Stabilizing Grand Cooperation of Machine Scheduling Game via Setup Cost Pricing

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Outline

Preliminaries

Cooperative Game

A **cooperative game** is defined by a pair (V, C):

- A set $V = \{1, 2, \ldots, v\}$ of players, **grand colaition**;
- A **characteristic function** $C(S) =$ the minimum total cost achieved by the cooperation of members in coalition $S \in \mathbb{S} = 2^V \setminus \{\emptyset\}$.

The game requires:

- A **cost allocation** $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_v] \in \mathbb{R}^v$, where $\alpha_k =$ the cost allocated to each player $k \in V$.

Core

Define $\alpha(S) = \sum_{k \in S} \alpha_k$.

A cost allocation $\alpha \in \mathbb{R}^v$ is in the **core** if it satisfies:

- **Budget Balance** Constraint: $\alpha(V) = C(V)$;
- **Coalition Stability** Constraints: $\alpha(S) \leq C(S)$ for each $S \in \mathbb{S}$.

$$\mathrm{Core}(V, C) = \Big\{\alpha : \ \alpha(V) = C(V), \\ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\}, \ \alpha \in \mathbb{R}^v\Big\}.$$

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$$\mathrm{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \right.$$
$$\left. \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\}, \ \alpha \in \mathbb{R}^V \right\}.$$

However, $\mathrm{Core}(V, C)$ can be empty.

Existing Instruments

$$\mathrm{Core}(V, C) = \left\{\alpha : \ \alpha(V) = C(V), \ \alpha(S) \le C(S), \ \forall S \in \mathbb{S} \setminus \{V\}\right\}$$

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- Penalization: $\alpha(S) \leq C(S) + z$, **least core**;

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- Simul. S & P: $\alpha(V) = C(V) - \theta$ and $\alpha(S) \le C(S) + z$, **PSF**;

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- Inv. Opt.: Changing c to d such that $\mathrm{Core}(V, D)$ is non-empty.

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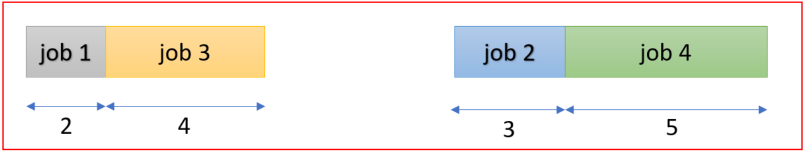
S.	Caprara and Letchford (2010, MP), Liu et al. (2016, IJOC)
P.	Faigle et al. (2001, IJGT), Schulz and Uhan (2010, OR)
P&S	Liu et al. (2018, OR)
Inv. Opt.	Liu et al. (2020, under review)

Illustrative Example

Example: Machine Scheduling Game (MSG)

Game of Parallel Machine Scheduling with Setup Cost:

- Grand coalition: $V = \{1, 2, 3, 4\}$;
- Processing times: $t_1 = 2,\ t_2 = 3,\ t_3 = 4,\ t_4 = 5$;
- Machine setup cost: $t_0 = 9.5$;
- $C(S)$ for $S \in \mathbb{S}$: minimizing the total completion time of jobs in S plus the machine setup cost;
- $C(V) = C(\{1,3\}) + C(\{2,4\}) = 8 + 11 + 9.5 \times 2 = 38.$



Example: Empty Core

Coalitions	Cost
$\{1\}$	11.5
$\{2\}$	12.5
$\{3\}$	13.5
$\{4\}$	14.5
$\{1,2\}$	16.5
$\{1,3\}$	17.5
$\{1,4\}$	18.5
$\{2,3\}$	19.5
$\{2,4\}$	20.5
$\{3,4\}$	22.5
$\{1,2,3\}$	25.5
$\{1,2,4\}$	26.5
$\{1,3,4\}$	28.5
$\{2,3,4\}$	31.5
$\{1,2,3,4\}$	38

