Inverse Optimization of Stabilizing Grand Coalitions via Cost Vector Adjustment

Lindong Liu

School of Management
University of Science and Technology of China

Co-authored with Xiangtong Qi (HKUST) and Zhou Xu (HKPU)

WHU, Dec. 10th, 2018

Outline

PRELIMINARIES

Cooperative Game

A cooperative game is defined by a pair (V, C):

- A set $V = \{1, 2, \ldots, v\}$ of players, **grand colaition**;
- A **characteristic function** $C(S)$ = the minimum total cost achieved by the cooperation of members in coalition $S \in \mathbb{S} = 2^V \setminus \{\emptyset\}$.

The game requires:

- A cost allocation $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_v] \in \mathbb{R}^v$, where α_k = the cost allocated to each player $k \in V$.

Core

Define $\alpha(S) = \sum_{k \in S} \alpha_k$.

A cost allocation $\alpha \in \mathbb{R}^V$ is in the **core** if it satisfies:

- Budget Balance Constraint: $\alpha(V) = C(V)$;
- Coalition Stability Constraints: $\alpha(S) \leq C(S)$ for each $S \in \mathbb{S}$.

$$\mathrm{Core}(V, C) = \Big\{ \alpha : \ \alpha(V) = C(V), \\ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\}, \ \alpha \in \mathbb{R}^V \Big\}.$$

Core

Define $\alpha(S) = \sum_{k \in S} \alpha_k$.

A cost allocation $\alpha \in \mathbb{R}^V$ is in the **core** if it satisfies:

- Budget Balance Constraint: $\alpha(V) = C(V)$;
- Coalition Stability Constraints: $\alpha(S) \leq C(S)$ for each $S \in \mathbb{S}$.

$$\mathrm{Core}(V, C) = \Big\{\alpha : \ \alpha(V) = C(V), \\ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\}, \ \alpha \in \mathbb{R}^V \Big\}.$$

However, $\mathrm{Core}(V, c)$ can be empty.

Instruments to Stabilize Grand Coalitions

Filling

Cutting

Simul. Filling & Cutting

Stretching

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \le C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;
- Penalization: $\alpha(S) \le C(S) + z$, **least core**;

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \le C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;
- Penalization: $\alpha(S) \le C(S) + z$, **least core**;
- Simul. S & P: $\alpha(V) = C(V) - \theta$ and $\alpha(S) \le C(S) + z$, **PSF**.

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \le C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;
- Penalization: $\alpha(S) \le C(S) + z$, **least core**;
- Simul. S & P: $\alpha(V) = C(V) - \theta$ and $\alpha(S) \le C(S) + z$, **PSF**.

S.	Caprara and Letchford (2010, MP), Liu et al. (2016, IJOC)
P.	Faigle et al. (2001, IJGT), Schulz and Uhan (2010, OR)
P&S	Liu et al. (2018, OR)

Instruments to Stabilize Grand Coalitions

Unbalanced Game

Subsidization

Penalization

Simultaneously Subsidization & Penalization

Instruments to Stabilize Grand Coalitions

Unbalanced Game

Subsidization

Penalization

Simultaneously Subsidization & Penalization

Cost Vector Adjustment

Motivation & Example

Braess's Paradox

4000 cars in total

Braess's Paradox

4000 cars in total

$$t = 0$$

$$START \rightarrow A \rightarrow B \rightarrow END$$

$$4000/100 + 4000/100 = 80min$$

Braess's Paradox

4000 cars in total

$$t = 0$$

$$START \rightarrow A \rightarrow B \rightarrow END$$

$$4000/100 + 4000/100 = 80min$$

$$t = \infty$$

$$START \rightarrow A \rightarrow END$$

$$START \rightarrow B \rightarrow END$$

$$A = B = 2000$$

$$2000/100 + 45 = 65min$$

Cost Vector Adjustment on IM Games

$C(S)$: need to solve an optimization problem, not given.

Integer Minimization Games: for each coalition $S \in \mathbb{S}$, an incidence vector $y^S \in \{0,1\}^v$, with $y_k^S = 1$ if $k \in S$, and with $y_k^S = 0$ otherwise, for all $k \in V$, such that

$$C(S) = \min_x \{cx : Ax \geq By^S + E,\ x \in \mathbb{Z}^t\}.$$

Examples: machine scheduling games, facility location games, travelling salesman games, etc.

