Assignment #3

UW-Madison MATH 421

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Exercise #1: Prove the following theorem by induction.

Theorem. *If n is a natural number, then* $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. We argue by induction.

Let $P(n) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Base case: $(n = 1)$ $P(1) =$

$$\begin{aligned}
\sum_{i=1}^{1} n^2 &= \frac{1(1+1)(2(1)+1)}{6} \\
1^2 &= \frac{1 * 2 * 3}{6} \\
1 &= \frac{6}{6} \\
1 &= 1
\end{aligned}$$

Induction step: $(n \implies n+1)$ Let $n \in \mathbb{N}$. Suppose $P(n)$ is true.

$P(n+1) =$

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \\
\sum_{i=1}^{n} i^2 + (n+1)^2 &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \\
\frac{n(n+1)(2n+1)}{6} + (n+1)^2 &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \\
\frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \\
\frac{n(n+1)(2n+1) + 6(n+1)^2}{6} &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \\
\frac{(n+1)(n(2n+1) + 6(n+1))}{6} &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \\
\frac{(n+1)(2n^2 + 7n + 6)}{6} &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \\
\frac{(n+1)(n+2)(2n+3)}{6} &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \\
\frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}
\end{aligned}$$

$\square$

Exercise #2: Prove the following theorem by induction.

Theorem (Bernoulli's inequality)**.** *Suppose n is a natural number and x is a real number. If $x > -1$, then*

$$(1+x)^n \geq 1 + nx.$$

Proof. We argue by strong induction.
Let $P(n) = (1+x)^n \geq 1 + nx$
Base case: $(n=1)$ $P(1) =$

$$(1+x)^1 \geq 1 + (1)x$$
$$1 + x \geq 1 + x$$

Base case: $(n=2)$ $P(1) =$

$$(1+x)^2 \geq 1 + (2)x$$
$$1 + 2x + x^2 \geq 1 + 2x$$
$$x^2 \geq 0$$

Induction step: $(1, 2, \cdots, n \implies n+1)$ Let $n \in \mathbb{N}$. Suppose $P(n)$ is true.

$$(1+x)^1 \geq 1 + (1)x$$
$$1 + x \geq 1 + x$$

□

Exercise #3: For this problem we need the following definition:

Definition. An integer n is *divisible* by an integer k if the ratio n/k is an integer.

For example: -3, 0, 3, 6 are all divisible by 3 while 1, 2, 4, 5 are not divisible by 3. Prove the following:

Theorem. *Suppose n is an integer. If n^2 is divisible by 3, then n is divisible by 3.*

Proof. We argue by contrapositive. Assume $n \nmid 3$. Thus, $n = 3k+1$ or $n = 3k+2$ for some integer k.
Case 1: $n = 3k+1$

$$\begin{aligned} n^2 &= (3k+1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1 \\ &\nmid 3 \end{aligned}$$

Case 2: $n = 3k+2$

$$\begin{aligned} n^2 &= (3k+2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 3(3k^2 + 4k + 1) + 1 \\ &\nmid 3 \end{aligned}$$

Thus, $n \nmid 3 \implies n^2 \nmid 3$, and by the contrapositive, $n \mid 3 \implies n^2 \mid 3$. □

Exercise #4: Prove that $\sqrt{3}$ is irrational.

Proof. We argue by contradiction.

Assume $\sqrt{3}$ is rational. Thus, $\sqrt{3}$ may be written of the form p/q, where p, q are coprime integers. Thus, at most one of p, q is divisible by 3.

$$(\sqrt{3})^2 = \frac{p^2}{q^2}$$
$$3 * q^2 = p^2$$

Thus, p^2 has a factor of 3, and likewise, p has a factor of 3. Let $p = 3r$ for some integer r.

$$\begin{aligned} 3 * q^2 &= p^2 \\ 3 * q^2 &= (3r)^2 \\ 3 * q^2 &= 9r^2 \\ q^2 &= 3r^2 \end{aligned}$$

Thus, q^2 has a factor of 3, and likewise, q has a factor of 3.

This breaks the statement from earlier that at most one of p, q is divisible by 3.

Thus, a contradiction arises from the statement that $\sqrt{3}$ is rational, and thus, $\sqrt{3}$ is irrational. □

Exercise #5: Prove that $\sqrt{2} + \sqrt{3}$ and $\sqrt{2} - \sqrt{3}$ are both irrational.

Proof. Notice that $\sqrt{2} + \sqrt{3} = \frac{-1}{\sqrt{2}-\sqrt{3}}$. Thus, either $\sqrt{2} + \sqrt{3}$ and $\sqrt{2} - \sqrt{3}$ are both rational or irrational.

We argue by contradiction.

Assume both $\sqrt{2} + \sqrt{3}$ and $\sqrt{2} - \sqrt{3}$ are rational.

Then $(\sqrt{2} + \sqrt{3}) + (\sqrt{2} - \sqrt{3})$ is rational. However, $(\sqrt{2} + \sqrt{3}) + (\sqrt{2} - \sqrt{3}) = 2 * \sqrt{2}$, which is irrational.

Thus, a contradiction arises, and both $\sqrt{2} + \sqrt{3}$ and $\sqrt{2} - \sqrt{3}$ are irrational. □

Exercise #6: Spivak, Chapter 3, Problem 14.

Solution. At any point where f and g are defined: Notice that $|f - g|$ is the absolute difference between the two functions. Adding this value to the smaller of the two functions the min into the max: $min(f, g) + |f - g| = max(f, g)$ Adding the other value (the max) to this total results in $2 * max(f, g$. Because addition is associative, this summation is the same regardless of whether $f \leq g$ or $f > g$.

Thus, $max(f, g) = \frac{f+g+|f-g|}{2}$ Likewise, $min(f, g) = max(f, g) - |f - g| = \frac{f+g+|f-g|}{2} - |f - g| = \frac{f+g-|f-g|}{2}$ □

Exercise #7: Spivak, Chapter 3, Problem 23.

Proof of (a). □

Proof of (b). □

Exercise #8: Spivak, Chapter 3, Problem 26.

Proof. □

1 Extra Credit Questions

Each extra credit question is worth 1 extra point.

$\boxed{\textbf{Exercise E.C.\#1:}}$ Spivak, Chapter 2, Problem 17

$\boxed{\textbf{Exercise E.C.\#2:}}$ Spivak, Chapter 3, Problem 16

$\boxed{\textbf{Exercise E.C.\#3:}}$ Spivak, Chapter 3, Problem 17

$\boxed{\textbf{Exercise E.C.\#4:}}$ Spivak, Chapter 3, Problem 20 (b)