Assignment #3

UW-Madison MATH 421

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Exercise #1: Spivak, Chapter 11, Problem 13

Theorem. $\forall x \in \mathbb{R}^+ : x + \frac{1}{x} \geq 2$

Proof. Let $f(x) = x + \frac{1}{x}$.
Then $f'(x) = 1 - \frac{1}{x^2}$.
By the algorithm for finding min values of f on $(0, \infty)$:
(1): Critical points are where

$$\begin{aligned} 0 &= 1 - \frac{1}{x^2} \\ 1 &= \frac{1}{x^2} \\ x^2 &= 1 \\ x &= 1 \end{aligned}$$

(2): No points of non-differentiability in $(0, \infty)$.
(3):

$$\begin{aligned} f(1) &= 2 \\ \lim_{x \to 0^+} f(x) &= \infty \\ \lim_{x \to \infty} f(x) &= \infty \end{aligned}$$

Thus f's minimum value is 2.
Hence $\forall x \in \mathbb{R}^+ : f(x) \geq 2$
$\implies \forall x \in \mathbb{R}^+ : x + \frac{1}{x} \geq 2$. $\square$

Exercise #2: Spivak, Chapter 11, Problem 26

Theorem. *Suppose f is a polynomial of degree n, with $f \geq 0$. (Hence, n must be even).*
$f + f' + f'' + \cdots + f^{(n)} \geq 0$.

Proof. Let $g = f + f' + \cdots + f^{(n)}$.
Because f is a polynomial,
$\forall n : f^{(n)}$ is a polynomial,
and g is a polynomial.
Hence g is differentiable and continuous everywhere.
Notice that $g' = f' + f'' + \cdots + f^{(n)} + f^{(n+1)} = f' + f'' + \cdots + f^{(n)}$.
Hence, $g = f + g'$.
Because f has an even degree, so does g, and
By a theorem proved in class or hwk, $\exists x : x$ is a minimum point of g on $\mathbb{R}$.
Thus $g'(x) = 0$.
Further, $g(y) = f(y) + g'(y) = f(y)$.
Because $f(y) \geq 0$, $g(y) \geq 0$, and since $g(y)$ is the minimum value, $\forall x : g(x) = f(x) + f'(x) + \cdots + f^{(n)}(x) \geq 0$. $\square$

$\boxed{\textbf{Exercise \#3:}}$ Spivak, Chapter 11, Problem 30 (a)

Theorem. *Suppose* $\forall x : f'(x) > g'(x)$, *and* $f(a) = g(a)$
$x > a \implies f(x) > g(x)$ $x < a \implies f(x) < g(x)$

Proof. Let $h = f - g$.
Notice that $h(a) = f(a) - g(a) = 0$.
Notice that $\forall x : f'(x) > g'(x) \implies \forall x : h'(x) > 0$.
For $x < a$, consider mean value theorem on $[x, a]$.
Thus, $\exists y \in (x, a) : h'(y) = \frac{h(a)-h(x)}{a-x} = \frac{-h(x)}{a-x}$.
Since $h'(y) > 0$ and $x < a \implies a - x > 0$, then $-h(x) > 0 \implies h(x) < 0 \implies f(x) < g(x)$.
Thus $x < a \implies f(x) < g(x)$.
For $x > a$, consider mean value theorem on $[a, x]$.
Thus, $\exists y \in (a, x) : h'(y) = \frac{h(x)-h(a)}{x-a} = \frac{h(x)}{x-a}$.
Since $h'(y) > 0$ and $x > a \implies x - a > 0$, then $h(x) > 0 \implies f(x) > g(x)$.
Thus $x > a \implies f(x) > g(x)$. □

$\boxed{\textbf{Exercise \#4:}}$ Spivak, Chapter 11, Problem 38

Theorem. *If* $\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$,
Then $\exists x \in (0, 1) : a_0 + a_1x + \cdots + a_nx^n = 0$.

Proof. Let $f(x)\frac{a_0}{1}x + \frac{a_1}{2}x^2 + \cdots + \frac{a_n}{n+1}x^{n+1}$,
Notice that $f(0) = 0$, and $f(1) = \frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$.
Notice that $f'(x) = a_0 + a_1x + \cdots + a_nx^n$.
By Rolles theorem, $\exists y \in (0, 1) : f'(y) = 0$.
$\implies \exists y \in (0, 1) : a_0 + a_1x + \cdots + a_nx^n = 0$.
□

$\boxed{\textbf{Exercise \#5:}}$ Spivak, Chapter 11, Problem 43

Theorem. *Suppose* f *is a function where* $\forall x > 0 : f'(x) = \frac{1}{x}$, *and* $f(1) = 0$.
$\forall x, y > 0 : f(xy) = f(x) + f(y)$

Proof. Fix $y > 0$.
Let $g(x) = f(xy)$
Then $g'(x) = y * f'(xy) = y * \frac{1}{xy} = \frac{1}{x} = f'(x)$.
By Corollary 2 of Thm 11-4, $f(x) = g(x) + c$ for some c.
Note that $f(1) = 0 = g(1) + c \implies g(1) = -c$, and $g(1) = f(1 * y) = f(y)$.
Thus $f(x) = g(x) - c = g(x) - f(y)$
$\implies g(x) = f(xy) = f(x) + f(y)$.
Since $y > 0$ was arbitrary,
$\forall x, y > 0 : f(xy) = f(x) + f(y)$ □

$\boxed{\textbf{Exercise \#6:}}$ Prove that $\frac{1}{21} < \sqrt{101} - 10 < \frac{1}{20}$

Proof. Let $f(x) = \sqrt{x}$.
Consider mean value theorem of f on $(100, 101)$.
Thus $\exists c : f'(c) = \frac{1}{2\sqrt{c}} = \frac{\sqrt{101}-\sqrt{100}}{101-100} = \sqrt{101} - 10$.
Thus,

$$
\begin{aligned}
&100 < x < 101 \\
\implies &100 < x < 110.25 \\
\implies &10 < \sqrt{x} < 10.5 \\
\implies &\frac{1}{10} < \frac{1}{\sqrt{x}} < \frac{1}{10.5} \\
\implies &\frac{1}{20} < \frac{1}{2\sqrt{x}} < \frac{1}{21} \\
\implies &\frac{1}{20} < \sqrt{101} - 10 < \frac{1}{21}
\end{aligned}
$$

□

Exercise #7: Spivak, Chapter 11, Problem 64

Theorem. *Suppose that $f(0) = 0$, and f' is increasing.*
$g(x) = f(x)/x$ is increasing on $(0, \infty)$

Proof. Let $g(x) = \frac{f(x)}{x}$.
Thus, $g'(x) = \frac{xf'(x) - f(x)}{x^2}$.
Fix $N > 0$.
Consider mean value theorem of f on $(0, N)$.
Thus, $\exists c \in (0, N) : f'(c) = \frac{f(N)-f(0)}{N-0} = \frac{f(N)}{N} \implies Nf'(c) = f(N)$.
Since $c < N$ and $N > 0$, $f'(c) < f'(N) \implies Nf'(c) < Nf'(N)$.
Hence, $f(N) < Nf'(N) \implies Nf'(N) - f(N) > 0$.
Since $N > 0$, $g'(N) = \frac{Nf'(N)-f(N)}{N^2} > 0$.
Since $N > 0$ was arbitrary, $\forall N > 0 : g'(N) > 0$.
By Corollary 3 of Them 11-4, $g(x) = \frac{f(x)}{x}$ is increasing on $(0, \infty)$. □