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$\{1,2,3,4\}$	38

Optimal Cost Allocation Problem

$$\begin{aligned} \max\ & \left(\alpha_1+\alpha_2+\alpha_3+\alpha_4\right) = 37.25 < 38 \\ s.t.\ & \alpha_1 \leq 11.5,\ \cdots,\ \alpha_4 \leq 14.5, \\ & \alpha_1+\alpha_2 \leq 16.5,\ \cdots,\ \alpha_3+\alpha_4 \leq 22.5, \\ & \cdots, \\ & \alpha_1+\alpha_2+\alpha_3+\alpha_4 \leq 38. \end{aligned}$$

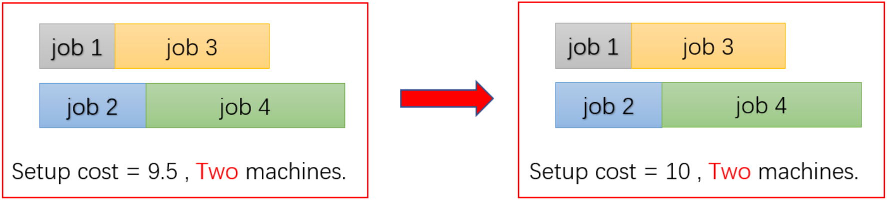
$$\alpha^* = \left[6; 8.75; 10.75; 11.75\right]$$

The minimum subsidy:

$$C(V) - \alpha(V) = 38 - 37.25 = 0.75$$

Example: Pricing Instrument

Increase the setup cost from 9.5 to 10.



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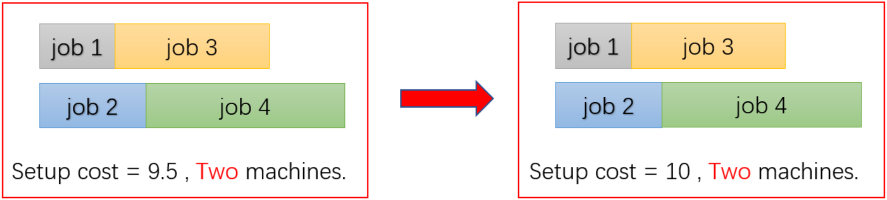
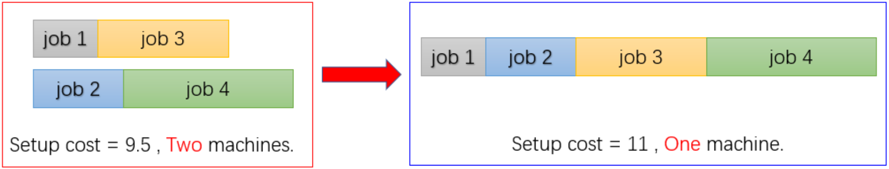


Setup cost	Increment	Num of Machines	Total pricing	C(V)	α(V)	Subsidy
9.5	0	2	0	38	37.25	0.75
10	0.5	2	1	39	38	1

The total pricing can exactly cover the gap, which means the grand coalition can be stabilized by the players themselves.

Example: Pricing Instrument

Increase the setup cost from 9.5 to 11.14.
For the grand coalition, it only needs one machine now.



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9.5	0	2	0	38	37.25	0.75
11.14	1.64	1	1.64	41.14	39.5	1.64

Models & Analyses

Problem Definition and Formulation

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Parallel Machine Scheduling Game with Setup Cost (MSG):

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- Identical machines: $M = \{1, 2, \ldots, m\}$;

Problem Definition and Formulation

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Parallel Machine Scheduling Game with Setup Cost (MSG):

- Grand coalition: $V = \{1, 2, \ldots, v\}$;
- Identical machines: $M = \{1, 2, \ldots, m\}$;
- Setup cost (unit price) of opening every single machine: P;
- Processing time of each job: t_k, $\forall k \in V$;
- Characteristic function: $C(S)$, denoting the total completion time and setup cost for each coalition $S \in \mathbb{S}$.

