Math. Program., Ser. A 104, 21–37 (2005)

Digital Object Identifier (DOI) 10.1007/s10107-004-0567-y

Carlo Filippi · Alessandro Agnetis

# An asymptotically exact algorithm for the high-multiplicity bin packing problem

Received: March 17, 2004 / Accepted: November 28, 2004
Published online: February 9, 2005 – 

**Abstract.** The bin packing problem consists of finding the minimum number of bins, of given capacity $D$, required to pack a set of objects, each having a certain weight. We consider the high-multiplicity version of the problem, in which there are only $C$ different weight values. We show that when $C = 2$ the problem can be solved in time $O(\log D)$. For the general case, we give an algorithm which provides a solution requiring at most $C - 2$ bins more than the optimal solution, i.e., an algorithm that is asymptotically exact. For fixed $C$, the complexity of the algorithm is $O(\text{poly}(\log D))$, where $\text{poly}(\cdot)$ is a polynomial function not depending on $C$.

## 1. Introduction

The bin packing problem (denoted BP) is defined as follows.

BP Given $n$ objects with integer weights $p_1, p_2, \ldots, p_n$ and an integer bin capacity $D$, what is the minimum number of bins needed to store all the $n$ objects?

We consider here the high-multiplicity version of the bin packing problem, i.e., the case in which only a (small) fixed number $C$ of different weights is allowed (the high-multiplicity concept was first introduced by Hochbaum and Shamir [7]):

BP$C$ Given $q_j$ objects with integer weight $p_j$ for all $j = 1, \ldots, C$, and an integer bin capacity $D$, what is the minimum number of bins needed to store all the $n \equiv \sum_{j=1}^{C} q_j$ objects?

We denote the corresponding feasibility problem as FBP$C$:

FBP$C$ Given $q_j$ objects with integer weight $p_j$ for all $j = 1, \ldots, C$, is there a feasible assignment of all the $n \equiv \sum_{j=1}^{C} q_j$ objects to $m$ bins of capacity $D$ each?

Observe that an instance of BP$C$ is described by $2C + 1$ positive integers only. More precisely, if $p$ denotes the $C$-vector of positive integer weights, $q$ the $C$-vector of the corresponding numbers of objects, and $D$ the integer capacity, an instance of BP$C$ is described as $(p, q, D)$. We assume that all values $p_j$ are different and that they are

C. Filippi: Department of Pure and Applied Mathematics, University of Padova, Via Belzoni 7, 35131 Padova, Italy. e-mail: carlo@math.unipd.it

A. Agnetis: Department of Information Engineering, University of Siena, Via Roma 56, 53100 Siena, Italy. e-mail: agnetis@dii.unisi.it

*Mathematics Subject Classification (2000): 20E28, 20G40, 20C20*

all strictly less than the bin capacity $D$. The input size of a BP$C$ instance is therefore $O(\log D + \log q_{\max})$, where $q_{\max} \equiv \max_j\{q_j\}$. A *feasible bin pattern* is an ordered $C$-tuple of integers $x = (x_1, x_2, \ldots, x_C)$ such that $x \geq 0$ and $p^T x \leq D$. It means that a bin can be packed by $x_j$ objects of weight $p_j$, $j = 1, \ldots, C$.

Note that problem BP$C$ can be formulated as a one-dimensional cutting stock problem with a fixed number $C$ of different object sizes to be cut.

Complexity results for FBP$C$ are related to those for BP$C$ via binary search. FBP$C$ was first addressed by Leung [10], who provided a pseudopolynomial algorithm for any $C$. The state of the art on FBP$C$ is represented by a paper by McCormick, Smallwood and Spieksma [12], who devised an $O((\log D)^2)$ algorithm, hence polynomial, for FBP2. They show that an instance of FBP2 is feasible if and only if there exist three feasible bin patterns describing a triangle in $\mathbb{R}^2$ such that: $(i)$ the triangle has area 1/2, and $(ii)$ the triangle contains the point $(q_1/m, q_2/m)$. As a consequence, any feasible FBP2 instance has a solution using at most three different bin patterns.

They also prove that, given an instance of FBP$C$, if there exist $C$ feasible bin patterns describing a simplex with volume $1/C!$, containing the point $(q_1/m, q_2/m, \ldots, q_C/m)$, then the instance is feasible. However, the converse may not be true and it does not seem that this result can be exploited to devise a polynomial algorithm for FBP$C$, which is not known in the literature. Like the approach in [12], ours starts from the polyhedral representation of the set of feasible bin patterns, but employs different notions and techniques.

The contribution of this paper is an algorithm for BP$C$ producing a solution that requires at most $C - 2$ bins more than the optimal solution. In particular, for $C = 2$ the algorithm is exact and has a time bound of $O(\log D)$. Applying the approach to the general BP$C$ problem, we get an algorithm which is asymptotically exact (as the total number of objects grows, for a fixed $C$), and whose complexity is $O(\text{poly}(\log D))$, where $\text{poly}(\cdot)$ is a polynomial function not depending on $C$, arising from the application of the ellipsoid method.

Our approach can be outlined as follows. We first characterize a class of special instances of BP$C$. These are instances in which there exist $C$ distinct feasible bin patterns that *completely* fill a bin, and such that the vector $q$ is in the cone generated by the bin patterns in $\mathbb{R}^C$. For these instances we can devise an approximate solution in constant time. Then, we show how to associate to any instance of BP$C$ a suitable special instance in polynomial time, preserving the approximation result.

In accordance with the literature, the time analysis is presented in terms of elementary arithmetic operations.

The structure of the paper is as follows. In Section 2, we introduce some further notation that will be used throughout the paper. Section 3 describes an exact algorithm for the case $C = 2$. In Section 4, we describe an approximate algorithm for any $C$. Conclusions follow in Section 5.

## 2. Preliminaries

Given an instance $(p, q, D)$ of BP$C$, a *feasible solution* is a set

$$\sigma \equiv \{(x^1, b_1), \ldots, (x^k, b_k)\}, \tag{1}$$

where $x^1, \dots, x^k$ are feasible bin patterns and $b_1, \dots, b_k$ are positive integers representing the number of bins filled with bin pattern $x^i$ ($i = 1, 2, \dots, k$). The pairs $(x^i, b_i)$ must be chosen so that the total number of packed objects with weight $p_j$ is exactly $q_j$, for all $j = 1, 2, \dots, C$. Let $b(\sigma) \equiv \sum_{i=1}^{k} b_i$ be the number of bins used by solution $\sigma$. An optimal solution $\sigma^\star$ to BP$C$ minimizes $b(\sigma)$. We denote by $b^\star$ such a minimum value.

Given a rational number $\alpha$, we denote by $\lfloor \alpha \rfloor$ the largest integer not greater than $\alpha$, and by $\lceil \alpha \rceil$ the smallest integer not less than $\alpha$. The following simple observation will be crucial in our development.

*Remark 1.* Given an instance of BP$C$, let $b^\star$ be its optimal value, and define $\overline{b} \equiv (p^T q)/D$. Then, $b^\star \geq \lceil \overline{b} \rceil$.

Note that the above remark implies that $b^\star$ is not polynomially bounded in the input size. Hence an optimal solution $\sigma^\star$ might contain a non-polynomial number of distinct bin patterns. To the best of our knowledge, when $C \geq 3$, whether or not an optimal solution to BP$C$ containing a polynomial number of distinct bin patterns always exists (i.e., whether or not FBP$C$ is in NP) is a remarkable open problem.

Given two feasible bin patterns $x^1$ and $x^2$, we say that $x^1$ is *dominated* by $x^2$ if $x^1 \leq x^2$.

