Stabilizing Grand Cooperation of Machine Scheduling Game via Setup Cost Pricing

Lindong Liu

School of Management; International Institute of Finance

University of Science and Technology of China

Co-authored with Zikang Li (PG Student, USTC)

NWPU-Online, July, 2020

Outline

Preliminaries

Cooperative Game

A **cooperative game** is defined by a pair (V, C):

- A set $V = \{1, 2, \ldots, v\}$ of players, **grand colaition**;
- A **characteristic function** $C(S)$ = the minimum total cost achieved by the cooperation of members in coalition $S \in \mathbb{S} = 2^V \setminus \{\emptyset\}$.

The game requires:

- A **cost allocation** $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_v] \in \mathbb{R}^v$, where α_k = the cost allocated to each player $k \in V$.

Core

Define $\alpha(S) = \sum_{k \in S} \alpha_k$.

A cost allocation $\alpha \in \mathbb{R}^v$ is in the **core** if it satisfies:

- **Budget Balance** Constraint: $\alpha(V) = C(V)$;
- **Coalition Stability** Constraints: $\alpha(S) \leq C(S)$ for each $S \in \mathbb{S}$.

$$\mathrm{Core}(V, C) = \Big\{\alpha : \ \alpha(V) = C(V), \\ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\}, \ \alpha \in \mathbb{R}^v\Big\}.$$

Core

Define $\alpha(S) = \sum_{k \in S} \alpha_k$.

A cost allocation $\alpha \in \mathbb{R}^v$ is in the **core** if it satisfies:

- **Budget Balance** Constraint: $\alpha(V) = C(V)$;
- **Coalition Stability** Constraints: $\alpha(S) \leq C(S)$ for each $S \in \mathbb{S}$.

$$\mathrm{Core}(V, C) = \Big\{ \alpha : \ \alpha(V) = C(V), \\ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\}, \ \alpha \in \mathbb{R}^v \Big\}.$$

However, $\mathrm{Core}(V, c)$ can be empty.

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;
- Penalization: $\alpha(S) \leq C(S) + z$, **least core**;

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \le C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;
- Penalization: $\alpha(S) \le C(S) + z$, **least core**;
- Simul. S & P: $\alpha(V) = C(V) - \theta$ and $\alpha(S) \le C(S) + z$, **PSF**;

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;
- Penalization: $\alpha(S) \leq C(S) + z$, **least core**;
- Simul. S & P: $\alpha(V) = C(V) - \theta$ and $\alpha(S) \leq C(S) + z$, **PSF**;
- Inv. Opt.: Changing c to d such that $\mathrm{Core}(V, D)$ is non-empty.

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;
- Penalization: $\alpha(S) \leq C(S) + z$, **least core**;
- Simul. S & P: $\alpha(V) = C(V) - \theta$ and $\alpha(S) \leq C(S) + z$, **PSF**;
- Inv. Opt.: Changing c to d such that $\mathrm{Core}(V, D)$ is non-empty.

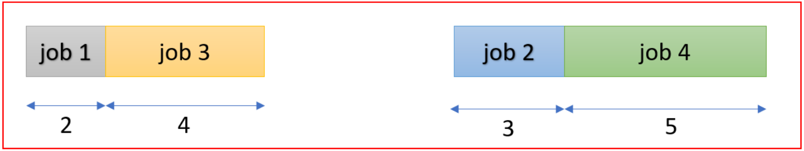
S.	Caprara and Letchford (2010, MP), Liu et al. (2016, IJOC)
P.	Faigle et al. (2001, IJGT), Schulz and Uhan (2010, OR)
P&S	Liu et al. (2018, OR)
Inv. Opt.	Liu et al. (2020, under review)

Illustrative Example

Example: Machine Scheduling Game (MSG)

Game of Parallel Machine Scheduling with Setup Cost:

- Grand coalition: $V = \{1, 2, 3, 4\}$;
- Processing times: $t_1 = 2,\ t_2 = 3,\ t_3 = 4,\ t_4 = 5$;
- Machine setup cost: $t_0 = 9.5$;
- $c(S)$ for $S \in \mathbb{S}$: minimizes the total completion time of jobs in S plus the machine setup cost;
- $\pi(N) = \pi(\{1, 3\}) + \pi(\{2, 4\}) = 38$ (SPT Rule).



