Dynamic Pricing for Hotel Rooms When Customers Request Multiple-Day Stays

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Abstract

Prominent hotel chains quote a booking price for a particular type of rooms on each day and dynamically update these prices over time. We present a novel Markov decision process (MDP) formulation that determines the optimal booking price for a single type of rooms under this strategy, while considering the availability of rooms throughout the multiple-day stays requested by customers. We analyze special cases of our MDP to highlight the importance of modeling multiple-day stays and provide guidelines to potentially simplify the implementation of pricing policies around peak-demand events such as public holidays and conferences. Since computing an optimal policy to our MDP is intractable in general, we develop heuristics based on a fluid approximation and approximate linear programming (ALP). We numerically benchmark our heuristics against a single-day decomposition approach (SDD) and an adaptation of a fixed-price heuristic. The ALP-based heuristic (i) outperforms the other methods; (ii) generates up to 7% and 6% more revenue than the SDD and the fixed-price heuristic respectively; and (iii) incurs a revenue loss of only less than 1% when using our pricing structure around peak-demand events, which supports the use of this simple pricing profile. Our findings are potentially relevant beyond the hotel domain for applications involving the dynamic pricing of capacitated resources.

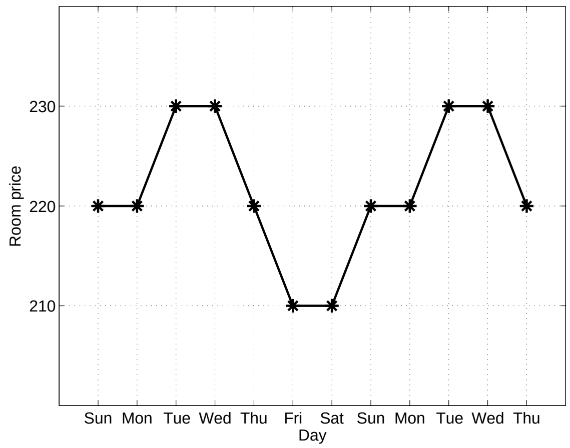
Keywords: hotel revenue management, resource pricing, Markov decision process, approximate dynamic programming

1 Introduction

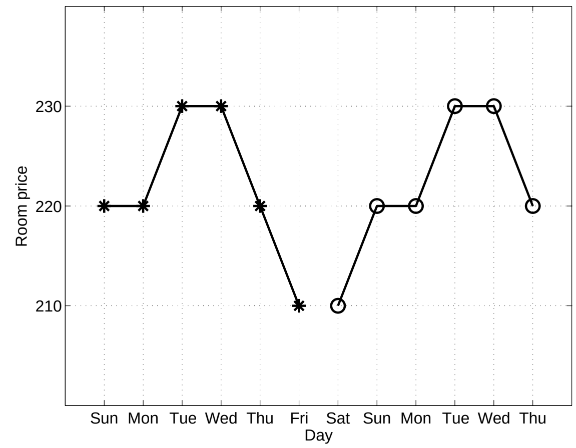
Many companies face the challenge of optimally setting the prices of their services or products at different stages of a selling season so that their revenue is maximized. For example, companies in the travel industry sell service capacity over a season, which includes seats on airline flights, cabins on cruises, and rooms in hotels. A good pricing policy is often vital to the survival of these companies. This problem is also common among retailers who sell perishable or fashion products. These products usually have a replenishment lead time much longer than the selling season, and must be discarded if they remain unsold after the season. The dynamic pricing problems faced by these companies have two common properties (Bitran and Caldentey, 2003, Elmaghraby and Keskinocak, 2003, Talluri and van Ryzin, 2004): (i) capacity or inventory for the selling season is fixed, and (ii) a deadline exists after which sales must cease.

In the domain of hotel revenue management, customers typically request rooms for multiple-day stays (products). If the hotel is out of rooms (resources) for any day within the requested duration, the customer will turn away. We study the observed room pricing practice of quoting only a single booking price (such as the one shown on its website) at any point in time to all customers for the same day of stay. These prices are dynamically updated over time. This pricing strategy falls under the umbrella of *best available rate* pricing (Carvell and Quan, 2005 and Rohlfs and Kimes, 2007). It is adopted, for example, by hotels owned by Marriott International Inc. and Intercontinental Hotels Group (IHG) with 19 and 9 hotel brands respectively. Prominent brands in these hotel chains include Courtyard, Crowne plaza, Holiday inn, Intercontinental hotels and resorts, Marriott, Renaissance, and Residence inn. Figure 1 provides an example of actual prices posted on Expedia.com for a deluxe room at the Courtyard Marriott located close to the Chicago Midway airport. Figures 1(a), 1(b), 1(c), and 1(d) correspond to bookings with lengths of stay equal to 12, 6, 4, and 3, respectively. For instance, the 12-day stay corresponds to a check-in on 10/23/2016 (Sunday) and a check-out on 11/04/2016 (Friday), with prices quoted for 12 nights spanning 10/23/2016 (Sunday) to 11/03/2016 (Thursday). As can be seen in these figures, the room prices on each day coincide regardless of the chosen start and length of stay.

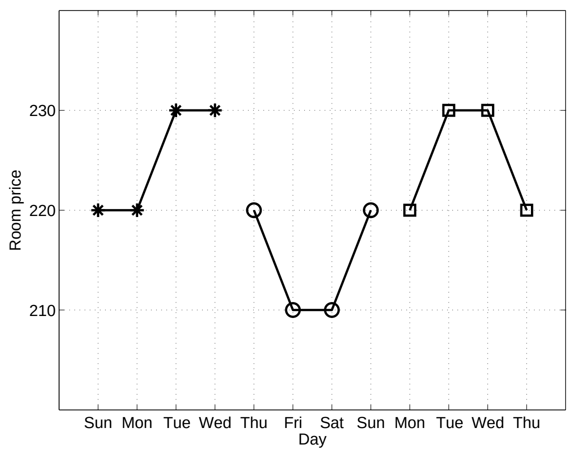
Hotels implementing the above pricing strategy determine the booking price for a room on each individual day, and charge a multiple-day stay the sum of these prices over the duration of stay.



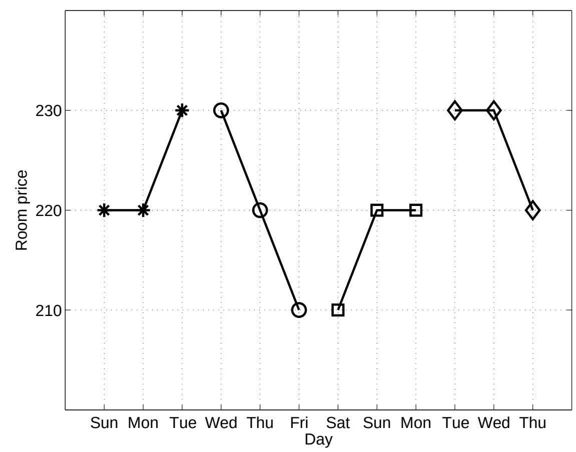
(a) 12-day stay



(b) 6-day stays



(c) 4-day stays



(d) 3-day stays

Figure 1: **Booking prices for a deluxe room at the Courtyard Marriott Chicago Midway Airport accessed on 10/15/2016 for different lengths of stay between 10/23/2016 (Sunday) and 11/03/2016 (Thursday).**

Nevertheless, it is the average price over the multiple-day stay that is prominently displayed on booking websites (for example, Expedia.com). The daily price breakdown is accessible with some additional clicks. The average price derived from the booking price on each day is perceived by customers as being more transparent, which disincentivizes them from looking at daily prices with different windows of stay. Further, quoting the booking price for each day also allows unambiguous adjustments in a statement if there is any extension or reduction in the length of stay.

When pricing rooms in this manner, a hotel may vary the booking prices across different days such that it can exploit the high demands for certain days, but maintain an attractive average daily booking price for customers willing to stay for multiple days. Is the modeling of multiple-day stays important when determining booking prices and does it affect the sensitivity of booking price to demand? If some days have significantly higher demand for rooms due to a peak-demand event (for example, a conference), what booking price should be quoted for each day? What methods can be used to compute dynamic room pricing policies?

Motivated to shed light on these questions, we formulate a Markov decision process to model a hotel with a single room type using the dynamic pricing practice discussed above. Customers check the booking prices and make reservations at random points during an extended period of time known as the *pricing-booking horizon*. Each reservation (usually through the Internet) spans multiple days in a *service horizon*, which is an extended duration of time in the future where a booking price for each day is specified. For the sake of simplicity, we assume these two horizons do not overlap. Figure 2 illustrates the relationship between a pricing-booking horizon and a service horizon. A reservation is made if there is at least one room available for *all* days during the multiple-day stay and the customer is happy with the total booking price.

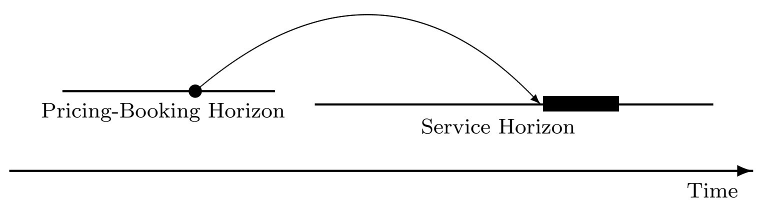


Figure 2: **The pricing-booking horizon and the service horizon**.

Customers having the same arrival and departure dates in the service horizon belong to the same *class*. Each class can have a different stochastic and non-stationary demand function. This allows our model to capture realistic features such as a surge in room requests for a conference approaching its start date, higher willingness to pay of customers that are attending a sports event, and increase in their willingness to pay after they learn that their favorite teams are playing in the event. Customers from different classes request rooms at random points in the pricing-booking horizon (that is, we do not assume one class requests after another). Given these random requests, our Markov decision process formulation determines the daily booking prices in the service horizon such that the expected revenue over the pricing-booking horizon is maximized.

We make the following contributions:

1. We provide a Markov decision process formulation that dynamically prices capacitated *resources*. In contrast, existing models study the dynamic pricing of *products*, predominantly in airline network revenue management (see, for example, §5.4 of Talluri and van Ryzin (2004)) and recently in the context of hotel room pricing (Zhang and Weatherford, 2016). Such models can be used to price multiple-day stays (products) directly, but do not represent all hotel pricing strategies. In particular, there are prominent hotels that price rooms on each day (resources) and charge the sum of these prices over a multiple-day stay. Our model of this unexplored dynamic pricing practice in the hotel industry is novel.

2. We show that optimizing over the booking prices is a concave quadratic maximization problem, which is critical for efficiently computing a dynamic pricing policy using the value function or its approximations. We compare the optimal pricing policy with a single-day decomposition approach and study the sensitivity of hotel room prices around peak-demand events in special cases that are analytically tractable. Our findings suggest the following insights: (i) The sensitivity of the room price to demand parameters differs under the optimal policy that accounts for multiple-day stays and a policy based on the single-day decomposition approach. (ii) To maximize revenue around peak-demand events, hotels should not only substantially raise the booking prices for some high-demand days, but also significantly lower the booking prices for the low-demand days that are *immediately adjacent* to these high-demand days. The former insight highlights the importance of modeling multiple-day stays and the latter insight can be used to reduce the number of prices around peak-demand events, thus making the pricing policies potentially easier to interpret and implement.

3. To overcome the intractability of computing an optimal pricing policy, we develop heuristic pricing policies based on a fluid approximation and approximate linear programming (ALP). Our fluid approximation differs from ones developed in the product-pricing setting. Our ALP-based pricing policy employs an affine value function approximation (Schweitzer and Seidmann, 1985, de Farias and Van Roy, 2003, Adelman, 2007). Our constraint generation algorithm to solve the ALP and compute this approximation is different from those in the literature due to model-specific nonlinearities arising from pricing resources (rooms) instead

of products.

We numerically benchmark our heuristics against the single-day decomposition approach and an adaptation of a fixed-price heuristic by Gallego and van Ryzin (1994). We obtain the following insights from our numerical study: (i) Pricing policies that incorporate multiple-day stays are significantly better than ones that do not include this feature by up to 7% on our instances. (ii) The ALP heuristic based on an affine approximation delivers the best performance despite the inherent non-convexities of the problem. (iii) Imposing the pricing structure for peak-demand events (see point (2) above) in the ALP heuristic leads to only a less than 1% loss in revenue. (iv) The fluid approximation heuristic performs very well when capacity is abundant but can be suboptimal when capacity is limited. Findings (i)–(iii) are new, while (iv) is consistent with analogous observations in the product-pricing setting (Gallego and van Ryzin, 1994, 1997).

Our model and insights are potentially relevant beyond the hotel domain for other applications involving dynamic pricing of capacitated resources used by multiple products. For example, when ordering customized laptops (products) from the Dell or Toshiba website, the product price is the sum of the prices of individual components (resources) used to build it. Our research also motivates further study on dynamic resource-pricing models, which appear to have received limited attention in the dynamic pricing literature.

This paper is outlined as follows. We review related literature in §2. We formulate the dynamic pricing problem for hotels and contrast it with product-pricing models in §3. We analyze the optimal policy in §4. We propose a fluid approximation heuristic in §5. We develop several approximate dynamic programming heuristics to obtain pricing policies that consider multiple-day stays in §6, and benchmark them against the optimal policy (if available) and the single-day decomposition approach in §7. We conclude in §8 and present all proofs in Online Appendix A.

2 Related literature

Kimes (1989) gives a nice overview of yield management in the hotel industry. The author highlights that the yield management techniques for airlines are not always applicable to hotels, for instance, due to the feature of multiple-day stays. Bitran and Mondschein (1995) study the problem of room allocation in the hotel industry. Given a fixed capacity, the problem is to find revenue-maximizing policies for renting hotel rooms when customers arrive in a stochastic and dynamic way from different market segments within a finite horizon. Bitran and Gilbert (1996) give an excellent description of the problem of managing reservations in hotels. They present a model that combines a tactical reservation control problem and an operational capacity allocation problem. Carvell and Quan (2005) and Rohlfs and Kimes (2007) discuss hotels using the best available rate strategies that share the feature modeled in our paper for quoting the room price for each day. Other papers studying the hotel reservation problem include Rothstein (1974), Ladany (1976, 1996), and Liberman and Yechiali (1978). None of these papers consider the dynamic pricing of hotel rooms.

Gallego and van Ryzin (1994) study a continuous-time dynamic pricing model for a single perishable product over a finite horizon. Extending this work, Gallego and van Ryzin (1997) model firms that sell a set of products over a finite horizon. They assume that firms have inventories of a set of resources that are used to produce the products. The problem is to price the finished products so that the expected revenue is maximized over the sales horizon. The formulations in Gallego and van Ryzin (1994, 1997) assume a one-to-one relationship between the demand rate and the price. This assumption ensures the existence of a null price for implicitly turning off a product's demand when one of its resources is unavailable. We refer the reader to Bitran and Caldentey (2003) and Elmaghraby and Keskinocak (2003) for excellent reviews on dynamic pricing.

Recently, Zhang and Weatherford (2016) investigate the applicability of the Gallego and van Ryzin (1997) model for pricing multiple-day stays in the hotel industry using data from a major hotel. In this model, stays with different arrival and departure dates correspond to different products, and room capacities of different days correspond to inventories of different resources. The authors benchmark several popular heuristics based on a fluid approximation and approximate dynamic programming. An approximate dynamic programming heuristic performs best and their results show that the length of multiple-day stays has an impact on the performance of the heuristics.

In contrast to the product-pricing models above, we introduce a resource-pricing formulation motivated by the observed dynamic pricing practice of some prominent hotels (see §1). Specifically, we study hotels that determine the booking prices of individual days (resources) in the service horizon and dynamically update these prices. For example, the total booking price for a customer staying from Monday to Wednesday is the sum of the daily booking prices for Monday, Tuesday, and Wednesday. The demand rate of a multi-day stay (product) thus depends on multiple resource prices. This one-to-many relationship between the product demand and resource prices generally voids the existence of a null price when using a linear demand model. Null prices do exist under nonlinear models (for example, exponential arrival intensity) but the optimization problem to find room prices under such models is nonconvex. These features are unique to resource pricing and make our model different from the product-pricing model studied in Gallego and van Ryzin (1997).

Deterministic fluid approximation heuristics have been proposed for approximating stochastic optimization formulations arising in the dynamic pricing literature. Gallego and van Ryzin (1994, 1997) propose early heuristics of this type, and provide a static pricing policy and an upper bound on the optimal revenue. Talluri and van Ryzin (2004) discuss related generalizations. There is also substantial work on using deterministic models in network revenue management where the product price is fixed (see, for example, Bertsimas and Popescu, 2003, and Jasin and Kumar, 2013). Our fluid approximation is non-convex and needs to explicitly account for the possibility of customer rejections, which makes it different from the one in Gallego and van Ryzin (1994, 1997).

Adelman (2007), Zhang and Adelman (2009), Tong and Topaloglu (2014), and Kirshner and Nediak (2015) use approximate dynamic programming, specifically approximate linear programming, for solving airline network revenue management problems. These papers formulate the stochastic dynamic program arising in their application as a linear program, and apply an affine approximation to the value function. Piecewise-linear and non-separable value function approximations have also been considered for network revenue management (Farias and Van Roy, 2007, Kunnumkal and Talluri, 2011, Zhang, 2011, and Zhang and Lu, 2011). Constraint generation or problem reductions are used to solve the resultant approximate linear program. We also use approximate linear programming with an affine value function approximation. However, our constraint generation strategy is different from the aforementioned papers due to model-specific nonlinearities.

3 Problem formulation

In §3.1 we present a Markov decision process formulation for the dynamic pricing problem for hotels and approach its solution via stochastic dynamic programming. In §3.2 we discuss differences between our formulation and dynamic product-pricing problems studied in the literature.

