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A Re-Solving Heuristic with Bounded Revenue Loss for Network Revenue Management with Customer Choice

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We consider a network revenue management problem with customer choice and exogenous prices. We study the performance of a class of certainty-equivalent heuristic control policies. These heuristics periodically re-solve the deterministic linear program (DLP) that results when all future random variables are replaced by their average values and implement the solutions in a probabilistic manner. We provide an upper bound for the expected revenue loss under such policies when compared to the optimal policy. Using this bound, we construct a schedule of re-solving times such that the resulting expected revenue loss, obtained by re-solving the DLP at these times and implementing the solution as a probabilistic scheme, is bounded by a constant that is independent of the size of the problem.

Key words: revenue management; customer choice; asymptotic optimality; reoptimization
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1. Introduction. We consider a finite-horizon revenue management (RM) problem where a decision maker offers collections of products over a finite period at pre-determined prices for customers to choose from. The sale of products consumes resources whose inventories are pre-determined. Meeting a customer request for a product can require multiple resources simultaneously, as in an itinerary that requires a seat on each of multiple flight legs in the airline context. That is, we consider a network RM problem with customer choice. Uncertainty in our setting arises from both the arrival of potential customers, modeled via Poisson processes, as well as their choices. The objective is to design a dynamic control policy that offers a collection of products, or simply an *offer* each time a customer arrives so as to maximize expected revenue earned over the entire duration of the problem. Our setting includes, but is not limited to, the classical application of seat allocation in airline RM with customer choice.

We remark that, in principle, this problem can be solved using dynamic programming (DP); but, the resulting formulation scales poorly with problem size, making its implementation impractical even for problems of moderate size. Consequently, the literature documents many attempts to find heuristic solutions that yield respectable performance and yet are computationally tractable (Adelman [1], Zhang and Adelman [17], Meissner and Strauss [11], Chaneton and Vulcano [3], Farias and Van Roy [6], Kunnumkal and Topaloglu [8]). These heuristics typically find approximations to the value function from the DP, and then use this approximation to construct control policies via either the so-called booking limits (offer products of a type up to a limit) or bid prices (offer products as long as the revenue earned from the sale exceeds the marginal cost determined via the approximate value function). Of course, one needs to define "respectable" performance in a way that allows one to determine (theoretically) whether a proposed heuristic meets the definition while remaining reasonable from the perspective of real-world application. An appealing definition of performance is to determine the order of expected revenue loss from using the heuristic when compared to the optimal revenue from DP policy as the *size* of the problem, measured, say, by the total inventory or the total demand for all products, grows without bound. We use this asymptotic expected revenue loss as the performance measure in this paper.

A large class of heuristics that has received significant attention in the literature is the one based on the solution of the deterministic linear program (DLP) (Talluri and van Ryzin [15], Liu and van Ryzin [9]). In this approach, first, a deterministic optimization problem, which happens to be a linear program, that results when all random variables in the problem are replaced by their average values is solved, and then its solution is used to construct a heuristic control. The DLP-based control has two obvious drawbacks—it not only ignores stochastic variability but also fails to correct for it as the variability is revealed. Despite this, it is known, in settings a bit more restrictive than ours, that when the size of the problem as represented by a number k becomes large, the expected revenue loss from using a DLP-based control, rather than DP, is no worse than $O(\sqrt{k})$

(Liu and van Ryzin [9], Cooper [5]). So, it performs adequately if $O(\sqrt{k})$ is an acceptable loss in revenue. But, in applications such as airline revenue management, the scale of the problem motivates the search for heuristics with better performance guarantees. The question is whether there is a cheap way to improve the performance of DLP-based controls.

A way to partly overcome the shortcomings of the DLP approach is to use it as a certainty-equivalent controller. To be specific, rather than solving the LP once and using that solution to determine a policy that is used over the entire period, one *re-solves* the LP that results when the current state is taken into account but all future variability is replaced by average behavior from time to time, and implements the solution until the next re-solve. While the benefits of re-solving are not *a priopri* obvious (Cooper [5] gave an example where re-solving may actually backfire and worsen performance), one expects that re-solving does as well as not re-solving at all. Indeed, such results have been established in various contexts. Maglaras and Meissner [10] showed in the context of dynamic pricing that re-solving should not result in an asympotically worse performance than that of the static pricing scheme. Chen and Homem-de-Mello [4] studied the impact of re-solving a two-stage stochastic programming formulation and proved that re-solving can never worsen expected revenue in their setting. Secomandi [13] established sufficient conditions which guarantee that re-solving will not deteriorate revenue performance (i.e., conditions under which re-solving will result in a monotonic increase in revenue).

Reiman and Wang [12] showed that, with a properly chosen re-solving time, a single re-solve is enough to reduce the order of revenue loss from $O(\sqrt{k})$ to $o(\sqrt{k})$. Their setting, which is more general in terms of interarrival time distributions than ours, does not involve customer choice as ours does. Moreover, they bound the revenue loss using the so-called hindsight bound which is a tighter bound in their setting than the LP bound used in this paper. They also showed that the actual implementation of the LP solution can have significant impact on performance, by establishing that $o(\sqrt{k})$ is unattainable if the solution to the DLP is implemented via booking limits. Motivated by this, they use a probabilistic implementation when determining the control policy. A key aspect of their work is that the re-solve time must be determined endogenously in a very subtle way. From a practical perspective, it may be preferable to have a re-solving scheme that can be exogenously specified upfront, and is simple, such as periodic re-solving at pre-determined intervals. It is also worth noting that it is not clear from their work whether more frequent re-solving can mitigate the impact of the implementation, i.e., whether the inability to reduce the order of revenue loss is a consequence of using booking limits per se or is simply because there is only one re-solve. In a paper written subsequent to this paper (Jasin and Kumar [7]), we show that this is a consequence of implementation—the effect observed by Reiman and Wang with booking limits persists regardless of re-solving frequency.

In this paper we study the benefit of re-solving the DLP at predetermined times while implementing the solutions of the DLP probabilistically as in Reiman and Wang [12]. We call the resulting heuristic *probabilistic allocation control* (*PAC*). The main result of this paper is the following. We construct a schedule of re-solving times such that the expected revenue loss of PAC obtained by re-solving the DLP at these times is bounded by a constant independent of the size of the problem. That is, if V_{opt}^k denotes the optimal expected revenue and $\mathbf{E}[R_{\text{PAC}}^k]$ denotes the expected revenue of PAC when the size of the problem is equal to k, then there exists a positive constant ρ independent of k such that

$$V_{\text{opt}}^k - \mathbf{E}[R_{\text{PAC}}^k] \le \rho \quad \text{for all } k \in \mathbb{Z}^+.$$

To belabor the obvious, PAC works well enough that it captures pretty much *all* the potential revenue as the problem size gets large. One cannot expect to improve on this result in the asymptotic sense: the constant revenue loss is the price one has to pay for implementing any scheme that ignores all future variability. We provide two re-solving schemes that achieve constant revenue loss. One trades off a larger constant for significantly fewer resolves. Both these schemes are obtained via a more general result: we provide a general bound on the expected revenue loss of PAC for a large class of re-solving schemes, not all of which would result in a bounded revenue loss. Such a bound endows a revenue manager with flexibility to optimize the trade-off between the number of re-solves, the choice of re-solving times, and the desired revenue performance.

The key ideas in the proof of the bound are the following. First, as long as the optimal basis in the DLP does not change, the DLP has corrective power. That is, when the actual sample paths inevitably deviate from their mean paths, the next re-solve of the DLP compensates for this deviation adequately. This correction can be computed using standard results in linear programming. Second, given the stochastic assumptions in our setting, which can be summarized loosely as "no big shocks," it takes a long time for the paths to deviate sufficiently for the optimal basis in the DLP to change. This is obtained as a consequence of Doob's maximal inequality for martingales in our setting. Consequently, for most of the period the revenue loss due to stochastic variability is not much worse than it would be for the optimal policy. Finally, since the basis changes so late, even if all future re-solves were to be ignored, not too much revenue is lost.

The remainder of this paper is organized as follows. In §2, we define the model and the proposed control. In §3, we provide numerical simulations to motivate our results. Section 4 gives a fairly complete analysis on a simple case of single-resource RM (often called single-leg RM in literature) to help the readers navigate through the proof of the general result that follows. Section 5 contains the main results of the paper in its most general form. Variants of PAC that re-solve at endogenously determined times are briefly discussed in §6. Finally, in §7, we discuss several potential directions for future research. Unless otherwise noted, the proof of the main results can be found in the appendix.

2. The model and the proposed policy. In this section we describe problem setting, modelling assumptions, and the definition of PAC.

2.1. The setting. We consider a finite selling horizon that is normalized to $[0, 1]$. There are N customer types (indexed by $q = 1, 2, \ldots$). Customer type q arrives according to a Poisson process of rate $\lambda_q \geq 0$.[1] Arrival processes of different types are assumed to be independent. Let $\Lambda_q(s, t)$ denote the number of arrivals of customer type q during time interval $[s, t)$, $0 \leq s < t \leq 1$. (We use λ and $\Lambda(.,.)$ to denote the vector of arrival rates and the number of arrivals, respectively; all vectors are to be understood as column vectors unless specified otherwise.)

There are m resources (indexed by $i = 1, \ldots, m$). Resource i has initial capacity (or inventory) C_i. Let C denote the vector of initial capacities. The decision-maker can make one of n offers (indexed by $j = 1, \ldots, n$) at any time. An offer corresponds to a collection of products and a product may consume multiple resources simultaneously. In the airline setting, a product could be an itinerary from San Francisco (SFO) to Los Angeles (LAX), and an offer could be a collection of different itineraries from SFO to LAX (e.g., morning discount fare, noon business fare, etc.). We dispense with the notion of "product" and simply use "offer" as the primitive. Without loss of generality, we assume that each offer j is uniquely associated with a single customer type $q(j)$. We define an incidence matrix P as $P_{q,j} = 1$ if and only if $q = q(j)$ for $j \in \{1, \ldots, n\}$, and $P_{q,j} = 0$ otherwise. Let $S_q = \{j: q(j) = q\}$.

