1 Problem 10 The RBF kernel

Consider the function:

$$k(x,x') = exp(-\frac{||x-x'||^2}{2\sigma^2}), x, x' \epsilon R^d$$

popularly known as the RBF kernel. Prove that it is a valid kernel. Hint: expand the square and express the kernel as the product of three terms.

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We can see that this is a valid kernel by expanding the square:

$$||x-x'||^2 = x^T x + (x')^T x' - 2x^T x'$$

which will give us:

$$k(x,x') = exp(\frac{-x^T x}{2\sigma^2}) exp(\frac{x^T x'}{\sigma^2}) exp(\frac{-(x')^T x'}{2\sigma^2})$$

We also know that given two valid kernels $k_1(x,x'), k_2(x,x')$, then following kernels are also valid:

$$k(x,x') = f(x) k_1(x,x') f(x')$$

we see that we only need to prove that $exp(\frac{x^T x}{\sigma^2})$ is a valid kernel. Since we can express k(x,x') as:

$$k(x,x') = f(x) k(x,x') f(x')$$

where

$$f(x) = exp(\frac{-x^T x}{2\sigma^2}); f(y) = exp(\frac{-(x')^T x'}{2\sigma^2}); k(x,x') = exp(\frac{x^T x'}{\sigma^2})$$

As we can see in the Bishop, $k(x,x') = exp(k(x,x'))$ is a valid kernel. Which means that $exp(\frac{x^T x}{\sigma^2})$ is valid kernel therefore RBF is valid.

This is because, we know that the sum and multiplication of valid kernels gives a valid kernel. Moreover, since exponential function is the sum of infinite polynomials (Taylor Series Expansion) and $exp(\frac{x^T x}{\sigma^2})$ is an inner product, we have that $exp(\frac{x^T x}{\sigma^2})$ is a valid kernel. This idea can be formally expressed as:

Using Taylor expansion around 0:

$$\exp(K) = \exp(0) + \exp(0)K + \frac{\exp(0)}{2!}K^2 + \frac{\exp(0)}{3!}K^3 + ...$$

$$\exp(K) = 1 + K + \frac{1}{2}K^2 + \frac{1}{6}K^3 + ...$$

we can see that the exponential of a kernel is just an infinite series of multiplications and additions of that kernel.

Using the fact that addition and multiplication of kernels yield valid kernels:

$$K' = \alpha K_1 + \beta K_2$$
$$K' = K_1 K_2$$

we can conclude that the exponential of a kernel is a kernel.