3.1 A Markov decision process formulation

Consider a hotel with a service horizon of N days. The hotel sells its available capacity over the service horizon in advance during a pricing-booking horizon. We divide the pricing-booking horizon into T periods. Each period t has length δ_t, which is assumed to be sufficiently small that at most one request for booking prices occurs in the period. This is an approximation of a continuous-time model in which decisions are made only at random instants when requests arrive. Under this approximate model, decisions are made at discrete time periods. This model serves practical purposes well because we can approach the results of the continuous-time model by reducing δ_t. As is standard in the revenue management literature, we assume that there are (i) no cancellations once a reservation is made, and (ii) no group reservations, that is, each reservation requires exactly one room per day of stay. For convenience, define $\mathcal{T} = \{1, 2, \ldots, T\}$, $\mathcal{N} = \{1, 2, \ldots, N\}$, and $\mathcal{UV} = \{(u, v)|(u, v) \in \mathcal{N} \times \mathcal{N}, v \geq u\}$.

Suppose the hotel has a total capacity of C rooms. We define *booking time* t as the start of period t in the pricing-booking horizon. The state of the hotel at booking time t can be represented by the capacity vector $\mathbf{x}_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,N})'$, where $x_{t,i}$ is the number of rooms available for day i at booking time t. Each capacity vector $\mathbf{x}_t$ falls in the state space $\mathcal{X} = \{\mathbf{x} \in \mathbb{Z}^N | 0 \leq x_i \leq C, i \in \mathcal{N}\}$. Let $p_{t,i}$ denote the price for a room on day i at booking time t. At booking time t we determine the booking prices $\mathbf{p}_t = (p_{t,1}, p_{t,2}, \ldots, p_{t,N})'$ based on the state $\mathbf{x}_t$.

Define class u-v, $1 \leq u \leq v \leq N$, as a group of customers that would like to stay from day u to day v (they will check out in the morning of day $v+1$). Let $\bar{\lambda}_t^{[u,v]}(\mathbf{p}_t)$ denote the arrival intensity of class u-v. A class u-v customer may make a reservation only if there is at least one room available for all days in the interval $[u, v]$. We use the linear arrival intensity function

$$\bar{\lambda}_t^{[u,v]}(\mathbf{p}_t) \quad := \quad a_t^{[u,v]} - b_t^{[u,v]} \times \sum_{i=u}^{v} p_{t,i}/(v - u + 1), \tag{1}$$

where $a_t^{[u,v]}$ denotes the base demand of class u-v in period t, and $b_t^{[u,v]}$ denotes the price sensitivity factor of class u-v in period t with respect to the average daily booking price. As discussed in §1, the dependence of the arrival rate on the average price over the multiple-day stay in Equation (1) is consistent with this average price being the main displayed price on booking websites, and the customer's perception of an average price being fair when it is based on room prices for each day.

We model room availability using an indicator function $A^{[u,v]}(\mathbf{x}_t)$ that equals 1 if there is at least one room available on *every* day in the interval $[u, v]$, and 0 otherwise. Given the state $\mathbf{x}_t$ and the booking prices $\mathbf{p}_t$ at the start of period t, the actual reservations follow a Poisson process with a rate of

$$\lambda_t^{[u,v]}(\mathbf{x}_t, \mathbf{p}_t) \quad := \quad A^{[u,v]}(\mathbf{x}_t)\left(a_t^{[u,v]} - b_t^{[u,v]} \times \sum_{i=u}^{v} p_{t,i}/(v-u+1)\right). \tag{2}$$

We maintain nonnegative reservation rates by defining the set of feasible booking price vectors as $\mathcal{P}_t(\mathbf{x}_t) = \{\mathbf{p}_t \in \mathbb{R}_+^N | \sum_{i=u}^{v} p_{t,i}/(v-u+1) \leq a_t^{[u,v]}/b_t^{[u,v]}, (u,v) \in \mathcal{UV}$ and $A^{[u,v]}(\mathbf{x}_t) = 1\}$. The constraints in this set enforce the nonnegativity of the arrival rate for each demand class with at least one room available for the entire duration of stay, which implies non-negative reservation rates because the indicator function $A^{[u,v]}(\mathbf{x}_t)$ automatically makes the reservation rate zero when the hotel runs out of rooms on any day within the duration of stay.

We are now ready to formulate the dynamic pricing problem. The total reservation rate in period t is $\Lambda_t(\mathbf{x}_t, \mathbf{p}_t) := \sum_{u=1}^{N}\sum_{v=u}^{N} \lambda_t^{[u,v]}(\mathbf{x}_t, \mathbf{p}_t)$. Recall our assumption that δ_t is sufficiently small such that we have at most one request (thus, at most one reservation) in each period t. Moreover, the probabilities of having 0 and 1 reservation in period t are $1 - \Lambda_t(\mathbf{x}_t, \mathbf{p}_t)\delta_t$ and $\Lambda_t(\mathbf{x}_t, \mathbf{p}_t)\delta_t$ respectively. Define $V_t(\mathbf{x}_t)$ as a value function that represents the maximum expected revenue from our Markov decision process, when starting with room capacities $\mathbf{x}_t$ at the start of period t and continuing until the end of the pricing-booking horizon. This value function can be determined by solving the following stochastic dynamic program (SDP):

$$V_t(\mathbf{x}_t) = \max_{\mathbf{p}_t \in \mathcal{P}_t(\mathbf{x}_t)} \left\{(1 - \Lambda_t(\mathbf{x}_t, \mathbf{p}_t)\delta_t)V_{t+1}(\mathbf{x}_t) + \sum_{u=1}^{N}\sum_{v=u}^{N} \lambda_t^{[u,v]}(\mathbf{x}_t, \mathbf{p}_t)\delta_t \left(\sum_{i=u}^{v} p_{t,i} + V_{t+1}(\mathbf{x}_t - \mathbf{e}_{u,v})\right)\right\}; \tag{3}$$

where $\mathbf{e}_{u,v}$ is an N-dimensional vector with all its entries equal to 0 except for the u-th to v-th entries equal to 1. The boundary conditions are $V_{T+1}(\mathbf{x}_t) = 0, \mathbf{x}_t \in \mathcal{X}; V_t(\mathbf{0}) = 0, t \in \mathcal{T}$.

The above stochastic dynamic program has multi-dimensional state and action spaces that grow with N. To better understand the challenges of solving Problem (3), we present it in a format that facilitates analysis:

$$V_t(\mathbf{x}_t) \quad = \quad V_{t+1}(\mathbf{x}_t) + \max_{\mathbf{p}_t \in \mathcal{P}_t(\mathbf{x}_t)} \{f_t(\mathbf{x}_t, \mathbf{p}_t)\}\,; \tag{4}$$

where

$$f_t(\mathbf{x}_t, \mathbf{p}_t) \quad := \quad \sum_{u=1}^{N} \sum_{v=u}^{N} \left(\sum_{i=u}^{v} p_{t,i} - \triangle V_{t+1}^{[u,v]}(\mathbf{x}_t) \right) \lambda_t^{[u,v]}(\mathbf{x}_t, \mathbf{p}_t) \delta_t, \tag{5}$$

and $\triangle V_{t+1}^{[u,v]}(\mathbf{x}_t) := V_{t+1}(\mathbf{x}_t) - V_{t+1}(\mathbf{x}_t - \mathbf{e}_{u,v})$. The term $\triangle V_{t+1}^{[u,v]}(\mathbf{x}_t)$ is called the *marginal value of capacity for interval* $[u, v]$, and can be interpreted as the expected future revenue that can be gained if the hotel does not sell a room for the interval. Thus, the term $\sum_{i=u}^{v} p_{t,i} - \triangle V_{t+1}^{[u,v]}(\mathbf{x}_t)$ in Equation (5) can be interpreted as the hotel's net gain if a room is reserved for interval $[u, v]$.

The benefit of the form of Problem (4) is that $f_t(\mathbf{x}_t, \mathbf{p}_t)$ is a quadratic function of the booking prices $\mathbf{p}_t$. We further establish the following property of the function $f_t(\mathbf{x}_t, \mathbf{p}_t)$.

Theorem 1. *For any given $\mathbf{x}_t$, the function $f_t(\mathbf{x}_t, \mathbf{p}_t)$ is concave in $\mathbf{p}_t$.*

For any given $\mathbf{x}_t$ at booking time t, Theorem 1 shows that we can efficiently compute optimal booking prices using standard optimization procedures. In other words, the multi-dimensional action space does not pose a challenge. Nevertheless, Problem (4) is still prohibitively hard to solve due to the multi-dimensional state space.

3.2 Differences between resource- and product-pricing models

The Markov decision process formulation (3) prices resources under a linear arrival rate model and differs conceptually from the product-pricing models found in Gallego and van Ryzin (1994, 1997) and Talluri and van Ryzin (2004) when employing an analogous linear model. A consequence of this difference is that $\bar{\lambda}_t^{[u,v]}(\mathbf{p}_t)$ is non-invertible (the price for a room on day i, $p_{t,i}$, appears in the arrival intensity functions of multiple u-v classes) when $\bar{\lambda}_t^{[u,v]}(\mathbf{p}_t)$ takes the linear form in Equation (1). Therefore, unlike in the product-pricing setting, a large "null" price is unavailable in general to implicitly shut off demand for a product when resource capacity is unavailable. This results in the need of an explicit indicator function $A^{[u,v]}(\mathbf{x}_t)$ in Equation (2) to model capacity availability. We explain using a special case with $N = 3$ why implicit capacity control can fail in the resource-pricing setting.

For $N = 3$, there are six classes 1-1, 2-2, 3-3, 1-2, 2-3, and 1-3. Equation (1) for classes 1-1, 2-2, and 3-3 becomes $\bar{\lambda}_t^{[i,i]}(p_{t,i}) = a_t^{[i,i]} - b_t^{[i,i]} p_{t,i}$, for $i = 1, 2, 3$. We can use these equations to eliminate prices from the formulation and obtain an intensity control problem as done in Gallego and van Ryzin (1994, 1997). Using the substitution $p_{t,i} = \dfrac{a_t^{[i,i]} - \bar{\lambda}_t^{[i,i]}}{b_t^{[i,i]}}$ in Equation (1), the demand for classes $[1, 2]$, $[2, 3]$, and $[1, 3]$ is

$$\bar{\lambda}_t^{[1,2]} = a_t^{[1,2]} - \frac{b_t^{[1,2]}}{2}\left(\frac{a_t^{[1,1]}}{b_t^{[1,1]}} + \frac{a_t^{[2,2]}}{b_t^{[2,2]}}\right) + \frac{b_t^{[1,2]}}{2}\left(\frac{\bar{\lambda}_t^{[1,1]}}{b_t^{[1,1]}} + \frac{\bar{\lambda}_t^{[2,2]}}{b_t^{[2,2]}}\right); \tag{6}$$

$$\bar{\lambda}_t^{[2,3]} = a_t^{[2,3]} - \frac{b_t^{[2,3]}}{2}\left(\frac{a_t^{[2,2]}}{b_t^{[2,2]}} + \frac{a_t^{[3,3]}}{b_t^{[3,3]}}\right) + \frac{b_t^{[2,3]}}{2}\left(\frac{\bar{\lambda}_t^{[2,2]}}{b_t^{[2,2]}} + \frac{\bar{\lambda}_t^{[3,3]}}{b_t^{[3,3]}}\right); \tag{7}$$

$$\bar{\lambda}_t^{[1,3]} = a_t^{[1,3]} - \frac{b_t^{[1,3]}}{3}\left(\frac{a^{[1,1]}}{b^{[1,1]}} + \frac{a^{[2,2]}}{b_t^{[2,2]}} + \frac{a_t^{[3,3]}}{b_t^{[3,3]}}\right) + \frac{b_t^{[1,3]}}{3}\left(\frac{\bar{\lambda}_t^{[1,1]}}{b_t^{[1,1]}} + \frac{\bar{\lambda}_t^{[2,2]}}{b_t^{[2,2]}} + \frac{\bar{\lambda}_t^{[3,3]}}{b_t^{[3,3]}}\right). \tag{8}$$

Consider the case where capacity on days 1 and 3 are zero but the capacity for day 2 is nonzero. Since capacity is zero on days 1 and 3, it follows that $\bar{\lambda}_t^{[1,1]} = \bar{\lambda}_t^{[3,3]} = \bar{\lambda}_t^{[1,2]} = \bar{\lambda}_t^{[2,3]} = \bar{\lambda}_t^{[1,3]} = 0$. This zeroing out of arrivals can be achieved by adding constraints $\bar{\lambda}_t^{[i,i]} \leq x_i$, for $i = 1, 2, 3$, and $\bar{\lambda}_t^{[u,v]} \leq \bar{\lambda}_t^{[i,i]}$, for $i = u, \ldots, v$ and $(u, v) \in \mathcal{UV}$. Enforcing these conditions in Equations (6)–(8) gives the following equalities involving $\bar{\lambda}_t^{[2,2]}$:

$$\bar{\lambda}_t^{[2,2]} = b_t^{[2,2]}\left(\left(\frac{a_t^{[1,1]}}{b_t^{[1,1]}} + \frac{a_t^{[2,2]}}{b_t^{[2,2]}}\right) - \frac{2a_t^{[1,2]}}{b_t^{[1,2]}}\right);$$

$$\bar{\lambda}_t^{[2,2]} = b_t^{[2,2]}\left(\left(\frac{a_t^{[2,2]}}{b_t^{[2,2]}} + \frac{a_t^{[3,3]}}{b_t^{[3,3]}}\right) - \frac{2a_t^{[2,3]}}{b_t^{[2,3]}}\right);$$

$$\bar{\lambda}_t^{[2,2]} = b_t^{[2,2]}\left(\left(\frac{a_t^{[1,1]}}{b_t^{[1,1]}} + \frac{a_t^{[2,2]}}{b_t^{[2,2]}} + \frac{a_t^{[3,3]}}{b_t^{[3,3]}}\right) - \frac{3a_t^{[1,3]}}{b_t^{[1,3]}}\right).$$

Clearly $\bar{\lambda}_t^{[2,2]}$ is over-specified in general, which shows that a set of "null" prices does not exist to handle the case of zero capacity on days 1 and 3 and nonzero capacity on day 2. In other words, an explicit use of the indicator function $A^{[u,v]}(\mathbf{x}_t)$ in our formulation to account for the availability of resource capacity is necessary even for this simple case of $N = 3$.

The absence of a null price above can be alleviated by replacing the linear arrival rate model (1) by a nonlinear model that, for example, truncates this linear arrival rate below at zero or is based on an exponential arrival intensity. Under these specifications, explicit capacity control is not necessary but the problem of finding the optimal room prices at a given period t and room capacities $\mathbf{x}_t$ becomes a nonconvex optimization problem. Instead, as shown in Theorem 1, computing room prices in our formulation requires solving a tractable concave optimization problem, which justifies our use of a linear arrival intensity model.

We briefly describe the room pricing formulation with an exponential intensity model of the form $\hat{\lambda}_t^{[u,v]}(\mathbf{p}_t) := e^{-\sum_{i=u}^{v} p_{t,i}}$ (our arguments carry over to other exponential forms as well) to highlight the aforementioned difficulty of computing room prices when using a nonlinear arrival intensity model. Under this specification, setting the booking price to infinity for a day i with no available rooms forces the arrival intensity of all classes including day i to zero. In period t, the condition that a customer can make a reservation only if there is at least one room available can be modeled by restricting prices to the following set: $\bar{\mathcal{P}}_t(\mathbf{x}_t) := \left\{\mathbf{p}_t \in R_+^N | \hat{\lambda}_t^{[u,v]}(\mathbf{p}_t)\delta_t \leq x_{t,i},\ \forall i = \{u, \ldots, v\}, (u,v) \in \mathcal{UV}\right\}$. These constraints are convex in $\mathbf{p}_t$ and linear in $\mathbf{x}_t$. Defining $\hat{\Lambda}_t(\mathbf{p}_t) := \sum_{u=1}^{N} \sum_{v=u}^{N} \hat{\lambda}_t^{[u,v]}(\mathbf{p}_t)$, the analogue of formulation (4) is $\bar{V}_t(\mathbf{x}_t) = \bar{V}_{t+1}(\mathbf{x}_t) + \max_{\mathbf{p}_t \in \bar{\mathcal{P}}_t(\mathbf{x}_t)} \left\{\bar{f}_t(\mathbf{x}_t, \mathbf{p}_t)\right\}$, where $\bar{V}_t(\mathbf{x})$ denotes the value function under the exponential arrival intensity at period t given room inventory vector $\mathbf{x}_t$; $\triangle \bar{V}_{t+1}^{[u,v]}(\mathbf{x}_t) := \bar{V}_{t+1}(\mathbf{x}_t) - \bar{V}_{t+1}(\mathbf{x}_t - \mathbf{e}_{u,v})$; and $\bar{f}_t(\mathbf{x}_t, \mathbf{p}_t) := \sum_{u=1}^{N} \sum_{v=u}^{N} \left(\sum_{i=u}^{v} p_{t,i} - \triangle \bar{V}_{t+1}^{[u,v]}(\mathbf{x}_t)\right) \hat{\lambda}_t^{[u,v]}(\mathbf{p}_t)\delta_t$. The boundary conditions are $\bar{V}_{T+1}(\mathbf{x}_t) = 0, \mathbf{x}_t \in \mathcal{X}; \bar{V}_t(\mathbf{0}) = 0, t \in \mathcal{T}$. Unfortunately, computing the prices under the exponential arrival intensity is intractable due to the following proposition.