## 3. Two different weights

We start by addressing the case $C = 2$. The motivation for doing so is twofold. First, some arguments that will be used for the general case are more intuitive and easier to understand in dimension two. Second, for this particular case we are able to obtain a deterministic exact algorithm with a better time bound than the one in the literature.

### 3.1. Two key lemmas

An instance of BP2 may be explicitly written as $(p_1, p_2, q_1, q_2, D)$, thus it is specified by only five integers. Consider the lines $L \equiv \{x \in \mathbb{R}^2 : p_1 x_1 + p_2 x_2 = D\}$ and $Q \equiv \{x \in \mathbb{R}^2 : x_1/q_1 = x_2/q_2\}$ on the Euclidean plane. They intersect in a unique point $(\psi_1, \psi_2)$, which we refer to as the *target point* of instance $(p_1, p_2, q_1, q_2, D)$ (see Figure 1). It is easy to see that

$$(\psi_1, \psi_2) = (q_1/\overline{b}, q_2/\overline{b}).$$

We say that a PB2 instance $(p_1, p_2, q_1, q_2, D)$ is *special* if there exist two feasible bin patterns $(y_1, y_2)$ and $(z_1, z_2)$ such that: $(i)$ $p_1 y_1 + p_2 y_2 = p_1 z_1 + p_2 z_2 = D$; $(ii)$ the target point $(\psi_1, \psi_2)$ lies on the line segment joining $(y_1, y_2)$ and $(z_1, z_2)$. We refer to $(y_1, y_2)$ and $(z_1, z_2)$ as the *spanning patterns* associated with $(p_1, p_2, q_1, q_2, D)$.

The following two lemmas show that in special BP2 instances the optimal solution can be easily characterized. These results are crucial to efficiently solve *any* instance of BP2, as shown in Section 3.2.

**Lemma 1.** *If the target point $(\psi_1, \psi_2)$ is integer then $b^\star = \lceil \overline{b} \rceil$ and there exists an optimal solution of BP2 using only bin patterns that are dominated by $(\psi_1, \psi_2)$.*

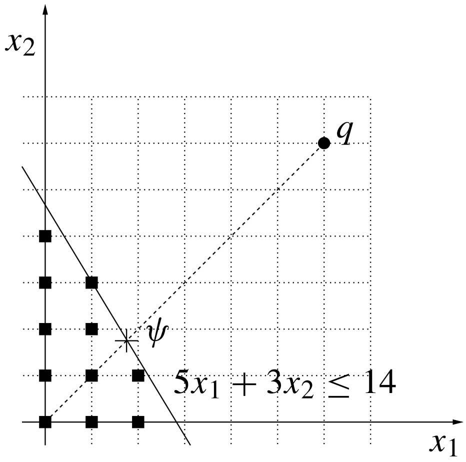


**Fig. 1.** A graphical representation of the BP2 instance (5, 3, 6, 6, 14), where black squares denote the feasible bin patterns.

*Proof.* We fill $\lfloor \overline{b} \rfloor$ bins by the feasible bin pattern $(\psi_1, \psi_2)$. If $\overline{b}$ is integer then we are done, else $0 < \overline{b} - \lfloor \overline{b} \rfloor < 1$ and we fill a further bin by the feasible bin pattern

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \equiv (\overline{b} - \lfloor \overline{b} \rfloor) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} - \lfloor \overline{b} \rfloor \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

In this way, we pack all $q_1 + q_2$ objects by using $\lceil \overline{b} \rceil$ bins. As a consequence of Remark 1, the obtained solution is optimal. Furthermore, all bin patterns used are convex combinations of $(0, 0)$ and $(\psi_1, \psi_2)$. □

**Lemma 2.** *If* $(p_1, p_2, q_1, q_2, D)$ *is a special PB2 instance with associated spanning patterns* $(y_1, y_2)$ *and* $(z_1, z_2)$*, then* $b^\star = \lceil \overline{b} \rceil$ *and there exists an optimal solution of BP2 using only feasible bin patterns that are convex combinations of* $(0, 0)$*,* $(y_1, y_2)$*, and* $(z_1, z_2)$*, or are dominated by either* $(y_1, y_2)$ *or* $(z_1, z_2)$*.*

*Proof.* By the hypothesis, the system

$$\begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \end{bmatrix} \cdot \begin{pmatrix} \lambda \\ \nu \end{pmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

has the unique solution

$$\lambda^\star = \frac{q_2 z_1 - q_1 z_2}{z_1 y_2 - y_1 z_2}, \quad \nu^\star = \frac{q_1 y_2 - q_2 y_1}{z_1 y_2 - y_1 z_2},$$

which is nonnegative. Indeed, since $(\psi_1, \psi_2)$ lies on the line segment joining $(y_1, y_2)$ and $(z_1, z_2)$, we have $y_1 < q_1/\overline{b} < z_1$ and $z_2 < q_2/\overline{b} < y_2$, that imply $q_2 z_1 - q_1 z_2 > 0$, $q_1 y_2 - q_2 y_1 > 0$, and $z_1 y_2 - y_1 z_2 > 0$.

We have

$$\lambda^\star + \nu^\star = \frac{\lambda^\star(p_1y_1 + p_2y_2) + \nu^\star(p_1z_1 + p_2z_2)}{D}$$
$$= \frac{p_1(\lambda^\star y_1 + \nu^\star z_1) + p_2(\lambda^\star y_2 + \nu^\star z_2)}{D}$$
$$= \frac{p_1q_1 + p_2q_2}{D}$$
$$= \overline{b}. \tag{2}$$

For any scalar $\alpha$, let $\phi(\alpha) \equiv \alpha - \lfloor\alpha\rfloor$. Define

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \equiv \phi(\lambda^\star)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \phi(\nu^\star)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$
$$= \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} - \lfloor\lambda^\star\rfloor\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \lfloor\nu^\star\rfloor\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

We distinguish three cases.

*Case 1.* $\phi(\lambda^\star) + \phi(\nu^\star) = 0$. Necessarily, $w_1 = w_2 = 0$, while $\lambda^\star$, $\nu^\star$, and $\overline{b}$ are all integers. We get the solution

$$\sigma' \equiv \{(y, \lambda^\star), (z, \nu^\star)\}$$

where $b(\sigma') = \lambda^\star + \nu^\star = \overline{b} = \lceil\overline{b}\rceil$.

*Case 2.* $0 < \phi(\lambda^\star) + \phi(\nu^\star) \leq 1$. The integer point $(w_1, w_2)$ is a convex combination of $(0, 0)$, $(y_1, y_2)$ and $(z_1, z_2)$, and thus it is a feasible bin pattern. We get the solution

$$\sigma'' \equiv \{(y, \lfloor\lambda^\star\rfloor), (z, \lfloor\nu^\star\rfloor), (w, 1)\}.$$

If $\phi(\lambda^\star)+\phi(\nu^\star) < 1$ then $\overline{b}$ is not integer and $b(\sigma'') = \lfloor\lambda^\star\rfloor+\lfloor\nu^\star\rfloor+1 = \lfloor\overline{b}\rfloor+1 = \lceil\overline{b}\rceil$. Otherwise, $\overline{b}$ is integer and $b(\sigma'') = \lfloor\lambda^\star\rfloor + \lfloor\nu^\star\rfloor + 1 = \overline{b} = \lceil\overline{b}\rceil$.

*Case 3.* $\phi(\lambda^\star) + \phi(\nu^\star) > 1$. If $w \geq y$ (resp. $w \geq z$) then set $w^1 \equiv y$ (resp. $w^1 \equiv z$). Further, set $w^2 \equiv w - w^1$. Hence,

$$p_1w_1^1 + p_2w_2^1 = D$$

and

$$p_1w_1^2 + p_2w_2^2 = (p_1w_1 + p_2w_2) - (p_1w_1^1 + p_2w_2^1) < D,$$

where the inequality follows from the fact that by definition $w < y + z$ and thus $p_1w_1 + p_2w_2 < 2D$.

