# Stabilizing Grand Cooperation in Unbalanced Cooperative Games

Xiangtong Qi

Department of Industrial Engineering and Decision Analytics,

The Hong Kong University of Science and Technology

Joint works with **Lindong Liu** (USTC), **Zhou Xu** (HKPU)

MISTA, Ningbo, Dec. 2019

# Introduction

# Cooperation

**Cooperation** is everywhere, from cells in your body, to people in a city, and partners in **a scheduling problem**.

# Cooperation

**Cooperation** is everywhere, from cells in your body, to people in a city, and partners in **a scheduling problem**.

- centralized decision making, to minimize the total cost (for social optimum);
- enforced by an external party, to minimize negative externalities (e.g. the number of machines used, etc).

# Cooperation

Is every player beneficial from the grand cooperation?

Is the grand cooperation stable?

# Cooperation

Is every player beneficial from the grand cooperation?

Is the grand cooperation stable?

Not Always.

If the grand cooperation is not stable, how to stabilize it?

# Preliminaries

# Cooperative Game

A **cooperative game** is defined by a pair $(N, \pi)$:

- A set $N = \{1, 2, \ldots, n\}$ of players, **grand colaition**;
- A **characteristic function** $\pi(S)$ = the minimum total cost achieved by the cooperation of members in coalition $S \in \mathbb{S} = 2^N \setminus \{\emptyset\}$.

The game requires:

- A **cost allocation** $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n] \in \mathbb{R}^n$, where $\alpha_j$ = the cost allocated to each player $j \in N$.

# Integer Minimization Games

$\pi(S)$: need to solve an optimization problem, not given.

**Integer Minimization (IM) Games:**

For each coalition $S \in \mathbb{S}$, an incidence vector $y^S \in \{0,1\}^n$, with $y^S_j = 1$ if $j \in S$, and with $y^S_j = 0$ otherwise, for all $j \in N$, such that

$$\pi(S) = \min_x \{cx : Ax \geq By^S + E,\ x \in \mathbb{Z}^q\}.$$

# Core

Denote $\alpha(S) = \sum_{j \in S} \alpha_j$.

A cost allocation $\alpha \in \mathbb{R}^n$ is in the **core** if it satisfies:

- **Budget Balance** Constraint: $\alpha(N) = \pi(N)$;
- **Coalition Stability** Constraints: $\alpha(S) \leq \pi(S)$ for each $S \in \mathbb{S}$.

$$\mathrm{Core}(N, \pi) = \Big\{ \alpha : \ \alpha(N) = \pi(N), \\ \alpha(S) \leq \pi(S), \ \forall S \in \mathbb{S} \setminus \{N\}, \ \alpha \in \mathbb{R}^n \Big\}.$$

# Core

Denote $\alpha(S) = \sum_{j \in S} \alpha_j$.

A cost allocation $\alpha \in \mathbb{R}^n$ is in the **core** if it satisfies:

- **Budget Balance** Constraint: $\alpha(N) = \pi(N)$;
- **Coalition Stability** Constraints: $\alpha(S) \le \pi(S)$ for each $S \in \mathbb{S}$.

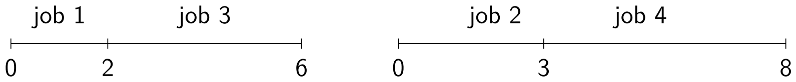
$$\mathrm{Core}(N, \pi) = \Big\{ \alpha : \ \alpha(N) = \pi(N), \ \alpha(S) \le \pi(S), \ \forall S \in \mathbb{S} \setminus \{N\}, \ \alpha \in \mathbb{R}^n \Big\}.$$

However, $\mathrm{Core}(N, \pi)$ can be empty.

# Example: Machine Scheduling Game (MSG)

Game of Single Machine Scheduling with Setup Cost:

- Grand coalition: $N = \{1, 2, 3, 4\}$;
- Processing times: $t_1 = 2$, $t_2 = 3$, $t_3 = 4$, $t_4 = 5$;
- Machine setup cost: $t_0 = 9.5$;
- $\pi(S)$ for $S \in \mathbb{S}$: minimizes the total completion time of jobs in $S$ plus the machine setup cost;
- $\pi(N) = \pi(\{1,3\}) + \pi(\{2,4\}) = 38$ (SPT Rule).



# Example: Empty Core

| Coalitions | Cost |
|---|---|
| $\{1\}$ | 11.5 |
| $\{2\}$ | 12.5 |
| $\{3\}$ | 13.5 |
| $\{4\}$ | 14.5 |
| $\{1,2\}$ | 16.5 |
| $\{1,3\}$ | 17.5 |
| $\{1,4\}$ | 18.5 |
| $\{2,3\}$ | 19.5 |
| $\{2,4\}$ | 20.5 |
| $\{3,4\}$ | 22.5 |
| $\{1,2,3\}$ | 25.5 |
| $\{1,2,4\}$ | 26.5 |
| $\{1,3,4\}$ | 28.5 |
| $\{2,3,4\}$ | 31.5 |
| $\{1,2,3,4\}$ | 38 |

