Stabilizing Grand Cooperation of Machine Scheduling Game via Setup Cost Pricing

Lindong Liu

School of Management; International Institute of Finance

University of Science and Technology of China

Co-authored with Zikang Li (PG Student, USTC)

NWPU-Online, July, 2020

Outline

PRELIMINARIES

Cooperative Game

A **cooperative game** is defined by a pair (V, C):

- A set $V = \{1, 2, \ldots, v\}$ of players, **grand colaition**;
- A **characteristic function** $C(S) =$ the minimum total cost achieved by the cooperation of members in coalition $S \in \mathbb{S} = 2^V \setminus \{\emptyset\}$.

The game requires:

- A **cost allocation** $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_v] \in \mathbb{R}^v$, where $\alpha_k =$ the cost allocated to each player $k \in V$.

Core

Define $\alpha(S) = \sum_{k \in S} \alpha_k$.

A cost allocation $\alpha \in \mathbb{R}^v$ is in the **core** if it satisfies:

- **Budget Balance** Constraint: $\alpha(V) = C(V)$;
- **Coalition Stability** Constraints: $\alpha(S) \leq C(S)$ for each $S \in \mathbb{S}$.

$$\mathrm{Core}(V, C) = \Big\{ \alpha : \ \alpha(V) = C(V), \\ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\}, \ \alpha \in \mathbb{R}^v \Big\}.$$

Core

Define $\alpha(S) = \sum_{k \in S} \alpha_k$.

A cost allocation $\alpha \in \mathbb{R}^v$ is in the **core** if it satisfies:

- **Budget Balance** Constraint: $\alpha(V) = C(V)$;
- **Coalition Stability** Constraints: $\alpha(S) \leq C(S)$ for each $S \in \mathbb{S}$.

$$\mathrm{Core}(V, C) = \Big\{ \alpha : \ \alpha(V) = C(V), \\ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\}, \ \alpha \in \mathbb{R}^v \Big\}.$$

However, $\mathrm{Core}(V, c)$ can be empty.

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;
- Penalization: $\alpha(S) \leq C(S) + z$, **least core**;

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;
- Penalization: $\alpha(S) \leq C(S) + z$, **least core**;
- Simul. S & P: $\alpha(V) = C(V) - \theta$ and $\alpha(S) \leq C(S) + z$, **PSF**;

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;
- Penalization: $\alpha(S) \leq C(S) + z$, **least core**;
- Simul. S & P: $\alpha(V) = C(V) - \theta$ and $\alpha(S) \leq C(S) + z$, **PSF**;
- Inv. Opt.: Changing c to d such that $\mathrm{Core}(V, D)$ is non-empty.

Existing Instruments

$$\text{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;
- Penalization: $\alpha(S) \leq C(S) + z$, **least core**;
- Simul. S & P: $\alpha(V) = C(V) - \theta$ and $\alpha(S) \leq C(S) + z$, **PSF**;
- Inv. Opt.: Changing c to d such that $\text{Core}(V, D)$ is non-empty.

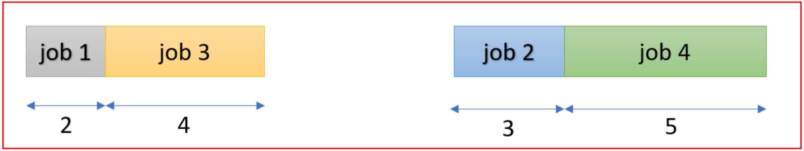
S.	Caprara and Letchford (2010, MP), Liu et al. (2016, IJOC)
P.	Faigle et al. (2001, IJGT), Schulz and Uhan (2010, OR)
P&S	Liu et al. (2018, OR)
Inv. Opt.	Liu et al. (2020, under review)

Illustrative Example

Example: Machine Scheduling Game (MSG)

Game of Parallel Machine Scheduling with Setup Cost:

- Grand coalition: $V = \{1, 2, 3, 4\}$;
- Processing times: $t_1 = 2$, $t_2 = 3$, $t_3 = 4$, $t_4 = 5$;
- Machine setup cost: $t_0 = 9.5$;
- $c(S)$ for $S \in \mathbb{S}$: minimizes the total completion time of jobs in S plus the machine setup cost;
- $\pi(N) = \pi(\{1, 3\}) + \pi(\{2, 4\}) = 38$ (SPT Rule).



