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LIPSCHITZ CONTINUITY OF SOLUTIONS OF LINEAR INEQUALITIES, PROGRAMS AND COMPLEMENTARITY PROBLEMS*

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Abstract. It is shown that solutions of linear inequalities, linear programs and certain linear complementarity problems (e.g. those with P-matrices or Z-matrices but *not* semidefinite matrices) are Lipschitz continuous with respect to changes in the right-hand side data of the problem. Solutions of linear programs are *not* Lipschitz continuous with respect to the coefficients of the objective function. The Lipschitz constant given here is a generalization of the role played by the norm of the inverse of a nonsingular matrix in bounding the perturbation of the solution of a system of equations in terms of a right-hand side perturbation.

Key words. linear inequalities, linear programming, linear complementarity problems, Lipschitz continuity, perturbation analysis

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1. Introduction. The purpose of this work is to show that solutions of linear inequalities, linear programs and certain linear complementarity problems are Lipschitz continuous with respect to changes in the right-hand side of the problem. Speaking in general and in somewhat loose terms, if we denote, by r^1 and r^2, two distinct right-hand sides, then there exist corresponding solutions x^1 and x^2 such that

$$\|x^1-x^2\| \leqq K\|r^1-r^2\| \tag{1.1}$$

where the Lipschitz constant K depends only on the matrix defining the problem, but not on the right-hand sides nor the objective function if there is one. A key role in determining the Lipschitz constant K is played by the condition number for linear inequalities, introduced in [11], which is a generalization of the very useful concept of a condition number for a nonsingular square matrix [3]. In [19] Robinson obtained local Lipschitz continuity results for generalized equations which include linear programs, convex quadratic programs and monotone linear complementarity problems. Robinson's Lipschitz constant [19, Thm. 2] involves a bound on the solution set which is assumed to be bounded. By contrast our Lipschitz constants are global, and our solution sets need not be bounded. In [18] Robinson obtained a Lipschitz constant for the perturbation of linear inequalities which is different from our constant (2.5).

We give now a summary of our principal results. Theorem 2.2 deals with a system of linear inequalities and equalities (2.1) and shows that if the system is solvable for right-hand sides r^1 and r^2, then for each solution x^1 for right-hand side r^1 there exists a solution x^2 for right-hand side r^2 such that (1.1) holds. The Lipschitz constant here plays the same role as the norm of the inverse of a nonsingular matrix does for a system of linear equations. Our Lipschitz constant for the system (2.1) defined by (2.5), is a minor variation of the constant (6) of [11] but is different from Robinson's [18]. Furthermore, the Lipschitz continuity Theorem 2.2 leads in a very elementary way to Theorem 2.2′ which is essentially equivalent to Theorem 1 of [11] and to Hoffman's theorem [8], [18] and which gives an estimate of the error in an approximate solution to the systems of linear inequalities and equalities (2.1) in terms of the residual of the

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approximate solution and the Lipschitz constant. Again the role played in Theorem 2.2′ by the Lipschitz constant is an extension of the same role played by the norm of the inverse of a matrix for a system of linear equations. Computation of the Lipschitz constant (2.5) for the system of linear inequalities and equalities (2.1) is quite difficult, but an important fact is that such a constant exists and is finite. For some special cases such as when we have strongly stable linear inequalities only (that is linear inequalities solvable for all right-hand sides) the Lipschitz constant can be computed by a single linear program as in (2.17) below. By using the Lipschitz constant for linear inequalities and equalities we show in Theorem 2.4 that solutions of linear programs are also Lipschitz continuous with respect to right-hand side perturbations only. Proposition 2.6 shows that our Lipschitz constant (2.20) for the linear program (2.18) is sharper than that of Cook et al. [4, Thm. 5]. By means of a simple example (2.26), we show that solutions of linear programs are *not* Lipschitz continuous with respect to perturbations in the objective function coefficients. Finally in § 3 by using the Lipschitz constant for linear inequalities and equalities we establish in Theorem 3.2 Lipschitz continuity of solutions of linear complementarity problems with respect to right-hand side perturbations that generate unique solutions along the line segment joining perturbed and unperturbed right-hand sides. A simple consequence of this result is Theorem 3.3 which shows that the solution of a linear complementarity problem with a P-matrix (that is a matrix with positive principal minors) is Lipschitz continuous with respect to right-hand side perturbations. Example 3.4 shows that solutions of positive semidefinite linear complementarity problems are not Lipschitz continuous with respect to their right-hand sides. Finally by exploiting the fact that for certain classes of matrices such as Z-matrices (real matrices with nonpositive off-diagonal elements) the linear complementarity problem can be solved as a linear program [10], Lipschitz continuity of solutions of such linear complementarity problems are obtained in Theorem 3.5.

A brief word about notation and some basic concepts employed. For a vector x in the n-dimensional real space R^n, $|x|$ and x_+ will denote the vectors in R^n with components $|x|_i := |x_i|$ and $(x_+)_i := \max\{x_i, 0\}$, $i = 1, \cdots, n$, respectively. For a norm $\|x\|_\beta$ on R^n, $\|x\|_{\beta^*}$ will denote the dual norm [9], [16] on R^n, that is $\|x\|_{\beta^*} := \max_{\|y\|_\beta = 1} xy$, where xy denotes the scalar product $\sum_{i=1}^n x_i y_i$. The generalized Cauchy-Schwarz inequality $|xy| \leqq \|x\|_\beta \cdot \|y\|_{\beta^*}$, for x and y in R^n, follows immediately from this definition of the dual norm. For $1 \leqq p, q \leqq \infty$, and $(1/p) + (1/q) = 1$, the p-norm $(\sum_{i=1}^n |x_i|^p)^{1/p}$ and the q-norm are dual norms on R^n [16]. If $\|\cdot\|_\beta$ is a norm on R^n, we shall, with a slight abuse of notation, let $\|\cdot\|_\beta$ also denote the corresponding norm on R^m for $m \neq n$. For an $m \times n$ real matrix A, A_i denotes the ith row, $A_{.j}$ denotes the jth column, $A_I := A_{i \in I}$, and $A_{.J} := A_{.j \in J}$, where $I \subset \{1, \cdots, m\}$ and $J \subset \{1, \cdots, n\}$. $\|A\|_\beta$ denotes the matrix norm [16], [20] subordinate to the vector norm $\|\cdot\|_\beta$, that is $\|A\|_\beta = \max_{\|x\|_\beta = 1} \|Ax\|_\beta$. The consistency condition $\|Ax\|_\beta \leqq \|A\|_\beta \|x\|_\beta$ follows immediately from this definition of a matrix norm. A monotonic norm on R^n is any norm $\|\cdot\|$ on R^n such that for a, b in R^n, $\|a\| \leqq \|b\|$ whenever $|a| \leqq |b|$ or equivalently if $\|a\| = \| |a| \|$ [9, p. 47]. The p-norm for $p \geqq 1$ is monotonic [16]. A vector of ones in any real space will be denoted by e. The identity matrix of any order will be denoted by I. The nonnegative orthant in R^n will be denoted by R^n_+. The abbreviation rhs will denote "right-hand side."

