UNIVERSITAT POLITÈCNICA DE CATALUNYA

BARCELONA SCHOOL OF INFORMATICS

MASTER IN INNOVATION AND RESEARCH IN INFORMATICS

Kernel-Based Machine Learning and Multivariate Modelling

- Week 1 exercise -

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1 Solving simple extremal problems 3

Consider the circle formed by the intersection of the unit sphere with the plane x + y + z = 0.5. Consider the three-dimensional optimization problem of finding the point on this circle that is closest to the point (1,2,3). Solve it using Lagrange multipliers.

We see from the problem that we have to find the circle formed by the intersection of these two equations:

$$\begin{aligned} UnitSphere: x^2 + y^2 + z^2 = 1 \\ Plane: x + y + z = 0.5 \end{aligned} \tag{1}$$

which is closest to the point (1,2,3) which represents a **minimization** problem that can be formulated as the following:

$$\begin{aligned} min(x-1)^2 + (y-2)^2 + (z-3)^2 \\ x^2 + y^2 + z^2 = 1 \\ x + y + z = 0.5 \end{aligned} \tag{2}$$

First lets plot the problem so that we have a better understanding of what we are trying to optimize.

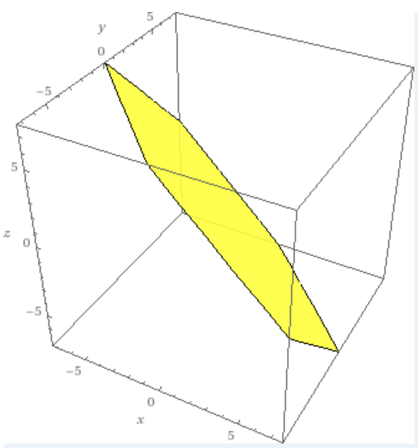

Figure 1: Plane corresponding to x+y+z = 0.5

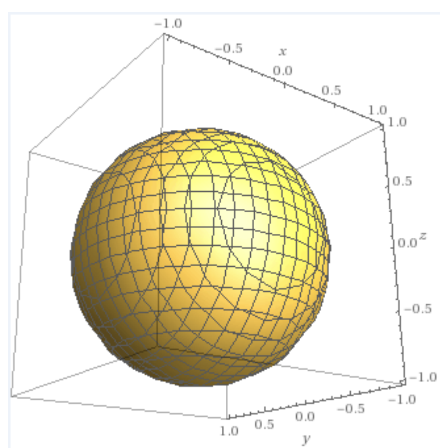

Figure 2: Unit Sphere

Once we have a better idea of which problem are we trying to minimize, let's proceed to solve it using Lagrange multipliers:

$$L_p(x,y,z,\alpha,\beta) = (x-1)^2+(y-2)^2+(z-3)^2+\alpha(x^2+y^2+z^2-1)+\beta(2x+2y+2z-1)$$

$$L_p(4-\alpha-\beta+(\alpha+1)x^2+2(\beta-1)x+(\alpha+1)y^2+2(\beta-2)y+(\alpha+1)z^2+2(\beta-3)z$$

Which if do the partial derivates we obtain:

$$\frac{\partial L_p}{\partial x} = 2(\alpha+1)x+2(\beta-1); \frac{\partial L_p}{\partial y} = 2(\alpha+1)x+2(\beta-2); \frac{\partial L_p}{\partial z} = 2(\alpha+1)x+2(\beta-3)$$

$$x = \frac{-\beta-1}{\alpha+1}; y = \frac{-\beta-2}{\alpha+1}; z = \frac{-\beta-3}{\alpha+1}$$

Applying the dual equation:

$$L_d(\alpha,\beta) = -\frac{(\beta-1)^2+(\beta-2)^2+(\beta-3)^2}{\alpha+1} - \alpha - \beta + 14$$

$$\frac{\partial L_d}{\partial \alpha} = \frac{-3\beta^2+12\beta-14}{(\alpha+1)^2} - 1$$
$$\frac{\partial L_d}{\partial \beta} = \frac{-6\beta+12}{\alpha+1} - 1$$

Which if we set to 0, we obtain the following values:

$$3\beta^2 - 12\beta + 14 = (\alpha+1)^2$$
$$-6\beta + 12 = \alpha + 1$$

$$\beta_1 = 2 - \sqrt{\frac{2}{33}}; \beta_2 = 2 + \sqrt{\frac{2}{33}}; \alpha_1 = -1 - 2\sqrt{\frac{6}{11}}; \alpha_2 = -1 + 2\sqrt{\frac{6}{11}}$$

once we obtain the values from β and α, we can substitute them in the previous x, y, z values to obtain their value:

$$x_1 = \frac{-1}{6} + \frac{1}{2}\sqrt{\frac{11}{6}}; x_2 = \frac{-1}{6} - \frac{1}{2}\sqrt{\frac{11}{6}}; y_1 = \frac{-1}{6}; y_2 = \frac{1}{6};$$
$$z_1 = \frac{-1}{6} - \frac{1}{2}\sqrt{\frac{11}{6}}; z_2 = \frac{-1}{6} + \frac{1}{2}\sqrt{\frac{11}{6}}$$

Which between (x_1, y_1, z_1) and (x_2, y_2, z_2), we can find that the optimum value is $(x_2, y_2, z_2) = 13 - 2\sqrt{\frac{22}{3}}$ so the optimum is:

$$x_2 = \frac{-1}{6} - \frac{1}{2}\sqrt{\frac{11}{6}}; y_2 = \frac{1}{6}; z_2 = \frac{-1}{6} + \frac{1}{2}\sqrt{\frac{11}{6}}$$