Problem Definition and Formulation

Definition

The characteristic function value, $C(S)$, of MSG is given by ILP

$$C(S,P) = \min \sum_{k \in V} \sum_{j \in O} c_{kj} x_{kj} + P \sum_{k \in S} x_{k1}$$

$$\begin{aligned}
s.t. \quad & \sum_{j \in O} x_{kj} - y_k^S = 0, \forall k \in V, \\
& \sum_{k \in V} x_{kj} \le m, \forall j \in O, \\
& x_{kj} \in \{0,1\}, \forall k \in V, \forall j \in O, \\
& y_k^S = 1, k \in S \,; y_k^S = 0, k \notin S.
\end{aligned}$$

Properties

Definition

- $\left[P_L(i,S), P_H(i,S)\right]$: Price range of using i machines for scheduling jobs among players S;

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Definition

- $\left[P_L(i,S), P_H(i,S)\right]$: Price range of using i machines for scheduling jobs among players S;
- $\left[0, P^*\right]$: Effective domain of pricing MSG for stabilization;
- P_i: For easy of exposition, let $P_1 = P^*$ and $P_i = P_H(i,V) = P_L(i-1,V)$. Thus, the effective domain of $\left[0, P^*\right]$ is divided into v non-overlapping sub-intervals by P_i, $\forall i = \{2, 3, ..., v\}$.

Note: P^* is the lowest price under which the grand cooperation of MSG is stable and it uses only one machine in the optimal scheduling decision, i.e., $P^* \in \left[P_L(1,V), P_H(1,V)\right]$ and MSG $\left(V, C(\ \cdot\ , P^*)\right)$ has non-empty core.

Properties

Gap: $\omega(P) \quad = \min_{\alpha}\{C(V,P) - \alpha(V) : \alpha(S) \leq C(S,P), \forall S \in \mathbb{S}\}$

Theorem 1

Function $\omega(P)$ is piecewise linear, and convex in P at each sub-interval $[P_{i+1}, P_i]$.

Lemma 1

Breaking price, P_i $(2 \leq i \leq v)$, can be obtained by SPT rules in polynomial time.

Theorem 2

Boundary price P^* equals $\sum_{i=2}^{v} P_i$, and can be obtained in polynomial time.

Theorem 3

Gap $\omega(P) = 0$ when the number of machines used by $C(V,P)$ is larger than $\frac{v}{2}$.

Properties

Single Machine Scheduling Game (SMSG):

$$C(S, P, \text{``}m = \text{''}1) = \min \sum_{k\in V} \sum_{j\in O} c_{kj}x_{kj} + P$$

$$s.t. \quad \sum_{j\in O} x_{kj} - y_k^S = 0, \ \forall k \in V,$$

$$\sum_{k\in V} x_{kj} \leq 1, \ \forall j \in O,$$

$$x_{kj} \in \{0, 1\}, \ \forall k \in V, \ \forall j \in O, \ y_k^S = 1, \ \forall k \in S\,; y_k^S = 0, \ \forall k \notin S.$$

Theorem 4

For SMSG (or, MSG with only one machine), given $P \in [0, P^*]$ (or, $[P_2, P_1]$), the slope of each segment for function $\omega(P)$ is within $\left(-1, -\frac{1}{v-1}\right]$, and the number of breakpoints for function $\omega(P)$ is within $O(v^2)$.

Properties

Definition

Define that

$$\omega_1(P) = \min_{\alpha} \left\{ C(V, P) - \alpha(V) : \alpha(S) \leq C(S, P, 1),\ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

Theorem 5

The original problem $\omega(P)$ is equivalent to $\omega_1(P)$, which is polynomially solvable by cutting plane (see CP Algorithm below).

