2.93    0 0.4010 6.604  18.8  6.2196   1 265
## 196 50.0  0.01381  80.0  0.46    0 0.4220 7.875  32.0  5.6484   4 255
## 197 33.3  0.04011  80.0  1.52    0 0.4040 7.287  34.1  7.3090   2 329
## 198 30.3  0.04666  80.0  1.52    0 0.4040 7.107  36.6  7.3090   2 329
## 199 34.6  0.03768  80.0  1.52    0 0.4040 7.274  38.3  7.3090   2 329
## 200 34.9  0.03150  95.0  1.47    0 0.4030 6.975  15.3  7.6534   3 402
## 202 24.1  0.03445  82.5  2.03    0 0.4150 6.162  38.4  6.2700   2 348
```

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## 203 42.3  0.02177  82.5  2.03    0 0.4150 7.610  15.7  6.2700   2 348
## 204 48.5  0.03510  95.0  2.68    0 0.4161 7.853  33.2  5.1180   4 224
## 205 50.0  0.02009  95.0  2.68    0 0.4161 8.034  31.9  5.1180   4 224
## 206 22.6  0.13642   0.0 10.59    0 0.4890 5.891  22.3  3.9454   4 277
## 208 22.5  0.25199   0.0 10.59    0 0.4890 5.783  72.7  4.3549   4 277
## 209 24.4  0.13587   0.0 10.59    1 0.4890 6.064  59.1  4.2392   4 277
## 212 19.3  0.37578   0.0 10.59    1 0.4890 5.404  88.6  3.6650   4 277
## 213 22.4  0.21719   0.0 10.59    1 0.4890 5.807  53.8  3.6526   4 277
## 214 28.1  0.14052   0.0 10.59    0 0.4890 6.375  32.3  3.9454   4 277
## 216 25.0  0.19802   0.0 10.59    0 0.4890 6.182  42.4  3.9454   4 277
## 218 28.7  0.07013   0.0 13.89    0 0.5500 6.642  85.1  3.4211   5 276
## 219 21.5  0.11069   0.0 13.89    1 0.5500 5.951  93.8  2.8893   5 276
## 220 23.0  0.11425   0.0 13.89    1 0.5500 6.373  92.4  3.3633   5 276
## 221 26.7  0.35809   0.0  6.20    1 0.5070 6.951  88.5  2.8617   8 307
## 223 27.5  0.62356   0.0  6.20    1 0.5070 6.879  77.7  3.2721   8 307
## 225 44.8  0.31533   0.0  6.20    0 0.5040 8.266  78.3  2.8944   8 307
## 226 50.0  0.52693   0.0  6.20    0 0.5040 8.725  83.0  2.8944   8 307
## 227 37.6  0.38214   0.0  6.20    0 0.5040 8.040  86.5  3.2157   8 307
## 228 31.6  0.41238   0.0  6.20    0 0.5040 7.163  79.9  3.2157   8 307
## 230 31.5  0.44178   0.0  6.20    0 0.5040 6.552  21.4  3.3751   8 307
## 231 24.3  0.53700   0.0  6.20    0 0.5040 5.981  68.1  3.6715   8 307
## 234 48.3  0.33147   0.0  6.20    0 0.5070 8.247  70.4  3.6519   8 307
## 235 29.0  0.44791   0.0  6.20    1 0.5070 6.726  66.5  3.6519   8 307
## 236 24.0  0.33045   0.0  6.20    0 0.5070 6.086  61.5  3.6519   8 307
## 237 25.1  0.52058   0.0  6.20    1 0.5070 6.631  76.5  4.1480   8 307
## 239 23.7  0.08244  30.0  4.93    0 0.4280 6.481  18.5  6.1899   6 300
## 240 23.3  0.09252  30.0  4.93    0 0.4280 6.606  42.2  6.1899   6 300
## 242 20.1  0.10612  30.0  4.93    0 0.4280 6.095  65.1  6.3361   6 300
## 243 22.2  0.10290  30.0  4.93    0 0.4280 6.358  52.9  7.0355   6 300
## 244 23.7  0.12757  30.0  4.93    0 0.4280 6.393   7.8  7.0355   6 300
## 246 18.5  0.19133  22.0  5.86    0 0.4310 5.605  70.2  7.9549   7 330
## 247 24.3  0.33983  22.0  5.86    0 0.4310 6.108  34.9  8.0555   7 330
## 248 20.5  0.19657  22.0  5.86    0 0.4310 6.226  79.2  8.0555   7 330
## 249 24.5  0.16439  22.0  5.86    0 0.4310 6.433  49.1  7.8265   7 330
## 250 26.2  0.19073  22.0  5.86    0 0.4310 6.718  17.5  7.8265   7 330
## 251 24.4  0.14030  22.0  5.86    0 0.4310 6.487  13.0  7.3967   7 330
## 252 24.8  0.21409  22.0  5.86    0 0.4310 6.438   8.9  7.3967   7 330
## 254 42.8  0.36894  22.0  5.86    0 0.4310 8.259   8.4  8.9067   7 330
## 255 21.9  0.04819  80.0  3.64    0 0.3920 6.108  32.0  9.2203   1 315
## 256 20.9  0.03548  80.0  3.64    0 0.3920 5.876  19.1  9.2203   1 315
## 258 50.0  0.61154  20.0  3.97    0 0.6470 8.704  86.9  1.8010   5 264
## 260 30.1  0.65665  20.0  3.97    0 0.6470 6.842 100.0  2.0107   5 264
## 262 43.1  0.53412  20.0  3.97    0 0.6470 7.520  89.4  2.1398   5 264
## 264 31.0  0.82526  20.0  3.97    0 0.6470 7.327  94.5  2.0788   5 264
## 265 36.5  0.55007  20.0  3.97    0 0.6470 7.206  91.6  1.9301   5 264
## 266 22.8  0.76162  20.0  3.97    0 0.6470 5.560  62.8  1.9865   5 264
## 267 30.7  0.78570  20.0  3.97    0 0.6470 7.014  84.6  2.1329   5 264
## 268 50.0  0.57834  20.0  3.97    0 0.5750 8.297  67.0  2.4216   5 264
## 269 43.5  0.54050  20.0  3.97    0 0.5750 7.470  52.6  2.8720   5 264
## 270 20.7  0.09065  20.0  6.96    1 0.4640 5.920  61.5  3.9175   3 223
## 271 21.1  0.29916  20.0  6.96    0 0.4640 5.856  42.1  4.4290   3 223
## 273 24.4  0.11460  20.0  6.96    0 0.4640 6.538  58.7  3.9175   3 223
## 274 35.2  0.22188  20.0  6.96    1 0.4640 7.691  51.8  4.3665   3 223
## 275 32.4  0.05644  40.0  6.41    1 0.4470 6.758  32.9  4.0776   4 254
```