Example: Empty Core

Coalitions	Cost
$\{1\}$	11.5
$\{2\}$	12.5
$\{3\}$	13.5
$\{4\}$	14.5
$\{1, 2\}$	16.5
$\{1, 3\}$	17.5
$\{1, 4\}$	18.5
$\{2, 3\}$	19.5
$\{2, 4\}$	20.5
$\{3, 4\}$	22.5
$\{1, 2, 3\}$	25.5
$\{1, 2, 4\}$	26.5
$\{1, 3, 4\}$	28.5
$\{2, 3, 4\}$	31.5
$\{1, 2, 3, 4\}$	38

Example: Empty Core

Coalitions	Cost
$\{1\}$	11.5
$\{2\}$	12.5
$\{3\}$	13.5
$\{4\}$	14.5
$\{1,2\}$	16.5
$\{1,3\}$	17.5
$\{1,4\}$	18.5
$\{2,3\}$	19.5
$\{2,4\}$	20.5
$\{3,4\}$	22.5
$\{1,2,3\}$	25.5
$\{1,2,4\}$	26.5
$\{1,3,4\}$	28.5
$\{2,3,4\}$	31.5
$\{1,2,3,4\}$	38

Optimal Cost Allocation Problem

$$
\begin{aligned}
\max \ & \left(\alpha_1+\alpha_2+\alpha_3+\alpha_4\right) = 37.25 < 38 \\
\textit{s.t.} \ & \alpha_1 \leq 11.5, \ \cdots, \ \alpha_4 \leq 14.5, \\
& \alpha_1+\alpha_2 \leq 16.5, \ \cdots, \ \alpha_3+\alpha_4 \leq 22.5, \\
& \cdots, \\
& \alpha_1+\alpha_2+\alpha_3+\alpha_4 \leq 38.
\end{aligned}
$$

$$\alpha^* = \left[6; 8.75; 10.75; 11.75\right]$$

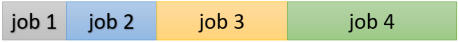
The minimum subsidy:

$c(V) - \alpha(V) = 38 - 37.25 = 0.75$

Example: Pricing Instrument

Increase the setup cost from 9.5 to 16.
For the grand coalition, it only needs one machine.

Setup cost = 9.5 , Two machines.



Setup cost = 16 , One machine.

Setup cost	Increment	Num of Machines	Total pricing	c(V)	α(V)	Subsidy	Empty core
9.5	0	2	0	38	37.25	0.75	Yes
10	0.5	2	1	39	38	1	Yes
16	6.5	1	6.5	46	46	0	No

The total pricing can cover the subsidy, which means the grand coalition can be stabilized by the players themselves.

Models & Analyses

Problem Definition and Formulation

Definition

An Identical Variable Parallel machine scheduling of Unweighted jobs game (IVPU):

Problem Definition and Formulation

Definition

An Identical Variable Parallel machine scheduling of Unweighted jobs game (IVPU):

- Grand coalition: $V = \{1, 2, \ldots, v\}$;

Problem Definition and Formulation

Definition

An Identical Variable Parallel machine scheduling of Unweighted jobs game (IVPU):

- Grand coalition: $V = \{1, 2, \ldots, v\}$;
- Identical machine: $M = \{1, 2, \ldots, m\}$;

Problem Definition and Formulation

Definition

An Identical Variable Parallel machine scheduling of Unweighted jobs game (IVPU):

- Grand coalition: $V = \{1, 2, \ldots, v\}$;
- Identical machine: $M = \{1, 2, \ldots, m\}$;
- Each machine Price: P and Each job processing time: t_k;

Problem Definition and Formulation

Definition

An Identical Variable Parallel machine scheduling of Unweighted jobs game (IVPU):

- Grand coalition: $V = \{1, 2, \ldots, v\}$;
- Identical machine: $M = \{1, 2, \ldots, m\}$;
- Each machine Price: P and Each job processing time: t_k;
- Characteristic function: $c(S) = \min(\sum_{k \in S} C_k + Pm_S)$,

  where C_k is the completion time of job $k \in S$ and m_S is the number of using machine for the sub-coalition S.