Algorithms & Computations

CP Algorithm

Cutting Plane Algorithm for Computing $\omega_1(P)$ under Given P

Step 1. Let $\mathbb{S}' \subseteq \mathbb{S} \backslash \{V\}$ indicates a restricted coalition set, which includes some initial coalitions, e.g.,$\{1\}, \{2\}, \ldots, \{v\}$.

Step 2. Find an optimal solution $\bar{\alpha}$ to LP

$$\max_{\alpha \in \mathbb{R}^n} \left\{ \alpha(V) : \alpha(S) \leq C(S, P, 1), \text{ for all } S \in \mathbb{S}' \right\}.$$

Step 3. Find an optimal solution S^* to separation problem

$$\delta = \min \left\{ C(S, P, 1) - \bar{\alpha}(S) : \forall S \in \mathbb{S} \setminus \{V\} \right\}.$$

Step 4. If $\delta < 0$, then add S^* to $\mathbb{S}'$, and go to step 2; otherwise, return $\omega(P) = \omega_1(P) = C(V, P) - \bar{\alpha}(V)$.

DP Algorithm

Dynamic Programming for Solving Separation Problem

- Without loss of generality, assuming $t_1 \geq t_2 \geq \ldots \geq t_v$.
- Recall that For the separation problem is given by

$$\delta = \min \left\{ C(S, P, 1) - \bar{\alpha}(S) : \forall S \in \mathbb{S} \setminus \{V\} \right\}.$$

- Note: If some player $k \notin S$ is added into S, where $|S| = u$, the increment of δ is $(u+1)t_k - \alpha_k$ (see recursion in Step 3).

Step 1. Initially, let $D(k, u)$ indicate the minimum objective value of the restricted problem of separation problem δ, where k is a player in the grand coalition and u is the number of players included in S. So $k \in \{1, 2, \ldots, v\}$ and $u \in \{0, 1, \ldots, v\}$.

DP Algorithm

Step 2. Given the initial conditions $D(1,0) = P$ and $D(1,1) = t_1 - \alpha_1 + P$. The boundary conditions are $D(k,u) = \infty$ if $u > k$, for all $k \in V$.

Step 3. Given the recursion:

$$D(k,u) = \min \begin{cases} D(k-1,u), \text{when } S^* \text{ does not contain } k, \\ D(k-1,u-1) + ut_k - \alpha_k, \text{when } S^* \text{ contains } k. \end{cases}$$

Step 4. Obtain the optimal objective value of the separation problem by $\delta = \min\{D(v,u) : u \in \{1,2,\ldots,v-1\}\}$. Return δ.

DP Algorithm can solve the separation problem δ in $O(v^2)$ time.

Note that $k \in \{1,2,\ldots,v\}$ and $u \in \{0,1,\ldots,v\}$ in $D(k,u)$, thus the time complexity of this DP is $O(v^2)$.

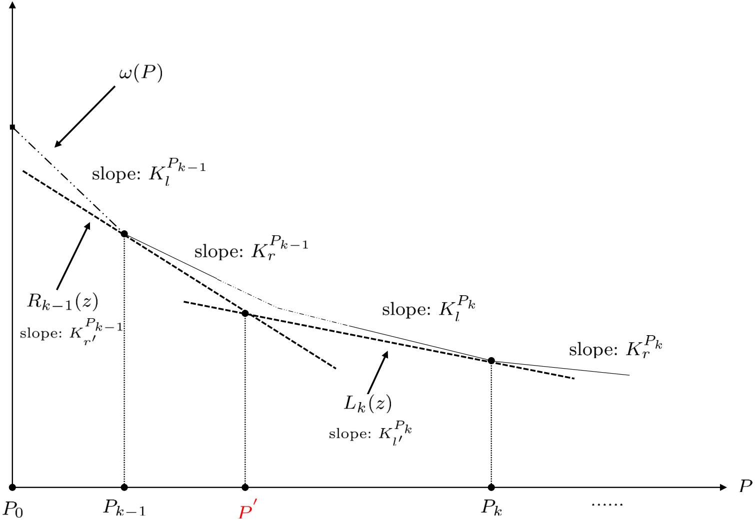
IPC Algorithm

DP (P-time) $\Rightarrow$ CP (P-time) $\Rightarrow$ $\omega(P)$ for given P (P-time)

