Stabilizing Grand Cooperation of Machine Scheduling Game via Setup Cost Pricing

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NWPU-Online, July, 2020

Outline

Preliminaries

Cooperative Game

A **cooperative game** is defined by a pair (V, C):

- A set $V = \{1, 2, \ldots, v\}$ of players, **grand colaition**;
- A **characteristic function** $C(S) =$ the minimum total cost achieved by the cooperation of members in coalition $S \in \mathbb{S} = 2^V \setminus \{\emptyset\}$.

The game requires:

- A **cost allocation** $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_v] \in \mathbb{R}^v$, where $\alpha_k =$ the cost allocated to each player $k \in V$.

Core

Define $\alpha(S) = \sum_{k \in S} \alpha_k$.

A cost allocation $\alpha \in \mathbb{R}^V$ is in the **core** if it satisfies:

- **Budget Balance** Constraint: $\alpha(V) = C(V)$;
- **Coalition Stability** Constraints: $\alpha(S) \leq C(S)$ for each $S \in \mathbb{S}$.

$$\text{Core}(V, C) = \Big\{ \alpha : \ \alpha(V) = C(V), \\ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\}, \ \alpha \in \mathbb{R}^V \Big\}.$$

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However, $\mathrm{Core}(V, C)$ can be empty.

Existing Instruments

$$\text{Core}(V, C) = \left\{ \alpha : \ \alpha(V) = C(V), \ \alpha(S) \leq C(S), \ \forall S \in \mathbb{S} \setminus \{V\} \right\}$$

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- Simul. S & P: $\alpha(V) = C(V) - \theta$ and $\alpha(S) \leq C(S) + z$, **PSF**;

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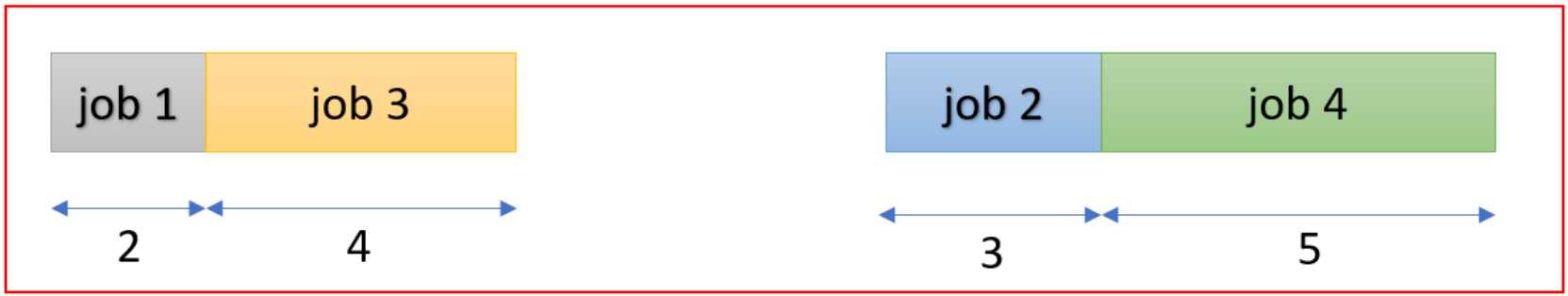
S.	Caprara and Letchford (2010, MP), Liu et al. (2016, IJOC)
P.	Faigle et al. (2001, IJGT), Schulz and Uhan (2010, OR)
P&S	Liu et al. (2018, OR)
Inv. Opt.	Liu et al. (2020, under review)

Illustrative Example

Example: Machine Scheduling Game (MSG)

Game of Parallel Machine Scheduling with Setup Cost:

- Grand coalition: $V = \{1, 2, 3, 4\}$;
- Processing times: $t_1 = 2,\ t_2 = 3,\ t_3 = 4,\ t_4 = 5$;
- Machine setup cost: $t_0 = 9.5$;
- $C(S)$ for $S \in \mathbb{S}$: minimizing the total completion time of jobs in S plus the machine setup cost;
- $C(V) = C(\{1, 3\}) + C(\{2, 4\}) = 8 + 11 + 9.5 \times 2 = 38.$



