

# Stabilizing Grand Cooperation of Machine Scheduling Game via Setup Cost Pricing

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# Outline

- 1 Preliminaries
- 2 Motivation and Illustrative Example
- 3 Models and Analyses
- 4 Algorithms and Computations
- 5 Extension and Generalization
- 6 Conclusion

# PRELIMINARIES

# Cooperative Game

A **cooperative game** is defined by a pair  $(V, C)$ :

- A set  $V = \{1, 2, \dots, v\}$  of players, **grand colalition**;
- A **characteristic function**  $C(S)$  = the minimum total cost achieved by the cooperation of members in coalition  $S \in \mathbb{S} = 2^V \setminus \{\emptyset\}$ .

The game requires:

- A **cost allocation**  $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_v] \in \mathbb{R}^v$ , where  $\alpha_k$  = the cost allocated to each player  $k \in V$ .

Define  $\alpha(S) = \sum_{k \in S} \alpha_k$ .

A cost allocation  $\alpha \in \mathbb{R}^V$  is in the **core** if it satisfies:

- **Budget Balance** Constraint:  $\alpha(V) = C(V)$ ;
- **Coalition Stability** Constraints:  $\alpha(S) \leq C(S)$  for each  $S \in \mathbb{S}$ .

$$\text{Core}(V, C) = \left\{ \alpha : \alpha(V) = C(V), \right. \\ \left. \alpha(S) \leq C(S), \forall S \in \mathbb{S} \setminus \{V\}, \alpha \in \mathbb{R}^V \right\}.$$

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However,  $\text{Core}(V, C)$  can be empty.

# Existing Instruments

$$\text{Core}(V, C) = \left\{ \alpha : \alpha(V) = C(V), \alpha(S) \leq C(S), \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

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- Simul. S & P:  $\alpha(V) = C(V) - \theta$  and  $\alpha(S) \leq C(S) + z$ , PSF;

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- Simul. S & P:  $\alpha(V) = C(V) - \theta$  and  $\alpha(S) \leq C(S) + z$ , PSF;
- Inv. Opt.: Changing  $c$  to  $d$  such that  $\text{Core}(V, D)$  is non-empty.

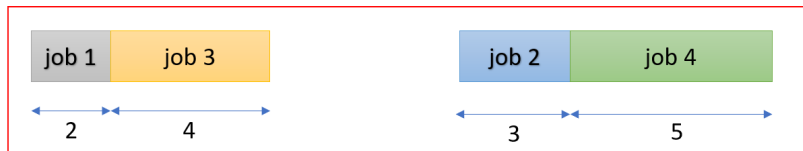


# ILLUSTRATIVE EXAMPLE

# Example: Machine Scheduling Game (MSG)

## Game of Parallel Machine Scheduling with Setup Cost:

- Grand coalition:  $V = \{1, 2, 3, 4\}$ ;
- Processing times:  $t_1 = 2$ ,  $t_2 = 3$ ,  $t_3 = 4$ ,  $t_4 = 5$ ;
- Machine setup cost:  $t_0 = 9.5$ ;
- $C(S)$  for  $S \in \mathbb{S}$ : minimizing the total completion time of jobs in  $S$  plus the machine setup cost;
- $C(V) = C(\{1, 3\}) + C(\{2, 4\}) = 8 + 11 + 9.5 \times 2 = 38$ .



# Example: Empty Core

Coalitions	Cost
$\{1\}$	11.5
$\{2\}$	12.5
$\{3\}$	13.5
$\{4\}$	14.5
$\{1, 2\}$	16.5
$\{1, 3\}$	17.5
$\{1, 4\}$	18.5
$\{2, 3\}$	19.5
$\{2, 4\}$	20.5
$\{3, 4\}$	22.5
$\{1, 2, 3\}$	25.5
$\{1, 2, 4\}$	26.5
$\{1, 3, 4\}$	28.5
$\{2, 3, 4\}$	31.5
$\{1, 2, 3, 4\}$	38

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{1}	11.5
{2}	12.5
{3}	13.5
{4}	14.5
{1, 2}	16.5
{1, 3}	17.5
{1, 4}	18.5
{2, 3}	19.5
{2, 4}	20.5
{3, 4}	22.5
{1, 2, 3}	25.5
{1, 2, 4}	26.5
{1, 3, 4}	28.5
{2, 3, 4}	31.5
{1, 2, 3, 4}	38

