Mathematical Derivation for week 7 AML.

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"The used Free energy (from replica method) can be found in the slides."

First we will derive H.

We know $H = -\frac{\partial F}{\partial T} = -\left(\frac{\partial F}{\partial \beta}\frac{\partial \beta}{\partial T}\right) = \frac{\partial F}{\partial \beta}\frac{1}{T^2} \boxed{= \beta^2 \frac{\partial F}{\partial \beta}}$

Where F is the free energy.

$\rightarrow H = \frac{\partial}{\partial \beta}(\beta^2 F) - 2\beta F$ (a product rule)

$\frac{\partial}{\partial \beta}(\beta^2 F) = \beta J_0 m^2 - \frac{3}{4}\beta^2 J^2 (q-1)^2 - \frac{1}{\sqrt{2\pi}} S(z) - \frac{\beta}{\sqrt{2\pi}}\frac{\partial}{\partial \beta}(S(z,\beta$

where $S(z) = S(z,\beta) = \int dz\, e^{-z^2/2} \log(2\cosh(\beta J_0 m + \beta J \sqrt{q} z))$

$2\beta F = J_0 \beta m^2 - \frac{1}{2}\beta^2 J^2 (q-1)^2 - \frac{2}{\sqrt{2\pi}} S(z,\beta)$

$\Rightarrow H = -\frac{\beta^2 J^2}{4}(q-1)^2 + \frac{1}{\sqrt{2\pi}} S(z,\beta) - \frac{\beta}{\sqrt{2\pi}}\frac{\partial}{\partial \beta}(S(z,\beta))$

"Thus we need to prove:"

$\beta/\sqrt{2\pi}\, \frac{\partial}{\partial \beta}(S(z,\beta)) = \beta J_0 m^2 + \beta^2 J^2 q(1-q)$

$\Rightarrow \frac{1}{\sqrt{2\pi}}\, \partial/\partial\beta (S(z,\beta)) = J_0 m^2 + \beta J^2 q(1-q)$

$\frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial \beta}(S(z,\beta)) = \frac{1}{\sqrt{2\pi}}\int dz\, e^{-z^2/2} \frac{2\sinh(\beta J_0 m + \beta J\sqrt{q} z)}{2\cosh(\beta J_0 m + \beta J \sqrt{q} z)}(J_0 m + J\sqrt{q} z)$

$= \underbrace{\frac{1}{\sqrt{2\pi}}\int dz\, e^{-z^2/2} \tanh(\beta J_0 m + \beta J\sqrt{q} z)}_{= m}(J_0 m + J\sqrt{q} z)$ ① ②

$\rightarrow$ ① $= J_0 m^2 \Rightarrow$ Only left to prove:

$\frac{1}{\sqrt{2\pi}}\int dz\, e^{-z^2/2} \tanh(\beta J_0 m + \beta J\sqrt{q} z) J\sqrt{q} z \overset{!}{=} \beta J^2 q(1-q)$

For this we use partial integration on the LHS:

with $f(z) = \tanh(\beta J_0 m + \beta J \sqrt{q} z)$

$g'(z) = e^{-z^2/2} J\sqrt{q} z \rightarrow g(z) = -e^{-z^2/2} J\sqrt{q}$

thus, ~~f(∞)~~ $g(\infty) = g(-\infty) = 0$ and $f(\infty) = 1, f(-\infty) = -1$

$\rightarrow \frac{1}{\sqrt{2\pi}}\int f(z) g'(z)\, dz = \frac{1}{\sqrt{2\pi}}\Big(\underbrace{f(z)g(z)\Big]_{-\infty}^{\infty}}_{\overset{!}{=} 0} - \int dz\, f'(z)\, g(z)\Big)$

with $f'(z) = \operatorname{sech}^2(\beta J_0 m + \beta J\sqrt{q}\,z)\,\beta J\sqrt{q}$

$$\rightarrow = \int dz\, e^{-z^2/2}\, \beta J^2 q\, \operatorname{sech}^2(\beta J_0 m + \beta J\sqrt{q}\,z)\, \frac{1}{\sqrt{2\pi}}$$

Further we know,

$$1-q = \frac{1}{\sqrt{2\pi}}\int dz\, e^{-\frac{z^2}{2}} \operatorname{sech}^2(\beta J_0 m + \beta J\sqrt{q}\,z)$$

And thus we find:

$$= \beta J^2 q(1-q)$$

which was the last thing needed to be proven to have:

$$H = -\frac{\partial F}{\partial T} = \frac{-\beta^2 J^2}{4}(q-1)^2 + \frac{1}{\sqrt{2\pi}}\int dz\, e^{-z^2/2}\log(2\cosh(\beta J_0 m + \beta J\sqrt{q}\,z))$$
$$-\beta J_0 m^2 - \beta^2 J^2 q(1-q)$$

Next we need to determine: $\langle E\rangle$:

we know: $H = \log(Z(\beta)) + \beta\langle E\rangle$

with $\log(Z(\beta)) = -\beta F$

$$\rightarrow \langle E\rangle = \frac{1}{\beta}H + \beta F\cdot\frac{1}{\beta} = \frac{1}{\beta}H + F$$

$$= \frac{-\beta J^2}{4}(q-1)^2 + \frac{1}{\sqrt{2\pi}\,\beta}\int dz\, S(z,\beta) - J_0 m^2 - \beta J^2 q(1-q)$$

$$+ \frac{1}{2}\cancel{\beta} J_0 m^2 - \frac{\beta J^2}{4}(q-1)^2 - \frac{1}{\sqrt{2\pi}\,\beta}\int dz\, S(z,\beta)$$

$$= \frac{-\beta J^2}{2}(q-1)^2 - \frac{1}{2}J_0 m^2 - \beta J^2 q(1-q)$$

$$= -\frac{1}{2}J_0 m^2 - \frac{\beta J^2}{2}\left((q-1)^2 + 2q(1-q)\right)$$

$$= -\frac{1}{2}J_0 m^2 - \frac{\beta J^2}{2}\left(q^2 - 2q + 1 + 2q - 2q^2\right)$$

$$= -\frac{1}{2}J_0 m^2 - \frac{\beta J^2}{2}(1-q^2) = \langle E\rangle$$

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