# Example: Empty Core

| Coalitions | Cost |
|---|---|
| $\{1\}$ | 11.5 |
| $\{2\}$ | 12.5 |
| $\{3\}$ | 13.5 |
| $\{4\}$ | 14.5 |
| $\{1,2\}$ | 16.5 |
| $\{1,3\}$ | 17.5 |
| $\{1,4\}$ | 18.5 |
| $\{2,3\}$ | 19.5 |
| $\{2,4\}$ | 20.5 |
| $\{3,4\}$ | 22.5 |
| $\{1,2,3\}$ | 25.5 |
| $\{1,2,4\}$ | 26.5 |
| $\{1,3,4\}$ | 28.5 |
| $\{2,3,4\}$ | 31.5 |
| $\{1,2,3,4\}$ | 38 |

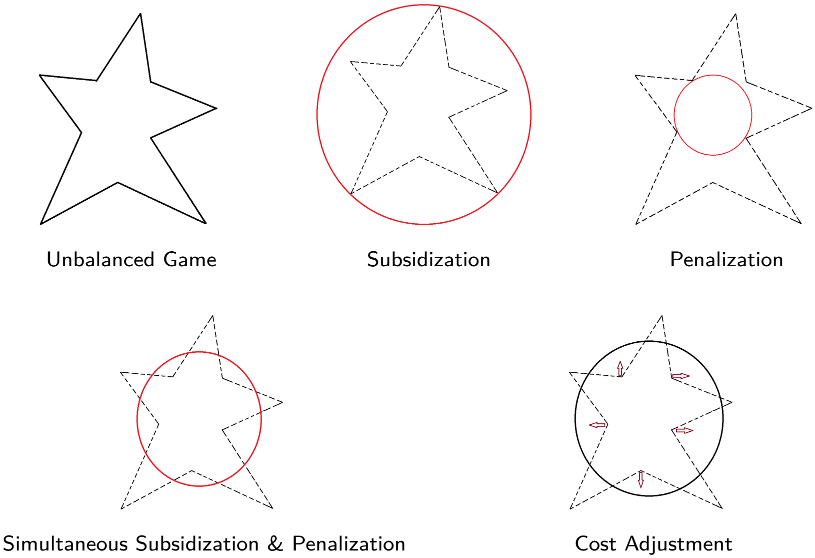
Optimal Cost Allocation Problem

$$\begin{aligned} \max \ & \left(\alpha_1+\alpha_2+\alpha_3+\alpha_4\right) = 37.25 < 38 \\ s.t. \ & \alpha_1 \le 11.5, \ \cdots, \ \alpha_4 \le 14.5, \\ & \alpha_1+\alpha_2 \le 16.5, \ \cdots, \ \alpha_3+\alpha_4 \le 22.5, \\ & \cdots, \\ & \alpha_1+\alpha_2+\alpha_3+\alpha_4 \le 38. \end{aligned}$$

$$\alpha^* = \left[6; 8.75; 10.75; 11.75\right]$$

# Instruments

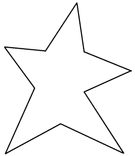
# Instruments to Stabilize Grand Coalitions



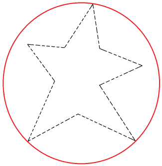
# Our Studies

| | |
|---|---|
| **S.** | Caprara and Letchford (2010, MP), Liu et al. (2016, IJOC) |
| **P.** | Schulz and Uhan (2010, OR; 2013, DO) |
| **P.&S.** | Liu et al. (2018, OR) |
| **C.A.** | Liu et al. (2019, under review) |

# 1. Instrument of Subsidization



Unbalanced Game



Subsidization

# Instrument 1: LRB Cost Allocation, 2016, IJOC

## Instrument of Subsidization ($\epsilon$-core)

$$\mathrm{Core}(N,\pi) = \left\{\alpha : \ \alpha(N) = \pi(N), \ \alpha(S) \le \pi(S), \ \forall S \in \mathbb{S} \setminus \{N\}, \ \alpha \in \mathbb{R}^n\right\}.$$

Relax Budget Balanced constraint: $\alpha(N) = \pi(N)$

- Minimum Subsidy to stabilize the grand coalition:

$$\omega^* = \min\left\{\pi(N) - \alpha(N) : \alpha(S) \le \pi(S), \ \forall S \in \mathbb{S}\right\}.$$

# Instrument 1: LRB Cost Allocation, 2016, IJOC

## Instrument of Subsidization ($\epsilon$-core)

$$\mathrm{Core}(N, \pi) = \left\{ \alpha : \; \alpha(N) = \pi(N), \; \alpha(S) \leq \pi(S), \; \forall S \in \mathbb{S} \setminus \{N\}, \; \alpha \in \mathbb{R}^n \right\}.$$

Relax Budget Balanced constraint: $\alpha(N) = \pi(N)$

- Minimum Subsidy to stabilize the grand coalition:

$$\omega^* = \min \left\{ \pi(N) - \alpha(N) : \alpha(S) \leq \pi(S), \; \forall S \in \mathbb{S} \right\}.$$

Minimum Subsidy needed for MSG Example:

$38 - 37.25 = 0.75$

# Instrument 1: LRB Cost Allocation, 2016, IJOC

## Motivation: Advantages of Lagrangian Relaxation over Linear Relaxation

**Optimal Cost Allocation Problem (OCAP)**:

$$\max_{\alpha} \left\{ \alpha(N) : \alpha(S) \leq \pi(S),\ \forall S \in \mathbb{S} \right\}.$$

**All existing methods are based on Linear Relaxation**

Lagrangian Relaxation v.s. Linear Relaxation

- Applicable in non-linear case;
- Generally more effective;
- New insights via decomposition;
- Adequate studies on LR for ILP.

