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Random maximal independent sets and the unfriendly theater seating arrangement problem

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ABSTRACT

People arrive one at a time to a theater consisting of m rows of length n. Being unfriendly they choose seats at random so that no one is in front of them, behind them or to either side. What is the expected number of people in the theater when it becomes full, i.e., it cannot accommodate any more unfriendly people? This is equivalent to the random process of generating a maximal independent set of an $m \times n$ grid by randomly choosing a node, removing it and its neighbors, and repeating until there are no nodes remaining. The case of $m = 1$ was posed by Freedman and Shepp [D. Freedman, L. Shepp, An unfriendly seating arrangement (problem 62-3), SIAM Rev. 4 (2) (1962) 150] and solved independently by Friedman, Rothman and MacKenzie [H.D. Friedman, D. Rothman, Solution to: An unfriendly seating arrangement (problem 62-3), SIAM Rev. 6 (2) (1964) 180–182; J.K. MacKenzie, Sequential filling of a line by intervals placed at random and its application to linear adsorption, J. Chem. Phys. 37 (4) (1962) 723–728] by proving the asymptotic limit $\frac{1}{2} - \frac{1}{2e^2}$. In this paper we solve the case $m = 2$ and prove the asymptotic limit $\frac{1}{2} - \frac{1}{4e}$. In addition, we consider the more general case of $m \times n$ grids, $m \geq 1$, and prove the existence of asymptotic limits in this general setting. We also make several conjectures based upon Monte Carlo simulations.

1. Introduction

Freedman and Shepp [11] posed the following "unfriendly seating arrangement" problem:

> There are n seats in a row at a luncheonette and people sit down one at a time at random. They are unfriendly and so never sit next to one another (no moving over). What is the expected number of persons to sit down?

This can be thought of as a special case of the following problem.

> Consider the following natural process for generating a maximal independent set of a graph. Randomly choose a node and place it in the independent set. Remove the node and all its neighbors from the graph. Repeat this process until no nodes remain. What is the expected size of the resulting maximal independent set?

The problem of Freedman and Shepp asks one to analyze this process for the case of a $1 \times n$ grid. Solutions to this problem were provided by Friedman, Rothman and MacKenzie [12,14] who show that as n tends to infinity, the expected fraction of the seats that are occupied goes to $\frac{1}{2} - \frac{1}{2e^2}$. (For a nice exposition on this and related problems see [10].)

In this paper we study the generalization of this problem to the $m \times n$ grid where $m > 0$ is fixed (see Fig. 1).

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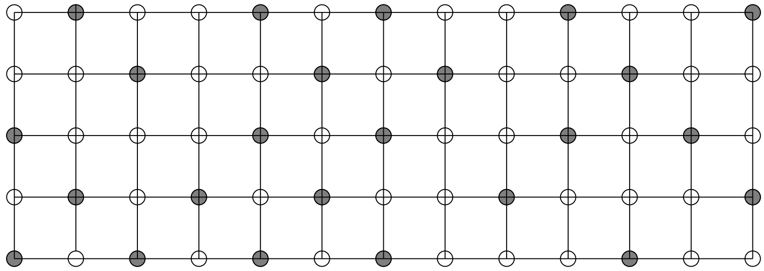

Fig. 1. A saturated seating arrangement on an $m \times n$ grid with mn seats (and $m = 5$, $n = 13$). Empty bullets represent unoccupied seats and gray bullets occupied seats.

In particular, we solve the following problem:

> There are n seats on either side of a long rectangular dining table at which people sit down at random, one at a time. They are unfriendly so that each person requires that their neighboring seats and the seat across from them is empty. What is the expected number of persons to sit down?

So, for the case of the $2 \times n$ grid, we show that the expected fraction of seats occupied goes to $\frac{1}{2} - \frac{1}{4e}$, as n tends to infinity. The limiting fraction is shown to exist for each m and we provide estimates on their values by way of Monte Carlo simulations. We refer to the general $m \times n$ case as the *Unfriendly Theater Seating Arrangement Problem* where people arrive at a movie theater with m rows of n seats each.

1.1. Related work

The original seating arrangement problem was generalized to the case where the number of seats left on either side of a new arrival must be at least b and solved by Rothman and MacKenzie [12]. (Clearly this can be thought of as the maximal independent set process on a $1 \times n$ grid where each node is connected to its b closest neighbors on either side.) They also discuss the relation of this problem to the well-known *Parking Problem* that was first studied by Renyi [16]: Given the closed interval $[0, x]$ with $x > 1$, let one-dimensional cars of unit length be parked (i.e., without overlap) randomly on the interval. What is the expected value of the number of cars as a function of x? Renyi shows that this is .748... asymptotically in x. A recent related paper on this topic is [5].