Proposition 1. *Given any $\mathbf{x}_t$, $\bar{f}_t(\mathbf{x}_t, \mathbf{p}_t)$ is non-concave in $\mathbf{p}_t$.*

Given the high dimensional state space and the non-convexity caused by explicit capacity control in formulation (3), a natural question is whether the existing deterministic models in revenue management (Gallego and van Ryzin, 1994, Bitran and Caldentey, 2003, §5.4 of Talluri and van Ryzin, 2004) can be leveraged to approximate our problem. These "fluid" approximations are substantially easier to solve than the original stochastic version of the problem because they replace the random arrivals of class u-v reservations by their respective expected arrival rates. For instance, consider the well-known fluid approximation in §5.4.1 of Talluri and van Ryzin (2004) for pricing products that require multiple resources with finite capacities. The dual variables of the resource capacity constraints in this formulation may be used as resource prices. However, as explained next, these dual prices are poor surrogates for the resource prices. Consider a situation where there is abundant capacity for each resource so that the demand of every class is satisfied. In this case, the marginal values of the resource capacities (the dual prices) will be zero but the optimal revenue from selling rooms is clearly nonzero. In other words, *the dual prices represent the marginal values of resource capacities as opposed to the resource prices that maximize the revenue from using the existing resource capacities.* Therefore, the resource prices in our formulation are conceptually different, which motivates us to propose a fluid approximation specific to our resource-pricing formulation (3) in §5.

4 Understanding the optimal pricing policy

We study the behavior of the optimal pricing policy in tractable special cases to provide some insights into room pricing on each day. In §4.1 we compare the optimal pricing policy and the pricing policy based on a single-day decomposition in a two-day setting. In §4.2 we investigate the room pricing profile around peak demand events assuming that the demand data is symmetric.

4.1 Comparison with a single-day decomposition

Consider a hypothetical situation in which a class u-v customer is willing to make a partial reservation when the hotel cannot fully accommodate all days on the itinerary $[u, v]$. We call this the *single-day decomposition* (SDD) because under this approach a class u-v customer is considered as a group of distinct customers indexed as $k = u, \ldots, v$. Each customer k in the group wants to stay only on a single day k in the service horizon, for $k = u, \ldots, v$.

Under the single-day decomposition, the problem of maximizing expected revenue can be separated into N independent sub-problems, each corresponding to a different day. The technical benefit of this approach is that each sub-problem is a one-dimensional stochastic dynamic program, and hence is considerably easier to solve than the original problem (4). Under this approach, the approximate model is described as follows. Given that there are $x_{t,i}$ rooms available and the booking price is $p_{t,i}$ for day i at the start of period t, we assume that the reservations for day i follow a Poisson process with a rate of $\lambda_t^i(x_{t,i}, p_{t,i}) := \mathbf{I}(x_{t,i} > 0) \sum_{u=1}^{i} \sum_{v=i}^{N} \left(a_t^{[u,v]} - b_t^{[u,v]} p_{t,i} \right)$, where $\mathbf{I}(\cdot)$ equals 1 if $\cdot$ is true, and 0 otherwise. Therefore, the probabilities of having 0 and 1 reservation, respectively, for day i in period t are $1 - \lambda_t^i(x_{t,i}, p_{t,i})\delta_t$ and $\lambda_t^i(x_{t,i}, p_{t,i})\delta_t$.

Let $p_{t,i}^{\max} := \sum_{u=1}^{i} \sum_{v=i}^{N} a_t^{[u,v]} / \sum_{u=1}^{i} \sum_{v=i}^{N} b_t^{[u,v]}$. Given that there are $x_{t,i}$ rooms available for day i at the start of period t, define the value function $V_t^i(x_{t,i})$ as the maximum expected revenue for day i under the single-day decomposition from period t until the end of the pricing-booking horizon. The value function $V_t^i(x_{t,i})$ can be determined by solving the following stochastic dynamic program:

$$V_t^i(x_{t,i}) = \max_{0 \le p_{t,i} \le p_{t,i}^{\max}} \left\{ \left(1 - \lambda_t^i(x_{t,i}, p_{t,i})\delta_t\right) V_{t+1}^i(x_{t,i}) + \lambda_t^i(x_{t,i}, p_{t,i})\delta_t \left(p_{t,i} + V_{t+1}^i(x_{t,i} - 1)\right) \right\}; \quad (9)$$

with boundary conditions $V_{T+1}^i(x_{t,i}) = 0, x_{t,i} \in [0, C]$ and $V_t^i(0) = 0, t \in \mathcal{T}$; $i \in \mathcal{N}$.

Problem (9) can be rewritten as $V_t^i(x_{t,i}) = V_{t+1}^i(x_{t,i}) + \max_{0 \le p_{t,i} \le p_{t,i}^{\max}} \left\{ f_t^i(x_{t,i}, p_{t,i}) \right\}$, where

$$f_t^i(x_{t,i}, p_{t,i}) \quad := \quad \left(p_{t,i} - \triangle V_{t+1}^i(x_{t,i})\right) \lambda_t^i(x_{t,i}, p_{t,i})\delta_t, \quad (10)$$

and $\triangle V_{t+1}^i(x_{t,i}) := V_{t+1}^i(x_{t,i}) - V_{t+1}^i(x_{t,i}-1)$, which is called the expected marginal value of capacity for day i. It is straightforward to see that $f_t^i(x_{t,i}, p_{t,i})$ is a concave quadratic function of $p_{t,i}$. Given a state $\mathbf{x}_t$ at the start of period t, the problem is to maximize $f_t^i(x_{t,i}, p_{t,i})$ subject to $0 \le p_{t,i} \le p_{t,i}^{\max}$, for $i \in \mathcal{N}$. Solving the first-order conditions $\partial f_t^i(x_{t,i}, p_{t,i})/\partial p_{t,i} = 0$, the booking prices in each period t under the single-day decomposition can be obtained as follows:

$$p_{t,i}^s \quad = \quad \frac{a_t^i + b_t^i \triangle V_{t+1}^i(x_{t,i})}{2b_t^i}; \quad (11)$$

where $a_t^i := \sum_{u=1}^{i} \sum_{v=i}^{N} a_t^{[u,v]}$ and $b_t^i := \sum_{u=1}^{i} \sum_{v=i}^{N} b_t^{[u,v]}$, for $i \in \mathcal{N}$. Since $f_t^i(x_{t,i}, p_{t,i})$ is a concave function of $p_{t,i}$, set $p_{t,i}^s = 0$ if the right hand side of Equation (11) is negative, and set $p_{t,i}^s = p_{t,i}^{\max}$ if the right hand side of Equation (11) is larger than $p_{t,i}^{\max}$, for $i \in \mathcal{N}$.

We now contrast the single-day decomposition pricing policy and the optimal pricing policy for the case of $N = 2$. Let $p_{t,1}^*$ and $p_{t,2}^*$ denote the optimal prices on days 1 and 2, respectively.

Proposition 2. *For $N = 2$ an optimal solution to Problem (3) in period $t \in \mathcal{T}$ has the following properties:*

(1) $p_{t,1}^*$ increases with $a_t^{[1,1]}$ and $a_t^{[1,2]}$, but decreases with $a_t^{[2,2]}$;

(2) $p_{t,2}^*$ increases with $a_t^{[2,2]}$ and $a_t^{[1,2]}$, but decreases with $a_t^{[1,1]}$.

Proposition 2 implies that, for $N = 2$, if the demand of *single-day stays* for day i increases, then the hotel should not only raise the booking price for day i, but also lower the booking price for day j ($\neq i$). This is to exploit the increasing demand for a certain day, but maintain an attractive average daily booking price for two-day stays. If the demand of *two-day stays* increases, then the hotel should raise the booking prices for both days.

In contrast, from Equations (11) the booking prices under the single-day decomposition for $N = 2$ are

$$
\begin{aligned}
p^s_{t,1} &= \frac{\sum_{v=1}^2 a_t^{[1,v]} + \sum_{v=1}^2 b_t^{[1,v]} \triangle V^1_{t+1}(x_{t,1})}{2\sum_{v=1}^2 b_t^{[1,v]}}, \\
p^s_{t,2} &= \frac{\sum_{u=1}^2 a_t^{[u,2]} + \sum_{u=1}^2 b_t^{[u,2]} \triangle V^2_{t+1}(x_{t,2})}{2\sum_{u=1}^2 b_t^{[u,2]}}.
\end{aligned}
$$

It is worth noting that $p^s_{t,1}$ is independent of $a_t^{[2,2]}$ and $p^s_{t,2}$ is independent of $a_t^{[1,1]}$. The optimal pricing policy characterized in Proposition 2 captures the interaction of $p^*_{t,1}$ and $a_t^{[2,2]}$, and the interaction of $p^*_{t,2}$ and $a_t^{[1,1]}$. These interactions are missing under the single-day decomposition. Similarly, both the optimal booking prices $p^*_{t,1}$ and $p^*_{t,2}$ at period t are nonlinear functions of the price sensitivity factors $b_t^{[1,1]}$, $b_t^{[1,2]}$, and $b_t^{[2,2]}$, whereas $p^s_{t,1}$ and $p^s_{t,2}$ are independent of $b_t^{[2,2]}$ and $b_t^{[1,1]}$, respectively, under the single-day decomposition.

Proposition 3 highlights a difference in the sensitivity of prices to changes in demand under the optimal policy and the single-day decomposition. It can be shown by simple algebra and thus the proof is omitted. Let $\triangle p^*_t := p^*_{t,2} - p^*_{t,1}$ and $\triangle p^s_t := p^s_{t,2} - p^s_{t,1}$.

Proposition 3. *For $N = 2$ if the hotel will not run out of rooms and $b_t^{[u,v]}$ equals a constant b for all $(u,v) \in \mathcal{UV}$, then (1) $\triangle p^*_t$ is proportional to $\frac{a_t^{[2,2]} - a_t^{[1,1]}}{2b}$; and (2) $\triangle p^s_t$ is proportional to $\frac{a_t^{[2,2]} - a_t^{[1,1]}}{4b}$.*

Proposition 3 implies that under both approaches, the gap between the booking prices for days 1 and 2 is proportional to the difference $a_t^{[2,2]} - a_t^{[1,1]}$. Furthermore, under the optimal policy the sensitivity of this gap to the demands for single-day stays ($a_t^{[1,1]}$ and $a_t^{[2,2]}$) is two times of that under the single-day decomposition.

In summary, the revenue generated by the single-day decomposition and the optimal policy may differ as a result of the missing interactions and sensitivity differences highlighted using the simple $N = 2$ case. In addition, the single-day decomposition does not correctly account for the capacity needed to satisfy a multi-day stay request and may thus drastically overestimate the expected revenue for the hotel. We expect this overestimation to happen particularly when the supply of rooms is tight or when demand exhibits significant inter-day variability. Our numerical study in §7 explores the revenue impact of these differences by comparing the single-day decomposition with heuristics discussed in §5–6 that account for multiple-day stays.

4.2 Room pricing for peak-demand events

Hotels often encounter a situation where the demands on certain days are substantially higher than others, for example, due to a conference, a sporting event, or public holidays. How should hotels price rooms when the service horizon includes peak-demand events? We provide analysis of special cases to shed light on this question, where we assume that the input data and room capacity vector are "reflective". Define $r(i) := N - i + 1$, for $i \in \mathcal{N}$. We say the input data is reflective if $a_t^{[u,v]} = a_t^{[r(v),r(u)]}$ and $b_t^{[u,v]} = b_t^{[r(v),r(u)]}$, for each $(t, u, v) \in \mathcal{T} \times \mathcal{UV}$. Further, we say a capacity vector $\mathbf{x}_t$ is reflective if $x_{t,i} = x_{t,r(i)}$, for all $i = 1, \ldots, \lfloor N/2 \rfloor$. This assumption allows us to isolate the impact of peak-demand days and ensures analytical tractability.

Proposition 4 establishes that the symmetry in the input data and capacity vector extends to the value function and optimal room prices.

Proposition 4. *If the input data is reflective then the following hold:*

1. *For capacity vectors $\mathbf{x}_t^1$ and $\mathbf{x}_t^2$, if $x_{t,i}^1 = x_{t,r(i)}^2$, for all $t \in \mathcal{T}$, $i \in \mathcal{N}$, then $V_t\left(\mathbf{x}_t^1\right) = V_t\left(\mathbf{x}_t^2\right)$.*
2. *For a given period t and reflective room capacity vector, the optimal room price vector is also reflective, that is, it satisfies $p_{t,i} = p_{t,r(i)}$, for $i = 1, \ldots, \lfloor N/2 \rfloor$.*

Proposition 4 supports the use of a reflective pricing policy over the service horizon when the demand is reflective, which reduces the number of prices computed for each period by roughly fifty percent. Indeed, the input demand data is unlikely to be reflective in practice. Nevertheless assuming reflective room prices around a peak-demand event could lead to a potentially effective heuristic when (i) the peak-demand event can be centered by choosing the service (planning) horizon appropriately; and (ii) the demand on the days of the event are relatively stable compared to non-peak days. We numerically test a heuristic that enforces reflective room prices in §7 and find that it works well.

We now turn to understanding how prices in a reflective pricing profile change as a result of a peak-demand event by focusing on the case of $N = 8$. We consider two events intended to model a public holiday and a conference. The public-holiday and conference events have peak demand classes 4-5 and 3-6 respectively. We model each of these events by increasing the base demand of the peak-demand class by a factor of $\alpha > 1$ and use a constant base demand of a for the other classes. Specifically, the base demands $a_t^{[4,5]}$ in the public-holiday case and $a_t^{[3,6]}$ in the conference case is αa. We assume the demand sensitivity $b_t^{[u,v]}$ is equal to a constant b for all classes. Proposition 5 highlights the sensitivity of prices to α for both events.

Proposition 5. *For $N = 8$, assume for a given t that the capacity vector $\mathbf{x}_t$ is reflective and strictly positive. The following hold:*

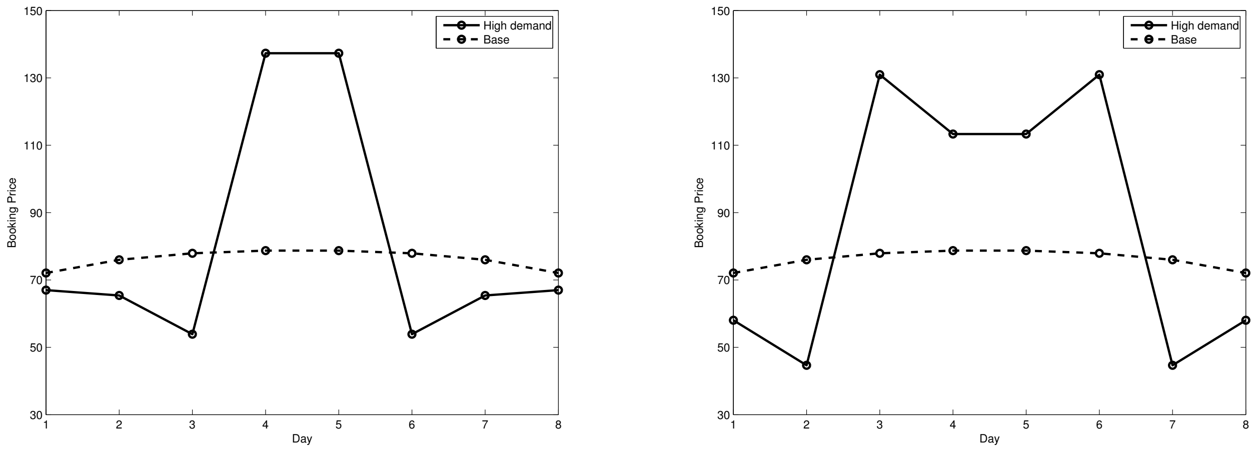
1. *Public holiday: If $b_t^{[u,v]} = b$ for all $(u,v) \in \mathcal{UV}$, $a_t^{[u,v]} = a$ for all $(u,v) \in \mathcal{UV} \setminus \{(4,5)\}$, $a_t^{[4,5]} = \alpha a$ with $\alpha > 1$, and the optimal room rate vector $\mathbf{p}_t$ is in the interior of $\mathcal{P}_t(\mathbf{x}_t)$, then $\partial p_{t,3}/\partial\alpha < \partial p_{t,2}/\partial\alpha < \partial p_{t,1}/\partial\alpha < 0$ and $\partial p_{t,4}/\partial\alpha > \max_{i=1,\ldots,3} |\partial p_{t,i}/\partial\alpha|$.*

2. *Conference: If $b_t^{[u,v]} = b$ for all $(u,v) \in \mathcal{UV}$, $a_t^{[u,v]} = a$ for all $(u,v) \in \mathcal{UV} \setminus \{(3,6)\}$, $a_t^{[3,6]} = \alpha a$ with $\alpha > 1$, and the optimal room rate vector $\mathbf{p}_t$ is in the interior of $\mathcal{P}_t(\mathbf{x}_t)$, then $\partial p_{t,2}/\partial\alpha < \partial p_{t,1}/\partial\alpha < 0$, and $\partial p_{t,3}/\partial\alpha > \partial p_{t,4}/\partial\alpha > \max_{i=1,2} |\partial p_{t,i}/\partial\alpha|$.*

For the public-holiday event, the booking prices increase with α for peak days and decrease with α for non-peak days. Interestingly, (i) the price increase on peak days is larger than the price decrease on non-peak days, and (ii) the decrease in the booking price is larger for the non-peak days closer to a peak day. The price sensitivity of the conference event also shares these features. However, a new pattern emerges with respect to the price changes on the peak days. Specifically, the prices increase by a larger amount on the first and the last peak days, but by a smaller amount on the intermediate peak days.

