Stabilizing Grand Cooperation of Machine Scheduling Game via Setup Cost Pricing

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Outline

PRELIMINARIES

Cooperative Game

A **cooperative game** is defined by a pair (V, C):

- A set $V = \{1, 2, \ldots, v\}$ of players, **grand colaition**;
- A **characteristic function** $C(S) =$ the minimum total cost achieved by the cooperation of members in coalition $S \in \mathbb{S} = 2^V \setminus \{\emptyset\}$.

The game requires:

- A **cost allocation** $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_v] \in \mathbb{R}^v$, where $\alpha_k =$ the cost allocated to each player $k \in V$.

Core

Define $\alpha(S) = \sum_{k \in S} \alpha_k$.

A cost allocation $\alpha \in \mathbb{R}^v$ is in the **core** if it satisfies:

- **Budget Balance** Constraint: $\alpha(V) = C(V)$;
- **Coalition Stability** Constraints: $\alpha(S) \leq C(S)$ for each $S \in \mathbb{S}$.

$$\mathrm{Core}(V, C) = \Big\{ \alpha : \ \alpha(V) = C(V), \\ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\}, \ \alpha \in \mathbb{R}^v \Big\}.$$

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$$\left. \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\}, \ \alpha \in \mathbb{R}^v \right\}.$$

However, $\mathrm{Core}(V, c)$ can be empty.

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

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- Simul. S & P: $\alpha(V) = C(V) - \theta$ and $\alpha(S) \leq C(S) + z$, **PSF**;

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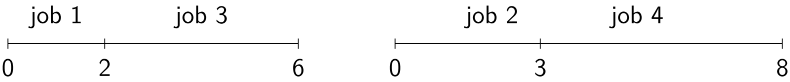
S.	Caprara and Letchford (2010, MP), Liu et al. (2016, IJOC)
P.	Faigle et al. (2001, IJGT), Schulz and Uhan (2010, OR)
P&S	Liu et al. (2018, OR)
Inv. Opt.	Liu et al. (2020, under review)

Illustrative Example

Example: Machine Scheduling Game (MSG)

Game of Parallel Machine Scheduling with Setup Cost:

- Grand coalition: $V = \{1, 2, 3, 4\}$;
- Processing times: $t_1 = 2$, $t_2 = 3$, $t_3 = 4$, $t_4 = 5$;
- Machine setup cost: $t_0 = 9.5$;
- $c(S)$ for $S \in \mathbb{S}$: minimizes the total completion time of jobs in S plus the machine setup cost;
- $\pi(N) = \pi(\{1, 3\}) + \pi(\{2, 4\}) = 38$ (SPT Rule).



Example: Empty Core

Coalitions	Cost
$\{1\}$	11.5
$\{2\}$	12.5
$\{3\}$	13.5
$\{4\}$	14.5
$\{1,2\}$	16.5
$\{1,3\}$	17.5
$\{1,4\}$	18.5
$\{2,3\}$	19.5
$\{2,4\}$	20.5
$\{3,4\}$	22.5
$\{1,2,3\}$	25.5
$\{1,2,4\}$	26.5
$\{1,3,4\}$	28.5
$\{2,3,4\}$	31.5
$\{1,2,3,4\}$	38

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Optimal Cost Allocation Problem

$$\begin{aligned} \max \ & \left(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\right) = 37.25 < 38 \\ s.t. \ & \alpha_1 \leq 11.5, \ \cdots, \ \alpha_4 \leq 14.5, \\ & \alpha_1 + \alpha_2 \leq 16.5, \ \cdots, \ \alpha_3 + \alpha_4 \leq 22.5, \\ & \cdots, \\ & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 38. \end{aligned}$$

$$\alpha^* = \left[6; 8.75; 10.75; 11.75\right]$$

Models & Analyses

Problem Definition and Formulation

Definition

An Identical Variable Parallel machine scheduling of Unweighted jobs game (IVPU):

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- Identical machine: $M = \{1, 2, \ldots, m\}$;
- Each machine Price: P and Each job processing time: t_k;
- Characteristic function: $c(S) = \min(\sum_{k \in S} C_k + Pm_S)$,

  where C_k is the completion time of job $k \in S$ and m_S is the number of using machine for the sub-coalition S.

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$$
\begin{aligned}
c(S,P) = & \min \sum_{k\in V}\sum_{j\in O} c_{kj}x_{kj} + P\sum_{k\in s} x_{k1} \\
s.t. \quad & \sum_{j\in O} x_{kj} - y_k^s = 0, \forall k \in V, \\
& \sum_{k\in V} x_{kj} \leq m, \forall j \in O, \\
& x_{kj} \in \{0,1\}, \forall k \in V, \forall j \in O, \\
& y_k^s = 1, k \in s \,; y_k^s = 0, k \notin s.
\end{aligned}
$$

Properties

Definition

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- For the grand coalition V, define $P_L(v, V) = 0, P_H(1, V) = P^*$. So the practical domain of the price is $[0, P^*]$ which is divided into v nonoverlapping subintervals by the number of using machines.
- Denote the right end of every subinterval as $P_i, 1 \leq i \leq v$, where $P_1 = P^*$.

