Assignment #3

UW-Madison MATH 421

GEOFF YOERGER

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Exercise #1: Assuming that the function $f(x) = e^x$ is continuous, prove that the equation $e^x = 4 - x^7$ has a solution.

Proof. Let $g(x) = e^x + x^7 - 4$. It follows from the theorems proven in class/homework, and the given assumption, that $g(x)$ is continuous.

Let $a = 0$. Then $g(a) = -3 < 0$.

Let $b = 2$. Then $g(b) = e^2 + 128 - 4 \approx 131.389 > 0$.

Because $g(a) < 0 < g(b)$, by Theorem 7.1, $\exists x : g(x) = e^x + x^7 - 4 = 0$, equivicantly, $\exists x : e^x = 4 - x^7$. $\square$

Exercise #2: Spivak, Chapter 7, Problem # 14 (b)

If f is a continuous function on $[0,1]$, let $||f||$ be the maximum value of $|f|$ on $[0,1]$.

Theorem. $||f + g|| \leq ||f|| + ||g||$

Proof. Let $h(x) = f(x) + g(x)$. It follows that $h(x)$ is continuous. By Theorem 7.3,

$$\begin{aligned}
&\exists y_f \in [0,1]\ \forall x \in [0,1] : |f(x)| \leq |f(y_f)| \\
&\exists y_g \in [0,1]\ \forall x \in [0,1] : |g(x)| \leq |g(y_g)| \\
&\exists y_h \in [0,1]\ \forall x \in [0,1] : |h(x)| \leq |h(y_h)| \implies |f(x) + g(x)| \leq |f(y_h) + g(y_h)|
\end{aligned}$$

Thus, $||f|| = |f(y_f)|$, $||g|| = |g(y_g)|$, $||f + g|| = |h(y_h)| = |f(y_h) + g(y_h)|$.

Because $\forall x \in [0,1] : |f(x)| \leq |f(y_f)|$, and $y_h \in [0,1]$, it follows that $|f(y_h)| \leq |f(y_f)|$.

Because $\forall x \in [0,1] : |g(x)| \leq |g(y_g)|$, and $y_h \in [0,1]$, it follows that $|g(y_h)| \leq |g(y_g)|$.

Then, $||f + g|| = |f(y_h) + g(y_h)| \leq |f(y_h)| + |g(y_g)| \leq |f(y_f)| + |g(y_g)| = ||f|| + ||g||$. $\square$

Example. Example where $||f + g|| \neq ||f|| + ||g||$

Let $f(x) = x$. Let $g(x) = -x$. Then $(f + g)(x) = x - x = 0$.

Then $||f + g|| = 0$, $||f|| = 1$, $||g|| = 0$, and $||f|| + ||g|| = 0 + 1 \neq 0 = ||f + g||$.

Exercise #3: Suppose f is continuous on $[a,b]$. If $f(x) \neq 0$ for all x in $[a,b]$, then either $f(x) > 0$ for all x in $[a,b]$ or $f(x) < 0$ for all x in $[a,b]$

Proof. We argue by contrapositive.

[Original: $(\forall x \in [a,b] : f(x) \neq 0) \implies (\forall x \in [a,b] : f(x) > 0) \vee (\forall x \in [a,b] : f(x) < 0)$]

[Contrapositive: $(\exists x_1 \in [a,b] : f(x_1) \leq 0) \wedge (\exists x_2 \in [a,b] : f(x_2) \geq 0) \implies \exists x_3 \in [a,b] : f(x_3) = 0$]

If $\exists x_1 \in [a,b] : f(x_1) \leq 0$ and $\exists x_2 \in [a,b] : f(x_2) \geq 0$, Then proceed by cases

Case 1. $f(x_1) = 0$

Then $\exists x = x_1 \in [a,b] : f(x) = 0$

Case 2. $f(x_2) = 0$

Then $\exists x = x_2 \in [a,b] : f(x) = 0$

Case 3. $f(x_1) < 0 \wedge f(x_2) > 0 \implies f(x_1) < 0 < f(x_2)$

By Theorem 7.1, $\exists x \in [x_1, x_2] \subset [a,b] : f(x) = 0$

$\square$

$\boxed{\textbf{Exercise \#4:}}$ Spivak, Chapter 7, Problem # 20 (a)

Suppose f is continuous on $[0,1]$ and $f(0) = f(1)$.