The number and size of random independent sets on grids (and other graphs) is of great interest in statistical physics. These studies consider the case of *hard* particles in lattices satisfying the exclusion rule that when a vertex of the lattice is occupied by a particle its neighbors must be vacant. Such *hard square* and *hard lattice* problems have been studied extensively both in physics and combinatorics [1–4,6,7,9,18,19]. Interestingly, we came to this problem by way of studying the number of saturated secondary structures of a random RNA sequence [13]. We note that in all of these studies the independent sets considered are *not* generated by the sequential process considered here and thus the results do not apply in our context.

1.2. Preliminary definitions

An $m \times n$ grid graph has vertex set $\{(i, j) \mid 1 \le i \le m, 1 \le j \le n\}$ and edge set $\{\{(i, j), (k, l)\} \mid |i - k| + |j - l| = 1\}$. An *independent set* is a subset of the vertices such that no two vertices are joined by an edge. An independent set is called *maximal* (or *saturated*) if no vertices can be added to it to form a larger independent set. For m fixed, define $F_{m,n}$ to be the expected size of a maximal independent set returned when the above process for generating a maximal independent set is applied to an $m \times n$ grid.

1.3. Results of the paper

In the sequel we study the asymptotic behavior of the expected size of a maximal independent set of an $m \times n$ grid. We prove in Section 2 that the double limit

$$\lim_{m,n\to\infty} \frac{F_{m,n}}{mn}$$

exists. We also prove various inequalities and identities concerning the relative sizes of the limits

$$f_m := \lim_{n\to\infty} \frac{F_{m,n}}{mn}$$

when m is fixed. For specific values of m, results of [12] show that f_1 is equal to $\frac{1}{2} - \frac{1}{2e^2}$. In Section 4 we show that f_2 is equal to $\frac{1}{2} - \frac{1}{4e}$. We finish by discussing the results of some Monte Carlo simulations for estimating f_m for small $m > 2$ along with some conjectures and open problems.

2. Asymptotics of the expected size of saturated configurations

In this section we prove the main result on the existence of asymptotic limits of the expected size of random saturated configurations. First we begin with some basic inequalities on saturated configurations, next we prove a basic lemma on weakly superadditive functions on the integers and conclude with proving the existence of the double limit $\lim_{m,n\to\infty}\frac{F_{m,n}}{mn}$.

2.1. Basic inequalities

Consider an undirected graph $G=(V,E)$ with V its set of vertices and E its set of edges.

Definition 1. A subset A of V is called *G-independent* (or *independent in G*) if for no two different vertices $u,v\in A$ is it true that $\{u,v\}\in E$.

Definition 2. For two subsets A,B of V we say that *A is G-independent of B* if the following conditions hold:
(1) $A\cap B=\emptyset$,
(2) for all $u\in A$ and $v\in B$ we have that $\{u,v\}\notin E$, i.e., no vertex of one set is adjacent with a vertex of the other.

If the graph G is easily understood from the context we will simply say that A is independent of B.

Let X_G be the random variable that counts the number of occupied seats of a saturated configuration in G, and let $X_A, X_B, X_{A\cup B}$ be the random variables that count the number of occupied seats of saturated configurations in the subgraphs induced on $A, B, A\cup B$, respectively.

Lemma 1. *Consider a graph G. If A is G-independent of B then*

$$E[X_G]\geq E[X_A]+E[X_B].$$

Moreover, if $A\cup B=V$ then we have equality $E[X_G]=E[X_A]+E[X_B]$.

Proof. It is clear that for every k, $\Pr[X_G\geq k]\geq\Pr[X_{A\cup B}\geq k]$, and therefore $E[X_G]\geq E[X_{A\cup B}]$. Additionally, if A is G-independent of B we have $X_{A\cup B}=X_A+X_B$, and hence $E[X_{A\cup B}]=E[X_A]+E[X_B]$. □

Lemma 2. *If H is an induced subgraph of G then*

$$E[X_H]\leq E[X_G],$$

where X_G, X_H are the random variables that count the number of occupied seats of saturated configurations in G and H, respectively,

Proof. Observe that if a set is independent in H it is also independent in G. Therefore for all k we have that

$$\Pr[X_H\geq k]\leq\Pr[X_G\geq k].$$

Hence, $E[X_H]\leq E[X_G]$. □

2.2. Weakly superadditive functions

Before proving the main limit theorem we will give the proof of a useful result which is an extension of a theorem due to [8] on superadditive functions. First we state the following definition (see also [17]).