Intersection Points Computation for Constructing Function $\omega(P)$

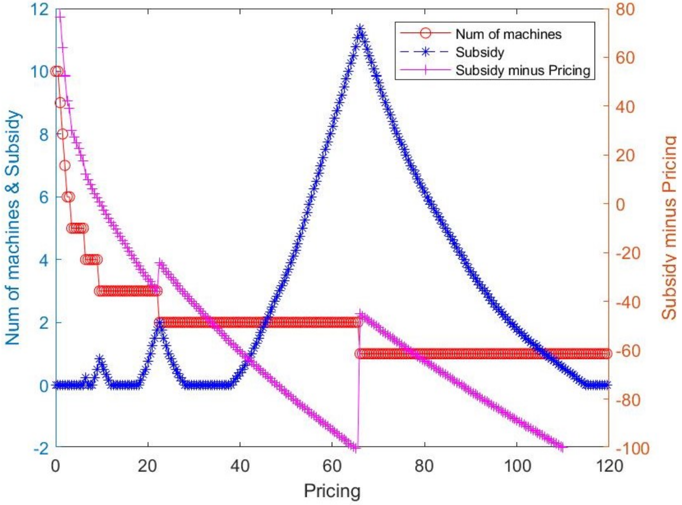
Since function $\omega(P)$ is shown to be piecewise linear, convex and have polynomial number of breakpoints within each interval $[P_{i+1}, P_i]$ ($i = v-1, v-2, ..., 1$), we claim that function $\omega(P)$ can be polynomially constructed by the IPC algorithm developed in Liu et.al. 2018 (OR).

IPC Algorithm



Computational Results

Processing time:[1, 1.5, 2, 2.5, 3.5, 4, 4, 6.5, 6.5, 7]



Extension & Generalization

Machine Scheduling Game with Weighted Jobs

Definition

Machine Scheduling Game with Weighted Jobs (WSGW):

- Each job $k \in V$ has a processing time, t_k, and a weight, w_k.

Properties

- $C(S, P)$ and P_i $(2 \leq i \leq v)$ can be obtained by analysing the order of t_k/ω_k instead of t_k.
- $\omega(P)$ is piecewise linear, convex in price P at each subinterval.
- IPC, CP Algorithms can be used to construct function $\omega(P)$ in polynomial time.

Pricing in General IM Games

Definition

General Integer Minimization Games:

- $C(S,P) = \min_x\{cx + Pm(x) : Ax \geq By^S + D, \beta x \leq m, x \in \mathbb{Z}^t\}$
- Let $H_i = C(V,P) - Pi$, where $i = m(x^*)$ and x^* solves $C(V,P)$.

Properties for MSG

- $H_{i-1} - H_i > 0 \Leftrightarrow P_i > 0, \ \forall i = 2, 3, \ldots, v.$
- $H_i - H_{i+1} < H_{i-1} - H_i \Leftrightarrow P_i > P_{i+1}, \ \forall i = 2, 3, \ldots, v-1.$
- Intervals $[P_v, P_{v-1}]$, $[P_{v-1}, P_{v-2}]$, ..., $[P_2, P_1]$, $[P_1, P^*]$ are non-overlapping.

Conclusions

⋆ Cooperative Game Theory:

– Stabilizing Grand Cooperation via Pricing.

⋆ Scheduling Problem:

– Parallel Machine Scheduling with Setup Cost.

⋆ Models, Solution Methods and Applications:

– Several ILP formulations;

– CP, DP and IPC algorithms for constructing function $\omega(P)$.

The End

Thank you!