Stabilizing Grand Cooperation of Machine Scheduling Game via Setup Cost Pricing

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Outline

PRELIMINARIES

Cooperative Game

A **cooperative game** is defined by a pair (V, C):

- A set $V = \{1, 2, \ldots, v\}$ of players, **grand colaition**;
- A **characteristic function** $C(S)$ = the minimum total cost achieved by the cooperation of members in coalition $S \in \mathbb{S} = 2^V \setminus \{\emptyset\}$.

The game requires:

- A **cost allocation** $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_v] \in \mathbb{R}^v$, where α_k = the cost allocated to each player $k \in V$.

Core

Define $\alpha(S) = \sum_{k \in S} \alpha_k$.

A cost allocation $\alpha \in \mathbb{R}^v$ is in the **core** if it satisfies:

- **Budget Balance** Constraint: $\alpha(V) = C(V)$;
- **Coalition Stability** Constraints: $\alpha(S) \leq C(S)$ for each $S \in \mathbb{S}$.

$$\mathrm{Core}(V, C) = \Big\{ \alpha : \ \alpha(V) = C(V), \\ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\}, \ \alpha \in \mathbb{R}^v \Big\}.$$

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A cost allocation $\alpha \in \mathbb{R}^v$ is in the **core** if it satisfies:

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$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \right.$$
$$\left. \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\}, \ \alpha \in \mathbb{R}^v \right\}.$$

However, $\mathrm{Core}(V, C)$ can be empty.

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

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- Subsidization: $\alpha(V) = C(V) - \theta$, $\epsilon-$**core**;

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- Penalization: $\alpha(S) \leq C(S) + z$, **least core**;
- Simul. S & P: $\alpha(V) = C(V) - \theta$ and $\alpha(S) \leq C(S) + z$, **PSF**;

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- Inv. Opt.: Changing c to d such that $\mathrm{Core}(V, D)$ is non-empty.

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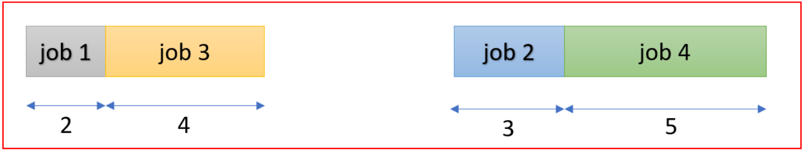
S.	Caprara and Letchford (2010, MP), Liu et al. (2016, IJOC)
P.	Faigle et al. (2001, IJGT), Schulz and Uhan (2010, OR)
P&S	Liu et al. (2018, OR)
Inv. Opt.	Liu et al. (2020, under review)

Illustrative Example

Example: Machine Scheduling Game (MSG)

Game of Parallel Machine Scheduling with Setup Cost:

- Grand coalition: $V = \{1, 2, 3, 4\}$;
- Processing times: $t_1 = 2,\ t_2 = 3,\ t_3 = 4,\ t_4 = 5$;
- Machine setup cost: $t_0 = 9.5$;
- $C(S)$ for $S \in \mathbb{S}$: minimizing the total completion time of jobs in S plus the machine setup cost;
- $C(V) = C(\{1,3\}) + C(\{2,4\}) = 8 + 11 + 9.5 \times 2 = 38.$



Example: Empty Core

Coalitions	Cost
$\{1\}$	11.5
$\{2\}$	12.5
$\{3\}$	13.5
$\{4\}$	14.5
$\{1, 2\}$	16.5
$\{1, 3\}$	17.5
$\{1, 4\}$	18.5
$\{2, 3\}$	19.5
$\{2, 4\}$	20.5
$\{3, 4\}$	22.5
$\{1, 2, 3\}$	25.5
$\{1, 2, 4\}$	26.5
$\{1, 3, 4\}$	28.5
$\{2, 3, 4\}$	31.5
$\{1, 2, 3, 4\}$	38

Example: Empty Core

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$\{1\}$	11.5
$\{2\}$	12.5
$\{3\}$	13.5
$\{4\}$	14.5
$\{1,2\}$	16.5
$\{1,3\}$	17.5
$\{1,4\}$	18.5
$\{2,3\}$	19.5
$\{2,4\}$	20.5
$\{3,4\}$	22.5
$\{1,2,3\}$	25.5
$\{1,2,4\}$	26.5
$\{1,3,4\}$	28.5
$\{2,3,4\}$	31.5
$\{1,2,3,4\}$	38