# Instrument 1: LRB Cost Allocation, 2016, IJOC

## Research Question

**Compute a cost allocation** based on **Lagrangian Relaxation** technique such that

- the cost assigned to each coalition **does not exceed the total cost incurred** if they deviate from grand coalition;
- the total assigned cost is **as large as possible**.

# Instrument 1: LRB Cost Allocation, 2016, IJOC

## Lagrangian Relaxation and Decomposition

$$\pi(S) = \min_x \left\{ f(x) : Ax \geq B\gamma^s + E,\ A'x \geq B'\gamma^s + E',\ x \in \{0,1\}^q \right\},$$

$$\pi^\lambda_{LR}(S) = \min_x \{ f(x) - \lambda(A'x - B'\gamma^s - E') : Ax \geq B\gamma^s + E,\ x \in \{0,1\}^q \}.$$

- Lagrangian sub-characteristic functions 1 and 2:

$$\pi^\lambda_{LR1}(S) = \lambda B'\gamma^s + \lambda E',\ \forall S \in \mathbb{S}.$$

$$\pi^\lambda_{LR2}(S) = \min_x \{ f(x) - \lambda A'x : Ax \geq B\gamma^s + E,\ x \in \{0,1\}^q \},\ \forall S \in \mathbb{S}.$$

### Theorem

Given any Lagrangian multiplier $\lambda$, if $\alpha^\lambda_{LR1}$ and $\alpha^\lambda_{LR2}$ are feasible cost allocations for sub-games $(N, \pi^\lambda_{LR1})$ and $(N, \pi^\lambda_{LR2})$, respectively,

then $\alpha^\lambda_{LR} = \alpha^\lambda_{LR1} + \alpha^\lambda_{LR2}$ is a feasible cost allocation for game $(N, \pi)$.

- For sub-game 1 $(N, \pi^{\lambda}_{LR1})$, a vector $\alpha^{\lambda}_{LR1}$ with

$$\{(\alpha^{\lambda}_{LR1})_j = (\lambda B')_j + \frac{1}{n}\lambda E' : \ \forall j \in N\}$$

  is a feasible cost allocation and lies in the core.

- For sub-game 2 $(N, \pi^{\lambda}_{LR2})$, its optimal cost allocation $\alpha^{\lambda}_{LR2}$ can be computed by
  - Primal-Dual-Type algorithm, if this game is sub-modular;
  - Column Generation Based algorithm, if this game is not.

# Instrument 1: LRB Cost Allocation, 2016, IJOC

In general, the lower bound obtained by Lagrangian relaxation technique is **as sharp as** the lower bound obtained by linear programming relaxation.

## Theorem

For game $(N, \pi)$, under the same ILP formulation for $\pi(S)$, the LRB cost allocation value is no less than the LPB cost allocation value, i.e., $\alpha^{\lambda}_{LR}(N) \geq \alpha_{LP}(N)$, when

(1) Given Lagrangian multiplier is optimal, i.e., $\lambda = \lambda^*$;

(2) $Core(N, \pi^{\lambda}_{LR2})$ is non-empty.

### Applications on Four Types of Facility Location Games

| | Sub-modular | Not Sub-modular | Applicable Method(s) |
|---|---|---|---|
| Linear | Type one: UFL | Type two: CFL | LPB, LRB |
| Non-linear | Type three: NLUFL | Type four: NLCFL | LRB |

# Instrument 1: LRB Cost Allocation, 2016, IJOC

## Uncapacitated Facility Location Game

### Theorem

LRB cost allocation and LPB cost allocation are both optimal.

The optimal cost allocations computed by different methods for the example

| Method | Player 1 | Player 2 | Player 3 | Player 4 | Total |
|---|---|---|---|---|---|
| LPB with Simplex | 5.00 | 6.50 | 8.50 | 6.50 | **26.5** |
| LPB with Interior Point | 6.58 | 6.50 | 8.50 | 4.92 | **26.5** |
| LRB | 6.87 | 6.50 | 8.50 | 4.63 | **26.5** |

- LRB cost allocation shares the same amount of cost as LPB cost allocation, in addition, LRB algorithm can sometimes generate cost allocations which are not easy to obtain by conventional LPB methods.