Cost Vector Adjustment on IM Games

$C(S)$: need to solve an optimization problem, not given.

Integer Minimization Games: for each coalition $S \in \mathbb{S}$, an incidence vector $y^S \in \{0,1\}^v$, with $y_k^S = 1$ if $k \in S$, and with $y_k^S = 0$ otherwise, for all $k \in V$, such that

$$C(S) = \min_x \{cx : Ax \geq By^S + E,\ x \in \mathbb{Z}^t\}.$$

Examples: machine scheduling games, facility location games, travelling salesman games, etc.

Cost Vector Adjustment: $c \to d$ and $C(S) \to D(S)$

An Illustrative Example on UFL Game

Social Optimum $C(V)$:

$$10 + 10 + 3 + 3 + 2 + 1 = 29$$

Optimal Cost Allocation Problem:

$$\begin{gathered} \max \ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ s.t. \ \ \alpha_1 \leq 13, \ \cdots, \ \alpha_4 \leq 11, \\ \alpha_1 + \alpha_2 \leq 16, \ \cdots, \ \alpha_3 + \alpha_4 \leq 17, \\ \cdots, \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 29. \end{gathered}$$

Optimal Cost Allocation:

$$7.5 + 6.5 + 8.5 + 4 = 26.5$$

An Illustrative Example on UFL Game

Social Optimum $C(V)$: 29

Maximum Shared Cost $\alpha(V)$: 26.5

$(c_{22} = 1 \rightarrow 100)$:
Optimal Cost Allocation

$3 + 11 + 13 + 2 = 29$ (*Stabilized*)

$(c_{14} = 2 \rightarrow 100)$:
Optimal Cost Allocation

$8 + 8 + 7 + 5 = 28$ (*Not stabilized*)

Models & Analyses

Properties of an IM Game

IM Game (V, C): $C(S) = \min_x\{cx : Ax \geq By^S + E,\ x \in \mathbb{Z}^t\}$

Parameters	Properties	Explanations
$B \geq \mathbf{0}$	Monotonicity	$C(S \cup \{k\}) \geq C(S)$ for all $S \in \mathbb{S},\ k \in V$: $S \cup \{k\} \in \mathbb{S}$;
$B = \mathbf{0}$	Coalitional-Independence	$C(S)$ for all $S \in \mathbb{S}$ are identical;
$E \geq \mathbf{0}$	Subadditivity	$C(S_1 \cup S_2) \leq C(S_1) + C(S_2)$ for all $S_1, S_2 \in \mathbb{S}$: $S_1 \cup S_2 \in \mathbb{S},\ S_1 \cap S_2 = \emptyset$;
$E = \mathbf{0}$	Homogeneity (Assignability)	$\alpha^*(V) \leq \min_x \left\{cx : Ax \geq B\mathbf{1} + E,\ x \in \mathbb{R}^q_+\right\}$;
$A = \mathbf{II}$, $B = \mathbf{I}$, $E = \mathbf{0}$	Complete Assignability	$\alpha^*(V) = \min_x \left\{cx : Ax \geq B\mathbf{1} + E,\ x \in \mathbb{R}^q_+\right\}$.

$\alpha^*(V) = \max_\alpha \left\{\alpha(V) : \alpha(S) \leq C(S),\ \forall S \in \mathbb{S}\right\}$; $\mathbf{0}$, $\mathbf{1}$, $\mathbf{II}$, $\mathbf{I}$: all-zeros, all-ones, binary, identity matrices

GCSP via CVA Instrument

Definition

Grand Coalition Stabilization Problem (GCSP) via Cost Vector Adjustment (CVA):

$$c \to d, \text{ such that } \textbf{OCB}$$

GCSP via CVA Instrument

Definition

Grand Coalition Stabilization Problem (GCSP) via Cost Vector Adjustment (CVA):

$$c \rightarrow d, \text{ such that } \textbf{OCB}$$

- Optimality: an initial feasible solution x^0 is optimal to $D(V)$;

GCSP via CVA Instrument

Definition

Grand Coalition Stabilization Problem (GCSP) via Cost Vector Adjustment (CVA):

$$c \to d, \text{ such that } \textbf{OCB}$$

- Optimality: an initial feasible solution x^0 is optimal to $D(V)$;
- Consistency: the social optimum is consistent $C(V) = D(V)$;

GCSP via CVA Instrument

Definition

Grand Coalition Stabilization Problem (GCSP) via Cost Vector Adjustment (CVA):

$$c \to d, \text{ such that } \textbf{OCB}$$

- Optimality: an initial feasible solution x^0 is optimal to $D(V)$;
- Consistency: the social optimum is consistent $C(V) = D(V)$;
- Balancedness: the updated IM game (V, D) is balanced.