Example: Empty Core

Coalitions	Cost
$\{1\}$	11.5
$\{2\}$	12.5
$\{3\}$	13.5
$\{4\}$	14.5
$\{1, 2\}$	16.5
$\{1, 3\}$	17.5
$\{1, 4\}$	18.5
$\{2, 3\}$	19.5
$\{2, 4\}$	20.5
$\{3, 4\}$	22.5
$\{1, 2, 3\}$	25.5
$\{1, 2, 4\}$	26.5
$\{1, 3, 4\}$	28.5
$\{2, 3, 4\}$	31.5
$\{1, 2, 3, 4\}$	38

Example: Empty Core

Coalitions	Cost
$\{1\}$	11.5
$\{2\}$	12.5
$\{3\}$	13.5
$\{4\}$	14.5
$\{1,2\}$	16.5
$\{1,3\}$	17.5
$\{1,4\}$	18.5
$\{2,3\}$	19.5
$\{2,4\}$	20.5
$\{3,4\}$	22.5
$\{1,2,3\}$	25.5
$\{1,2,4\}$	26.5
$\{1,3,4\}$	28.5
$\{2,3,4\}$	31.5
$\{1,2,3,4\}$	38

Optimal Cost Allocation Problem

$$\max \ \left(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\right) = 37.25 < 38$$
$$s.t. \ \ \alpha_1 \leq 11.5, \ \cdots, \ \alpha_4 \leq 14.5,$$
$$\alpha_1 + \alpha_2 \leq 16.5, \ \cdots, \ \alpha_3 + \alpha_4 \leq 22.5,$$
$$\cdots,$$
$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 38.$$

$$\alpha^* = \left[6; 8.75; 10.75; 11.75\right]$$

The minimum subsidy:

$$c(V) - \alpha(V) = 38 - 37.25 = 0.75$$

Example: Pricing Instrument

Increase the setup cost from 9.5 to 10.
For the grand coalition, it only needs one machine.

Setup cost = 9.5 , Two machines.

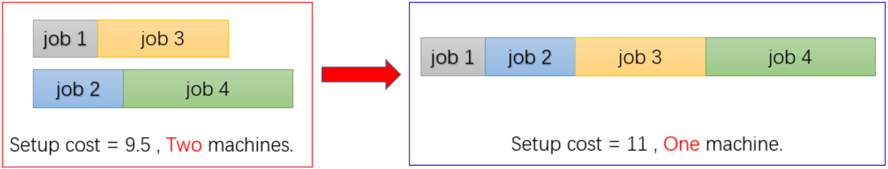
Setup cost = 10 , Two machines.

Setup cost	Increment	Num of Machines	Total pricing	c(V)	α(V)	Subsidy
9.5	0	2	0	38	37.25	0.75
10	0.5	2	1	39	38	1

The total pricing can cover the subsidy, which means the grand coalition can be stabilized by the players themselves.

Example: Pricing Instrument

Increase the setup cost from 9.5 to 11.
For the grand coalition, it only needs one machine.



Setup cost	Increment	Num of Machines	Total pricing	c(V)	α(V)	Subsidy
9.5	0	2	0	38	37.25	0.75
11.14	1.64	1	1.64	41.14	39.5	1.64

The total pricing can cover the subsidy, which means the grand coalition can be stabilized by the players themselves.

Models & Analyses

Problem Definition and Formulation

Definition

An Identical Variable Parallel machine scheduling of Unweighted jobs game (IVPU):

Problem Definition and Formulation

Definition

An Identical Variable Parallel machine scheduling of Unweighted jobs game (IVPU):

- Grand coalition: $V = \{1, 2, \ldots, v\}$;

Problem Definition and Formulation

Definition

An Identical Variable Parallel machine scheduling of Unweighted jobs game (IVPU):

- Grand coalition: $V = \{1, 2, \ldots, v\}$;
- Identical machine: $M = \{1, 2, \ldots, m\}$;

Problem Definition and Formulation

Definition

An Identical Variable Parallel machine scheduling of Unweighted jobs game (IVPU):