2. Linear inequalities and programs. We shall first be concerned with Lipschitz continuity of solutions of the following set of linear inequalities with respect to changes in the right-hand side

$$Ax \leqq b, \qquad Cx = d \tag{2.1}$$

where b and d are given points in R^m and R^k, respectively, $A \in R^{m\times n}$, that is an $m \times n$ real matrix and $C \in R^{k\times n}$. We shall employ a slight variation of the condition constant introduced in [11, Eq. (6)] for linear inequalities and programs as our Lipschitz constant for the linear inequalities (2.1) and subsequently for the linear program (2.18) and the linear complementarity problem (3.1).

We begin with a simple extension of the fundamental theorem on basic solutions [6, Thm. 2.11] to unrestricted as well as nonnegative variables.

LEMMA 2.1 (Basic solutions). *Let $A \in R^{m\times n}$, $C \in R^{k\times n}$ and $p \in R^n$. The system*

$$A^T u + C^T v = p, \qquad u \geqq 0 \tag{2.2}$$

has a solution $(u, v) \in R^{m+k}$ if and only if it has a basic solution, that is a solution (u, v) such that the rows of $\binom{A}{C}$ corresponding to nonzero components of (u, v) are linearly independent.

Proof. The system (2.2) having a solution (u, v) implies that

$$A^T u + \tilde{C}^T v = p, \qquad (u, v) \geqq 0 \tag{2.3}$$

has a solution where $\tilde{C}$ is obtained from C by multiplying by -1 those rows of C corresponding to negative components of v. It follows from the fundamental theorem on basic solutions [6, Thm. 2.11] that (2.3) has a basic solution and consequently so does (2.2). □

We proceed now to establish Lipschitz continuity of solutions of (2.1) with respect to right-hand side perturbations. Robinson [18, Cor. 2.2] gives this result with a different Lipschitz constant.

THEOREM 2.2 (Lipschitz continuity of feasible points of linear inequalities and equalities). *Let the linear inequalities and equalities* (2.1) *have nonempty feasible sets S^1 and S^2 for the right-hand sides (b^1, d^1) and (b^2, d^2), respectively. For each $x^1 \in S^1$ there exists an $x^2 \in S^2$ closest to x^1 in the ∞-norm such that*

$$\|x^1 - x^2\|_\infty \leqq \mu_\beta(A; C) \left\| \begin{matrix} b^1 - b^2 \\ d^1 - d^2 \end{matrix} \right\|_\beta \tag{2.4}$$

where $\|\cdot\|_\beta$ is some norm on R^{m+k} and

$$\mu_\beta(A; C) := \sup_{u,v} \left\{ \left\| \begin{matrix} u \\ v \end{matrix} \right\|_{\beta^*} \;\middle|\; \begin{matrix} \|uA + vC\|_1 = 1, \qquad u \geqq 0. \\ \textit{Rows of } \binom{A}{C} \textit{ corresponding to nonzero} \\ \textit{elements of } \binom{u}{v} \textit{ are linear independent} \end{matrix} \right\}. \tag{2.5}$$

Proof. We note that $\mu_\beta(A; C)$ is finite. For if not, there would exist *fixed* subsets I and J of $\{1, \cdots, m\}$ and $\{1, \cdots, k\}$, respectively, and a sequence $\{u_I^i, v_J^i\}$ such that $\{\|u_I^i, v_J^i\|\} \to \infty$ and the rows of $\binom{A_I}{C_J}$ are linearly independent. Hence a subsequence

$$\left\{ \frac{(u_I^{i_j}, v_J^{i_j})}{\|u_I^{i_j}, v_J^{i_j}\|} \right\}$$

converges to $(\bar{u}_I, \bar{v}_J)$ satisfying $\bar{u}_I A_I + \bar{v}_J C_J = 0$, $\|\bar{u}_I, \bar{v}_J\| = 1$, which contradicts the linear independence of the rows of $\binom{A_I}{C_J}$.

Now let $x^1 \in S^1$. Choose $x^2 \in S^2$ which is closest to x^1 in the ∞-norm. Thus x^2 must solve

$$\min_x \|x - x^1\|_\infty \quad \text{s.t. } Ax \leqq b^2, \quad Cx = d^2 \tag{2.6}$$

which is equivalent to the linear program

$$\min_{x,\delta} \delta \quad \text{s.t. } Ax \leqq b^2, \quad Cx = d^2, \quad x + e\delta \geqq x^1, \quad -x + e\delta \geqq -x^1. \tag{2.7}$$