We model customer choice via random consumption of resources. Upon arrival we observe the arriving customer's type q.[2] Taking into account all the accumulated information up to now, we then decide which offer $j \in S_q$ should be offered. Then, A_{ij} units of resource i are consumed, where A_{ij} is a bounded, nonnegative, random variable with mean equal to $\bar{A}_{ij}$. We use the notations $A = [A_{ij}]$, $\bar{A} = \mathbf{E}[A]$, and A^j for the jth column of matrix A (we call it the *consumption vector* for offer j). We assume that the realizations of A^j (at arrival times when offer j is made) are independent and identically distributed for each j, and the realizations of A^j when j is offered are independent of the realizations of $A^{j'}$ when j' is offered ($j \neq j'$). The distributions of A^j are not essential for our analysis. Finally, we assume that there are n revenue functions, one for each offer. Depending on the realization of consumption vector A^j, we earn revenue $r_j(A^j)$ for presenting offer j. Note that the realization $A^j = 0$ corresponds to the case of a customer deciding not to buy anything from offer j, and we set $r_j(0) = 0$. We remark here that our model subsumes the choice model in Talluri and van Ryzin [14]. The reader familiar with their model can relate back to ours by equating our notion of an *offer* to their notion of a *choice set*.

We use $R_\pi(s, t)$ to denote the revenue earned under control π during time interval $[s, t)$, $0 \leq s < t \leq 1$. For brevity, we will write $R_\pi(0, 1) = R_\pi$. Let $N_{j,\pi}(s, t)$ denote the number of times offer j is made under control π during time interval $[s, t)$, $0 \leq s < t \leq 1$. Also, let $A^{j,u}$, $u = 1, 2, \ldots$, be the successive consumption vectors A^j that are realized each time offer j is made. Then, the stochastic optimization formulation of network RM is given by

$$\begin{aligned} V_{\text{opt}} = \sup_{\pi \in \Pi} \; & \mathbf{E}\left[\sum_{j=1}^{n} \sum_{u=1}^{N_{j,\pi}(0,1)} r_j(A^{j,u})\right] \\ \text{s.t.} \; & \sum_{j=1}^{n} \sum_{u=1}^{N_{j,\pi}(0,1)} A^{j,u} \leq C \text{ (a.s)}, \\ & \sum_{j=1}^{n} \sum_{u=1}^{N_{j,\pi}(0,t)} P^j \leq \Lambda(0, t), \quad \forall\, t \in [0, 1) \text{ (a.s)}, \end{aligned} \tag{1}$$

[1] Although we assume that customers arrive according to homogeneous Poisson processes, the analysis presented in this paper can be generalized to other cases including nonhomogeneous (time-dependent) Poisson arrivals and compound Poisson arrivals.

[2] The customer's type does not necessarily tell us the choice that the customer will make. Rather, the type gives us partial information about the choices.

where the constraints must be satisfied with probability 1. Here, Π is the set of admissible controls. To be precise, Π is the set of all controls that determine offers at time t based only on the observed type of the customer arriving at time t and all the available information acquired up to time t. That is, Π is the set of controls π which are $\Im_t^\pi$-adapted, where $\Im_t^\pi$ is the sigma algebra generated by $\sigma(N_{j,\pi}(0,s), j=1,\ldots,n, 0\le s\le t)$ and $\bigcap_{s>t}\sigma(\Lambda_q(0,s), q=1,\ldots,N)$. The definition of Π does not include an explicit check of whether there is adequate capacity for any customer choice resulting from the offer to be realized. Rather, this check is done after the fact using the constraints in (1). Practically speaking, our set-up corresponds to the assumption that a customer may be allowed to make a choice that is not capacity-feasible, and if she does so, she is told that the choice is not available, and the customer is lost. But the heuristic that we propose will not end up doing this, because it is particularly conservative with respect to capacity constraints.

2.2. The DLP. Formally taking expectation of all the random quantities in (1), we arrive at the following linear program:

$$\begin{aligned}\mathbf{DLP}[C,\lambda]\text{: maximize }\ & \bar r' x\\ \text{s.t. }\ & \bar A x\le C,\ Px\le\lambda,\quad x\ge 0.\end{aligned}\tag{2}$$

We call the above formulation the *static* DLP formulation of network RM. Intuitively, the variable x_j can be interpreted as the expected number offer j has presented to customer type $q(j)$, the first constraint can be interpreted as the supply constraint (i.e., the amount of resources used cannot exceed the available capacity, on average), and the second constraint can be interpreted as the demand constraint (i.e., the total number of offers made should not exceed the number of arrived customers, on average).

We now state two modeling assumptions that will facilitate our analysis in §3.

Assumption 2.1. $\mathbf{DLP}[C,\lambda]$ *is nondegenerate and has a unique optimal solution* Y.[3]

Assumption 2.2. *For all optimal solutions z of the following DLP*:

$$\begin{aligned}\text{maximize }\ & \bar r' x\\ \textit{s.t. }\ & Px\le\lambda,\quad x\ge 0,\end{aligned}\tag{3}$$

we have: $\bar A z\nleq C$.

Assumption 2.2 says that we do not have sufficient supply to satisfy expected demand. That is, if we are to compute the optimal allocation without taking into account the amount of available resources, then the resulting allocation will violate capacity constraints.

2.3. The proposed PAC heuristic. As discussed earlier in the introduction, we intend to re-solve the DLP at various times and use the resulting solutions to construct a heuristic control. To this end, let $C_\pi(t)$ denote the vector of available capacity at time $t, 0\le t<1$, under control π, and let $Y_\pi(t)$ denote the optimal solution to $\mathbf{DLP}[C_\pi(t),(1-t)\lambda]$; that is,

$$\begin{aligned}\mathbf{DLP}[C_\pi(t),(1-t)\lambda]\text{: maximize }\ & \bar r' x\\ \text{s.t. }\ & \bar A x\le C_\pi(t),\ Px\le(1-t)\lambda,\quad x\ge 0.\end{aligned}$$

By definition, $C_\pi(0)=C$ and $Y_\pi(0)=Y$ (and this is true for all π). Also, $(1-t)\lambda$ is the expected number of future arrivals from time t onward. So, $\mathbf{DLP}[C_\pi(t),(1-t)\lambda]$ is the optimization problem that we have to solve if we decide to re-solve the DLP at time t. Let $\Gamma=\{0<t_1<t_2<\cdots<t_M<1\}$ denote a set of increasing sequences of re-solving times. We assume that these times are deterministic and pre-specified at the beginning of the selling horizon. We use $\ell=1,\ldots,M$ to index re-solving times, and we also use $t_0=0$ and $t_{M+1}=1$.

Probabilistic Allocation Control (PAC). We now define the proposed heuristic.

(i) Start with $\ell=0$.

(ii) At time t_ℓ, solve $\mathbf{DLP}[C_{\text{PAC}}(t_\ell),(1-t_\ell)\lambda]$. When a customer of type q arrives at time $t\in[t_\ell,t_{\ell+1})$, do the following.

(a) Pick an offer $j\in S_q$ with probability $p_{j,\text{PAC}}(t_\ell)\equiv Y_{j,\text{PAC}}(t_l)/[(1-t_l)\lambda_q]$.

[3] It is possible to relax this assumption by requiring only that $\mathbf{DLP}[C,\lambda]$ be nondegenerate. We assume that the DLP has a unique solution for simplicity of analysis. Also, since we intend to re-solve the DLP several times during the selling horizon, it is not difficult to imagine that at some points during the selling horizon the corresponding DLP may be neither nondegenerate nor have a unique optimal solution. This is fine. We simply require that the first DLP solved at time 0 have a unique optimal solution.

(b) Let ξ_j be such that $\xi_j \geq A_{ij}$ for all i almost surely. Check whether $C_{\text{PAC}}(t) - \xi_j \geq 0$. If yes, present the offer; otherwise, present nothing, that is, we do not sell any product.[4]

(iii) Increment ℓ and repeat at the next t_ℓ until either $\ell = M+1$ or $C_{\text{PAC}}(t) - \xi_j < 0$ for all j.

Two types of re-solving schedules will be discussed in greater detail in this paper. The first one leads to *periodic* PAC, which re-solves the DLP periodically with a specified period $h > 0$. That is, we use: $t_l = lh$, $l = 1, 2, \ldots$. The second one leads to *mid-point* PAC because it re-solves the DLP at the mid-points of the remaining selling horizon. That is, we use: $t_l = 1 - 2^{-l}$, $l = 1, 2, \ldots$.

The following result is routine and the proof can be found in Appendix A. Let V_{DLP} and R_{PAC} denote the optimal value of $\mathbf{DLP}[C, \lambda]$ and the revenue earned under PAC during time interval $[0, 1)$,[5] respectively. Then,

THEOREM 2.1. $\mathbf{E}[R_{PAC}] \leq V_{opt} \leq V_{DLP}$.

We will use Theorem 2.1 as a convenient way to bound the revenue loss of the PAC when compared to the optimal control. We will simply bound the revenue loss by $V_{\text{DLP}} - \mathbf{E}[R_{\text{PAC}}]$. In the case without customer choice a tighter bound can be obtained using the so-called hindsight bound, which assumes an omniscient decision-maker who can predict the future exactly. This is the bound used by Reiman and Wang [12].

2.4. Asymptotic setting. Throughout the paper, we will consider a sequence of increasing problems with capacity $C^k = kC$ and arrival rate vectors $\lambda^k = k\lambda$, $k = 1, 2, \ldots$. The superscript k can be interpreted as the *size* or the *scale* of the problem. The kth problem is a problem with parameters $[C^k, \lambda^k]$. The capacity consumption matrix, incidence matrix, and revenue vector, which we still write as $\bar{A}$, P, and $\bar{r}$, do not depend on k. On the other hand, the number of arrivals of customer type q during time interval $[s, t)$, $0 \leq s < t \leq 1$, does depend on k and is denoted $\Lambda_q^k(s, t)$. Similarly, the choice of re-solving times can depend on k and so we let t_l^k and Γ^k denote the lth re-solving time and the set of re-solving times, respectively. In an obvious way, V_{DLP}^k denotes the optimal value of $\mathbf{DLP}[C^k, k\lambda]$, and R_{PAC}^k denotes the revenue earned under PAC during the entire interval $[0, 1)$ in the kth problem.