Theorem. $\forall n \in \mathbb{N}\ \exists x : f(x) = f(x + \frac{1}{n})$

Proof. Fix $n \in \mathbb{N}$. Let $g(x) = f(x) - f(x + \frac{1}{n})$. It follows that g is continuous on $[0, 1 - \frac{1}{n}]$.

If $\exists x \in [0, 1 - \frac{1}{n}] : g(x) = 0$, then $\exists x \in [0, 1 - \frac{1}{n}] : f(x) = f(x + \frac{1}{n})$.

Otherwise, $\forall x \in [0, 1 - \frac{1}{n}] : g(x) \neq 0$. We argue by contradiction.

By exercise 3, either $\forall x \in [0, 1 - \frac{1}{n}] : g(x) > 0$ or $\forall x \in [0, 1 - \frac{1}{n}] : g(x) < 0$.

If $\forall x \in [0, 1 - \frac{1}{n}] : g(x) > 0$, then $\forall x \in [0, 1 - \frac{1}{n}] : f(x) > f(x + \frac{1}{n})$. Then, $f(0) > f(\frac{1}{n}) > \cdots > f(\frac{n-1}{n}) > f(\frac{n}{n}) = f(1)$. Thus $f(0) \neq f(1)$, which is a contradiction.

Likewise, if $\forall x \in [0, 1 - \frac{1}{n}] : g(x) < 0$, then $\forall x \in [0, 1 - \frac{1}{n}] : f(x) < f(x + \frac{1}{n})$. Then, $f(0) < f(\frac{1}{n}) < \cdots < f(\frac{n-1}{n}) < f(\frac{n}{n}) = f(1)$. Thus $f(0) \neq f(1)$, which is a contradiction.

Thus, $\forall x \in [0, 1 - \frac{1}{n}] : g(x) \neq 0$ leads to a contradiction, and $\exists x \in [0, 1 - \frac{1}{n}] \subset \mathbb{R} : g(x) = 0$.

Since n was aribtrary, $\forall n \in \mathbb{N}\ \exists x : f(x) = f(x + \frac{1}{n})$

□

The next three problems involve infinite limits which are defined as follows.

Definition.

1. We write $\lim_{x\to\infty} f(x) = \infty$ if for every number $M > 0$ there exists $N > 0$ such that: if $x > N$, then $f(x) > M$.
2. We write $\lim_{x\to\infty} f(x) = -\infty$ if for every number $M > 0$ there exists $N > 0$ such that: if $x > N$, then $f(x) < -M$.
3. We write $\lim_{x\to-\infty} f(x) = \infty$ if for every number $M > 0$ there exists $N > 0$ such that: if $x < -N$, then $f(x) > M$.
4. We write $\lim_{x\to-\infty} f(x) = -\infty$ if for every number $M > 0$ there exists $N > 0$ such that: if $x < -N$, then $f(x) < -M$.

Remark. In the definition above, we should think about M as a very large number.

$\boxed{\textbf{Exercise \#5:}}$ Suppose $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial. Prove:

(a) $\lim_{x\to\infty} f(x) = \infty$.

(b) If n is even, then $\lim_{x\to-\infty} f(x) = \infty$.

(c) If n is odd, then $\lim_{x\to-\infty} f(x) = -\infty$.

Proof of (a). Fix $M > 0$. Let $N = max\{1, 2M, 2n|a_0|, 2n|a_1|, \cdots, 2n|a_{n-1}|\}$.

