Assignment #2

UW-Madison MATH 421

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February 9, 2021

Exercise #1: Prove the following theorem by cases.

Theorem. *If x is an integer, then $x^2 + 3x - 9$ is odd.*

Proof. Suppose x is an integer.
Consider the two parities of x.

Case 1. *x is odd, by definition $x = 2n + 1$ for some integer n*

$$\begin{aligned}
x^2 + 3x - 9 &= (2n + 1)^2 + 3(2n + 1) - 9 \\
&= 4n^2 + 4n + 1 + 6n + 3 - 9 \\
&= 4n^2 + 10n - 5 \\
&= 2 * 2n^2 + 2 * 5n + 2 * (-3) + 1 \\
&= 2(2n^2 + 5n - 3) + 1
\end{aligned}$$

is odd because $2n^2 + 5n - 3$ is an integer.

Case 2. *x is even, by definition $x = 2m$ for some integer m*

$$\begin{aligned}
x^2 + 3x - 9 &= (2m)^2 + 3(2m) - 9 \\
&= 4m^2 + 6m - 9 \\
&= 2 * 2m^2 + 2 * 3m + 2 * (-5) + 1 \\
&= 2(2m^2 + 3m - 5) + 1
\end{aligned}$$

is odd because $2m^2 + 3m - 5$ is an integer.

□

Exercise #2: Prove the following theorem in two ways: by contrapositive and by contradiction.

Theorem. *Suppose x is an integer. If x^2 is even, then x is even.*

Proof by contrapositive. Suppose x is an integer.
Suppose x is odd, by definition $x = 2n + 1$ for some integer n

$$\begin{aligned}
x^2 &= (2n + 1)^2 \\
&= 4n^2 + 4n + 1 \\
&= 2 * 2n^2 + 2 * 2n + 1 \\
&= 2(2n^2 + 2n) + 1
\end{aligned}$$

is odd because $2n^2 + 2n$ is an integer. Hence, x is odd $\implies$ x^2 is odd. Likewise, the contrapositive, x^2 is even $\implies$ x is even.

□

Proof by contradiction. Suppose x is an integer.

Suppose x^2 is even $\not\Longrightarrow$ x is even. Equivicantly, x^2 is even $\Longrightarrow$ x is odd.

Suppose x^2 is even. By definition, $x^2 = 2n$ for some integer n, and $x = 2m+1$ for some integer m. Then

$$\begin{aligned} x^2 &= 2n = (2m+1)^2 \\ &= 2n = 4m^2 + 4m + 1 \\ &= 2n = 2 * 2m^2 + 2 * 2m + 1 \\ &= 2n = 2(2m^2 + 2m) + 1 \end{aligned}$$

Which is impossible because n and $2m^2 + 2m$ are both integers. Hence x^2 is even $\Longrightarrow$ x is odd is a contradiction, and x^2 is even $\Longrightarrow$ x is even is true.

□

Exercise #3: Prove the following theorem.

Theorem. *If the name of a month has 5 or more characters, then a 4-letter word can be formed using those characters.*

Proof. Consider the months with more than 5 characters. (Cases compacted for space)

$$\begin{aligned} JANUARY &\to JURY \\ FEBRUARY &\to FEAR \\ AUGUST &\to STAG \\ SEPTEMBER &\to STEM \\ OCTOBER &\to ROOT \\ NOVEMBER &\to NORM \\ DECEMBER &\to DEER \end{aligned}$$

Hence, every month with more than 5 characters has a corrisponding 4-letter word formable from its characters. □

Exercise #4: Prove the following theorem.

Theorem. *For all numbers x and y, $(x+y)^2 = x^2 + y^2$ if and only if $x = 0$ or $y = 0$.*

Proof. Suppose x and y are numbers ($\Rightarrow$): Suppose $(x+y)^2 = x^2 + y^2$.

$$(x+y)^2 = x^2 + xy + y^2$$
$$x^2 + xy + y^2 = x^2 + y^2 \Rightarrow xy = 0$$
$$xy = 0 \Rightarrow (x = 0 \vee y = 0)$$
$$\text{Therefore, } (x+y)^2 = x^2 + y^2 \Rightarrow (x = 0 \vee y = 0)$$

($\Leftarrow$): Suppose $x = 0$ or $y = 0$.

Case 1. *$x = 0$*

$$\begin{aligned} (x+y)^2 &= (0+y)^2 \\ &= y^2 \\ &= 0^2 + y^2 \\ &= x^2 + y^2 \end{aligned}$$

Case 2. *$y = 0$*

$$\begin{aligned}(x+y)^2 &= (x+0)^2 \\ &= x^2 \\ &= x^2 + 0^2 \\ &= x^2 + y^2\end{aligned}$$

Hence, $(x+y)^2 = x^2 + y^2 \Leftrightarrow (x = 0 \vee y = 0)$

$\square$

$\boxed{\textbf{Exercise \#5:}}$ Using only properties P1-P12 and noting every time you use one, prove the following theorem.

Theorem. *Suppose a and b are numbers. If $ab = 1$, then $b = a^{-1}$.*

Proof. Suppose a and b are numbers. Suppose $ab = 1$.