GCSP via CVA Instrument

Definition

Grand Coalition Stabilization Problem (GCSP) via Cost Vector Adjustment (CVA):

$$c \to d, \text{ such that } \mathbf{OCB}$$

- Optimality: an initial feasible solution x^0 is optimal to $D(V)$;
- Consistency: the social optimum is consistent $C(V) = D(V)$;
- Balancedness: the updated IM game (V, D) is balanced.

$$\min_{\delta} \left\{ f(\delta) : \ d \in \mathbb{O}, \ D(V) = C(V), \ dx^0 = D(V), \ \delta = d - c \right\},$$

where $f(\delta) = \omega \times |\delta|$ (L_1 norm); $\mathbb{O}$ is the Balanced Cost Vector Set.

GCSP via CVA $\equiv$ CIOP

Constrained Inverse Optimization Problem (CIOP)

$$\min\left\{\omega\times(\tau+\eta): d\in\mathbb{O},\ D(V)=C(V),\ dx^0=D(V),\ \tau-\eta=d-c,\right.$$
$$\left. d\in\mathbb{R}^q,\ \tau\in\mathbb{R}^q_+,\ \eta\in\mathbb{R}^q_+\right\}.$$

- Only Optimality: Inverse Optimal Solution Problem;
- Only Consistency: Inverse Optimal Value Problem;
- Only Balancedness: Optimal Cost Allocation Problem.

Feasibility Analyses

$$Q^{xy} = \left\{(x,y):\ Ax \geq By + E,\ y = y(S) \text{ for some } S \in \mathbb{S},\ x \in \mathbb{Z}^t,\ y \in \{0,1\}^v\right\},$$

$$\mathbb{C}^x = \text{proj}_x\left(\text{cone } Q^{xy} \cap \left\{(x,y) \in \mathbb{R}^{q+v} : y = \mathbf{1}\right\}\right) = \left\{x:\ \mathbb{A}x \geq \mathbb{B}\right\}.$$

Lemma

EQUIVALENCE 1: The CIOP is equivalent to the following LP.

$$\begin{aligned} \min\ \ & \omega \times \left(\tau + \eta\right)^T \\ s.t.\ \ & dx \geq cx^*,\ \forall x \in \mathbb{C}^x, \\ & dx^0 = cx^*, \\ & d - c = \tau - \eta, \text{ and } d \in \mathbb{R}^q,\ \tau,\ \eta \in \mathbb{R}^q_+. \end{aligned}$$

Feasibility Analyses

Theorem

Feasibility – Sufficient and Necessary Conditions

Theorem

Feasibility – Necessary Conditions

Theorem

Feasibility – Sufficient Conditions

Feasibility Analyses

$$C(V) = \min_x \{cx : Ax \geq B\mathbf{1} + E,\ x \in \mathbb{Z}^t\}$$

Corollary

Two special sufficient conditions for the feasibility of the CIOP *are as follows: in the ILP of* $C(V)$

- *there exists some coalitional independent constraint* $ax = e$ *with* $e > 0$ *and* $cx^* > 0$*; or,*
- *there exists some homogeneous constraint* $ax = b\mathbf{1}$.

How to Handle an Infeasible CIOP

Motivated by the Instrument of Subsidization

$$\min \Big\{ \omega \times (\tau + \eta) : d \in \mathbb{O}(\theta),\ D(V) = C(V),\ dx^0 = D(V),\ \tau - \eta = d - c,\\ d \in \mathbb{R}^q,\ \tau \in \mathbb{R}^q_+,\ \eta \in \mathbb{R}^q_+ \Big\}.$$

- $\mathbb{O}(\theta_1) \subseteq \mathbb{O}(\theta_2)$ for all $\theta_1 \leq \theta_2$;
- $lim_{\theta \to \infty} \mathbb{O}(\theta) = \mathbb{R}^q$, the CIOP is feasible when θ is large;
- balancedness $\iff$ $\theta^* \leq 0$ ($f(\delta) = 0$ for all $\theta \geq \theta^*$).