As for the two re-solving schedules mentioned in the previous subsection, formally, we have the following. For periodic PAC with period $h > 0$, we use $\Gamma^k = \{h, 2h, \ldots, M^k h\}$, where M^k is the unique integer satisfying $1 - h \leq M^k h < 1$; whereas, for mid-point PAC we use $t_l^k = 1 - 2^{-l}$, $l = 1, 2, \ldots, M^k$, where M^k is the smallest integer such that $2^{-M^k} \leq 1/k$. (Note that, with periodic PAC, the value of M^k depends on the choice of h. However, we suppress this dependence for notational brevity.) Effectively, one does not have to re-solve the DLP when the length of the remaining selling horizon is less than $1/k$. Hence, a very frequent periodic PAC would re-solve no more than $O(k)$ times and mid-point PAC would re-solve no more than $O(\log_2 k)$ times.

3. Numerical examples. The purpose of this section is to first motivate our results in §5 through numerical simulations. We give two examples: one with independent demand and the other involving customer choice. On each example, we will run simulations using periodic and mid-point PAC.

3.1. Example 1—customer choice specified by observable type. This example falls in the standard setting called independent demand (see Talluri and van Ryzin [15, Chapter 3]) in airline revenue management. Our example has four cities, denoted by (P, Q, R, S); four flights, denoted by (F_1, F_2, F_3, F_4); and six products that represent six different routes. F_1 is a flight connecting city P and city Q; F_2, a flight connecting city Q and city R; F_3, a flight connecting city P and city R; and F_4, a flight connecting city R and city S. Without loss of generality we normalize the capacity of each flight to 1; i.e., $C = (1, 1, 1, 1)'$. The details of the six products are given in Table 1.

In the independent demand model, the arriving customer's type immediately tells us the only product that the customer will choose, if she chooses one at all. So, the six customer types correspond directly to the six products. Since the customer's choice is predetermined by type, we only need to make one of six offers. The distinction

[4] As defined, PAC is somewhat wasteful in that it first decides on an offer and then checks if there is enough capacity to make the offer feasible, losing the customer if it is not. But this is not an issue of significant consequence in our setting. For the vast majority of the customers, the offers made remain capacity feasible. In fact, even in the asymptotic setting to be discussed shortly, where the arrival rates and capacities are increasing without bound, the number of offers made that are capacity infeasible remain bounded, as evidenced by the revenue loss which is proven to be bounded, with sufficiently frequent re-solving.

[5] To be more precise, we should have written $R_{\text{PAC}}(\Gamma)$ to express the dependence of the value of R_{PAC} on the choice of Γ. However, for simplicity of notation, whenever it is clear from the context which Γ is being used, we will use R_{PAC} instead of $R_{\text{PAC}}(\Gamma)$.

TABLE 1. Details of six products.

Product no.	Route	Price	Demand rate (λ)
1	$P \to Q \to R$	200	1
2	$P \to Q$	150	0.5
3	$Q \to R$	100	1
4	$P \to R$	250	0.5
5	$P \to R \to S$	400	1
6	$R \to S$	250	1

between offers and types is essentially moot. We assume that each product uses up one seat on each component flight. So, in notation of §2, we have

$$\bar{r} = [200, 150, 100, 250, 400, 250]', \qquad C = [1, 1, 1, 1]', \qquad \lambda = [1, 0.5, 1, 0.5, 1, 1]',$$

$$A = \bar{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \text{and} \qquad P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We simulate PAC in two scenarios.

SCENARIO 1. We fix $k = 200$ (this corresponds to the case where all flights have capacity 200 seats) and simulate the system under PAC with a periodic re-solving scheme. We plot a bound of expected revenue loss for different re-solving frequencies. We compute and report $(V_{\text{DLP}}^k - \hat{\mathbf{E}}[R_{\text{PAC}}^k])/V_{\text{DLP}}^k \times 100$, with $\hat{\mathbf{E}}$ corresponding to the empirical expectation computed via simulation, and report it as %Expected Loss. This quantity gives an upper bound on the deviation from optimality of periodic PAC and serves as a convenient device to avoid computing the optimal revenue. The plot can be seen in Figure 1 and the complete numerical results can be seen in Table 2.

SCENARIO 2. We simulate the system varying $k = 50, 100, 150, 200, \ldots, 500$. For each k we run periodic PAC with k re-solves, and then we plot the same simulation-based bound of expected revenue loss as a function of k. As a comparison we also plot the resulting simulation-based bound of expected revenue loss under PAC without re-solving as well as mid-point PAC. The plot can be seen in Figure 2 and the complete numerical results can be seen in Table 3.

We make two observations:

(i) In the first scenario, simulation results suggest that there is a decreasing marginal return to increasing re-solving frequency. That is, frequent re-solving of PAC seems to result in a better performance (although the "better" here may not be in the monotonic sense). Observe that re-solving at a frequency beyond $M = 200$, i.e., once per arriving customer on average, has no significant impact on performance.

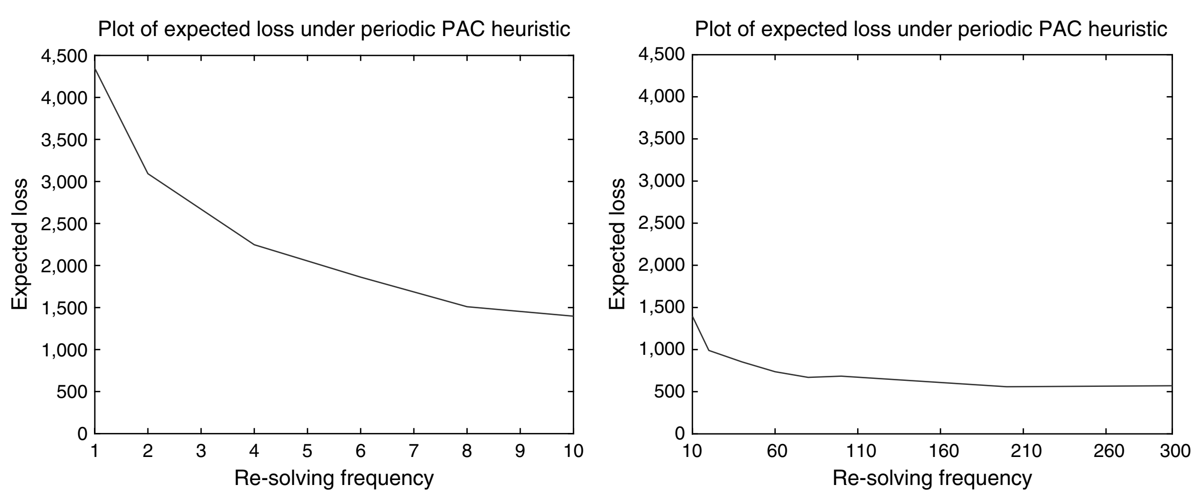

FIGURE 1. Plot of expected loss under periodic PAC heuristic.

TABLE 2. Example 1—Expected revenue loss under periodic PAC.

M	Expected loss	Std dev	%Loss
1	4,346	96	3.22
2	3,093	63	2.29
4	2,247	48	1.66
6	1,861	36	1.38
8	1,510	33	1.12
10	1,397	30	1.03
20	990	20	0.73
40	854	18	0.63
60	737	16	0.55
80	670	16	0.50
100	684	14	0.51
200	558	14	0.41
300	571	14	0.42

(ii) In the second scenario, the size of expected loss for both periodic and mid-point PAC *do not* appear to scale with the size of the problem. That is, they can be bounded by a constant independent of the size of the problem. As for mid-point PAC, the simulation suggests that with an intelligent choice of re-solving times, one can still have an excellent performance with lesser effort (as discussed in §2.4, effectively, we only require $O(\log_2 k)$ re-solvings with mid-point PAC in comparison to $O(k)$ re-solvings with periodic PAC). Observe that the constant bound for mid-point PAC is evidently larger than that of periodic PAC.

3.2. Example 2—customer choice. In this example, we will show how to incorporate customer choice into our model. Still using an example of a seat-allocation problem from the airline industry, suppose now that we have only two flights going from San Francisco (SFO) to Los Angeles (LAX). Customer preferences can no longer be predicted with certainty upon arrival as in the previous example. Rather, it is impossible in this example to tell customers apart when they arrive, and so we assume that we see a single stream of customers arriving with rate $\lambda = 2$. Flight 1 is assumed to be less popular on average than flight 2. Flight 1 has only one fare class, labeled ticket 1, and flight 2 has two different fare classes (e.g., for business and leisure customers), which we call tickets 2 and 3. The fare for ticket 1 is \$100, for ticket 2 is \$200, and for ticket 3 is \$100.

Unlike Example 1, offering different tickets to an arriving customer will cause the customer to purchase differently. For example, if tickets 1 and 3 are offered, then a customer who would have originally purchased ticket 1 were it the only ticket in the offer may switch to purchasing ticket type 3. So, the choice of offer becomes the crucial decision. We further specify customer choice by specifying a set of purchase probabilities (see Table 4). So, for example, the probability that a customer will buy ticket 2 given that only tickets 1 and 2

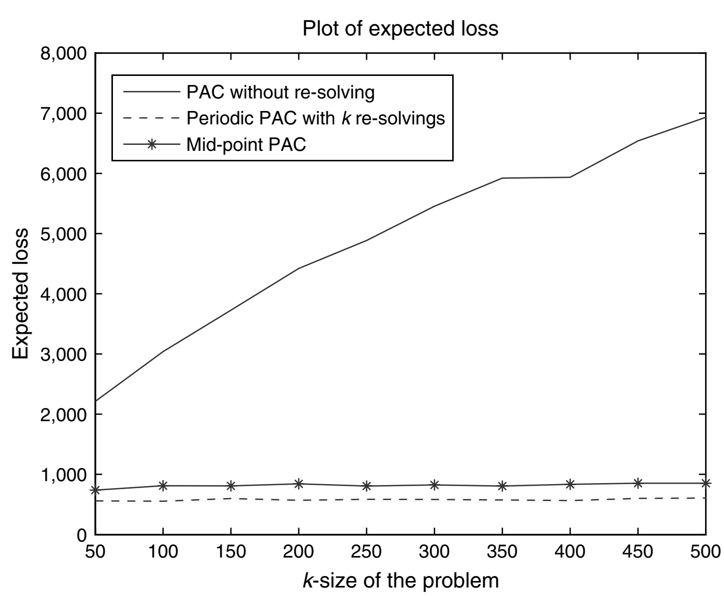

FIGURE 2. Plot of expected loss under periodic PAC versus mid-point PAC.