# Instrument 1: LRB Cost Allocation, 2016, IJOC

## Single Source Capacitated Facility Location Game

| Q | Average (%) | | LRCA - LPCA (%) | | Total time (s) | | |
|---|---|---|---|---|---|---|---|
| | LPCA | LRCA | Max | Min | Avg. | Max | Min |
| 10 | **97.15** | **98.79** | 2.38 | 1.00 | 19 | 21 | 18 |
| 20 | **97.20** | **98.31** | 1.51 | 0.88 | 24 | 27 | 23 |
| 30 | **94.70** | **95.25** | 0.75 | 0.38 | 15 | 28 | 21 |
| 40 | **94.11** | **94.25** | 0.28 | 0.07 | 24 | 24 | 23 |
| 50 | **93.87** | **93.88** | 0.04 | -0.02 | 31 | 35 | 27 |

- Effectiveness of LPB and LRB cost allocation methods applied in CFL.
- Sharpness of LRB cost allocation compared with LPB cost allocation.
- Convergence of LPB and LRB cost allocation as capacity increases.
- Time efficiency of LRB cost allocation method.

# Instrument 1: LRB Cost Allocation, 2016, IJOC

## Non-linear UFL Game

| $\theta$ | LRCA / BFSC (%) | | | Total Time (s) | | |
|---|---|---|---|---|---|---|
| | Avg. | Max | Min | Avg. | Max | Min |
| UFL | 74.37 | 78.27 | 69.52 | - | - | - |
| 0.01 | **78.31** | 82.27 | 74.65 | 379 | 394 | 343 |
| 0.10 | **87.75** | 91.13 | 84.18 | 415 | 484 | 382 |
| 0.50 | **95.83** | 96.49 | 95.02 | 446 | 519 | 381 |
| 1.00 | **97.82** | 98.41 | 97.37 | 478 | 550 | 383 |

- Effectiveness of NLUFL LRB cost allocations.
- Time efficiency of LRB cost allocation method applied in NLUFL.

# Instrument 1: LRB Cost Allocation, 2016, IJOC

## Non-linear CFL Game

| Q | $\theta$ | LRB / BFSC (%) | | | Total time(s) | | |
|---|---|---|---|---|---|---|---|
| | | Avg | Max | Min | Avg | Max | Min |
| | 0.01 | **99.64** | 99.70 | 99.55 | 5683 | 6838 | 4987 |
| | 0.10 | **99.87** | 99.89 | 99.78 | 5690 | 6834 | 4980 |
| 10 | 0.50 | **99.90** | 99.92 | 99.87 | 5742 | 6814 | 5036 |
| | 0.01 | **99.61** | 99.76 | 99.48 | 9925 | 10478 | 9485 |
| | 0.10 | **99.83** | 99.85 | 99.82 | 9835 | 10458 | 9322 |
| 20 | 0.50 | **99.85** | 99.88 | 99.84 | 9825 | 10487 | 9315 |
| | 0.01 | **99.02** | 99.15 | 98.82 | 11686 | 12831 | 10410 |
| | 0.10 | **99.73** | 99.77 | 99.67 | 11755 | 12816 | 10421 |
| 30 | 0.50 | **99.81** | 99.87 | 99.78 | 11485 | 13064 | 10277 |

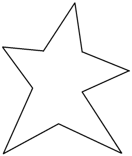
- Effectiveness of LRB cost allocations.
- Time efficiency of LRB cost allocation method applied in NLCFL.

# Instrument 1: LRB Cost Allocation, 2016, IJOC

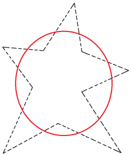
## Conclusions

- Cooperative Game Theory:
  - New Cost Allocation Method via Lagrangian Relaxation;
  - Generic framework applicable to linear, non-linear cases;
  - Effective method which in general can generate better cost allocations than LPB method.
- Models, Solution Methods:
  - Lagrangian relaxation, Lagrangian decomposition, Sub-games analyses;
- Applications:
  - Implementations on four types of facility location games.

# 2. Instrument of Simult. P&S



Unbalanced Game



Penalization & Subsidization

# Instrument 2: Simultaneous P&S, 2018, OR

## Instrument of Penalization (Least Core)

$$\mathrm{Core}(N,\pi) = \Big\{\alpha : \ \alpha(N) = \pi(N), \ \alpha(S) \le \pi(S), \ \forall S \in \mathbb{S} \setminus \{N\}, \ \alpha \in \mathbb{R}^n \Big\}.$$

Relax Coalition Stability constraints: $\alpha(S) \le \pi(S)$

- Minimum Penalty to stabilize the grand coalition:

$$z^* = \min \big\{ z : \alpha(N) = \pi(N), \ \alpha(S) \le \pi(S) + z, \ \forall S \in \mathbb{S} \setminus \{N\} \big\}.$$

# Instrument 2: Simultaneous P&S, 2018, OR

## Instrument of Penalization (Least Core)

$$\mathrm{Core}(N,\pi) = \left\{ \alpha : \ \alpha(N) = \pi(N), \ \alpha(S) \leq \pi(S), \ \forall S \in \mathbb{S} \setminus \{N\}, \ \alpha \in \mathbb{R}^n \right\}.$$

Relax Coalition Stability constraints: $\alpha(S) \leq \pi(S)$

- Minimum Penalty to stabilize the grand coalition:

$$z^* = \min \left\{ z : \alpha(N) = \pi(N), \ \alpha(S) \leq \pi(S) + z, \ \forall S \in \mathbb{S} \setminus \{N\} \right\}.$$

Minimum Penalty needed for MSG Example:

$z^* = 0.5$

# Instrument 2: Simultaneous P&S, 2018, OR

## Penalty-Subsidy Pair

$$\mathrm{Core}(N, \pi) = \left\{ \alpha : \ \alpha(N) = \pi(N), \ \alpha(S) \leq \pi(S), \ \forall S \in \mathbb{S} \setminus \{N\}, \ \alpha \in \mathbb{R}^n \right\}.$$

Relax Coalition Stability and Budget Balance constraints

- Penalty-Subsidy Pair to stabilize the grand coalition:

$$\omega(z) = \min_{\alpha} \left\{ \pi(N) - \alpha(N) : \ \alpha(S) \leq \pi(S) + z, \ \forall S \in \mathbb{S} \setminus \{N\} \right\}$$

# Instrument 2: Simultaneous P&S, 2018, OR

## Penalty-Subsidy Pair

$$\mathrm{Core}(N,\pi) = \left\{\alpha :\ \alpha(N) = \pi(N),\ \alpha(S) \leq \pi(S),\ \forall S \in \mathbb{S} \setminus \{N\},\ \alpha \in \mathbb{R}^n\right\}.$$

Relax Coalition Stability and Budget Balance constraints

- Penalty-Subsidy Pair to stabilize the grand coalition:

$$\omega(z) = \min_{\alpha} \left\{\pi(N) - \alpha(N) :\ \alpha(S) \leq \pi(S) + z,\ \forall S \in \mathbb{S} \setminus \{N\}\right\}$$

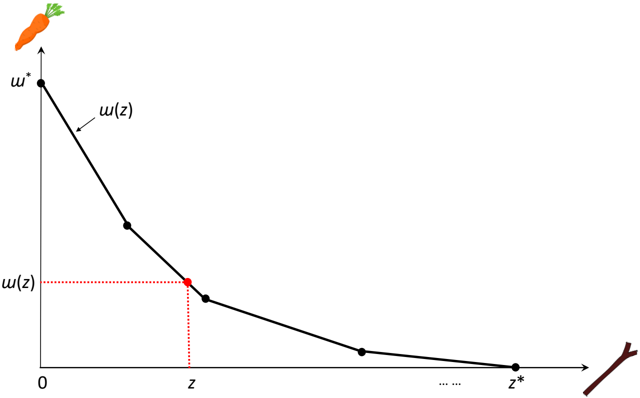
Penalty-Subsidy Pair for MSG Example:

E.g., combination of penalty 0.1667 and subsidy 0.5

# Instrument 2: Simultaneous P&S, 2018, OR

## Research Questions

How to use penalty and subsidy simultaneously to stabilize the grand cooperation? What's the trade-off?



# Instrument 2: Simultaneous P&S, 2018, OR

## Penalty Subsidy Function $\omega(z)$ – Properties

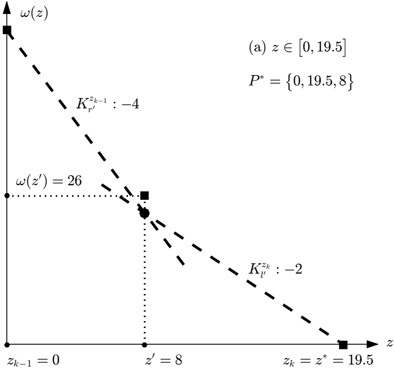
### Theorem

- $\omega(z)$ is decreasing, piecewise linear, and convex in $z \in \left[0, z^*\right]$.
- For each segment of $\omega(z)$, the slope $\omega'(z) \in \left[-n, -\frac{n}{n-1}\right]$.

- Decreasing of $\omega(z)$: trade-off between penalty and subsidy;
- Convexity of $\omega(z)$: diminishing effect on increasing the penalty to reduce the minimum subsidy desired.

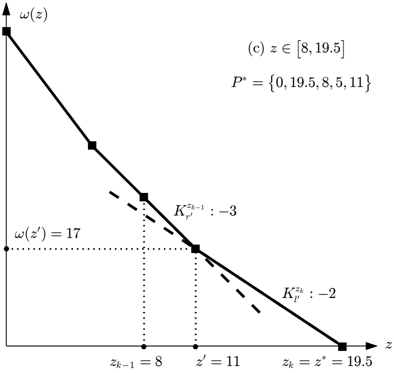
# Instrument 2: Simultaneous P&S, 2018, OR

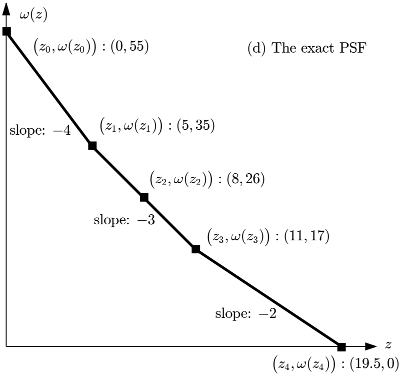
## Intersection Points Computation Algorithm: Illustrations on Scheduling Game