Figures 3(a) and 3(b) illustrate the insights on price sensitivity using a simple example with $a = 1.0$, $b = 0.01$, $\alpha = 8$, $\delta_t = 0.001$, and $\Delta V_{t+1}^{[u,v]}(\mathbf{x}_t) = 0$, for $(t,u,v) \in \mathcal{T} \times \mathcal{UV}$, that is, the price profile corresponds to a state with abundant room inventory: $x_{t,i} \geq T$ for $i \in \mathcal{N}$. We compute the prices using their closed-form expressions in the proof of Proposition 5 in Online Appendix A. The dotted line in each figure corresponds to the pricing profile when there is no peak-demand class. The solid lines in Figures 3(a) and 3(b) show the pricing profiles in the presence of the public-holiday and conference events respectively. Comparing the dotted and solid lines shows that the difference in booking price can be substantial.

To summarize, our analysis suggests that if the base demand of a class u-v, where $v > u + 1$, is especially strong, the hotel should significantly increase the booking prices for the "boundary"



(a) (b)

Figure 3: **Booking price sensitivities for (a) public-holiday and (b) conference events.**

peak days u and v, but should increase by less the booking prices for the intermediate peak days $u+1$ to $v-1$. This is to exploit the strong demand of class u-v but to attract customers staying on intermediate peak days such as classes u'-v', where $u' \geq u+1$ and $v' \leq v-1$. On the other hand, the booking prices for the immediately-adjacent, non-peak days $u-1$ and $v+1$ should be substantially lowered. This is to attract customers staying on both peak and non-peak days such as classes u'-u, where $u' \leq u-1$, and classes v-v', where $v' \geq v+1$. We thus recommend the rules in Table 1 for hotel managers. Although these rules do not determine the booking prices in detail, they provide a simple guideline to handle a demand surge from a specific customer class. They are potentially easier to interpret and implement. Moreover, this structure can be imposed on a pricing heuristic, which we describe in §6.2.

Table 1: **Simple rules to handle a high-demand customer class u-v.**

To maximize the revenue, follow both of the rules below.

Increase booking prices : Significantly raise the booking prices for days u and v. If the interval $[u, v]$ spans more than two days ($v > u+1$), then raise by less the booking prices for days $u+1$ to $v-1$.

Decrease booking prices : Significantly lower the booking prices for days $u-1$ and $v+1$.

5 Fluid approximation heuristic: FAH

Computing the optimal booking prices using the exact formulation (3) quickly becomes intractable as the size of the problem grows. Motivated by the discussion at the end of §3.2, we introduce here a fluid approximation for our SDP (3) that directly price the resources. Recall that the arrival intensity of class u-v in period t is $\bar{\lambda}_t^{[u,v]}(\mathbf{p}_t) \equiv a_t^{[u,v]} - b_t^{[u,v]} \frac{\sum_{i=u}^{v} p_{t,i}}{v-u+1}$. We use $\eta_t^{[u,v]}$ to represent the rate of class u-v requests that are rejected in period t. In other words, the effective number of class u-v reservations in period t is $\left(\bar{\lambda}_t^{[u,v]}(\mathbf{p}_t) - \eta_t^{[u,v]}\right)\delta_t$. Let $\boldsymbol{\eta}_t$ denote the vector $\left(\bar{\eta}_t^{[u,v]}, \forall (u,v) \in \mathcal{UV}\right)$. Let $\mathbf{p}$ and $\boldsymbol{\eta}$ denote $(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_T)$ and $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \ldots, \boldsymbol{\eta}_T)$ respectively. Our fluid approximation is the following mathematical program:

$$\max_{\mathbf{p},\boldsymbol{\eta}} \sum_{t=1}^{T} \sum_{u=1}^{N} \sum_{v=u}^{N} \left(\sum_{i=u}^{u} p_{t,i}\right) \left(a_t^{[u,v]} - \frac{b_t^{[u,v]}}{v-u+1} \sum_{i=u}^{v} p_{t,i} - \eta_t^{[u,v]}\right) \delta_t \tag{12}$$

$$\text{s.t.} \quad \sum_{t=1}^{T} \sum_{u=1}^{i} \sum_{v=i}^{N} \left(a_t^{[u,v]} - \frac{b_t^{[u,v]}}{v-u+1} \sum_{i=u}^{v} p_{t,i} - \eta_t^{[u,v]}\right) \delta_t \le C, \quad \forall i \in \mathcal{N}, \tag{13}$$

$$\eta_t^{[u,v]} \le \left(a_t^{[u,v]} - \frac{b_t^{[u,v]}}{v-u+1} \sum_{i=u}^{v} p_{t,i}\right), \quad \forall t \in \mathcal{T}, (u,v) \in \mathcal{UV}, \tag{14}$$

$$0 \le \eta_t^{[u,v]} \le a_t^{[u,v]}, \quad \forall t \in \mathcal{T}, (u,v) \in \mathcal{UV}, \tag{15}$$

$$0 \le p_{t,i}, \quad \forall t \in \mathcal{T}, i \in \mathcal{N}. \tag{16}$$

The decision variables in (12)-(16) are the price $p_{t,i}$ charged in period t for a room on day i and the rejection rate $\eta_t^{[u,v]}$ of class u-v in period t. The objective function (12) is the total revenue, where the class u-v revenue component is the product of (i) the sum of prices charged during the period of stay and (ii) the corresponding effective number of reservations. Constraints (13) make sure that the number of rooms sold for day i is no larger than the number of rooms available. Conditions (14) ensure that the expected rate of rejected requests in each class is no larger than its arrival intensity. Constraints (15)–(16) specify bounds on the decision variables. We highlight that the role of the variables $\eta_t^{[u,v]}$ in the objective function and constraints is akin to the indicator function $A^{[u,v]}(\mathbf{x})$ that we use in (3) to control capacity explicitly. That is, both $\eta_t^{[u,v]}$ and $A^{[u,v]}(\mathbf{x})$ allow their respective models to reject reservations.

Formulation (12)–(16) has a non-concave objective function because of the product between the rejection variables $\eta_t^{[u,v]}$ and the price variables $p_{t,i}$. Removing the rejection variables from the formulation may cause it to become infeasible because controlling the prices alone is insufficient to switch off the demand for the days that run out of capacity, as discussed in §3.2. We thus fix the rejection variables $\eta_t^{[u,v]}$ to some pre-determined values $\bar{\eta}_t^{[u,v]}$ that ensure feasibility, and solve the resultant restricted model, which is a convex program. For a given candidate rejection vector $\bar{\boldsymbol{\eta}} = (\bar{\boldsymbol{\eta}}_1, \bar{\boldsymbol{\eta}}_2, \ldots, \bar{\boldsymbol{\eta}}_T)$, we solve the following convex program:

$$\max_{\mathbf{p}} \sum_{t=1}^{T}\sum_{u=1}^{N}\sum_{v=u}^{N}\left[\left(\sum_{i=u}^{v} p_{t,i}\right)\left(a_t^{[u,v]} - \frac{b_t^{[u,v]}}{v-u+1}\sum_{i=u}^{v} p_{t,i}\right)\delta_t - \bar{\eta}_t^{[u,v]}\delta_t \sum_{i=u}^{v} p_{t,i}\right] \tag{17}$$

$$\text{s.t.} \sum_{t=1}^{T}\sum_{u=1}^{i}\sum_{v=i}^{N}\left(a_t^{[u,v]} - \frac{b_t^{[u,v]}}{v-u+1}\sum_{i=u}^{v} p_{t,i} - \bar{\eta}_t^{[u,v]}\right)\delta_t \le C, \quad \forall i \in \mathcal{N}, \tag{18}$$

$$\bar{\eta}_t^{[u,v]} \le \left(a_t^{[u,v]} - \frac{b_t^{[u,v]}}{v-u+1}\sum_{i=u}^{v} p_{t,i}\right), \quad \forall t \in \mathcal{T}, (u,v) \in \mathcal{UV}, \tag{19}$$

$$0 \le p_{t,i}, \quad \forall t \in \mathcal{T}, i \in \mathcal{N}. \tag{20}$$

Formulation (17)–(20) has linear constraints. It has a concave objective function because its first term is the same as the immediate reward in SDP (3), which we show to be concave in the proof of Theorem 1, and its second term is linear in the prices.

We now discuss the intuition behind our approach for generating the candidate rejection vectors, and then present a linear program for generating such vectors. The effective number of reservations of class u-v can be reduced by either (i) increasing the relevant room prices and/or (ii) increasing the number of rejected requests. The non-convex formulation (12)–(16) can be interpreted as trying to determine a revenue-maximizing balance between these two strategies while satisfying the capacity constraints. Our generation of candidate rejection vectors mimics this trade off using a weight $\kappa \in [0,1]$ embedded in the following linear program:

$$\max_{\mathbf{p},\boldsymbol{\eta}} \sum_{t=1}^{T}\sum_{u=1}^{N}\sum_{v=u}^{N}\left[\kappa\left(\frac{b_t^{[u,v]}}{v-u+1}\sum_{i=u}^{v} p_{t,i}\right) + (1-\kappa)\eta_t^{[u,v]}\right] \tag{21}$$

$$s.t. \quad (13) - (16). \tag{22}$$

The first and second terms in the objective (21) correspond to the reduction in class u-v demand due to higher room prices and a higher rejection rate, respectively. We formalize our fluid approximation heuristic described above in Algorithm 1.

Algorithm 1: Fluid approximation heuristic

Inputs: Discretization size M.

Initialization: Set $\Delta\kappa = 1/M$, $\kappa = 0$, $\mathrm{E} = \emptyset$, $\mathbf{p}^* = 0$, and $R^* = 0$.

While $\kappa < 1$ **do**:

(i) Solve the linear program (21)–(22) with weight κ to obtain a rejection vector $\bar{\boldsymbol{\eta}}$.

(ii) **If** $\bar{\boldsymbol{\eta}} \notin \mathrm{E}$ **then** add $\bar{\boldsymbol{\eta}}$ to set E.

(iii) $\kappa \leftarrow \kappa + \Delta\kappa$.

For each $\bar{\boldsymbol{\eta}} \in \mathrm{E}$ **do**:

(i) Solve the convex program (17)–(20) and obtain a price vector $\bar{\mathbf{p}}$ and optimal revenue $\bar{R}$.

(ii) **If** $\bar{R} > R^*$ **then** set $\mathbf{p}^* = \bar{\mathbf{p}}$ and $R^* = \bar{R}$.

Return $\mathbf{p}^*$.

---

6 Approximate dynamic programming based heuristics

In this section, we consider two heuristics that account for the uncertain evolution of room availability over time. These heuristics are based on the approximate dynamic programming approach that computes and uses an approximation $\tilde{V}_t(\mathbf{x}_t)$ instead of the exact value function $V_t(\mathbf{x}_t)$ to determine decisions (Bertsekas, 2012). In other words, the room price vector $\mathbf{p}_t$ is computed by solving

$$\max_{\mathbf{p}_t \in \mathcal{P}_t(\mathbf{x}_t)} \left\{ \tilde{f}_t(\mathbf{x}_t, \mathbf{p}_t) \right\}, \tag{23}$$

where $\tilde{f}_t(\mathbf{x}_t, \mathbf{p}_t) := \sum_{u=1}^{N} \sum_{v=u}^{N} \left(\sum_{i=u}^{v} p_{t,i} - \tilde{V}_{t+1}(\mathbf{x}_t) + \tilde{V}_{t+1}(\mathbf{x}_t - \mathbf{e}_{u,v}) \right) \lambda_t^{[u,v]}(\mathbf{x}_t, \mathbf{p}_t)\delta_t$. Problem (23) is a convex quadratic program because $\mathcal{P}_t(\mathbf{x}_t)$ is convex and $\tilde{f}_t(\mathbf{x}_t, \mathbf{p}_t)$ remains concave in $\mathbf{p}_t$ by Theorem 1 after replacing $V_{t+1}(\mathbf{x}_{t+1})$ by $\tilde{V}_{t+1}(\mathbf{x}_{t+1})$. Thus, we can compute the booking prices efficiently via Problem (23) once we have a value function approximation for each period.

6.1 A multiple-day heuristic: MDH

We consider a heuristic MDH that computes an approximation $\tilde{V}_{t+1}(\mathbf{x}_{t+1})$ by assuming that the booking prices are kept fixed at a value from period $t+1$ until the end of the pricing-booking horizon (a similar strategy is adopted in the fixed-price heuristic by Gallego and van Ryzin (1994)). Assume that the demand for day i in each period $\tau = t+1, \ldots, T$ follows a Poisson process with rate $\tilde{\lambda}^i_\tau(p_{\tau,i}) := \sum_{u=1}^{i}\sum_{v=i}^{N} a^{[u,v]}_\tau - b^{[u,v]}_\tau p_{\tau,i}$. The demand for day i in period τ is a Poisson random variable with mean $\tilde{\lambda}^i_\tau(p_{\tau,i})\delta_\tau$. Since the sum of Poisson random variables is still a Poisson random variable (see Ross, 2000, page 58), the demand for day i from period $t+1$ to period T can be represented by a Poisson random variable Y^i_{t+1} with mean $\sum_{\tau=t+1}^{T}\tilde{\lambda}^i_\tau(p_{\tau,i})\delta_\tau$.

Let $\tilde{V}^i_{t+1}(x_{t+1,i}, p_{t+1,i})$ denote the expected revenue generated by these Poisson arrivals from period $t+1$ until the end of the pricing-booking horizon, given that $x_{t+1,i}$ rooms are available for day i and its booking price is $p_{t+1,i}$ at the start of period $t+1$. We have $\tilde{V}^i_{t+1}(x_{t+1,i}, p_{t+1,i}) = p_{t+1,i}E\left[min\left\{x_{t+1,i}, Y^i_{t+1}\right\}\right]$. Let $\tilde{V}_{t+1}(\mathbf{x}_{t+1}, \mathbf{p}_{t+1})$ be the sum of expected revenue generated by the demands for all days, given the state $\mathbf{x}_{t+1}$ and the booking prices $\mathbf{p}_{t+1}$ at the start of period $t+1$. That is, $\tilde{V}_{t+1}(\mathbf{x}_{t+1}, \mathbf{p}_{t+1}) = \sum_{i=1}^{N}\tilde{V}^i_{t+1}(x_{t+1,i}, p_{t+1,i})$. Our goal is to find $\mathbf{p}^*{}_{t+1} \in \mathcal{P}_{t+1}(\mathbf{x}_{t+1})$ such that $\tilde{V}_{t+1}\left(\mathbf{x}_{t+1}, \mathbf{p}^*{}_{t+1}\right) = \sum_{i=1}^{N}\tilde{V}^i_{t+1}(x_{t+1,i}, p^*_{t+1,i}) \geq \tilde{V}_{t+1}(\mathbf{x}_{t+1}, \mathbf{p}_{t+1})$, for all $\mathbf{p}_{t+1} \in \mathcal{P}_{t+1}(\mathbf{x}_{t+1})$. Since there is no closed-form expression for $\mathbf{p}^*{}_{t+1}$, we find $\mathbf{p}^*{}_{t+1}$ numerically (see Gallego and van Ryzin, 1994). We then set $\tilde{V}_{t+1}(\mathbf{x}_{t+1}) = \tilde{V}_{t+1}\left(\mathbf{x}_{t+1}, \mathbf{p}^*{}_{t+1}\right)$.

6.2 Approximate linear programming heuristics: ALPH and ALPH-3P

We also compute $\tilde{V}_t(\mathbf{x}_t)$ by solving an *approximate linear program* (ALP) (Schweitzer and Seidmann, 1985, de Farias and Van Roy, 2003) of the SDP (3). We first derive this ALP and then discuss heuristics to solve it.

6.2.1 Approximate linear program

We first formulate the SDP (3) as the following (infinite) linear program (Manne, 1960, Herńandez-Lerma and Lasserre, 1996):

$$\min_{\bar{\boldsymbol{V}}} \bar{V}_1(C\mathbb{1}_N) \tag{24}$$

$$\text{s.t. } \bar{V}_t(\mathbf{x}_t) \geq (1-\Lambda_t(\mathbf{x}_t,\mathbf{p}_t)\delta_t)\bar{V}_{t+1}(\mathbf{x}_t) + \sum_{u=1}^{N}\sum_{v=u}^{N}\lambda^{[u,v]}_t(\mathbf{x}_t,\mathbf{p}_t)\delta_t\left(\sum_{i=u}^{v}p_i + \bar{V}_{t+1}(\mathbf{x}_t - \mathbf{e}_{u,v})\right),$$
$$\forall(t,\mathbf{x}_t,\mathbf{p}_t) \in \mathcal{T}\times\mathcal{X}\times\mathcal{P}_t(\mathbf{x}_t), \tag{25}$$

$$\bar{V}_{T+1}(\mathbf{x}_{T+1}) = 0, \quad \forall \mathbf{x}_{T+1} \in \mathcal{X}. \tag{26}$$

The variables $\bar{V}_t(\mathbf{x}_t)$ are surrogates for the value functions, and an optimal solution $\bar{V}^*_t(\mathbf{x}_t)$ satisfies $\bar{V}^*_t(\mathbf{x}_t) = V_t(\mathbf{x}_t)$, for all $(t,\mathbf{x}_t) \in \mathcal{T}\times\mathcal{X}$. The objective function minimizes the value function at the initial state $C\mathbb{1}_N$, where $\mathbb{1}_N$ is a vector of N ones (the initial state can be changed as required).

Constraints (25)–(26) are closely related to the recursion of the SDP (3). They can be obtained by changing the equality in (3) to an inequality and modeling the maximization over $\mathbf{p}_t$ by an infinite set of inequalities.