Definition 3. A function f defined on the nonnegative integers is called superadditive if

$$f(s)+f(s')\leq f(s+s'),\quad\text{for all } s,s'. \tag{1}$$

An extension of this definition that will be useful in the sequel is the following.

Definition 4. A function f defined on the nonnegative integers is called weakly superadditive if there is an integer constant $c\geq 1$ such that

$$f(s)+f(s')\leq f(s+s'+c),\quad\text{for all } s,s'. \tag{2}$$

Lemma 3. *For any monotone function $g(n)\geq n$ and for any weakly superadditive function f for which $f(n)/g(n)$ is bounded, the limit*

$$\lim_{n\to\infty}\frac{f(n)}{g(n)}$$

exists.

Proof. Let f be a weakly superadditive function f satisfying (2) for some integer constant $c\geq 1$ and set $\limsup_{n\to\infty}\frac{f(n)}{g(n)}=a$. Then, there exists a strictly increasing sequence (a_k) such that $\lim_{k\to\infty}\frac{f(a_k)}{g(a_k)}=a$. For some fixed k, and for any n, we can find $0\leq s\leq g(a_k)+c-1$ such that $g(n)=s+i(g(a_k)+c)$. Using induction on i and (2) we see that

$$\begin{aligned} f(n) &= f(s+i(a_k+c)) \\ &= f(s+(i-1)(a_k+c)+a_k+c) \\ &\geq f(s+(i-1)(a_k+c))+f(a_k) \\ &\ \ \vdots \\ &\geq f(s)+if(a_k). \end{aligned}$$

Therefore

$$\frac{f(n)}{g(n)} \geq \frac{if(a_k)+f(s)}{g(n)} = \frac{f(a_k)}{g(a_k)+c}\left(1-\frac{s}{g(n)}\right)+\frac{f(s)}{g(n)}.$$

It follows that $\liminf_{n\to\infty} \frac{f(n)}{g(n)} \geq \frac{f(a_k)}{g(a_k)+c}$, which yields

$$\liminf_{n\to\infty} \frac{f(n)}{g(n)} \geq \lim_{k\to\infty} \frac{f(a_k)}{g(a_k)+c}. \tag{3}$$

Since the sequence $(g(a_k))$ is strictly increasing, we have $\lim_{k\to\infty} \frac{f(a_k)}{g(a_k)+c} = a$ and the lemma follows. $\square$

2.3. Existence of double limit

Next we concentrate on the proof of the existence of the double limit. First we prove a useful lemma.

Lemma 4. *Let t be an increasing integer valued function such that*

$$\max\{t(n), t(n')\} \leq t(n+n'+1), \quad \textit{for all } n, n' \geq 1. \tag{4}$$

Then the limit

$$\lim_{n\to\infty} \frac{F_{t(n),n}}{t(n)n}$$

exists.

Proof. Consider a $t(n+n'+1) \times (n+n'+1)$ grid and the following two subsets of vertices to the left and right of the $(n+1)$st column:

- A consists of the first $1, 2, \ldots, n$ columns (which form a $t(n+n'+1) \times n$ grid), and
- B consists of the last n' columns $n+2, n+3, \ldots, n+n'+1$ (which form a $t(n+n'+1) \times n'$ grid).

It is clear that A is independent of B in the $t(n+n'+1) \times (n+n'+1)$ grid. In view of the main property in (4) of the function t and Lemmas 1 and 2 we have

$$\begin{aligned} F_{t(n),n} + F_{t(n'),n'} &\leq F_{\max\{t(n),t(n')\},n} + F_{\max\{t(n),t(n')\},n'} \\ &\leq F_{t(n+n'+1),n} + F_{t(n+n'+1),n'} \\ &\leq F_{t(n+n'+1),n+n'+1}. \end{aligned}$$

Therefore the function $f(s) := F_{t(s),s}$ satisfies the hypothesis of Lemma 3. This implies that the limit $\lim_{n\to\infty} \frac{F_{t(n),n}}{t(n)n}$ exists and completes the proof of the Lemma. $\square$

For the time being we will use Lemma 4 to conclude that for various functions t satisfying (4) it makes sense to define the limit

$$f_t := \lim_{n\to\infty} \frac{F_{t(n),n}}{t(n)n}. \tag{5}$$

The following definition provides a useful notation.

Definition 5. We use the following notation for the limit defined by (5) when

(1) $t(n) = m$ is the constant function such that $t(n) = m$, for all n, and $m \geq 1$,

$$f_m := \lim_{n\to\infty} \frac{F_{m,n}}{mn}.$$

(2) if $t := id$ is the identity function such that $id(n) = n$, for all n,

$$f_{id} := \lim_{n\to\infty} \frac{F_{n,n}}{n^2}.$$

Now we can prove the main theorem.