Example: Empty Core

Coalitions	Cost
$\{1\}$	11.5
$\{2\}$	12.5
$\{3\}$	13.5
$\{4\}$	14.5
$\{1,2\}$	16.5
$\{1,3\}$	17.5
$\{1,4\}$	18.5
$\{2,3\}$	19.5
$\{2,4\}$	20.5
$\{3,4\}$	22.5
$\{1,2,3\}$	25.5
$\{1,2,4\}$	26.5
$\{1,3,4\}$	28.5
$\{2,3,4\}$	31.5
$\{1,2,3,4\}$	38

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Optimal Cost Allocation Problem

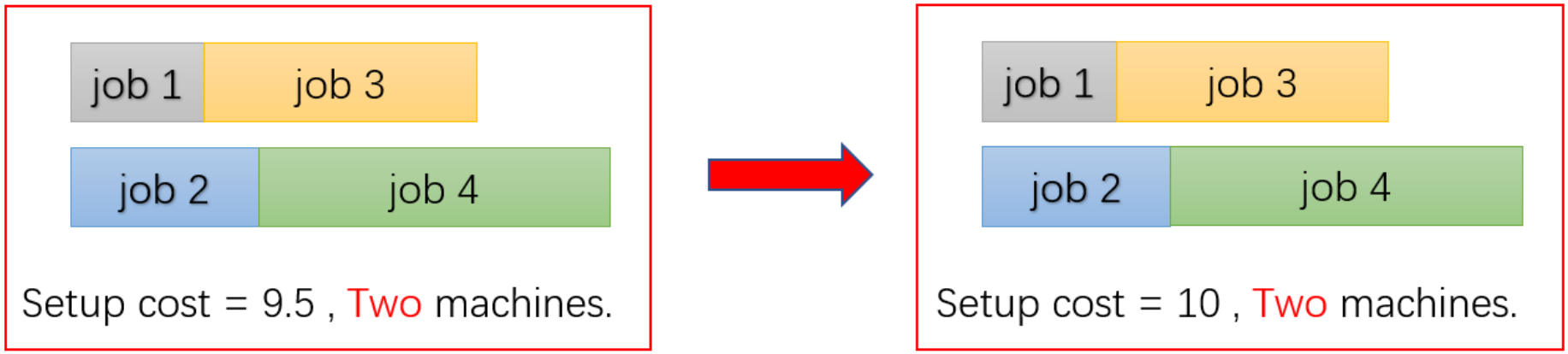
$$\begin{aligned} \max \ & \left(\alpha_1+\alpha_2+\alpha_3+\alpha_4\right) = 37.25 < 38 \\ s.t. \ & \alpha_1 \le 11.5, \ \cdots, \alpha_4 \le 14.5, \\ & \alpha_1+\alpha_2 \le 16.5, \ \cdots, \ \alpha_3+\alpha_4 \le 22.5, \\ & \cdots, \\ & \alpha_1+\alpha_2+\alpha_3+\alpha_4 \le 38. \end{aligned}$$

$$\alpha^* = \left[6; 8.75; 10.75; 11.75\right]$$

The minimum subsidy:
$C(V) - \alpha(V) = 38 - 37.25 = 0.75$

Example: Pricing Instrument

Increase the setup cost from 9.5 to 10.



