Assignment #3
UW-Madison MATH 421

GEOFF YOERGER
April 22, 2021

Exercise #1: Spivak, Chapter 11, Problem 53

Theorem. *Suppose*

$$f(x) = \begin{cases} \frac{g(x)}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and $g(0) = g'(0) = 0$ *and* $g''(0) = 17$.
Find $f'(0)$

Proof. Notice that $\lim_{h\to 0} g'(h) = 0$ and $\lim_{h\to 0} 2h = 0$, and $\lim_{h\to 0} \frac{g''(h)}{2} = \frac{17}{2}$ exists.
By L'Hopitals rule, $\lim_{h\to 0} \frac{g'(h)}{2h}$ exists, and is equal to $\lim_{h\to 0} \frac{g''(h)}{2}$.
Notice that $\lim_{h\to 0} g(h) = 0$ and $\lim_{h\to 0} h^2 = 0$.
By L'Hopitals rule, $\lim_{h\to 0} \frac{g(h)}{h^2}$ exists, and is equal to $\lim_{h\to 0} \frac{g'(h)}{2h}$.
Then,

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{f(h)}{h} \\ &= \lim_{h\to 0} \frac{g(h)}{h^2} \\ &= \lim_{h\to 0} \frac{g'(h)}{2h} \\ &= \lim_{h\to 0} \frac{g(h)}{2} \\ &= \frac{17}{2} \end{aligned}$$

□

Exercise #2: Spivak, Chapter 11, Problem 65

Theorem. *If* $n \geq 1$, *then* $(-1 < x < 0 \vee 0 < x) \implies (1+x)^n > 1 + nx$.

Proof. Let $g(x) = (1+x)^n - 1 - nx$.
Then $g'(x) = n(1+x)^{n-1} - n = n((1+x)^{n-1} - 1)$.
Notice that $n - 1 \geq 0$.
Cases on x:
If $-1 < x < 0$, then $(1+x) < 1 \implies (1+x)^{n-1} < 1 \implies g'(x) < 0$.
If $0 < x$, then $(1+x) > 1 \implies (1+x)^{n-1} > 1 \implies g'(x) > 0$.
By corollary, g is decreasing on $(-1, 0)$, and increasing on $(0, \infty)$.
Thus, by definition, $-1 < x < 0 \implies g(x) > 0$ and $x > 0 \implies g(x) > 0$.
$\equiv -1 < x < 0 \vee 0 < x \implies (1+x)^n > 1 + nx$. □

Exercise #3: If P, Q are partitions of $[a, b]$, $P \subset Q$, Q has one more element than P, and f is bounded on $[a, b]$, then $U(f, P) \geq U(f, Q)$.

Proof. By assumption, $P = \{t_0, \cdots, t_n\}$, $Q = \{t_0, \cdots, t_{k-1}, u, t_k, \cdots, t_n\}$

Let $M_i = \sup\{f(x) : t_{i-1} \leq x \leq t_i\}$.
Let $M' = \sup\{f(x) : t_{k-1} \leq x \leq u\}$.
Let $M'' = \sup\{f(x) : u \leq x \leq t_k\}$.
Then $U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$.
Then $U(f, Q) = (\sum_{i=1}^{k-1} M_i(t_i - t_{i-1})) + m'(u - t_{k-1}) + m''(t_k - u) + \sum_{j=k+1}^{n} M_j(t_j - t_{j-1})$.
So $U(f, Q) - U(f, P) = M'(u - t_{k-1}) + M''(t_k - u) - M_k(t_k - t_{k+1})$.
By defn, $M' \leq M_k$, $M'' \leq M_k$.
So $U(f, Q) - U(f, P) \leq M_k(u - t_{k-1}) + M_k(t_k - u) - M_k(t_k - t_{k-1}) = 0$.
So $U(f, Q) \leq U(f, P)$. □

$\boxed{\textbf{Exercise \#4:}}$ Suppose f is integrable on $[a, b]$. Prove: if $c \in \mathbb{R}$, then cf is integrable on $[a, b]$ and $\int_a^b cf = c \int_a^b f$.

(Hint: how to $L(f, P)$, $L(cf, P)$, $U(f, P)$, and $U(cf, P)$ relate? It is a good idea to treat the cases $c \leq 0$ and $c \geq 0$ separately.)

Proof. Assuming several properties of sup and inf.