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## 276 32.0  0.09604  40.0  6.41     0 0.4470 6.854  42.8  4.2673   4 254
## 277 33.2  0.10469  40.0  6.41     1 0.4470 7.267  49.0  4.7872   4 254
## 279 29.1  0.07978  40.0  6.41     0 0.4470 6.482  32.1  4.1403   4 254
## 280 35.1  0.21038  20.0  3.33     0 0.4429 6.812  32.2  4.1007   5 216
## 281 45.4  0.03578  20.0  3.33     0 0.4429 7.820  64.5  4.6947   5 216
## 282 35.4  0.03705  20.0  3.33     0 0.4429 6.968  37.2  5.2447   5 216
## 283 46.0  0.06129  20.0  3.33     1 0.4429 7.645  49.7  5.2119   5 216
## 285 32.2  0.00906  90.0  2.97     0 0.4000 7.088  20.8  7.3073   1 285
## 286 22.0  0.01096  55.0  2.25     0 0.3890 6.453  31.9  7.3073   1 300
## 287 20.1  0.01965  80.0  1.76     0 0.3850 6.230  31.5  9.0892   1 241
## 288 23.2  0.03871  52.5  5.32     0 0.4050 6.209  31.3  7.3172   6 293
## 289 22.3  0.04590  52.5  5.32     0 0.4050 6.315  45.6  7.3172   6 293
## 290 24.8  0.04297  52.5  5.32     0 0.4050 6.565  22.9  7.3172   6 293
## 291 28.5  0.03502  80.0  4.95     0 0.4110 6.861  27.9  5.1167   4 245
## 292 37.3  0.07886  80.0  4.95     0 0.4110 7.148  27.7  5.1167   4 245
## 294 23.9  0.08265   0.0 13.92     0 0.4370 6.127  18.4  5.5027   4 289
## 296 28.6  0.12932   0.0 13.92     0 0.4370 6.678  31.1  5.9604   4 289
## 297 27.1  0.05372   0.0 13.92     0 0.4370 6.549  51.0  5.9604   4 289
## 299 22.5  0.06466  70.0  2.24     0 0.4000 6.345  20.1  7.8278   5 358
## 301 24.8  0.04417  70.0  2.24     0 0.4000 6.871  47.4  7.8278   5 358
## 303 26.4  0.09266  34.0  6.09     0 0.4330 6.495  18.4  5.4917   7 329
## 304 33.1  0.10000  34.0  6.09     0 0.4330 6.982  17.7  5.4917   7 329
## 305 36.1  0.05515  33.0  2.18     0 0.4720 7.236  41.1  4.0220   7 222
## 306 28.4  0.05479  33.0  2.18     0 0.4720 6.616  58.1  3.3700   7 222
## 307 33.4  0.07503  33.0  2.18     0 0.4720 7.420  71.9  3.0992   7 222
## 308 28.2  0.04932  33.0  2.18     0 0.4720 6.849  70.3  3.1827   7 222
## 309 22.8  0.49298   0.0  9.90     0 0.5440 6.635  82.5  3.3175   4 304
## 311 16.1  2.63548   0.0  9.90     0 0.5440 4.973  37.8  2.5194   4 304
## 313 19.4  0.26169   0.0  9.90     0 0.5440 6.023  90.4  2.8340   4 304
## 314 21.6  0.26938   0.0  9.90     0 0.5440 6.266  82.8  3.2628   4 304
## 316 16.2  0.25356   0.0  9.90     0 0.5440 5.705  77.7  3.9450   4 304
## 317 17.8  0.31827   0.0  9.90     0 0.5440 5.914  83.2  3.9986   4 304
## 318 19.8  0.24522   0.0  9.90     0 0.5440 5.782  71.7  4.0317   4 304
## 319 23.1  0.40202   0.0  9.90     0 0.5440 6.382  67.2  3.5325   4 304
## 320 21.0  0.47547   0.0  9.90     0 0.5440 6.113  58.8  4.0019   4 304
## 321 23.8  0.16760   0.0  7.38     0 0.4930 6.426  52.3  4.5404   5 287
## 322 23.1  0.18159   0.0  7.38     0 0.4930 6.376  54.3  4.5404   5 287
## 323 20.4  0.35114   0.0  7.38     0 0.4930 6.041  49.9  4.7211   5 287
## 324 18.5  0.28392   0.0  7.38     0 0.4930 5.708  74.3  4.7211   5 287
## 327 23.0  0.30347   0.0  7.38     0 0.4930 6.312  28.9  5.4159   5 287
## 328 22.2  0.24103   0.0  7.38     0 0.4930 6.083  43.7  5.4159   5 287
## 330 22.6  0.06724   0.0  3.24     0 0.4600 6.333  17.2  5.2146   4 430
## 331 19.8  0.04544   0.0  3.24     0 0.4600 6.144  32.2  5.8736   4 430
## 332 17.1  0.05023  35.0  6.06     0 0.4379 5.706  28.4  6.6407   1 304
## 333 19.4  0.03466  35.0  6.06     0 0.4379 6.031  23.3  6.6407   1 304
## 334 22.2  0.05083   0.0  5.19     0 0.5150 6.316  38.1  6.4584   5 224
## 335 20.7  0.03738   0.0  5.19     0 0.5150 6.310  38.5  6.4584   5 224
## 336 21.1  0.03961   0.0  5.19     0 0.5150 6.037  34.5  5.9853   5 224
## 337 19.5  0.03427   0.0  5.19     0 0.5150 5.869  46.3  5.2311   5 224
## 338 18.5  0.03041   0.0  5.19     0 0.5150 5.895  59.6  5.6150   5 224
## 340 19.0  0.05497   0.0  5.19     0 0.5150 5.985  45.4  4.8122   5 224
## 341 18.7  0.06151   0.0  5.19     0 0.5150 5.968  58.5  4.8122   5 224
## 343 16.5  0.02498   0.0  1.89     0 0.5180 6.540  59.7  6.2669   1 422
## 345 31.2  0.03049  55.0  3.78     0 0.4840 6.874  28.1  6.4654   5 370
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## 346 17.5  0.03113   0.0  4.39    0 0.4420 6.014  48.5  8.0136   3 352
## 347 17.2  0.06162   0.0  4.39    0 0.4420 5.898  52.3  8.0136   3 352
## 349 24.5  0.01501  80.0  2.01    0 0.4350 6.635  29.7  8.3440   4 280
## 350 26.6  0.02899  40.0  1.25    0 0.4290 6.939  34.5  8.7921   1 335
## 351 22.9  0.06211  40.0  1.25    0 0.4290 6.490  44.4  8.7921   1 335
## 352 24.1  0.07950  60.0  1.69    0 0.4110 6.579  35.9 10.7103   4 411
## 353 18.6  0.07244  60.0  1.69    0 0.4110 5.884  18.5 10.7103   4 411
## 355 18.2  0.04301  80.0  1.91    0 0.4130 5.663  21.9 10.5857   4 334
## 356 20.6  0.10659  80.0  1.91    0 0.4130 5.936  19.5 10.5857   4 334
## 358 21.7  3.84970   0.0 18.10    1 0.7700 6.395  91.0  2.5052  24 666
## 359 22.7  5.20177   0.0 18.10    1 0.7700 6.127  83.4  2.7227  24 666
## 360 22.6  4.26131   0.0 18.10    0 0.7700 6.112  81.3  2.5091  24 666
## 361 25.0  4.54192   0.0 18.10    0 0.7700 6.398  88.0  2.5182  24 666
## 363 20.8  3.67822   0.0 18.10    0 0.7700 5.362  96.2  2.1036  24 666
## 365 21.9  3.47428   0.0 18.10    1 0.7180 8.780  82.9  1.9047  24 666
## 366 27.5  4.55587   0.0 18.10    0 0.7180 3.561  87.9  1.6132  24 666
## 367 21.9  3.69695   0.0 18.10    0 0.7180 4.963  91.4  1.7523  24 666
## 369 50.0  4.89822   0.0 18.10    0 0.6310 4.970 100.0  1.3325  24 666
## 370 50.0  5.66998   0.0 18.10    1 0.6310 6.683  96.8  1.3567  24 666
## 373 50.0  8.26725   0.0 18.10    1 0.6680 5.875  89.6  1.1296  24 666
## 374 13.8 11.10810   0.0 18.10    0 0.6680 4.906 100.0  1.1742  24 666
## 375 13.8 18.49820   0.0 18.10    0 0.6680 4.138 100.0  1.1370  24 666
## 376 15.0 19.60910   0.0 18.10    0 0.6710 7.313  97.9  1.3163  24 666
## 379 13.1 23.64820   0.0 18.10    0 0.6710 6.380  96.2  1.3861  24 666
## 380 10.2 17.86670   0.0 18.10    0 0.6710 6.223 100.0  1.3861  24 666
## 382 10.9 15.87440   0.0 18.10    0 0.6710 6.545  99.1  1.5192  24 666
## 383 11.3  9.18702   