# Instrument 2: Simultaneous P&S, 2018, OR

## Intersection Points Computation Algorithm: Illustrations on Scheduling Game



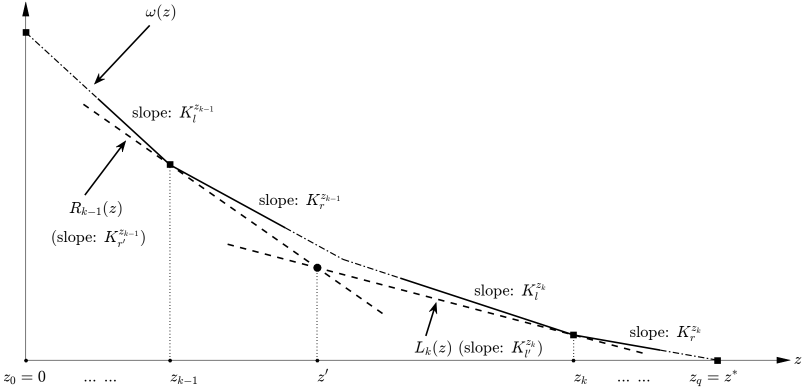



# Instrument 2: Simultaneous P&S, 2018, OR

## Efficiency of the IPC Algorithm

### Theorem

If function $\omega(z)$ has $\hat{q} \geq 2$ linear segments, then the IPC algorithm will terminate after at most $4\hat{q} - 1$ iterations.



**Step 1.** Evenly divide $\left[0, z^*\right]$ into $\lceil 2v/\epsilon \rceil$ sub-intervals;

**Step 2.** For each sub-interval, compute $\left(z_i, \omega(z_i)\right)$;

**Step 3.** Obtain upper bound $U_\epsilon(z)$ for $\omega(z)$ by linking these points.

## Theorem

$E_c \le (\epsilon/2)\left(z^*\right)^2 \le \epsilon \int_0^{z^*} \omega(z)\,\mathrm{d}z$ and $E_{\max} \le \left(\epsilon\, z^*\right)/2$, $\left(\epsilon > 0\right)$.

Cumulative error: $E_c = \int_0^{z^*} \left|U_\epsilon(z) - \omega(z)\right| \mathrm{d}z,$
Maximum error: $E_{\max} = \max_{z \in \left[0, z^*\right]} \left\{\left|U_\epsilon(z) - \omega(z)\right|\right\}.$

# Instrument 2: Simultaneous P&S, 2018, OR

## How to compute $\omega(z)$ for given penalty $z$?

$$\omega(z) = \min_{\alpha} \Big\{ \pi(N) - \alpha(N) : \alpha(S) \leq \pi(S) + z \text{ for all } S \in \mathbb{S} \setminus \{N\},\ \alpha \in \mathbb{R}^n \Big\}.$$

Define:

$$\tau(z) = \max_{\alpha} \Big\{ \alpha(N) : \alpha(S) \leq \pi(S) + z \text{ for all } S \in \mathbb{S} \setminus \{N\},\ \alpha \in \mathbb{R}^n \Big\}.$$

We obtain that $\omega(z) = c(N) - \tau(z)$.

# Instrument 2: Simultaneous P&S, 2018, OR

## Cutting-Plane (CP) Approach to Computing $\omega(z)$ for Given $z$

**Step 1.** $\mathbb{S}' \subseteq \mathbb{S} \setminus \{N\}$ indicates a restricted coalition set.

**Step 2.** Find an optimal $\bar{\alpha}(\,\cdot\,, z)$ to a relaxed LP of $\tau(z)$:

$$\max_{\alpha \in \mathbb{R}^n} \left\{ \alpha(N, z) : \alpha(S, z) \leq \pi(S) + z, \text{ for all } S \in \mathbb{S}' \right\}.$$

**Step 3.** Find an optimal $S^*$ to the separation problem:

$$\delta = \min \left\{ \pi(S) + z - \bar{\alpha}(S, z) : \forall S \in \mathbb{S} \setminus \{N\} \right\}.$$

**Step 4.** If $\delta < 0$, then add $S^*$ to $\mathbb{S}'$, and go to step 2; otherwise, return $\omega(z) = \pi(N) - \bar{\alpha}(N, z)$.

Generate a lower bound when serving as a heuristic

Inspired by Caprara & Letchford (2010) for $\tau(0)$, define:

$$Q^{x\mu y} = \Big\{(x, \mu, y) : Ax \geq By + D\mu, y = y^s \text{ for some } s \in S \setminus \{N\},$$
$$\mu = 1,\ x \in \mathbb{Z}^t,\ y \in \{0,1\}^n\Big\},$$
$$C^{x\mu} = \mathrm{proj}_{x\mu}\Big(\Big\{(x, \mu, y) \in \mathbb{R}^{t+1+n} : y = \mathbf{1}\Big\} \cap \mathrm{cone}\ Q^{x\mu y}\Big).$$

## Theorem

$$\tau(z) = \min\Big\{cx + z\mu : (x, \mu) \in C^{x\mu}\Big\}.$$

Generate an upper bound when serving as a heuristic

Parallel Machine Scheduling Games:

- Players $N = \{1, \ldots, n\}$; Machines $M = \{1, \ldots, m\}$.
- Each coalition $S$ minimizes the total (weighted) completion time of jobs of players in $S$.