The CIOP is NP-hard in general

How to Handle an Infeasible CIOP

Motivated by the Instrument of Penalization

$$\hat{C}(S) = \min_{x,\sigma} \left\{ cx + 0 \times \sigma : Ax \geq By(S) + E,\ \sigma \geq 1 - |S|/|V|,\ \sigma \in \mathbb{Z}_+,\ x \in \mathbb{Z}_+^q \right\}$$

$$\min \left\{ \hat{\omega} \times \left(\hat{\tau} + \hat{\eta} \right) : \hat{d} \in \hat{\mathbb{O}}(\theta),\ \hat{D}(V) = \hat{C}(V),\ \hat{d}x^0 = \hat{D}(V),\ \hat{\tau} - \hat{\eta} = \hat{d} - \hat{c}, \right.$$
$$\left. \hat{d} \in \mathbb{R}^q,\ \hat{\tau} \in \mathbb{R}_+^q,\ \hat{\eta} \in \mathbb{R}_+^q \right\}.$$

- the resulting CIOP is feasible;
- least core value $\iff$ $f(\delta)$, when $\omega_c = \infty$.

The CIOP is NP-hard in general regardless of the computational complexity of checking the core non-emptiness

Methods & Applications

Computational Complexity

Lemma

For an IM game (V, C), the corresponding CIOP is NP-hard to solve.

Lemma

For an IM game (V, C), the corresponding CIOP is NP-hard to solve, even when checking the non-emptiness of $\mathrm{Core}(V, C)$ is in polynomial time.

Column Generation Method

Lemma

EQUIVALENCE 2: The CIOP is equivalent to the following LP.

$$
\begin{gathered}
\min \ \omega \times \left(\tau + \eta\right)^{T} \\
s.t. \ \ \alpha \mathbf{1} = cx^{*}, \\
\alpha y \leq dx, \ \forall (x, y) \in Q^{xy}, \\
dx^{0} = cx^{*}, \\
d - c = \tau - \eta, \ \text{and} \ d \in \mathbb{R}^{q}, \ \tau \in \mathbb{R}^{q}_{+}, \ \eta \in \mathbb{R}^{q}_{+}.
\end{gathered}
$$

Column Generation Method

Step 1. Let $Q^{'xy}$ be a subset of Q^{xy}, which includes some initial feasible solutions in Q^{xy}.

Step 2. Find an optimal solution $\left[\tau'; \eta'; d'; \alpha'\right]$ to a relaxed LP of (1), where Q^{xy} is replaced by $Q^{'xy}$.

Step 3. Find an optimal solution $\left[x'; y'\right]$ to separation problem $\epsilon = \min\left\{d'x - \alpha' y : \forall (x, y) \in Q^{xy}\right\}$.

Step 4. If $\epsilon < 0$, then add $\left[x'; y'\right]$ to $Q^{'xy}$, go to step 2; otherwise, return (i) the updated cost coefficients d'; and (ii) the total minimum perturbation $\omega \times \left(\tau + \eta\right)^T$.

Generate a lower bound when serving as a heuristic

Cone Optimization Method

Lemma

EQUIVALENCE 3: The CIOP is equivalent to the following LP.

$$\min_{\tau,\eta,d,\rho} \Big\{\omega \times \big(\tau+\eta\big)^T : \mathbb{B}^T\rho \geq cx^*;\ \mathbb{A}^T\rho = d^T;\ dx^0 = cx^*;$$
$$\tau-\eta = d-c;\ \text{and } d \in \mathbb{R}^q,\ \tau,\eta,\rho \in \mathbb{R}^q_+\Big\},$$

where $\mathbb{C}^x = \{x:\ \mathbb{A}x \geq \mathbb{B}\}$ and ρ is the associating dual variable.

Cone Optimization Method

Step 1. Derive an expression of $\mathbb{C}^x$, denoted as $\left\{x : \mathbb{A}x \geq \mathbb{B}\right\}$, with finite number of constraints for IM game (V, C).

Step 2. Find an optimal solution $\left[\tau; \eta^*; d^*; \rho^*\right]$ to the CIOP.

Step 3. Return (i) the optimal cost coefficients d^*; and (ii) the total minimum adjustment cost $\omega \times \left(\tau^* + \eta^*\right)^T$.