Problem Definition and Formulation

Definition

A cooperative TU game (V,c) is called an IVPU game if it satisfies the following formulations:

Problem Definition and Formulation

Definition

A cooperative TU game (V,c) is called an IVPU game if it satisfies the following formulations:

$$
\begin{aligned}
c(S,P) = &\min \sum_{k\in V}\sum_{j\in O} c_{kj}x_{kj} + P\sum_{k\in S} x_{k1} \\
s.t. \quad & \sum_{j\in O} x_{kj} - y_k^S = 0, \forall k\in V, \\
& \sum_{k\in V} x_{kj} \le m, \forall j\in O, \\
& x_{kj}\in\{0,1\}, \forall k\in V, \forall j\in O, \\
& y_k^S = 1, k\in S\,; y_k^S = 0, k\notin S.
\end{aligned}
$$

Properties

Definition

- The interval $[P_L(m, S), P_H(m, S)]$ denotes the value range of price when the number of using machines m and the coalition S are given.

Properties

Definition

- The interval $[P_L(m,S), P_H(m,S)]$ denotes the value range of price when the number of using machines m and the coalition S are given.
- For the grand coalition V, define $P_L(v,V) = 0, P_H(1,V) = P^*$. So the practical domain of the price is $[0, P^*]$ which is divided into v nonoverlapping subintervals by the number of using machines.

Properties

Definition

- The interval $[P_L(m,S), P_H(m,S)]$ denotes the value range of price when the number of using machines m and the coalition S are given.
- For the grand coalition V, define $P_L(v,V) = 0, P_H(1,V) = P^*$. So the practical domain of the price is $[0, P^*]$ which is divided into v nonoverlapping subintervals by the number of using machines.
- Denote the right end of every subinterval as $P_i, 1 \leq i \leq v$, where $P_1 = P^*$.

Properties

$$\omega(P) = \min_{\alpha}\{c(V, m(V, P)) - \alpha(V) : \alpha(s) \leq c(s, m(s, P)), \forall s \in S, \alpha \in \mathbb{R}^v\};$$

Theorem 1

$\omega(P)$ is piecewise linear, and convex in price P at each subinterval $[P_L(m, V), P_H(m, V)]$ in $[0, P^*]$.

Lemma 1

$P_i, 2 \leq i \leq v$ can be obtained by SPT rules.

Theorem 2

$P_1 = P_2 + \cdots + P_n = \sum_{i=2}^{n} P_i$.

Theorem 3

$\omega(P)$ can be bounded by zero when the number of using machines, m, is larger than $\frac{n}{2}$.

Properties

- When the number of using machines is 1 for the grand coalition, the range of slopes of the line segments in the interval is $\left(-1, -\frac{1}{n-1}\right]$, and the number of break-points is $O(v^2)$;

Properties

- When the number of using machines is 1 for the grand coalition, the range of slopes of the line segments in the interval is $\left(-1, -\frac{1}{n-1}\right]$, and the number of breakpoints is $O(v^2)$;
- Define that

$$\begin{aligned}\omega_1(P) = \quad & \min_{\alpha}\{c(V, m(V, P)) - \alpha(V) : \\ & \alpha(s) \leq c(s) + P, \forall s \in S, \alpha \in \mathbb{R}^v\}\end{aligned}$$

Then the original problem $\omega(P)$ is equivalent to $\omega_1(P)$ which means that all sub-coalitions only use one machine.

Algorithms & Computations

IPC Algorithm

The Intersection Points Computation(IPC) Algorithm to Construct $\omega(P)$ Function.

Step 1. Initially, set $I^* = \{P_L, P_H\}$ and $\mathbb{I} = \{[P_L, P_H]\}$.

Step 2. If $\mathbb{I}$ is not empty, update I^* and $\mathbb{I}$ by the following steps:

Step 3. Sort values in I^* by $P_0 < P_1 < \cdots < P_q$, where $P_0 = P_L, P_q = P_H$ and $q = |I^*| - 1$.

Step 4. Select any interval from $\mathbb{I}$, denoted by $[P_{k-1}, P_k]$ with $1 \le k \le q$.

