Advanced Machine Learning week 1

Exercise 27.1

We use the change of variables formula:

$$p(\vec{y}) = p(\vec{x} = f^{-1}(\vec{y}))\left|\det\left(\frac{\partial \vec{f}}{\partial \vec{x}}\right)\right|^{-1}$$

Since $\vec{x}$ is drawn from a uniform distribution on $[0,1]$, we have $p(\vec{x} = f^{-1}(\vec{y})) = 1$. For the entries of the Jacobian we have:

$$\frac{\partial y_1}{\partial x_1} = -\frac{1}{\sqrt{-2\log x_1}} \cdot \frac{1}{x_1} \cdot \cos 2\pi x_2 \qquad \frac{\partial y_1}{\partial x_2} = -\sqrt{-2\log x_1} \cdot \sin 2\pi x_2 \cdot 2\pi$$

$$\frac{\partial y_2}{\partial x_1} = \sqrt{-2\log x_2} \cdot \cos 2\pi x_2 \cdot 2\pi \qquad \frac{\partial y_2}{\partial x_2} = -\frac{1}{\sqrt{-2\log x_1}} \cdot \frac{1}{x_1} \cdot \sin 2\pi x_2$$

For the determinant we get $\frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2} \cdot \frac{\partial y_2}{\partial x_1} = -\frac{2\pi}{x_1}$

Then $\left|\det\left(\frac{\partial \vec{f}}{\partial \vec{x}}\right)\right|^{-1} = \frac{x_1}{2\pi}$. Since $y_1^2 + y_2^2 = -2\log x_1$, we have $x_1 = e^{-\frac{1}{2}(y_1^2+y_2^2)}$.

$$\text{Thus, } p(\vec{y}) = \frac{1}{2\pi} \cdot e^{-\frac{1}{2}(y_1^2+y_2^2)} = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}y_1^2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}y_2^2}$$

$$= N(y_1 | 0, 1)\, N(y_2 | 0, 1)$$

Now, we can generate samples from $N(0,1)$ by drawing x_1 and x_2 ~~un~~ uniform and then apply the transformation to y_1 and y_2, which will then be normally distributed.