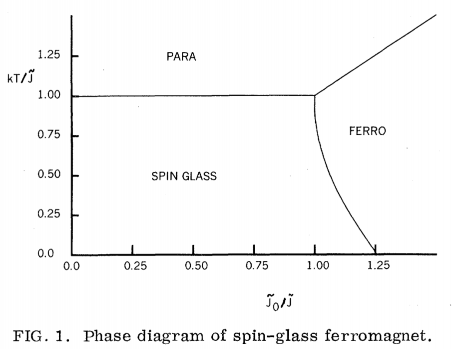

Figure 1: Phase portrait of SK model in the RS approximation.

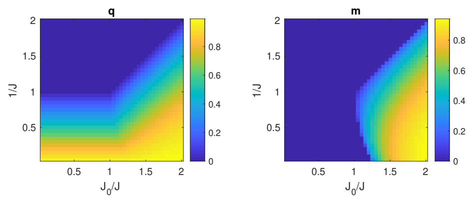

Figure 2: The solution q and m of the RS equations as a function of J_0/J and $1/J$ ($\beta = 1$). $dx = 0.05$.

Phase portrait of the SK model

The phase portrait of the SK model in the replica symmetric approximation is given in fig. 1 [Sherrington and Kirkpatrick, 1975].

1. In this exercise you will reproduce this phase plot, as follows. Without loss of generality we can assume $\beta = 1$. Make a 2d grid equally spaced grid (x, y) with $x = dx, 2dx, \ldots, 2$ with some suitable dx and the same for y. For each x, y define $J = 1/y, J_0 = x/y$. Subsequently solve the RS mean field equations

$$
\begin{aligned}
q &= 1 - \frac{1}{\sqrt{2\pi}} \int dz e^{-z^2/2} \mathrm{sech}^2 \left(\beta J \sqrt{q} z + \beta J_0 m\right) \\
m &= \frac{1}{\sqrt{2\pi}} \int dz e^{-z^2/2} \tanh \left(\beta J \sqrt{q} z + \beta J_0 m\right)
\end{aligned}
$$

   using fixed point iteration. [1] Store the results in matrices $q(x, y)$ and $m(x, y)$ and display as a function of x, y. The solutions $q(J_0/J, 1/J)$ and $m(J_0/J, 1/J)$ should look like fig. 2, with the paramagnetic phase $m = q = 0$, the ferro magnetic phase $m > 0, q > 0$ and the spin glass phase $m = 0, q > 0$.

2. Compute the entropy of the solution in the different phases.

[1] There exist standard numerical routines to compute the definite integral using Gaussian quadrature (in Matlab the routine `integral(fun,a,b)` integrates the function `fun` between a and b).

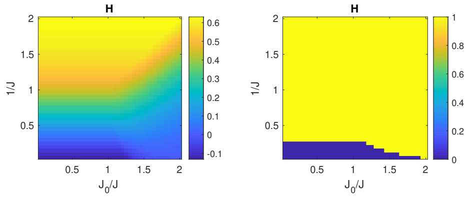

Figure 3: The entropy H of the system in the RS approximation as a function of J_0/J and $1/J$ (left) and the thresholded value ($H > 0$) (right) showing the AT line ($\beta = 1$).

(a) Use the result of MacKay exercise 31.1 to derive an expression for the entropy and the average energy from the replica symmetric free energy. The result is

$$\begin{aligned} H &= -\frac{\beta^2}{4}J^2(q-1)^2 + \frac{1}{\sqrt{2\pi}}\int dz e^{-z^2/2}\log[2\cosh(\beta J_0 m + \beta J\sqrt{q}z)] \\ &\quad -\beta J_0 m^2 - \beta^2 J^2 q(1-q) \\ \langle E\rangle &= -\frac{1}{2}J_0 m^2 - \frac{1}{2}\beta J^2(1-q^2) \end{aligned}$$

(b) Compute the entropy in the phase plot similar to fig. 3. Hint: I encountered problems evaluating numerically the integral $\frac{1}{\sqrt{2\pi}}\int dz e^{-z^2/2}\log[2\cosh\left(\beta J_0 m + \beta J\sqrt{q}z\right)]$. I resolved it by writing $\log 2\cosh(f(z)) = |f(z)| + \log(1 + e^{-2|f(z)|})$ with $f(z) = .\beta J_0 m + \beta J\sqrt{q}z$. This removes the numerical overflow of $e^{|f(z)|}$ for large z and the underflow $e^{-|f(z)|}$ is harmless since it evaluates to zero.

References

[Sherrington and Kirkpatrick, 1975] Sherrington, D. and Kirkpatrick, S. (1975). Solvable model of Spin-Glass. *Physical review letters*, 35:1792–1796.