Otherwise, we have necessarily $y_1 < w_1 < z_1$ and $z_2 < w_2 < y_2$. Set $w_1^1 \equiv w_1$ and $w_2^1 \equiv 0$. Further, set $w^2 \equiv w - w^1$. Hence, by construction

$$p_1w_1^1 + p_2w_2^1 = p_1w_1^1 < p_1z_1 \leq D,$$

and analogously,

$$p_1w_1^2 + p_2w_2^2 = p_2w_2^2 < p_2y_2 \leq D.$$

In both cases, $w^1$ and $w^2$ are feasible bin patterns, and we get the solution

$$\sigma''' \equiv \{(y, \lfloor\lambda^\star\rfloor), (z, \lfloor\nu^\star\rfloor), (w^1, 1), (w^2, 1)\},$$

where $b(\sigma''') = \lfloor\lambda^\star\rfloor + \lfloor\nu^\star\rfloor + 2 = \lambda^\star + \nu^\star + 2 - (\phi(\lambda^\star) + \phi(\nu^\star)) < \overline{b} + 1$. Since $1 < \phi(\lambda^\star) + \phi(\nu^\star) < 2$, in this case $\overline{b}$ is not integer and therefore $b(\sigma''') \le \lceil\overline{b}\rceil$. As a consequence of Remark 1, $b(\sigma''') = \lceil\overline{b}\rceil$. □

In view of the above lemmas, an instance of BP2 can be straightforwardly solved provided we are given two nonnegative solutions $y$ and $z$ of the diophantine equation

$$p_1x_1 + p_2x_2 = D \tag{3}$$

such that

$$y_1 \le \psi_1 < z_1 \text{ and } z_2 < \psi_2 \le y_2. \tag{4}$$

The general solution of equation (3) is (cf. [9])

$$(D/\omega)\begin{bmatrix}\alpha\\ \beta\end{bmatrix} + k\begin{bmatrix}\gamma\\ \delta\end{bmatrix} \qquad \text{(for all } k \in \mathbb{Z}), \tag{5}$$

where $\omega$ is the greatest common divisor of $p_1$ and $p_2$, and $\alpha, \beta, \gamma, \delta$ are integers such that

$$p_1\alpha + p_2\beta = \omega$$
$$p_1\gamma + p_2\delta = 0.$$

Using an algorithm by Kertzner [9], the five integers $\omega, \alpha, \beta, \gamma, \delta$ can be found in $O(\log p_1 + \log p_2)$ time (cf. [15], Theorem 5.1). Clearly, the solution (5) is indeed integer if and only if $D$ is a multiple of $\omega$. The largest integer multiple of $\omega$ not greater than $D$ is $\overline{D} \equiv \omega\lfloor D/\omega\rfloor$. Since every feasible bin pattern uses no more than $\overline{D}$ units of the available capacity, the instance $(p_1, p_2, q_1, q_2, D)$ is equivalent to $(p_1, p_2, q_1, q_2, \overline{D})$. By using this equivalence relation and (5), we reformulate conditions (4) as

$$\lfloor D/\omega\rfloor\alpha + k\gamma \le \psi_1 < \lfloor D/\omega\rfloor\alpha + (k+1)\gamma.$$

The above conditions are satisfied if and only if $k = k^\star \equiv \lfloor(\psi_1 - \alpha\lfloor D/\omega\rfloor)/\gamma\rfloor$. So the required points are

$$\begin{pmatrix}y_1\\ y_2\end{pmatrix} \equiv \lfloor D/\omega\rfloor\begin{bmatrix}\alpha\\ \beta\end{bmatrix} + k^\star\begin{bmatrix}\gamma\\ \delta\end{bmatrix}, \tag{6a}$$

$$\begin{pmatrix}z_1\\ z_2\end{pmatrix} \equiv \lfloor D/\omega\rfloor\begin{bmatrix}\alpha\\ \beta\end{bmatrix} + (k^\star + 1)\begin{bmatrix}\gamma\\ \delta\end{bmatrix}. \tag{6b}$$

Hence, $(y_1, y_2)$ and $(z_1, z_2)$ satisfying equations (3) and (4) can always be found in $O(\log p_1 + \log p_2) = O(\log D)$ time. The BP2 instance $(p_1, p_2, q_1, q_2, D)$ is special, and thus solvable via Lemma 1 or Lemma 2, if and only if the integer vectors defined in (6) are nonnegative. Clearly, this is not always the case.

In the next section we show how to exploit the above lemmas to efficiently solve also non-special instances.

*3.2. An $O(\log D)$ Algorithm for BP2*

On the Euclidean plane, consider the triangle

$$T \equiv \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, p_1x_1 + p_2x_2 \leq D\}.$$

The set of feasible bin patterns is $T \cap \mathbb{Z}^2$. Let $P$ be the convex hull of the set of feasible bin patterns, i.e.,

$$P \equiv \text{conv}(T \cap \mathbb{Z}^2).$$

Harvey [6] proposed an algorithm for computing the smallest set of linear inequalities that define the integer hull of a convex two-dimensional polygon in $O(m \log D_{\max})$ time, where $m$ is the number of input inequalities and $D_{\max}$ is the largest input coefficient. By using Harvey's algorithm, a smallest set of linear inequalities representing $P$ can be found in $O(\log D)$ time.

Consider the following algorithm.

**Algorithm 1**

*Step 1.* Compute a smallest set of linear inequalities defining $P$ by Harvey's algorithm; call them $a_{k1}x_1 + a_{k2}x_2 \leq d_k$ $(k = 1, \ldots, K)$.

*Step 2.* Find an edge of $P$ intersecting the line $Q = \{x \in \mathbb{R}^2 : x_1/q_1 = x_2/q_2\}$. This is equivalent to finding, among the lines $\{x \in \mathbb{R}^2 : a_{k1}x_1 + a_{k2}x_2 = d_k\}$ $(k = 1, \ldots, K)$, one having intersection with line $Q$ closest to the origin. Let $\widetilde{k}$ be its index and $(\widetilde{\psi}_1, \widetilde{\psi}_2)$ be the corresponding intersection point.

*Step 3.* Define the special BP2 instance $(\widetilde{p}_1, \widetilde{p}_2, q_1, q_2, \widetilde{D}) \equiv (a_{\widetilde{k}1}, a_{\widetilde{k}2}, q_1, q_2, d_{\widetilde{k}})$, having $(\widetilde{\psi}_1, \widetilde{\psi}_2)$ as target point.

Case $(i)$: Point $(\widetilde{\psi}_1, \widetilde{\psi}_2)$ is integer. Find an optimal solution $\widetilde{\sigma}^\star$ of $(\widetilde{p}_1, \widetilde{p}_2, q_1, q_2, \widetilde{D})$ by using Lemma 1.

Case $(ii)$: Point $(\widetilde{\psi}_1, \widetilde{\psi}_2)$ is not integer. By means of (6), find two nonnegative integer solutions $(\widetilde{y}_1, \widetilde{y}_2)$ and $(\widetilde{z}_1, \widetilde{z}_2)$ of $\widetilde{p}_1x_1 + \widetilde{p}_2x_2 = \widetilde{D}$ such that (4) holds, and find an optimal solution $\widetilde{\sigma}^\star$ of $(\widetilde{p}_1, \widetilde{p}_2, q_1, q_2, \widetilde{D})$ by using Lemma 2.

*Step 4.* Return $\widetilde{\sigma}^\star$.