Hence (x^2, δ^2) and some $(u^2, v^2, r^2, s^2) \in R^{m+k+2n}$ satisfy the following Karush-Kuhn-Tucker conditions for (2.7)

$$\begin{aligned} & Ax^2 \leqq b^2, \quad Cx^2 = d^2, \quad \|x^1 - x^2\|_\infty = \delta^2, \\ (2.8) \qquad & u^2(-Ax^2 + b^2) = 0, \quad r^2(x^2 + e\delta^2 - x^1) = 0, \quad s^2(-x^2 + e\delta^2 + x^1) = 0, \\ & -u^2A + v^2C + r^2 - s^2 = 0, \quad e(r^2 + s^2) = 1, \quad (u^2, r^2, s^2) \geqq 0. \end{aligned}$$

Note that if $0 = \delta^2 = \|x^1 - x^2\|_\infty$, then (2.4) is trivially true. So assume that $\delta^2 > 0$. It follows from $\delta^2 > 0$ and $r_j^2(x^2 + e\delta^2 - x^1)_j = 0$ and $s_j^2(-x^2 + e\delta^2 + x^1)_j = 0$ that $r_j^2 s_j^2 = 0$, for $j = 1, \cdots, n$. Hence

$$(2.9) \qquad -u^2A + v^2C + r^2 - s^2 = 0, \quad e(r^2 + s^2) = 1, \quad r^2 s^2 = 0, \quad (u^2, r^2, s^2) \geqq 0.$$

By Lemma 2.1 and $u^2(-Ax^2 + b^2) = 0$ it follows that we may take

$$u^2 = \begin{pmatrix} u_I^2 \\ 0 \end{pmatrix} \geqq 0 \quad \text{and} \quad v^2 = \begin{pmatrix} v_J^2 \\ 0 \end{pmatrix}$$

such that the rows of $\binom{A_I}{C_J}$ are linearly independent and $u_I^2(-A_Ix^2 + b_I^2) = 0$. Hence (2.9) becomes

$$\begin{aligned} & \|-u_I^2 A_I + v_J^2 C_J\|_1 = \|r^2 - s^2\|_1 = e(r^2 + s^2) = 1, \qquad u_I^2 \geqq 0, \\ & \text{Rows of } \tbinom{A_I}{C_J} \text{ linear independent.} \end{aligned}$$

Hence by (2.5) we have

$$(2.10) \qquad \left\| \begin{matrix} u^2 \\ v^2 \end{matrix} \right\|_{\beta^*} \leqq \mu_\beta(A; C).$$

We now have

$$\begin{aligned} \|x^1 - x^2\|_\infty &= \delta^2 = -b^2u^2 + d^2v^2 + x^1(r^2 - s^2) \\ &= -b^2u^2 + d^2v^2 + x^1(A^Tu^2 - C^Tv^2) \\ &= u^2(Ax^1 - b^2 + b^1 - b^1) + v^2(-Cx^1 + d^2 + d^1 - d^1) \\ (2.11) \qquad &\leqq u^2(b^1 - b^2) + v^2(d^2 - d^1) \\ &\leqq \left\| \begin{matrix} u^2 \\ v^2 \end{matrix} \right\|_{\beta^*} \left\| \begin{matrix} b^1 - b^2 \\ d^1 - d^2 \end{matrix} \right\|_\beta \\ &\leqq \mu_\beta(A; C) \left\| \begin{matrix} b^1 - b^2 \\ d^1 - d^2 \end{matrix} \right\|_\beta \qquad \text{(by (2.10))}. \qquad \square \end{aligned}$$

Note that the Lipschitz constant $\mu_\beta(A; C)$ of (2.4) plays the same role as that of the norm of the inverse of a nonsingular matrix of a system of linear equations. This fact can be seen more clearly from the following corollary to Theorem 2.2 applied to systems solvable for all right-hand sides (i.e. strongly stable) systems. Note also that we can get a sharper result by replacing $(b^1 - b^2)$, in (2.4) and (2.11) onward, by $(b^1 - b^2)_+$.

COROLLARY 2.3 (Lipschitz continuity of feasible points of strongly stable linear inequalities). *Let $A \in R^{m \times n}$ and $C \in R^{k \times n}$ be such that*

$$(2.12) \qquad \begin{aligned} & \textit{Rows of } C \textit{ are linearly independent and} \\ & Ax < 0, \quad Cx = 0 \quad \textit{has a solution } x. \end{aligned}$$

Then the linear inequalities (2.1) *are solvable for all right-hand sides* $(b, d) \in R^{m+k}$. *For each* x^1 *in the solution set of* (2.1) *with right-hand sides* (b^1, d^1), *there exists an* x^2 *in the solution set of* (2.1) *with right-hand sides* (b^2, d^2) *such that*

$$\|x^1 - x^2\|_\infty \leqq \bar{\mu}_\beta(A; C)\left\|\begin{matrix} b^1 - b^2 \\ d^1 - d^2 \end{matrix}\right\|_\beta \tag{2.13}$$

where $\|\cdot\|_\beta$ *is some norm on* R^{m+k} *and*

$$\bar{\mu}_\beta(A; C) := \max_{(u,v)\in R^{m+k}} \left\{ \left\|\begin{matrix} u \\ v \end{matrix}\right\|_{\beta^*} \middle| \begin{matrix} \|uA + vC\|_1 = 1 \\ u \geqq 0 \end{matrix} \right\}. \tag{2.14}$$

Proof. That (2.1) is solvable for any right-hand side (b, d) follows from solving $Cx = d$ for x^d for any given d and then taking as the desired solution $x^d + \lambda\bar{x}$ for sufficiently large positive λ, where $\bar{x}$ solves $Ax < 0$, $Cx = 0$. The rest of the proof of the corollary is similar to the proof of Theorem 2.2, except that u^2 and v^2 are not decomposed into

$$\begin{pmatrix} u_I^2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_J^2 \\ 0 \end{pmatrix}.$$

The finiteness of $\bar{\mu}_\beta(A; C)$ of (2.14) follows from the boundedness of the feasible region of (2.14). For if it were unbounded, there would exist $\{u^i, v^i\}$ such that $\{\|u^i, v^i\|\} \to \infty$, and consequently an accumulation point $(\bar{u}, \bar{v})$ would exist such that

$$\bar{u}A + \bar{v}C = 0, \quad \bar{u} \geqq 0, \quad (\bar{u}, \bar{v}) \neq 0. \tag{2.15}$$