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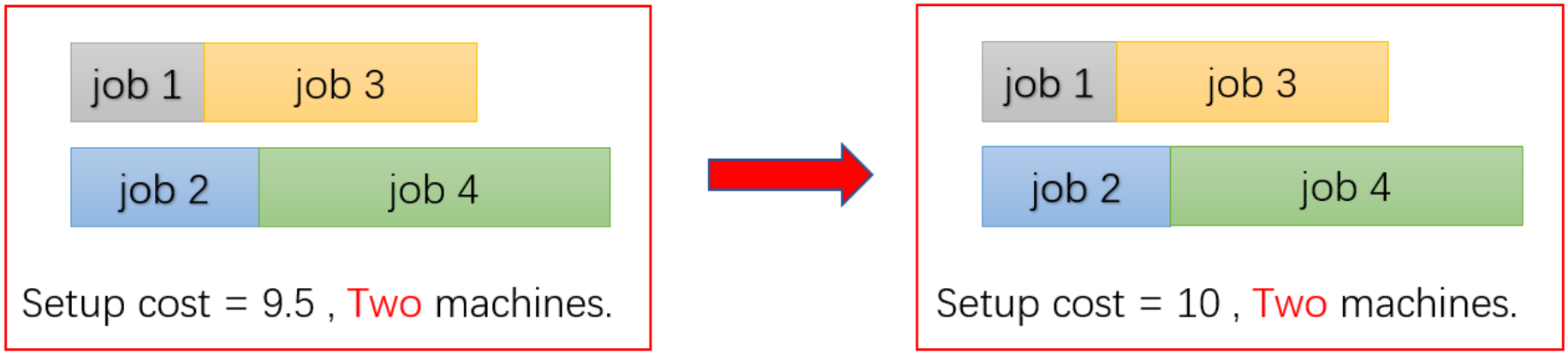
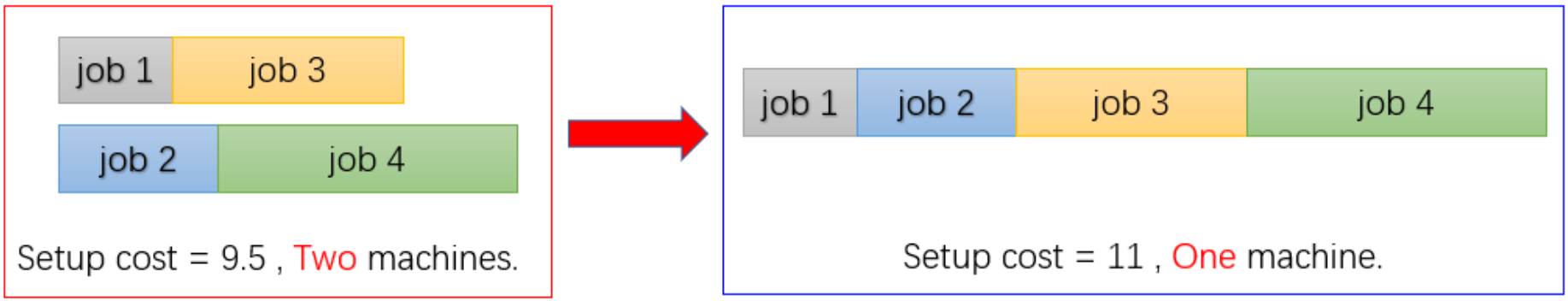


Setup cost	Increment	Num of Machines	Total pricing	c(V)	α(V)	Subsidy
9.5	0	2	0	38	37.25	0.75
10	0.5	2	1	39	38	1

The total pricing can exactly cover the gap, which means the grand coalition can be stabilized by the players themselves.

Example: Pricing Instrument

Increase the setup cost from 9.5 to 11.14.
For the grand coalition, it only needs one machine now.



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11.14	1.64	1	1.64	41.14	39.5	1.64

Models & Analyses

Problem Definition and Formulation

Definition

Parallel Machine Scheduling Game with Setup Cost (MSG):

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Parallel Machine Scheduling Game with Setup Cost (MSG):

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- Identical machines: $M = \{1, 2, \ldots, m\}$;
- Setup cost (unit price) of opening every single machine: P;
- Processing time of each job: t_k, $\forall k \in V$;
- Characteristic function: $C(S)$, denoting the total completion time and setup cost for each coalition $S \in \mathbb{S}$.

Problem Definition and Formulation

Definition

The characteristic function value, $C(S)$, of MSG is given by ILP

$$C(S,P) = \min \sum_{k\in V}\sum_{j\in O} c_{kj}x_{kj} + P\sum_{k\in S} x_{k1}$$

$$s.t. \quad \sum_{j\in O} x_{kj} - y_k^S = 0, \forall k \in V,$$

$$\sum_{k\in V} x_{kj} \leq m, \forall j \in O,$$

$$x_{kj} \in \{0,1\}, \forall k \in V, \forall j \in O,$$

$$y_k^S = 1, k \in S\ ; y_k^S = 0, k \notin S.$$

Properties

Definition

- $\left[P_L(i,S), P_H(i,S)\right]$: Price range of using i machines for scheduling jobs among players S;

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- $\left[0, P^*\right]$: Effective domain of pricing MSG for stabilization;

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- $\left[P_L(i,S), P_H(i,S)\right]$: Price range of using i machines for scheduling jobs among players S;
- $\left[0, P^*\right]$: Effective domain of pricing MSG for stabilization;
- P_i: For easy of exposition, let $P_1 = P^*$ and $P_i = P_H(i,V) = P_L(i-1,V)$. Thus, the effective domain of $\left[0, P^*\right]$ is divided into v non-overlapping sub-intervals by P_i, $\forall i = \{2,3,...,v\}$.