If $x > N$,

$$\begin{aligned}
|\frac{a_{n-j}}{x^j}| &= \frac{|a_{n-j}|}{|x|^j} \\
&< \frac{|a_{n-j}|}{|x|} && \text{Since } |x| > N > 1 \\
&< \frac{|a_{n-j}|}{2n|a_{n-j}|} && \text{Since } |x| > N > 2n|a_{n-j}| \\
&= \frac{1}{2n}
\end{aligned}$$

Thus,

$$\begin{aligned}
1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n} &\geq 1 - |\frac{a_{n-1}}{x}| - \cdots - |\frac{a_0}{x^n}| \\
&> 1 - \frac{1}{2n} - \cdots - \frac{1}{2n} \\
&= 1 - n * \frac{1}{2n} \\
&= 1 - \frac{n}{2n} \\
&= 1 - \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Since $f(x) = x^n(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n})$,
If $x > N$, Then $f(x) > x^n * \frac{1}{2} = \frac{x^n}{2} > \frac{x}{2} > \frac{N}{2} \geq \frac{2M}{2} = M$.
Since M was arbitrary, $\forall M > 0\ \exists N > 0 : x > N \implies f(x) > M$, equivicantly, $\lim_{x\to\infty} f(x) = \infty$. □

Theorem (Lemma 1)**.** *If $x < -N \leq -1$ and n is even, then $x^n > N$.*

Theorem (Lemma 2)**.** *If $x < -N \leq -1$ and n is odd, then $x^n < -N$.*

Proof of Lemma 1. Notice $x < -N \implies x^2 > N^2$.
Since $n = 2m$ is even, $x^n = (x^2)^m > (N^2)^m = N^{2m} > N$, since $N > 1$. □

Proof of Lemma 2. Using Lemma 1,
Since $n = 2m + 1$ is odd and $x < 0$, $x^n = n * n^{n-1} = n * n^{2m} < x * N < -N$. □

Proof of (b). Fix $M > 0$. Let $N = max\{1, 2M, 2n|a_0|, 2n|a_1|, \cdots, 2n|a_{n-1}|\}$.
If $x < -N$,

$$\begin{aligned}
|\frac{a_{n-j}}{x^j}| &= \frac{|a_{n-j}|}{|x|^j} \\
&< \frac{|a_{n-j}|}{|x|} && \text{Since } |x| > N > 1 \\
&< \frac{|a_{n-j}|}{2n|a_{n-j}|} && \text{Since } |x| > N > 2n|a_{n-j}| \\
&= \frac{1}{2n}
\end{aligned}$$

Thus,

$$\begin{aligned}
1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n} &\geq 1 - |\frac{a_{n-1}}{x}| - \cdots - |\frac{a_0}{x^n}| \\
&> 1 - \frac{1}{2n} - \cdots - \frac{1}{2n} \\
&= 1 - n * \frac{1}{2n} \\
&= 1 - \frac{n}{2n} \\
&= 1 - \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Since $f(x) = x^n(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n})$,
If $x < -N$, then $f(x) > x^n * \frac{1}{2} > \frac{N}{2} \geq \frac{2M}{2} = M$.
Since M was arbitrary, $\forall M > 0\ \exists N > 0 : x < -N \implies f(x) > M$, equivicantly, $\lim_{x\to-\infty} f(x) = \infty$. $\square$

Proof of (c). Fix $M > 0$. Let $N = max\{1, 2M, 2n|a_0|, 2n|a_1|, \cdots, 2n|a_{n-1}|\}$.
If $x < -N$,

$$\begin{aligned}
|\frac{a_{n-j}}{x^j}| &= \frac{|a_{n-j}|}{|x|^j} \\
&< \frac{|a_{n-j}|}{|x|} && \text{Since } |x| > N > 1 \\
&< \frac{|a_{n-j}|}{2n|a_{n-j}|} && \text{Since } |x| > N > 2n|a_{n-j}| \\
&= \frac{1}{2n}
\end{aligned}$$