Theorem 1. *The double limit exists and the following identities hold*

$$\lim_{m,n\to\infty}\frac{F_{m,n}}{mn}=\lim_{m\to\infty}f_m=f_{id}.$$

Proof. Before proving the existence of the double limit we prove the following inequalities for all integers $m\geq 1$,

Claim 1. *If id is the identity function then*

$$\frac{m}{m+1}\cdot f_m\leq f_{id},$$

Claim 2.

$$\frac{F_{m,m}}{m^2}\leq\frac{m+1}{m}\cdot f_m.$$

Let $m\geq 1$ be a given integer. In view of Lemma 4 the quantities f_m are well defined. Therefore without loss of generality we may assume throughout that m divides n.

First we prove Claim 1. Consider n/m rectangular grids each having dimensions $m\times(n+n/m-1)$ and separated from each other by $n/m-1$ many rows (i.e., $1\times(n+n/m-1)$ grids). There results a square with dimensions $(n+n/m-1)\times(n+n/m-1)$. Since the $m\times(n+n/m-1)$ grids above are independent of each other we can apply Lemma 1 in order to derive

$$\frac{n}{m}F_{m,n+n/m-1}\leq F_{n+n/m-1,n+n/m-1}.$$

If we divide both sides by $(n+n/m-1)^2$ we get

$$\begin{aligned}\frac{\frac{n}{m}F_{m,n+n/m-1}}{(n+n/m-1)^2}&=\frac{n}{n+n/m-1}\cdot\frac{F_{m,n+n/m-1}}{m(n+n/m-1)}\\&=\frac{1}{1+1/m-1/n}\cdot\frac{F_{m,n+n/m-1}}{m(n+n/m-1)}\\&\leq\frac{F_{n+n/m-1,n+n/m-1}}{(n+n/m-1)^2}\end{aligned}$$

and therefore the desired inequality in Claim 1 above follows by passing to the limit as $n\to\infty$ for m constant.

Next we prove Claim 2. Take n/m square grids each of size $m\times m$ and separated from one another by columns (i.e, $m\times 1$ grids). The resulting grid has dimensions

$$m\times\left(\frac{n}{m}\cdot m+\left(\frac{n}{m}-1\right)\right)=m\times\left(n+\frac{n}{m}-1\right).$$

Since the square grids above are independent of each other, Lemma 1 applies to show that

$$\frac{n}{m}F_{m,m}\leq F_{m,n+\frac{n}{m}-1}.$$

Therefore if we divide both sides by $m(n+n/m-1)$ we derive

$$\frac{\frac{n}{m}F_{m,m}}{m(n+\frac{n}{m}-1)}\leq\frac{F_{m,n+\frac{n}{m}-1}}{m(n+\frac{n}{m}-1)}.$$

Hence,

$$\frac{F_{m,m}}{m^2}\leq\frac{n+\frac{n}{m}-1}{n}\cdot\frac{F_{m,n+\frac{n}{m}-1}}{m(n+\frac{n}{m}-1)},$$

which implies the desired inequality by passing to the limit as $n\to\infty$.

It remains to prove the identities concerning the double limit. Indeed since $\lim_{m\to\infty}\frac{m+1}{m}=1$ we have that

$$\begin{aligned}f_{id}&=\lim_{m\to\infty}\frac{F_{m,m}}{m^2}\quad\text{(by definition)}\\&\leq\lim_{m\to\infty}f_m\quad\text{(by Claim 2)}\\&=\lim_{m,n\to\infty}\frac{F_{m,n}}{mn}\quad\text{(by definition)}\\&\leq f_{id}\quad\text{(by Claim 1)},\end{aligned}$$

as desired. This completes the proof of Theorem 1. □

The existence of the double limit can also be shown to imply that the square grid is the asymptotic limit of rectangular $m \times n$ grids in which m is a function of n. More precisely, Theorem 1 implies the following corollary which generalizes Lemma 4.

Corollary 1. *Let t be an increasing integer valued function such that $t(n) \to \infty$ as $n \to \infty$. Then the limit*

$$f_t = \lim_{n\to\infty} \frac{F_{t(n),n}}{t(n)n},$$

exists and is independent of t, namely $f_t = f_{id}$.

3. Inequalities on the asymptotic limits

In this section we prove several inequalities which compare the relative sizes of the asymptotic limits $\{f_m : m \geq 1\}$.

