Assignment #3

UW-Madison MATH 421

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Exercise #1: Assuming that $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$, prove using the limit definition that: if $g(x) = \sqrt{x^2+1}$, then $g'(x) = \frac{x}{\sqrt{x^2+1}}$ when $x > 0$.

Proof. Notice that because $g(x) = f(x^2+1)$, and $\forall x : x^2 + 1 > 0$, both $g(a)$ and $g(a+h)$ are continuous, and so is $\frac{g(a+h)-g(a)}{h}$ when $h \neq 0$. Thus,

$$\begin{aligned}
&\lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{a^2+2ah+h^2+1} - \sqrt{a^2+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{a^2+2ah+h^2+1} - \sqrt{a^2+1}}{h} * \frac{\sqrt{a^2+2ah+h^2+1} + \sqrt{a^2+1}}{\sqrt{a^2+2ah+h^2+1} + \sqrt{a^2+1}} \\
&= \lim_{h\to 0} \frac{a^2+2ah+h^2+1-a^2+1}{h(\sqrt{a^2+2ah+h^2+1} + \sqrt{a^2+1})} \\
&= \lim_{h\to 0} \frac{2ah+h^2}{h(\sqrt{a^2+2ah+h^2+1} + \sqrt{a^2+1})} \\
&= \lim_{h\to 0} \frac{2a+h}{(\sqrt{a^2+2ah+h^2+1} + \sqrt{a^2+1})} \\
&= \frac{2a}{(\sqrt{a^2+1} + \sqrt{a^2+1})} \\
&= \frac{a}{\sqrt{a^2+1}}
\end{aligned}$$

Thus, the limit exists, and $g'(a) = \frac{a}{\sqrt{a^2+1}}$. □

Exercise #2: Suppose f and g are differentiable at $x = a$. If $f(a) = g(a)$, $f'(a) = g'(a)$, and

$$k(x) = \begin{cases} f(x) & \text{if } x \leq a \\ g(x) & \text{if } x \geq a \end{cases},$$

then k is differentiable at $x = a$ and $k'(a) = f'(a) = g'(a)$.

Proof. Notice that $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$,

and $g'(a) = \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} = \lim_{h\to 0^+} \frac{g(a+h)-g(a)}{h}$.

Then, $\lim_{h\to 0^-} \frac{k(a+h)-k(a)}{h} = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = f'(a)$.

Then, $\lim_{h\to 0^+} \frac{k(a+h)-k(a)}{h} = \lim_{h\to 0^+} \frac{g(a+h)-g(a)}{h} = g'(a)$.

Because $f'(a) = g'(a)$, $\lim_{h\to 0} \frac{k(a+h)-k(a)}{h} = f'(a) = g'(a)$ exists, k is differentiable at a, and $k'(a) = \lim_{h\to 0} \frac{k(a+h)-k(a)}{h}$. □

Exercise #3: Spivak, Chapter 9, Problem 15 (a)

Theorem. *Let f be a function such that $\forall x : |f(x)| \leq x^2$.*
f is differentiable at 0.

Proof. Let $a = 0$.

Notice that $|f(0)| \leq 0^2 \implies f(0) = 0$.

So, $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$.

Notice that $|\frac{f(h)}{h}| \leq \frac{h^2}{|h|} = |\frac{h^2}{h}| = |h|$.

Thus, $\lim_{h\to 0} \frac{f(h)}{h} \leq \lim_{h\to 0} |h| = 0$, thus $\lim_{h\to 0} \frac{f(h)}{h} = 0$.

Thus, the limit exists, and f is differentiable at 0. ☐

Exercise #4: Spivak, Chapter 10, Problem 16 (a), (b)

Theorem. *If f is differentiable at a and $f(a) \neq 0$, then $|f|$ is differentiable at a.*

4a. Because $f(a) \neq 0$, $\exists\delta : f(a)$ is either positive or negative on $(a-\delta, a+\delta)$.

By considering when $|h| \leq \delta$, $f(a+h)$ and $f(a)$ will either both be positive or both be negative.

If $f(a) > 0$, then $\lim_{h\to 0} \frac{|f(a+h)|-|f(a)|}{h} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$ exists. If $f(a) < 0$, then $\lim_{h\to 0} \frac{|f(a+h)|-|f(a)|}{h} = \lim_{h\to 0} \frac{-f(a+h)+f(a)}{h} = -f'(a)$ exists.

Thus, $|f|$ is differentiable at a. ☐

4b. Let $f(x) = x$.

Then $\lim_{h\to 0+} \frac{|f(a+h)|-|f(a)|}{h} \neq \lim_{h\to 0-} \frac{|f(a+h)|-|f(a)|}{h}$, because the $f(a+h)$ terms differ in each limit.

So $\lim_{h\to 0} \frac{|f(a+h)|-|f(a)|}{h}$ does not exist, and $|x|$ is not differentiable at 0. ☐

Exercise #5: Spivak, Chapter 10, Problem 22 (a)

Theorem. *If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $\exists g, h : g' = h' = f$, and $g \neq h$.*

Proof. Let $g(x) = \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \cdots + \frac{a_0}{1}x + 1$.

Let $h(x) = g(x) + 1$.

By applying theorems 10-1 through 10-6,

$g'(x) = \frac{a_n(n+1)}{n+1}x^n + \frac{a_{n-1}(n)}{n}x^{n-1} + \cdots + a_0 = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = f(x)$.

Likewise,

$h'(x) = \frac{a_n(n+1)}{n+1}x^n + \frac{a_{n-1}(n)}{n}x^{n-1} + \cdots + a_0 = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = f(x)$. ☐