- Grand coalition: $V = \{1, 2, \ldots, v\}$;
- Identical machine: $M = \{1, 2, \ldots, m\}$;
- Each machine Price: P and Each job processing time: t_k;

Problem Definition and Formulation

Definition

An Identical Variable Parallel machine scheduling of Unweighted jobs game (IVPU):

- Grand coalition: $V = \{1, 2, \ldots, v\}$;
- Identical machine: $M = \{1, 2, \ldots, m\}$;
- Each machine Price: P and Each job processing time: t_k;
- Characteristic function: $c(S) = \min(\sum_{k \in S} C_k + Pm_S)$,

  where C_k is the completion time of job $k \in S$ and m_S is the number of using machine for the sub-coalition S.

Problem Definition and Formulation

Definition

A cooperative TU game (V,c) is called an IVPU game if it satisfies the following formulations:

Problem Definition and Formulation

Definition

A cooperative TU game (V,c) is called an IVPU game if it satisfies the following formulations:

$$
\begin{aligned}
c(S,P) = & \min \sum_{k\in V}\sum_{j\in O} c_{kj}x_{kj} + P\sum_{k\in S} x_{k1} \\
s.t. \quad & \sum_{j\in O} x_{kj} - y_k^S = 0, \forall k \in V, \\
& \sum_{k\in V} x_{kj} \leq m, \forall j \in O, \\
& x_{kj} \in \{0,1\}, \forall k \in V, \forall j \in O, \\
& y_k^S = 1, k \in S \,; y_k^S = 0, k \notin S.
\end{aligned}
$$

Properties

Definition

- The interval $[P_L(m, S), P_H(m, S)]$ denotes the value range of price when the number of using machines m and the coalition S are given.

Properties

Definition

- The interval $[P_L(m,S), P_H(m,S)]$ denotes the value range of price when the number of using machines m and the coalition S are given.
- For the grand coalition V, define $P_L(v,V) = 0, P_H(1,V) = P^*$. So the practical domain of the price is $[0, P^*]$ which is divided into v nonoverlapping subintervals by the number of using machines.

Properties

Definition

- The interval $[P_L(m,S), P_H(m,S)]$ denotes the value range of price when the number of using machines m and the coalition S are given.
- For the grand coalition V, define $P_L(v,V) = 0, P_H(1,V) = P^*$. So the practical domain of the price is $[0, P^*]$ which is divided into v nonoverlapping subintervals by the number of using machines.
- Denote the right end of every subinterval as $P_i, 1 \leq i \leq v$, where $P_1 = P^*$.

Properties

$$\begin{aligned}\color{red}{\omega(P)} \quad &= \min_\alpha\{c(V, m(V,P)) - \alpha(V) : \\ &\alpha(s) \le c(s, m(s,P)), \forall s \in S, \alpha \in \mathbb{R}^v\};\end{aligned}$$

Theorem 1

$\omega(P)$ is piecewise linear, and convex in price P at each subinterval $[P_L(m,V), P_H(m,V)]$ in $[0, P^*]$.

Lemma 1

$P_i, 2 \le i \le v$ can be obtained by SPT rules.

Theorem 2

$P_1 = P_2 + \cdots + P_n = \sum_{i=2}^n P_i$.

Theorem 3

$\omega(P)$ can be bounded by zero when the number of using machines, m, is larger than $\frac{n}{2}$.

Properties

Theorem 4

When the number of using machines is 1 for the grand coalition, the range of slopes of the line segments in the interval is $\left(-1, -\frac{1}{n-1}\right]$, and the number of breakpoints is $O(v^2)$.

Theorem 5

Define that

$$\begin{aligned} \omega_1(P) = \quad & \min_{\alpha}\{c(V, m(V, P)) - \alpha(V) : \\ & \alpha(s) \le c(s) + P, \forall s \in S, \alpha \in \mathbb{R}^v\} \end{aligned}$$

Then the original problem $\omega(P)$ is equivalent to $\omega_1(P)$ which means that all sub-coalitions only use one machine.

Algorithms & Computations

IPC Algorithm

The Intersection Points Computation Algorithm to Construct $\omega(P)$ Function.