Theorem 2. *For any integers $m, m' \geq 0$ we have*

$$\frac{mf_m}{m+m'+1} + \frac{m'f_{m'}}{m+m'+1} \leq f_{m+m'+1}.$$

Proof. First consider the case where both $m, m' \geq 1$. Consider the $(m+m'+1) \times n$ grid and the following two subsets of vertices separated by the $(m'+1)$st row:

- A consists of the top m rows (which is an $m \times n$ grid), and
- B consists of the bottom m' rows (which is an $m' \times n$ grid).

Clearly, A is independent of B in the $(m+m'+1) \times n$ grid. In view of Lemma 1 we have

$$F_{m,n} + F_{m',n} \leq F_{m+m'+1,n}.$$

It follows that

$$\frac{F_{m,n}}{(m+m'+1)n} + \frac{F_{m',n}}{(m+m'+1)n} \leq \frac{F_{m+m'+1,n}}{(m+m'+1)n}.$$

Hence, passing to the limit as $n \to \infty$ we derive that for all $m, m' \geq 1$

$$\frac{mf_m}{m+m'+1} + \frac{m'f_{m'}}{m+m'+1} \leq f_{m+m'+1}.$$

A similar proof will work if either $m = 0$ or $m' = 0$. Details are left to the reader. This completes the proof of the theorem. □

As an immediate consequence of Theorem 2 we have the following inequality,

$$\frac{m}{m+1} \cdot f_m \leq f_{m+1},$$

for any integer $m \geq 1$, obtained from Theorem 3 when $m' = 0$. This in turn is improved in Theorem 3 by using a more careful analysis.

Theorem 3. *For all integers $m \geq 0$ we have that*

$$\frac{1}{4(m+1)} \leq f_{m+1} - \frac{m}{m+1} \cdot f_m \leq \frac{1}{2(m+1)}.$$

Proof. First consider the case $m \geq 1$. Consider the $(m+1) \times n$ grid G and the $m \times n$ grid H. Let X_G and X_H be the random variables that count the number of occupied seats in saturated configurations of G and H, respectively. We can assume that H is obtained from G by eliminating the top row of G. Since occupied seats cannot be adjacent, when eliminating the top row, at most $\lceil n/2 \rceil$ occupied seats are being removed. Therefore, for every k we have that

$$\Pr[X_G \geq k] \leq \Pr[X_H + \lceil n/2 \rceil \geq k]. \tag{6}$$

Similarly, it is easy to see that we can never have more than three consecutive unoccupied seats in the top row in any saturated configuration of G. This means that the top row contains at least $\lfloor n/4 \rfloor$ occupied seats. Therefore, for every k we have that

$$\Pr[X_H + \lfloor n/4 \rfloor \geq k] \leq \Pr[X_G \geq k]. \tag{7}$$

If we take the expected values of the corresponding random variables on both sides of (6) and (7) it follows that

$$\lfloor n/4 \rfloor + F_{m,n} \leq F_{m+1,n} \leq \lceil n/2 \rceil + F_{m,n},$$

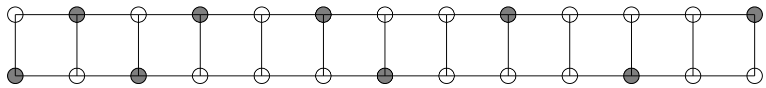

Fig. 2. A seating arrangement on a $2 \times n$ grid with $2n$ seats. Empty bullets represent unoccupied seats and gray bullets occupied seats.

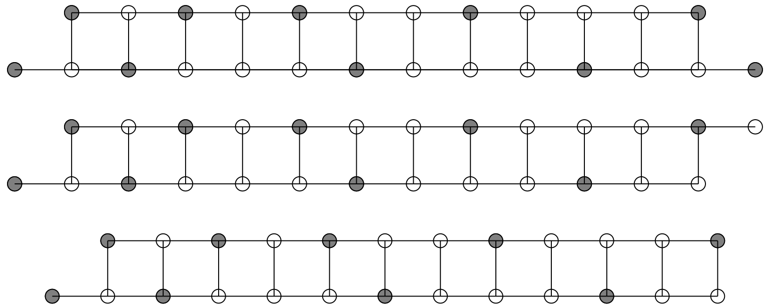

Fig. 3. A seating arrangement on a $2 \times (n+2)$ grid with $2n+2$ seats used in computing a_n(top), b_n(middle), respectively, and a seating arrangement on a $2 \times (n+1)$ grid with $2n+1$ seats used in computing c_n(bottom), for $n = 12$. Empty bullets represent unoccupied seats and gray bullets occupied seats.

which yields

$$\frac{\lfloor n/4 \rfloor}{(m+1)n} + \frac{F_{m,n}}{(m+1)n} \leq \frac{F_{m+1,n}}{(m+1)n} \leq \frac{\lceil n/2 \rceil}{(m+1)n} + \frac{F_{m,n}}{(m+1)n}.$$

Hence the result follows by passing to the limit as $n \to \infty$. A similar proof will work for the case $m = 0$. Details are left to the reader. $\square$

Observe that when m increases, the number of nodes of degree less than four, as a fraction of the total number of nodes of the grid, drops. Therefore f_m should be a non-increasing function of m. Motivated by the result of Theorem 3 we state the following conjecture.