Note: P^* is the lowest price under which the grand cooperation of MSG is stable and it uses only one machine in the optimal scheduling decision, i.e., $P^* \in \left[P_L(1,V), P_H(1,V)\right]$ and MSG $\left(V, C(\ \cdot\ , P^*)\right)$ has non-empty core.

Properties

$$\omega(P) = \min_{\alpha}\{C(V,P) - \alpha(V) : \alpha(S) \leq C(S,P), \forall S \in \mathbb{S}, \alpha \in \mathbb{R}^{v}\};$$

Theorem 1

$\omega(P)$ is piecewise linear, and convex in price P at each sub-interval $[P_{i+1}, P_i]$, where $i = \{1, 2, ..., v-1\}$.

Lemma 1

$P_i, 2 \leq i \leq v$ can be obtained by SPT rules.

Theorem 2

$P_1 = P_2 + \cdots + P_v = \sum_{i=2}^{v} P_i$.

Theorem 3

$\omega(P)$ can be bounded by zero when the number of using machines, m_V, is larger than $\frac{n}{2}$.

Properties

Theorem 4

When the number of using machines is 1 for the grand coalition, the range of slopes of the line segments in the interval is $\left(-1, -\frac{1}{v-1}\right]$, and the number of breakpoints is $O(v^2)$.

Define characteristic function of the single machine scheduling game:

$$C'(S,P) = \min \sum_{k\in V}\sum_{j\in O} c_{kj}x_{kj} + P$$

$$s.t. \quad \sum_{j\in O} x_{kj} - y_k^S = 0, \forall k \in V,$$

$$\sum_{k\in V} x_{kj} \leq 1, \forall j \in O,$$

$$x_{kj} \in \{0,1\}, \forall k \in V, \forall j \in O,$$

$$y_k^S = 1, k \in S\,; y_k^S = 0, k \notin S.$$

Properties

Theorem 5

Define that

$$\omega_1(P) = \min_\alpha\{C(V,P) - \alpha(V) : \alpha(S) \leq C'(S,P), \forall S \in \mathbb{S} \setminus \{V\}, \alpha \in \mathbb{R}^v\}$$

Then the original problem $\omega(P)$ is equivalent to $\omega_1(P)$, where all sub-coalitions only use one machine.

Algorithms & Computations

IPC Algorithm

The Intersection Points Computation Algorithm to Construct $\omega(P)$ Function.

Step 1. Initially, set $I^* = \{P_L, P_H\}$ and $\mathbb{I} = \{[P_L, P_H]\}$.

Step 2. If $\mathbb{I}$ is not empty, update I^* and $\mathbb{I}$ by the following steps:

Step 3. Sort values in I^* by $P_0 < P_1 < \cdots < P_q$, where $P_0 = P_L, P_q = P_H$ and $q = |I^*| - 1$.

Step 4. Select any interval from $\mathbb{I}$, denoted by $[P_{k-1}, P_k]$ with $1 \leq k \leq q$.

Step 5. Construct two linear function $R_{k-1}(P)$ and $L_k(P)$ so that $R_{k-1}(P)$ passes $(P_{k-1}, \omega(P_{k-1}))$ with a slope equal to a right derivative $K_r^{P_{k-1}}$ of $\omega(P)$ at P_{k-1}, and that $L_k(z)$ passes $(P_k, \omega(P_k))$ with a slope equal to a left derivative $K_l^{P_k}$ of $\omega(P)$ at P_k.