We next show that the solution built in Step 3 of Algorithm 1 is optimal for the original BP2 instance $(p_1, p_2, q_1, q_2, D)$. Consider the special BP2 instance $(\widetilde{p}_1, \widetilde{p}_2, q_1, q_2, \widetilde{D})$, and let $\widetilde{b}^\star$ be its optimal value. Again, let $b^\star$ be the optimal value of the original BP2 instance $(p_1, p_2, q_1, q_2, D)$. A feasible bin pattern for the original BP2 instance is feasible also for the associated special one. In fact, line $\widetilde{L} = \{x \in \mathbb{R}^2 : \widetilde{p}_1x_1 + \widetilde{p}_2x_2 = \widetilde{D}\}$ supports the integer hull $P$ of the original instance (see Figure 2). Thus, $b(\widetilde{\sigma}^\star) = \widetilde{b}^\star \leq b^\star$.

If Case $(i)$ of Step 3 applies, $(\widetilde{\psi}_1, \widetilde{\psi}_2)$ is a feasible bin pattern for both the associated and the original instances, and, from Lemma 1, the bin patterns used by $\widetilde{\sigma}^\star$ are all dominated by $(\psi_1, \psi_2)$. Similarly, in Case $(ii)$, points $(\widetilde{y}_1, \widetilde{y}_2)$ and $(\widetilde{z}_1, \widetilde{z}_2)$ are feasible bin patterns for both instances, and, from Lemma 2, the bin patterns used by $\widetilde{\sigma}^\star$ are all convex combinations of $(0, 0)$, $(\widetilde{y}_1, \widetilde{y}_2)$ and $(\widetilde{z}_1, \widetilde{z}_2)$, or dominated by either $(\widetilde{y}_1, \widetilde{y}_2)$ or $(\widetilde{z}_1, \widetilde{z}_2)$. In both cases, $\widetilde{\sigma}^\star$ is feasible also for the original instance, and thus $b(\widetilde{\sigma}^\star) \geq b^\star$.

We conclude that $b(\widetilde{\sigma}^\star) = b^\star$ and so the solution $\widetilde{\sigma}^\star$ returned by Algorithm 1 is optimal for the original BP2 instance.

Concerning the complexity of Algorithm 1, note that each step is performed exactly once. Step 1 requires $O(\log D)$ time [6]. In Step 2, in order to find the facet of $P$ intersecting $Q$, we only need to compute

$$\widetilde{\psi}_1 = \frac{q_1 d_{\widetilde{k}}}{q_1 a_{\widetilde{k}1} + q_2 a_{\widetilde{k}2}} = \min_{k=1,\dots,K} \left\{ \frac{q_1 d_k}{q_1 a_{k1} + q_2 a_{k2}} \right\}$$

and

$$\widetilde{\psi}_2 = \frac{q_2}{q_1} \; \widetilde{\psi}_1$$

Since $K = O(\log D)$, this can be done in $O(\log D)$ time. In Step 3, we build the solution $\widetilde{\sigma}^\star$ of the associated special instance by means of either Lemma 1 (Case $(i)$) or Lemma 2 (Case $(ii)$). In Case $(i)$, this requires constant time. In Case $(ii)$, as observed at the end of Section 3, computing $(\widetilde{y}_1, \widetilde{y}_2)$ and $(\widetilde{z}_1, \widetilde{z}_2)$ requires $O(\log D)$ time, after which $\widetilde{\sigma}^\star$ can be built in constant time. Thus, in the worst case, also Step 3 takes $O(\log D)$ time.

The above discussion is summarized in the following result.

**Theorem 1.** *Algorithm 1 correctly solves any BP2 instance in $O(\log D)$ time.*

In order to illustrate Algorithm 1, we consider the BP2 instance $(5, 3, 6, 6, 14)$, already represented in Figure 1. The polygon $P$, computed in Step 1, is depicted in Figure 2(a), where the edge of $P$ identified in Step 2 is also evidenced. The associated special instance considered in Step 3 is $(2, 1, 6, 6, 5)$, and it is depicted in Figure 2(b), together with the spanning patterns $y = (1, 3)^T$ and $z = (2, 1)^T$. Note that the convex hull $\widetilde{P}$ of the feasible bin patterns contains $P$. We fall in Case $(ii)$ of Step 3, so Lemma 2 applies. We get $\lambda^\star = 1.2$, $\nu^\star = 2.4$, and $w = (1, 1)^T$. Since $\phi(\lambda^\star) + \phi(\nu^\star) = 0.6 < 1$, the returned solution is $\sigma = \{(y, 1), (z, 2), (w, 1)\}$, with $b(\sigma) = 4$, depicted by dotted lines in Figure 2(b). In Figure 2(c) we illustrate the fact that the solution built up according to Lemma 2 uses only bin patterns contained in the set $R$ and thus feasible for both $(5, 3, 6, 6, 14)$ and $(2, 1, 6, 6, 5)$.

## 4. The general case

The concept of "special instance" can be generalized as follows. We say that a BP$C$ instance $(p, q, D)$ is *special* if there exist $C$ linearly independent, nonnegative integer $C$-vectors $v^1, \dots, v^C$ such that: $(i)$ $p^T v^i = D$ for all $i = 1, \dots, C$; $(ii)$ the unique $C$-vector $\lambda^\star$ such that $\sum_{i=1}^{C} v^i \lambda_i^\star = q$ is nonnegative. As before, we refer to $v^1, \dots, v^C$ as the *spanning patterns* associated with $(p, q, D)$.

The approach used for solving BP2 is based on the fact that the straightforward lower bound $\lceil \overline{b} \rceil$ is always tight for a special BP2 instance. Unfortunately, this fact cannot be generalized to the case $C > 2$. Consider, e.g., the BP3 instance $(p, q, D)$ where $p = (91, 26, 14)^T$, $q = (1, 4, 12)^T$, and $D = 182$. Here, the points $v^1 = (2, 0, 0)^T$, $v^2 = (0, 7, 0)^T$, and $v^3 = (0, 0, 13)$ are feasible bin patterns such that $p^T v^i = D$ for $i = 1, 2, 3$, and $q$ is clearly in the cone generated by $v^1, v^2, v^3$. So we have a special