Thus,

$$\begin{aligned}
1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n} &\geq 1 - |\frac{a_{n-1}}{x}| - \cdots - |\frac{a_0}{x^n}| \\
&> 1 - \frac{1}{2n} - \cdots - \frac{1}{2n} \\
&= 1 - n * \frac{1}{2n} \\
&= 1 - \frac{n}{2n} \\
&= 1 - \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Since $f(x) = x^n(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n})$,
If $x < -N$, then $f(x) < x^n * \frac{1}{2} < \frac{-N}{2} \leq \frac{-2M}{2} = -M$
Since M was arbitrary, $\forall M > 0\ \exists N > 0 : x < -N \implies f(x) < -M$, equivicantly, $\lim_{x\to-\infty} f(x) = -\infty$. $\square$

Exercise #6: Suppose f is continuous on $\mathbb{R}$. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, then there exists a number x such that $f(x) = 0$.

Proof. If $lim_{x\to\infty} f(x) = \infty$, then $\forall M > 0\ \exists N > 0 : x > N \implies f(x) > M$.
If $lim_{x\to-\infty} f(x) = -\infty$, then $\forall M > 0\ \exists N > 0 : x < -N \implies f(x) < -M$.
Let $M = 1$. Then $\exists N_1 > 0 : x_1 > N_1 \implies f(x_1) > 1$, and $\exists N_2 > 0 : x_2 < -N_2 \implies f(x_2) < -1$.
Fix $x_1 > N_1 > 0$. Then $f(x_1) > 1 > 0$.
Fix $x_2 < -N_2 < 0$. Then $f(x_2) < -1 < 0$.
By Theorem 7.1, $\exists x \in [x_2, x_1] \subset \mathbb{R} : f(x) = 0$.
Since x_1, x_2 were arbitrary, $\exists x \in \mathbb{R} : f(x) = 0$. $\square$

Exercise #7: Suppose f is continuous on $\mathbb{R}$. If $\lim\limits_{x\to\infty} f(x) = \infty = \lim\limits_{x\to-\infty} f(x)$, then there exists a number y such that $f(y) \leq f(x)$ for all x.

Proof. If $lim_{x\to\infty} f(x) = \infty$, then $\forall M > 0\ \exists N > 0 : x > N \implies f(x) > M$.

If $lim_{x\to-\infty} f(x) = \infty$, then $\forall M > 0\ \exists N > 0 : x < -N \implies f(x) > M$.

Fix $M > 0$. Then $\exists N_1 > 0 : x_1 > N_1 \implies f(x_1) > M$, and $\exists N_2 > 0 : x_2 < -N_2 \implies f(x_2) > M$.

Fix $x_1 > N_1$. $x_2 < -N_2$. By Theorem 7.4, $\exists y \in [x_2, x_1]\ \forall x \in [x_2, x_1] : f(y) \leq f(x)$.

Since M, x_1, x_2 were arbitrary, $\exists y \in \mathbb{R}\ \forall x \in \mathbb{R}\ \ f(y) \leq f(x)$.

□

1 Extra Credit Questions

Each extra credit question is worth 1 extra point.

Exercise E.C.#2: Spivak, Chapter 7, Problem 17

Suppose $f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

Theorem. $\exists y\ \forall x : |f(y)| \leq |f(x)|$

Proof. Proceed by cases

Case 1. $f(x) = a_0$

Let $y = 0$.

Then $\forall x : |f(y)| = |a_0| = |f(x)|$.

Thus $\forall x : |f(y)| \leq |f(x)|$.

Case 2. $f(x)$ has some factor of the form $a_j x^j$ where $a_j \neq 0$ and $j \in \mathbb{Z} \geq 1$.

Let $g(x) = |f(x)|$.

It follows from properties of infinite limits of polynomials that $\lim_{x\to\infty} |f(x)| = lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to-\infty} |f(x)| = \lim_{x\to-\infty} g(x) = \infty$.

Applying proof from exercise (7), $\exists y\ \forall x : g(y) \leq g(x)$, equivicantly, $\exists y\ \forall x : |f(y)| \leq |f(x)|$

□