Solving (24)–(26) is not practical due to its exponentially many variables and infinitely many constraints. We can reduce the number of variables in (24)–(26) by restricting their possible values. Let $\boldsymbol{\theta}$ and $\boldsymbol{\alpha}$ denote the vectors $(\theta_t, \forall t)$ and $(\alpha_{t,i}, \forall t, i)$ respectively. We consider an affine restriction $\bar{V}_t(\mathbf{x}_t) = \theta_t + \sum_{i=1}^{N} \alpha_{t,i} x_{t,i}$, which results in the following ALP:

$$\min_{\boldsymbol{\theta},\boldsymbol{\alpha}} \theta_1 + C \sum_{i=1}^{N} \alpha_{1,i} \tag{27}$$

$$\text{s.t. } \theta_t + \sum_{i=1}^{N} \alpha_{t,i} x_{t,i} \geq \left(\theta_{t+1} + \sum_{i=1}^{N} \alpha_{t+1,i} x_{t,i} \right) + \sum_{u=1}^{N} \sum_{v=u}^{N} \lambda_t^{[u,v]}(\mathbf{x}_t, \mathbf{p}_t) \delta_t \sum_{i=u}^{v} (p_{t,i} - \alpha_{t+1,i}),$$

$$\forall (t, \mathbf{x}_t, \mathbf{p}_t) \in \{1, \ldots, T-1\} \times \mathcal{X} \times \mathcal{P}_t(\mathbf{x}_t), \tag{28}$$

$$\theta_T + \sum_{i=1}^{N} \alpha_{T,i} x_{T,i} \geq \sum_{u=1}^{N} \sum_{v=u}^{N} \lambda_T^{[u,v]}(\mathbf{x}_T, \mathbf{p}_T) \delta_T \sum_{i=u}^{v} p_{T,i}, \ \forall (\mathbf{x}_T, \mathbf{p}_T) \in \mathcal{X} \times \mathcal{P}_T(\mathbf{x}_T). \tag{29}$$

ALP (27)–(29) has $T(N+1)$ variables: θ_t, $\forall t$ and $\alpha_{t,i}$, $\forall (t,i)$. However, it has infinitely many constraints. Thus, its solution can be approached by generating only a subset of constraints (see, for example, Adelman, 2004, 2007) using Algorithm 2. The termination criteria in Algorithm 2 could include a maximum number of iterations and/or a minimum improvement in the ALP objective. If one of the termination criteria is met before the ALP is solved to optimality (while violated constraints exist), we still obtain a value function approximation. Thus, we can interpret Algorithm 2 as an ALP solution heuristic that delivers an affine value function approximation $\tilde{V}_t(\mathbf{x}_t) = \theta_t^* + \sum_{i=1}^{N} \alpha_{t,i}^* x_{t,i}$.

6.2.2 Constraint generation

A key element in the above ALP heuristic is the constraint generation problem. For a given solution $(\boldsymbol{\theta}, \boldsymbol{\alpha})$, define the following functions that compute the difference between the right and the left hand sides of the ALP constraints (28)–(29):

$$g_t(\mathbf{x}_t, \mathbf{p}_t; \boldsymbol{\theta}, \boldsymbol{\alpha}) = (\theta_{t+1} - \theta_t) + \sum_{i=1}^{N} (\alpha_{t+1,i} - \alpha_{t,i}) x_{t,i}$$

Algorithm 2: ALP heuristic

Inputs: Termination criteria

Initialization: Set $k = 0$.

While termination criteria are not met **do**:

(i) Solve a relaxation of ALP defined by a finite subset $\mathcal{C}_t$ of its constraints for each period t.

(ii) Given a solution $\theta_t^*(k)$ and $\alpha_{t,i}^*(k)$, $\forall t, i$, to this relaxation, solve a *constraint generation problem* for each period t to identify the most violated constraint in (28)–(29) that is not in $\mathcal{C}_t$.

(iii) If there are no violated constraints, stop and return $\theta_t^*(k)$ and $\alpha_{t,i}^*(k)$, $\forall t, i$, as the ALP optimal solution; otherwise, add the violated constraints to $\mathcal{C}_t$, $\forall t$, and set $k \leftarrow k + 1$.

$$+\sum_{u=1}^{N}\sum_{v=u}^{N}\lambda_t^{[u,v]}(\mathbf{x}_t, \mathbf{p}_t)\delta_t\sum_{i=u}^{v}(p_{t,i} - \alpha_{t+1,i}), \forall (t, \mathbf{x}_t, \mathbf{p}_t) \in \{1, \ldots, T-1\} \times \mathcal{X} \times \mathcal{P}_t(\mathbf{x}_t);$$

$$g_T(\mathbf{x}_T, \mathbf{p}_T; \boldsymbol{\theta}, \boldsymbol{\alpha}) = -\theta_T - \sum_{i=1}^{N}\alpha_{T,i}x_{T,i} + \sum_{u=1}^{N}\sum_{v=u}^{N}\lambda_T^{[u,v]}(\mathbf{x}_T, \mathbf{p}_T)\delta_T\sum_{i=u}^{v}p_{T,i}, \quad \forall (\mathbf{x}_T, \mathbf{p}_T) \in \mathcal{X} \times \mathcal{P}_T(\mathbf{x}_T).$$

Given a solution $(\boldsymbol{\theta}, \boldsymbol{\alpha})$ in an iteration, the most violated constraint for each period $t \in \mathcal{T}$ can be generated by solving the constraint generation problem:

$$l_t(\boldsymbol{\theta}, \boldsymbol{\alpha}) = \max_{(\mathbf{x}_t, \mathbf{p}_t)} g_t(\mathbf{x}_t, \mathbf{p}_t; \boldsymbol{\theta}, \boldsymbol{\alpha}), \tag{30}$$

where $l_t(\boldsymbol{\theta}, \boldsymbol{\alpha})$ is the optimal objective function value. The above constraint generation problem is non-convex in $\mathbf{x}_t$ due to the indicator function in $\lambda_t^{[u,v]}(\mathbf{x}_t, \mathbf{p}_t)$. Using binary variables to model this indicator function leads to a mixed-integer program with cubic terms in the objective function. Such optimization problems are highly intractable, and solving them exactly at each iteration of the ALP heuristic is not practical. In contrast, to the best of our knowledge, ALPs formulated with an affine value function approximation in existing revenue management applications give rise to constraint generation problems that are either linear programs or mixed-integer linear programs (see, for example, Adelman, 2004, 2007, and Zhang and Adelman, 2009). Thus, our application appears to be more challenging.

Since finding the most violated constraint is too challenging, we approach the solution of problem (30) by solving convex quadratic programs and integer linear programs sequentially. These mathematical programs can be solved using off-the-shelf commercial solvers such as CPLEX and GUROBI. At each step of the sequential procedure either $\mathbf{x}_t$ or $\mathbf{p}_t$ is fixed in $g_t(\mathbf{x}_t, \mathbf{p}_t)$, and the maximization is performed over the domain of the remaining variable. For a given $\hat{\mathbf{x}}_t$, the maximization problem over $\mathbf{p}_t$ is the following simple quadratic program $\max_{\mathbf{p}_t \in \mathcal{P}_t(\hat{\mathbf{x}}_t)} g_t(\hat{\mathbf{x}}_t, \mathbf{p}_t)$, which we label as $P_t(\hat{\mathbf{x}}_t)$. For a given $\hat{\mathbf{p}}_t$, the maximization problem over $\mathbf{x}_t$ is $\max_{\mathbf{x}_t \in \mathcal{X}} g_t(\mathbf{x}_t, \hat{\mathbf{p}}_t)$. Defining $f_t^{[u,v]}(\mathbf{p}_t) := \left(a_t^{[u,v]} - b_t^{[u,v]} \frac{\sum_{i=u}^{v} p_{t,i}}{v-u+1}\right) \delta_t \sum_{i=u}^{v}(p_{t,i} - \alpha_{t+1,i})$, we formulate this problem as the following integer linear program labelled as $X_t(\hat{\mathbf{p}}_t)$:

$$
\begin{aligned}
\max_{\mathbf{x}_t, \mathbf{y}_t, \boldsymbol{\gamma}_t} \ & \sum_{i=1}^{N} (\alpha_{t+1,i} - \alpha_{t,i}) x_{t,i} + \sum_{u=1}^{N} \sum_{v=u}^{N} \gamma_t^{[u,v]} f_t^{[u,v]}(\hat{\mathbf{p}}_t) \\
\text{s.t.} \ & y_{t,i} \geq \frac{x_{t,i}}{C}, \quad \forall i \in \mathcal{N}, && (31) \\
& y_{t,i} \leq x_{t,i}, \quad \forall i \in \mathcal{N}, && (32) \\
& \gamma_t^{[u,v]} \geq \sum_{i=u}^{v} y_{t,i} + u - v, \quad \forall (u,v) \in \mathcal{UV}, && (33) \\
& \gamma_t^{[u,v]} \leq y_{t,i}, \ \forall i \in \{u, \ldots, v\}, (u,v) \in \mathcal{UV}, && (34) \\
& x_{t,i} \in \{0, \ldots, C\}, \quad \forall i \in \mathcal{N}, && (35) \\
& y_{t,i} \in \{0,1\}, \quad \forall i \in \mathcal{N}, && (36) \\
& \gamma_t^{[u,v]} \in \{0,1\}, \quad \forall (u,v) \in \mathcal{UV}. && (37)
\end{aligned}
$$

The objective function of $X_t(\hat{\mathbf{p}}_t)$ models the function $g_t(\mathbf{x}_t, \hat{\mathbf{p}}_t)$ by virtue of constraints (31)–(37). Constraints (31)–(32) ensure that y_i is equal to 1 if and only if $x_{t,i}$ is at least 1. Constraints (33)–(34) require $\gamma_t^{[u,v]}$ to equal 1 if and only if all days in the interval $[u,v]$ are not out of rooms. Finally, constraints (35)–(37) define the variable domains.

Our constraint generation heuristic identifies violated constraints by iteratively solving $P_t(\hat{\mathbf{x}}_t)$ and $X_t(\hat{\mathbf{p}}_t)$ as outlined in Algorithm 3. We set the termination criteria to be a maximum number of iterations and/or a minimum increase in constraint violation across successive iterations. We refer to the ALP heuristic using this constraint generation heuristic as ALPH.

We also consider a modified version of ALPH, denoted as ALPH-3P, to validate the simple pricing profile described in Table 1. We implement this profile using only three price levels $\bar{p}$, $\bar{p} + \Delta p_1$, and $\bar{p} - \Delta p_2$, where $\Delta p_1, \Delta p_2 \geq 0$: (i) prices on days u and v equal $\bar{p} + \Delta p_1$, (ii) prices on

Algorithm 3: Constraint generation heuristic

Inputs: Termination criteria

Initialization: Set $k = 0$ and $\mathbf{x}_t^{*,0} = C\mathbb{1}_N$.

While termination criteria are not met **do**:

(i) Increment k: $k \leftarrow k + 1$.

(ii) Solve $P\left(\mathbf{x}_t^{*,k-1}\right)$ and compute $\mathbf{p}_t^{*,k}$.

(iii) Solve $X\left(\mathbf{p}_t^{*,k}\right)$ and compute $\mathbf{x}_t^{*,k}$.

Return $\left(\mathbf{x}_t^{*,k}, \mathbf{p}_t^{*,k}\right)$.

days $u-1$ and $v+1$ equal $\bar{p} - \Delta p_2$, (iii) prices on all the other days in the service horizon equal $\bar{p}$. Specifically, we add constraints enforcing this structure to $\mathcal{P}_t(\mathbf{x}_t)$, both when solving $P_t(\hat{\mathbf{x}}_t)$ in Algorithm 3 and computing the booking prices in problem (23).

7 Numerical study

We numerically compare SDD, FAH, MDH, ALPH, and ALPH-3P on thirty-six instances described below. We choose a service horizon with $N = 7$ (one week) and a pricing-booking horizon with $T = 300$. We set the period length δ_t equal to $1/T$. We consider the capacity C equal to 5, 50, and 200. For each of these capacity values, we design two sets of instances: The first models a case of two public holidays within the service horizon by having peak demand from class 4-5, while the second corresponds to a case of a 4-day conference, which we model with peak demand from class 4-7.

For a given C, the instances in each set are obtained by choosing the base demand of the peak class to be a multiple of a constant base demand assigned to all the non-peak classes. We call this multiple the *demand spike factor* (DSF) and choose its value from the set $\{1, 4, 8, 12, 16, 20\}$. We set the non-peak base demand as 4.4 for $C = 5$ and 10 for $C = 50$ and 200. This choice ensures that the probability of an arrival in every period is less than 1.

We use simulations to evaluate our heuristics. Customers make reservations according to a Poisson process with rate determined by Equation (2). We first substitute the booking prices given by the heuristics into Equation (2), and then compute the average revenue over 1,000 simulation runs. This average revenue is then compared across different heuristics, and against the optimal policy for small instances. We use CPLEX V12.6 (IBM ILOG CPLEX 2014) to solve the linear, quadratic, and mixed-integer linear programs arising from our heuristics. When implementing ALPH and ALPH-3P, we set the termination criteria of Algorithm 2 to a maximum iteration limit of 500 and a minimum ALP objective function improvement of 0.01%. In addition, we set the termination criteria of Algorithm 3 to a maximum iteration limit of 5 and a minimum increase in the constraint violation of 0.01%. When implementing FAH, we set the discretization parameter M equal to 50.

Tables 2 and 3 show the simulation results for the small public-holiday and conference instances, respectively, with $C = 5$. The Exact-DP column in these tables display the average revenues when using the optimal pricing policy. The remaining columns display the average revenues from the remaining heuristics as an absolute value and as a percentage (in parentheses) relative to the average revenue of Exact-DP. The standard errors of all the reported average revenue estimates in Tables 2 and 3 are within 0.05% of the corresponding Exact-DP value.

Table 2: **Average revenues for small public-holiday instances.**

DSF	**Exact-DP**	**ALPH** (% **Exact-DP**)	**ALPH-3P** (% **Exact-DP**)	**MDH** (% **Exact-DP**)	**SDD** (% **Exact-DP**)	**FAH** (% **Exact-DP**)
1	2597.24	2400.24 (92.41)	2317.74 (92.25)	2472.74 (95.21)	2476.43 (95.35)	2233.37 (86.00)
4	2416.00	2353.99 (97.43)	2355.70 (97.35)	2249.25 (93.10)	2234.55 (92.49)	1627.61 (67.37)
8	2336.34	2282.70 (97.70)	2264.58 (96.92)	2142.86 (91.72)	2115.79 (90.56)	1524.92 (65.27)
12	2280.68	2243.78 (98.38)	2229.66 (97.76)	2103.40 (92.23)	2072.25 (90.86)	1469.72 (64.44)
16	2258.57	2214.58 (98.05)	2210.80 (97.88)	2088.09 (92.45)	2048.08 (90.68)	1477.88 (65.43)
20	2262.85	2197.30 (97.10)	2177.20 (96.21)	2078.90 (91.87)	2037.62 (90.05)	1441.89 (63.72)

Table 3: **Average revenues for small conference instances.**

DSF	**Exact-DP**	**ALPH** (% **Exact-DP**)	**ALPH-3P** (% **Exact-DP**)	**MDH** (% **Exact-DP**)	**SDD** (% **Exact-DP**)	**FAH** (% **Exact-DP**)
1	2595.58	2390.31 (92.09)	2375.76 (91.53)	2469.36 (95.14)	2472.06 (95.24)	2212.35 (85.23)
4	2796.44	2741.52 (98.04)	2747.09 (98.24)	2690.88 (96.23)	2639.74 (94.40)	2115.05 (75.63)
8	2807.94	2752.10 (98.01)	2748.31 (97.88)	2668.53 (95.04)	2656.28 (94.60)	2119.39 (75.48)
12	2787.52	2750.15 (98.66)	2752.15 (98.73)	2680.42 (96.16)	2657.47 (95.33)	2135.14 (76.60)
16	2797.91	2740.92 (97.96)	2750.42 (98.30)	2674.21 (95.58)	2657.39 (94.98)	2196.50 (78.51)
20	2802.37	2740.82 (97.80)	2758.92 (98.44)	2672.77 (95.38)	2665.82 (95.13)	2185.62 (78.00)

The average revenue of MDH is at most 2% higher than that of SDD in Tables 2 and 3. ALPH generally dominates the other heuristics on both the public-holiday and conference instances, except for DSF = 1, where the SDD average revenue is roughly 3% higher than that of ALPH. As DSF increases, the average revenues of MDH and SDD deteriorate significantly. The average revenue of ALPH is up to 7% and 6% higher than that of SDD and MDH, respectively, on the small public-holiday instances, while these differences are between 2-3% on the small conference instances. ALPH outperforms FAH by a substantially larger margin: up to 33% and 22% on the small public-holiday and conference instances respectively.

It is interesting that the ALPH-3P average revenues are within 1% of the ALPH average revenues, which provides some support for using the simple price profile in Table 1. Overall, the dynamic heuristics that account for multiple-day stays (ALPH, ALPH-3P, and MDH) outperform the dynamic heuristic based on the single-day decomposition (SDD) and the static fluid approximation (FAH) on the small instances. The superior performance of ALPH and ALPH-3P over MDH can be attributed to the fixed-price assumption made in MDH, but not in ALPH and ALPH-3P. The high sub-optimality of FAH suggests that dynamically accounting for uncertainty is critical when capacity is limited.

Tables 4 and 5 report results for the larger public-holiday and conference instances, respectively, with $C = 50$ and 200. Since we are unable to compute the optimal booking prices for these larger instances, we use the average revenue of ALPH as our reference. Standard errors of the reported average revenues are below 0.05% of the corresponding ALPH values. The performance of ALPH-3P is within 1% of ALPH, which corroborates the use of the simple pricing profile in Table 1.