## Optimal Cost Allocation Problem

$$\begin{aligned} \max \quad & (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 37.25 < 38 \\ \text{s.t.} \quad & \alpha_1 \leq 11.5, \dots, \alpha_4 \leq 14.5, \\ & \alpha_1 + \alpha_2 \leq 16.5, \dots, \alpha_3 + \alpha_4 \leq 22.5, \\ & \dots, \\ & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 38. \end{aligned}$$

$$\alpha^* = [6; 8.75; 10.75; 11.75]$$

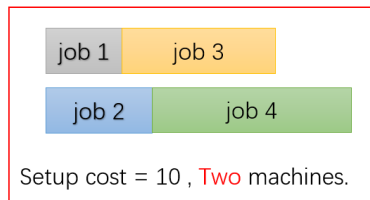
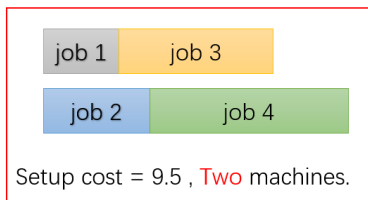
The minimum subsidy:

$$C(V) - \alpha(V) = 38 - 37.25 = 0.75$$



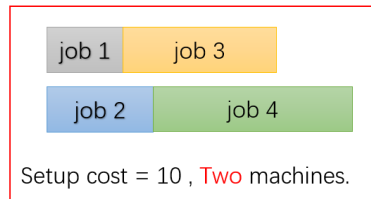
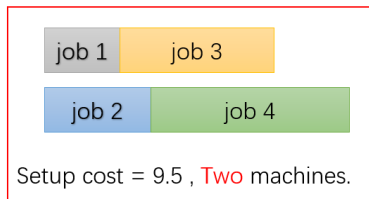
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Increase the setup cost from 9.5 to 10.



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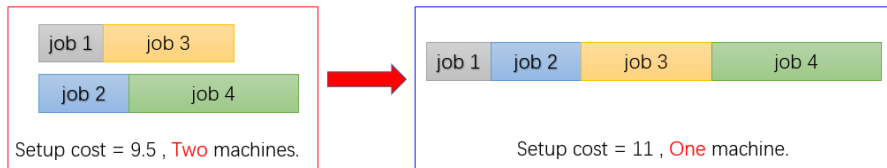


Setup cost	Increment	Num of Machines	Total pricing	$c(V)$	$\alpha(V)$	Subsidy
9.5	0	2	0	38	37.25	0.75
10	0.5	2	1	39	38	1

The total pricing can exactly cover the gap, which means the grand coalition can be stabilized by the players themselves.

# Example: Pricing Instrument

Increase the setup cost from 9.5 to 11.14.  
For the grand coalition, it only needs one machine now.



Setup cost	Increment	Num of Machines	Total pricing	$c(V)$	$\alpha(V)$	Subsidy
9.5	0	2	0	38	37.25	0.75
11.14	1.64	1	1.64	41.14	39.5	1.64

# MODELS & ANALYSES

# Problem Definition and Formulation

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- **Setup cost (unit price) of opening every single machine:**  $P$ ;
- **Processing time of each job:**  $t_k, \forall k \in V$ ;
- **Characteristic function:**  $C(S)$ , denoting the total completion time and setup cost for each coalition  $S \in \mathbb{S}$ .

# Problem Definition and Formulation

## Definition

The characteristic function value,  $C(S)$ , of MSG is given by ILP

$$C(S, P) = \min \sum_{k \in V} \sum_{j \in O} c_{kj} x_{kj} + P \sum_{k \in S} x_{k1}$$

$$\text{s.t.} \quad \sum_{j \in O} x_{kj} - y_k^S = 0, \forall k \in V,$$

$$\sum_{k \in V} x_{kj} \leq m, \forall j \in O,$$

$$x_{kj} \in \{0, 1\}, \forall k \in V, \forall j \in O,$$

$$y_k^S = 1, k \in S; y_k^S = 0, k \notin S.$$

## Definition

- $[P_L(i, S), P_H(i, S)]$ : Price range of using  $i$  machines for scheduling jobs among players  $S$ ;