This, however, would contradict the linear independence of the rows of C if $\bar{u} = 0$, and if $\bar{u} \neq 0$ would contradict the solvability of $Ax < 0$, $Cx = 0$, because then $0 = \bar{u}Ax + \bar{v}Cx = \bar{u}Ax < 0$. □

Note that if A is vacuous and C is a nonsingular square matrix, then

$$\bar{\mu}_\infty(\phi; C) = \max_{v\in R^k} \{\|v\|_1 \,|\, \|vC\|_1 = 1\} = \|(C^T)^{-1}\|_1 = \|C^{-1}\|_\infty. \tag{2.16}$$

This was already pointed out in [11, Remark 2]. Note also that (2.14) can be written in the equivalent form

$$\bar{\mu}_\beta(A; C) = \max_{(u,v,z)\in R^{m+k+n}} \left\{ \left\|\begin{matrix} u \\ v \end{matrix}\right\|_{\beta^*} \middle| \begin{matrix} -z \leqq uA + vC \leqq z \\ u \geqq 0, \quad ez = 1 \end{matrix} \right\}. \tag{2.14$'$}$$

This is a difficult convex-function *maximization* problem on a polyhedral set which is closely related to the *NP*-complete problem of a norm-maximization problem on a polyhedral set for positive integer β^* [12]. However for $\beta^* = \infty$, that is $\beta = 1$, it can be shown, as in [12], that (2.14′) is in *P*. In addition a good bound for $\bar{\mu}_\beta(A; C)$ for any β can be obtained by solving a single linear program [12]. When C is empty and $\beta = \infty$, (2.14′) degenerates to the following linear program:

$$\bar{\mu}_\infty(A; \phi) = \max_{(u,z)\in R^{m+n}} \{eu \,|\, -z \leqq uA \leqq z, \ u \geqq 0, \ ez = 1\}. \tag{2.17}$$

We note that the Lipschitz constants $\mu_\beta(A; C)$ and $\bar{\mu}_\beta(A; C)$, which play the role of the norm of the inverse of a nonsingular matrix of a system of linear equations, can also be used, just as the norm of the inverse can, to obtain a bound on the error in an approximate solution in terms of the residual. Thus, if we assume for the moment that A is vacuous and that C is $n \times n$ and nonsingular, then $\bar{\mu}_\beta(\phi; C) = \|C^{-1}\|_\infty$ by

(2.16). Thus (2.5) and (2.13) are the extensions to a system of linear inequalities and equalities of the following simple Lipschitz continuity property of $Cx=d$

$$\|x^1-x^2\|_\infty \leqq \|C^{-1}\|_\infty \|d^1-d^2\|_\infty$$

where $x^1=C^{-1}d^1$ and $x^2=C^{-1}d^2$. Since $\|C^{-1}\|_\infty$ can also be used to estimate the error in an approximate solution x to $Cx^1=d^1$ in terms of its residual $\|Cx-d^1\|_\infty$ as follows:

$$\|x-x^1\|_\infty = \|C^{-1}(Cx-d^1)\|_\infty \leqq \|C^{-1}\|_\infty \|Cx-d^1\|_\infty,$$

it follows that the Lipschitz constants $\mu_\beta(A;\,C)$ and $\bar{\mu}_\beta(A;\,C)$ can be similarly used to give an estimate on the error in an approximate solution to (2.1) in terms of its residual. In fact this estimate has been given in [11, Thm. 1] and by Hoffman [8], [18] with a different constant. It also follows very easily from Theorem 2.2 above as follows.

THEOREM 2.2′ (Error bound for approximate solution of linear inequalities and equalities). *Let the linear inequalities and equalities* (2.1) *have a nonempty feasible set* S^1 *for the right-hand side* (b^1, d^1). *For each* x in R^n *there exists an* $x^1 \in S^1$ *such that*

$$\|x-x^1\|_\infty \leqq \mu_\beta(A;\,C) \left\| \begin{matrix} (Ax-b^1)_+ \\ Cx-d^1 \end{matrix} \right\|_\beta$$

where $\mu_\beta(A;\,C)$ *is defined by* (2.5).

Proof. Since for each $x \in R^n$

$$Ax \leqq b^1 + (Ax-b^1)_+, \qquad Cx = d^1 + (Cx-d^1)$$

it follows by Theorem 2.2 that there exists an $x^1 \in S^1$ such that the conclusion of the theorem holds. □

A similar error bound holds for strongly stable linear inequalities which is based on (2.13).

It is interesting to note that Theorem 2.2 is stronger than Theorem 2.2′ in the sense that the latter follows directly from the former as was demonstrated above, whereas the converse holds with the additional assumption that the norm $\|\cdot\|_\beta$ is a monotonic norm [9], [16]. Thus to obtain Theorem 2.2 from Theorem 2.2′, we have from Theorem 2.2′ that for each $x^1 \in S^1$ there exists an $x^2 \in S^2$ such that

$$\|x^2-x^1\|_\infty \leqq \mu_\beta(A;\,C) \left\| \begin{matrix} (Ax^1-b^2)_+ \\ Cx^1-d^2 \end{matrix} \right\|_\beta \leqq \mu_\beta(A;\,C) \left\| \begin{matrix} b^1-b^2 \\ d^1-d^2 \end{matrix} \right\|_\beta$$

where the last inequality follows from

$$(Ax^1-b^2)_+ = (Ax^1-b^2+b^1-b^1)_+ \leqq (b^1-b^2)_+ \leqq |b^1-b^2|,$$
$$|Cx^1-d^2| = |Cx^1-d^2+d^1-d^1| = |d^1-d^2|$$

and the monotonicity of the norm $\|\cdot\|_\beta$.