Table 4: **Average revenues for large public-holiday instances.**

Capacity	**DSF**	**ALPH**	**ALPH-3P** (%**ALPH**)	**MDH** (%**ALPH**)	**SDD** (%**ALPH**)	**FAH** (%**ALPH**)
50	1	17591.84	17523.23 (99.61)	17529.31 (99.64)	17568.39 (99.87)	17604.21 (100.00)
50	4	17877.73	17875.58 (99.99)	17682.58 (98.91)	17365.58 (97.13)	16756.60 (93.73)
50	8	16385.32	16325.21 (99.63)	16354.10 (99.81)	16057.61 (98.00)	14420.50 (88.00)
50	12	15634.71	15611.63 (99.85)	15581.32 (99.66)	15293.87 (97.82))	14015.96 (89.65)
50	16	15220.59	15160.37 (99.60)	15172.40 (99.68)	14855.30 (97.60)	13573.24 (89.18)
50	20	14910.04	14878.44 (99.79)	14853.16 (99.62)	14878.44 (99.79)	13189.28 (88.46)
200	1	21080.70	20901.35 (99.15)	21032.65 (99.77)	21074.70 (99.97)	20999.45 (99.61)
200	4	25129.45	25081.02 (99.81)	25070.88 (99.77)	24439.89 (97.26)	25284.30 (100.62)
200	8	32926.07	32886.24 (99.88)	32864.99 (99.81)	30378.00 (92.26)	32826.62 (99.70)
200	12	40915.11	40892.95 (99.95)	40758.92 (99.62)	37945.36 (92.74)	40990.92 (100.19)
200	16	47382.67	47112.17 (99.43)	46737.36 (98.64)	46325.84 (97.77)	47243.92 (99.71)
200	20	48476.60	48465.46 (99.98)	48058.03 (99.14)	48376.70 (99.79)	46162.52 (95.23)

The average revenue of MDH is up to 3% worse than ALPH on some large instances but its performance for the most part does not deteriorate as much with DSF compared to the trend observed on the small instances. In contrast, the performance of SDD continues to vary significantly with DSF, and is up to 7% and 4% worse than ALPH on the large public-holiday and conference instances respectively. FAH continues to exhibit significant sub-optimality, up to 12%, on the

Table 5: **Average revenues for large conference instances.**

Capacity	DSF	ALPH	ALPH-3P (%ALPH)	MDH (%ALPH)	SDD (%ALPH)	FAH (%ALPH)
50	1	17554.91	17516.29 (99.78)	17543.21 (99.93)	17645.83 (100.52)	17639.08 (100.48)
50	4	21083.49	20921.15 (99.23)	20882.91 (99.05)	20454.71 (97.02)	20729.52 (98.32)
50	8	22750.95	22711.88 (99.83)	22156.49 (97.39)	21941.21 (96.44)	20566.59 (90.40)
50	12	22875.00	22782.89 (99.60)	22221.75 (97.14)	22079.38 (96.52)	20576.91 (89.95)
50	16	22822.24	22760.96 (99.73)	22269.79 (97.58)	22127.13 (96.95)	20588.22 (90.21)
50	20	22762.67	22744.11 (99.92)	22315.74 (98.04)	22247.97 (97.74)	20752.00 (91.17)
200	1	20968.15	21107.85 (100.67)	20967.75 (100.00)	21047.30 (100.38)	21086.75 (100.57)
200	4	28393.82	28170.66 (99.21)	28429.14 (100.12)	27973.61 (98.52)	28259.00 (99.53)
200	8	42149.43	41816.57 (99.21)	42142.81 (99.98)	40301.61 (95.62)	42057.65 (99.78)
200	12	57586.13	57503.04 (99.86)	57507.92 (99.86)	55178.85 (95.82)	57613.84 (100.05)
200	16	72363.25	72465.23 (100.14)	72312.95 (99.93)	71426.54 (98.71)	72353.65 (99.99)
200	20	80277.06	80700.56 (100.53)	79232.40 (98.70)	80426.35 (100.19)	79847.58 (99.52)

instances with $C = 50$, but is comparable to ALPH on the instances with $C = 200$.

Overall, our ALP-based heuristics outperform the remaining methods. Moreover, ALPH, ALPH-3P, MDH dominate SDD once again underscoring the importance of accounting for multiple-day stays. The former heuristics also outperform FAH for $C = 50$, which indicates that accounting for the uncertain evolution of room availability is important even at this moderate capacity value. In contrast, the excellent performance of FAH for $C = 200$ suggests that it can be used for pricing when capacity is substantially larger than the base demand. This observation is consistent with the existing theory on fluid approximations in the product-pricing setting (Gallego and van Ryzin, 1994). Finally, the performance improvement of MDH on the large instances relative to the small instances (Tables 2 and 3) suggests that the fixed-price assumption in MDH becomes benign as the total capacity C increases.

8 Conclusion

Dynamic pricing of rooms on each day (resources) used by multiple-day stays (products) requested by customers is a practice adopted by major hotel chains. We present a novel Markov decision process formulation of this understudied problem. Analysis of tractable special cases of our model sheds light on the sensitivity of room prices to demand parameters, and suggests room pricing guidelines around peak-demand events such as public holidays and conferences. We develop heuristics based on a fluid approximation and approximate linear programming to overcome the intractability of computing an optimal pricing policy. We obtain the following insights based on numerically benchmarking these heuristics against a single-day decomposition and a fixed-price heuristic: (i) Heuristics considering multiple-day stays outperform the single-day decomposition by up to 7%, which underscores the importance of accounting for multiple-day stays. (ii) The approximate linear programming heuristic based on an affine value function outperforms the fixed-price heuristic by up to 6% despite the original problem being non-convex. (iii) The revenue loss from using a pricing profile with three distinct price levels is less than 1%, thus suggesting that the pricing policy can be simplified around peak-demand events consistent with our analysis. (iv) The fluid approximation is suboptimal when room capacity is low, but progressively improves and becomes near optimal as the capacity increases, an observation in line with the performance of such models in the product-pricing setting.

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Online Appendix

A Proofs

Proof of Theorem 1.

For convenience, we omit the subscript t of $\mathbf{x}_t$ and $\mathbf{p}_t$. Rewriting Equation (5), we have the objective function

$$f_t(\mathbf{x},\mathbf{p}) \;=\; \delta_t \sum_{u=1}^{N}\sum_{v=u}^{N}\left(\sum_{i=u}^{v} p_i - \triangle V_{t+1}^{[u,v]}(\mathbf{x})\right)\lambda_t^{[u,v]}(\mathbf{x},\mathbf{p}).$$

Since $\mathbf{x}$ are constant, they can be omitted in the analysis. As a result, we have

$$f_t(\mathbf{p}) \;=\; \delta_t \sum_{u=1}^{N}\sum_{v=u}^{N}\left(\sum_{i=u}^{v} p_i - \triangle V_{t+1}^{[u,v]}\right)\lambda_t^{[u,v]}(\mathbf{p}).$$

Taking the partial derivative of $f_t(\mathbf{p})$ with respect to p_j, we have

$$\begin{aligned}\frac{\partial f_t(\mathbf{p})}{\partial p_j} \;=\; \delta_t \sum_{u=1}^{j}\sum_{v=j}^{N}\Bigg[& A^{[u,v]}\left(a_t^{[u,v]} + \frac{b_t^{[u,v]}}{v-u+1}\triangle V_{t+1}^{[u,v]}\right) \\ & - 2A^{[u,v]}\frac{b_t^{[u,v]}}{v-u+1}\sum_{i=u}^{v} p_i\Bigg].\end{aligned}$$

Taking the partial derivative of $\partial f_t(\mathbf{p})/\partial p_j$ with respect to p_k gives the second-order partial derivative

$$\begin{aligned}\frac{\partial^2 f_t(\mathbf{p})}{\partial p_j \partial p_k} &= \begin{cases} -2\delta_t \sum_{u=1}^{j}\sum_{v=k}^{N} A^{[u,v]}\frac{b_t^{[u,v]}}{v-u+1}, & \text{if } j \le k; \\ -2\delta_t \sum_{u=1}^{k}\sum_{v=j}^{N} A^{[u,v]}\frac{b_t^{[u,v]}}{v-u+1}, & \text{otherwise;}\end{cases} \\ &= -2\delta_t \sum_{u=1}^{\min\{j,k\}}\;\sum_{v=\max\{j,k\}}^{N} A^{[u,v]}\frac{b_t^{[u,v]}}{v-u+1}.\end{aligned}$$

If day i is out of rooms, the problem can be partitioned into two separate subproblems. Each subproblem can be solved individually. Therefore, without loss of generality, we assume $A^{[u,v]} = 1$ for all u and v.

The Hessian matrix $H(N)$ of the objective function has elements

$$h_{jk} = -2\delta_t \sum_{u=1}^{\min\{j,k\}}\;\sum_{v=\max\{j,k\}}^{N} \phi_{uv}, \qquad j,k = 1,\ldots,N;$$

where $\phi_{uv} = b_t^{[u,v]}/(v-u+1) \ge 0$. Let $G(N)$ be a matrix that has elements

$$g_{jk} = \sum_{u=1}^{\min\{j,k\}}\;\sum_{v=\max\{j,k\}}^{N} \phi_{uv}, \qquad j,k = 1,\ldots,N. \tag{38}$$

To show that the objective function is concave, it is equivalent to show that all principal minors of matrix $G(N)$ are nonnegative (see Winston (1994)).

We will prove by induction. For $N = 1$, $|G(1)| = \phi_{11} \geq 0$. The objective function is concave.

For $N = 2$,

$$G(2) = \begin{bmatrix} \phi_{11} + \phi_{12} & \phi_{12} \\ \phi_{12} & \phi_{12} + \phi_{22} \end{bmatrix}.$$

All first principal minors are nonnegative. The second principal minor is $(\phi_{11} + \phi_{12})(\phi_{12} + \phi_{22}) - \phi_{12}^2 \geq 0$. Thus, the objective function is concave for $N = 2$.

For $N = 3$,

$$G(3) = \begin{bmatrix} \phi_{11} + \phi_{12} + \phi_{13} & \phi_{12} + \phi_{13} & \phi_{13} \\ \phi_{12} + \phi_{13} & \phi_{12} + \phi_{13} + \phi_{22} + \phi_{23} & \phi_{13} + \phi_{23} \\ \phi_{13} & \phi_{13} + \phi_{23} & \phi_{13} + \phi_{23} + \phi_{33} \end{bmatrix}.$$

It is straightforward to show that all first and second principal minors are nonnegative. The third principal minor is the determinant $|G(3)|$. By decomposing the first column of $|G(3)|$ into two parts, we have

$$\begin{aligned} |G(3)| = & \begin{vmatrix} \phi_{11} & \phi_{12} + \phi_{13} & \phi_{13} \\ 0 & \phi_{12} + \phi_{13} + \phi_{22} + \phi_{23} & \phi_{13} + \phi_{23} \\ 0 & \phi_{13} + \phi_{23} & \phi_{13} + \phi_{23} + \phi_{33} \end{vmatrix} + \\ & \begin{vmatrix} \phi_{12} + \phi_{13} & \phi_{12} + \phi_{13} & \phi_{13} \\ \phi_{12} + \phi_{13} & \phi_{12} + \phi_{13} + \phi_{22} + \phi_{23} & \phi_{13} + \phi_{23} \\ \phi_{13} & \phi_{13} + \phi_{23} & \phi_{13} + \phi_{23} + \phi_{33} \end{vmatrix}. \end{aligned}$$

It is straightforward to show that the first determinant on the right hand side is nonnegative. For the second determinant, subtracting row 1 from row 2 and subtracting column 1 from column 2, we have

$$|G(3)| \geq \begin{vmatrix} \phi_{12} + \phi_{13} & 0 & \phi_{13} \\ 0 & \phi_{22} + \phi_{23} & \phi_{23} \\ \phi_{13} & \phi_{23} & \phi_{13} + \phi_{23} + \phi_{33} \end{vmatrix}.$$

Decomposing the first column of the determinant on the right hand side into two parts, we have

$$\begin{aligned} |G(3)| \geq & \begin{vmatrix} \phi_{12} & 0 & \phi_{13} \\ 0 & \phi_{22} + \phi_{23} & \phi_{23} \\ 0 & \phi_{23} & \phi_{13} + \phi_{23} + \phi_{33} \end{vmatrix} + \\ & \begin{vmatrix} \phi_{13} & 0 & \phi_{13} \\ 0 & \phi_{22} + \phi_{23} & \phi_{23} \\ \phi_{13} & \phi_{23} & \phi_{13} + \phi_{23} + \phi_{33} \end{vmatrix}. \end{aligned}$$

It is straightforward to show that the first determinant on the right hand side is nonnegative. For the second determinant, subtracting row 1 from row 3, we have

$$|G(3)| \geq \begin{vmatrix} \phi_{13} & 0 & \phi_{13} \\ 0 & \phi_{22} + \phi_{23} & \phi_{23} \\ 0 & \phi_{23} & \phi_{23} + \phi_{33} \end{vmatrix} \geq 0.$$

Thus, the objective function is concave for $N = 3$.

We have shown that $|G(1)|$, $|G(2)|$, and $|G(3)|$ are nonnegative. We make the following induction hypothesis:

$$|G(n)| \geq 0, \qquad n = 1, \ldots, N-1. \tag{39}$$

We will show that $|G(N)| \geq 0$. The proof is a generalization of the proof for the $N = 3$ case. Equation (38) gives

$$|G(N)| \quad = \quad \begin{vmatrix} \sum\limits_{v=1}^{N} \phi_{1v} & \sum\limits_{v=2}^{N} \phi_{1v} & \sum\limits_{v=3}^{N} \phi_{1v} & \cdots & \phi_{1N} \\ \sum\limits_{v=2}^{N} \phi_{1v} & \sum\limits_{u=1}^{2}\sum\limits_{v=2}^{N} \phi_{uv} & \sum\limits_{u=1}^{2}\sum\limits_{v=3}^{N} \phi_{uv} & \cdots & \sum\limits_{u=1}^{2} \phi_{uN} \\ \sum\limits_{v=3}^{N} \phi_{1v} & \sum\limits_{u=1}^{2}\sum\limits_{v=3}^{N} \phi_{uv} & \sum\limits_{u=1}^{3}\sum\limits_{v=3}^{N} \phi_{uv} & \cdots & \sum\limits_{u=1}^{3} \phi_{uN} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{1N} & \sum\limits_{u=1}^{2} \phi_{uN} & \sum\limits_{u=1}^{3} \phi_{uN} & \cdots & \sum\limits_{u=1}^{N} \phi_{uN} \end{vmatrix}.$$

Call the above determinant the *full* determinant. Decomposing the first column into two parts, we have

$$|G(N)| =$$

$$\begin{vmatrix} \phi_{11} & \sum\limits_{v=2}^{N} \phi_{1v} & \sum\limits_{v=3}^{N} \phi_{1v} & \cdots & \phi_{1N} \\ 0 & \sum\limits_{u=1}^{2}\sum\limits_{v=2}^{N} \phi_{uv} & \sum\limits_{u=1}^{2}\sum\limits_{v=3}^{N} \phi_{uv} & \cdots & \sum\limits_{u=1}^{2} \phi_{uN} \\ 0 & \sum\limits_{u=1}^{2}\sum\limits_{v=3}^{N} \phi_{uv} & \sum\limits_{u=1}^{3}\sum\limits_{v=3}^{N} \phi_{uv} & \cdots & \sum\limits_{u=1}^{3} \phi_{uN} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \sum\limits_{u=1}^{2} \phi_{uN} & \sum\limits_{u=1}^{3} \phi_{uN} & \cdots & \sum\limits_{u=1}^{N} \phi_{uN} \end{vmatrix} + \begin{vmatrix} \sum\limits_{v=2}^{N} \phi_{1v} & \sum\limits_{v=2}^{N} \phi_{1v} & \sum\limits_{v=3}^{N} \phi_{1v} & \cdots & \phi_{1N} \\ \sum\limits_{v=2}^{N} \phi_{1v} & \sum\limits_{u=1}^{2}\sum\limits_{v=2}^{N} \phi_{uv} & \sum\limits_{u=1}^{2}\sum\limits_{v=3}^{N} \phi_{uv} & \cdots & \sum\limits_{u=1}^{2} \phi_{uN} \\ \sum\limits_{v=3}^{N} \phi_{1v} & \sum\limits_{u=1}^{2}\sum\limits_{v=3}^{N} \phi_{uv} & \sum\limits_{u=1}^{3}\sum\limits_{v=3}^{N} \phi_{uv} & \cdots & \sum\limits_{u=1}^{3} \phi_{uN} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{1N} & \sum\limits_{u=1}^{2} \phi_{uN} & \sum\limits_{u=1}^{3} \phi_{uN} & \cdots & \sum\limits_{u=1}^{N} \phi_{uN} \end{vmatrix}.$$

For the second determinant on the right hand side, subtracting row 1 from row 2 and subtracting column 1 from column 2, we have

$$|G(N)| =$$

$$\phi_{11} \begin{vmatrix} \sum\limits_{u=1}^{2}\sum\limits_{v=2}^{N} \phi_{uv} & \sum\limits_{u=1}^{2}\sum\limits_{v=3}^{N} \phi_{uv} & \cdots & \sum\limits_{u=1}^{2} \phi_{uN} \\ \sum\limits_{u=1}^{2}\sum\limits_{v=3}^{N} \phi_{uv} & \sum\limits_{u=1}^{3}\sum\limits_{v=3}^{N} \phi_{uv} & \cdots & \sum\limits_{u=1}^{3} \phi_{uN} \\ \vdots & \vdots & \vdots & \vdots \\ \sum\limits_{u=1}^{2} \phi_{uN} & \sum\limits_{u=1}^{3} \phi_{uN} & \cdots & \sum\limits_{u=1}^{N} \phi_{uN} \end{vmatrix} + \begin{vmatrix} \sum\limits_{v=2}^{N} \phi_{1v} & 0 & \sum\limits_{v=3}^{N} \phi_{1v} & \cdots & \phi_{1N} \\ 0 & \sum\limits_{v=2}^{N} \phi_{2v} & \sum\limits_{v=3}^{N} \phi_{2v} & \cdots & \phi_{2N} \\ \sum\limits_{v=3}^{N} \phi_{1v} & \sum\limits_{v=3}^{N} \phi_{2v} & \sum\limits_{u=1}^{3}\sum\limits_{v=3}^{N} \phi_{uv} & \cdots & \sum\limits_{u=1}^{3} \phi_{uN} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{1N} & \phi_{2N} & \sum\limits_{u=1}^{3} \phi_{uN} & \cdots & \sum\limits_{u=1}^{N} \phi_{uN} \end{vmatrix}.$$

The first determinant on the right hand side can be transformed into a new determinant of a corresponding $(N-1)$-day problem. Each element of the new determinant is defined as

$$g_{jk} = \sum_{u=1}^{\min\{j,k\}} \sum_{v=\max\{j,k\}}^{N-1} \Phi_{uv}, \qquad j, k = 1, \ldots, N-1;$$

where

$$\Phi_{uv} \quad = \quad \begin{cases} \phi_{1,v+1} + \phi_{2,v+1}, & \text{if } u = 1; \\ \phi_{u+1,v+1}, & \text{if } u = 2, \ldots, N-1; \end{cases}$$

for $v = 1, \ldots, N-1$. According to the induction hypothesis (39), this new determinant is nonnegative. Thus, we have

$$|G(N)| \quad \geq \quad \begin{vmatrix} \sum_{v=2}^{N} \phi_{1v} & 0 & \sum_{v=3}^{N} \phi_{1v} & \cdots & \phi_{1N} \\ 0 & \sum_{v=2}^{N} \phi_{2v} & \sum_{v=3}^{N} \phi_{2v} & \cdots & \phi_{2N} \\ \sum_{v=3}^{N} \phi_{1v} & \sum_{v=3}^{N} \phi_{2v} & \sum_{u=1}^{3}\sum_{v=3}^{N} \phi_{uv} & \cdots & \sum_{u=1}^{3} \phi_{uN} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{1N} & \phi_{2N} & \sum_{u=1}^{3} \phi_{uN} & \cdots & \sum_{u=1}^{N} \phi_{uN} \end{vmatrix}.$$

Note that we have removed ϕ_{11} from the analysis. Let the determinant on the right hand side above be the full determinant for the subsequent analysis.