Optimal Cost Allocation Problem

$$\begin{aligned}
\max \ & \big(\alpha_1+\alpha_2+\alpha_3+\alpha_4\big) = 37.25 < 38 \\
s.t. \ & \alpha_1 \le 11.5, \ \cdots, \ \alpha_4 \le 14.5, \\
& \alpha_1+\alpha_2 \le 16.5, \ \cdots, \ \alpha_3+\alpha_4 \le 22.5, \\
& \cdots, \\
& \alpha_1+\alpha_2+\alpha_3+\alpha_4 \le 38.
\end{aligned}$$

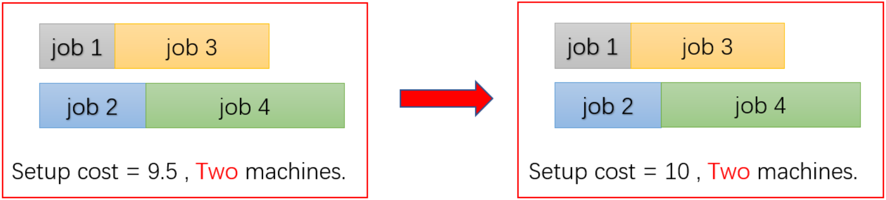
$$\alpha^* = \big[6; 8.75; 10.75; 11.75\big]$$

The minimum subsidy:

$C(V) - \alpha(V) = 38 - 37.25 = 0.75$

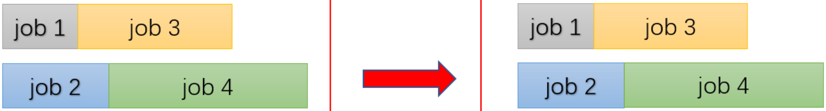
Example: Pricing Instrument

Increase the setup cost from 9.5 to 10.



Example: Pricing Instrument

Increase the setup cost from 9.5 to 10.



Setup cost = 9.5 , Two machines.

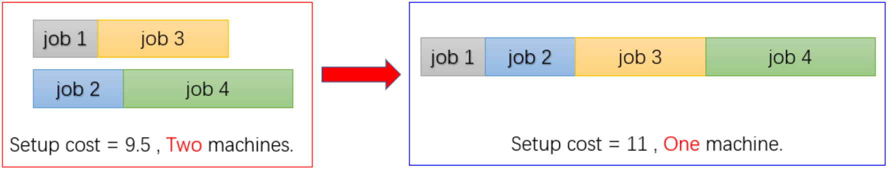
Setup cost = 10 , Two machines.

Setup cost	Increment	Num of Machines	Total pricing	c(V)	α(V)	Subsidy
9.5	0	2	0	38	37.25	0.75
10	0.5	2	1	39	38	1

The total pricing can exactly cover the gap, which means the grand coalition can be stabilized by the players themselves.

Example: Pricing Instrument

Increase the setup cost from 9.5 to 11.14.
For the grand coalition, it only needs one machine now.



Setup cost	Increment	Num of Machines	Total pricing	c(V)	α(V)	Subsidy
9.5	0	2	0	38	37.25	0.75
11.14	1.64	1	1.64	41.14	39.5	1.64

Models & Analyses

Problem Definition and Formulation

Definition

Parallel Machine Scheduling Game with Setup Cost (MSG):

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- Identical machines: $M = \{1, 2, \ldots, m\}$;

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- Processing time of each job: $t_k, \ \forall k \in V$;

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Parallel Machine Scheduling Game with Setup Cost (MSG):

- Grand coalition: $V = \{1, 2, \ldots, v\}$;
- Identical machines: $M = \{1, 2, \ldots, m\}$;
- Setup cost (unit price) of opening every single machine: P;
- Processing time of each job: t_k, $\forall k \in V$;
- Characteristic function: $C(S)$, denoting the total completion time and setup cost for each coalition $S \in \mathbb{S}$.