# Properties

## Definition

- $[P_L(i, S), P_H(i, S)]$ : Price range of using  $i$  machines for scheduling jobs among players  $S$ ;
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- $[P_L(i, S), P_H(i, S)]$ : Price range of using  $i$  machines for scheduling jobs among players  $S$ ;
- $[0, P^*]$ : Effective domain of pricing MSG for stabilization;
- $P_i$ : For easy of exposition, let  $P_1 = P^*$  and  $P_i = P_H(i, V) = P_L(i - 1, V)$ . Thus, the effective domain of  $[0, P^*]$  is divided into  $v$  non-overlapping sub-intervals by  $P_i, \forall i = \{2, 3, \dots, v\}$ .

Note:  $P^*$  is the **lowest** price under which the grand cooperation of MSG is stable and it uses only one machine in the optimal scheduling decision, i.e.,  $P^* \in [P_L(1, V), P_H(1, V)]$  and MSG  $(V, C(\cdot, P^*))$  has non-empty core.

# Properties

$$\begin{aligned}\omega(P) &= \min_{\alpha} \{C(V, P) - \alpha(V) : \\ &\alpha(S) \leq C(S, P), \forall S \in \mathbb{S}, \alpha \in \mathbb{R}^v\};\end{aligned}$$

## Theorem 1

$\omega(P)$  is piecewise linear, and convex in price  $P$  at each sub-interval  $[P_{i+1}, P_i]$ , where  $i = \{1, 2, \dots, v-1\}$ .

## Lemma 1

$P_i, 2 \leq i \leq v$  can be obtained by SPT rules.

## Theorem 2

$$P_1 = P_2 + \dots + P_v = \sum_{i=2}^v P_i.$$

## Theorem 3

$\omega(P)$  can be bounded by zero when the number of using machines,  $m_V$ , is larger than  $\frac{n}{2}$ .

# Properties

## Theorem 4

When the number of using machines is 1 for the grand coalition, the range of slopes of the line segments in the interval is  $(-1, -\frac{1}{n-1}]$ , and the number of breakpoints is  $O(n^2)$ .

## Definition

$$C(S, P) = \min \sum_{k \in V} \sum_{j \in O} c_{kj} x_{kj} + P \sum_{k \in S} x_{k1}$$

$$s.t. \quad \sum_{j \in O} x_{kj} - y_k^S = 0, \forall k \in V,$$

$$\sum_{k \in V} x_{kj} \leq m, \forall j \in O,$$

$$x_{kj} \in \{0, 1\}, \forall k \in V, \forall j \in O,$$



## Theorem 5

Define that

$$\omega_1(P) = \min_{\alpha} \{c(V, P) - \alpha(V) : \\ \alpha(S) \leq C(S) + P, \forall S \in \mathbb{S}, \alpha \in \mathbb{R}^V\}$$

Then the original problem  $\omega(P)$  is equivalent to  $\omega_1(P)$  which means that all sub-coalitions only use **one** machine.

# ALGORITHMS & COMPUTATIONS

# IPC Algorithm

The Intersection Points Computation Algorithm to Construct  $\omega(P)$  Function.

- Step 1.** Initially, set  $I^* = \{P_L, P_H\}$  and  $\mathbb{I} = \{[P_L, P_H]\}$ .
- Step 2.** If  $\mathbb{I}$  is not empty, update  $I^*$  and  $\mathbb{I}$  by the following steps:
- Step 3.** Sort values in  $I^*$  by  $P_0 < P_1 < \dots < P_q$ , where  $P_0 = P_L, P_q = P_H$  and  $q = |I^*| - 1$ .
- Step 4.** Select any interval from  $\mathbb{I}$ , denoted by  $[P_{k-1}, P_k]$  with  $1 \leq k \leq q$ .
- Step 5.** Construct two linear function  $R_{k-1}(P)$  and  $L_k(P)$  so that  $R_{k-1}(P)$  passes  $(P_{k-1}, \omega(P_{k-1}))$  with a slope equal to a right derivative  $K_r^{P_{k-1}}$  of  $\omega(P)$  at  $P_{k-1}$ , and that  $L_k(z)$  passes  $(P_k, \omega(P_k))$  with a slope equal to a left derivative  $K_l^{P_k}$  of  $\omega(P)$  at  $P_k$ .