Results on computing $\omega(z)$ for given $z$:

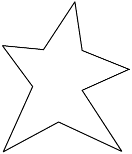
| | Machines | Jobs | CP Approach | LP Approach |
|---|---|---|---|---|
| IPU | Identical | Unweighted | P-time | P-time |
| UPU | Unrelated | Unweighted | – | P-time |
| IPW | Identical | Weighted | Pseudo P-time (fixed $m$) | – |
| UPW | Unrelated | Weighted | Lower Bound | Upper Bound |

# Instrument 2: Simultaneous P&S, 2018, OR

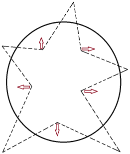
## Conclusions

- ⋆ Cooperative Game Theory:
  - New Instrument for Stabilization via Simultaneous P&S.
- ⋆ Models, Solution Methods:
  - Characterize properties of the penalty-subsidy function $\omega(z)$, revealing the trade-off;
  - Develop two algorithms to construct function $\omega(z)$;
  - Develop two solution approaches to computing the values of $\omega(z)$ for any given $z$.
- ⋆ Applications:
  - Implementations on several machine scheduling games.

# 3. Instrument of Cost Adjustment



Unbalanced Game



Cost Adjustment

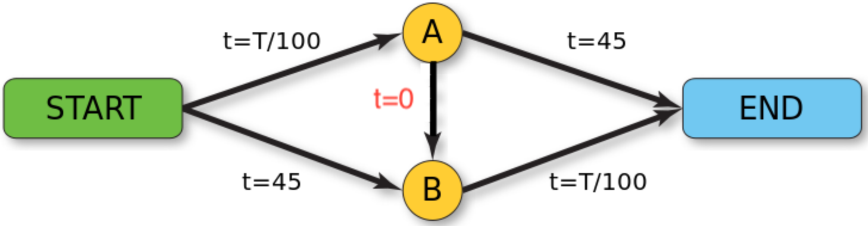
# Instrument 3: Cost Adjustment, 2019, Under Review

## Ex-post Actions

**Ex-post actions** on $\mathrm{Core}(N, \pi)$

$$\mathrm{Core}(N, \pi) = \left\{ \alpha : \ \alpha(N) = \pi(N), \ \alpha(S) \leq \pi(S), \ \forall S \in \mathbb{S} \setminus \{N\} \right\}$$

- Subsidization: $\alpha(N) = \pi(N) - \theta$, $\epsilon-$**core**;
- Penalization: $\alpha(S) \leq \pi(S) + z$, **least core**;
- Simult. S & P: $\alpha(N) = \pi(N) - \theta$ and $\alpha(S) \leq \pi(S) + z$, **PSF**.




4000 cars in total

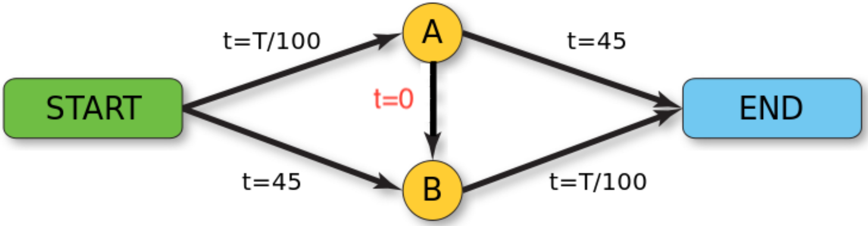
$$t = 0$$

$$START \rightarrow A \rightarrow B \rightarrow END$$

$$4000/100 + 4000/100 = 80min$$

# Instrument 3: Cost Adjustment, 2019, Under Review

## Illustrative Example: Braess's Paradox




4000 cars in total

$$t = 0$$

$$START \rightarrow A \rightarrow B \rightarrow END$$

$$4000/100 + 4000/100 = 80min$$

$$t = \infty$$

$$START \rightarrow A \rightarrow END$$

$$START \rightarrow B \rightarrow END$$

$$A = B = 2000$$

$$2000/100 + 45 = 65min$$

# Instrument 3: Cost Adjustment, 2019, Under Review

Ex-ante Action

$$\mathrm{Core}(N, \pi) = \left\{\alpha : \ \alpha(N) = \pi(N; c), \ \alpha(S) \leq \pi(S; c), \ \forall S \in \mathbb{S}\right\}.$$

$$\pi(S; c) = \min_x \{cx : Ax \geq By^S + E, \ x \in \mathbb{Z}^q\}.$$

**Ex-ante Action**

Cost Adjustment: $c \to d$ and $\pi(S; c) \to \pi(S; d)$

## Definition

Grand Coalition Stabilization Problem (GCSP) via Cost Adjustment (CA):

$$c \to d, \text{ such that } \textbf{BCC}$$

- Balancedness: the updated IM game $\big(N, \pi(\cdot; d)\big)$ is balanced;
- Cooperation Scheme: an initial $x^0$ is optimal to $\pi(N; d)$;
- Cost Sharing: total cost $\pi(N; d)$ to share is within $\big[l, u\big]$.