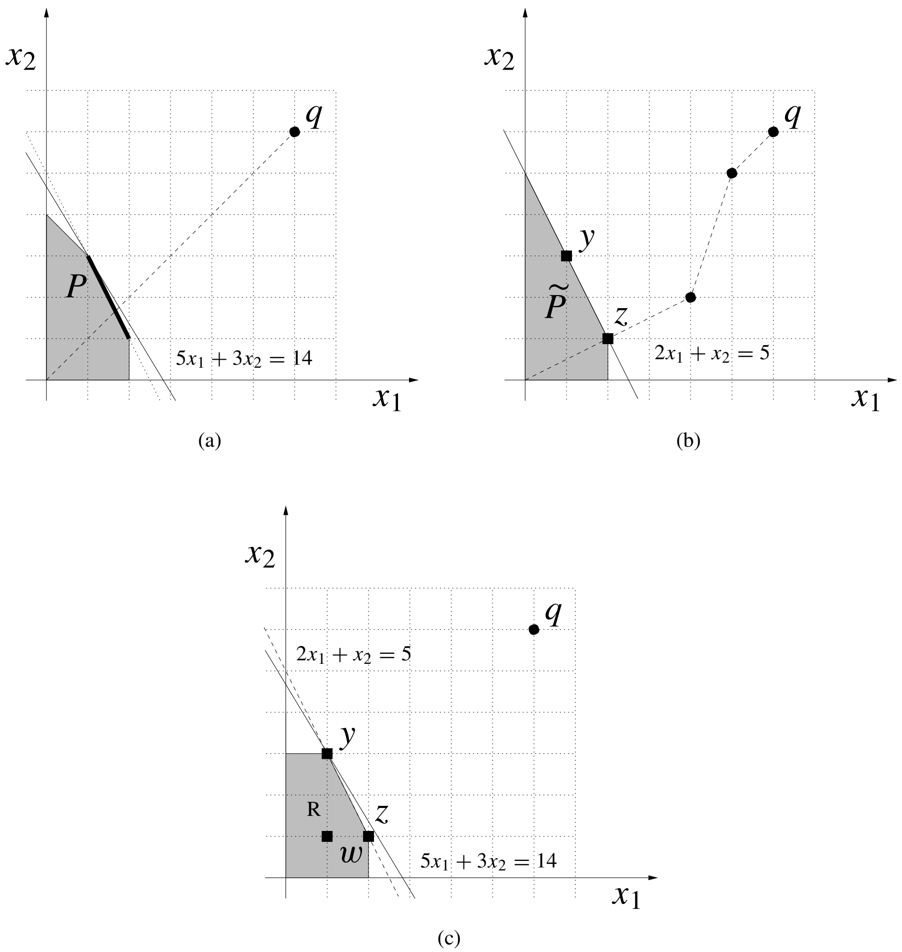


**Fig. 2.** A graphical illustration of Algorithm 1 on the BP2 instance $(5, 3, 6, 6, 14)$.

BP3 instance. It is easy to see that $\lceil \overline{b} \rceil = 2$. However, *it is not possible to decompose q into the sum of* 2 *feasible bin patterns*. Indeed, in this case $b^\star = 3 > \lceil \overline{b} \rceil$.

Nonetheless, for any $C \geq 2$ we are able to prove the following result.

**Lemma 3.** *If* $(p, q, D)$ *is a special BPC instance with associated spanning patterns* $v^1, \ldots, v^C$*, then we can build in constant time a solution*

$$\sigma^H = \{(w^1, b_1), \ldots, (w^K, b_K)\}$$

*such that:*

$(I)$ $b(\sigma^H) \leq \lceil \overline{b} \rceil + C - 2$*;*
$(II)$ *for all* $k = 1, \ldots, K$*, the feasible bin pattern* $w^k$ *is a convex combination of* $v^0 \equiv 0, v^1, \ldots, v^C$ *or* $w^k \leq v^i$ *for some* $i = 1, \ldots, C$.

*Proof.* Let $V \equiv [v^1, \dots, v^C]$, so that $\lambda^\star \equiv V^{-1}q$. We may write:

$$D\overline{b} = p^T q = p^T V\lambda^\star = [p^T v^1 \dots p^T v^C]\lambda^\star = [D \dots D]\lambda^\star.$$

Hence,

$$\overline{b} = \sum_{i=1}^{C} \lambda_i^\star. \tag{7}$$

Let $\phi = [\phi_i]$ be a $C$-vector whose entries are the fractional parts of the corresponding entries of $\lambda^\star$:

$$\phi_i \equiv \lambda_i^\star - \lfloor \lambda_i^\star \rfloor \qquad \text{for all } i = 1, \dots, C.$$

Define

$$w \equiv V\phi.$$

Note that $w$ is integer since $w = \sum_{i=1}^{C} v^i(\lambda_i^\star - \lfloor \lambda_i^\star \rfloor) = q - \sum_{i=1}^{C} v^i \lfloor \lambda_i^\star \rfloor$. Finally, let $\Phi$ be the sum of the entries of $\phi$, and observe that $0 \le \Phi < C$.

We distinguish three cases:

*Case 1.* $\Phi = 0$. Necessarily, $w = 0$, while $\lambda^\star$ is a nonnegative integer vector and $\overline{b}$ is integer. Consider the solution

$$\sigma^H \equiv \{(v^i, \lambda_i^\star) : i = 1, \dots, C\}.$$

Since $b(\sigma^H) = \sum_{i=1}^{C} \lambda_i^\star = \overline{b} = \lceil \overline{b} \rceil$, $\sigma^H$ is optimal and both $(I)$ and $(II)$ trivially hold.

*Case 2.* $0 < \Phi \le 1$. The integer point $w$ is a convex combination of $v^0, v^1, \dots, v^C$, and thus it is a feasible bin pattern. Consider the solution

$$\sigma^H \equiv \{(v^i, \lfloor \lambda_i^\star \rfloor) : i = 1, \dots, C\} \cup \{(w, 1)\}.$$

If $\Phi < 1$ then $\overline{b}$ is not integer and $b(\sigma^H) = \sum_{i=1}^{C} \lfloor \lambda_i^\star \rfloor + 1 = \lfloor \overline{b} \rfloor + 1 = \lceil \overline{b} \rceil$. Otherwise, $\overline{b}$ is integer and $b(\sigma^H) = \sum_{i=1}^{C} \lfloor \lambda_i^\star \rfloor + 1 = \overline{b} = \lceil \overline{b} \rceil$. In both cases $\sigma^H$ is optimal and both $(I)$ and $(II)$ hold.

*Case 3.* $\Phi > 1$. Since by construction $w < \sum_{i=1}^{C} v^i$, the following method decomposes $w$ into at most $C$ nonzero feasible bin patterns.

**for all** $k = 1, \dots, C$
  **for all** $j = 1, \dots, C$
    $w_j^k = \min\{w_j, v_j^k\}$;
    $w_j = w_j - w_j^k$;

Let $F$ be the largest index $k$ such that $w^k \neq 0$, and consider the solution

$$\sigma^H \equiv \{(v^i, \lfloor\lambda_i^\star\rfloor) : i = 1, \ldots, C\} \cup \left(\bigcup_{k=1}^{F} \{(w^k, 1)\}\right).$$

Clearly, $b(\sigma^H) = \sum_{i=1}^{C} \lfloor\lambda_i^\star\rfloor + F$. Since $\lceil\bar{b}\rceil = \lceil\sum_{i=1}^{C} \lambda_i^\star\rceil = \sum_{i=1}^{C} \lfloor\lambda_i^\star\rfloor + \lceil\Phi\rceil$, we have $b(\sigma^H) - \lceil\bar{b}\rceil = F - \lceil\Phi\rceil \leq C - 2$. Then, $(I)$ holds. Since by construction $w^k \leq v^k$ for all $k = 1, \ldots, F$, condition $(II)$ also holds. □

Note that the solution provided by the above lemma employs no more than $2C$ different bin patterns, and hence is polynomial in size. Moreover, it is optimal under Case 1 and Case 2, and also under Case 3 when $\lceil\Phi\rceil = C$.

If we apply Lemma 3 to the instance given at the beginning of this section, we fall in Case 3 with $\lceil\Phi\rceil = C - 1$. We get the solution

$$\sigma^H = \{((1, 0, 0)^T, 1), ((0, 4, 0)^T, 1), ((0, 0, 12)^T, 1)\}$$

with $b(\sigma^H) = \lceil\bar{b}\rceil + 1 = 3$. (Recalling that in this case $b^\star = 3$, we see that the obtained solution is indeed optimal, though its optimality is not guaranteed by Lemma 3.)

### *4.1. An asymptotically exact algorithm*

Given any BP$C$ instance $(p, q, D)$, consider the simplex:

$$S \equiv \{x \in \mathbb{R}^C : p^T x \leq D, x \geq 0\}.$$

The set of feasible bin patterns is $S \cap \mathbb{Z}^C$. The *knapsack polytope* $P$ is the convex hull of the set of feasible bin patterns, i.e.,

$$P \equiv \mathrm{conv}(S \cap \mathbb{Z}^C).$$

Let $v^0 \equiv 0, v^1, \ldots, v^K$ denote the vertices of $P$. The following observation will be useful in the sequel.