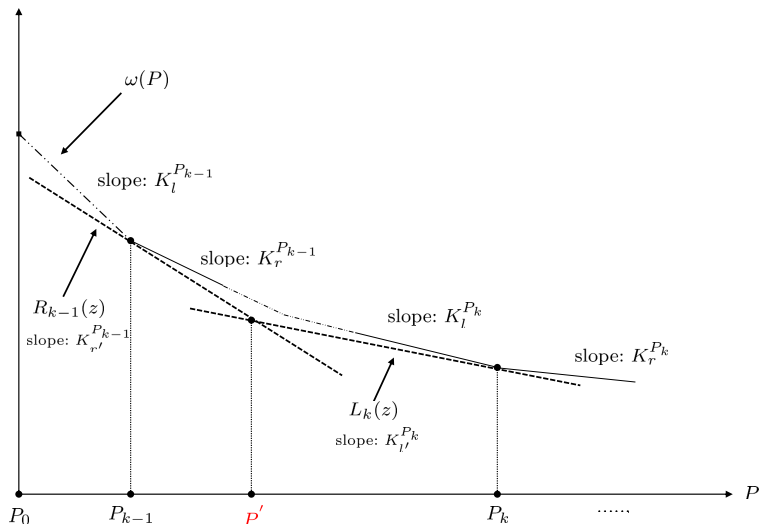
# IPC Algorithm

**Step 6.** If  $R_{k-1}(P)$  passes  $(P_k, \omega(P_k))$  or  $L_k(P)$  passes  $(P_{k-1}, \omega(P_{k-1}))$ , then update  $\mathbb{I}$  by removing  $[P_{k-1}, P_k]$ . Otherwise,  $R_{k-1}(P)$  and  $L_k(P)$  must have a unique intersection point at  $P = P'$  for some  $P' \in (P_{k-1}, P_k)$ . Update  $I^*$  by adding  $P'$ , and update  $\mathbb{I}$  by removing  $[P_{k-1}, P_k]$ , adding  $[P_l, P']$  and  $[P', P_r]$ .

**Step 7.** Go to step 2.

**Step 8.** Return a piecewise linear function by connecting points  $(P, \omega(P))$  for all  $P \in I^*$ .

# IPC Algorithm



The Cutting Plane Algorithm to compute  $\omega(P)$  for a given  $P$ .

**Step 1.** Let  $\mathbb{S}' \subseteq \mathbb{S} \setminus \{N\}$  indicates a restricted coalition set, which includes some initial coalitions, e.g.,  $\{1\}, \{2\}, \dots, \{v\}$ .

**Step 2.** Find an optimal solution  $\bar{\alpha}(\cdot, P)$  to LP  $\tau(P)$ :

$$\max_{\alpha \in \mathbb{R}^n} \left\{ \alpha(N, P) : \alpha(s, P) \leq c(s) + P, \text{ for all } s \in \mathbb{S}' \right\}.$$

**Step 3.** Find an optimal solution  $s^*$  to **the separation problem**:

$$\delta = \min \left\{ c(s) + P - \bar{\alpha}(s, z) : \forall s \in \mathbb{S} \setminus \{N\} \right\}.$$

**Step 4.** If  $\delta < 0$ , then add  $s^*$  to  $\mathbb{S}'$ , and go to step 2; otherwise, return  $\omega(P) = c(N) - \bar{\alpha}(N, P)$ .

# DP Algorithm

The Dynamic Programming Algorithm to solve the separation problem.

**Step 1.** Initially, let  $D(k, u)$  indicate the minimum objective value of the restricted problem of separation problem, where  $k \in \{1, 2, \dots, v\}$  and  $u \in \{0, 1, \dots, v\}$ .

**Step 2.** Given the initial conditions  $D(1, 0) = P$  and  $D(1, 1) = t_1 - \beta_1 + P$ . The boundary conditions are  $D(k, u) = \infty$  if  $u > k$ , for all  $k \in V$ .