1. $c \geq 0 : \inf\{c * a : a \in A\} = c * \inf\{a : a \in A\}$
2. $c \geq 0 : \sup\{c * a : a \in A\} = c * \sup\{a : a \in A\}$
3. $c \leq 0 : \inf\{c * a : a \in A\} = c * \sup\{a : a \in A\}$
4. $c \leq 0 : \sup\{c * a : a \in A\} = c * \inf\{a : a \in A\}$

Notice that "f is bounded on $[a, b]$" $\implies$ "cf is bounded on $[a, b]$".
If $c \geq 0$, because f is integrable on $[a, b]$, and $\int_a^b f = \sup\{L(f, P)\} = \inf\{U(f, P)\}$,
Then,

$$\begin{aligned}
&\implies c * \int_a^b f && \\
&\equiv c * \sup\{L(f, P)\} && = c * \inf\{U(f, P)\} \\
&\equiv \sup\{c * L(f, P)\} && = \inf\{c * U(f, P)\} \\
&\equiv \sup\{c * \sum_{i=1}^{n} \inf\{f(x) : t_{i-1} \leq x \leq t_i\}(t_i - t_{i-1})\} && = \inf\{c * \sum_{i=1}^{n} \sup\{f(x) : t_{i-1} \leq x \leq t_i\}(t_i - t_{i-1})\} \\
&\equiv \sup\{\sum_{i=1}^{n} \inf\{c * f(x) : t_{i-1} \leq x \leq t_i\}(t_i - t_{i-1})\} && = \inf\{\sum_{i=1}^{n} \sup\{c * f(x) : t_{i-1} \leq x \leq t_i\}(t_i - t_{i-1})\} \\
&\equiv \sup\{L(cf, P)\} && = \inf\{U(cf, P)\} \\
&\equiv \int_a^b cf && \\
&\implies cf \text{ is integrable on } [a, b] &&
\end{aligned}$$

If $c \leq 0$, because f is integrable on $[a, b]$, and $\int_a^b f = \sup\{L(f, P)\} = \inf\{U(f, P)\}$,
Then,

$$
\begin{aligned}
&\implies c * \int_a^b f \\
&\quad \equiv c * \sup\{L(f,P)\} && = c * \inf\{U(f,P)\} \\
&\quad \equiv \inf\{c * L(f,P)\} && = \sup\{c * U(f,P)\} \\
&\quad \equiv \inf\{c * \sum_{i=1}^{n} \inf\{f(x) : t_{i-1} \leq x \leq t_i\}(t_i - t_{i-1})\} && = \sup\{c * \sum_{i=1}^{n} \sup\{f(x) : t_{i-1} \leq x \leq t_i\}(t_i - t_{i-1})\} \\
&\quad \equiv \inf\{\sum_{i=1}^{n} \sup\{c * f(x) : t_{i-1} \leq x \leq t_i\}(t_i - t_{i-1})\} && = \sup\{\sum_{i=1}^{n} \inf\{c * f(x) : t_{i-1} \leq x \leq t_i\}(t_i - t_{i-1})\} \\
&\quad \equiv \inf\{U(cf,P)\} && = \sup\{L(cf,P)\} \\
&\quad \equiv \int_a^b cf \\
&\implies cf \text{ is integrable on } [a,b]
\end{aligned}
$$

□

Exercise #5: Spivak, Chapter 13, Problem 20 (a), (b), and (c)

Theorem. *Suppose that f is nondecreasing on $[a,b]$.*
If $P = t_0, \cdots, t_n$ is a partition of $[a,b]$, what is $L(f,P)$ and $U(f,P)$?

Proof of (a). By definition, $m_i = \inf\{f(x) : t_{i-1} \leq x \leq t_i\}$.
By definition, $M_i = \sup\{f(x) : t_{i-1} \leq x \leq t_i\}$.
Since f is nondecreasing on $[a,b]$, $t_{i-1} \leq x \implies f(t_{i-1}) \leq f(x)$.
Thus, $m_i = f(t_{i-1})$, and $L(f,P) = \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1})$.
Thus, $M_i = f(t_i)$, and $U(f,P) = \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$. □

Theorem. *Suppose $\forall i : t_i - t_{i-1} = \delta$.*
Prove $U(f,P) - L(f,P) = \delta * (f(b) - f(a))$.