Problem Definition and Formulation

Definition

The characteristic function value, $C(S)$, of MSG is given by ILP

$$
\begin{aligned}
C(S,P) = \min & \sum_{k\in V}\sum_{j\in O} c_{kj}x_{kj} + P\sum_{k\in S} x_{k1} \\
s.t. \quad & \sum_{j\in O} x_{kj} - y_k^S = 0, \forall k \in V, \\
& \sum_{k\in V} x_{kj} \leq m, \forall j \in O, \\
& x_{kj} \in \{0,1\}, \forall k \in V, \forall j \in O, \\
& y_k^S = 1, k \in S \,;\, y_k^S = 0, k \notin S.
\end{aligned}
$$

Properties

Definition

- $\big[P_L(i,S), P_H(i,S)\big]$: Price range of using i machines for scheduling jobs among players S;

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- $\left[0, P^*\right]$: Effective domain of pricing MSG for stabilization;

Properties

Definition

- $\left[P_L(i,S), P_H(i,S)\right]$: Price range of using i machines for scheduling jobs among players S;
- $\left[0, P^*\right]$: Effective domain of pricing MSG for stabilization;
- P_i: For easy of exposition, let $P_1 = P^*$ and $P_i = P_H(i,V) = P_L(i-1,V)$. Thus, the effective domain of $\left[0, P^*\right]$ is divided into v non-overlapping sub-intervals by P_i, $\forall i = \{2, 3, ..., v\}$.

Note: P^* is the lowest price under which the grand cooperation of MSG is stable and it uses only one machine in the optimal scheduling decision, i.e., $P^* \in \left[P_L(1,V), P_H(1,V)\right]$ and MSG $\left(V, C(\ \cdot\ , P^*)\right)$ has non-empty core.

Properties

$$\textcolor{red}{\omega(P)} = \min_\alpha \{ C(V,P) - \alpha(V) : \alpha(S) \le C(S,P), \forall S \in \mathbb{S}, \alpha \in \mathbb{R}^v \};$$

Theorem 1

$\omega(P)$ is piecewise linear, and convex in price P at each sub-interval $[P_{i+1}, P_i]$, where $i = \{1, 2, ..., v-1\}$.

Lemma 1

$P_i, 2 \le i \le v$ can be obtained by SPT rules.

Theorem 2

$P_1 = P_2 + \cdots + P_v = \sum_{i=2}^{v} P_i$.

Theorem 3

$\omega(P)$ can be bounded by zero when the number of using machines, m_V, is larger than $\frac{n}{2}$.

Properties

Theorem 4

When the number of using machines is 1 for the grand coalition, the range of slopes of the line segments in the interval is $\left(-1, -\frac{1}{n-1}\right]$, and the number of breakpoints is $O(v^2)$.

Define characteristic function of the single machine scheduling game:

$$
\begin{aligned}
C'(S,P) = \min & \sum_{k\in V}\sum_{j\in O} c_{kj}x_{kj} + P \\
s.t. \quad & \sum_{j\in O} x_{kj} - y_k^S = 0, \forall k \in V, \\
& \sum_{k\in V} x_{kj} \le 1, \forall j \in O, \\
& x_{kj} \in \{0,1\}, \forall k \in V, \forall j \in O, \\
& y_k^S = 1, k \in S\,;\, y_k^S = 0, k \notin S.
\end{aligned}
$$

Properties

Theorem 5

Define that

$$\omega_1(P) = \min_\alpha\{C(V,P) - \alpha(V) : \alpha(S) \le C'(S,P), \forall S \in \mathbb{S} \setminus \{V\}, \alpha \in \mathbb{R}^v\}$$

Then the original problem $\omega(P)$ is equivalent to $\omega_1(P)$, where all sub-coalitions only use one machine.

Algorithms & Computations

IPC Algorithm

The Intersection Points Computation Algorithm to Construct $\omega(P)$ Function.

Step 1. Initially, set $I^* = \{P_L, P_H\}$ and $\mathbb{I} = \{[P_L, P_H]\}$.

Step 2. If $\mathbb{I}$ is not empty, update I^* and $\mathbb{I}$ by the following steps:

Step 3. Sort values in I^* by $P_0 < P_1 < \cdots < P_q$, where $P_0 = P_L, P_q = P_H$ and $q = |I^*| - 1$.