Step 5. Construct two linear function $R_{k-1}(P)$ and $L_k(P)$ so that $R_{k-1}(P)$ passes $(P_{k-1}, \omega(P_{k-1}))$ with a slope equal to a right derivative $K_r^{P_{k-1}}$ of $\omega(P)$ at P_{k-1}, and that $L_k(z)$ passes $(P_k, \omega(P_k))$ with a slope equal to a left derivative $K_l^{P_k}$ of $\omega(P)$ at P_k.

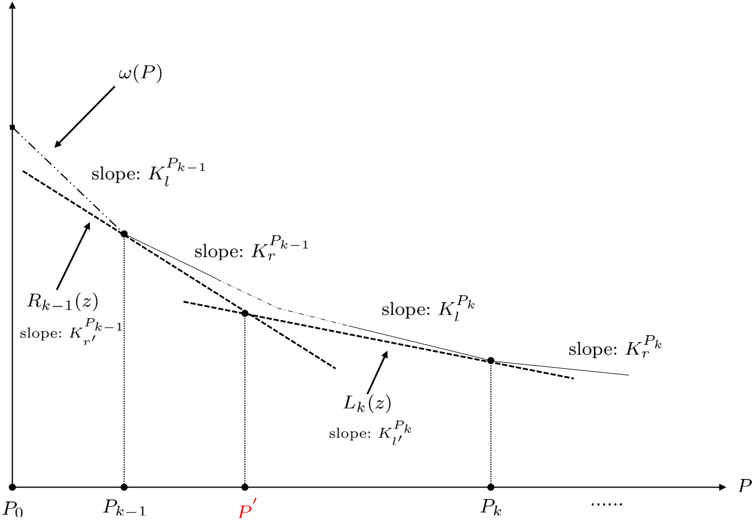
IPC Algorithm

Step 6. If $R_{k-1}(P)$ passes $(P_k, \omega(P_k))$ or $L_k(P)$ passes $(P_{k-1}, \omega(P_{k-1}))$, then update $\mathbb{I}$ by removing $[P_{k-1}, P_k]$. Otherwise, $R_{k-1}(P)$ and $L_k(P)$ must have a unique intersection point at $P = P'$ for some $P' \in (P_{k-1}, P_k)$. Update I^* by adding $P^{'}$, and update $\mathbb{I}$ by removing $[P_{k-1}, P_k]$, adding $[P_l, P']$ and $[P', P_r]$.

Step 7. Go to step 2.

Step 8. Return a piecewise linear function by connecting points $(P, \omega(P))$ for all $P \in I^*$.

IPC Algorithm



CP Algorithm

The Cutting Plane(CP) Algorithm to compute $\omega(P)$ for a given P.

Step 1. Let $\mathbb{S}' \subseteq \mathbb{S} \setminus \{N\}$ indicates a restricted coalition set, which includes some initial coalitions, e.g.,$\{1\}, \{2\}, \ldots, \{v\}$.

Step 2. Find an optimal solution $\bar{\alpha}(\,\cdot\,, P)$ to LP $\tau(P)$:

$$\max_{\alpha \in \mathbb{R}^n} \left\{ \alpha(N, P) : \alpha(s, P) \leq c(s) + P, \text{ for all } s \in \mathbb{S}' \right\}.$$

Step 3. Find an optimal solution s^* to the separation problem:

$$\delta = \min \left\{ c(s) + P - \bar{\alpha}(s, z) : \forall s \in \mathbb{S} \setminus \{N\} \right\}.$$

Step 4. If $\delta < 0$, then add s^* to $\mathbb{S}'$, and go to step 2; otherwise, return $\omega(P) = c(N) - \bar{\alpha}(N, P)$.

DP Algorithm

The Dynamic Programming(DP) Algorithm to solve the separation problem.

Step 1. Initially, let $D(k, u)$ indicate the minimum objective value of the restricted problem of separation problem, where $k \in \{1, 2, \ldots, v\}$ and $u \in \{0, 1, \ldots, v\}$.