Proof of (b). Then $L(f,P) = \sum_{i=1}^{n} f(t_{i-1}) * \delta$.
Then $U(f,P) = \sum_{i=1}^{n} f(t_i) * \delta$.
Then $U(f,P) - L(f,P) = \delta * (f(t_1) + \cdots + f(t_n) - f(t_0) - \cdots - f(t_{n-1})) = \delta * (f(t_n) - f(t_0)) = \delta * (f(b) - f(a))$. □

Theorem. *Prove f is integrable.*

Proof of (c). Fix $\epsilon > 0$.
By archimedian property, $\exists n \in \mathbb{N} : 0 < \frac{b-a}{n} < \frac{\epsilon}{f(b)-f(a)}$.
Let $\delta = \frac{b-a}{n}$.
Let $P = \{a, a+\delta, a+2\delta, \cdots, a+n\delta = a+b-a = b\}$. Notice that $\forall i : t_i - t_{i-1} = \delta$.
Thus,

$$
\begin{aligned}
U(f,P) - L(f,P) &= \delta(f(b) - f(a)) \\
&= \frac{b-a}{n}(f(b) - f(a)) \\
&< \frac{\epsilon}{f(b) - f(a)}(f(b) - f(a)) \\
&= \epsilon
\end{aligned}
$$

Since $\epsilon > 0$ was arbitrary, by Theorem 13-2, f is integrable on $[a,b]$. □

Exercise #6: Prove: if f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$.
(Hint: show that $U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P)$)

Proof. Fix $P = \{t_0, \cdots, t_n\}$ as some partition of $[a, b]$.
Consider each interval: Fix $1 \leq i \leq n$.
Cases on $m_i(f), M_i(F)$:
If $m_i(f) < 0, M_i(f) < 0$, then $m_i(f) = -M_i(|f|)$, and $M_i(f) = -m_i(|f|)$. Then $M_i(|f|) - m_i(|f|) = M_i(f) - m_i(f)$.
If $m_i(f) > 0, M_i(f) > 0$, then $m_i(f) = m_i(|f|)$, and $M_i(f) = M_i(|f|)$. Then $M_i(|f|) - m_i(|f|) = M_i(f) - m_i(f)$.
If $m_i(f) < 0, M_i(f) > 0, |M_i(f)| \geq |m_i(f)|$, then $m_i(|f|) = 0$ and $M_i(|f|) = M_i(f)$. Then $M_i(|f|) - m_i(|f|) = M_i(f) \leq M_i(f) - m_i(f)$.
If $m_i(f) < 0, M_i(f) > 0, |M_i(f)| \leq |m_i(f)|$, then $m_i(|f|) = 0$ and $M_i(|f|) = -m_i(f)$. Then $M_i(|f|) - m_i(|f|) = -m_i(f) \leq M_i(f) - m_i(f)$.
Since i was arbitrary, and from cases, $\forall i : M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$.
Then, $U(f, P) - L(f, P) = \sum_{i=1}^{n}(M_i(f) - m_i(f))(t_i - t_{i-1}) \geq \sum_{i=1}^{n}(M_i(|f|) - m_i(|f|))(t_i - t_{i-1}) = U(|f|, P) - L(|f|, P)$.
Since P was arbitrary, $\forall P : U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P)$.
Because f is integrable on $[a, b]$, by theorem 13-2, $\forall \epsilon > 0\ \exists P : \epsilon > U(f, P) - L(f, P) \geq U(|f|, P) - L(|f|, P)$.
Thus, $\forall \epsilon > 0\ \exists P : U(|f|, P) - L(|f|, P) < \epsilon$.
Thus, by theorem 13-2, $|f|$ is integrable on $[a, b]$. □

Exercise #7: Prove: if f and g are integrable on $[a, b]$, then $\max\{f, g\}$ and $\min\{f, g\}$ are integrable on $[a, b]$.
(Hint: use a past HW problem for expressing the minimum and maximum in terms of absolute values).

Proof. By definition, $\max\{f, g\} = \frac{f+g+|f-g|}{2}$, and $\min\{f, g\} = \frac{f+g-|f-g|}{2}$.
By Thm 6, $-g$ is integrable on $[a, b]$.
By Thm 5, $f - g$ is integrable on $[a, b]$.
By result of exercise 6, $|f - g|$ is integrable on $[a, b]$.
By Thm 6, $-|f - g|$ is integrable on $[a, b]$.
By Thm 5, $f + g$ is integrable on $[a, b]$.
By Thm 5, $f + g + |f - g|$ and $f + g - |f - g|$ are both integrable on $[a, b]$.
By Thm 6, $\frac{f+g+|f-g|}{2}$ and $\frac{f+g-|f-g|}{2}$ are both integrable on $[a, b]$.
Equivicantly, $\max\{f, g\}$ and $\min\{f, g\}$ are both integrable on $[a, b]$. □