TABLE 3. Example 1—Comparison of expected loss.

	PAC without re-solving			Periodic PAC with k re-solvings			Mid-point PAC		
k	Exp. loss	Std dev	%Loss	Exp. loss	Std dev	%Loss	Exp. loss	Std dev	%Loss
50	2,214	53	6.56	590	13	1.75	770	17	2.28
100	3,039	71	4.50	574	13	0.85	833	19	1.23
150	3,727	90	3.68	598	15	0.59	810	19	0.80
200	4,421	103	3.27	531	15	0.39	802	19	0.59
250	4,886	117	2.90	575	14	0.34	798	19	0.47
300	5,455	125	2.69	642	14	0.32	884	20	0.44
350	5,922	138	2.51	656	14	0.28	886	19	0.37
400	5,936	142	2.20	526	14	0.19	794	19	0.29
450	6,541	151	2.15	602	15	0.20	853	21	0.28
500	6,931	161	2.05	677	15	0.20	924	19	0.27

are offered is 0.4 as opposed to 0.2 when all tickets are offered. Similarly, the probability that a customer will buy ticket 3 given that only ticket types 2 and 3 are offered is 0.6, and so on.

In recasting this problem into the DLP formulation in §2, note that we have three products, which correspond to the three tickets; seven offers, which correspond to seven different combinations of open tickets; and one customer type. We will number the offers in the following order: offer 1 is an open set $\{1\}$, offer 2 is $\{2\}$, offer 3 is $\{3\}$, offer 4 is $\{1,2\}$, offer 5 is $\{1,3\}$, offer 6 is $\{2,3\}$, and offer 7 is $\{1,2,3\}$. The price vector $\bar{r}$ can be computed as follows: $\bar{r}_1 = 0.5\times\$100 = \$50$, $\bar{r}_7 = 0.3\times\$100 + 0.2\times\$200 + 0.5\times\$100 = \$120$, and so on.

We assume that ticket one requires one seat from flight 1, and each of tickets 2 and 3 requires one seat from flight 2. Let $\bar{A}^j$ denote the jth column of $\bar{A}$. The capacity consumption matrix $\bar{A}$ can be computed as follows: $\bar{A}^1 = 0.5\times[1,0]' = [0.5,0]'$ (this is the first column of $\bar{A}$), $\bar{A}^7 = 0.3\times[1,0]' + 0.2\times[0,1]' + 0.5\times[0,1]' = [0.3,0.7]'$ (this is the seventh column of $\bar{A}$), etc.

Finally, assuming a normalized initial capacity $C = [0.5, 1]$, we have:

$$\bar{r} = [50, 60, 60, 110, 90, 100, 120]', \qquad C = [0.5, 1]', \qquad \lambda = 2,$$

$$\bar{A} = \begin{bmatrix} 0.5 & 0 & 0 & 0.3 & 0.3 & 0 & 0.3 \\ 0 & 0.3 & 0.6 & 0.4 & 0.6 & 0.8 & 0.7 \end{bmatrix}, \qquad \text{and} \qquad P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

As in Example 1, we simulate two scenarios.

SCENARIO 1. We fix $k = 100$, which corresponds to the case where flight 1 has capacity 50 seats and flight 2 has capacity 100 seats, and plot a simulation-based bound of expected revenue loss under periodic PAC at different re-solving frequencies. The plot can be seen in Figure 3 and the complete numerical results can be seen in Table 5.

SCENARIO 2. We vary $k = 50, 100, 150, 200, \ldots, 500$. For each k we simulate periodic PAC with k re-solves, and then we plot the simulation-based bound on expected revenue loss as a function of k. As a comparison we also plot the resulting bounds on expected revenue loss under PAC without re-solving as well as mid-point PAC. The plot can be seen in Figure 4, and the complete numerical results can be seen in Table 6.

We make several observations:

(i) As in Example 1, in the first scenario, the simulation results suggest that there is a decreasing marginal return to increasing re-solving frequency. However, the improvement in revenue performance due to increasingly frequent re-solving is not as significant as in Example 1. This suggests that adding another degree of uncertainty into the problem by allowing customer choice makes it more difficult to extract revenue via re-solving-based heuristics.

(ii) Again, as in Example 1, in scenario 2, the size of expected loss *does not* appear to significantly scale with the size of the problem for both periodic and mid-point PAC. However, while the size of expected loss under

TABLE 4. Purchase probability table.

Offer	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
Buy type 1	0.5	0	0	0.3	0.3	0	0.3
Buy type 2	0	0.3	0	0.4	0	0.2	0.2
Buy type 3	0	0	0.6	0	0.6	0.6	0.5

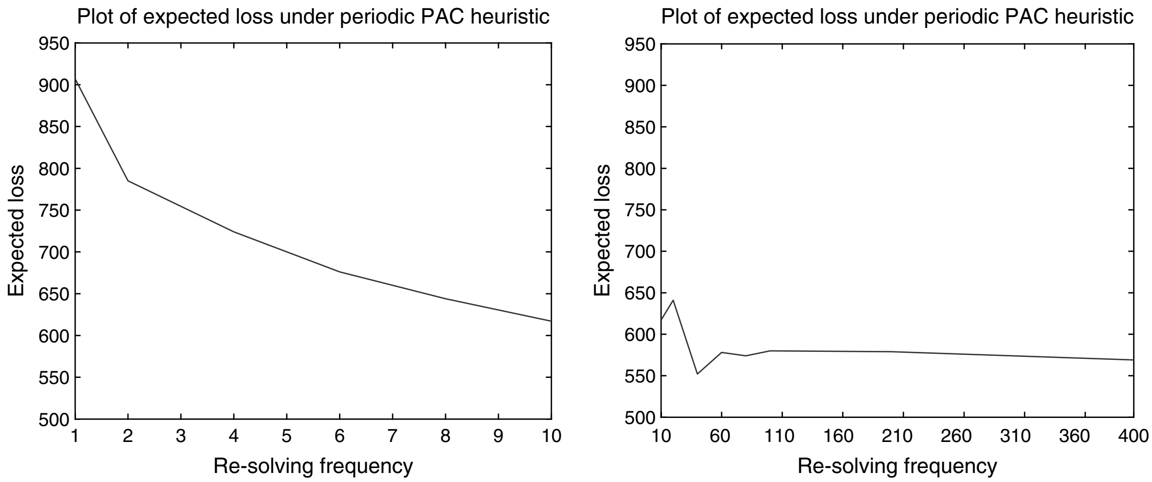

FIGURE 3. Plot of expected loss under periodic PAC heuristic.

periodic PAC seems evidently of order $O(1)$, the size of expected loss under mid-point PAC appears to be better described as $O(\log k)$ rather than $O(1)$. As noted in the previous comment, adding another degree of uncertainty by allowing customer choice makes it more difficult to extract revenue via re-solving schemes. Since mid-point PAC adapts more slowly than periodic PAC during the early segment of the selling horizon, we expect that this additional uncertainty effect will show up even more clearly for mid-point PAC than periodic PAC.

4. Special case: A single resource with two products. In this section we will give a fairly complete analysis of periodic PAC on a simple, nontrivial case, corresponding to the independent demand model in Example 1 (see §3). The goal here is to help readers navigate through the key ideas behind the proof of the general results in §5 (see §5.3).

4.1. Model and setting. We assume that we have a single resource of capacity C and two types of customers that arrive according to independent Poisson processes of rates λ_1 and λ_2. Customers will always choose one of two products that corresponds to her type if that product is offered. Product 1 earns more revenue than product 2, i.e., $r_1 > r_2$ and each product sold consumes exactly one unit of the resource.

The primal deterministic LP corresponding to this problem is:

$$\begin{aligned} \text{maximize} \quad & r_1 x_1 + r_2 x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq C, \quad 0 \leq x \leq \lambda. \end{aligned}$$

We further restrict attention to the one interesting case, where $\lambda_1 < C$ and $\lambda_1 + \lambda_2 > C$ (this condition satisfies Assumption 2.2). In this case, the LP solution is given by $Y_1(0) = \lambda_1 = Y_1$ and $Y_2(0) = C - \lambda_1 = Y_2$. We state our result for this special case.

TABLE 5. Example 2—Expected loss under periodic PAC.

M	Expected loss	Std dev	%Loss
1	907	34	4.14
2	785	36	3.58
4	724	37	3.31
6	676	38	3.09
8	644	35	2.94
10	617	35	2.82
20	641	37	2.93
40	552	37	2.52
60	578	36	2.64
80	574	38	2.62
100	580	35	2.65
200	579	34	2.65
400	569	29	2.60

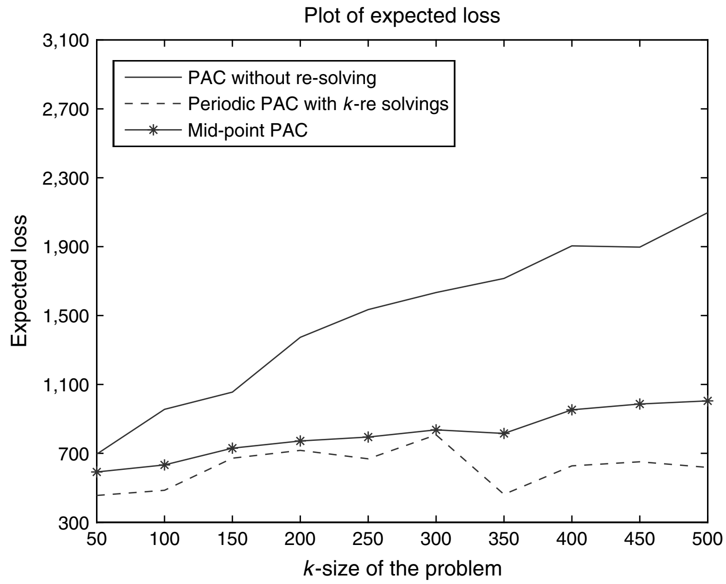

FIGURE 4. Plot of expected loss under periodic PAC versus mid-point PAC heuristic.