*Remark 2.* If $v \in P$ and $0 \leq w \leq v$ then $w \in P$.

Lemma 3 can be directly applied only if $(i)$ the hyperplane $Q \equiv \{x \in \mathbb{R}^C : p^T x = D\}$ supports a facet $R$ of $P$ intersecting the half-line $L \equiv \{x \in \mathbb{R}^C : x = \rho q, \rho \geq 0\}$, and $(ii)$ we know $C$ linearly independent integer points of $R$, possibly vertices of $P$, which generate a cone containing $q$. Despite this very special situation, Lemma 3 can be used to get a solution using at most $b^\star + C - 2$ bins for *any* BP$C$ instance. The approach may be outlined as follows.

**Algorithm 2**

*Step 1.* Find an integer, nonnegative $C$-vector $\widetilde{p}$ and a positive integer $\widetilde{D}$ such that $\{x \in \mathbb{R}^C : \widetilde{p}^T x = \widetilde{D}\}$ is the supporting hyperplane of a facet $R$ of $P$ that intersects $L$, and $C$ linearly independent vertices $v^{i_1}, \ldots, v^{i_C}$ of $R$ whose convex hull intersects $L$;
*Step 2.* Find a solution $\sigma^H$ of the special BP$C$ instance $(\widetilde{p}, q, \widetilde{D})$ with associated spanning patterns $v^{i_1}, \ldots, v^{i_C}$ by applying Lemma 3.
*Step 3.* Return $\sigma^H$.

**Theorem 2.** *Given any BPC instance* $(p, q, D)$*, Algorithm 2 returns a solution* $\sigma^H$ *of* $(p, q, D)$ *such that* $b(\sigma^H) - b^\star \leq C - 2$.

*Proof.* Let $(\widetilde{p}, q, \widetilde{D})$ be the special instance considered in Step 1 of Algorithm 2, and let $\widetilde{b} \equiv \widetilde{p}^T q/\widetilde{D}$. Then from Lemma 3, $\sigma^H$ is a solution of $(\widetilde{p}, q, \widetilde{D})$ such that $b(\sigma^H) - \lceil \widetilde{b} \rceil \leq C - 2$. Hence, in order to prove the theorem it suffices to prove that: $(i)$ $\sigma^H$ is feasible also for $(p, q, D)$; $(ii)$ $b^\star \geq \lceil \widetilde{b} \rceil$.

$(i)$ Let $h$ be any bin pattern used by $\sigma^H$. Then $h$ belongs to the convex hull of $v^0 \equiv 0, v^{i_1}, \ldots, v^{i_C}$ or $h \leq v^{i_\ell}$ for some $\ell = 1, \ldots, C$. It follows from Remark 2 that $h$ is contained in the knapsack polytope $P$, and thus $\sigma^H$ is feasible also for the original problem.

$(ii)$ The set of feasible bin patterns of the given instance $(p, q, D)$ is, by construction, contained in the set of feasible bin patterns of the associated special instance $(\widetilde{p}, q, \widetilde{D})$. As a consequence, the optimal value of $(\widetilde{p}, q, \widetilde{D})$ cannot be greater than the optimal value of $(p, q, D)$. Thus, Remark 1 implies that $\lceil \widetilde{b} \rceil$ is a valid lower bound for $b^\star$. □

### 4.2. *Complexity issues*

In this section we show that Algorithm 2 can be implemented to run in polynomial time.

**Theorem 3.** *Algorithm 2 may be run in* $O\,(poly(\log D))$ *time, where* $poly(\cdot)$ *is a polynomial that does not depend on* $C$.

As a consequence of Lemma 3, Step 2 of Algorithm 2 requires constant time. Hence, to prove the above result we need only to analyze Step 1.

In order to apply Lemma 3 we need $C$ linearly independent vertices of the knapsack polytope $P$, all belonging to the same facet of $P$, and such that $q$ is contained in their conical hull. To this aim, we look for the point of $P$, lying on $L$, farthest from the origin. Such a point is determined by the following primal-dual pair of linear programs, where $v^0 \equiv 0, v^1, \ldots, v^K$ are the vertices of $P$.

$$\begin{array}{rll} \max & \rho & \\ \text{subject to} & \sum_{k=0}^{K} \mu_k v_j^k = \rho q_j & (j = 1, \ldots, C) \\ & \sum_{k=0}^{K} \mu_k = 1 & \\ & \mu_k \geq 0 & (k = 0, 1, \ldots, K) \\ & \rho \geq 0 & \end{array} \tag{8}$$

$$\begin{aligned} &\min \pi_0 \\ &\text{subject to } \textstyle\sum_{j=1}^{C} v_j^k \pi_j + \pi_0 \geq 0 \ (k = 0, 1, \ldots, K) \\ &\qquad\qquad -\textstyle\sum_{j=1}^{C} q_j \pi_j \geq 1 \end{aligned} \tag{9}$$

It is easy to see that both problems (8) and (9) are feasible, so that an optimal basic primal-dual solution exists, and the optimal value is strictly positive. A basis is *nontrivial* if in the corresponding solution $\rho > 0$. Note that any nontrivial basis of (8) is identified by a subset $B = \{i_1, \ldots, i_C\} \subseteq \{0, 1, \ldots, K\}$ of cardinality $C$. The corresponding basis matrix may be written as:

$$V^B \equiv \begin{bmatrix} -q & v^{i_1} & \cdots & v^{i_C} \\ 0 & 1 & \cdots & 1 \end{bmatrix}. \tag{10}$$

**Lemma 4.** *Let $B^\star$ be an optimal basis of problem (8). Define $\widetilde{p} \equiv -|\det(V^{B^\star})|\pi^\star$ and $\widetilde{D} \equiv |\det(V^{B^\star})|\pi_0^\star$. The following hold:*

- $(a)$ *$\widetilde{p}^T v^{i_\ell} = \widetilde{D}$ for all $\ell = 1, \ldots, C$;*
- $(b)$ *$q$ belongs to the cone generated by $v^{i_1}, \ldots, v^{i_C}$;*
- $(c)$ *$\widetilde{p} \geq 0$ and $\widetilde{D} > 0$;*
- $(d)$ *$\widetilde{p}$ and $\widetilde{D}$ are integer;*
- $(e)$ *$\widetilde{p}^T x \leq \widetilde{D}$ is valid for $P$.*

*Proof.* First note that $-\sum_{j=1}^{C} v_j^k \pi_j^\star - \pi_0^\star$ is the reduced cost coefficient of the primal variable $\mu_k$ with respect to $B^\star$. Property $(a)$ then follows from the fact that basic variables have zero reduced cost, whereas property $(e)$ follows from the optimality of $B^\star$. Property $(b)$ follows from the fact that $(\rho^\star, \mu^\star)$ is feasible for (8), so that $q = \sum_{k \in B^\star} (\mu_k^\star / \rho^\star) v^k$.