Conjecture 1. *$f_m > f_{m+1}$, for all $m \geq 1$.*

Table 1 in Section 5 reports the results of Monte Carlo simulations which seem to confirm the conjecture on the monotonicity of the sequence $\{f_m : m \geq 1\}$.

4. The 2 × *n* Grid

In this section, we are interested in the number of occupied seats when unfriendly people arrive to sit at a long rectangular table with n chairs on each side. See Fig. 2.

We prove the following theorem:

Theorem 4.

$$f_2 = \lim_{n\to\infty} \frac{F_{2,n}}{2n} = \frac{1}{2} - \frac{1}{4e}.$$

The proof of the theorem will follow after first proving Lemmas 5 and 6 below.

Towards a proof of Theorem 4 we define three quantities a_n, b_n, c_n. For $n > 0$, let a_n be the expected number of occupied seats on a $2 \times (n+2)$ grid with the nodes $(1, 1)$ and $(1, n+2)$ missing. Let b_n be the expected number of occupied seats on a $2 \times (n+2)$ grid with the nodes $(1, 1)$ and $(2, n+2)$ missing. Let c_n be the expected number of occupied seats on $2 \times (n+1)$ grid with node $(1, 1)$ missing. Note that by symmetry a_n is also the expected number of occupied seats on a $2 \times (n+2)$ grid with the nodes $(2, 1)$ and $(2, n+2)$ missing, b_n is also the expected number on a grid with $(2, 1)$ and $(1, n+2)$ missing, and c_n is also the expected number on a grid with $(2, 1)$, $(1, n+1)$, $(2, n+1)$ missing, respectively. Fig. 3 shows examples of the structures counted by a_n, b_n, c_n for the case $n = 12$.

We first show the following lemma:

Lemma 5.

$$\lim_{n\to\infty} \frac{a_n}{2n+2} = \lim_{n\to\infty} \frac{b_n}{2n+2} = \frac{1}{2} - \frac{1}{4e}.$$

Proof. By elementary case analysis it is easy to derive the following recurrences for a_n and b_n for $n > 5$:

$$a_n = 1 + \frac{1}{2n+2}\left(\sum_{k=1}^{n-4}(b_k + b_{n-k-3}) + 2(b_{n-1} + b_{n-2} + b_{n-3}) + 4 + \sum_{k=1}^{n-4}(a_k + a_{n-k-3}) + 2(a_{n-2} + a_{n-3}) + 4\right)$$
$$b_n = 1 + \frac{1}{2n+2}\left(2\sum_{k=1}^{n-4}(a_{n-k-3} + b_k) + 2(a_{n-1} + a_{n-2} + a_{n-3}) + 2(b_{n-3} + b_{n-2}) + 4\right).$$

with initial conditions for (a_n, b_n), for $n \le 5$, computed directly from the definition as follows:

$$(a_n, b_n) = \begin{cases} (5/2, 2) & \text{for } n = 1 \\ (3, 19/6) & \text{for } n = 2 \\ (47/12, 31/8) & \text{for } n = 3 \\ (113/24, 283/60) & \text{for } n = 4 \\ (3981/720, 3980/720) & \text{for } n = 5. \end{cases} \tag{8}$$

From this, using elementary calculations, we derive the recurrences (for $n > 5$):

$$(n+1)a_n = 1 + na_{n-1} + a_{n-2} + b_{n-1} \tag{9}$$
$$(n+1)b_n = 1 + nb_{n-1} + b_{n-2} + a_{n-1}. \tag{10}$$

Letting $u_n := a_{n+4} + b_{n+4}$ we get

$$(n+5)u_n = 2 + (n+5)u_{n-1} + u_{n-2}, \tag{11}$$

for $n > 1$ where $u_0 = 1131/120$ and $u_1 = 7961/720$.