THEOREM 4.1. *Consider periodic PAC with period $h > 0$. There exists positive constants ρ and $\hat{\rho}$ independent of k such that*

$$V_{DLP}^{k} - \mathbf{E}[R_{PAC}^{k}] \le \rho + \hat{\rho}\sqrt{kh} \quad \textit{for all } k \in \mathbb{Z}^{+}. \tag{4}$$

Observe, in particular, that choosing $h = 1/k$ results in a constant bound independent of k. This confirms our numerical results in §4.

4.2. Key ideas and roadmap to the proof. The proof of Theorem 4.1 can be essentially broken down into five main steps as follows.

Step 1. A key difficulty with the analysis is that the optimal basis of the LP can change when the LP is re-solved. If the basis does not change, then the solution of the re-solved LP can be explicitly expressed in terms of the remaining capacity vector $C(t)$. (In our special case, we would have $Y_1(t) = (1-t)\lambda_1$ and $Y_2(t) = C(t) - (1-t)\lambda_1$.) In §4.3, we introduce a relaxation of PAC that we call online PAC (OPAC) that uses this explicit formula. OPAC and PAC are equivalent until the LP basis changes, that is, until the stopping time τ when the optimal basis of the LP changes for the first time.

Step 2. In §4.4, we precisely establish the equivalence between OPAC and PAC. In addition, we also show that OPAC, thus PAC, has *corrective* power. It corrects for past under- or over-allocation of resources when compared to the DLP.[6]

Step 3. The analysis of OPAC allows us to provide a bound for $\mathbf{E}[\tau]$ in §4.5. The key result, which is a consequence of the corrective power of OPAC, is that it takes a *long* time before the LP finally changes its basis. Hence, OPAC is equivalent to PAC throughout most of the selling period. (To properly characterize what we mean by "long," we resort to the asymptotic setting.)

Step 4. Another feature of OPAC is that it ignores capacity constraints. This is fine until τ but not beyond. In the fourth step in §4.6, we introduce another relaxation of PAC called MPAC that solves the LP one last time at τ and implements it by ignoring capacity constraints. That is, MPAC pretends that there is enough capacity to implement OPAC until the end of the horizon. Given that in MPAC the capacity constraints were ignored and the DLP solution was implemented throughout the horizon, we can establish that $V_{\mathrm{DLP}} - \mathbf{E}[R_{\mathrm{MPAC}}] = 0$. So, MPAC attains an upper bound on all feasible expected revenue.

Step 5. In §4.7, we complete the rest of the argument by bounding the revenue difference between MPAC and PAC. They have the same revenue until τ by definition, and so the revenue difference can be bounded by simply ignoring all revenue under PAC obtained after τ. Given that τ happens sufficiently late, using the bound for $\mathbf{E}[\tau]$ computed in Step 3, even this crude bound on the revenue loss of PAC when compared to MPAC and thus the DLP is tight enough to establish Theorem 4.1.

[6] It is worth emphasizing that an arbitrary DLP-based heuristic may not have the same corrective power. This result is discussed in more detail in a subsequent paper (Jasin and Kumar [7]).

TABLE 6. Example 2—Comparison of expected loss under different heuristics.

	PAC without re-solving			Periodic PAC with k re-solvings			Mid-point PAC		
k	Exp. loss	Std dev	%Loss	Exp. loss	Std dev	%Loss	Exp. loss	Std dev	%Loss
50	659	27	6.02	456	28	4.16	592	35	5.41
100	956	41	4.37	486	43	2.22	634	35	2.89
150	1,055	51	3.21	673	55	2.05	730	36	2.22
200	1,373	61	3.14	718	61	1.64	772	37	1.76
250	1,535	63	2.80	669	69	1.22	795	37	1.45
300	1,634	72	2.49	807	73	1.23	837	37	1.27
350	1,715	75	2.24	462	81	0.60	816	38	1.06
400	1,905	73	2.18	628	80	0.72	953	38	1.09
450	1,897	53	1.93	651	60	0.66	987	41	1.00
500	2,097	59	1.92	618	64	0.56	1,005	39	0.92

Before we proceed, we will introduce some notation. Let s and t be such that $ih \le s < t \le (i+1)h$ for some i, and let $\Im_s$ denote the accumulated information up to time s. Then, for any heuristic π,

$$N_{j,\pi}(s,t) = \text{amounts of allocated resources to customer type } j \text{ during } [s,t),$$

$$\delta_{j,\pi}(s,t) = N_{j,\pi}(s,t) - \mathbf{E}[N_{j,\pi}(s,t) \mid \Im_s], \quad \text{and}$$

$$\tilde{\Delta}_\pi(s,t) = \delta_{1,\pi}(s,t) + \delta_{2,\pi}(s,t).$$

We will suppress the superscript k and subscript π whenever this can be done without confusion.

4.3. Online PAC (OPAC). The definition of the OPAC heuristic, which we will establish is *almost* equivalent to PAC, is as follows:

(i) Start with $\ell = 0$.

(ii) At time ℓh, compute $\hat{Y}(\ell h)$ according to the formula below.

$$\hat{Y}_1(\ell h) = (1-\ell h)Y_1, \quad \text{and}$$

$$\hat{Y}_2(\ell h) = \left[(1-\ell h)Y_2 - \left(\tilde{\Delta}((\ell-1)h, \ell h) + \sum_{i=1}^{\ell-1} \tilde{\Delta}((i-1)h, ih)\frac{1-\ell h}{1-ih}\right)\right]^+,$$

where $Y = (Y_1, Y_2)$ is the optimal solution to $\mathbf{DLP}[C, \lambda]$.

(iii) For a customer type j arriving at time $t \in [\ell h, (\ell+1)h)$, grant the request with probability $p_j(\ell h) = \hat{Y}_j(\ell h)/[(1-\ell h)\lambda_j]$. Ignore the capacity constraint while granting the request.

(iv) Increment n and repeat until the end of the selling horizon.

The motivation behind formulas for $\hat{Y}(t)$ is that if the optimal basis for the DLP solved by PAC did not change in $[0,t]$, then $\hat{Y}(t)$ would give the solution to the DLP solved at time t explicitly. This will be made precise in the next subsection. Finally, it should be noted that the implementation of OPAC ignores the capacity constraint. That is, we allow the seller to collect revenue by selling "imaginary" products. All of this holds in the general case as well, except that the explicit formula is more involved.

4.4. Relationship between OPAC, PAC, and deviation correction. The following lemma identifies a condition that enables us to write the solution of the DLP solved by PAC at time t explicitly. Moreover, this explicit solution is identical to the formula used by OPAC while this condition holds. Finally, the formula implies the DLP has corrective power.

LEMMA 4.1. *Consider OPAC and fix a time $t \in [0,1)$. Define $Q = \min\{Y_2, \lambda_2 - Y_2\}$. If the following condition holds for any $s \in [0,t]$,*

Condition (†):

$$\left|\tilde{\Delta}(mh, s) + \sum_{i=1}^{m} \tilde{\Delta}((i-1)h, ih)\frac{1-t}{1-ih}\right| \le (1-s)Q - 1,$$

where m is such that $s \in [mh, (m+1)h)$, the solution to $\mathbf{DLP}[C(t), (1-t)\lambda]$ is given by $Y_1(t) = (1-t)\lambda_1$, and

$$Y_2(t) = (1-t)Y_2 - \left(\tilde{\Delta}(\ell h, t) + \sum_{i=1}^{\ell} \tilde{\Delta}((i-1)h, ih)\frac{1-t}{1-ih}\right),$$

where ℓ is such that $t \in [\ell h, (\ell+1)h)$. In addition, we have $\hat{Y}(s) = Y(s)$ for all $s \in [0,t]$.

Some comments are in order here. First, Lemma 4.1 tells us that *if* condition (†) is satisfied during $[0, t)$, then implementing OPAC during this period is "equivalent" to implementing PAC (because $\hat{Y}(t) = Y(t)$, which leads to the same allocation probabilities, i.e. $\hat{Y}_j(t)/[(1-t)\lambda_j] = Y_j(t)/[(1-t)\lambda_j]$). To be precise, if we apply either heuristics to the same realization of customer arrivals and the same realizations of the probabilistic allocation rule, then the sample paths realized under OPAC and PAC are identical as long as (†) is satisfied; i.e., $N_{\mathrm{PAC}}(0, t) = N_{\mathrm{OPAC}}(0, t)$. Since the key to the proof of Theorem 4.1 is to analyze what happens until (†) is violated (more on this in §§4.5 and 4.7), we can analyze the performance of PAC by analyzing the performance of OPAC, which is easier to analyze because $N_{\mathrm{OPAC}}(\cdot,\cdot)$ is an easily characterized binomial random variable (multinomial in the general case), whereas $N_{\mathrm{PAC}}(\cdot,\cdot)$ is not because of the capacity constraint.

The Proof of Lemma 4.1. The result can be proved in a straightforward way via induction. We will provide the basis of this induction argument in a way that lays out the insight behind the result. We start at time 0. By definition, $\hat{Y}(0) = Y(0)$. The allocation rule that results is "admit every type 1 customer and admit type 2 customers with probability $p_2(0) = \hat{Y}_2(0)/\lambda_2 = Y_2/\lambda_2$" until time h. The remaining capacity at time h is given by $C(h) = C - N_1(0, h) - N_2(0, h) = C - p_1(0)\lambda_1 h - p_2(0)\lambda_2 h - \tilde{\Delta}(0, h) = (1-h)C - \tilde{\Delta}(0, h)$. Note that if there had been no deviation from the expected capacity usage during $[0, h)$, we would have $C(h) = (1-h)C$. So, the capacity would have only been reduced at rate C per unit time.