Next we establish the Lipschitz continuity with respect to right-hand side perturbation of solutions of the linear program

$$\max_x px \quad \text{s.t. } Ax \leqq b, \quad Cx = d \tag{2.18}$$

where $p \in R^n$ and A, b, C, d are as in (2.1). For the Lipschitz continuity results for linear programs we have to restrict the norms employed to monotonic norms [9], [16] and have to drop $u \geqq 0$ from (2.5). Lipschitz continuity results for more general optimization problems are given in [1], [7].

THEOREM 2.4 (Lipschitz continuity of solutions of linear programs with respect to right-hand side perturbation). *Let the linear program* (2.18) *have nonempty solution sets* S^1 *and* S^2 *for right-hand sides* (b^1, d^1) *and* (b^2, d^2), *respectively. For each* $x^1 \in S^1$ *there exists an* $x^2 \in S^2$ *such that*

$$\|x^1 - x^2\|_\infty \leqq \nu_\beta(A; C)\left\|\begin{matrix} b^1 - b^2 \\ d^1 - d^2 \end{matrix}\right\|_\beta \tag{2.19}$$

where $\|\cdot\|_\beta$ *is some monotonic norm on* R^{m+k} *and*

$$\nu_\beta(A; C) := \sup_{u,v}\left\{\left\|\begin{matrix} u \\ v \end{matrix}\right\|_{\beta^*} \;\middle|\; \begin{array}{l} \|uA + vC\|_1 = 1 \\ \textit{Rows of } \binom{A}{C} \textit{ corresponding to nonzero} \\ \textit{elements of } \binom{u}{v} \textit{ are linear independent} \end{array}\right\}. \tag{2.20}$$

Proof. Given $x^1 \in S^1$, let

$$A_I x^1 = b_I^1, \qquad A_J x^1 < b_J^1$$

where $I \cup J = \{1, 2, \cdots, m\}$. Fix any $\bar{x}^2 \in S^2$ and let $I = I_1 \cup I_2$ where

$$I_1 := \{i \in I \mid A_i \bar{x}^2 = b_i^2\}, \qquad I_2 := \{i \in I \mid A_i \bar{x}^2 < b_i^2\}.$$

Since $x = \bar{x}^2$ satisfies the system of constraints

$$\begin{array}{lll} \text{(i)} & A_{I_1} x = b_{I_1}^2, & \\ \text{(ii)} & A_{I_2} \bar{x}^2 \leqq A_{I_2} x, & A_{I_2} x \leqq b_{I_2}^2, \\ \text{(iii)} & A_J x \leqq b_J^2, & Cx = d^2, \end{array} \tag{2.21}$$

it follows that (2.21) is nonvacuous. As in Theorem 2.2, let x^2 be a solution of

$$\min \|x - x^1\|_\infty \quad \text{s.t. } (2.21). \tag{2.22}$$

Since (2.22) is a convex program, x^2 remains optimal after we remove any number of inactive constraints. For each $i \in I_2$, at least one of the two constraints of (2.21)(ii) is inactive because $A_i \bar{x}^2 < b_i^2$. So we can remove one inactive constraint for each $i \in I_2$ thus obtaining

$$\|x^2 - x^1\|_\infty = \min \|x - x^1\|_\infty \quad \text{s.t. } (2.24) = \min \|x - x^1\|_\infty \quad \text{s.t. } (2.21) \tag{2.23}$$

where

$$\begin{array}{lll} \text{(i)} & A_{I_1} x = b_{I_1}^2, & \\ \text{(iia)} & A_K \bar{x}^2 \leqq A_K x, & \\ \text{(iib)} & A_L x \leqq b_L^2, & \\ \text{(iic)} & A_J x \leqq b_J^2, & Cx = d^2 \end{array} \tag{2.24}$$

where $K \cup L = I_2$, $K \cap L = \phi$. So $I_1 \cup K \cup L \cup J = \{1, 2, \cdots, m\}$ and I_1, K, L and J are all disjoint. On the other hand, since

$$A_K x^1 = b_K^1 - b_K^2 + b_K^2 \geqq b_K^1 - b_K^2 + A_k \bar{x}^2,$$

it follows that $x = x^1$ satisfies the following system:

$$\begin{array}{lll} \text{(i)} & A_{I_1} x = b_{I_1}^1, & \\ \text{(iia)} & b_K^1 - b_K^2 + A_K \bar{x}^2 \leqq A_K x, & \\ \text{(iib)} & A_L x \leqq b_L^1, & \\ \text{(iic)} & A_J x \leqq b_J^1, & Cx = d^1. \end{array} \tag{2.24'}$$

It follows by (2.23), (2.24′), Theorem 2.2 and the norm monotonicity that

$$\|x^1-x^2\|_\infty \leqq \mu_\beta\left(\begin{pmatrix}-A_K\\ A_L\\ A_J\end{pmatrix};\ \begin{pmatrix}A_{I_1}\\ C\end{pmatrix}\right)\left\|\begin{matrix}-b_K^1+b_K^2\\ b_H^1-b_H^2\\ d^1-d^2\end{matrix}\right\|_\beta$$

$$\leqq \nu_\beta(A;\ C)\left\|\begin{matrix}b^1-b^2\\ d^1-d^2\end{matrix}\right\|_\beta$$

where $H=I_1\cup L\cup J$ is the complement of K.

It remains to show that $x^2\in S^2$. Since $x^1\in S^1$, we have by the Karush–Kuhn–Tucker optimality conditions that

$$A_I^Tu_I^1+C^Tv^1=p \quad \text{for some } u_I^1\geqq 0 \quad \text{and some } v^1. \tag{2.25}$$

Since both $\bar{x}^2$ and x^2 satisfy (2.21) it follows that

$$px^2=u_I^1A_Ix^2+v^1Cx^2\geqq u_I^1A_I\bar{x}^2+v^1C\bar{x}^2=p\bar{x}^2$$

and the proof is complete. □

Remark 2.5. We note that Cook, Gerards, Schrijver and Tardos [4, Thm. 5] have a similar result to Theorem 2.4 for *integer* entries for A but without the equality constraints $Cx=d$. However their Lipschitz constant is bigger than or equal to our Lipschitz constant. In fact their Lipschitz constant $n\Delta(A)$ is only for $\beta=\infty$, where ΔA is the maximum of the absolute values of the determinants of the square submatrices of A. We formalize the relation between the two Lipschitz constants as follows.