Properties

$$\begin{aligned} \omega(P) \quad &= \min_{\alpha}\{c(V, m(V,P)) - \alpha(V) : \\ &\alpha(s) \le c(s, m(s,P)), \forall s \in S, \alpha \in \mathbb{R}^{v}\}; \end{aligned}$$

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- $P_1 = P_2 + \cdots + P_n = \sum_{i=2}^{n} P_i$.

Properties

- When the number of using machines is 1 for the grand coalition, the range of slopes of the line segments in the interval is $\left(-1, -\frac{1}{n-1}\right]$, and the number of break-points is $O(v^2)$;

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- When the number of using machines is 1 for the grand coalition, the range of slopes of the line segments in the interval is $\left(-1, -\frac{1}{n-1}\right]$, and the number of break-points is $O(v^2)$;
- Define that

$$\omega_1(P) = \min_\alpha\{c(V, m(V, P)) - \alpha(V) : \alpha(s) \leq c(s) + P, \forall s \in S, \alpha \in \mathbb{R}^v\}$$

Then the original problem $\omega(P)$ is equivalent to $\omega_1(P)$ which means that all sub-coalitions only use one machine.

Algorithms & Computations

The Intersection Points Computation(IPC) Algorithm to Construct $\omega(P)$ Function.

Step 1. Initially, set $I^* = \{P_L, P_H\}$ and $\mathbb{I} = \{[P_L, P_H]\}$.

Step 2. If $\mathbb{I}$ is not empty, update I^* and $\mathbb{I}$ by the following steps:

Step 3. Sort values in I^* by $P_0 < P_1 < \cdots < P_q$, where $P_0 = P_L, P_q = P_H$ and $q = |I^*| - 1$.

Step 4. Select any interval from $\mathbb{I}$, denoted by $[P_{k-1}, P_k]$ with $1 \leq k \leq q$.

Step 5. Construct two linear function $R_{k-1}(P)$ and $L_k(P)$ so that $R_{k-1}(P)$ passes $(P_{k-1}, \omega(P_{k-1}))$ with a slope equal to a right derivative $K_r^{P_{k-1}}$ of $\omega(P)$ at P_{k-1}, and that $L_k(z)$ passes $(P_k, \omega(P_k))$ with a slope equal to a left derivative $K_l^{P_k}$ of $\omega(P)$ at P_k.

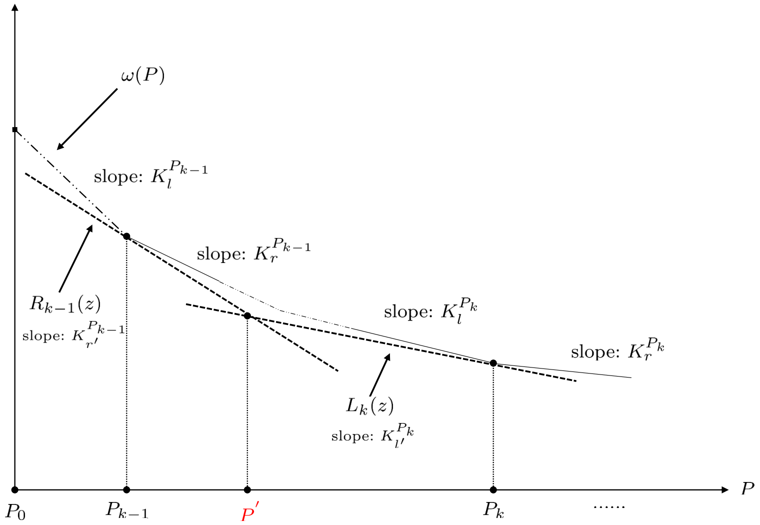
IPC Algorithm

Step 6. If $R_{k-1}(P)$ passes $(P_k, \omega(P_k))$ or $L_k(P)$ passes $(P_{k-1}, \omega(P_{k-1}))$, then update $\mathbb{I}$ by removing $[P_{k-1}, P_k]$. Otherwise, $R_{k-1}(P)$ and $L_k(P)$ must have a unique intersection point at $P = P'$ for some $P' \in (P_{k-1}, P_k)$. Update I^* by adding $P^{'}$, and update $\mathbb{I}$ by removing $[P_{k-1}, P_k]$, adding $[P_l, P']$ and $[P', P_r]$.

Step 7. Go to step 2.

Step 8. Return a piecewise linear function by connecting points $(P, \omega(P))$ for all $P \in I^*$.

IPC Algorithm



CP Algorithm

The Cutting Plane(CP) Algorithm to compute $\omega(P)$ for a given P.

Step 1. Let $\mathbb{S}' \subseteq \mathbb{S} \setminus \{N\}$ indicates a restricted coalition set, which includes some initial coalitions, e.g.,$\{1\}, \{2\}, \ldots, \{v\}$.