The new DLP at time h is:

$$\begin{aligned} \text{maximize} \quad & r_1 x_1 + r_2 x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le C(h), \quad 0 \le x \le (1-h)\lambda. \end{aligned}$$

As long as $\tilde{\Delta}(0, h)$ is small enough in magnitude, i.e., $-(1-h)(\lambda_2 - Y_2) < \tilde{\Delta}(0, h) < (1-h)Y_2$, the structure of the LP solution does not change.[7] Note that if (†) is satisfied at $s = h$, then the previous condition is satisfied—it is weaker than (†). To be specific, we have: $Y_1(h) = (1-h)\lambda_1$ and $Y_2(h) = C(h) - (1-h)\lambda_1 = (1-h)Y_2 - \tilde{\Delta}(0, h)$. By definition, we have $\hat{Y}(h) = Y(h)$ and the new allocation probabilities are given by

$$p_1(h) = 1 \qquad \text{and} \qquad p_2(h) = \frac{Y_2(h)}{(1-h)\lambda_2} = \frac{Y_2}{\lambda_2} - \frac{\tilde{\Delta}(0, h)}{(1-h)\lambda_2}.$$

Now, consider the situation at time $2h$ when the policy is revised again. The allocation rule that results is "admit every type 1 customer and admit type 2 customers with probability $p_2(h)$" until time $2h$, and the remaining capacity at time $2h$ is given by

$$\begin{aligned} C(2h) &= C(h) - N_1(h, 2h) - N_2(h, 2h) \\ &= (1-h)C - \tilde{\Delta}(0, h) - p_1(h)\lambda_1 h - p_2(h)\lambda_2 h - \tilde{\Delta}(h, 2h) \\ &= (1-2h)C - \tilde{\Delta}(h, 2h) - \frac{1-2h}{1-h}\tilde{\Delta}(0, h). \end{aligned}$$

If there had been no deviation from the expected capacity usage during $[h, 2h)$, we would have $C(2h) = (1-2h)C - ((1-2h)/(1-h))\tilde{\Delta}(0, h)$. Observe that the term $\tilde{\Delta}(0, h)$ is multiplied by constant $(1-2h)/(1-h)$. This suggests that the deviation in capacity usage during $[0, h)$ has been corrected by a factor of $h/(1-h)$ under PAC and OPAC. This is the observation: the LP corrects the accumulated errors at a uniform rate over the remaining horizon. So, with $1-h$ left to go, the error is corrected at rate $1/1-h$ over the period of length h. This insight holds in the general case.

Arguing as before, as long as $\tilde{\Delta}(h, 2h) + ((1-2h)/(1-h))\tilde{\Delta}(0, h)$ is small enough in magnitude, i.e.,

$$-(1-2h)(\lambda_2 - Y_2) < \tilde{\Delta}(h, 2h) + \frac{1-2h}{1-h}\tilde{\Delta}(0, h) < (1-2h)Y_2$$

(once again, the condition above is weaker than (†)), the structure of the LP solution does not change. To be specific, we have: $Y_1(2h) = (1-2h)\lambda_1$ and $Y_2(2h) = (1-2h)Y_2 - \tilde{\Delta}(h, 2h) - ((1-2h)/(1-h))\tilde{\Delta}(0, h)$. By definition of $\hat{Y}(\cdot)$, we again have $\hat{Y}(2h) = Y(2h)$. The new allocation probabilities are then given by

$$p_1(2h) = 1 \qquad \text{and} \qquad p_2(2h) = \frac{Y_2(2h)}{(1-2h)\lambda_2} = \frac{Y_2}{\lambda_2} - \frac{\tilde{\Delta}(h, 2h)}{(1-2h)\lambda_2} - \frac{\tilde{\Delta}(0, h)}{(1-h)\lambda_2}.$$

[7] The right inequality ensures that all of type 1 demand can be met, and the left ensures that the remaining capacity is fully exhausted, as is to be expected from the DLP's solution.

Applying the new admission policy, "admits every type 1 customer and admits type 2 customers with probability $p_2(2h)$," until time $3h$, it is not difficult to see that the remaining capacity at time $3h$ is given by $C(3h) = C(2h) - N_1(2h, 3h) - N_2(2h, 3h) = (1-3h)C - \tilde{\Delta}(2h, 3h) - ((1-3h)/(1-2h))\tilde{\Delta}(h, 2h) - ((1-3h)/(1-h))\tilde{\Delta}(0, h)$, and the same argument can be continued inductively to complete the proof.

Remark. In every step, (†) provides a convenient sufficient condition for the actual condition that guarantees that the basis does not change. (Intuitively, this means that we have sufficient capacity to accept all type 1 customers.) In the general case, an additional condition is necessary to ensure that those resources whose capacity is not fully exhausted in the optimal allocation of the static DLP are also not exhausted in the DLP solved by PAC at various times. Other than that, and the more general formula for $\hat{Y}(t)$, the proof remains essentially the same. The important thing to note here is that the deviation from the expected capacity usage during $[0, h)$ has been further corrected by a factor of $h/(1-h)$ and the deviation during $[h, 2h)$ has been corrected by a factor of $h/(1-2h)$. The successively larger correction of past deviations in future periods is the first key idea toward the proof of Theorem 4.1. The next subsection will discuss how good this correction actually is.

4.5. Hitting time τ. Define τ to be the minimum of 1 and the first time t such that condition (†) is violated under OPAC. Intuitively, τ can be interpreted as the time when the size of cumulative deviations has became sufficiently large that the LP will change its basis, that is, the time beyond which we will not be able to write the solution of the LP explicitly as we did above. The purpose of this subsection is to compute an upper bound for $\mathbf{E}[1-\tau]$. To do this, we turn to the asymptotic set-up of §3 and denote this hitting time in the kth system by τ^k.

Lemma 4.2. *Consider periodic OPAC with period $h > 0$. There exist positive constants ρ and ρ' independent of $k \geq 1$ and h such that*

$$\mathbf{E}[1-\tau^k] \leq \frac{\rho}{k} + \rho'\sqrt{\frac{h}{k}}.$$

The Proof of Lemma 4.2. We proceed in several steps.

Step 1. Since $\tau^k \in [0, 1]$ we have

$$\mathbf{E}[\tau^k] = 1 - \int_0^1 P(\tau^k < t)\, dt.$$

Define $\theta^k = 2/(kQ)$. We will split the integral into two; $\int_0^{1-\theta^k} P(\tau^k < t)\, dt$ and $\int_{1-\theta^k}^1 P(\tau^k < t)\, dt$. The latter can be bounded trivially by θ^k. The main thrust of the proof then is to find an upper bound for the term $P(\tau^k < t)$ for $t \in [0, 1-\theta^k)$.

Step 2. Since τ^k is defined by the violation of (†), it is convenient to define

$$E^k(t) = \tilde{\Delta}^k(nh, t) + \sum_{j=1}^{n} \frac{1-t}{1-jh} \tilde{\Delta}^k((j-1)h, jh), \quad nh \leq t < (n+1)h.$$

Fix $t \in [0, 1)$. Define $L(t)$ to be the unique integer n such that $nh \leq t < (n+1)h$. Now, we define two stochastic processes $X^k(s)$ and $\hat{X}^k(s)$ for $s \in [0, t]$ as follows:

$$X^k(s) = \frac{E^k(s)}{1-s} = \frac{\tilde{\Delta}^k(nh, s)}{1-s} + \sum_{j=1}^{n} \frac{\tilde{\Delta}^k((j-1)h, jh)}{1-jh}, \quad \text{for all } s \in [nh, (n+1)h),$$

$$\hat{X}^k(s) = \begin{cases} \dfrac{\tilde{\Delta}^k(nh, s)}{1-(n+1)h} + \displaystyle\sum_{j=1}^{n} \frac{\tilde{\Delta}^k((j-1)h, jh)}{1-jh} & \text{for all } s \in [nh, (n+1)h), \quad n < L(t), \\[2ex] \dfrac{\tilde{\Delta}^k(L(t)h, s)}{1-t} + \displaystyle\sum_{j=1}^{L(t)} \frac{\tilde{\Delta}^k((j-1)h, jh)}{1-jh} & \text{for all } s \in [L(t)h, t]. \end{cases}$$

Since $t \leq 1-\theta^k$ implies $kQ(1-t) - 1 \geq kQ(1-t)/2$, for any nonnegative function $v(\cdot,\cdot)$ integrable with respect to its second argument, we have:

$$\begin{aligned} P(\tau^k < t) &= P(|E^k(s)| \geq kQ(1-s) - 1 \text{ for some } s \in [0, t)) \\ &\leq P(|E^k(s)| \geq kQ(1-s)/2 \text{ for some } s \in [0, t)) \\ &= P(|X^k(s)| \geq kQ/2 \text{ for some } s \in [0, t)) \end{aligned}$$

$$
\begin{aligned}
&= P\Big(\sup_{0\le s\le t} |X^k(s)| \ge kQ/2\Big)\\
&\le P\Big(\sup_{0\le s\le t} |\hat{X}^k(s)| \ge kQ/2\Big)\\
&\le P\Big(\sup_{0\le s\le t} v(k,t)(1-t)|\hat{X}^k(s)| \ge v(k,t)kQ(1-t)/2\Big)\\
&= P\Big(\sup_{0\le s\le t} \exp[v(k,t)(1-t)|\hat{X}^k(s)|] \ge \exp[v(k,t)kQ(1-t)/2]\Big).
\end{aligned}
$$

REMARK. The second inequality follows because $\sup_{0\le s\le t} |X_i^k(s)| \ge kQ/2$ implies $\sup_{0\le s\le t} |\hat{X}_i^k(s)| \ge kQ/2$. Note that $|X_i^k(s)| \ge kQ/2$ does not always imply $|\hat{X}_i^k(s)| \ge kQ/2$.