Generate an upper bound when serving as a heuristic

CG Method on Weighted Matching Game

Definition

A WMP game is defined as (V, C_{WMP}) with players being V and characteristic function C_{WMP} determined by the following ILP,

$$-C_{\mathrm{WMP}}(S) = \min_{x} \Bigg\{ \sum_{e\in E} -w_e x_e : \sum_{e\in\varphi(k)} x_e \leq y_k(S), \\ \sum_{e\in E} x_e \geq 1,\ x_e \in \{0,1\},\ \forall k \in V,\ \forall e \in E \Bigg\}.$$

CG Method on Weighted Matching Game

Separation problem:

$$\epsilon^* = \min_{(x,y)\in Q^{xy}_{\mathrm{WMP}}} w'x - \alpha' y = \min_{S\in\mathbb{S},\ |S|\geq 2} \left\{ D'_{\mathrm{WMP}}(S) - \alpha'(S) \right\}.$$

Solve the separation problem by letting $w''_e = w'_e - \alpha'_i - \alpha'_k$ for each $e : (i,k) \in E$, and then finding the maximum weighted matching with respect to w'' by some polynomial algorithms (see Gabow 1990).

Lemma

The CIOP for a WMP game (V, C_{WMP}) is feasible and can be solved in polynomial time by the column generation method.

CO Method on Uncapacitated FL Game

Definition

A UFL game (V, C_{UFL}) is defined with the players being the customers in V and the characteristic function $C_{\mathrm{UFL}}(S)$ determined by

$$C_{\mathrm{UFL}}(S) = \min_{v,u} \Big\{ \sum_{i \in M} f_i v_i + \sum_{i \in M} \sum_{k \in V} r_{ik} u_{ik} : \sum_{i \in M} u_{ik} = y_k(S), \\ 0 \leq u_{ik} \leq v_i \leq 1, v_i, u_{ik} \in \{0,1\}, \ \forall i \in M, \forall k \in V. \Big\}$$

Lemma

The CIOP for a UFL game (V, C_{UFL}) is feasible and can be solved in polynomial time by the cone optimization method.

CO Method on Uncapacitated FL Game

$$\min \sum_{i\in M} \omega_i^f(\tau_i^f + \eta_i^f) + \sum_{i\in M}\sum_{k\in V} \omega_{ik}^r(\tau_{ik}^r + \eta_{ik}^r)$$

$$s.t. \quad \sum_{k\in V} \pi_k \geq \sum_{i\in M} f_i v_i^* + \sum_{i\in M}\sum_{k\in V} r_{ik} u_{ik}^*,$$

$$\sum_{k\in V} \varrho_{ik} = \bar{f}_i, \ \forall i \in M,$$

$$\pi_k - \varrho_{ik} + \varsigma_{ik} = \bar{r}_{ik}, \ \forall i \in M, \ \forall k \in V,$$

$$\sum_{i\in M} \bar{f}_i v_i^0 + \sum_{i\in M}\sum_{k\in V} \bar{r}_{ik} u_{ik}^0 = \sum_{i\in M} f_i v_i^* + \sum_{i\in M}\sum_{k\in V} r_{ik} u_{ik}^*,$$

$$\tau_i^f - \eta_i^f = \bar{f}_i - f_i, \ \forall i \in M \text{ and } \tau_{ik}^r - \eta_{ik}^r = \bar{r}_{ik} - r_{ik}, \ \forall i \in M, \ \forall k \in V,$$

$$\tau^f, \eta^f, \pi_k \geq 0, \ \forall k \in V, \ \tau^r, \eta^r, \varrho_{ik}, \varsigma_{ik} \geq 0, \ \forall i \in M, \ \forall k \in V.$$

CO Method on Uncapacitated FL Game

Table: Computational Results of the CIOP for the UFL Game

Problem size	Number of blocked arcs			
	Max.	Min.	Avg.	Avg.(%)
30×30	12	2	3.46	0.427
40×40	16	2	7.91	0.475
50×50	19	3	9.92	0.389
60×60	24	6	14.13	0.401

Conclusions

- ⋆ Cooperative Game Theory:
  - New Instrument for Stabilization via Cost Adjustment.
- ⋆ Inverse Problem:
  - Constrained Inverse Optimization Problem.
- ⋆ Models, Solution Methods and Applications:
  - Several equivalent LP formulations;
  - Feasibility analyses & How to handle infeasibility;
  - Implementations on WMG and UFL games.

The End

Thank you!