Assignment #3

UW-Madison MATH 421

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March 10, 2021

Exercise #1: Prove: If $\lim_{x\to a} f(x) = \infty$, then $\lim_{x\to a} \frac{1}{f(x)} = 0$.

Proof. Fix $\epsilon > 0$. Because $\lim_{x\to a} f(x) = \infty$, $\exists \delta > 0 : 0 < |x-a| < \delta \implies f(x) > \frac{1}{\epsilon}$.

Then if $0 < |x-a| < \delta$, then $f(x) > \frac{1}{\epsilon} \implies 0 < \frac{1}{f(x)} < \epsilon \implies |\frac{1}{f(x)} - 1| = \frac{1}{f(x)} < \epsilon$.

Since $\epsilon > 0$ was arbitrary, $\lim_{x\to a} \frac{1}{f(x)} = 0$ □

Exercise #2: Suppose $\lim_{x\to a} f(x) = 0$ and there exists $\delta_0 > 0$ such that f is positive on $(a-\delta_0, a+\delta_0)$. Prove that $\lim_{x\to a} \frac{1}{f(x)} = \infty$.

Proof. Fix $M > 0$. Because $\lim_{x\to a} f(x) = 0$, $\exists \delta_1 : 0 < |x-a| < \delta_1 \implies |f(x) - 0| < \frac{1}{M}$.

Let $\delta = min\{\delta_0, \delta_1\}$.

Notice that $0 < |x-a| < \delta \implies -\delta < x-a < \delta \implies a-\delta < x < a+\delta \implies a-\delta_0 \le a-\delta < x < a+\delta \le a+\delta_0 \implies f(x) > 0 \implies \frac{1}{f(x)} > 0$.

Then if $0 < |x-a| < \delta$, then $|f(x) - 0| = |f(x)| < \frac{1}{M} \implies |\frac{1}{f(x)}| > M \implies \frac{1}{f(x)} > M$

Since $\epsilon > 0$ was arbitrary, $\lim_{x\to a} \frac{1}{f(x)} = \infty$. □

Exercise #3: If $\lim_{x\to\infty} f(x) = L$, then

$$\lim_{x\to 1^+} f\left(\frac{1}{x-1}\right) = L.$$

Proof. Fix $\epsilon > 0$. Because $\lim x \to \infty f(x) = L$, $\exists N > 0 : x > N \implies |f(x) - L| < \epsilon$.

Let $\delta = \frac{1}{N}$.

Then if $x > 1 \wedge 0 < |x-1| < \delta$, then $0 < x-1 < \delta$, and $0 < x-1 < \delta \implies \frac{1}{x-1} > \frac{1}{\delta} = N \implies |f(\frac{1}{x-1}) - L| < \epsilon$.

Since $\epsilon > 0$ was arbitrary, $\lim_{x\to 1^+} f\left(\frac{1}{x-1}\right) = L$. □

Exercise #4: If $\lim_{x\to\infty} f(x) = L$, then

$$\lim_{x\to 1^+} f\left(\frac{1}{x^2-1}\right) = L.$$

Proof. Fix $\epsilon > 0$. Because $\lim_{x\to\infty} f(x) = L$, $\exists N > 0 : x > N \implies |f(X) - L| < \epsilon$.

Let $\delta = min\{1, \frac{1}{3N}\}$.

If $x > 0 \wedge 0 < |x-1| < \delta$, then $1 < x < 1+\delta \le 2$.

Then $0 < x-1 < \delta \le \frac{1}{3N}$, and $2 < x+1 \le 3$.

So, $\frac{1}{x^2-1} = \frac{1}{x-1} * \frac{1}{x+1} > \frac{1}{\frac{1}{3N}} * \frac{1}{3} = N \implies |f(\frac{1}{x^2-1}) - L| < \epsilon$.

Since $\epsilon > 0$ was arbitrary, $\lim_{x\to 1^+} f\left(\frac{1}{x^2-1}\right) = L$. □

Exercise #5: If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = L$, then

$$\lim_{x\to\infty} g\left(f(x)\right) = L.$$

Proof. Fix $\epsilon > 0$.

Since $\lim_{x\to\infty} g(x) = L$, $\exists N_2 > 0 : x > N_2 \implies |g(x) - L| < \epsilon$.

Since $\lim_{x\to\infty} f(x) = \infty$, $\exists N_1 > 0 : x > N_1 \implies f(x) > N_2$.

So, $x > N_1 \implies f(x) > N_2 \implies |g(f(x)) - L| < \epsilon$.

Since $\epsilon > 0$ was arbitrary, $\lim_{x\to\infty} g\left(f(x)\right) = L$. □

Exercise #6: Suppose f is continuous at $x = 2$ and $f(2) = 5$.

(a) Prove, using the results in class, that

$$\lim_{x\to 1} f(3x-1) = 5.$$

(b) Prove, using an ϵ/δ proof, that

$$\lim_{x\to 1} f(3x-1) = 5.$$

Proof of (a). $g(x) = 3x - 1$ is a polynomial, that is continunous on $\mathbb{R}$. By the property of continuous functions proven in class, $\lim_{x\to 1} f(3x-1) = \lim_{x\to 1} f(g(x)) = f(g(1)) = f(2) = 5$. □

Proof of (b). Fix $\epsilon > 0$. Since $f(x)$ is continuous at $x = 2$, $\exists \delta_1 : |x-2| < \delta_1 \implies |f(x) - 5| < \epsilon$.