Notice that, because of the theorem proved in week 2, $a = 0 \vee b = 0 \iff ab = 0$. Equivicantly, $a \neq 0 \wedge b \neq 0 \iff ab \neq 0$. Because $ab = 1 \neq 0$, $a \neq 0$ and $b \neq 0$. This enables the use of P7 below.

$$\begin{aligned}
& & ab &= 1 \\
& & a^{-1} * a * b &= a^{-1} * 1 \\
&\text{By P5...} & (a^{-1} * a) * b &= a^{-1} * 1 \\
&\text{By P7...} & (1) * b &= a^{-1} * 1 \\
&\text{By P6...} & b &= a^{-1}
\end{aligned}$$

$\square$

$\boxed{\textbf{Exercise \#6:}}$ Using only properties P1-P12 and noting every time you use one, prove the following theorem.

Theorem. *Suppose a and b are numbers. If $a \neq 0$ and $b \neq 0$, then $(ab)^{-1} = a^{-1}b^{-1}$.*

Proof. Suppose a and b are numbers. Suppose $a \neq 0$ and $b \neq 0$

Notice that, because of the theorem proved in week 2, $a = 0 \vee b = 0 \iff ab = 0$. Equivicantly, $a \neq 0 \wedge b \neq 0 \iff ab \neq 0$. Because $a \neq 0$ and $b \neq 0$, $ab \neq 0$. This enables the use of P7 below.

$$\begin{aligned}
&\text{By P7...} & (ab)^{-1} * (ab) &= 1 \\
& & (ab)^{-1} * (ab) * a^{-1} &= 1 * a^{-1} \\
& & (ab)^{-1} * (ab) * a^{-1} * b^{-1} &= 1 * a^{-1} * b^{-1} \\
&\text{By many P5s...} & (ab)^{-1} * (a * a^{-1}) * (b * b^{-1}) &= 1 * (a^{-1} * b^{-1}) \\
&\text{By P7...} & (ab)^{-1} * (1) * (1) &= 1 * (a^{-1} * b^{-1}) \\
&\text{By P6...} & (ab)^{-1} &= a^{-1} * b^{-1}
\end{aligned}$$

$\square$

$\boxed{\textbf{Exercise \#7:}}$ Using only properties P1-P12 and noting every time you use one, prove the following theorem.

Theorem. *Suppose a, b, and c are numbers. If $a < b$ and $0 < c$, then $ac < bc$.*

Proof. Suppose a, b, and c are numbers. Suppose $a < b$ and $0 < c$.
Notice that by definition of inequalities, $a < b \iff b - a \in \mathbb{R}_{>0}$, and $0 < c \iff c - 0 \in \mathbb{R}_{>0}$.

$$\begin{aligned}
\text{By definition of subtraction...}\ & c - 0 \in \mathbb{R}_{>0} \implies c + (-0) \in Rp \implies c + (0) \in \mathbb{R}_{>0} \\
\text{By P2...}\ & c + 0 \in \mathbb{R}_{>0} \implies c \in \mathbb{R}_{>0} \\
\text{By P12...}\ & c * (b - a) \in \mathbb{R}_{>0} \\
\text{By P9...}\ & (c * b - c * a) \in \mathbb{R}_{>0} \\
\text{By P8...}\ & (b * c - a * c) \in \mathbb{R}_{>0} \\
\text{By definition of inequalities...}\ & ac < bc
\end{aligned}$$

□

Exercise #8: Prove the following: if x and y are numbers, then

1. $|xy| = |x|\,|y|$,
2. $|x - y| \leq |x| + |y|$,
3. $|x| - |y| \leq |x - y|$.

Hint: you can give a short proof of (2) and (3) by reducing to the triangle inequality. You do not have to reference properties P1-P12.

Proof of (1). Consider signs of x and y

Case 1. $x \geq 0, y \geq 0$, *hence* $xy \geq 0$

$$\begin{aligned}
|xy| &= xy \\
&= |x||y|
\end{aligned}$$

Case 2. $x \geq 0, y < 0$, *hence* $xy \leq 0$

$$\begin{aligned}
|xy| &= -(x * y) \\
&= x * -y \\
&= |x||y|
\end{aligned}$$

Case 3. $x < 0, y \geq 0$, *hence* $xy \leq 0$

$$\begin{aligned}
|xy| &= -(x * y) \\
&= -x * y \\
&= |x||y|
\end{aligned}$$

Case 4. $x < 0, y < 0$, *hence* $xy > 0$

$$\begin{aligned}
|xy| &= x * y \\
&= -1 * -1 * x * y \\
&= -1 * x * -1 * y \\
&= -x * -y \\
&= |x||y|
\end{aligned}$$

$\square$

Proof of (2). Let $z = -y$, then z is a number. Then

$$\begin{aligned} |x - y| \leq |x| + |y| &\iff |x + z| \leq |x| + |-z| \\ &\iff |x + z| \leq |x| + |z| \end{aligned}$$

Hence $|x - y| \leq |x| + |y|$ is logically equivalent to the proven triangle inequality, $\square$

Proof of (3). Suppose $|x + y| \leq |x| + |y|$
Let $z = y + x$, then $y = z - x$

$$\begin{aligned} |x + y| \leq |x| + |y| &\iff |z| \leq |x| + |z - x| \\ &\iff |z| - |x| \leq |x| + |z - x| - |x| \\ &\iff |z| - |x| \leq |z - x| \end{aligned}$$

$\square$