Next we proceed to find an asymptotic formula for u_n using MAPLE [15]. After some simplifications we get

$$u_n = 2n + 12 - \frac{\Gamma(n+8, -1)}{e(n+6)!},$$

where $\Gamma(n, -1) := \int_{-1}^{\infty} \exp(-t)t^{n-1}\mathrm{d}t$ is the incomplete gamma function. From this it is easily seen that

$$\lim_{n\to\infty} \frac{a_n + b_n}{4n+4} = \frac{1}{2} - \frac{1}{4e}. \tag{12}$$

Define $d_n := a_n - b_n$. Using (9) and (10) we get that

$$d_n = \frac{n-1}{n+1}d_{n-1} + \frac{d_{n-2}}{n+1}.$$

The last recursion easily gives rise to the following explicit formula

$$d_n = \frac{(-1)^{n+1}}{(n+1)!}$$

which implies trivially $\lim_{n\to\infty} \frac{a_n - b_n}{4n+4} = 0$. This completes the proof of Lemma 5. □

Let A_n be the $2 \times (n+2)$ grid with the nodes $(1, 1)$ and $(1, n+2)$ missing (see top grid depicted in Fig. 3). We define by X_{A_n} the random variable that counts the number of occupied seats in a saturated seating arrangement of A_n. We define similarly the grids B_n, C_n for the middle and bottom grids depicted in Fig. 3, respectively, and the associated random variables X_{B_n}, X_{C_n}. In addition, let X_n be the random variable that counts the number of occupied seats in a saturated seating arrangement on a $2 \times n$ grid. Observe that by definition $F_{2,n} = E[X_n]$, $a_n = E[X_{A_n}]$, $b_n = E[X_{B_n}]$, $c_n = E[X_{C_n}]$. We now prove the following lemma.

Lemma 6.

$$\lim_{n\to\infty} \frac{F_{2,n}}{2n} = \lim_{n\to\infty} \frac{a_n}{2n+2} = \lim_{n\to\infty} \frac{b_n}{2n+2} = \lim_{n\to\infty} \frac{c_n}{2n+1}.$$

Proof. We prove only $\lim_{n\to\infty} \frac{F_{2,n}}{2n} = \lim_{n\to\infty} \frac{a_n}{2n+2}$. The other identities are proved similarly. Observe that A_n is an induced subgraph of the $2 \times (n+2)$ grid. Therefore by Lemma 2 we have that $E[X_{A_n}] \le E[X_{n+2}]$. Passing to the limit it follows that

$$\lim_{n\to\infty} \frac{a_n}{2n+2} \le \lim_{n\to\infty} \frac{F_{2,n+2}}{2(n+2)} = \lim_{n\to\infty} \frac{F_{2,n}}{2n}.$$

Similarly, the $2 \times n$ grid is an induced subgraph of A_n. Therefore by Lemma 2 we have that $E[X_n] \le E[X_{A_n}]$. Passing to the limit we see that

$$\lim_{n\to\infty} \frac{F_{2,n}}{2n} \le \lim_{n\to\infty} \frac{a_n}{2n+2}.$$

This completes the proof of Lemma 6. □

The proof of Theorem 4 is now an immediate consequence of Lemmas 5 and 6. As a corollary of the proof of Theorem 4 we also derive the following result.

Corollary 2.

$$\lim_{n\to\infty} (F_{2,n+1} - F_{2,n}) = 2f_2.$$

Proof. For simplicity, we use the notation $x_n := F_{2,n}$. By elementary case analysis it is easy to derive the following recurrences, for $n \ge 2$, involving x_n, a_n, b_n, c_n, in addition to the recurrences derived at the beginning of the proof of Lemma 5 (see Fig. 3).

$$x_n = 1 + \frac{1}{n}\left(c_{n-2} + \sum_{k=2}^{n-1}(c_{k-2} + c_{n-k-1}) + c_{n-2}\right).$$

From these identities, using elementary calculations, we derive the following recurrence (for $n \ge 2$):

$$(n+1)x_{n+1} = 1 + nx_n + 2c_{n-1}. \tag{13}$$

Collecting terms in (13) we see that

$$x_{n+1} - x_n = \frac{1}{n+1} - \frac{1}{n+1}x_n + \frac{2}{n+1}c_{n-1}. \tag{14}$$

By Lemma 6, $\lim_{n\to\infty} \frac{c_n}{2n+1}$ exists and is equal to f_2. Using (14) and passing to the limit as $n \to \infty$ we see that

$$\lim_{n\to\infty} (F_{2,n+1} - F_{2,n}) = 2f_2,$$

as desired. □

5. Experimental results and open problems

We do not know how to calculate the exact value of f_m for $m > 2$. See Table 1 for the approximate predicted values of f_m for $m \le 15$ based upon extensive Monte Carlo simulations.

Table 1
Experimental values for the asymptotic limit of the expected number of occupied seats on $m \times n$ grids. The results are derived from the average of 100 experiments on an $m \times 10,000$ grid.