Step 1. Initially, set $I^* = \{P_L, P_H\}$ and $\mathbb{I} = \{[P_L, P_H]\}$.

Step 2. If $\mathbb{I}$ is not empty, update I^* and $\mathbb{I}$ by the following steps:

Step 3. Sort values in I^* by $P_0 < P_1 < \cdots < P_q$, where $P_0 = P_L, P_q = P_H$ and $q = |I^*| - 1$.

Step 4. Select any interval from $\mathbb{I}$, denoted by $[P_{k-1}, P_k]$ with $1 \le k \le q$.

Step 5. Construct two linear function $R_{k-1}(P)$ and $L_k(P)$ so that $R_{k-1}(P)$ passes $(P_{k-1}, \omega(P_{k-1}))$ with a slope equal to a right derivative $K_r^{P_{k-1}}$ of $\omega(P)$ at P_{k-1}, and that $L_k(z)$ passes $(P_k, \omega(P_k))$ with a slope equal to a left derivative $K_l^{P_k}$ of $\omega(P)$ at P_k.

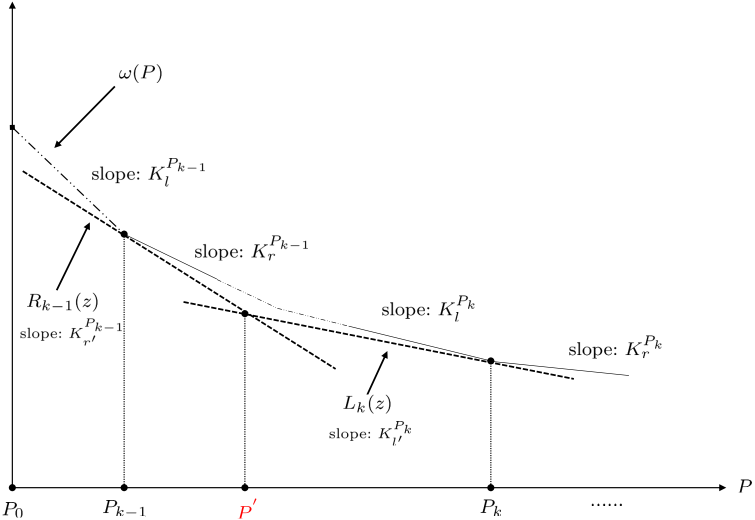
IPC Algorithm

Step 6. If $R_{k-1}(P)$ passes $(P_k, \omega(P_k))$ or $L_k(P)$ passes $(P_{k-1}, \omega(P_{k-1}))$, then update $\mathbb{I}$ by removing $[P_{k-1}, P_k]$. Otherwise, $R_{k-1}(P)$ and $L_k(P)$ must have a unique intersection point at $P = P'$ for some $P' \in (P_{k-1}, P_k)$. Update I^* by adding $P^{'}$, and update $\mathbb{I}$ by removing $[P_{k-1}, P_k]$, adding $[P_l, P']$ and $[P', P_r]$.

Step 7. Go to step 2.

Step 8. Return a piecewise linear function by connecting points $(P, \omega(P))$ for all $P \in I^*$.

IPC Algorithm



CP Algorithm

The Cutting Plane Algorithm to compute $\omega(P)$ for a given P.

Step 1. Let $\mathbb{S}' \subseteq \mathbb{S} \setminus \{N\}$ indicates a restricted coalition set, which includes some initial coalitions, e.g.,$\{1\}, \{2\}, \ldots, \{v\}$.

Step 2. Find an optimal solution $\bar{\alpha}(\,\cdot\,, P)$ to LP $\tau(P)$:

$$\max_{\alpha \in \mathbb{R}^n} \left\{ \alpha(N, P) : \alpha(s, P) \leq c(s) + P, \ \text{for all } s \in \mathbb{S}' \right\}.$$

Step 3. Find an optimal solution s^* to the separation problem:

$$\delta = \min \left\{ c(s) + P - \bar{\alpha}(s, z) : \forall s \in \mathbb{S} \setminus \{N\} \right\}.$$

Step 4. If $\delta < 0$, then add s^* to $\mathbb{S}'$, and go to step 2; otherwise, return $\omega(P) = c(N) - \bar{\alpha}(N, P)$.