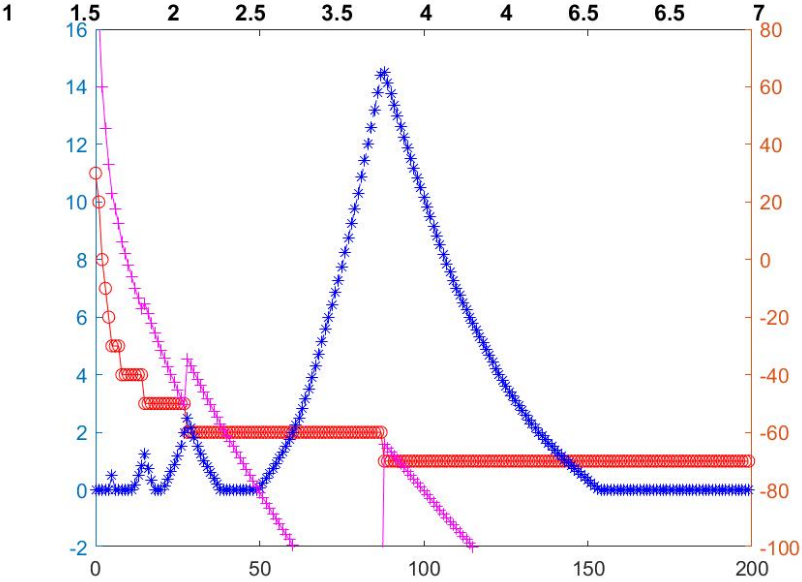
Step 2. Given the initial conditions $D(1, 0) = P$ and $D(1, 1) = t_1 - \beta_1 + P$. The boundary conditions are $D(k, u) =$ if $u > k$, for all $k \in V$.

Step 3. Given the recursion:

$$D(k, u) = \min \begin{cases} D(k-1, u), \text{for the case when } s^* \text{ does not contain } k, \\ D(k-1, u-1) + ut_k - \alpha_k, \text{for the case when } s^* \text{ contains } k. \end{cases}$$

Step 4. Obtain the optimal objective value of separation problem by $\delta_{AIPU} = \min\{D(v, u) : u \in \{1, 2, \ldots, v-1\}\}$. return δ_{AIPU}.

Computational Results



Extension & Generalization

Machine Scheduling Game with Weighted Jobs

- Each job $k \in V$ has a processing time and a weight denoted by t_k, ω_k, respectively.
- $c(S)$ can be obtained by assuming that
  $t_1/\omega_1 \leq t_2/\omega_2 \leq \ldots \leq t_v/\omega_v$.
- $P_m = c_0(V, m) - c_0(V, m-1), 2 \leq m \leq v$.
- $\omega(P)$ is piecewise linear, and convex in price P at each subinterval.
- The separation problem can be solved in polynomial time.

Pricing in General IM Games

- $ILP:$ $c(S, m(S, P)) = \min_x\{cx + Pm(x) : Ax \geq By^s + D, \tilde{\alpha}x \leq m, x \in \mathbb{Z}^{t\times 1}\}$
- Decompose $c(S, m(S, P))$ into $c_0(S, m(S)) + Pm$.
- $c_0(V, m) - c_0(V, m-1) > 0 \Leftrightarrow P_m > 0, m = 2, \ldots, v.$
- $c_0(V, m) - c_0(V, m+1) < c_0(V, m-1) - c_0(V, m) \Leftrightarrow P_m < P_{m+1}, m = 2, 3, \ldots, v-1.$
- $c_0(S_1, m-1) - c_0(S_1, m) \geq c_0(S_1, m) - c_0(S_1, m+1) \quad m = 2, \ldots, v-1.$
- $c_0(S_1, m-1) - c_0(S_1, m) \leq c_0(S_2, m-1) - c_0(S_2, m)$

m=2,...,v, *where* $S_1 \subset S_2$.

Conclusions

- ⋆ Cooperative Game Theory:
  - New Instrument for Stabilization via Setup cost Pricing.
- ⋆ Scheduling Problem:
  - Parallel Machine Scheduling with Setup Cost.
- ⋆ Models, Solution Methods and Applications:
  - Several ILP formulations;
  - Cutting Plane to solve the seperation problem;
  - Implementations on the MSGW game.

The End

Thank you!