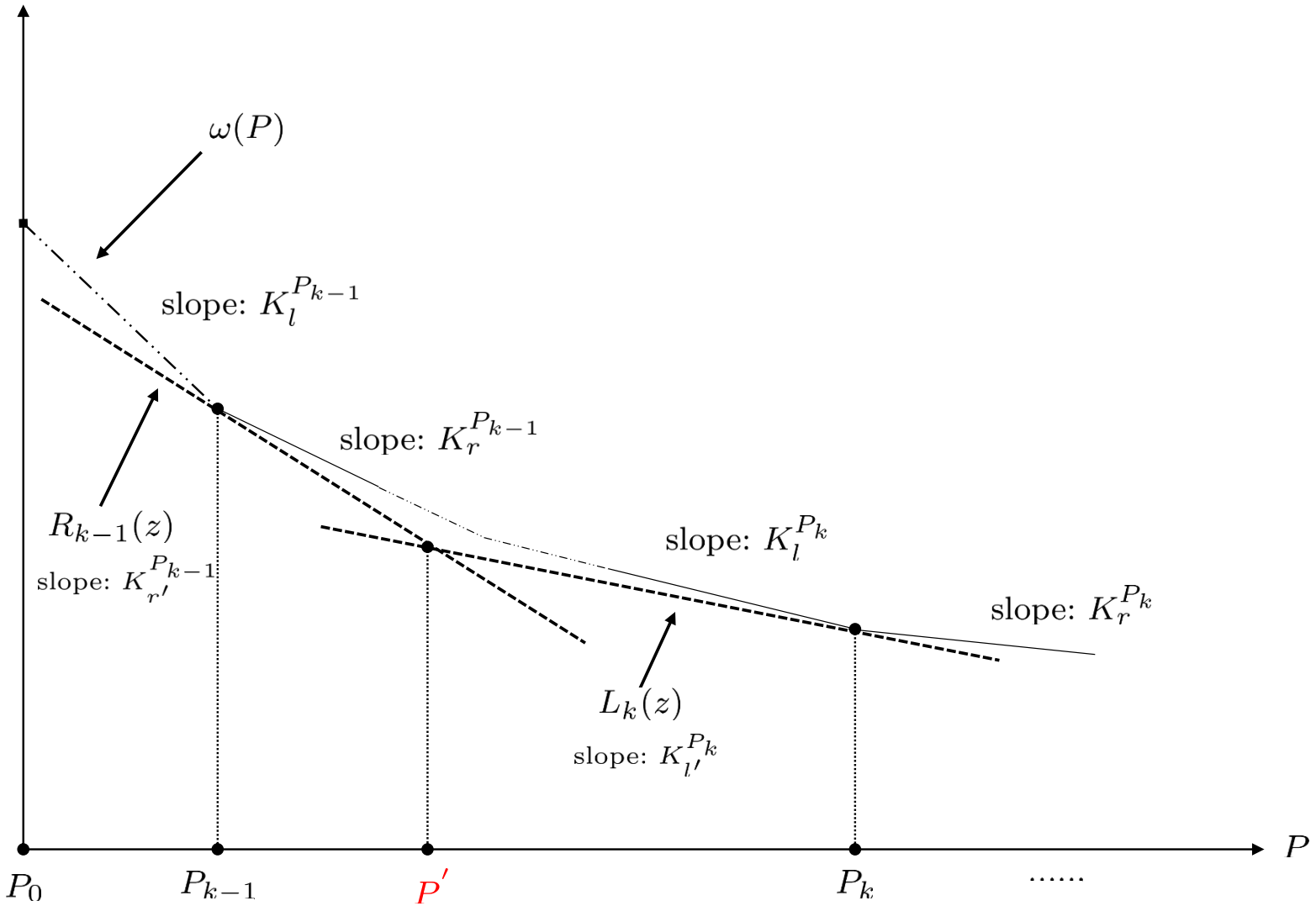
IPC Algorithm

Step 6. If $R_{k-1}(P)$ passes $(P_k, \omega(P_k))$ or $L_k(P)$ passes $(P_{k-1}, \omega(P_{k-1}))$, then update $\mathbb{I}$ by removing $[P_{k-1}, P_k]$. Otherwise, $R_{k-1}(P)$ and $L_k(P)$ must have a unique intersection point at $P = P'$ for some $P' \in (P_{k-1}, P_k)$. Update I^* by adding P', and update $\mathbb{I}$ by removing $[P_{k-1}, P_k]$, adding $[P_l, P']$ and $[P', P_r]$.

Step 7. Go to step 2.

Step 8. Return a piecewise linear function by connecting points $(P, \omega(P))$ for all $P \in I^*$.

IPC Algorithm



CP Algorithm

The Cutting Plane Algorithm to compute $\omega(P)$ for a given P.

Step 1. Let $\mathbb{S}' \subseteq \mathbb{S} \setminus \{V\}$ indicates a restricted coalition set, which includes some initial coalitions, e.g.,$\{1\}, \{2\}, \ldots, \{v\}$.