0.0 18.10    0 0.7000 5.536 100.0  1.5804  24 666
## 384 12.3  7.99248   0.0 18.10    0 0.7000 5.520 100.0  1.5331  24 666
## 385  8.8 20.08490   0.0 18.10    0 0.7000 4.368  91.2  1.4395  24 666
## 386  7.2 16.81180   0.0 18.10    0 0.7000 5.277  98.1  1.4261  24 666
## 387 10.5 24.39380   0.0 18.10    0 0.7000 4.652 100.0  1.4672  24 666
## 388  7.4 22.59710   0.0 18.10    0 0.7000 5.000  89.5  1.5184  24 666
## 389 10.2 14.33370   0.0 18.10    0 0.7000 4.880 100.0  1.5895  24 666
## 390 11.5  8.15174   0.0 18.10    0 0.7000 5.390  98.9  1.7281  24 666
## 392 23.2  5.29305   0.0 18.10    0 0.7000 6.051  82.5  2.1678  24 666
## 394 13.8  8.64476   0.0 18.10    0 0.6930 6.193  92.6  1.7912  24 666
## 396 13.1  8.71675   0.0 18.10    0 0.6930 6.471  98.8  1.7257  24 666
## 397 12.5  5.87205   0.0 18.10    0 0.6930 6.405  96.0  1.6768  24 666
## 398  8.5  7.67202   0.0 18.10    0 0.6930 5.747  98.9  1.6334  24 666
## 399  5.0 38.35180   0.0 18.10    0 0.6930 5.453 100.0  1.4896  24 666
## 400  6.3  9.91655   0.0 18.10    0 0.6930 5.852  77.8  1.5004  24 666
## 401  5.6 25.04610   0.0 18.10    0 0.6930 5.987 100.0  1.5888  24 666
## 402  7.2 14.23620   0.0 18.10    0 0.6930 6.343 100.0  1.5741  24 666
## 404  8.3 24.80170   0.0 18.10    0 0.6930 5.349  96.0  1.7028  24 666
## 405  8.5 41.52920   0.0 18.10    0 0.6930 5.531  85.4  1.6074  24 666
## 407 11.9 20.71620   0.0 18.10    0 0.6590 4.138 100.0  1.1781  24 666
## 408 27.9 11.95110   0.0 18.10    0 0.6590 5.608 100.0  1.2852  24 666
## 409 17.2  7.40389   0.0 18.10    0 0.5970 5.617  97.9  1.4547  24 666
## 410 27.5 14.43830   0.0 18.10    0 0.5970 6.852 100.0  1.4655  24 666
## 411 15.0 51.13580   0.0 18.10    0 0.5970 5.757 100.0  1.4130  24 666
## 412 17.2 14.05070   0.0 18.10    0 0.5970 6.657 100.0  1.5275  24 666
## 417  7.5 10.83420   0.0 18.10    0 0.6790 6.782  90.8  1.8195  24 666
## 418 10.4 25.94060   0.0 18.10    0 0.6790 5.304  89.1  1.6475  24 666
## 419  8.8 73.53410   0.0 18.10    0 0.6790 5.957 100.0  1.8026  24 666
```

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## 421 16.7 11.08740   0.0 18.10    0 0.7180 6.411 100.0  1.8589  24 666
## 422 14.2  7.02259   0.0 18.10    0 0.7180 6.006  95.3  1.8746  24 666
## 423 20.8 12.04820   0.0 18.10    0 0.6140 5.648  87.6  1.9512  24 666
## 425 11.7  8.79212   0.0 18.10    0 0.5840 5.565  70.6  2.0635  24 666
## 426  8.3 15.86030   0.0 18.10    0 0.6790 5.896  95.4  1.9096  24 666
## 427 10.2 12.24720   0.0 18.10    0 0.5840 5.837  59.7  1.9976  24 666
## 428 10.9 37.66190   0.0 18.10    0 0.6790 6.202  78.7  1.8629  24 666
## 429 11.0  7.36711   0.0 18.10    0 0.6790 6.193  78.1  1.9356  24 666
## 430  9.5  9.33889   0.0 18.10    0 0.6790 6.380  95.6  1.9682  24 666
## 431 14.5  8.49213   0.0 18.10    0 0.5840 6.348  86.1  2.0527  24 666
## 432 14.1 10.06230   0.0 18.10    0 0.5840 6.833  94.3  2.0882  24 666
## 433 16.1  6.44405   0.0 18.10    0 0.5840 6.425  74.8  2.2004  24 666
## 434 14.3  5.58107   0.0 18.10    0 0.7130 6.436  87.9  2.3158  24 666
## 435 11.7 13.91340   0.0 18.10    0 0.7130 6.208  95.0  2.2222  24 666
## 436 13.4 11.16040   0.0 18.10    0 0.7400 6.629  94.6  2.1247  24 666
## 437  9.6 14.42080   0.0 18.10    0 0.7400 6.461  93.3  2.0026  24 666
## 438  8.7 15.17720   0.0 18.10    0 0.7400 6.152 100.0  1.9142  24 666
## 439  8.4 13.67810   0.0 18.10    0 0.7400 5.935  87.9  1.8206  24 666
## 440 12.8  9.39063   0.0 18.10    0 0.7400 5.627  93.9  1.8172  24 666
## 441 10.5 22.05110   0.0 18.10    0 0.7400 5.818  92.4  1.8662  24 666
## 442 17.1  9.72418   0.0 18.10    0 0.7400 6.406  97.2  2.0651  24 666
## 443 18.4  5.66637   0.0 18.10    0 0.7400 6.219 100.0  2.0048  24 666
## 444 15.4  9.96654   0.0 18.10    0 0.7400 6.485 100.0  1.9784  24 666
## 446 11.8 10.67180   0.0 18.10    0 0.7400 6.459  94.8  1.9879  24 666
## 447 14.9  6.28807   0.0 18.10    0 0.7400 6.341  96.4  2.0720  24 666
## 449 14.1  9.32909   0.0 18.10    0 0.7130 6.185  98.7  2.2616  24 666
## 450 13.0  7.52601   0.0 18.10    0 0.7130 6.417  98.3  2.1850  24 666
## 451 13.4  6.71772   0.0 18.10    0 0.7130 6.749  92.6  2.3236  24 666
## 452 15.2  5.44114   0.0 18.10    0 0.7130 6.655  98.2  2.3552  24 666
## 454 17.8  8.24809   0.0 18.10    0 0.7130 7.393  99.3  2.4527  24 666
## 455 14.9  9.51363   0.0 18.10    0 0.7130 6.728  94.1  2.4961  24 666
## 457 12.7  4.66883   0.0 18.10    0 0.7130 5.976  87.9  2.5806  24 666
## 458 13.5  8.20058   0.0 18.10    0 0.7130 5.936  80.3  2.7792  24 666
## 459 14.9  7.75223   0.0 18.10    0 0.7130 6.301  83.7  2.7831  24 666
## 461 16.4  4.81213   0.0 18.10    0 0.7130 6.701  90.0  2.5975  24 666
## 462 17.7  3.69311   0.0 18.10    0 0.7130 6.376  88.4  2.5671  24 666
## 463 19.5  6.65492   0.0 18.10    0 0.7130 6.317  83.0  2.7344  24 666
## 466 19.9  3.16360   0.0 18.10    0 0.6550 5.759  48.2  3.0665  24 666
## 467 19.0  3.77498   0.0 18.10    0 0.6550 5.952  84.7  2.8715  24 666
## 468 19.1  4.42228   0.0 18.10    0 0.5840 6.003  94.5  2.5403  24 666
## 469 19.1 15.57570   0.0 18.10    0 0.5800 5.926  71.0  2.9084  24 666
## 470 20.1 13.07510   0.0 18.10    0 0.5800 5.713  56.7  2.8237  24 666
## 471 19.9  4.34879   0.0 18.10    0 0.5800 6.167  84.0  3.0334  24 666
## 472 19.6  4.03841   0.0 18.10    0 0.5320 6.229  90.7  3.0993  24 666
## 473 23.2  3.56868   0.0 18.10    0 0.5800 6.437  75.0  2.8965  24 666
## 474 29.8  4.64689   0.0 18.10    0 0.6140 6.980  67.6  2.5329  24 666
## 475 13.8  8.05579   0.0 18.10    0 0.5840 5.427  95.4  2.4298  24 666
## 476 13.3  6.39312   0.0 18.10    0 0.5840 6.162  97.4  2.2060  24 666
## 477 16.7  4.87141   0.0 18.10    0 0.6140 6.484  93.6  2.3053  24 666
## 478 12.0 15.02340   0.0 18.10    0 0.6140 5.304  97.3  2.1007  24 666
## 479 14.6 10.23300   0.0 18.10    0 0.6140 6.185  96.7  2.1705  24 666
## 480 21.4 14.33370   0.0 18.10    0 0.6140 6.229  88.0  1.9512  24 666
## 483 25.0  5.73116   0.0 18.10    0 0.5320 7.061  77.0  3.4106  24 666
## 484 21.8  2.81838   0.0 18.10    0 0.5320 5.762  40.3  4.0983  24 666
```