Repeating a similar procedure, we can remove ϕ_{1j}, for $j = 2, \ldots, N$, from the analysis one at a time. In short, for each iteration j, we remove ϕ_{1j} by performing the following steps:

1. Decompose the first column of the full determinant into two parts. The first part corresponds to a column that contains ϕ_{1j} as its first element and zero for the remaining elements. Thus, the full determinant can be decomposed into two determinants.

2. The first determinant can be transformed into a determinant corresponding to an $(N-1)$-day problem. Under the induction hypothesis (39), this determinant is nonnegative and can be removed from the analysis.

3. For the second determinant, subtract row 1 from row $j+1$ and subtract column 1 from column $j+1$. Let the resultant determinant be the full determinant for the next iteration.

For the last iteration N, the second determinant is zero and this concludes that $|G(N)| \geq 0$.

Lastly, all n-th principal minors of $G(N)$, for $n = 1, \ldots, N-1$, can be transformed into a determinant corresponding to an n-day problem. According to the induction hypothesis (39), all these principal minors are nonnegative. This proves that the objective function for an N-day problem is concave. □

Proof of Proposition 1.
To prove the claim, it suffices to show that $\frac{\partial^2 \bar{f}_t(\mathbf{x}_t, \mathbf{p}_t)}{\partial p_{t,1}^2}$ can be positive. We have

$$\frac{\partial \bar{f}_t(\mathbf{x}_t, \mathbf{p}_t)}{\partial p_{t,1}} = \delta_t \sum_{v=1}^{N} \left[e^{-\sum_{i=1}^{v} p_{t,i}} \left(1 + \triangle \bar{V}_{t+1}^{[1,v]}(\mathbf{x}_t) - \sum_{i=1}^{v} p_{t,i} \right) \right],$$

and

$$\frac{\partial^2 \bar{f}_t(\mathbf{x}_t, \mathbf{p}_t)}{\partial p_{t,1}^2} = \delta_t \sum_{v=1}^{N} \left[-e^{-\sum_{i=1}^{v} p_{t,i}} \left(2 + \triangle \bar{V}_{t+1}^{[1,v]}(\mathbf{x}_t) - \sum_{i=1}^{v} p_{t,i} \right) \right].$$

Since $\mathbf{p}_t \in R_+^N$, there exists $\hat{\mathbf{p}}_t$ such that the term $2 + \triangle \bar{V}_{t+1}^{[1,v]}(\mathbf{x}_t) - \sum_{i=1}^{v} p_{t,i} < 0$ for $\mathbf{p}_t \geq \hat{\mathbf{p}}_t$, which implies that $\frac{\partial^2 \bar{f}_t(\mathbf{x}_t, \mathbf{p}_t)}{\partial p_{t,1}^2}$ is strictly positive in this range. $\square$

Proof of Proposition 2.
For convenience, we omit the subscript t of $\mathbf{x}_t$ and $\mathbf{p}_t$. By solving the equations $\partial f_t(\mathbf{x}, \mathbf{p})/\partial p_1 = 0$ and $\partial f_t(\mathbf{x}, \mathbf{p})/\partial p_2 = 0$, we have

$$\begin{aligned} p_1^* \;=\; & \Big\{ A^{[1,2]}(\mathbf{x}) b_t^{[1,2]} \Big[A^{[1,1]}(\mathbf{x}) \Big(a_t^{[1,1]} + b_t^{[1,1]} \triangle V_{t+1}^{[1,1]}(\mathbf{x}) \Big) - A^{[2,2]}(\mathbf{x}) \Big(a_t^{[2,2]} + b_t^{[2,2]} \triangle V_{t+1}^{[2,2]}(\mathbf{x}) \Big) \Big] + \\ & 2A^{[2,2]}(\mathbf{x}) b_t^{[2,2]} \times \sum_{v=1}^{2} A^{[1,v]}(\mathbf{x}) \left(a_t^{[1,v]} + \frac{b_t^{[1,v]}}{v} \triangle V_{t+1}^{[1,v]}(\mathbf{x}) \right) \Big\} / d(\mathbf{x}), \end{aligned}$$

$$\begin{aligned} p_2^* \;=\; & \Big\{ A^{[1,2]}(\mathbf{x}) b_t^{[1,2]} \Big[A^{[2,2]}(\mathbf{x}) \Big(a_t^{[2,2]} + b_t^{[2,2]} \triangle V_{t+1}^{[2,2]}(\mathbf{x}) \Big) - A^{[1,1]}(\mathbf{x}) \Big(a_t^{[1,1]} + b_t^{[1,1]} \triangle V_{t+1}^{[1,1]}(\mathbf{x}) \Big) \Big] + \\ & 2A^{[1,1]}(\mathbf{x}) b_t^{[1,1]} \times \sum_{u=1}^{2} A^{[u,2]}(\mathbf{x}) \left(a_t^{[u,2]} + \frac{b_t^{[u,2]}}{3-u} \triangle V_{t+1}^{[u,2]}(\mathbf{x}) \right) \Big\} / d(\mathbf{x}), \end{aligned}$$

where $d(\mathbf{x}) = A^{[1,2]}(\mathbf{x}) b_t^{[1,2]} \times 2 \sum_{i=1}^{2} A^{[i,i]}(\mathbf{x}) b_t^{[i,i]} + 4 \prod_{i=1}^{2} A^{[i,i]}(\mathbf{x}) b_t^{[i,i]}$. The theorem follows. $\square$

Lemma 1 below is needed in our proofs of Part 2 of Proposition 4 and Proposition 5. Recall that $r(u) := N - u + 1$. Define $A_t(i) = \sum_{u=1}^{i} \sum_{v=i}^{N} a_t^{[u,v]} \delta_t$, $g_t(i) := \sum_{u=1}^{i} \sum_{v=i}^{N} b_t^{[u,v]} \delta_t/(v-u+1)$, $G_t(i, \mathbf{x}_t) = \sum_{u=1}^{i} \sum_{v=i}^{N} b_t^{[u,v]} \Delta V_{t+1}^{[u,v]}(\mathbf{x}_t) \delta_t/(v-u+1)$, and $h_t(i,j) := \sum_{u=1}^{j} \sum_{v=i}^{N} b_t^{[u,v]} \delta_t/(v-u+1)$.

Lemma 1. *If $a_t^{[u,v]} = a_t^{[r(v),r(u)]}$ and $b_t^{[u,v]} = b_t^{[r(v),r(u)]}$ for each $t = 1, \ldots T$ and $(u,v) \in \mathcal{UV}$, then $A_t(i) = A_t(r(i))$ and $g_t(i) = g_t(r(i))$ for $i \in \mathcal{N}$; and $h_t(i,j) = h_t(r(j), r(i))$ for each $(i,j) \in \mathcal{UV}$. In addition, at period t, if $\Delta V_t^{[u,v]}(\mathbf{x}_t) = \Delta V_t^{[r(v),r(u)]}(\mathbf{x}_t)$ for each $(u,v) \in \mathcal{UV}$, then $G_t(i, \mathbf{x}_t) = G_t(r(i), \mathbf{x}_t)$.*

Proof. For a given $i \in \{1, \ldots, N\}$, the equality $A_t(i) = A_t(r(i))$ follows directly from our assumption $a_t^{[u,v]} = a_t^{[r(v),r(u)]}$. The equalities $g_t(i) = g_t(r(i))$ and $h_t(i,j) = h_t(r(j), r(i))$ hold because the term $b_t^{[u,v]} \delta_t/(v-u+1)$ corresponding to class u-v in both $g_t(i)$ and $h_t(i,j)$ is equal to the term $b_t^{[r(u),r(v)]} \delta_t/(r(u) - r(v) + 1)$ corresponding to $[r(v), r(u)]$ in both $g_t(r(i))$ and $h_t(r(j), r(i))$. The equality $G_t(i, \mathbf{x}_t) = G_t(r(i), \mathbf{x}_t)$ is true for analogous reasons after accounting for $\Delta V_t^{[u,v]}(\mathbf{x}_t) = \Delta V_t^{[r(v),r(u)]}(\mathbf{x}_t)$. $\square$

Proof of Proposition 4.
1. We proceed by backward induction on the number of stages. For $t = T$, we have that $\Delta V_{T+1}^{[u,v]}(\mathbf{x}_{T+1}) = 0$. Therefore,

$$V_T(\mathbf{x}_T) = \max_{\mathbf{p}_T \in \mathcal{P}_T(\mathbf{x}_T)} \sum_{u=1}^{N} \sum_{v=u}^{N} \left(\sum_{i=u}^{v} p_{T,i} \right) \left(a_T^{[u,v]} - \frac{b_T^{[u,v]} \sum_{i=u}^{v} p_{T,i}}{(v-u+1)} \right) A^{[u,v]}(\mathbf{x}_T).$$

Note that the objective function defining $V_T(\mathbf{x}_T)$ above has a term for every class u-v. Consider the terms corresponding to classes u-v and $r(v)$-$r(u)$ in the objective functions of $V_T(\mathbf{x}_T^1)$ and $V_T(\mathbf{x}_T^2)$, respectively.

$$(V_T(\mathbf{x}_T^1)\ [u,v] \text{ term}):\ \left(\sum_{i=u}^{v} p_{T,i}\right)\left(a_T^{[u,v]} - \frac{b_T^{[u,v]}\sum_{i=u}^{v} p_{T,i}}{(v-u+1)}\right) A^{[u,v]}(\mathbf{x}_T^1). \tag{40}$$

$$(V_T(\mathbf{x}_T^2)\ [r(v),r(u)] \text{ term}):\ \left(\sum_{i=r(v)}^{r(u)} p_{T,i}\right)\left(a_T^{[r(v),r(u)]} - \frac{b_T^{[r(v),r(u)]}\sum_{i=r(v)}^{r(u)} p_{T,i}}{(r(u)-r(v)+1)}\right) A^{[r(v),r(u)]}(\mathbf{x}_T^2). \tag{41}$$

Since we assume $x_{T,i}^1 = x_{T,r(i)}^2$ for all $i = 1,\ldots,N$, we have that $A^{[u,v]}(\mathbf{x}_T^1) = A^{[r(v),r(u)]}(\mathbf{x}_T^2)$. Moreover, given our assumption that $a_T^{[u,v]} = a_T^{[r(v),r(u)]}$ and $b_T^{[u,v]} = b_T^{[r(v),r(u)]}$ for each $(u,v) \in \mathcal{UV}$, and $(v-u+1) = (r(u)-r(v)+1)$, (41) is equivalent to

$$\left(\sum_{i=r(v)}^{r(u)} p_{T,i}\right)\left(a_T^{[u,v]} - \frac{b_T^{[u,v]}\sum_{i=r(v)}^{r(u)} p_{T,i}}{(v-u+1)}\right) A^{[u,v]}(\mathbf{x}_T^1).$$

Therefore, relabeling the variables $p_{T,i}$ as $p_{T,N-i+1}$ (i.e. permuting them) makes the above term equivalent to (40). Since we chose class u-v arbitrarily, it follows that the objective functions of $V_T(\mathbf{x}_T^1)$ and $V_T(\mathbf{x}_T^2)$ are equivalent under a relabeling of variables. Moreover, constraints in $\mathcal{P}_T(\mathbf{x}_T)$ remain equivalent under this relabeling. Therefore, the optimization problems determining $V_T(\mathbf{x}_T^1)$ and $V_T(\mathbf{x}_T^2)$ are equivalent and $V_T(\mathbf{x}_T^1) = V_T(\mathbf{x}_T^2)$.

Assume that for $t = t'+1,\ldots,T$ we have $V_t(\mathbf{x}_t^1) = V_t(\mathbf{x}_t^2)$ for all pairs $(\mathbf{x}_t^1, \mathbf{x}_t^2)$ such that $x_{t,i}^1 = x_{t,r(i)}^2$ for all $i = 1,\ldots,N$. We now show this property to hold for period t'. Recall that

$$V_{t'}(\mathbf{x}_{t'}) = \max_{\mathbf{p}_{t'} \in \mathcal{P}_{t'}(\mathbf{x}_{t'})} \sum_{u=1}^{N}\sum_{v=u}^{N}\left(\sum_{i=u}^{v} p_{t',i} - \Delta V_{t'+1}^{[u,v]}(\mathbf{x}_{t'+1})\right)\left(a_{t'}^{[u,v]} - \frac{b_{t'}^{[u,v]}\sum_{i=u}^{v} p_{t',i}}{(v-u+1)}\right) A^{[u,v]}(\mathbf{x}_{t'}). \tag{42}$$

If $\Delta V_{t'+1}^{[u,v]}(\mathbf{x}_{t'+1}^1) = \Delta V_{t'+1}^{[r(v),r(u)]}(\mathbf{x}_{t'+1}^2)$, we can show that $V_{t'}(\mathbf{x}_{t'}^1) = V_{t'}(\mathbf{x}_{t'}^2)$ by mirroring the arguments used in our proof for stage T, that is, we can establish that the optimization problems determining $V_{t'}(\mathbf{x}_{t'}^1)$ and $V_{t'}(\mathbf{x}_{t'}^2)$ are equivalent. The equality $\Delta V_{t'+1}^{[u,v]}(\mathbf{x}_{t'+1}^1) = \Delta V_{t'+1}^{[r(v),r(u)]}(\mathbf{x}_{t'+1}^2)$ follows because (i) $V_{t'+1}(\mathbf{x}_{t'}^1)$ is equal to $V_{t'+1}(\mathbf{x}_{t'}^2)$ and (ii) $V_{t'+1}(\mathbf{x}_{t'}^1 - \mathbf{e}_{u,v})$ is equal to $V_{t'+1}(\mathbf{x}_{t'}^2 - \mathbf{e}_{r(v),r(u)})$. Condition (i) is a direct consequence of our induction hypothesis. Condition (ii) also follows from the induction hypothesis because the i-th components of $\mathbf{x}_{t'}^1$ and $\mathbf{e}_{u,v}$ coincide with the $r(t)$-th components of $\mathbf{x}_{t'}^2$ and $\mathbf{e}_{r(v),r(u)}$, respectively, which implies that the vectors $\mathbf{x}_{t'}^{[1,u,v]} := \mathbf{x}_{t'}^1 - \mathbf{e}_{u,v}$ and $\mathbf{x}_{t'}^{[2,r(v),r(u)]} := \mathbf{x}_{t'}^2 - \mathbf{e}_{r(v),r(u)}$ satisfy $x_{t',i}^{[1,u,v]} = x_{t',r(i)}^{[2,r(v),r(u)]}$. Thus, $V_{t'}(\mathbf{x}_{t'}^1) = V_{t'}(\mathbf{x}_{t'}^2)$ and the proof is complete based on the principle of mathematical induction.

2. We provide an overview of our proof strategy before discussing the details. Consider the optimization problem (3), which we label P, determining $\mathbf{p}_t^*$ in period t and state $\mathbf{x}_t$, where $x_{t,i} = x_{t,N-i+1}$ for all $i = 1,\ldots,\lfloor N/2 \rfloor$ by assumption. We also construct a problem P' starting from P by relabeling its variable $p_{t,i}$ as $p'_{t,r(i)}$ for each $i = 1,\ldots,N$. Our focus will be to show that the vector $\mathbf{p}'^*_t$ defined as $p'^{,*}_{t,i} := p^*_{t,i}$ for all $i = 1,\ldots,N$ is an optimal solution to P'. This relationship plus the transformation we applied ($p_{t,i} = p'_{t,r(i)}$ for each $i = 1,\ldots,N$) implies that $p^*_{t,i} = p^*_{t,r(i)}$ for all

$i = 1, \ldots, \lfloor N/2 \rfloor$.