Step 2. Find an optimal solution $\bar{\alpha}(\,\cdot\,, P)$ to LP $\tau(P)$:

$$\max_{\alpha \in \mathbb{R}^n} \left\{ \alpha(V, P) : \alpha(S, P) \leq C'(S, P), \text{ for all } S \in \mathbb{S}' \right\}.$$

Step 3. Find an optimal solution S^* to the separation problem:

$$\delta = \min \left\{ C'(S, P) - \bar{\alpha}(S, P) : \forall S \in \mathbb{S} \setminus \{V\} \right\}.$$

Step 4. If $\delta < 0$, then add S^* to $\mathbb{S}'$, and go to step 2; otherwise, return $\omega(P) = C(V, P) - \bar{\alpha}(V, P)$.

DP Algorithm

The Dynamic Programming Algorithm to solve the separation problem.

- Assume that the processing jobs satisfy $t_1 \geq t_2 \geq \ldots \geq t_v$.
- For the separation problem $\delta_{AIPU} = \min\left\{C'(S,P) - \bar{\alpha}(S,P) : \forall S \in \mathbb{S} \setminus \{V\}\right\}$, where $C'(S,P)$ can be obtained by the SPT rule.
- That means if add a new player $k \notin S$ into S, where $|S| = u$, the increment for the restricted separation problem is $(u+1)t_k - \alpha_k$, which will be shown in the recursion in **Step 3**.

Step 1. Initially, let $D(k,u)$ indicate the minimum objective value of the restricted problem of separation problem δ_{AIPU}, where k is a player in the grand coalition and u is the number of players included in S. So $k \in \{1,2,\ldots,v\}$ and $u \in \{0,1,\ldots,v\}$.

DP Algorithm

Step 2. Given the initial conditions $D(1,0) = P$ and $D(1,1) = t_1 - \alpha_1 + P$. The boundary conditions are $D(k,u) = \infty$ if $u > k$, for all $k \in V$.

Step 3. Given the recursion:

$$D(k,u) = \min \begin{cases} D(k-1,u), \text{for the case when } S^* \text{ does not contain } k, \\ D(k-1,u-1) + ut_k - \alpha_k, \text{for the case when } S^* \text{ contains } k. \end{cases}$$