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## 485 20.6  2.37857   0.0 18.10    0 0.5830 5.871  41.9  3.7240  24 666
## 486 21.2  3.67367   0.0 18.10    0 0.5830 6.312  51.9  3.9917  24 666
## 488 20.6  4.83567   0.0 18.10    0 0.5830 5.905  53.2  3.1523  24 666
## 490  7.0  0.18337   0.0 27.74    0 0.6090 5.414  98.3  1.7554   4 711
## 492 13.6  0.10574   0.0 27.74    0 0.6090 5.983  98.8  1.8681   4 711
## 493 20.1  0.11132   0.0 27.74    0 0.6090 5.983  83.5  2.1099   4 711
## 494 21.8  0.17331   0.0  9.69    0 0.5850 5.707  54.0  2.3817   6 391
## 495 24.5  0.27957   0.0  9.69    0 0.5850 5.926  42.6  2.3817   6 391
## 496 23.1  0.17899   0.0  9.69    0 0.5850 5.670  28.8  2.7986   6 391
## 497 19.7  0.28960   0.0  9.69    0 0.5850 5.390  72.9  2.7986   6 391
## 498 18.3  0.26838   0.0  9.69    0 0.5850 5.794  70.6  2.8927   6 391
## 499 21.2  0.23912   0.0  9.69    0 0.5850 6.019  65.3  2.4091   6 391
## 500 17.5  0.17783   0.0  9.69    0 0.5850 5.569  73.5  2.3999   6 391
## 501 16.8  0.22438   0.0  9.69    0 0.5850 6.027  79.7  2.4982   6 391
## 502 22.4  0.06263   0.0 11.93    0 0.5730 6.593  69.1  2.4786   1 273
## 503 20.6  0.04527   0.0 11.93    0 0.5730 6.120  76.7  2.2875   1 273
## 505 22.0  0.10959   0.0 11.93    0 0.5730 6.794  89.3  2.3889   1 273
## 506 11.9  0.04741   0.0 11.93    0 0.5730 6.030  80.8  2.5050   1 273
##     PTRATIO      B LSTAT
## 1      15.3 396.90  4.98
## 2      17.8 396.90  9.14
## 3      17.8 392.83  4.03
## 4      18.7 394.63  2.94
## 5      18.7 396.90  5.33
## 6      18.7 394.12  5.21
## 8      15.2 396.90 19.15
## 9      15.2 386.63 29.93
## 10     15.2 386.71 17.10
## 12     15.2 396.90 13.27
## 16     21.0 395.62  8.47
## 17     21.0 386.85  6.58
## 19     21.0 288.99 11.69
## 21     21.0 376.57 21.02
## 23     21.0 396.90 18.72
## 27     21.0 376.88 14.81
## 28     21.0 306.38 17.28
## 29     21.0 387.94 12.80
## 30     21.0 380.23 11.98
## 31     21.0 360.17 22.60
## 32     21.0 376.73 13.04
## 33     21.0 232.60 27.71
## 34     21.0 358.77 18.35
## 35     21.0 248.31 20.34
## 36     19.2 396.90  9.68
## 37     19.2 377.56 11.41
## 38     19.2 396.90  8.77
## 39     19.2 393.43 10.13
## 41     18.3 395.62  1.98
## 42     17.9 385.41  4.84
## 44     17.9 394.46  7.44
## 48     17.9 392.74 18.80
## 51     16.8 395.56 13.45
## 53     16.8 396.90  5.28
## 56     17.9 395.93  4.81
```

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## 57      17.3 396.90  5.77
## 58      15.1 392.90  3.95
## 59      19.7 390.68  6.86
## 60      19.7 396.90  9.22
## 61      19.7 395.11 13.15
## 62      19.7 378.08 14.44
## 63      19.7 396.90  6.73
## 64      19.7 395.58  9.50
## 66      16.1 396.90  4.67
## 67      16.1 396.90 10.24
## 68      18.9 396.21  8.10
## 69      18.9 396.90 13.09
## 72      19.2 376.94  9.88
## 73      19.2 390.91  5.52
## 74      19.2 377.17  7.54
## 75      18.7 394.92  6.78
## 77      18.7 373.66 11.97
## 79      18.7 386.40 12.34
## 80      18.7 396.06  9.10
## 81      19.0 396.90  5.29
## 82      19.0 395.63  7.22
## 83      19.0 396.90  6.72
## 84      19.0 390.64  7.51
## 86      18.5 392.30  6.53
## 88      18.5 395.15  8.44
## 89      17.8 396.90  5.50
## 91      17.8 392.18  8.81
## 92      17.8 393.55  8.20
## 94      18.2 396.33  6.21
## 95      18.2 396.90 10.59
## 98      18.0 396.90  4.21
## 99      18.0 393.53  3.57
## 100     18.0 396.90  6.19
## 102     20.9 395.58  7.67
## 104     20.9 394.47 13.44
## 107     20.9 395.67 18.66
## 108     20.9 387.69 14.09
## 109     20.9 395.24 12.27
## 110     20.9 391.23 15.55
## 111     20.9 393.49 13.00
## 112     17.8 395.59 10.16
## 113     17.8 394.95 16.21
## 114     17.8 396.90 17.09
## 115     17.8 388.74 10.45
## 116     17.8 344.91 15.76
## 118     17.8 394.51 10.30
## 119     17.8 338.63 15.37
## 120     17.8 391.50 13.61
## 121     19.1 389.15 14.37
## 122     19.1 377.67 14.27
## 123     19.1 378.09 17.93
## 124     19.1 370.31 25.41
## 125     19.1 379.38 17.58
## 126     19.1 385.02 14.81
```