**Ex-post actions on** $\mathrm{Core}(N, \pi)$ **V.S. Ex-ante action on** $\pi(\cdot, c)$

## Constrained Inverse Optimization Problem (CIOP)

$$\min\left\{f(\delta) : \mathrm{Core}\left|\left(N, \pi(\cdot; d)\right)\right| \geq 1,\ dx^0 = \pi(N; d),\ l \leq \pi(N; d) \leq u,\ d \in \mathbb{R}^q\right\}.$$

- Only Optimality: Inverse Optimal Solution Problem;
- Only Consistency: Inverse Optimal Value Problem;
- Only Balancedness: Optimal Cost Allocation Problem.

# Instrument 3: Cost Adjustment, 2019, Under Review

## $\mathcal{NP}$-hardness and Feasibility

### Theorem

Solving the CIOP is in general $\mathcal{NP}$-hard.

### Theorem

Feasibility – Sufficient and Necessary Conditions

### Theorem

Feasibility – Necessary Conditions

### Theorem

Feasibility – Sufficient Conditions

## Lemma

The CIOP is equivalent to the following LP.

$$
\begin{gathered}
\min \;\; \omega \times \left(\tau + \eta\right)^{T} \\
s.t. \;\; \alpha \mathbf{1} = dx^{0}, \\
\alpha y \leq dx, \; \forall (x, y) \in Q_{xy}, \\
l \leq dx^{0} \leq u, \\
d - c = \tau - \eta, \text{ and } d \in \mathbb{R}^{q}, \; \tau \in \mathbb{R}^{q}_{+}, \; \eta \in \mathbb{R}^{q}_{+}.
\end{gathered}
$$

**Step 1.** Let $\hat{Q}_{xy}$ be a subset of $Q_{xy}$;

**Step 2.** Find an optimal solution $\left[\hat{\tau}; \hat{\eta}; \hat{d}; \hat{\alpha}\right]$ to a relaxed LP of (1), where $Q_{xy}$ is replaced by $\hat{Q}_{xy}$;

**Step 3.** Find an optimal solution $\left[x'; y'\right]$ to separation problem $\epsilon = \min\left\{\hat{d}x - \hat{\alpha}y : \forall (x, y) \in Q_{xy}\right\}$;

**Step 4.** If $\epsilon < 0$, then add $\left[x'; y'\right]$ to $\hat{Q}_{xy}$, go to step 2; otherwise, return (i) the updated cost coefficients $\hat{d}$; and (ii) the total minimum perturbation $\omega \times \left(\tau + \eta\right)^T$.

Generate a lower bound when serving as a heuristic

## Lemma

The CIOP is equivalent to the following LP.

$$\min_{\tau,\eta,d,\rho} \left\{ \omega \times \left(\tau + \eta\right)^T : \mathbb{B}^T \rho \geq dx^0;\ \mathbb{A}^T \rho = d^T;\ l \leq dx^0 \leq u; \right.$$
$$\left. \tau - \eta = d - c;\ \text{and } d \in \mathbb{R}^q,\ \tau, \eta, \rho \in \mathbb{R}^q_+ \right\},$$

where $C_x = \{x : \ \mathbb{A}x \geq \mathbb{B}\}$ and $\rho$ is the associating dual variable.

**Step 1.** Derive an expression of $C_x$, denoted as $\left\{x : \mathbb{A}x \geq \mathbb{B}\right\}$, with finite number of constraints for IM game $\big(N, \pi(\cdot; c)\big)$;

**Step 2.** Find an optimal solution $\left[\tau'; \eta'; d'; \rho'\right]$ to the CIOP;

**Step 3.** Return (i) the optimal cost coefficients $d'$; and (ii) the total minimum adjustment cost $\omega \times \left(\tau' + \eta'\right)^T$.

Generate an upper bound when serving as a heuristic

# Instrument 3: Cost Adjustment, 2019, Under Review

## Computational Results: weighted matching and uncapacitated facility location

| $(\|N\|, \|E\|)$ | U | DV% | Number of Adjusted Arcs | | |
|---|---|---|---|---|---|
| | | Avg. | Avg. | Max. | Min. |
| $(30, 435)$ | 16 | 0.018 | 1.87 | 4 | 1 |
| $(40, 780)$ | 21 | 0.005 | 1.57 | 3 | 1 |
| $(50, 1225)$ | 33 | 0.003 | 1.64 | 4 | 1 |
| $(60, 1770)$ | 26 | 0.002 | 1.96 | 6 | 1 |

| $(\|M\|, \|N\|)$ | U | DV% | Number of Adjusted Arcs | | | F |
|---|---|---|---|---|---|---|
| | | Avg. | Avg. | Max. | Min. | |
| $(20, 20)$ | 75 | 0.226 | 5.32 | 17 | 1 | 0 |
| $(40, 40)$ | 99 | 0.168 | 18.66 | 35 | 2 | 0 |
| $(60, 60)$ | 100 | 0.133 | 35.32 | 81 | 3 | 0 |
| $(80, 80)$ | 100 | 0.129 | 63.31 | 100 | 19 | 0 |

- ⋆ Cooperative Game Theory:
  - – New Instrument for Stabilization via Cost Adjustment.
- ⋆ Inverse Optimization:
  - – Constrained Inverse Optimization Problem.
- ⋆ Models, Solution Methods and Applications:
  - – Several Equivalent LP Formulations;
  - – Time complexity & Feasibility analyses;
  - – Implementations on WM and UFL games.

# The End

Thank You!