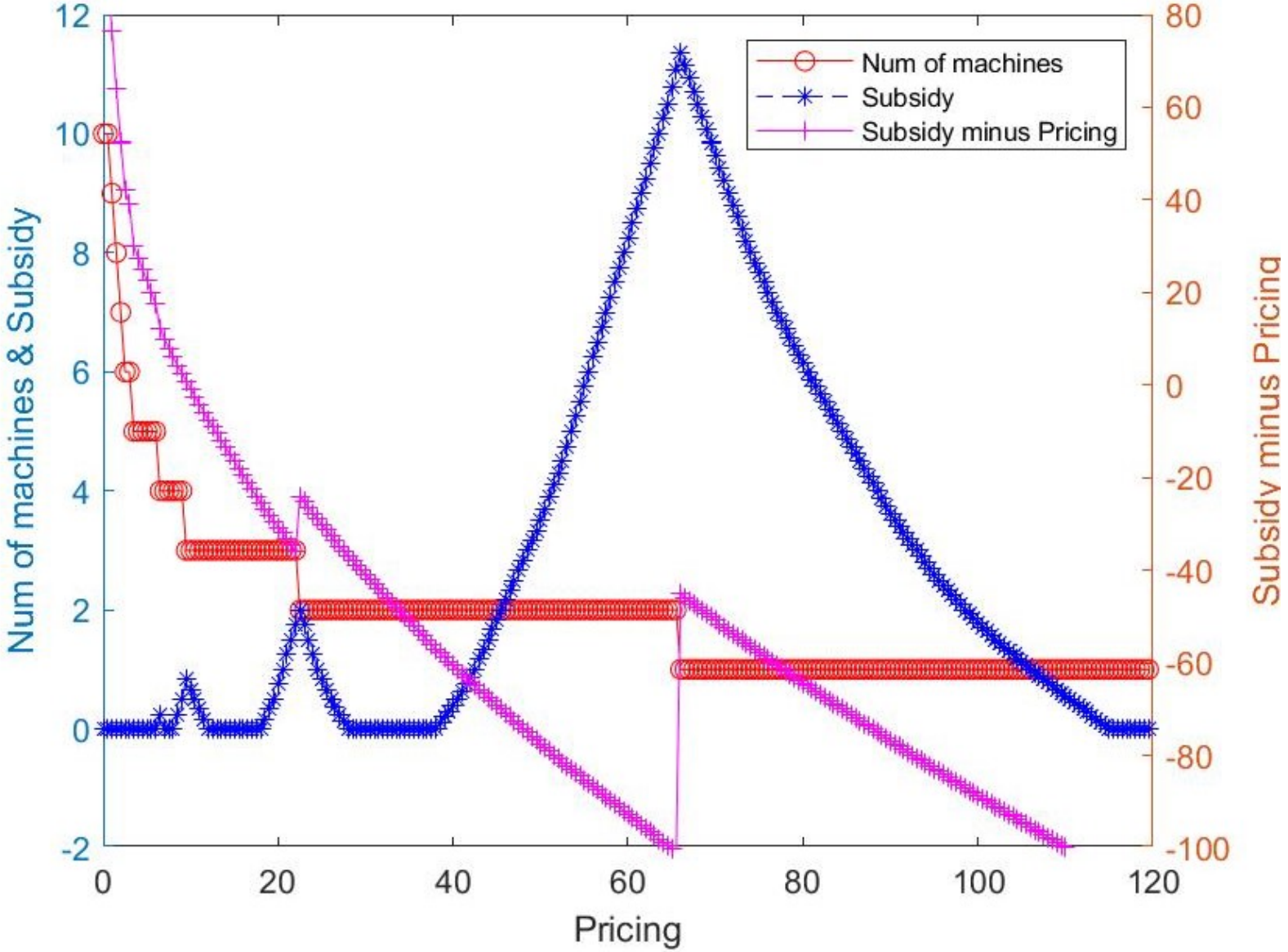
Step 4. Obtain the optimal objective value of separation problem by $\delta_{AIPU} = \min\{D(v,u) : u \in \{1,2,\ldots,v-1\}\}$. return δ_{AIPU}.

The DP Algorithm can solve the separation problem δ_{AIPU} in $O(v^2)$ time.

Notice that $k \in \{1,2,\ldots,v\}$ and $u \in \{0,1,\ldots,v\}$ for the $D(k,u)$, so the total running time is in $O(v^2)$.

Computational Results

Processing time:[1, 1.5, 2, 2.5, 3.5, 4, 4, 6.5, 6.5, 7]



Extension & Generalization

Machine Scheduling Game with Weighted Jobs

Definition

A Machine Scheduling Game with Weighted Jobs:

- Each job $k \in V$ has a processing time, t_k, and a weight, w_k.

Properties

- $C(S)$ and $P_i, 2 \leq i \leq v$ can also be obtained by assuming that $t_1/\omega_1 \leq t_2/\omega_2 \leq \ldots \leq t_v/\omega_v$.
- $\omega(P)$ is also piecewise linear, and convex in price P at each subinterval.
- IPC, CP, DP Algorithms can also be uesd to construct $\omega(P)$ function.

Pricing in General IM Games

Definition

General Integer Minimization Games:

- $C(S,P) = \min_x\{cx + Pm(x) : Ax \geq By^S + D, \beta x \leq m, x \in \mathbb{Z}^t\}$
- Define $H_0(V,i) = C(V,P) - Pi$, where $m(x^*) = i$

Properties

The following properties illustrate that P_i is in descending order:

- $C_0(V,i-1) - C_0(V,i) > 0 \Leftrightarrow P_i > 0, i = 2,\ldots,v.$
- $C_0(V,i) - C_0(V,i+1) < C_0(V,i-1) - C_0(V,i) \Leftrightarrow P_i > P_{i+1}, i = 2,3,\ldots,v-1.$

Conclusions

- ⋆ Cooperative Game Theory:
  - New Instrument for Stabilization via Setup cost Pricing.
- ⋆ Scheduling Problem:
  - Parallel Machine Scheduling with Setup Cost.
- ⋆ Models, Solution Methods and Applications:
  - Several ILP formulations;
  - Cutting Plane to solve the seperation problem;
  - Implementations on the MSGW game.

The End

Thank you!