Assignment #3
UW-Madison MATH 421

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Exercise #1: Prove the following theorem:

Theorem. *If $A \subset \mathbb{R}$, $A \neq \emptyset$, and A is bounded below, then A has a greatest lower bound.*

Proof. Consider $-A = \{-a : a \in A\}$. It follows that $A \subset R \implies -A \subset R$, and $A \neq \emptyset \implies -A \neq \emptyset$.
If A is bounded below, then

$$\begin{aligned}
\exists x\ \forall a \in A : x \leq a &\implies \exists x\ \forall a \in A :\ -x \geq -a \\
&\implies \exists x\ \forall a \in -A :\ -x \geq a \\
&\implies \exists y\ \forall a \in -A : y \geq a \\
&\implies \exists y\ \forall a \in -A : a \leq y
\end{aligned}$$

Thus, by P13, $-A$ has a least upper bound, say $-u = \sup(A)$.

Claim. *u is a lower bound of A.*

$$\begin{aligned}
&-u \text{ is an upper bound of } -A \\
&\implies \forall a \in -A : a \leq -u \\
&\implies \forall a \in -A :\ -a \geq u \\
&\implies \forall a \in A : a \geq u \\
&\implies \forall a \in A : u \leq a \\
&\implies u \text{ is a lower bound of } A
\end{aligned}$$

Claim. *x is a lower bound of $A \implies x \leq u$*

$$\begin{aligned}
&w \text{ is an upper bound of } -A \implies -u \leq w \\
&\equiv -w \text{ is a lower bound of } A \implies -u \leq w \\
&\equiv -w \text{ is a lower bound of } A \implies -w \leq u
\end{aligned}$$

Thus, by definition, u is the greatest lower bound of $A \implies A$ has a greatest lower bound. □

Exercise #2: Prove: If $A, B \subset \mathbb{R}$, then

$$\sup(A \cap B) \leq \min\{\sup(A), \sup(B)\}$$

and

$$\inf(A \cap B) \geq \max\{\inf(A), \inf(B)\}.$$

Find an example where $\sup(A \cap B) < \min\{\sup(A), \sup(B)\}$ and $\inf(A \cap B) < \max\{\inf(A), \inf(B)\}$.

Proof. We proceed by cases.

Case 1. $A = \emptyset$

Then $\sup(A \cap B) = \sup(\emptyset) = -\infty \leq min\{\sup(A), \sup(B)\} = min\{-\infty, \sup(B)\} = -\infty.$

Then $\inf(A \cap B) = \inf(\emptyset) = \infty \geq max\{\inf(A), \inf(B)\} = max\{\infty, \inf(B)\} = \infty.$

Case 2. $B = \emptyset$

Then $\sup(A \cap B) = \sup(\emptyset) = -\infty \leq min\{\sup(A), \sup(B)\} = min\{\sup(A), -\infty\} = -\infty$

Then $\inf(A \cap B) = \inf(\emptyset) = \infty \geq max\{\inf(A), \inf(B)\} = max\{\inf(A), \infty\} = \infty$

Case 3. *A, B both not bounded above*

Then $\sup(A \cap B) = \infty \leq min\{\sup(A), \sup(B)\} = min\{\infty, \infty\} = \infty$

Then $\inf(A \cap B) = -\infty \geq max\{\inf(A), \inf(B)\} = max\{-\infty, -\infty\} = -\infty$

Case 4. *A not bounded above, B bounded above*

Then $\sup(A \cap B) = \sup(B) \leq min\{\sup(A), \sup(B)\} = min\{\infty, \sup(B)\} = \sup(B)$

Then $\inf(A \cap B) = \inf(B) \geq max\{\inf(A), \inf(B)\} = max\{-\infty, \inf(B)\} = \inf(B)$

Case 5. *A bounded above, B not bounded above*

Then $\sup(A \cap B) = \sup(A) \leq min\{\sup(A), \sup(B)\} = min\{\sup(A), \infty\} = \sup(A)$

Then $\inf(A \cap B) = \inf(A) \geq max\{\inf(A), \inf(B)\} = max\{\inf(A), -\infty\} = \inf(A)$

Case 6. *A, B both bounded above*

Concerning sup*:*

Let $\alpha = min\{sup(A), sup(B)\}$.

Claim. *α is an upper bound of $A \cap B$.*

Suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$.

If $\alpha = \sup(A)$, *then* $x \leq \sup(A) = \alpha$

If $\alpha = \sup(B)$, *then* $x \leq \sup(B) = \alpha$

Since $x \in A \cap B$ was arbitrary, α is an upper bound for $A \cap B$.

Claim. *α is the least upper bound of $A \cap B$.*

Suppose x is an upper bound of $A \cap B$. Then x is an upper bound for A or an upper bound for B.

If x is an upper bound for A, then $x \leq \sup(A) \leq min\{\sup(A), \sup(B)\} = \alpha$.

If x is an upper bound for B, then $x \leq \sup(B) \leq min\{\sup(A), \sup(B)\} = \alpha$.

Thus $x \leq \alpha$.

Since x was arbitrary, α is the least upper bound of $A \cap B$.