Step 2. Find an optimal solution $\bar{\alpha}(\,\cdot\,, P)$ to LP $\tau(P)$:

$$\max_{\alpha \in \mathbb{R}^n} \left\{ \alpha(N, P) : \alpha(s, P) \le c(s) + P, \text{ for all } s \in \mathbb{S}' \right\}.$$

Step 3. Find an optimal solution s^* to the separation problem:

$$\delta = \min \left\{ c(s) + P - \bar{\alpha}(s, z) : \forall s \in \mathbb{S} \setminus \{N\} \right\}.$$

Step 4. If $\delta < 0$, then add s^* to $\mathbb{S}'$, and go to step 2; otherwise, return $\omega(P) = c(N) - \bar{\alpha}(N, P)$.

DP Algorithm

The Dynamic Programming(DP) Algorithm to Calculate c(S).

Step 1. Initially, let $D(k,u)$ indicate the minimum objective value of the restricted problem of separation problem, where $k \in \{1,2,\ldots,v\}$ and $u \in \{0,1,\ldots,v\}$.

Step 2. Given the initial conditions $D(1,0) = P$ and $D(1,1) = t_1 - \beta_1 + P$. The boundary conditions are $D(k,u) =$ if $u > k$, for all $k \in V$.

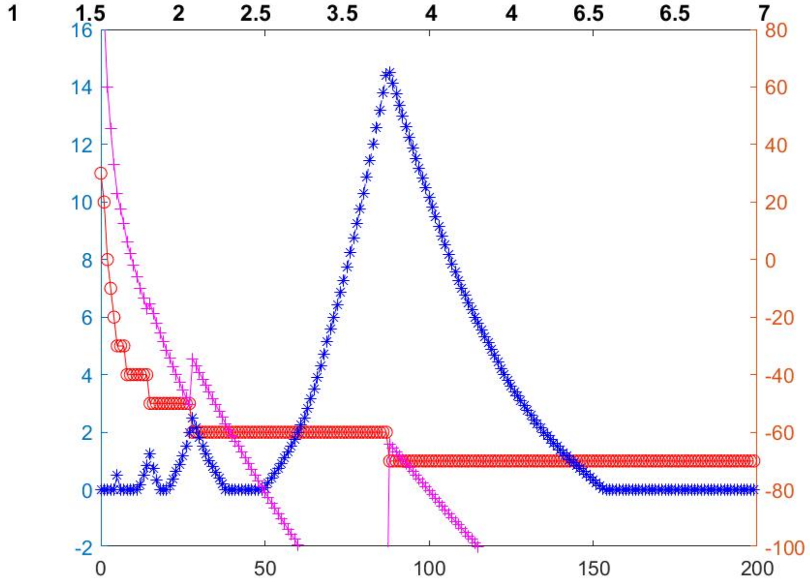
Step 3. Given the recursion:

$$D(k,u) = \min \begin{cases} D(k-1,u), \text{for the case when } s^* \text{ does not contain } k, \\ D(k-1,u-1) + ut_k - \alpha_k, \text{for the case when } s^* \text{ contains } k. \end{cases}$$

Step 4. Obtain the optimal objective value of separation problem by $\delta_{AIPU} = \min\{D(v,u) : u \in \{1,2,\ldots,v-1\}\}$. return δ_{AIPU}.

Computational Results

Image



1
1.5
2
2.5
3.5
4
4
6.5
6.5
7
16
14
12
10
8
6
4
2
0
-2
80
60
40
20
0
-20
-40
-60
-80
-100
0
50
100
150
200

Extension & Generalization

Machine Scheduling Game with Weighted Jobs

- $\omega(P)$ is piecewise linear, and convex in price P at each subinterval;
- $\omega(P)$ can be bounded by zero when the number of using machines, m, is larger than $\frac{n}{2}$.
- $P_i, 2 \leq i \leq v$ can be obtained by SPT rules.
- $P_1 = P_2 + \cdots + P_n = \sum_{i=2}^{n} P_i$.

Pricing in General IM Games

- $c_0(V, m) - c_0(V, m-1) > 0 \Leftrightarrow P_m > 0, m = 2, \ldots, n.$;
-
$$\begin{aligned} & c_0(V, m) - c_0(V, m+1) < c_0(V, m-1) - c_0(V, m) \\ \Leftrightarrow \quad & P_m < P_{m+1}, m = 2, 3, \ldots, n. \end{aligned}$$

Conclusions

- ⋆ Cooperative Game Theory:
  - New Instrument for Stabilization via Setup cost Pricing.
- ⋆ Scheduling Problem:
  - Constrained Inverse Optimization Problem.
- ⋆ Models, Solution Methods and Applications:
  - Several equivalent LP formulations;
  - Feasibility analyses & How to handle infeasibility;
  - Implementations on MSGW game.

The End

Thank you!