In order to prove property $(c)$, we first note that $\widetilde{D} > 0$ trivially holds from the definition, since $B^\star$ is a nontrivial basis and so $\pi_0^\star > 0$. We next show that $\widetilde{p} \geq 0$, proving by contradiction that $\pi^\star \leq 0$. Let $x^\star \equiv \sum_{k \in B^\star} \mu_k^\star v^k$. Since $\sum_{k \in B^\star} \mu_k^\star = 1$ and since $-\sum_{j=1}^{C} v_j^k \pi_j^\star = \pi_0^\star$ for all $k \in B^\star$, we have $-\sum_j \pi_j^\star x_j^\star = \pi_0^\star$. Suppose that $\pi_{\hat{j}}^\star > 0$ for some index $\hat{j}$, and consider the point $x^\epsilon = [x_j^\epsilon]$ where

$$x_j^\epsilon \equiv \begin{cases} x_j^\star - \epsilon & \text{if } j = \hat{j}, \\ x_j^\star & \text{otherwise.} \end{cases}$$

Since $x^\star > 0$, for $\epsilon > 0$ sufficiently small, $x^\epsilon \geq 0$. Furthermore, $x^\star \in P$ and $x^\epsilon \leq x^\star$ imply $x^\epsilon \in P$. However,

$$-\sum_j \pi_j^\star x_j^\epsilon = \pi_{\hat{j}}^\star \epsilon - \sum_{j \neq \hat{j}} \pi_j^\star x_j^\epsilon = \pi_{\hat{j}}^\star \epsilon + \pi_0^\star > \pi_0^\star,$$

contradicting the fact that $-\sum_j \pi_j^\star x_j \leq \pi_0^\star$ is a valid inequality for $P$.

Finally, property $(d)$ follows from Cramer's rule, since $(\pi^\star, \pi_0^\star)$ is the unique solution of the nonsingular system

$$(\pi_1 \ \ldots \ \pi_C \ \pi_0) V^{B^\star} = (0 \ \ldots \ 0 \ 1).$$

□

Points $(a)$–$(d)$ of Lemma 4 ensure that $(\widetilde{p}, q, \widetilde{D})$ is a special BP$C$ instance, and that $v^{i_1}, \ldots, v^{i_C}$ are spanning patterns associated with it. Point $(e)$ ensures that the feasible bin patterns of $(p, q, D)$ are feasible also for $(\widetilde{p}, q, \widetilde{D})$. We thus say that $(\widetilde{p}, q, \widetilde{D})$ is a special instance *associated* with $(p, q, D)$.

We now show that an optimal basis of problem (8) can be found in polynomial time. This can be done by using one of the main results in [5]. In order to state it, we need some definitions. A polyhedron $P \subset \mathbb{R}^n$ has *facet-complexity at most* $\varphi$ if there exists a system of inequalities with rational coefficients that has solution set $P$ and such that the encoding length of each inequality of the system is at most $\varphi$. A *well-described polyhedron* is a triple $(P; n, \varphi)$ where $P \subset \mathbb{R}^n$ is a polyhedron with facet complexity at most $\varphi$. The *strong separation problem* associated with polyhedron $P \subset \mathbb{R}^n$ is the following. Given a vector $y \in \mathbb{R}^n$, decide whether $y \in P$, and if not, find a vector $c \in \mathbb{R}^n$ such that $c^T y < \min\{c^T x : x \in P\}$. The polyhedron $P$ is *specified by a strong separation oracle* if an oracle solving the above separation problem is known. Finally, given a vector $c \in \mathbb{Q}^n$, we denote by size$(c)$ the length of the (binary) encoding of $c$. Now Theorem 6.6.5 in [5] can be stated as follows. (See [5] for further details.)

**Theorem 4.** *There exists an algorithm that, for any well-described polyhedron* $(P; n, \varphi)$ *specified by a strong separation oracle, and for any given vector* $c \in \mathbb{Q}^n$*, finds a basic optimum standard dual solution of problem* $\min\{c^T x : x \in P\}$ *if one exists. The number of calls on the separation oracle, and the number of elementary arithmetic operations executed by the algorithm are bounded by a polynomial in* $\varphi$*. All arithmetic operations are performed on numbers whose encoding length is bounded by a polynomial in* $\varphi + size(c)$.

Let $P \subset \mathbb{R}^{C+1}$ be the feasible region of problem (9). For any given vector $\pi \in \mathbb{R}^C$, we may assume that $-q^T \pi \geq 1$, since otherwise the strong separation problem associated with $P$ is trivially solved. The coefficients of the $k$th inequality defining $P$ are $[(v^k)^T, 1]$. Since for all $j = 1, \ldots, C$,

$$D \geq p^T v^k \geq \sum_j v_j^k \geq v_j^k,$$

we have size$([(v^k)^T, 1]) \leq C \log D + 1 = O(\log D)$. Hence, $\varphi = O(\log D)$.

In order to decide whether a given vector $(\pi, \pi_0) \in \mathbb{R}^{C+1}$ is feasible, it suffices to solve the following knapsack problem in fixed dimension:

$$\max\{\pi^T x : p^T x \leq D, x \in \mathbb{Z}^C\}.$$

By using a preprocessing algorithm proposed by Frank and Tardos in [4], the above knapsack problem can be transformed in constant time (i.e., polynomial in $C$), into the equivalent problem

$$\max\{\overline{\pi}^T x : p^T x \leq D, x \in \mathbb{Z}^C\}, \tag{11}$$

where size$(\overline{\pi})$ is polynomial in $C$. Recently Eisenbrand [3] showed that an integer program with a fixed number of both variables and constraints can be solved in $O(s)$ time, where $s$ is the length of the binary encoding. By using Eisenbrand's algorithm, problem (11) and thus the separation problem associated with the feasible region of (9) can be solved in $O(\log D)$ time.

It follows from Theorem 4 that an optimal basic solution of problem (8) can be found in $O(\mathrm{poly}(\log D))$ time. If primal degeneracy occurs, an optimal basic solution may not completely identify an optimal basis. To overcome this obstacle, it is sufficient to perturb the cost vector of problem (9) by adding to it a vector $(\epsilon, \epsilon^2, \dots, \epsilon^{C+1})$, where $\epsilon > 0$ is sufficiently small. Standard lower bounds on the norm of the nonzero roots of polynomials ensure that we can assume $\mathrm{size}(\epsilon) = O(\log D)$. We are thus able to get an optimal basis of (8) in $O(\mathrm{poly}(\log D))$ time, proving Theorem 3.

### 4.3. A better lower bound

It is interesting to note that $\lceil \widetilde{b} \rceil$ is a better lower bound for $b^\star$ than $\lceil \overline{b} \rceil$.

**Lemma 5.** $\lceil \widetilde{b} \rceil \geq \lceil \overline{b} \rceil$.

*Proof.* Let $(\rho^\star, \mu^\star)$ and $(\pi_0^\star, \pi^\star)$ be the solutions of problems (8) and (9) corresponding to an optimal basis $B^\star$ of (8). From the definitions of $\widetilde{p}$ and $\widetilde{D}$, we have

$$\widetilde{b} = (-\sum_{j=1}^{C} q_j \pi_j^\star)/\pi_0^\star. \tag{12}$$

Since $\pi_0^\star = \rho^\star$ and $\sum_{k=0}^{K} \mu_k^\star v^k = \rho^\star q$, we have

$$D = D\left(\sum_{k=0}^{K} \mu_k^\star\right) \geq \sum_{k=0}^{K} \mu_k^\star \left(p^T v^k\right) = p^T \left(\sum_{k=0}^{K} \mu_k^\star v^k\right) = p^T q \pi_0^\star$$

Hence, recalling that $-\sum_{j=1}^{C} q_j \pi_j^\star \geq 1$, we obtain from (12)

$$\widetilde{b} \geq 1/\pi_0^\star \geq p^T q/D = \overline{b}. \qquad \square$$

It is not hard to obtain instances where the gap between $\lceil \widetilde{b} \rceil$ and $\lceil \overline{b} \rceil$ is positive and arbitrarily large. Just consider the BP2 instance $(5, 3, 12, 12, 14)$, i.e., the one used in Figure 2 with $q$ doubled. The associated special instance is $(2, 1, 12, 12, 5)$, and is easy to see that in this case $\lceil \widetilde{b} \rceil = 8 > 7 = \lceil \overline{b} \rceil$. Moreover, if we consider the parametric instance $(5, 3, n, n, 14)$ and its special counterpart $(2, 1, n, n, 5)$, we see that $\lceil \overline{b} \rceil = \lceil 4n/7 \rceil$, whereas $\lceil \widetilde{b} \rceil = \lceil 3n/5 \rceil$. Since $3/5 - 4/7 = 1/35$, if we set $n = 12 + 35k$ with $k$ positive integer, then we obtain $\lceil \widetilde{b} \rceil = \lceil \overline{b} \rceil + 1 + k$.

