Assignment #3

UW-Madison MATH 421

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$\boxed{\textbf{Exercise \#1:}}$ Sketch the set of all points (x, y) in the plane satisfying

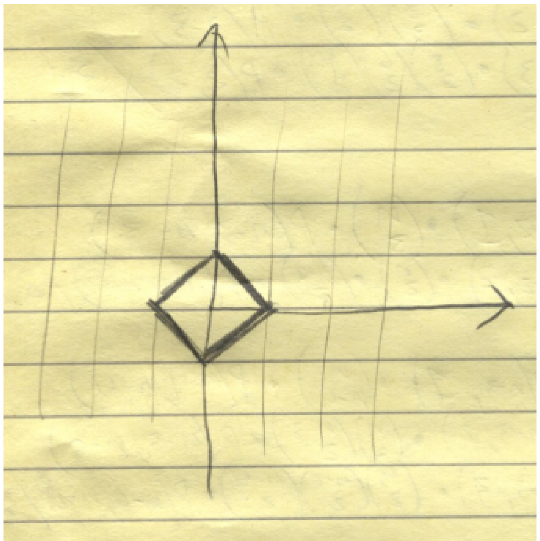

Figure 1: $|x| + |y| = 1$: A diamond

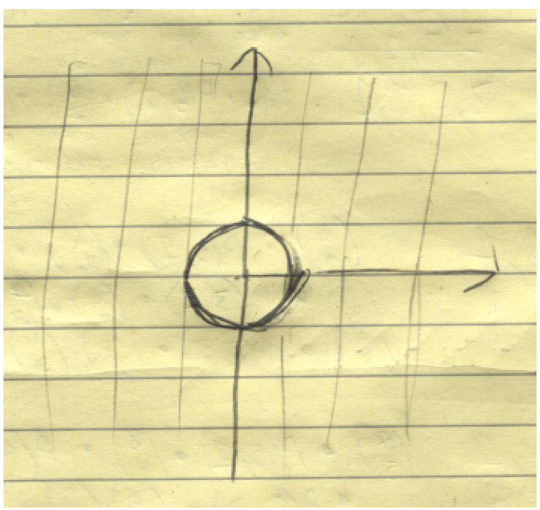

Figure 2: $x^2 + y^2 = 1$: A circle

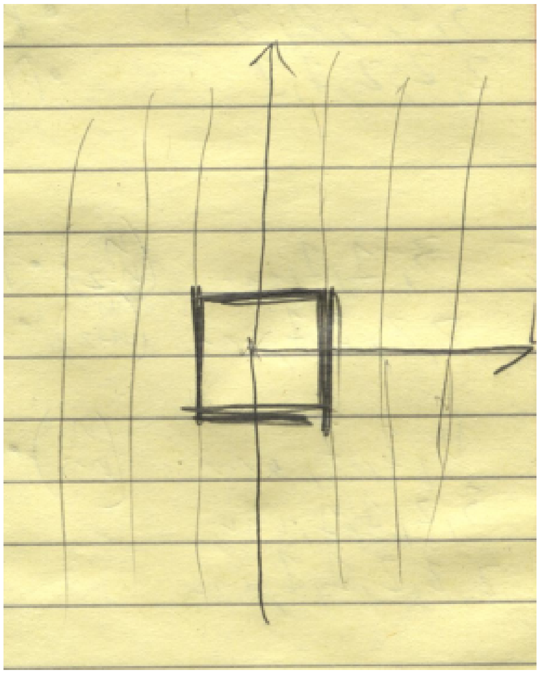

Figure 3: $\max\{|x|, |y|\} = 1$: A square

$\boxed{\textbf{Exercise \#2:}}$ Following the instructions on the previous problem: Spivak, Chapter 4, Problem 17 (i) and (ii).

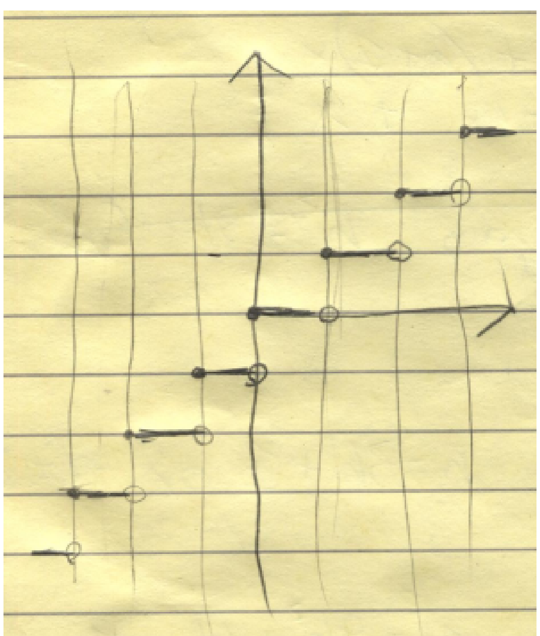

Figure 4: $f(x) = \lfloor x \rfloor$: (i) Steps

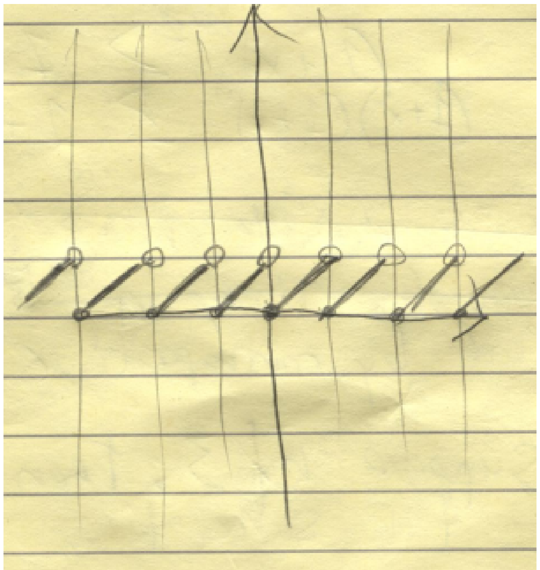

Figure 5: $f(x) = x - \lfloor x \rfloor$: (ii) Sawteeth

Exercise #3: Prove, using only the definition, that $\lim_{x\to 3} 5x = 15$.