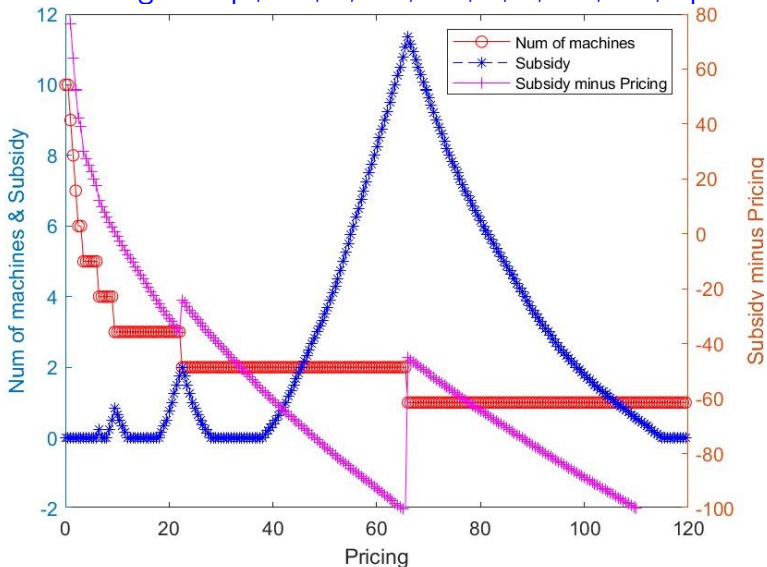
**Step 3.** Given the recursion:

$$D(k, u) = \min \begin{cases} D(k-1, u), & \text{for the case when } s^* \text{ does not contain } k, \\ D(k-1, u-1) + ut_k - \alpha_k, & \text{for the case when } s^* \text{ contains } k. \end{cases}$$

**Step 4.** Obtain the optimal objective value of separation problem by  $\delta_{AIPU} = \min\{D(v, u) : u \in \{1, 2, \dots, v-1\}\}$ . return  $\delta_{AIPU}$ .

# Computational Results

Processing time:[1, 1.5, 2, 2.5, 3.5, 4, 4, 6.5, 6.5, 7]





# EXTENSION & GENERALIZATION

# Machine Scheduling Game with Weighted Jobs

## Definition

A Machine Scheduling Game with Weighted Jobs:

- Each job  $k \in V$  has a processing time,  $t_k$ , and a weight,  $w_k$ .

## Properties

- $C(S)$  and  $P_i, 2 \leq i \leq v$  can also be obtained by assuming that  $t_1/\omega_1 \leq t_2/\omega_2 \leq \dots \leq t_v/\omega_v$ .
- $\omega(P)$  is also piecewise linear, and convex in price  $P$  at each subinterval.
- IPC, CP, DP Algorithms can also be used to construct  $\omega(P)$  function.

# Pricing in General IM Games

## Definition

The General Integer Minimization Games:

- **ILP**:  $C(S, m'(S, P)) = \min_x \{cx + Pm'(x) : Ax \geq By^S + D, \tilde{\alpha}x \leq m', x \in \mathbb{Z}^{t \times 1}\}$
- **Decompose**  $c(S, m'(S, P))$  into  $c_0(S, m'(S)) + Pm'$ .
- $c_0(V, i) - c_0(V, i-1) > 0 \Leftrightarrow P_i > 0, i = 2, \dots, v$ .
- $c_0(V, i) - c_0(V, i+1) < c_0(V, i-1) - c_0(V, i) \Leftrightarrow P_i < P_{i+1}, i = 2, 3, \dots, v-1$ .
- $c_0(S_1, i-1) - c_0(S_1, i) \geq c_0(S_1, i) - c_0(S_1, i+1) \quad i = 2, \dots, v-1$ .
- $c_0(S_1, i-1) - c_0(S_1, i) \leq c_0(S_2, i-1) - c_0(S_2, i)$   
 $i=2, \dots, v$ , where  $S_1 \subset S_2$ .

## CONCLUSIONS

- ★ **Cooperative Game Theory:**
  - New Instrument for Stabilization via Setup cost Pricing.
- ★ **Scheduling Problem:**
  - Parallel Machine Scheduling with Setup Cost.
- ★ **Models, Solution Methods and Applications:**
  - Several ILP formulations;
  - Cutting Plane to solve the separation problem;
  - Implementations on the MSGW game.

Thank you!