Step 4. Select any interval from $\mathbb{I}$, denoted by $[P_{k-1}, P_k]$ with $1 \leq k \leq q$.

Step 5. Construct two linear function $R_{k-1}(P)$ and $L_k(P)$ so that $R_{k-1}(P)$ passes $(P_{k-1}, \omega(P_{k-1}))$ with a slope equal to a right derivative $K_r^{P_{k-1}}$ of $\omega(P)$ at P_{k-1}, and that $L_k(z)$ passes $(P_k, \omega(P_k))$ with a slope equal to a left derivative $K_l^{P_k}$ of $\omega(P)$ at P_k.

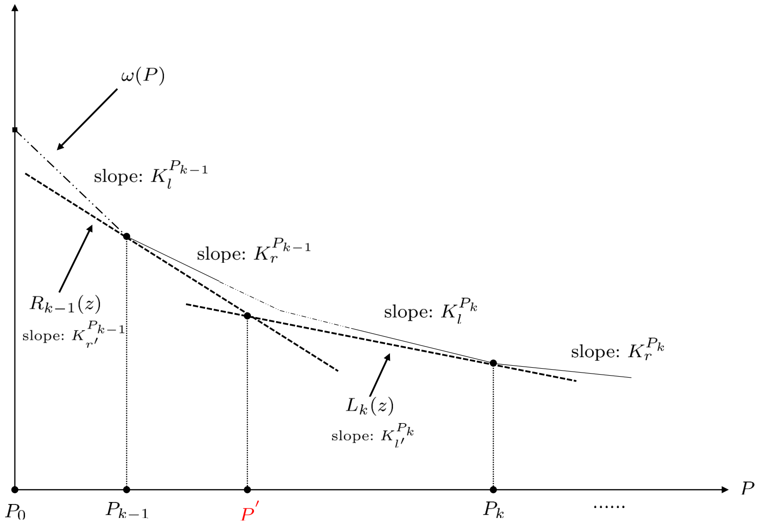
IPC Algorithm

Step 6. If $R_{k-1}(P)$ passes $(P_k, \omega(P_k))$ or $L_k(P)$ passes $(P_{k-1}, \omega(P_{k-1}))$, then update $\mathbb{I}$ by removing $[P_{k-1}, P_k]$. Otherwise, $R_{k-1}(P)$ and $L_k(P)$ must have a unique intersection point at $P = P'$ for some $P' \in (P_{k-1}, P_k)$. Update I^* by adding P', and update $\mathbb{I}$ by removing $[P_{k-1}, P_k]$, adding $[P_l, P']$ and $[P', P_r]$.

Step 7. Go to step 2.

Step 8. Return a piecewise linear function by connecting points $(P, \omega(P))$ for all $P \in I^*$.

IPC Algorithm



CP Algorithm

The Cutting Plane Algorithm to compute $\omega(P)$ for a given P.

Step 1. Let $\mathbb{S}' \subseteq \mathbb{S} \setminus \{N\}$ indicates a restricted coalition set, which includes some initial coalitions, e.g.,$\{1\}, \{2\}, \ldots, \{v\}$.

Step 2. Find an optimal solution $\bar{\alpha}(\,\cdot\,, P)$ to LP $\tau(P)$:

$$\max_{\alpha \in \mathbb{R}^n} \left\{ \alpha(N, P) : \alpha(s, P) \leq c(s) + P, \text{ for all } s \in \mathbb{S}' \right\}.$$

Step 3. Find an optimal solution s^* to the separation problem:

$$\delta = \min \left\{ c(s) + P - \bar{\alpha}(s, z) : \forall s \in \mathbb{S} \setminus \{N\} \right\}.$$

Step 4. If $\delta < 0$, then add s^* to $\mathbb{S}'$, and go to step 2; otherwise, return $\omega(P) = c(N) - \bar{\alpha}(N, P)$.

DP Algorithm

The Dynamic Programming Algorithm to solve the separation problem.

Step 1. Initially, let $D(k, u)$ indicate the minimum objective value of the restricted problem of separation problem, where $k \in \{1, 2, \ldots, v\}$ and $u \in \{0, 1, \ldots, v\}$.