Proof. Fix $\epsilon > 0$. Let $\delta = \frac{\epsilon}{5}$. If $0 < |x-3| < \delta$, then

$$\begin{aligned}
|f(x) - 15| &= |5x - 15| \\
&= 5|x-3| \\
&< 5\delta \\
&= 5\frac{\epsilon}{5} \\
&= \epsilon
\end{aligned}$$

Since ϵ was arbitrary, $\lim_{x\to 3} 5x = 15$ $\square$

Exercise #4: Prove, using only the definition, that $\lim_{x\to 2} x^2 + 2x = 8$.

Proof. Fix $\epsilon > 0$. Let $\delta = min\left\{1, \frac{\epsilon}{7}\right\}$. If $0 < |x-2| < \delta$, then

$$\begin{aligned}
|f(x) - 8| &= |x^2 + 2x - 8| \\
&= |x+4||x-2| \\
&= |(x-2)+6||x-2| \\
&\leq (|x-2|+6)|x-2| \\
&< (1+6)|x-2| && \text{Since } |x-2| < 1 \\
&< 7\frac{\epsilon}{7} && \text{Since } |x-2| < \frac{\epsilon}{5} \\
&= \epsilon
\end{aligned}$$

Since ϵ was arbitrary, $\lim_{x\to 2} x^2 + 2x = 8$ $\square$

$\boxed{\textbf{Exercise \#5:}}$ Prove the following theorem:

Theorem. *If x and y are numbers, then $||x| - |y|| \leq |x - y|$.*

Proof. Notice that $\forall x \forall y : |x| - |y| \leq |x - y|$. This was proved in HW2 §8.3. Swapping variables,

$$\begin{aligned} |y| - |x| \leq |y - x| &\Leftrightarrow |y| - |x| \leq |x - y| \\ &\Leftrightarrow -(|x| - |y|) \\ &\Leftrightarrow -(|x| - |y|) \leq |x - y| \wedge |x| - |y| \leq |x - y| \\ &\Leftrightarrow ||x| - |y|| \leq |x - y| \end{aligned}$$

□

$\boxed{\textbf{Exercise \#6:}}$ Spivak, Chapter 5, 16 (a)

Theorem. *If $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} |f|(x) = |L|$.*

Proof. Fix $\epsilon > 0$. By definition, $\exists \delta$ such that $0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$. Suppose $0 < |x - a| < \delta$ So,

$$\begin{aligned} ||f|(x) - |L|| &= ||f(x)| - |L|| \\ &\leq |f(x) - L| \\ &< \epsilon \end{aligned}$$

Thus, $0 < |x - a| < \delta \implies |f(x) - L| < \epsilon \implies ||f|(x) - |L|| < \epsilon$.
Since ϵ was arbitrary, $\lim_{x \to a} |f|(x) = |L|$ □

$\boxed{\textbf{Exercise \#7:}}$ Spivak, Chapter 5, Problem 12 (a)

Theorem. *If $\forall x (f(x) \leq g(x))$, $\lim_{x \to a} f(x) = L$ exists, and $\lim_{x \to a} g(x) = M$ exists, then $\lim_{x \to a} f(x) \leq lim_{x \to a} g(x)$, equivicantly $L \leq M$*

Proof. Suppose for a contradiction that $L > M$. By definition, $\lim_{x \to a}(g(x) - f(x)) = M - L$. Let $\epsilon = L - M$.
By definition, $\exists \delta > 0$ such that $0 < |x - a| < \delta \implies |g(x) - f(x) + L - M| < \epsilon = L - M$. Thus,

$$\begin{aligned} g(x) - f(x) + L - M &< L - M \\ g(x) - f(x) &< 0 \\ g(x) &< f(x) \\ f(x) &> g(x) \end{aligned}$$

But, $f(x) \leq g(x)$, thus a contradiction arises, and $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$ □

$\boxed{\textbf{Exercise \#8:}}$ Spivak, Chapter 5, Problem 37 (a)
Define $\lim_{x \to a} f(x) = \infty$ as $\forall N, \exists \delta > 0$ such that $0 < |x - a| < \delta \implies f(x) > N$.

Theorem. $\lim_{x \to 3} \frac{1}{(x-3)^2} = \infty$

Proof. Fix N. Let $\delta = \frac{1}{\sqrt{N}}$. If $0 < |x-3| < \delta = \frac{1}{\sqrt{N}}$, then

$$\begin{aligned} |x-3| = \sqrt{(x-3)^2} &< \frac{1}{\sqrt{N}} \\ \frac{1}{\sqrt{(x-3)^2}} &> \sqrt{N} \\ \frac{1}{(x-3)^2} &> N \end{aligned}$$

Since N was arbitrary, $0 < |x-3| < \delta \implies \frac{1}{(x-3)^2} > N$, so $lim_{x->a} \frac{1}{(x-3)^2} = \infty$ □