PROPOSITION 2.6. *For integer* A, $\nu_\infty(A;\ \phi)\leqq n\Delta(A)$.

Proof. For any u_I for which $\|u_IA_I\|_1=1$ and the rows of A_I are linearly independent, we can assume that

$$A_I=[B \quad N]$$

where B is a nonsingular square submatrix.

Let $q:=u_IB$, then $\|q\|_1\leqq\|u_IA_I\|_1=1$ since u_IB is a subvector of u_IA_I. It follows that

$$\|u_I\|_1=\|(B^T)^{-1}q\|_1\leqq\|(B^T)^{-1}\|_1\|q\|_1\leqq\|(B^T)^{-1}\|_1=\max_i\sum_j|h_{ij}|$$

where h_{ij} is the (i,j) entry of B^{-1} [16, p. 22]. Hence

$$h_{ij}=\frac{1}{\det B}(-1)^{i+j}B_{ji}$$

where B_{ij} is the (i,j) cofactor of B which is the determinant of a square submatrix of A. Hence

$$|B_{ji}|\leqq\Delta(A).$$

If A is integral $|\det B|\geqq 1$ is an integer; hence

$$|h_{ij}|\leqq\frac{1}{|\det B|}|B_{ji}|\leqq\Delta(A).$$

Consequently,

$$\|u_I\|_1\leqq\|(B^T)^{-1}\|_1=\max_i\sum_j|h_{ij}|\leqq n\Delta(A).$$

Since u_I is arbitrary, we have

$$\nu_\infty(A;\ \phi)=\sup\{\|u_I\|_1 \mid \|u_IA_I\|_1=1,\ \text{rows of } A_I \text{ linear independent}\}\leqq n\Delta(A). \quad □$$

Remark 2.7. Note that it is not true that solutions of linear programs are Lipschitzian with respect to perturbations in the objective function coefficients as evidenced by the following simple example:

$$\max\,(1+\delta)x_1+x_2 \quad \text{s.t. } x_1+x_2\leqq 1, \quad (x_1,x_2)\geqq 0. \tag{2.26}$$

The solution to this problem is:

$$x(\delta)=\begin{cases}(1,0) & \text{for } \delta>0,\\ (0,1) & \text{for } \delta<0.\end{cases}$$

Hence

$$\lim_{\delta\to 0+}\frac{\|x(\delta)-x(-\delta)\|}{2\delta}=\infty$$

and hence $x(\delta)$ is not Lipschitzian with respect to δ.

3. Linear complementarity problems. In this section we shall employ the Lipschitz constant $\mu_\beta(A;\,C)$ developed in Theorem 2.2 for linear inequalities and equalities to obtain a Lipschitz constant for linear complementarity problems with matrices that have positive principal minors [5] or which are hidden Z-matrices [17]. We will show by means of Example 3.4 that solutions of linear complementarity problems with a positive semidefinite matrix are not Lipschitz continuous with respect to right-hand side perturbations.

We consider the linear complementarity problem (M,q) of finding an x in R^n such that

$$Mx+q\geqq 0, \quad x\geqq 0, \quad x(Mx+q)=0 \tag{3.1}$$

where $M\in R^{n\times n}$ and $q\in R^n$. Note that given $J\subset\{1,\cdots,n\}$, any solution of the following system of $2n$ linear inequalities and equalities

$$\begin{aligned} &M_jx+q_j\geqq 0, \quad x_j=0, \quad j\in J,\\ &M_jx+q_j=0, \quad x_j\geqq 0, \quad j\notin J, \end{aligned} \tag{3.2}$$

is a solution of (M,q). For $J\subset\{1,\cdots,n\}$ let $Q(J)$ denote the set of all q vectors for which (3.2) has a solution. It is easy to verify that $Q(J)$ is a closed convex cone. In fact it is called a *complementary cone* of (M,q) [14, p. 482]. It is also obvious that $\bigcup Q(J)_{J\subset\{1,\cdots,n\}}$ is the set of all q for which (M,q) is solvable. Define

$$\sigma_\beta(M):=\max_{J\subset\{1,\cdots,n\}}\mu_\beta\left(\begin{pmatrix}-M_J\\-I_{\bar J}\end{pmatrix};\begin{pmatrix}I_J\\M_{\bar J}\end{pmatrix}\right) \tag{3.3}$$

where μ_β is defined by (2.5) and $\bar J$ is the complement of J in $\{1,\cdots,n\}$. We shall prove (Theorem 3.3) that $\sigma_\beta(M)$ will serve as a Lipschitz constant for solutions of (M,q) when M is a P-matrix, that is a matrix with positive principal minors [5], [2], or more generally (Theorem 3.2) for perturbations of q such that the linear complementarity problem is uniquely solvable along the line joining the original q and the perturbed q. We will also establish Lipschitz continuity for solutions of (M,q) when M is a hidden Z-matrix (Theorem 3.5). We begin with a lemma. A related result to this lemma appears in [15].