Concerning inf*:*

Let $\alpha = max\{\inf(A), \inf(B)\}$.

Claim. *α is a lower bound of $A \cap B$.*

Suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$.

If $\alpha = \inf(A)$, *then* $x \geq \inf(A) = \alpha$

If $\alpha = \inf(B)$, *then* $x \geq \inf(B) = \alpha$

Since $x \in A \cap B$ was arbitrary, α is a lower bound for $A \cap B$.

Claim. *α is the greatest lower bound of $A \cap B$.*

Suppose x is a lower bound of $A \cap B$. Then x is a lower bound for A or a lower bound for B.

If x is a lower bound for A, then $x \geq \inf(A) \geq max\{\inf(A), \inf(B)\} = \alpha$.

If x is a lower bound for B, then $x \geq \inf(B) \geq max\{\inf(A), \inf(B)\} = \alpha$.

Thus $x \geq \alpha$.

Since x was arbitrary, α is the greatest lower bound of $A \cap B$.

$\square$

Example. $A = \{0, 2, 4\}$, $B = \{1, 2, 3\}$, $A \cap B = \{2\}$

$\sup(A \cap B) = 2 < 3 = min\{3, 4\} = min\{\sup(A), \sup(B)\}$

$\inf(A \cap B) = 2 > 1 = max\{0, 1\} = max\{\inf(A), \inf(B)\}$

$\boxed{\textbf{Exercise \#3:}}$ Prove the following lemma from class:

Lemma. *If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there exist $\delta_1, \delta_2 > 0$ such that*

1. *f is negative on $[a, a + \delta_1)$*
2. *f is positive on $(b - \delta_2, b]$.*

Proof of (1). By a theorem proved in class, if $f(a) < 0$, then $\exists\delta_1 > 0\ \forall x : |x - a| < \delta_1 \implies f(x) < 0$.
$|x-a| < \delta_1 \implies -\delta_1 < x-a < \delta_1 \implies a-\delta_1 < x < a+\delta_1 \implies x \in (a-\delta_1, a+delta_1) \supset [a, a+delta_1)$.
Thus, $\exists\delta_1\ \forall x : x \in [a, a + \delta_1) \implies f(x) < 0$. □

Proof of (2). By a theorem proved in class, if $f(b) > 0$, then $\exists\delta_2 > 0\ \forall x : |x - b| < \delta_2 \implies f(x) > 0$.
$|x - b| < \delta_2 \implies -\delta_2 < x - b < \delta_2 \implies b - \delta_2 < x < b + \delta_2 \implies x \in (b - \delta_2, b + delta_2) \supset (b - \delta_2, b]$
Thus, $\exists\delta_2\ \forall x : x \in (b - \delta_2, b] \implies f(x) > 0$. □

$\boxed{\textbf{Exercise \#4:}}$ Prove the following theorem from class (in Chapter 6):

Theorem. *If $a < b$, then there exists an irrational number x with $a < x < b$.*

Proof. Since $b - a > 0$, $\exists n \in \mathbb{N}$:

$$\begin{aligned}
&n > \frac{\sqrt{2}}{b-a} \\
\implies &\frac{1}{n} < \frac{b-a}{\sqrt{2}} \\
\implies &\frac{\sqrt{2}}{n} < b - a \\
\implies &\frac{1}{n} < \frac{\sqrt{2}}{n} < b - a \\
\implies &a + \frac{1}{n} < a + \frac{\sqrt{2}}{n} < b
\end{aligned}$$

$\frac{\sqrt{2}}{n} \notin \mathbb{Q}$, $\frac{1}{n} \in \mathbb{Q}$.
If $a \in \mathbb{Q}$, then $a + \frac{\sqrt{2}}{n} \notin \mathbb{Q}$.
If $a \notin \mathbb{Q}$, then $a + \frac{1}{n} \notin \mathbb{Q}$.
Thus, $\forall a, b \in \mathbb{R}\ \exists x \notin \mathbb{Q} : a < x < b$. □

$\boxed{\textbf{Exercise \#5:}}$ Spivak, Chapter 8, Problem 3 (b)

Theorem. *Theorem 7-1 is provable using consideration of the set $B = \{x \in [a, b] \colon f(x) < 0\}$.*

Proof. Let $B = \{x \in [a, b] \colon f(x) < 0\}$. $a \in B \implies B \neq \emptyset$.
B is bounded above by b, thus $\sup(B)$ exists, $\sup(B) \in [a, b]$, and $\sup(B)$ is the least upper bound of B.

Claim. $a < \sup(B) < b$
By the lemma discussed in the proof of 7-1 in class,
$\exists\delta_1 : f$ is negative on $[a, a + \delta_1)$.
and $\exists\delta_2 : f$ is positive on $(b - \delta_2, b]$.
$[a, a - \delta_1) \subset B \implies \sup(B) \geq a + \delta_1 > a$.
$B \subset [a, b - \delta_2] \implies \sup(B) \leq b - \delta_2 < b$.

