Example of Cooperative Game

- There are 3 players, each having a job to do
  - The cost of each working individually
    $$V(\{1\}) = V(\{2\}) = V(\{3\}) = 10$$
  - The cost of any two working collaboratively
    $$V(\{1\}) = V(\{2\}) = V(\{3\}) = 14$$
  - The cost of all three working collaboratively
    $$V(\{1, 2, 3\}) = 18$$
- Question: are the three willing to work collaboratively?
  - We need a way of sharing the cost $V(\{1, 2, 3\}) = 18$ among the players
  - An easy solution $(6, 6, 6)$
- How about the following ways of sharing cost?
  $\{7, 7, 4\}, \{8, 6, 4\}, \{4, 4, 10\}, \ldots$

The Formulation

- We need a way of sharing the cost $V(\{1,2,3\}) = 18$ among the players, $\{x_1, x_2, x_3\}$, satisfying

$$
\begin{aligned}
x_1 &\leq 10, \\
x_2 &\leq 10, \\
x_3 &\leq 10, \\
x_1 + x_2 &\leq 14, \\
x_1 + x_3 &\leq 14, \\
x_2 + x_3 &\leq 14, \\
x_1 + x_2 + x_3 &= 18.
\end{aligned}
$$

- All are feasible solutions
  - $\{6,6,6\}, \{7,7,4\}, \{8,6,4\}, \{4,4,10\}, \ldots$
  - They are said to be in the **core** of the game

The Core May Be Empty

- Suppose that

$$\begin{aligned}&V(\{1\}) = V(\{2\}) = V(\{3\}) = 10\\&V(\{1,2\}) = V(\{1,3\}) = V(\{2,3\}) = 14\\&\underline{V(\{1,2,3\}) = 22}\end{aligned}$$

- There is no feasible solution to the following constraints

$$\begin{aligned}&x_1 \leq 10, x_2 \leq 10, x_3 \leq 10,\\&x_1 + x_2 \leq 14, x_1 + x_3 \leq 14, x_2 + x_3 \leq 14\\&x_1 + x_2 + x_3 = 22\end{aligned}$$

- Core is empty
- Centralized optimal decision cannot be reached

Scheduling with Machine Activation Cost

activation cost K= 9.5

p_1=2 p_2=3 p_3=4 p_4=5

No cooperation

C=11.5 p_1=2

C=12.5 p_2=3

C=13.5 p_3=4

C=14.5 p_4=5

Coalition of p_1 and p_2

p_1=2 p_2=3

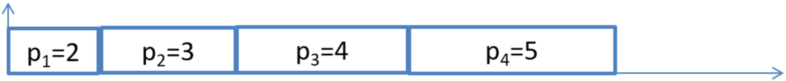
C = 9.5 + 2 + 5 = 16.5
To be shared by the two

Coalition of p_1, p_2, p_3

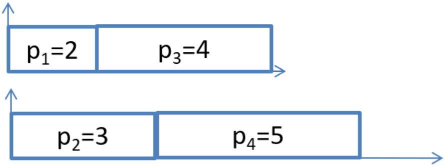
p_1=2 p_2=3 p_3=4

C = 9.5 + 2 + 5 + 9= 25.5
To be shared by the three

The Grand Coalition



C = 9.5 + 2 + 5 + 9 + 14 = 39.5 (**Optimal?**)



C = 9.5 + 2 + 6 + 9.5 +3 + 8 = 38

To be shared by the four (**How?**)

Related Concepts

- Approximate core
  - ???
- Least core
  - ???
- ...

- Focusing on bounds
- How to help making decisions?

Subsidization

Optimal Cost Allocation Problem

$$\max \ (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 37.25 < 38$$
$$s.t. \ \ \alpha_1 \leq 11.5, \ \cdots, \alpha_4 \leq 14.5,$$
$$\alpha_1 + \alpha_2 \leq 16.5, \ \cdots, \ \alpha_3 + \alpha_4 \leq 22.5,$$
$$\cdots,$$
$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 38.$$

$$\alpha^* = [6; 8.75; 10.75; 11.75]$$

Subsidy

An outside party chips in $\omega = 0.75$, making up the deficit.

Penalization

Coalitions	Cost
$\{1\}$	11.5
$\{2\}$	12.5
$\{3\}$	13.5
$\{4\}$	14.5
$\{1,2\}$	16.5
$\{1,3\}$	17.5
$\{1,4\}$	18.5
$\{2,3\}$	19.5
$\{2,4\}$	20.5
$\{3,4\}$	22.5
$\{1,2,3\}$	25.5
$\{1,2,4\}$	26.5
$\{1,3,4\}$	28.5
$\{2,3,4\}$	31.5
$\{1,2,3,4\}$	38

The outside party as the rule maker:

For any coalition that does not join the grand coalition, please pay a penalty of z.

$$\min z$$
$$s.t.\quad \alpha_1 \leq 11.5 + z, \quad \cdots, \alpha_4 \leq 14.5 + z,$$
$$\alpha_1 + \alpha_2 \leq 16.5 + z, \quad \cdots, \quad \alpha_3 + \alpha_4 \leq 22.5 + z,$$
$$\cdots,$$
$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \omega \leq 38.$$

$$Z_{\min} = 0.5$$

Result: No one pays the penalty

Simultaneous Subsidization and Penalization

Coalitions	Cost
$\{1\}$	11.5
$\{2\}$	12.5
$\{3\}$	13.5
$\{4\}$	14.5
$\{1,2\}$	16.5
$\{1,3\}$	17.5
$\{1,4\}$	18.5
$\{2,3\}$	19.5
$\{2,4\}$	20.5
$\{3,4\}$	22.5
$\{1,2,3\}$	25.5
$\{1,2,4\}$	26.5
$\{1,3,4\}$	28.5
$\{2,3,4\}$	31.5
$\{1,2,3,4\}$	38

Outside party as the rule maker:

(1) For any coalition that does not join the grand coalition, please pay a penalty of z.

(2) Outside party subsidizes grand coalition ω.

$$s.t.\quad \alpha_1 \le 11.5 + z, \quad \cdots, \alpha_4 \le 14.5 + z,$$
$$\alpha_1 + \alpha_2 \le 16.5 + z, \quad \cdots, \quad \alpha_3 + \alpha_4 \le 22.5 + z,$$
$$\cdots,$$
$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \omega \le 38.$$

For example, $\omega = 1/2, z = 1/6$