DP Algorithm

The Dynamic Programming Algorithm to solve the separation problem.

Step 1. Initially, let $D(k,u)$ indicate the minimum objective value of the restricted problem of separation problem, where $k \in \{1,2,\dots,v\}$ and $u \in \{0,1,\dots,v\}$.

Step 2. Given the initial conditions $D(1,0) = P$ and $D(1,1) = t_1 - \beta_1 + P$. The boundary conditions are $D(k,u) =$ if $u > k$, for all $k \in V$.

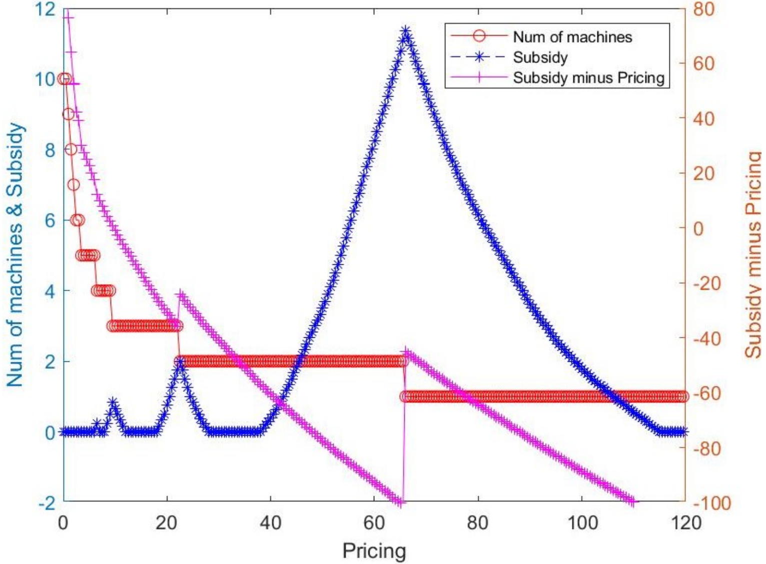
Step 3. Given the recursion:

$$D(k,u) = \min \begin{cases} D(k-1,u), \text{for the case when } s^* \text{ does not contain } k, \\ D(k-1,u-1) + ut_k - \alpha_k, \text{for the case when } s^* \text{ contains } k. \end{cases}$$

Step 4. Obtain the optimal objective value of separation problem by $\delta_{AIPU} = \min\{D(v,u) : u \in \{1,2,\dots,v-1\}\}$. return δ_{AIPU}.

Computational Results

Processing time:[1, 1.5, 2, 2.5, 3.5, 4, 4, 6.5, 6.5, 7]



Extension & Generalization

Machine Scheduling Game with Weighted Jobs

Definition

A Machine Scheduling Game with Weighted Jobs:

- Each job $k \in V$ has a processing time and a weight denoted by t_k, ω_k, respectively.
- $c(S)$ can be obtained by assuming that
  $t_1/\omega_1 \le t_2/\omega_2 \le \ldots \le t_v/\omega_v$.
- $P_m = c_0(V, m) - c_0(V, m-1), 2 \le m \le v$.
- The properties of $\omega(P)$ and the corresponding Algorithms are similar to unweighted one.

Pricing in General IM Games

Definition

The General Integer Minimization Games:

- *ILP* : $c(S, m(S,P)) = \min_x\{cx + Pm(x) : Ax \geq By^s + D, \tilde{\alpha}x \leq m, x \in \mathbb{Z}^{t\times 1}\}$
- Decompose $c(S, m(S,P))$ into $c_0(S, m(S)) + Pm$.
- Analyse the properties of $c_0(S, m(S))$.

Conclusions

- ⋆ Cooperative Game Theory:
  - New Instrument for Stabilization via Setup cost Pricing.
- ⋆ Scheduling Problem:
  - Parallel Machine Scheduling with Setup Cost.
- ⋆ Models, Solution Methods and Applications:
  - Several ILP formulations;
  - Cutting Plane to solve the seperation problem;
  - Implementations on the MSGW game.

The End

Thank you!