Claim. $f(\sup(B)) = 0$
Suppose for a contradiction that $f(\sup(B)) \neq 0$
We proceed by cases

Case 1. $f(\sup(B)) > 0$

By Theorem 6.3, $\exists\delta > 0 : f$ *is positive on* $(\sup(B) - \delta, \sup(B) + \delta)$.

Then $\sup(B) - \delta$ *is an upper bound, but* $\sup(B)$ *is the least upper bound. This is a contradiction.*

Case 2. $f(\sup(B)) < 0$

By Theorem 6.3, $\exists\delta > 0 : f$ *is negative on* $(\sup(B) - \delta, \sup(B) + \delta)$.

By a result in class, $\exists x \in B : \sup(B) - \delta < x \leq \sup(B)$.

But by properties of the supremum, $x \in B$.

So f *is negative on* $B \cup (\sup(B) - \delta, \sup(B) + \delta)$.

Thus $B \cup (\sup(B) - \delta, \sup(B) + \delta) \subset B$*, but this means* $\sup(B)$ *is not an upper bound of* B*. This is a contraction.*

Thus $f(\sup(B)) = 0$

(This proof occurs in the vicinity of $x = \sup(B)$) □

$\boxed{\textbf{Exercise \#6:}}$ Spivak, Chapter 8, Problem 8 (a)

Theorem. *Suppose* f *is a function such that* $a < b \implies f(a) \leq f(b)$

$\lim_{x\to a^-} f(x)$ *and* $\lim_{x\to a^+} f(x)$ *both exist.*

Proof. $\lim_{x\to a^-} f(x)$ exists. Since $x < a \implies f(x) \leq f(a)$, the set $A = \{f(x) : x < a\}$ is bounded above (one upper bound is $f(a)$).

Let $L = \sup(A)$.

Fix $\epsilon > 0$.

$x < a \implies 0 < a - x \implies f(x) < L \implies f(x) < L + \epsilon$.

By a theorem in class, $\exists y : L - \epsilon < f(y) \leq L \implies L - \epsilon < f(y)$.

Let $\delta = a - y$. It follows that

$$\begin{aligned}
&\forall x : y < x < a : L - \epsilon < f(y) \leq f(x) \leq \sup(A) = L \\
\implies &\forall x : y < x < a : L - \epsilon < f(y) \leq f(x) \leq \sup(A) = L < L + \epsilon \\
\implies &\forall x : y < x < a : L - \epsilon < f(x) < L + \epsilon \\
\implies &\forall x : -a < -x < -y : L - \epsilon < f(x) < L + \epsilon \\
\implies &\forall x : 0 < a - x < a - y : L - \epsilon < f(x) < L + \epsilon \\
\implies &\forall x : 0 < a - x < \delta : L - \epsilon < f(x) < L + \epsilon
\end{aligned}$$

Thus $\lim_{x\to a^-} f(x) = \sup(A)$ exists. □

Proof. $\lim_{x\to a^+} f(x)$ exists. Since $x < a \implies f(x) \leq f(a)$, the set $A = \{f(x) : x > a\}$ is bounded below.

Let $L = \inf(A)$.

Fix $\epsilon > 0$.

$x > a \implies 0 < x - a \implies L < f(x) \implies L - \epsilon < f(x)$.

By a theorem in class, $\exists y : L \leq f(y) < L + \epsilon \implies f(y) < L + \epsilon$.

Let $\delta = y - a$. It follows that

$$\begin{aligned}
&\forall x : a < x < y : L = \sup(A) \leq f(x) < f(y) < L + \epsilon \\
\implies &\forall x : a < x < y : L - \epsilon < f(x) < L + \epsilon \\
\implies &\forall x : 0 < x - a < y - a : L - \epsilon < f(x) < L + \epsilon \\
\implies &\forall x : 0 < x - a < \delta : L - \epsilon < f(x) < L + \epsilon
\end{aligned}$$

Thus $\lim_{x\to a^+} f(x) = \inf(A)$ exists. □

1 Extra Credit Questions

Each extra credit question is worth 1 extra point.

$\boxed{\textbf{Exercise E.C.\#1:}}$ Spivak, Chapter 8, Problem 14 (a)

Theorem. *Consider a sequence of closed intervals* $I_1 = [a_1, b_1], I_2 = [a_2, b_2], \cdots$.
Suppose that $\forall n : a_n \le a_{n+1}$ *and* $\forall n : b_{n+1} \le b_n$
$\exists x\ \forall n : x \in I_n$

Proof. Notice that $\forall n : a_1 \le a_2 \le \cdots \le a_{n-1} \le a_n \le b_n \le b_{n-1} \le \cdots \le b_2 \le b_1$.
Let $A = \{a_n : n \in \mathbb{N}\}$
Because $a_1 \in A \implies A \ne \emptyset$, A is bounded above (b_1 is an upper bound), then $x = \sup(A)$ exists.
By definition, $\forall n : a_n \le x$.
Because $\forall n, m : a_n \le b_m$, then $\forall m : b_m$ is an upper bound of A. Thus, $\forall m : x \le b_m$.
Thus $\forall n : a_n \le x \le b_n \implies \forall n : x \in [a_n, b_n] \implies \forall n : x \in I_n$ $\square$