Step 3. The reason we are interested in process $\hat{X}^k(s)$ is because, for a fixed t, $\{\hat{X}^k(s)\}_{0\le s\le t}$ is a Martingale with respect to filtration $\{\Im_s\}$, which implies that $\{\exp[v(k,t)(1-t)|\hat{X}^k(s)|]\}_{0\le s\le t}$ is a sub-Martingale (for all $v(k,t) > 0$) with respect to the same filtration. So, we can apply Doob's sub-Martingale inequality to the last term in Step 2 and get

$$
\begin{aligned}
&P\Big(\sup_{0\le s\le t} \exp[v(k,t)(1-t)|\hat{X}^k(s)|] \ge \exp[v(k,t)kQ(1-t)/2]\Big)\\
&\quad\le \min\left\{1, \frac{\mathbf{E}(\exp[v(k,t)(1-t)|\hat{X}^k(t)|])}{\exp[v(k,t)kQ(1-t)/2]}\right\}\\
&\quad= \min\left\{1, \frac{\mathbf{E}(\exp[v(k,t)|E^k(t)|])}{\exp[v(k,t)kQ(1-t)/2]}\right\}.
\end{aligned}
$$

Step 4. The last nontrivial task is to compute the term $\mathbf{E}(\exp[v(k,t)|E^k(t)|])$. Since $E^k(t)$ is a sum of $\tilde{\Delta}^k(\cdot,\cdot)$'s, and since $N_j^k(s,t) = \text{Binomial}(\Lambda_j^k(s,t), p_j^k(nh))$, $nh \le s < t \le (n+1)h$, the term $\mathbf{E}(\exp[v(k,t)|E^k(t)|])$ can be computed using the moment-generating functions of binomial and Poisson random variables. We defer the details of such computation to Appendix C.2, where we prove the result in the general multinomial case. Finally, using an appropriate choice of $v(k,t)$ (see Appendix C.5), it can be shown that $\mathbf{E}[1-\tau^k] \le \rho/k + \rho'\sqrt{h/k}$ for some constants ρ and ρ' independent of h and k. This completes the proof.

4.6. Modified PAC (MPAC). Define a relaxation of PAC labeled MPAC that works as follows. MPAC follows PAC, i.e., it re-solves the LP at time $h, 2h, \ldots,$ and interprets its solution as allocation probabilities until τ. At τ, MPAC re-solves the LP for one *last* time and applies the resulting probabilities throughout $[\tau, 1)$ ignoring the capacity constraint. Alternatively, MPAC can be seen as a relaxation of OPAC. That is, it follows OPAC until τ and then re-solves the LP once-and-for-all at time τ and applies the resulting allocation probabilities throughout the rest of the selling horizon by ignoring the capacity constraint.

We remark here that although hitting time τ in §4.5 is defined with respect to OPAC, there should be no confusion in adapting it to either PAC or MPAC. Per our discussion in §4.4, under the same realization of customer arrivals and the same realizations of the probabilistic allocation rule, the sample-path induced by OPAC during $[0,\tau)$ might as well be considered as the sample-path induced by either PAC or MPAC during the same period.

Let $R_{\text{MPAC}}(s,t)$ denote the revenue earned under MPAC during interval $[s,t)$. Now, a key result, which is proven in generality in Appendix C.3 (see Lemma C.2), is that

$$V_{\text{DLP}} - \mathbf{E}[R_{\text{MPAC}}(0,1)] = 0.$$

That is, in the idealized situation where the capacity constraint can be ignored while implementing the policy, the upper bound on revenue given by the DLP can be attained in expectation using MPAC.

4.7. Completing the proof. Going back to the asymptotic setting, by the equivalence established in Lemma 4.1, we have $R^k_{\text{OPAC}}(0,\tau^k) = R^k_{\text{PAC}}(0,\tau^k) = R^k_{\text{MPAC}}(0,\tau^k)$ for all $k \ge 1$. So,

$$
\begin{aligned}
V^k_{\text{DLP}} - \mathbf{E}[R^k_{\text{PAC}}(0,1)] &= V^k_{\text{DLP}} - \mathbf{E}[R^k_{\text{MPAC}}(0,1)] + \mathbf{E}[R^k_{\text{MPAC}}(0,1)] - \mathbf{E}[R^k_{\text{PAC}}(0,1)]\\
&= \mathbf{E}[R^k_{\text{MPAC}}(\tau^k,1)] - \mathbf{E}[R^k_{\text{PAC}}(\tau^k,1)]\\
&\le \mathbf{E}[R^k_{\text{MPAC}}(\tau^k,1)].
\end{aligned}
$$

That is, the expected loss in revenue under PAC when compared to the LP upper bound can be bounded by the expected revenue earned by an idealized modification of PAC after τ^k. In particular, we can bound $\mathbf{E}[R^k_{\mathrm{MPAC}}(\tau^k, 1)]$ crudely by $\mathbf{E}[r_1\Lambda^k_1(\tau^k, 1) + r_2\Lambda^k_2(\tau^k, 1)] = O(k\mathbf{E}[1-\tau^k])$, where we assume every arriving customer is allocated a product under MPAC. This allows us to simply ignore all requests coming after τ^k. By Lemma 4.2, we conclude that $V^k_{\mathrm{DLP}} - \mathbf{E}[R^k_{\mathrm{PAC}}] = O(1+\sqrt{kh})$. This completes the proof of Theorem 4.1.

5. General results. This section contains the main results of the paper. We start by giving the asymptotic result for periodic and mid-point PAC in §5.1 and then we provide the most general form in §5.2. An outline for the proof of the main theorem is given in §5.3.

5.1. Two re-solving schedules with bounded asymptotic revenue loss. We begin by stating the result for periodic PAC, established for the single resource case in Theorem 4.1, in the general setting of §2.

Theorem 5.1. *Consider periodic PAC with period $h > 0$. There exists positive constants ρ, $\hat{\rho}$, and ρ' independent of k such that*

$$V^k_{DLP} - \mathbf{E}[R^k_{PAC}] \le \rho + \hat{\rho}\sqrt{kh} < \rho + \hat{\rho}\sqrt{k/M} \quad \textit{for all } k \in \mathbb{Z}^+. \tag{5}$$

In particular, choosing $M = k$ yields

$$V^k_{DLP} - \mathbf{E}[R^k_{PAC}] \le \rho + \hat{\rho}\sqrt{kh} < \rho' \quad \textit{for all } k \in \mathbb{Z}^+. \tag{6}$$

Theorem 5.1 gives a bound on the asymptotic loss in revenue for any periodic PAC. Moreover, it quantifies the trade-off between the performance guarantee and the re-solving frequency. The expected revenue loss is no worse than inversely proportional to the square-root of re-solving frequency. That is, there is a diminishing return to increasingly frequent re-solving. Also, for a problem with size k, the bound in (5) suggests that re-solving more than k times will have no impact on the scale of the revenue loss. This confirms the simulation result in §3. If we re-solve the DLP and update the offer every time a new customer arrives, then the expected revenue loss of the PAC heuristic is independent of the size of the problem. In particular, this implies for the case without customer choice that the hindsight (the bound used in Reiman and Wang [12]), the optimal, and the DLP upper bound are within $O(1)$ from each other, and so are asymptotically equivalent.

Of course, re-solving every time a customer arrives is computationally wasteful. In the early part of the selling period, we can afford to be a bit "off" from the optimal policy because we have enough time to correct for deviations. As we get closer to the end of the selling horizon, our ability to correct for deviations decreases, and so we need to make sure that the deviations do not get too big. This is the underlying idea behind mid-point PAC.

Theorem 5.2. *Consider mid-point PAC. There exists a positive constant ρ independent of k such that*

$$V^k_{DLP} - \mathbf{E}[R^k_{PAC}] \le \rho \quad \textit{for all } k \in \mathbb{Z}^+. \tag{7}$$

Theorem 5.2 shows that one can still achieve $O(1)$ revenue loss with only $O(\log_2 k)$ re-solves. This is to be compared with periodic re-solves in Theorem 5.1, where $O(k)$ re-solves are needed to guarantee $O(1)$ loss in performance. An intelligent choice of re-solving times gives us roughly the same revenue performance as periodic re-solvings while saving significant computational effort. However, there is a caveat. As observed in the simulation result in §3, the constant bound in (7) can be much larger than the constant bound in (6). Also, for moderate k, a slowly growing function such as $O(\log k)$ would be a better description for the bound on the expected revenue loss than the asymptotic bound $O(1)$. Hence, the decision whether to use mid-point PAC or periodic PAC must be handled with caution for problems of moderate size.

5.2. A general bound on asymptotic revenue loss for PAC heuristic. Both Theorems 5.1 and 5.2 follow immediately from a more general result below. Let ξ_j be a constant such that $A_{ij} < \xi_j$ for all i (this is possible because we assume that A_{ij} is a bounded random variable), and let $\xi_{\max} = \max_j \xi_j$. Then,

Theorem 5.3. *There exist positive constants ρ, $\hat{\rho}$, and ρ' independent of k such that, for all choice of re-solve times $\Gamma^k = \{t^k_l\colon l = 1, \ldots, M^k\}$ and for any function $v(\cdot,\cdot)$ such that $0 < v(k,t) \le \min\{1, 1/\xi_{\max}\}$ for all $k \in \mathbb{Z}^+$ and $t \in [0,1)$, we have*:

$$V^k_{DLP} - \mathbf{E}[R^k_{PAC}] \le \rho + \hat{\rho}k\int_0^1 \min\{1, \rho' F(k,t)\}\, dt, \tag{8}$$

where $F(k,t) = \exp(kv(k,t)^2 G(k,t) - kv(k,t)(1-t))$ and

$$G(k,t) = (t - t^k_{i-1}) + \sum_{j=1}^{i-1}(t^k_j - t^k_{j-1})\frac{(1-t)^2}{(1-t^k_j)^2}, \qquad t^k_{i-1} \le t < t^k_i, \quad 0 \le t \le 1.$$

One could use Theorem 5.3 to evaluate any choice of re-solving times with regard to the trade-off between re-solving frequency and the quality of the performance guarantee. With suitable weights on both sides, an "optimal" scheme can be constructed. Moreover, by choosing a re-solving strategy with some special structure, the upper bound formula in Theorem 5.3 can be made to take a much simpler form which makes it more amenable to optimization. Indeed, this is what is done in Theorems 5.1 and 5.2.

For the result in Theorem 5.3 to be useful, the function $v(\cdot,\cdot)$ must be adjusted to match the re-solve times Γ^k. The explicit choice of $v(\cdot,\cdot)$ used for periodic PAC in Theorem 5.1 and mid-point PAC in Theorem 5.2 can be found in Appendix C.

One corollary of Theorem 5.3 is worth stating.

Corollary 5.1. *There exists positive constants ρ and $\hat{\rho}$ independent of k such that for any choice of re-solving times $\Gamma^k = \{t_l^k\colon l = 1, \ldots, M^k\}$, we have*:

$$V_{DLP}^k - \mathbf{E}[R_{PAC}^k(0,1)] \le \rho + \hat{\rho}\sqrt{k} \quad \text{for all } k \in \mathbb{Z}^+. \tag{9}$$

If we did not re-solve at all, then the bound on the revenue loss would be the expression on the right-hand side above. Corollary 5.1 says that, asymptotically speaking, implementing PAC with multiple re-solves should not result in a worse performance bound than that of PAC without re-solve. Although this looks obvious, as will be briefly discussed in §7, depending on how the DLP solution is used to construct the prevailing heuristic control, frequent re-solves may negatively impact performance. Corollary 5.1 guarantees that this is not the case with any PAC heuristic.