Let $\delta = \frac{\delta_1}{3}$.

If $0 < |x-1| < \delta = \frac{\delta_1}{3}$, then $|(3x-1)-2| = |3x-3| = 3|x-1| < 3\delta = \delta_1$, then $|f(3x-1) - 5| < \epsilon$.

Since $\epsilon > 0$ was arbitrary, $\lim_{x\to 1} f(3x-1) = 5$. □

Exercise #7: Suppose f is continuous at $x = 3$ and $f(3) = 7$.

(a) Prove, using the results in class, that

$$\lim_{x\to 1} f(x^2+x+1) = 7.$$

(b) Prove, using an ϵ/δ proof, that

$$\lim_{x\to 1} f(x^2+x+1) = 7.$$

Proof of (a). $g(x) = x^2 + x + 1$ is a polynomial, that is continunous on $\mathbb{R}$. By the property of continuous functions proven in class, $\lim_{x\to 1} f(x^2+x+1) = \lim_{x\to 1} f(g(x)) = f(g(1)) = f(3) = 7$.

□

Proof of (b). Fix $\epsilon > 0$. Since $f(x)$ is continuous at $x = 3$, $\exists \delta_1 : |x-3| < \delta_1 \implies |f(x) - 7| < \epsilon$.

Let $\delta = min\{1, \frac{\delta_1}{4}\}$.

If $0 < |x-1| < \delta$, then $|x^2+x+1-3| = |x^2+x-2| = |x+2||x-1| = |x-1+3||x-1| \le (|x-1|+3)|x-1| \le (1+3)\frac{\delta_1}{4} = \delta_1 \implies |f(x^2+x+1) - 7| < \epsilon$.

Since $\epsilon > 0$ was arbitrary, $\lim_{x\to 1} f(x^2+x+1) = 7$. □

Exercise #8: Find an example where $\lim_{x\to 3} f(x) = 7$, $\lim_{x\to 1} g(x) = 3$, and $\lim_{x\to 1} f(g(x)) \neq 7$.

Example. Let $f(x) = \{x + 4\colon x \neq 3, 8\colon x = 3\}$. Then $\lim_{x\to 3} f(x) = 7$.

Let $g(x) = 3$. Then $\lim_{x\to 1} g(x) = 3$

Then $\lim_{x\to 1} f(g(x)) = \lim_{x\to 1} f(3) = 8 \neq 7$.

□

Exercise #9: Prove: If $\lim_{x\to 3} f(x) = 7$, $\lim_{x\to 1} g(x) = 3$, and $g(x) \neq 3$ for all x, then $\lim_{x\to 1} f(g(x)) = 7$.

Proof. Fix $\epsilon > 0$.

Because $\lim_{x\to 3} f(x) = 7$, $\exists \delta_1 > 0 : 0 < |x-3| < \delta_1 \implies |f(x) - 7| < \epsilon$.
Because $\lim_{x\to 1} g(x) = 3$, $\exists \delta_2 > 0 : 0 < |x-1| < \delta_2 \implies |g(x) - 3| < \delta_1$.
Let $\delta = \delta_2$.
If $0 < |x-1| < \delta = \delta_2$, then $|g(x) - 3| < \delta_1$.
Since $g(x) \neq 3$, $0 < |g(x) - 3|$.
So, $0 < |g(x) - 3| < \delta_1 \implies |f(g(x)) - 7| < \epsilon$.
Since $\epsilon > 0$ was arbitrary, $\lim_{x\to 1} f(g(x)) = 7$.

□

Exercise #10:

(a) Prove: If f is continuous on $\mathbb{R}$, $\lim_{x\to -\infty} f(x) = L_1$, and $\lim_{x\to\infty} f(x) = L_2$ (where L_1, L_2 are real numbers), then f is bounded above on $\mathbb{R}$.

(b) Find an example where: f is continuous on $\mathbb{R}$, $\lim_{x\to -\infty} f(x) = L_1$, $\lim_{x\to\infty} f(x) = L_2$ (where L_1, L_2 are real numbers), and there does not exist a number $y \in \mathbb{R}$ such that $f(x) \leq f(y)$ for all $x \in \mathbb{R}$.

Proof of (a). Since $\lim_{x\to -\infty} f(x) = L_1$, $\exists N_1 > 0 : x < -N_1 \implies |f(x) - L_1| < 1$.
Since $\lim_{x\to\infty} f(x) = L_2$, $\exists N_2 > 0 : x > N_2 \implies |f(x) - L_2| < 1$.
By Theorem 7.2, f is bounded above on $[-N_1, N_2]$.
So, $\exists M\ \forall x \in [-N_1, N_2] : f(x) \leq M$.
Then f is bounded above by $max\{L_1 + 1, L_2 + 1, M\}$. □

Example for (b). Let $f(x) = -\frac{1}{x^2+1}$.
Then $\lim_{x\to\infty} f(x) = \lim_{x\to -\infty} f(x) = \sup(range(f)) = 0$, and $\forall x : f(x) \neq 0$. □