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## 128    21.2 392.11 17.19
## 131    21.2 395.04 12.60
## 132    21.2 396.90 12.26
## 133    21.2 385.76 11.12
## 134    21.2 388.69 15.03
## 136    21.2 394.67 16.96
## 137    21.2 378.25 16.90
## 140    21.2 396.90 18.46
## 142    21.2 396.90 34.41
## 143    14.7 396.90 26.82
## 144    14.7 396.90 26.42
## 145    14.7 396.90 29.29
## 146    14.7 172.91 27.80
## 147    14.7 169.27 16.65
## 148    14.7 391.71 29.53
## 149    14.7 356.99 28.32
## 150    14.7 351.85 21.45
## 152    14.7 341.60 13.28
## 153    14.7 343.28 12.12
## 156    14.7  88.01 15.02
## 157    14.7  88.63 16.14
## 158    14.7 363.43  4.59
## 159    14.7 353.89  6.43
## 162    14.7 374.43  1.73
## 163    14.7 389.61  1.92
## 165    14.7 395.11 11.64
## 166    14.7 240.16  9.81
## 167    14.7 369.30  3.70
## 168    14.7 227.61 12.14
## 169    14.7 297.09 11.10
## 170    14.7 330.04 11.32
## 171    14.7 292.29 14.43
## 172    14.7 348.13 12.03
## 173    16.6 396.90 14.69
## 174    16.6 395.50  9.04
## 175    16.6 393.23  9.64
## 176    16.6 390.96  5.33
## 178    16.6 395.60  6.29
## 179    16.6 391.27  6.92
## 180    17.8 396.90  5.04
## 181    17.8 395.56  7.56
## 182    17.8 396.90  9.45
## 184    17.8 396.90  5.68
## 186    17.8 387.11 13.15
## 187    17.8 392.63  4.45
## 188    15.2 393.87  6.68
## 189    15.2 382.84  4.56
## 190    15.2 396.90  5.39
## 191    15.2 377.68  5.10
## 192    15.2 389.71  4.69
## 193    15.2 390.49  2.87
## 195    15.6 376.70  4.38
## 196    14.4 394.23  2.97
## 197    12.6 396.90  4.08
```

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## 198     12.6 354.31  8.61
## 199     12.6 392.20  6.62
## 200     17.0 396.90  4.56
## 202     14.7 393.77  7.43
## 203     14.7 395.38  3.11
## 204     14.7 392.78  3.81
## 205     14.7 390.55  2.88
## 206     18.6 396.90 10.87
## 208     18.6 389.43 18.06
## 209     18.6 381.32 14.66
## 212     18.6 395.24 23.98
## 213     18.6 390.94 16.03
## 214     18.6 385.81  9.38
## 216     18.6 393.63  9.47
## 218     16.4 392.78  9.69
## 219     16.4 396.90 17.92
## 220     16.4 393.74 10.50
## 221     17.4 391.70  9.71
## 223     17.4 390.39  9.93
## 225     17.4 385.05  4.14
## 226     17.4 382.00  4.63
## 227     17.4 387.38  3.13
## 228     17.4 372.08  6.36
## 230     17.4 380.34  3.76
## 231     17.4 378.35 11.65
## 234     17.4 378.95  3.95
## 235     17.4 360.20  8.05
## 236     17.4 376.75 10.88
## 237     17.4 388.45  9.54
## 239     16.6 379.41  6.36
## 240     16.6 383.78  7.37
## 242     16.6 394.62 12.40
## 243     16.6 372.75 11.22
## 244     16.6 374.71  5.19
## 246     19.1 389.13 18.46
## 247     19.1 390.18  9.16
## 248     19.1 376.14 10.15
## 249     19.1 374.71  9.52
## 250     19.1 393.74  6.56
## 251     19.1 396.28  5.90
## 252     19.1 377.07  3.59
## 254     19.1 396.90  3.54
## 255     16.4 392.89  6.57
## 256     16.4 395.18  9.25
## 258     13.0 389.70  5.12
## 260     13.0 391.93  6.90
## 262     13.0 388.37  7.26
## 264     13.0 393.42 11.25
## 265     13.0 387.89  8.10
## 266     13.0 392.40 10.45
## 267     13.0 384.07 14.79
## 268     13.0 384.54  7.44
## 269     13.0 390.30  3.16
## 270     18.6 391.34 13.65
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## 271     18.6 388.65 13.00
## 273     18.6 394.96  7.73
## 274     18.6 390.77  6.58
## 275     17.6 396.90  3.53
## 276     17.6 396.90  2.98
## 277     17.6 389.25  6.05
## 279     17.6 396.90  7.19
## 280     14.9 396.90  4.85
## 281     14.9 387.31  3.76
## 282     14.9 392.23  4.59
## 283     14.9 377.07  3.01
## 285     15.3 394.72  7.85
## 286     15.3 394.72  8.23
## 287     18.2 341.60 12.93
## 288     16.6 396.90  7.14
## 289     16.6 396.90  7.60
## 290     16.6 371.72  9.51
## 291     19.2 396.90  3.33
## 292     19.2 396.90  3.56
## 294     16.0 396.90  8.58
## 296     16.0 396.90  6.27
## 297     16.0 392.85  7.39
## 299     14.8 368.24  4.97
## 301     14.8 390.86  6.07
## 303     16.1 383.61  8.67
## 304     16.1 390.43  4.86
## 305     18.4 393.68  6.93
## 306     18.4 393.36  8.93
## 307     18.4 396.90  6.47
## 308     18.4 396.90  7.53
## 309     18.4 396.90  4.54
## 311     18.4 350.45 12.64
## 313     18.4 396.30 11.72
## 314     18.4 393.39  7.90
## 316     18.4 396.42 11.50
## 317     18.4 390.70 18.33
## 318     18.4 396.90 15.94
## 319     18.4 395.21 10.36
## 320     18.4 396.23 12.73
## 321     19.6 396.90  7.20
## 322     19.6 396.90  6.87
## 323     19.6 396.90  7.70
## 324     19.6 391.13 11.74
## 327     19.6 396.90  6.15
## 328     19.6 396.90 12.79
## 330     16.9 375.21  7.34
## 331     16.9 368.57  9.09
## 332     16.9 394.02 12.43
## 333     16.9 362.25  7.83
## 334     20.2 389.71  5.68
## 335     20.2 389.40  6.75
## 336     20.2 396.90  8.01
## 337     20.2 396.90  9.80
## 338     20.2 394.81 10.56
```

