Assignment #3
UW-Madison MATH 421

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Exercise #1: In this problem we investigate left and right hand limits which are defined as follows.

Definition. We write $\lim_{x\to a^+} f(x) = L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that: if $x > a$ and $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.

Definition. We write $\lim_{x\to a^-} f(x) = L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that: if $x < a$ and $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.

Prove the following.

Theorem. $\lim_{x\to a} f(x) = L$ *if and only if* $\lim_{x\to a^+} f(x) = L$ *and* $\lim_{x\to a^-} f(x) = L$.

Proof. ($\lim_{x\to a} f(x) = L \implies \lim_{x\to a^+} f(x) = L \wedge \lim_{x\to a^-} f(x) = L$):

Because $\lim_{x\to a} f(x) = L$, $\forall \epsilon > 0\ \exists \delta > 0 : 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$

If $0 < |x-a| < \delta$, it follows that $x \neq a$. Thus either $x > a$ or $x < a$. Then,

whenever $x > a$, $x > a \wedge 0 < |x-a| < \delta \equiv 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$. Thus, $\lim_{x\to a^+} f(x) = L$. Since $x \not< a$, $x < a \wedge 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$, holds because of the falsity of the implicator. Thus, $\lim_{x\to a^-} f(x) = L$.

Likewise, whenever $x < a$, $x < a \wedge 0 < |x-a| < \delta \equiv 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$. Thus, $\lim_{x\to a^-} f(x) = L$. Since $x \not> a$, $x > a \wedge 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$ holds, because of the falsity of the implicator. Thus, $\lim_{x\to a^+} f(x) = L$.

Thus, ($\lim_{x\to a} f(x) = L \implies \lim_{x\to a^+} f(x) = L \wedge \lim_{x\to a^-} f(x) = L$).

($\lim_{x\to a^+} f(x) = L \wedge \lim_{x\to a^-} f(x) = L \implies \lim_{x\to a} f(x) = L$):

Because $\lim_{x\to a^+} f(x) = L$, $\forall \epsilon > 0\ \exists \delta > 0 : x > a \wedge 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$.

Because $\lim_{x\to a^-} f(x) = L$, $\forall \epsilon > 0\ \exists \delta > 0 : x < a \wedge 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$.

If $0 < |x-a| < \delta$, it follows that $x \neq a$. Thus either $x > a$ or $x < a$. Then,

whenever $x > a$, $x > a \wedge 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon \equiv 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$. Thus, $\lim_{x\to a} f(x) = L$.

Likewise, whenever $x < a$, $x < a \wedge 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon \equiv 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$. Thus, $\lim_{x\to a} f(x) = L$.

Thus, ($\lim_{x\to a} f(x) = L$.

□

Exercise #2: Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational.} \end{cases}$$

Using the ϵ/δ definition, prove that $\lim_{x\to a} f(x)$ does not exist for every real number a (and hence f is discontinuous at every real number).

Proof. Suppose for a contradiction that $\exists a : \lim_{x\to a} f(x) = L$.

Hence $\forall \epsilon > 0\ \exists \delta > 0 : 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$.

Notice that $\forall x : f(x) \in \{0, 1\}$.

Let $\epsilon = min\{|L|, |L-1|\}$.

When $f(x) = 1$, $|f(x) - L| = |1 - L| = |L - 1| \not< \epsilon = |L - 1|$.

When $f(x) = 0$, $|f(x) - L| = |0 - L| = |L| \not< \epsilon = |L|$.

Thus, $\exists\epsilon$ where $\forall\delta\ \ 0 < |x-a| < \delta \nrightarrow |f(x)-L| < \epsilon$.
Hence, $\exists a : \lim_{x\to a} f(x) = L$ is a contradiction, and $\forall a : \lim_{x\to a} f(x)$ DNE.

$\square$

$\boxed{\textbf{Exercise \#3:}}$ Prove the following

(a) The function $f(x) = x$ is continuous.

(b) If n is a natural number, then the function $f_n(x) = x^n$ is continuous.

(c) If g is a polynomial, then g is continuous on $\mathbb{R}$.

(d) If h is a rational function, then h is continuous at every point in its domain.

(Hint: b,c,d should follow quickly from results in class)

Proof of (a). "$f(x) = x$ is continuous" means $\forall a\ \forall\epsilon > 0\ \exists\delta > 0 : 0 < |x-a| < \delta \implies |f(x)-f(a)| < \epsilon$
Fix a. Fix ϵ. Let $\delta = \epsilon$. If $0 < |x-a| < \delta$, then $|f(x)-f(a)| = |x-a| < \delta = \epsilon$.
Since ϵ was arbitrary, $\forall\epsilon > 0\ \exists\delta > 0 : 0 < |x-a| < \delta \implies |f(x)-f(a)| < \epsilon$.
Since a was arbitrary, $\forall a\ \forall\epsilon > 0\ \exists\delta > 0 : 0 < |x-a| < \delta \implies |f(x)-f(a)| < \epsilon$. $\square$

Proof of (b). Because $f(x) = x$ is continuous, and using the theorem proved in class, it follows from induction that $f(x) = x^2 = x \cdot x$ is continuous, $f(x) = x^3 = x^2 \cdot x$ is continuous, $\cdots$. Hence, $\forall n \in \mathbb{N} : f(x) = x^n$ is continuous. $\square$

Proof of (c). By definition, $g(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n$ for some numbers $a_0, a_1, \cdots, a_n$.

Because the functions $f(x) = a_0$, $f(x) = a_1$, $\cdots$, $f(x) = a_n$ are continuous, and the functions $f(x) = x$, $f(x) = x^2$, $\cdots$, $f(x) = x^n$ are continuous, it follows from the theorem proved in class that the functions $f(x) = a_0$, $f(x) = a_1 \cdot x$, $f(x) = a_2 \cdot x^2$, $\cdots$, $f(x) = a_n \cdot x^n$ are continuous.

It follows from induction using the theorem proved in class that the functions $f(x) = a_0 + a_1 \cdot x$, $f(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 = (a_0 + a_1 \cdot x) + a_2 \cdot x^2$, $\cdots$, $f(x) = (a_0 + a_1 \cdot x + \cdots + a_{n-1} \cdot x^{n-1}) + a_n \cdot x^n = a_0 + a_1 \cdot x + \cdots + a_n \cdot x^n$ are continuous. $\square$

Proof of (d). Notice that, by definintion, the domain of a rational function $f(x) = \frac{g(x)}{h(x)}, g, h$ are polynomials is wherever $h(x) \neq 0$.

It follows from the theorem proved in the class, because $g(x)$ and $h(x)$ are polynomials that are continuous, that $f(x)$ is continuous in its domain. $\square$

$\boxed{\textbf{Exercise \#4:}}$ Spivak, Chapter 6, Problem 3 (a).

Theorem. *Given a function f where $\forall x : |f(x)| \leq |x|$, f is continuous at 0.*

Proof. Notice that when $x = 0$, $|f(0)| \leq |0| \implies f(0) = 0$
$\lim_{x\to 0} f(x) = 0$ means $\forall\epsilon\ \exists\delta : |x| < \delta \implies |f(x)| < \epsilon$.
Fix ϵ. Let $\delta = \epsilon$.
If $|x| < \delta$, then $|f(x)| \leq |x| < \delta = \epsilon$.
Since ϵ was arbitrary, $\lim_{x\to 0} f(x) = 0$. $\square$