Since the room capacity vector $\mathbf{x}_t$ is reflective, we have that $A^{[u,v]}(\mathbf{x}_t) = A^{[r(v),r(u)]}(\mathbf{x}_t)$. We therefore suppress the indicator function $A^{[u,v]}(\mathbf{x}_t)$ in the discussion below to ease exposition. Consider the original and transformed problems:

$$\mathbf{p}_t^* := \underset{\mathbf{p}_t \in \mathcal{P}_t(\mathbf{x}_t)}{\operatorname{argmax}} \sum_{i=1}^{N} p_{t,i} \Bigg[(A_t(i) + G_t(i, \mathbf{x}_t)) - \sum_{i=1}^{N} p_{t,i} g_t(i) \\ - \left(\sum_{j=1}^{i-1} p_{t,j} h_t(i,j) + \sum_{j=i+1}^{N} p_{t,j} h_t(j,i) \right) \Bigg]; \tag{43}$$

$$\mathbf{p'}_t^* := \underset{\mathbf{p'}_t \in \mathcal{P}_t(\mathbf{x}_t)}{\operatorname{argmax}} \sum_{i=1}^{N} p'_{t,i} \Bigg[(A_t(r(i)) + G_t(r(i), \mathbf{x}_t)) - \sum_{i=1}^{N} p'_{t,i} g_t(r(i)) \\ - \left(\sum_{j=1}^{r(i)-1} p'_{t,r(j)} h_t(i,j) + \sum_{j=r(i)+1}^{N} p'_{t,r(j)} h_t(j,i) \right) \Bigg]. \tag{44}$$

From Lemma 1 it immediately follows that the objective function coefficients of the linear and quadratic terms of $p_{t,i}$ and $p'_{t,i}$ are the same. Since $h_t(i,j) = h_t(r(i), r(j))$, also by Lemma 1, the sums $\sum_{j=1}^{i-1} p_{t,j} h_t(i,j)$ and $\sum_{j=i+1}^{N} p_{t,j} h_t(j,i)$ are identical to the sums $\sum_{j=r(i)+1}^{N} p'_{t,r(j)} h_t(j,i)$ and $\sum_{j=1}^{r(i)-1} p'_{t,r(j)} h_t(i,j)$, respectively. Thus, the objective function terms corresponding to $p_{t,i}$ and $p'_{t,i}$ in (43) and (44), respectively, are the same (except for the cosmetic label p and p'). It is also easy to verify that the constraints involving $p_{t,i}$ and $p'_{t,i}$ are the same. Thus, it follows that $\mathbf{p'}_t^*$ is an optimal solution to (44). $\square$

Proof of Proposition 5.
Since $\mathbf{x}_t > 1$, we have $A^{[u,v]}(\mathbf{x}_t) = 1$, for all $(u,v) \in \mathcal{UV}$. Define $\bar{G}_t(i, \mathbf{x}_t) = \sum_{u=1}^{i} \sum_{v=i}^{N} \Delta V_{t+1}^{[u,v]}(\mathbf{x}_t)/(v-u+1)$, $\bar{g}_t(i) := \sum_{u=1}^{i} \sum_{v=i}^{N} 1/(v-u+1)$ and $\bar{h}_t(i,j) := \sum_{u=1}^{j} \sum_{v=i}^{N} 1/(v-u+1)$. By assumption the optimal room rate vector is in the interior of $\mathcal{P}_t(\mathbf{x}_t)$. Therefore, these optimal room rates solve

$$\mathbf{p}_t^* := \underset{\mathbf{p}_t}{\operatorname{argmax}} \sum_{i=1}^{N} p_{t,i} \Bigg[i \times a\delta_t \times r(i) + b\delta_t \bar{G}_t(i, \mathbf{x}_t) \\ - \sum_{i=1}^{N} b\delta_t p_{t,i} \bar{g}_t(i) - b\delta_t \left(\sum_{j=1}^{i-1} p_{t,j} \bar{h}_t(i,j) + \sum_{j=i+1}^{N} p_{t,j} \bar{h}_t(j,i) \right) \Bigg].$$

The objective function is concave (by virtue of Theorem 1), $\mathbf{p}_t^*$ is determined by the following first order conditions of this function with respect to the room prices:

$$2p_i \bar{g}_t(i) + \sum_{j=1}^{i-1} p_j \bar{h}_t(i,j) + \sum_{j=i+1}^{N} p_j h_t(j,i) = i \times \frac{a}{b} \times r(i) + \bar{G}_t(i, \mathbf{x}_t), \forall i \in \{1, \ldots, N\}. \tag{45}$$

We now focus on the case of $N = 8$. To ease notation we suppress dependence on t, and define $I_{1,1} = 2\bar{g}(1) + \bar{h}(8,1)$, $I_{1,2} = \bar{h}(2,1) + \bar{h}(7,1)$, $I_{1,3} = \bar{h}(3,1) + \bar{h}(6,1)$, $I_{1,4} = \bar{h}(4,1) + \bar{h}(5,1)$, $I_{2,2} = 2\bar{g}(2) + \bar{h}(7,2)$, $I_{2,3} = \bar{h}(3,2) + \bar{h}(6,2)$, $I_{2,4} = \bar{h}(4,2) + \bar{h}(5,2)$, $I_{3,3} = 2\bar{g}(3) + \bar{h}(6,3)$, $I_{3,4} = \bar{h}(4,3) + \bar{h}(5,3)$, and $I_{1,1} = 2\bar{g}(4) + \bar{h}(5,4)$. Since we assume a reflective capacity vector, by

Proposition 4, $p_i = p_{r(i)}$ for $i = 1, \dots, \lfloor N/2 \rfloor$ in the solution of the first order equations. Further, by applying this condition and Lemma 1 (with $a\delta_t$ and $b\delta_t$ equal to 1) we obtain the following set of four equations that determine p_1, p_2, p_3, and p_4:

$$\begin{aligned}
I_{1,1}p_1 + I_{1,2}p_2 + I_{1,3}p_3 + I_{1,4}p_4 &= \frac{a}{b}N + \bar{G}_t(1, \mathbf{x}_t);\\
I_{1,2}p_1 + I_{2,2}p_2 + I_{2,3}p_3 + I_{2,4}p_4 &= 2\frac{a}{b}(N-1) + \bar{G}_t(2, \mathbf{x}_t);\\
I_{1,3}p_1 + I_{2,3}p_2 + I_{3,3}p_3 + I_{3,4}p_4 &= 3\frac{a}{b}(N-2) + \bar{G}_t(3, \mathbf{x}_t);\\
I_{1,4}p_1 + I_{2,4}p_2 + I_{3,4}p_3 + I_{4,4}p_4 &= 4\frac{a}{b}(N-3) + \bar{G}_t(4, \mathbf{x}_t).
\end{aligned}$$

Applying Gaussian elimination, the expression for $p_1, \dots, p_4$ can be obtained in closed form (and thus by Proposition 4 the remaining prices as well). Define

$$\begin{aligned}
D := {} & I_{1,4}^2 I_{2,3}^2 - 2I_{1,3}I_{1,4}I_{2,3}I_{2,4} + I_{1,3}^2 I_{2,4}^2 - I_{1,4}^2 I_{2,2}I_{3,3} + 2I_{1,2}I_{1,4}I_{2,4}I_{3,3} - I_{1,1}I_{2,4}^2 I_{3,3}\\
& + 2I_{1,3}I_{1,4}I_{2,2}I_{3,4} - 2I_{1,2}I_{1,4}I_{2,3}I_{3,4} - 2I_{1,2}I_{1,3}I_{2,4}I_{3,4} + 2I_{1,1}I_{2,3}I_{2,4}I_{3,4} + I_{1,2}^2 I_{3,4}^2\\
& - I_{1,1}I_{2,2}I_{3,4}^2 - I_{1,3}^2 I_{2,2}I(4,4) + 2I_{1,2}I_{1,3}I_{2,3}I(4,4) - I_{1,1}I_{2,3}^2 I(4,4) - I_{1,2}^2 I_{3,3}I(4,4)\\
& + I_{1,1}I_{2,2}I_{3,3}I(4,4);\\
C_{1,1} := {} & -(I_{2,4}^2 I_{3,3} - 2I_{2,3}I_{2,4}I_{3,4} + I_{2,2}I_{3,4}^2 + I_{2,3}^2 I_{4,4} - I_{2,2}I_{3,3}I_{4,4});\\
C_{1,2} := {} & -(-I_{1,4}I_{2,4}I_{3,3} + I_{1,4}I_{2,3}I_{3,4} + I_{1,3}I_{2,4}I_{3,4} - I_{1,2}I_{3,4}^2 - I_{1,3}I_{2,3}I_{4,4} + I_{1,2}I_{3,3}I_{4,4});\\
C_{1,3} := {} & -(I_{1,4}I_{2,3}I_{2,4} - I_{1,3}I_{2,4}^2 - I_{1,4}I_{2,2}I_{3,4} + I_{1,2}I_{2,4}I_{3,4} + I_{1,3}I_{2,2}I_{4,4} - I_{1,2}I_{2,3}I_{4,4});\\
C_{1,4} := {} & -(-I_{1,4}I_{2,3}^2 + I_{1,3}I_{2,3}I_{2,4} + I_{1,4}I_{2,2}I_{3,3} - I_{1,2}I_{2,4}I_{3,3} - I_{1,3}I_{2,2}I_{3,4} + I_{1,2}I_{2,3}I_{3,4});\\
C_{2,1} := {} & -(-I_{1,4}I_{2,4}I_{3,3} + I_{1,4}I_{2,3}I_{3,4} + I_{1,3}I_{2,4}I_{3,4} - I_{1,2}I_{3,4}^2 - I_{1,3}I_{2,3}I_{4,4} + I_{1,2}I_{3,3}I_{4,4});\\
C_{2,2} := {} & -(I_{1,4}^2 I_{3,3} - 2I_{1,3}I_{1,4}I_{3,4} + I_{1,1}I_{3,4}^2 + I_{1,3}^2 I_{4,4} - I_{1,1}I_{3,3}I_{4,4});\\
C_{2,3} := {} & -(-I_{1,4}^2 I_{2,3} + I_{1,3}I_{1,4}I_{2,4} + I_{1,2}I_{1,4}I_{3,4} - I_{1,1}I_{2,4}I_{3,4} - I_{1,2}I_{1,3}I_{4,4} + I_{1,1}I_{2,3}I_{4,4});\\
C_{2,4} := {} & -(I_{1,3}I_{1,4}I_{2,3} - I_{1,3}^2 I_{2,4} - I_{1,2}I_{1,4}I_{3,3} + I_{1,1}I_{2,4}I_{3,3} + I_{1,2}I_{1,3}I_{3,4} - I_{1,1}I_{2,3}I_{3,4});\\
C_{3,1} := {} & (-I_{1,4}I_{2,3}I_{2,4} + I_{1,3}I_{2,4}^2 + I_{1,4}I_{2,2}I_{3,4} - I_{1,2}I_{2,4}I_{3,4} - I_{1,3}I_{2,2}I_{4,4} + I_{1,2}I_{2,3}I_{4,4});\\
C_{3,2} := {} & (I_{1,4}^2 I_{2,3} - I_{1,3}I_{1,4}I_{2,4} - I_{1,2}I_{1,4}I_{3,4} + I_{1,1}I_{2,4}I_{3,4} + I_{1,2}I_{1,3}I_{4,4} - I_{1,1}I_{2,3}I_{4,4});\\
C_{3,3} := {} & (-I_{1,4}^2 I_{2,2} + 2I_{1,2}I_{1,4}I_{2,4} - I_{1,1}I_{2,4}^2 - I_{1,2}^2 I_{4,4} + I_{1,1}I_{2,2}I_{4,4});\\
C_{3,4} := {} & (I_{1,3}I_{1,4}I_{2,2} - I_{1,2}I_{1,4}I_{2,3} - I_{1,2}I_{1,3}I_{2,4} + I_{1,1}I_{2,3}I_{2,4} + I_{1,2}^2 I_{3,4} - I_{1,1}I_{2,2}I_{3,4});\\
C_{4,1} := {} & (I_{1,4}I_{2,3}^2 - I_{1,3}I_{2,3}I_{2,4} - I_{1,4}I_{2,2}I_{3,3} + I_{1,2}I_{2,4}I_{3,3} + I_{1,3}I_{2,2}I_{3,4} - I_{1,2}I_{2,3}I_{3,4});\\
C_{4,2} := {} & (-I_{1,3}I_{1,4}I_{2,3} + I_{1,3}^2 I_{2,4} + I_{1,2}I_{1,4}I_{3,3} - I_{1,1}I_{2,4}I_{3,3} - I_{1,2}I_{1,3}I_{3,4} + I_{1,1}I_{2,3}I_{3,4});\\
C_{4,3} := {} & (I_{1,3}I_{1,4}I_{2,2} - I_{1,2}I_{1,4}I_{2,3} - I_{1,2}I_{1,3}I_{2,4} + I_{1,1}I_{2,3}I_{2,4} + I_{1,2}^2 I_{3,4} - I_{1,1}I_{2,2}I_{3,4});\\
C_{4,4} := {} & (-I_{1,3}^2 I_{2,2} + 2I_{1,2}I_{1,3}I_{2,3} - I_{1,1}I_{2,3}^2 - I_{1,2}^2 I_{3,3} + I_{1,1}I_{2,2}I_{3,3}).
\end{aligned}$$

The closed form expression for the room price on day $i \in \mathcal{N}$ is

$$\begin{aligned}
p_i \equiv p_{r(i)} = {} & \frac{C_{i,1}}{D}\left(\frac{a}{b}N + \bar{G}_t(1, \mathbf{x}_t)\right) + \frac{C_{i,2}}{D}\left(2\frac{a}{b}(N-1) + \bar{G}_t(2, \mathbf{x}_t)\right)\\
& + \frac{C_{i,3}}{D}\left(3\frac{a}{b}(N-2) + \bar{G}_t(3, \mathbf{x}_t)\right) + \frac{C_{i,4}}{D}\left(4\frac{a}{b}(N-3) + \bar{G}_t(4, \mathbf{x}_t)\right). \qquad (46)
\end{aligned}$$

In the public holiday case, the class $(4,5)$ demand is αa, while the demand of the remaining classes remains at a. Accordingly inflating the base demand in the term corresponding to day four in equation (46) gives

$$\begin{aligned} p_i \equiv p_{r(i)} = &\frac{C_{i,1}}{D}\left(\frac{a}{b}N + \bar{G}_t(1, \mathbf{x}_t)\right) + \frac{C_{i,2}}{D}\left(2\frac{a}{b}(N-1) + \bar{G}_t(2, \mathbf{x}_t)\right) \\ &+ \frac{C_{i,3}}{D}\left(3\frac{a}{b}(N-2) + \bar{G}_t(3, \mathbf{x}_t)\right) + \frac{C_{i,4}}{D}\left(4\frac{a}{b}(N-3+(\alpha-1)) + \bar{G}_t(4, \mathbf{x}_t)\right). \end{aligned}$$

The derivative of price with respect to α is

$$\partial p_i/\partial\alpha \equiv \partial p_{r(i)}/\partial\alpha = \frac{a}{b} \times \frac{C_{i,4}}{D} \text{ for each } i = 1, \ldots, \lfloor N/2 \rfloor.$$

The ratio a/b does not affect either the sign or the ordering of derivatives $\partial p_i/\partial\alpha$, $i = 1, \ldots, \lfloor N/2 \rfloor$ because it is strictly positive and appears in each derivate. Therefore, it suffices to check $C_{1,4}/D$, $C_{2,4}/D$, $C_{3,4}/D$, and $C_{4,4}/D$, which evaluate to -0.0073, -0.0151, -0.0343, and 0.0837, respectively, for N equals to eight. The sign and ordering of these numbers confirms our claimed results for the public holiday setting.

For the conference setting, the high-demand class is $(3, 6)$ and equation (46) transforms to

$$\begin{aligned} p_i \equiv p_{r(i)} = &\frac{C_{i,1}}{D}\left(\frac{a}{b}N + \bar{G}_t(1, \mathbf{x}_t)\right) + \frac{C_{i,2}}{D}\left(2\frac{a}{b}(N-1) + \bar{G}_t(2, \mathbf{x}_t)\right) \\ &+ \frac{C_{i,3}}{D}\left(3\frac{a}{b}(N-2+(\alpha-1)) + \bar{G}_t(3, \mathbf{x}_t)\right) + \frac{C_{i,4}}{D}\left(4\frac{a}{b}(N-3+(\alpha-1)) + \bar{G}_t(4, \mathbf{x}_t)\right). \end{aligned}$$

The derivative of price with respect to α is

$$\partial p_i/\partial\alpha \equiv \partial p_{r(i)}/\partial\alpha = \frac{a}{b} \times \frac{C_{i,3} + C_{i,4}}{D} \text{ for each } i = 1, \ldots, \lfloor N/2 \rfloor.$$

Using the same arguments as in the public holiday case, we can verify the order/sign of the derivatives for the conference case by computing the ratios $\frac{C_{1,3} + C_{1,4}}{D}$, $\frac{C_{2,3} + C_{2,4}}{D}$, $\frac{C_{3,3} + C_{3,4}}{D}$, and $\frac{C_{4,3} + C_{4,4}}{D}$, which evaluate to -0.0201, -0.0447, 0.0758, and 0.0494, respectively. Hence, our claimed results hold. □