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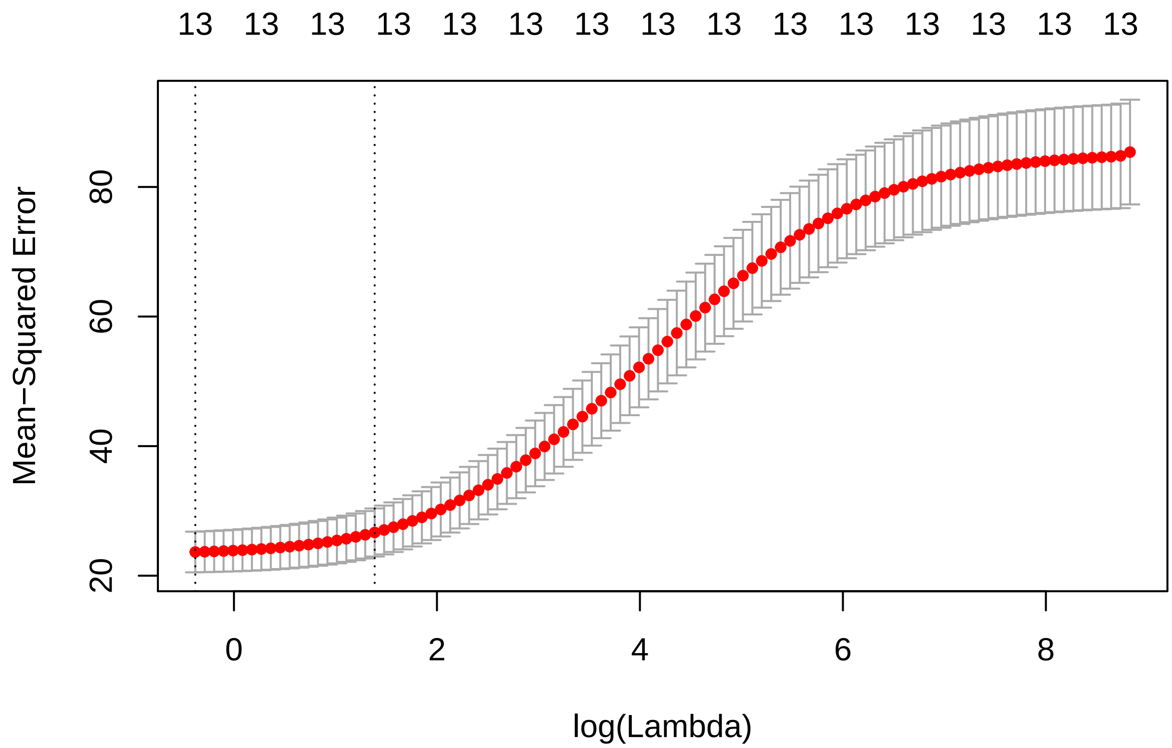
## 340     20.2 396.90  9.74
## 341     20.2 396.90  9.29
## 343     15.9 389.96  8.65
## 345     17.6 387.97  4.61
## 346     18.8 385.64 10.53
## 347     18.8 364.61 12.67
## 349     17.0 390.94  5.99
## 350     19.7 389.85  5.89
## 351     19.7 396.90  5.98
## 352     18.3 370.78  5.49
## 353     18.3 392.33  7.79
## 355     22.0 382.80  8.05
## 356     22.0 376.04  5.57
## 358     20.2 391.34 13.27
## 359     20.2 395.43 11.48
## 360     20.2 390.74 12.67
## 361     20.2 374.56  7.79
## 363     20.2 380.79 10.19
## 365     20.2 354.55  5.29
## 366     20.2 354.70  7.12
## 367     20.2 316.03 14.00
## 369     20.2 375.52  3.26
## 370     20.2 375.33  3.73
## 373     20.2 347.88  8.88
## 374     20.2 396.90 34.77
## 375     20.2 396.90 37.97
## 376     20.2 396.90 13.44
## 379     20.2 396.90 23.69
## 380     20.2 393.74 21.78
## 382     20.2 396.90 21.08
## 383     20.2 396.90 23.60
## 384     20.2 396.90 24.56
## 385     20.2 285.83 30.63
## 386     20.2 396.90 30.81
## 387     20.2 396.90 28.28
## 388     20.2 396.90 31.99
## 389     20.2 372.92 30.62
## 390     20.2 396.90 20.85
## 392     20.2 378.38 18.76
## 394     20.2 396.90 15.17
## 396     20.2 391.98 17.12
## 397     20.2 396.90 19.37
## 398     20.2 393.10 19.92
## 399     20.2 396.90 30.59
## 400     20.2 338.16 29.97
## 401     20.2 396.90 26.77
## 402     20.2 396.90 20.32
## 404     20.2 396.90 19.77
## 405     20.2 329.46 27.38
## 407     20.2 370.22 23.34
## 408     20.2 332.09 12.13
## 409     20.2 314.64 26.40
## 410     20.2 179.36 19.78
## 411     20.2   2.60 10.11
```