## 5. Conclusions and further research needs

We described a polynomial, asymptotically exact algorithm for the high-multiplicity bin packing problem. We conclude comparing our approach with previous literature.

- For $C = 2$, our algorithm has complexity $O(\log D)$ and provides an optimal solution to BP2. The previously best known result for this case is the algorithm by McCormick, Smallwood, and Spieksma [12] for FBP2. Their algorithm finds, if it exists, a packing of all the objects in $m$ bins in time $O((\log D)^2)$ [12]. Such an algorithm finds a triangle spanned by integer points of the knapsack polytope, containing point $(q_1/m, q_2/m)$. A straightforward application of a recent result by Althaus et al. [1] to FBP2 shows that such a triangle, if it exists, can be found in $O(\log D)$ expected time. So, the algorithm by McCormick et al. can be actually implemented to run in $O(\log D)$ expected time. Besides running in deterministic, worst-case $O(\log D)$ time, our approach directly solves the optimization problem BP2, thus saving the binary search. (On the other hand, McCormick et al. show that their algorithm may be generalized to other problems than FBP2, such as problems with uniform or unrelated machines, which are typically not addressed in the context of bin packing.)
- All the above time bounds, involving only the size of $D$, are expressed in terms of elementary arithmetic operations. It should be noted that the size of the numbers involved in such operations is polynomial in both the size of $D$ and the size of $q_{\max}$. This observation applies also to the size of the output, and may become relevant if the size of $q_{\max}$ is not polynomially bounded in the size of $D$.
- Our approach always returns a solution to BP$C$ using no more than $2C$ different bin patterns (attainable in Case 3 of Lemma 3) and such that at most $C$ of them are used more than once. The value $2C$ may not be the minimum (recall that for $C = 2$, the solution provided in [12] for FBP2 uses at most 3 bin patterns).
- The complexity of the problem for $C \geq 3$ remains open. Similar to McCormick, Smallwood, and Spieksma, we also observe that the structure of the problem when $C \geq 3$ seems very different from the case $C = 2$. Remarkably, it is not even known if FBP$C$, with $C \geq 3$, is in NP.
- By looking at the proof of Lemma 3, we already noticed that the solution returned is proved to be optimal when the sum of the fractional parts $\Phi$ is such that either $\Phi \leq 1$ or $\Phi > C - 1$. When $1 < \Phi \leq C - 1$ we suggest a trivial way to decompose the residual vector $w$. In fact, the residual BP$C$ instance $(p, w, D)$ can be formulated as an integer program with fixed numbers of variables and constraints, as $C$ is an upper bound on the number of bins, and thus it can be solved in polynomial time by Eisenbrand's algorithm [3]. However, solving at optimality such an instance would not affect the worst case analysis carried out in the proof of Lemma 3.
- The crucial role of the value of $\Phi$ suggests a connection between our results and the *integer round-up property* [2] for the integer programming formulation of BP$C$. In our setting, it is easy to prove that this property holds if and only if $b^\star = \lceil \widetilde{b} \rceil$. Starting from Marcotte [11], a few authors have studied the integer round-up property for one-dimensional cutting stock problems, mainly identifying classes of instances where such a property does not hold, see [14] for details and references. The analysis of the relationship between our approach and the integer round-up property is worth of further investigation.
- Another issue of interest is the connection between BP$C$ and the *Pallet Loading Problem* (PLP). Here, the objective is to pack as many $a \times b$ boxes into an $A \times B$ rectangle as possible, allowing orthogonal rotations (see Nelißen [13] for an excellent survey). In PLP, the number of boxes that is possible to pack in a solution is not polynomially

bounded in the input size, which is very compact, since it simply consists of the four integers $a, b, A, B$. A feasible solution specifies, for each packed box, the position of its lower left corner and whether the box is packed horizontally or vertically. As a consequence, like FBP$C$, it is not known whether the feasibility version of PLP is in NP (and the complexity status of the optimization version is also unknown).
As reported in [13], in order to derive an upper bound for PLP, Isermann [8] proposed the linear relaxation of an interesting integer linear programming model, itself a relaxation of the original PLP instance. It is possible to show that an optimal solution to the integer model can be found by solving a pseudopolynomial number of instances of BP2, for different values of $q_1$ and $q_2$, whereas in all instances we set $p_1 = a$, $p_2 = b$, and either $D = A$ or $D = B$. An interesting research issue would be the evaluation of an approach to PLP based on the exact solution of Isermann's integer model, both in terms of strength of the bound as well as computational feasibility.

*Acknowledgements.* The authors wish to thank an anonymous referee for the helpful comments and suggestions.

## References

1. Althaus, E., Eisenbrand, F., Funke, S., Mehlhorn, K.: Point containment in the integer hull of a polyhedron. In: Proceedings of the 15th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA-04), New Orleans, USA , 2004
2. Baum, S., Trotter, L.E. Jr.: Integer rounding for polymatroid and branching problems. SIAM J. Algebraic Discrete Methods **2** (4), 416–425 (1981)
3. Eisenbrand, F.: Fast integer programming in fixed dimension. Technical Report MPI–I–2003–NWG2-002, Max-Planck-Institut für Informatik Saarbrücken, Germany, February 2003
4. Frank, A., Tardos, É.: An application of simultaneous diophantine approximation in combinatorial optimization. Combinatorica **7**, 49–65 (1987)
5. Grötschel, M., Lovász, L., Schrijver, A.: Geometric algorithms and combinatorial optimization. Springer-Verlag, Berlin, Germany, 1988
6. Harvey, W.: Computing two-dimensional integer hulls. SIAM J. Comput. **28** (6), 2285–2299 (1999)
7. Hochbaum, D.S., Shamir, R.: Strongly polynomial algorithms for the high multiplicity scheduling problem. Oper. Res. **39**, 648–653 (1991)
8. Isermann, H.: Ein Planungssystem zur Optimierung der Palettenbeladung mit kongruenten rechteckigen Versandgebinden. OR Spektrum **9**, 235–249 (1987)
9. Kertzner, S.: The linear diophantine equation. Am. Math. Mon. **88**, 200–203 (1981)
10. Leung, J.Y.-T.: On scheduling independent tasks with restricted execution times. Oper. Res. **30**, 163–171 (1982)
11. Marcotte, O.: The cutting stock problem and integer rounding. Math. Program. **33**, 82–92 (1985)
12. McCormick, S.T., Smallwood, S.R., Spieksma, F.C.R.: A polynomial algorithm for multiprocessor scheduling with two job lengths. Math. Oper. Res. **26**, 31–49 (2001)
13. Nelißen, J.: New Approaches to the Pallet Loading Problem. Working Paper, RWTH Aachen, 1993. Available via anonymous ftp from ftp://ftp.informatik.rwth-aachen.de (137.226.225.3) as /pub/reports/others/pallet.ps.gz
14. Rietz, J., Scheithauer, G., Terno, J.: Families of non-IRUP instances of the one-dimensional cutting stock problem. Discrete Appl. Math. **121**, 229–245 (2002)
15. Schrijver, A.: Theory of Linear and Integer Programming. Wiley-Interscience, Chichester, 1986