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
f_m	0.4323	0.4078	0.3915	0.3847	0.3807	0.3778	0.3759	0.3744	0.3733	0.3724	0.3716	0.3710	0.3705	0.3700	0.3696

If we let $m = n$ we get the interesting case of the expected size of a maximal independent set generated by the above sequential process on a square grid. Although we do not know of a way to compute analytically $f_{id} = \lim_{n\to\infty} \frac{F_{n,n}}{n^2}$, Monte Carlo simulations (the average of 100 trials on a 400×400 grid) suggest that f_{id} is approximately equal to.3645.

Other interesting open problems arise when considering the above seating arrangement process on other graphs or graph families. It is easy to show that the case of the n node cycle yields the same asymptotic result as the $1 \times n$ grid. It is also not difficult to derive that if we add links to the $2 \times n$ grid so as to form rings on the rows (i.e., a $2 \times n$ torus) the results again do not change. In addition, for the case of the $3 \times n$ torus it is easy to show that the limit is $1/3$ since every triangle must contain exactly one occupied seat. However, in general we do not know what happens for the case of the $m \times n$ torus, for $m \ge 4$. Some other simple cases to analyze are cliques, stars and complete bipartite graphs, but beyond these all questions appear to be open.

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References

[1] R.J. Baxter, Planar lattice gases with nearest-neighbour exclusion, Ann. Comb. 3 (1999) 191–203.
[2] J. Bouttier, P.D. Francesco, E. Guitte, Critical and tricritical hard objects on bicolorable random lattices: Exact solutions, J. Phys. A 35 (2002) 3821–3854. Also available at: arXiv:cond-mat/0201213.
[3] J. Bouttier, P.D. Francesco, E. Guitte, Combinatorics of hard particles on planar graphs, J. Phys. A 38 (2005) 4529–4559. Also available at: arXiv:math/0501344v2.
[4] N.J. Calkin, H.S. Wilf, The number of independent sets in a grid graph, SIAM J. Discret. Math. 11 (1) (1998) 54–60.
[5] E.G. Coffman, C.L. Mallows, B. Poonen, Parking arcs on the circle with applications to one-dimensional communication networks, Ann. Appl. Probab. 4 (4) (1994) 1098–1111.
[6] K. Engel, On the Fibonacci number of an $m \times n$ lattice, Fibonacci Quart. (1990) 72–78.
[7] R. Euler, The Fibonacci number of a grid graph and a new class of integer sequences, J. Integer Seq. 8 (2005) Article 05.2.6.
[8] M. Fekete, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, Math. Z. 17 (1923) 228–249.
[9] S.R. Finch, Several constants arising in statistical mechanics, Ann. Comb. (1999) 323–335.
[10] P. Flajolet, A seating arrangement problem, (Accessed in 2007) (January 15, 2007). URL: http://algo.inria.fr/libraries/autocomb/fatmen-html/fatmen1.html.
[11] D. Freedman, L. Shepp, An unfriendly seating arrangement (problem 62-3), SIAM Rev. 4 (2) (1962) 150.
[12] H.D. Friedman, D. Rothman, Solution to: An unfriendly seating arrangement (problem 62-3), SIAM Rev. 6 (2) (1964) 180–182.
[13] W. Lorenz, Y. Ponty, P. Clote, Asymptotics of RNA shapes, J. Comput. Biol. 15 (1) (2008) 31–63.
[14] J.K. MacKenzie, Sequential filling of a line by intervals placed at random and its application to linear adsorption, J. Chem. Phys. 37 (4) (1962) 723–728.
[15] M.B. Monagan, K.O. Geddes, K.M. Heal, G. Labahn, S.M. Vorkoetter, J. McCarron, P. DeMarco, Maple 10 Programming Guide, Maplesoft, Waterloo ON, Canada, 2005.
[16] A. Rényi, On a one-dimensional problem concerning random space-filling, Publ. Math. Inst. Hung. Acad. Sci. 3 (1958) 109–127.
[17] J.M. Steele, Probability Theory and Combinatorial Optimization, in: CBMS-NSF Regional Conference Series in Applied Mathematics, 69, SIAM, 1997.
[18] F.H. Stillinger, H. Sakai, S. Torquato, Critical and tricritical hard objects on bicolorable random lattices: Exact solutions, Phys. Rev. E 67 (2003) 031107 1–10.
[19] S. Torquato, A. Donev, F. Stillinger, Breakdown of elasticity theory for jammed hard-particle packings: Conical nonlinear constitutive theory, Internat. J. Solids Struct. 40 (2003) 7143–7153.