```
## 412    20.2  35.05 21.22
## 417    20.2  21.57 25.79
## 418    20.2 127.36 26.64
## 419    20.2  16.45 20.62
## 421    20.2 318.75 15.02
## 422    20.2 319.98 15.70
## 423    20.2 291.55 14.10
## 425    20.2   3.65 17.16
## 426    20.2   7.68 24.39
## 427    20.2  24.65 15.69
## 428    20.2  18.82 14.52
## 429    20.2  96.73 21.52
## 430    20.2  60.72 24.08
## 431    20.2  83.45 17.64
## 432    20.2  81.33 19.69
## 433    20.2  97.95 12.03
## 434    20.2 100.19 16.22
## 435    20.2 100.63 15.17
## 436    20.2 109.85 23.27
## 437    20.2  27.49 18.05
## 438    20.2   9.32 26.45
## 439    20.2  68.95 34.02
## 440    20.2 396.90 22.88
## 441    20.2 391.45 22.11
## 442    20.2 385.96 19.52
## 443    20.2 395.69 16.59
## 444    20.2 386.73 18.85
## 446    20.2  43.06 23.98
## 447    20.2 318.01 17.79
## 449    20.2 396.90 18.13
## 450    20.2 304.21 19.31
## 451    20.2   0.32 17.44
## 452    20.2 355.29 17.73
## 454    20.2 375.87 16.74
## 455    20.2   6.68 18.71
## 457    20.2  10.48 19.01
## 458    20.2   3.50 16.94
## 459    20.2 272.21 16.23
## 461    20.2 255.23 16.42
## 462    20.2 391.43 14.65
## 463    20.2 396.90 13.99
## 466    20.2 334.40 14.13
## 467    20.2  22.01 17.15
## 468    20.2 331.29 21.32
## 469    20.2 368.74 18.13
## 470    20.2 396.90 14.76
## 471    20.2 396.90 16.29
## 472    20.2 395.33 12.87
## 473    20.2 393.37 14.36
## 474    20.2 374.68 11.66
## 475    20.2 352.58 18.14
## 476    20.2 302.76 24.10
## 477    20.2 396.21 18.68
## 478    20.2 349.48 24.91
```