LEMMA 3.1. *Let q^1 and q^2 be fixed distinct vectors in R^n and let $q(t):=(1-t)q^1+tq^2$ for $t\in[0,1]$. Assume that $(M,q(t))$ is solvable for $t\in[0,1]$. Then there exists a partition $0=t_0<t_1<\cdots<t_N=1$ such that for $1\leqq i\leqq N$*

$$q(t_{i-1})\in Q(J_i),\ q(t_i)\in Q(J_i) \quad \textit{for some } J_i\subset\{1,\cdots,n\}. \tag{3.4}$$

Proof. Let

$$T(J) := \{t \mid t \in [0, 1], q(t) \in Q(J)\}$$

for $J \subset \{1, \cdots, n\}$. It is easy to see that $T(J)$ is closed and convex and hence it is a closed interval which may degenerate to a single point or to the empty set. Since $(M, q(t))$ is solvable for $t \in [0, 1]$ it follows that

$$[0, 1] \subset \bigcup_{J \subset \{1, \cdots, n\}} T(J).$$

Let

$$L := \{[l_1, u_1], \cdots, [l_K, u_K]\}$$

be the set of *maximal* intervals in $\{T(J) \mid J \subset \{1, \cdots, n\}\}$, that is there is no other interval $T(J)$, $J \subset \{1, \cdots, n\}$ that properly contains $[l_i, u_i]$. By removing duplicates from L if needed, we can assume that $[l_i, u_i], \cdots, [l_K, u_K]$ are distinct and that $l_i < l_2 < \cdots < l_K$. Since each $t \in [0, 1]$ belongs to $T(J)$ (for some $J \subset \{1, \cdots, n\}$) which is either in L or contained in some interval of L, we have that

$$[0, 1] \subset \bigcup_{i=1}^{K} [l_i, u_i].$$

Thus $l_i \leqq u_{i-1}$, otherwise (u_{i-1}, l_i) would be an uncovered gap of $[0, 1]$. Also $u_{i-1} < u_i$, otherwise $[l_i, u_i]$ would not be maximal because it would be contained in $[l_{i-1}, u_{i-1}]$.

Hence $l_1 = 0$, $l_{i-1} < l_i \leqq u_{i-1} < u_i$ and $u_K = 1$. Let $0 = t_0 < t_i < \cdots < t_N = 1$ be the sorted numbers of $\{l_1, u_1, l_2, u_2, \cdots, l_K, u_K\}$ with duplicates removed. Then each interval $[t_{i-1}, t_i]$ is contained in some interval $T(J_i)$ in L and so

$$q(t_{i-1}) \in Q(J_i) \quad \text{and} \quad q(t_i) \in Q(J_i). \qquad \square$$

We establish now the Lipschitz continuity of linear complementarity problems with unique solutions along the line segment $q(t) := (1-t)q^1 + tq^2$, $t \in [0, 1]$.

THEOREM 3.2 (Lipschitz continuity of uniquely solvable linear complementarity problems). *Let q^1 and q^2 be points in R^n such that the linear complementarity problem $(M, q(t))$ with $q(t) := (1-t)q^1 + tq^2$ has a unique solution for each $t \in [0, 1]$. Then the unique solutions x^1 of (M, q^1) and x^2 of (M, q^2) satisfy*

$$\|x^1 - x^2\|_\infty \leqq \sigma_\beta(M) \|q^1 - q^2\|_\beta$$

where $\sigma_\beta(M)$ is defined by (3.3).

Proof. There exist $0 = t_0 < t_1 < \cdots < t_N = 1$ with properties stated in Lemma 3.1. Let $x(t_i)$ be the unique solution of $(M, q(t_i))$. Since for $1 \leqq i \leqq N$, $q(t_{i-1})$ and $q(t_i)$ belong to $Q(J_i)$ for some $J_i \subset \{1, \cdots, n\}$, there exists a solution $y(t_{i-1})$ of $(M, q(t_{i-1}))$ such that by (2.4) and (3.3) it follows that

$$\begin{aligned} \|x(t_i) - y(t_{i-1})\|_\infty &\leqq \mu_\beta \begin{pmatrix} -M_{J_i}; & I_{J_i} \\ -I_{\bar{J}_i;} & M_{\bar{J}_i} \end{pmatrix} \|q(t_i) - q(t_{i-1})\|_\beta \\ &\leqq \sigma_\beta(M)(t_i - t_{i-1}) \|q^1 - q^2\|_\beta \end{aligned} \tag{3.5}$$

where $\bar{J}_i$ is the complement of J_i in $\{1, \cdots, n\}$. Summing up for $i = 1, \cdots, N$ gives

$$\sum_{i=1}^{N} \|x(t_i) - y(t_{i-1})\|_\infty \leqq \sigma_\beta(M) \|q^1 - q^2\|_\beta.$$

Since $(M, q(t_{i-1}))$ has a unique solution, $y(t_{i-1}) = x(t_{i-1})$. Hence

$$\|x^1 - x^2\|_\infty \leqq \sum_{i=1}^{N} \|x(t_i) - x(t_{i-1})\|_\infty \leqq \sigma_\beta(M) \|q^1 - q^2\|_\beta. \qquad \square$$

Since for P-matrix M, the linear complementarity problem (M, q) has a unique solution for each $q \in R^n$ [13], the following theorem is an immediate corollary to Theorem 3.2.

THEOREM 3.3 (Lipschitz continuity of solutions of linear complementarity problems with P-matrices). *Let M be a P-matrix. For each q^1 and q^2 in R^n the corresponding unique solutions x^1 and x^2 of (M, q^1) and (M, q^2), respectively, satisfy*

$$\|x^1 - x^2\|_\infty \leqq \sigma_\beta(M)\|q^1 - q^2\|_\beta$$

where $\sigma_\beta(M)$ is defined by (3.3).

The following example shows that solutions of positive semidefinite linear complementarity problems may not be Lipschitzian.

Example 3.4.

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad q^1 = \begin{pmatrix} -\varepsilon \\ 1 \end{pmatrix}, \quad q^2 = \begin{pmatrix} \varepsilon \\ 1 \end{pmatrix}, \quad \varepsilon > 0,$$

$$q(t) = \begin{pmatrix} -\varepsilon + 2\varepsilon t \\ 1 \end{pmatrix}, \quad t_0 = 0, \quad t_1 = \frac{1}{2}, \quad t_2 = 1,$$

$$J_1 = \phi, \qquad J_2 = \{1, 2\},$$

$$q(t_0) \quad \text{and} \quad q(t_1) \quad \text{are in } Q(J_1) = \{q \in R^2 \mid q_1 \leqq 0, q_2 \geqq 0\},$$

$$q(t_1) \quad \text{and} \quad q(t_2) \quad \text{are in } Q(J_2) = R^2_+,$$

$$y(t_0) = x(t_0) = \begin{pmatrix} 1 \\ \varepsilon - 2\varepsilon t_0 \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix},$$

$$x(t_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In order to satisfy (3.5), $y(t_1)$ must be $\binom{0}{0}$. However (3.5) also requires that

$$x(t_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence $x(t_1) \neq y(t_1)$ and the proof of Theorem 3.2 fails. In fact, since

$$\lim_{\varepsilon \to 0} \frac{\|x(t_2) - x(t_0)\|_\infty}{\|q^2 - q^1\|_\infty} = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} = \infty,$$

the solutions of the problem cannot be Lipschitzian.

