David Morán Pomés

SMDE – Third Assignment [Notes]

Let A be the number of arrivals per minute. Then, we have:

$$A \sim P\left(\frac{1}{3}\right)$$

$$P(A = k) = \frac{1}{3^k} \cdot \frac{1}{k!} \cdot e^{-\frac{1}{3}} \qquad \mathrm{E}(A) = \frac{1}{3}$$

$$P(A = 0) = e^{-\frac{1}{3}} = 0.717$$
$$P(A = 1) = \frac{1}{3} e^{-\frac{1}{3}} = 0.239$$
$$P(A = 2) = \frac{1}{18} e^{-\frac{1}{3}} = 0.040$$
$$P(A = 3) = \frac{1}{162} e^{-\frac{1}{3}} = 0.004$$
$$P(A \geq 4) \cong 0$$

Let S be the service time of a single car. Then, we have:

$$S \sim Exp\left(\frac{1}{3}\right)$$

$$P(S \leq k) = 1 - e^{-\frac{k}{3}} \qquad \mathrm{E}(S) = 3$$
$$P(S \leq 1) = 1 - e^{-\frac{1}{3}} = 0.283$$

In general, we will consider $P(S) = P(S \leq 1)$, so that S means that the current car has left the station before the current minute has finished. We will also assume that a new car enters the station only when a minute starts.

Let B_i be the balking of one car at state i. Then, we have:

$$P(B_i) = \frac{i}{4}$$

The main formula for a state increase $(j \geq i)$ is:

$$P_{i,j} = \big(1 - P(S)\big) \cdot \sum_{k=j-i}^{\infty} P(A = k) \cdot P(B_i)^{k-(j-i)} \big(1 - P(B_i)\big)^{j-i} + P(S) \cdot \sum_{k=j-i+1}^{\infty} P(A = k) \cdot P(B_i)^{k-(j-i+1)} \big(1 - P(B_i)\big)^{j-i+1}$$

The first term means that, if no car leaves the station, an arrival of exactly $j - i$ cars is required. For each possible number k of arrivals, we require that $k - (j - i)$ balk and that $j - i$ stay at the queue. The second term means that, if a car leaves the station, the same as above is required, but with one more car.

The main formula for a state stay $(j = i)$ is:

$$P_{i,i} = \big(1 - P(S)\big) \cdot \sum_{k=0}^{\infty} P(A = k) \cdot P(B_i)^{k} + P(S) \cdot \sum_{k=1}^{\infty} P(A = k) \cdot P(B_i)^{k-1} \big(1 - P(B_i)\big)$$

The first term means that, if no car leaves the station, it is required that all cars that come, balk. The second term means that, if a car leaves the station, the same as above is required, but allowing one single car no to balk.

The main formula for a state decrease $(j = i - 1)$ is:

$$P_{i,i-1} = P(S) \cdot \sum_{k=0}^{\infty} P(A = k) \cdot P(B_i)^{k}$$

This means that the only way to decrease a state is that some car leaves and all cars that come, balk.

The main formula for a 2-state decrease $(j < i - 1)$ is:

$$P_{i,j} = 0$$

The first state is special, since no leaving can be produced:

$$P_{0,j} = P(A = j)$$

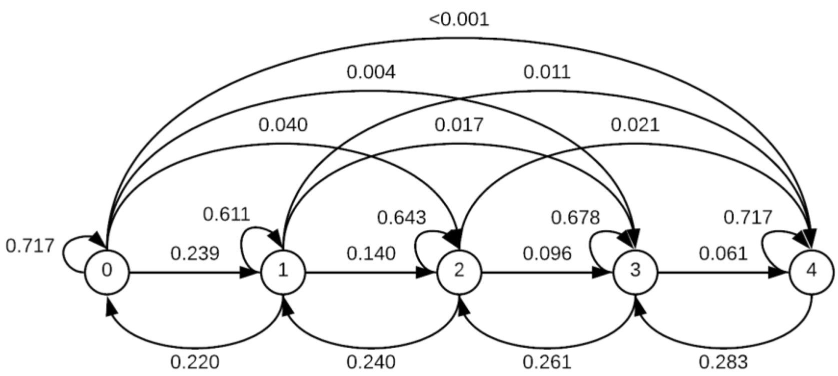
The last state is also special, since all states bigger than 4 can be considered as state 4. The easiest way to compute it is:

$$P_{i,4} = 1 - P_{i,0} - P_{i,1} - P_{i,2} - P_{i,3}$$

So, for the occurrence rates of the problem statement, we have the following transition probability matrix:

$$P = \begin{pmatrix} 0.717 & 0.239 & 0.040 & 0.004 & < 0.001 \\ 0.220 & 0.611 & 0.140 & 0.017 & 0.011 \\ 0 & 0.240 & 0.643 & 0.096 & 0.021 \\ 0 & 0 & 0.261 & 0.678 & 0.061 \\ 0 & 0 & 0 & 0.283 & 0.717 \end{pmatrix}$$

This transition probability matrix corresponds to the following graph:



This graph has a single class and is aperiodic. Therefore, it is an ergodic chain.

Since at the start of the system the gas station is always empty, we have:

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then, we can compute:

$$p(n) = (P^n)^T p(0)$$

And we define the steady state probabilities as:

$$\boldsymbol{\pi} = \lim_{n \to \infty} p(n)$$

For this problem, we obtained:

$$\boldsymbol{\pi} = \begin{pmatrix} 0.234 \\ 0.300 \\ 0.254 \\ 0.150 \\ 0.062 \end{pmatrix}$$

In fact, since this Markov Chain is ergodic, it is not relevant which the initial state is.

Now, we define the sojourn time when you arrive at state i as:

$$t_i = (i+1) \cdot S \sim (i+1) \cdot Exp\left(\frac{1}{3}\right) = Exp\left(\frac{1}{3 \cdot (i+1)}\right)$$

Them we can compute:

$$\mathrm{E}(t_i) = 3 \cdot (i+1)$$

So, the vector of expected times is:

$$\mathrm{E}(t) = \begin{pmatrix} 3 \\ 6 \\ 9 \\ 12 \\ 15 \end{pmatrix}$$

We can obtain the average sojourn time by multiplying by the steady probabilities:

$$\boldsymbol{\pi}^T \mathrm{E}(t) = 7.52$$

Therefore, the average sojourn time per client in the gas station is 7 minutes and 31 seconds.