```
## 479    20.2 379.70 18.03
## 480    20.2 383.32 13.11
## 483    20.2 395.28  7.01
## 484    20.2 392.92 10.42
## 485    20.2 370.73 13.34
## 486    20.2 388.62 10.58
## 488    20.2 388.22 11.45
## 490    20.1 344.05 23.97
## 492    20.1 390.11 18.07
## 493    20.1 396.90 13.35
## 494    19.2 396.90 12.01
## 495    19.2 396.90 13.59
## 496    19.2 393.29 17.60
## 497    19.2 396.90 21.14
## 498    19.2 396.90 14.10
## 499    19.2 396.90 12.92
## 500    19.2 395.77 15.10
## 501    19.2 396.90 14.33
## 502    21.0 391.99  9.67
## 503    21.0 396.90  9.08
## 505    21.0 393.45  6.48
## 506    21.0 396.90  7.88
```

```r
x <- model.matrix(MEDV~., train)[,-1]
y = train$MEDV
y.test = y[-trainIndex]

#perform cross-validation to choose tuning parameter lambda
cv.out <- cv.glmnet(x, y, alpha = 0)
plot(cv.out)
```




```
bestlambda <- cv.out$lambda.min #lambda that results in lowest cross validation error
grid <- 10^seq(10,-2,length = 100) #from 10^10 to 10^-2
ridge.mod <- glmnet(x,y,alpha= 0, lambda = grid, thresh = 1e-12)
#make predictions for lambda = bestlambda
ridge.pred <- predict(ridge.mod, s=bestlambda, newx = x[-trainIndex,])
ridge.MSE <- mean((ridge.pred - y.test)^2)
```

```
library(plotmo)

glmcoef<-coef(ridge.mod,bestlambda )
coef.increase<-dimnames(glmcoef[glmcoef[,1]>0,0])[[1]]
coef.decrease<-dimnames(glmcoef[glmcoef[,1]<0,0])[[1]]

#get ordered list of variables as they appear at smallest lambda
allnames<-names(coef(ridge.mod)[,
            ncol(coef(ridge.mod))][order(coef(ridge.mod)[,
            ncol(coef(ridge.mod))],decreasing=TRUE)])

#remove intercept
allnames<-setdiff(allnames,allnames[grep("Intercept",allnames)])

#assign colors
cols<-rep("gray",length(allnames))
cols[allnames %in% coef.increase]<-"green"      # higher medv is good
cols[allnames %in% coef.decrease]<-"red"        # lower medv is not
```

```r
plot_glmnet(ridge.mod,label=TRUE,s=bestlambda,col=cols)
```