We conclude by showing that other linear complementarity problems that can be formulated as linear programs [10] have solutions which are Lipschitzian with respect to their right-hand sides as a consequence of Theorem 2.4. In particular if M satisfies the condition of Theorem 2 of [10] with $c = 0$, that is

$$MZ_1 = Z_2, \quad rZ_1 + sZ_2 > 0, \quad (r, s) \geqq 0 \tag{3.6}$$

for some $n \times n$ Z-matrices Z_1 and Z_2, and some n-vectors r and s, then a solution to such a linear complementarity problem is obtained by solving the single linear program

$$\min px \quad \text{s.t. } Mx + q \geqq 0, \quad x \geqq 0$$

where $p = r + M^T s$, and hence p is independent of q. In the terminology of [17], such a matrix M is called a hidden Z-matrix and is a generalization of Z-matrix which includes such matrices as those with a strictly dominant diagonal, and all matrices of Table 1 in [10] except cases 12 to 14.

THEOREM 3.5 (Lipschitz continuity of solutions of linear complementarity problems with hidden Z-matrices). *Let M be a hidden Z-matrix, that is M satisfies* (3.6). *For each q^1 and q^2 in R^n for which (M, q^1) and (M, q^2) are solvable, there exist solutions x^1 of (M, q^1) and x^2 of (M, q^2) such as*

$$\|x^1-x^2\|_\infty \leqq \nu_\beta\begin{pmatrix} M \\ I \end{pmatrix}; \quad \phi\Big)\|q^1-q^2\|_\beta$$

where $\|\cdot\|_\beta$ is some norm on R^n and ν_β is defined by (2.20).

Proof. By [10], there exist solutions of (M, q^1) and (M, q^2) which are obtained by solving the linear programs

$$\min\{px \mid Mx+q^1 \geqq 0,\ x\geqq 0\},$$

$$\min\{px \mid Mx+q^2 \geqq 0,\ x\geqq 0\}$$

where p is a fixed vector independent of q^1 and q^2. The conclusion of the theorem follows immediately from Theorem 2.4. □

We note that for the case of a strictly diagonally dominant positive definite matrix M, (M, q) is uniquely solvable for each q in R^n, and the Lipschitz continuity of the solution follows also from either Theorem 3.5 or Theorem 3.3.

REFERENCES

[1] J.-P. AUBIN, *Lipschitz behavior of solutions to convex optimization problems*, Math. Oper. Res., 9 (1984), pp. 87–111.

[2] A. BERMAN AND R. J. PLEMMONS, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.

[3] S. D. CONTE AND C. DE BOOR, *Elementary Numerical Analysis*, third edition, McGraw-Hill, New York, 1980.

[4] W. COOK, A. M. H. GERARDS, A. SCHRIJVER AND É. TARDOS, *Sensitivity results in integer linear programming*, Math. Programming, 34 (1986), pp. 251–264.

[5] R. W. COTTLE AND G. B. DANTZIG, *Complementary pivot theory in mathematical programming*, Linear Algebra Appl., 1 (1968), pp. 103–125.

[6] D. GALE, *The Theory of Linear Economic Models*, McGraw-Hill, New York, 1960.

[7] W. W. HAGER, *Lipschitz continuity for constrained processes*, this Journal, 17 (1979), pp. 321–338.

[8] A. J. HOFFMAN, *On approximate solutions of systems of linear inequalities*, J. Res. Nat. Bur. Standards, 49 (1952), pp. 263–265.

[9] A. S. HOUSEHOLDER, *The Theory of Matrices in Numerical Analysis*, Blaisdell Publishing, New York, 1964.

[10] O. L. MANGASARIAN, *Characterization of linear complementarity problems as linear programs*, Math. Programming Stud., 7 (1978), pp. 74–87.

[11] ———, *A condition number for linear inequalities and linear programs*, in Methods of Operations Research 43, Proceedings of 6. Symposium über Operations Research, Universität Augsburg, September 7–9, 1981, G. Bamberg and O. Opitz, eds., Verlagsgruppe Athenäum/Hain/Scriptor/Hanstein, Konigstein 1981, pp. 3–15.

[12] O. L. MANGASARIAN AND T.-H. SHIAU, *A variable-complexity norm maximization problem*, SIAM J. Algebraic Discrete Methods, 7 (1986), pp. 455–461.

[13] K. G. MURTY, *On the number of solutions of the complementarity problem and spanning properties of complementarity cones*, Linear Algebra Appl., 5 (1972), pp. 65–108.

[14] ———, *Linear and Combinatorial Programming*, John Wiley, New York, 1976.

[15] ———, *Linear Complementarity, Linear and Nonlinear Programming*, Heldermann Verlag, West Berlin, 1985.

[16] J. M. ORTEGA, *Numerical Analysis: A Second Course*, Academic Press, New York, 1972.

[17] J.-S. PANG, *Hidden Z-matrices with positive principal minors*, Linear Algebra Appl., 23 (1979), pp. 201–215.

[18] S. M. ROBINSON, *Bounds for error in the solution set of a perturbed linear program*, Linear Algebra Appl., 6 (1973), pp. 69–81.

[19] ———, *Generalized equations and their solutions, Part* I*: Basic theory*, Math. Programming Stud., 10 (1979), pp. 128–141.

[20] G. W. STEWART, *Introduction to Matrix Computations*, Academic Press, New York, 1973.