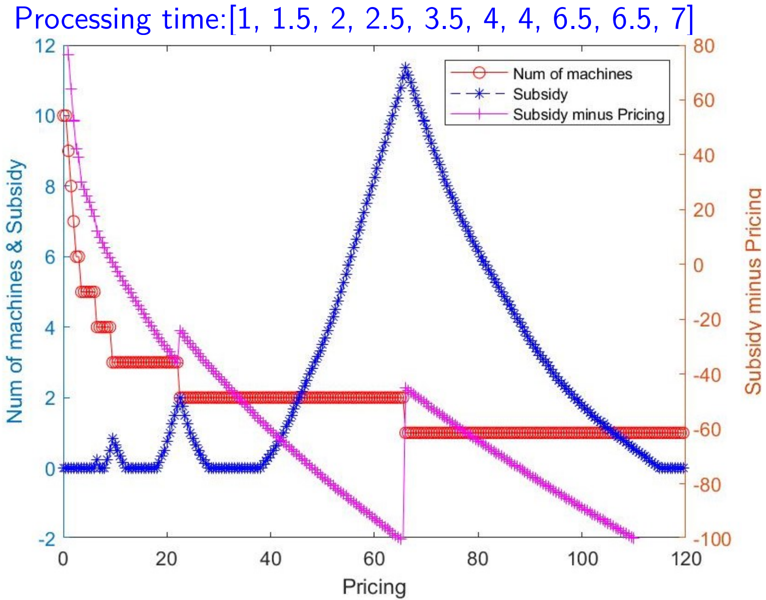
Step 2. Given the initial conditions $D(1, 0) = P$ and $D(1, 1) = t_1 - \beta_1 + P$. The boundary conditions are $D(k, u) = \infty$ if $u > k$, for all $k \in V$.

Step 3. Given the recursion:

$$D(k,u) = \min \begin{cases} D(k-1,u), \text{for the case when } s^* \text{ does not contain } k, \\ D(k-1,u-1) + ut_k - \alpha_k, \text{for the case when } s^* \text{ contains } k. \end{cases}$$

Step 4. Obtain the optimal objective value of separation problem by $\delta_{AIPU} = \min\{D(v, u) : u \in \{1, 2, \ldots, v-1\}\}$. return δ_{AIPU}.

Computational Results



Extension & Generalization

Machine Scheduling Game with Weighted Jobs

Definition

A Machine Scheduling Game with Weighted Jobs:

- Each job $k \in V$ has a processing time, t_k, and a weight, w_k.

Properties

- $C(S)$ and $P_i, 2 \leq i \leq v$ can also be obtained by assuming that $t_1/\omega_1 \leq t_2/\omega_2 \leq \ldots \leq t_v/\omega_v$.
- $\omega(P)$ is also piecewise linear, and convex in price P at each subinterval.
- IPC, CP, DP Algorithms can also be uesd to construct $\omega(P)$ function.

Pricing in General IM Games

Definition

The General Integer Minimization Games:

- *ILP* : $C(S, m'(S,P)) = \min_x\{cx + Pm'(x) : Ax \geq By^S + D, \tilde{\alpha}x \leq m', x \in \mathbb{Z}^{t\times 1}\}$
- Decompose $C(S, m'(S,P))$ into $C_0(S, m'(S)) + Pm'$.

Properties

The following properties illustrate that P_i is in descending order:

- $C_0(V, i-1) - C_0(V, i) > 0 \Leftrightarrow P_i > 0, i = 2, \ldots, v.$
- $C_0(V, i) - C_0(V, i+1) < C_0(V, i-1) - C_0(V, i) \Leftrightarrow P_i > P_{i+1}, i = 2, 3, \ldots, v-1.$

Conclusions

- ⋆ Cooperative Game Theory:
  - New Instrument for Stabilization via Setup cost Pricing.
- ⋆ Scheduling Problem:
  - Parallel Machine Scheduling with Setup Cost.
- ⋆ Models, Solution Methods and Applications:
  - Several ILP formulations;
  - Cutting Plane to solve the seperation problem;
  - Implementations on the MSGW game.

The End

Thank you!