5.3. Outline of proof of Theorem 5.3. The following outline will follow the five steps introduced in §4 closely. Below we will simply state the key lemmas and defer all the proofs in the appendix. We will also point out to relevant sections in the appendix where the details can be found. Note the statement of lemmas below may contain notation not previously introduced in the paper. The reader is advised to use Table 7 at the end of the paper to keep track of notation.

Step 1. As in §4.3, we begin by defining a relaxation of PAC that we call OPAC. For a given set of re-solving times $\Gamma^k = \{t_1^k, \ldots, t_{M^k}^k\}$, in the kth system, the implementation of OPAC is as follows:

(i) Start with $l = 0$.

(ii) At time t_l^k, compute $\hat{Y}^k(t_l^k)$ according to the prescription in Appendix B.3.

Table 7. Notation.

	Key notation
$\Im_s$	Accumulated information up to time s
$N_{j,\pi}(s,t)$	Number of times offer j is presented during $[s,t)$ under heuristic π
$\xi_\pi^j(s,t)$	Total capacity usage of offer j during $[s,t)$ under heuristic π
$\delta_{j,\pi}(s,t)$	$= N_{j,\pi}(s,t) - \mathbf{E}[N_{j,\pi}(s,t) \mid \Im_s]$
$\Delta_\pi^j(s,t)$	$= \xi_\pi^j(s,t) - \mathbf{E}[\xi_\pi^j(s,t) \mid \Im_s]$
$\tilde{\Delta}_\pi(s,t)$	$= \sum_j \Delta_\pi^j(s,t)$
$L(s)$	The smallest integer L such that $t_L \le s < t_{L+1}$
$C(t)$	Remaining capacity at time t (we write: $C(0) = C$)
λ	Customer arrival rate vector
$Y(t)$	Optimal solution to $\mathbf{DLP}[C(t), \lambda(1-t)]$ (we write: $Y(0) = Y$)
$\hat{Y}(t)$	Used for OPAC heuristic, defined in Appendix B.3
S_q	$= \{j\colon q(j) = q\}$
$R_\pi(s,t)$	Total revenue earned during $[s,t)$ under heuristic π
V_{DLP}	Optimal value of $\mathbf{DLP}[C, \lambda]$
	Some matrix-related notations (for other matrix notations, see Appendix B.1)
$\xi_{\max}$	Uniform upper bound for the realization of A_{ij}
H	The first $\lvert C_B\rvert$ columns of matrix M^{-1}
Ψ	Maximum absolute value of elements of $P_{N,y}H$, H, $-\bar{A}_{N,y}H$, $\bar{A}_B$, y, and I
Q	Minimum value of elements of $\lambda_N - P_N Y$, $\bar{A}_{B,y}Y_y$, Y_y, and $C_N - \bar{A}_N Y$
$\mathbf{\Psi}, \mathbf{Q}, \boldsymbol{\xi}_{\max}$	Matrix (vector) of appropiate size whose components are all equal to Ψ, Q, $\xi_{\max}$, respectively
$\tilde{H}$	$= [H, O]$, O is a zero matrix with appropiate size
Z	$= [-\bar{A}_{N,y}H, I]$, I is an identity matrix with appropriate size
N, B	When used as a subscript, denotes the submatrix (or vector) corresponding to binding and nonbinding constraints, respectively (see Appendix B.1)

(iii) For a customer type q arriving at time $t \in [t_l^k, t_{l+1}^k)$, present offer $j \in S_q$ with probability $p^k_{j,\text{OPAC}}(t_l^k) = \hat{Y}_j^k(t_l^k)/[(1-t_l^k)k\lambda_q]$.

(iv) Increment l and repeat at the next t_l^k until $l = M^k + 1$.

A couple of remarks regarding OPAC that we made in the special case of §4 are worth repeating here in the general case for completeness. The definition of $\hat{Y}^k(t)$ is much more complicated, of course, in terms of the algebra involved. But conceptually, it is simply providing an explicit formula for the solution of the re-solved LP assuming that the optimal basis did not change. As in §4.3, we never check capacity with OPAC and the customer is allowed to purchase an offer for which we do not have sufficient resources. Indeed, OPAC is a *relaxation* of PAC. Using OPAC simplifies obtaining a bound on $\mathbf{E}[\tau^k]$, where τ^k is the first time the basis of the LP solution changes in the kth system.

To proceed, we need to define a few random variables. Suppose that $0 \le t_l^k \le s < t \le t_{l+1}^k < 1$, and let $\Im_s$ denote the information acquired up to time s under PAC. We define:

$$\{N^k_{j,\text{OPAC}}(s,t)\}_{j\in S_q} = \text{multinomial}(\Lambda_q^k(s,t), \{p^k_{j,\text{OPAC}}(t_l^k)\}_{j\in S_q}),$$

$$\zeta^{j,k}_{\text{OPAC}}(s,t) = \sum_{u=1}^{N^k_{j,\text{OPAC}}(s,t)} A^{j,u},$$

$$\delta^k_{j,\text{OPAC}}(s,t) = N^k_{j,\text{OPAC}}(s,t) - \mathbf{E}[N^k_{j,\text{OPAC}}(s,t) \mid \Im_s],$$

$$\Delta^{j,k}_{\text{OPAC}}(s,t) = \zeta^{j,k}_{\text{OPAC}}(s,t) - \mathbf{E}[\zeta^{j,k}_{\text{OPAC}}(s,t) \mid \Im_s],$$

$$\tilde{\Delta}^k_{\text{OPAC}}(s,t) = \sum_j \Delta^{j,k}_{\text{OPAC}}(s,t),$$

where $A^{j,u}$, $u = 1, 2, \ldots,$ are each i.i.d. realizations of A^j. By definition, $N^k_{j,\text{OPAC}}(s,t)$ is the number of times offer j is presented to customer type $q(j)$ under allocation probability $p^k_{j,\text{OPAC}}(t_l^k)$ during time interval $[s,t)$, and $\zeta^{j,k}_{\text{OPAC}}(s,t)$ is its corresponding capacity usage vector (we use the superscript j in ζ^k_{OPAC} to denote the corresponding column vector). Note that, for each q, $\{N^k_{j,\text{OPAC}}(s,t)\}_{j:q(j)=q}$ has a joint multinomial distribution. That is, conditioned on $\Lambda_q^k(s,t) = \Lambda$, we have

$$P(N^k_{j,\text{OPAC}}(s,t) = n_j, j \in S_q) = \frac{\Lambda!}{n_0! \prod_{j\in S_q} n_j!}\left(1 - \sum_{j\in S_q} p^k_{j,\text{OPAC}}(t_l^k)\right)^{n_0} \prod_{j\in S_q} (p^k_{j,\text{OPAC}}(t_l^k))^{n_j},$$

where $n_0 = \Lambda - \sum_{j\in S_q} n_j$ is the number of times we decide not to present anything.

Step 2. The following lemma from Appendix B.4 is the analog of Lemma 4.1. Here, we need an additional condition, beyond the (†) of Lemma 4.1, to ensure that partial allocations are preserved, a situation that does not arise in the single-resource, two-product type case.

Lemma B.4. *Fix a set of re-solving times* $\Gamma^k = \{t_1^k, \ldots, t_{M^k}^k\}$, *and fix a time* $t \in [0,1)$. *Suppose the following conditions hold throughout time interval* $[0,t]$,

Condition (†):

$$\Psi\left|\tilde{\Delta}^k_{OPAC}(t^k_{L(s)}, s) + \sum_{i=1}^{L(s)} \tilde{\Delta}^k_{OPAC}(t^k_{i-1}, t^k_i)\frac{1-s}{1-t_i^k}\right| < (1-s)k\mathbf{Q} - \xi_{\mathbf{max}};$$

Condition (‡):

$$\Psi|\tilde{\Delta}^k_{OPAC}(s)| < k\mathbf{Q} - \xi_{\mathbf{max}},$$

where Ψ, $\mathbf{Q}$, *and* $\xi_{\mathbf{max}}$ *are constant matrices (or vectors) defined in Appendix B.4. Then we have*: (i) $C^k_{OPAC}(s) \ge \xi_{\mathbf{max}}$ *for all time* $s \in [0,t]$, (ii) *the optimal solution to* $\mathbf{DLP}[C^k_{OPAC}(t), (1-t)k\lambda]$ *is given by* $Y^k_{OPAC}(t) = \hat{Y}^k(t)$, *and* (iii) *total capacity usage under the OPAC heuristic up to time* t *is equal to*

$$\sum_j \zeta^{j,k}_{OPAC}(t) = \bar{A}kYt + \tilde{\Delta}^k_{OPAC}(t^k_{L(t)}, t) + \sum_{i=1}^{L(t)}\left(I - \bar{A}_y\tilde{H}\frac{t-t_i^k}{1-t_i^k}\right)\tilde{\Delta}^k_{OPAC}(t^k_{i-1}, t^k_i),$$

where $\tilde{H} = [H, O]$, *with* O *being a zero matrix with an appropriate size.*

The proof of the above lemma can be found in Appendix B.4. Let $N^k_{j,\,\mathrm{PAC}}(s,t)$ denote the number of times offer j is presented to customer type $q(j)$ during time interval $[s,t)$ under PAC, and let $\zeta^{j,k}_{\mathrm{PAC}}(s,t)$ be its corresponding capacity usage. By abuse of notation, we will write $N^k_{j,\,\mathrm{OPAC}}(t)$ to denote the number of times offer j is presented during time interval $[0,t)$. Similarly, we will also write $\zeta^{j,k}_{\mathrm{OPAC}}(t)$, $\delta^k_{j,\,\mathrm{OPAC}}(t)$, $\Delta^{j,k}_{\mathrm{OPAC}}(t)$, $\tilde{\Delta}^k_{\mathrm{OPAC}}(t)$, with the corresponding restriction to $[0,t)$. For example, for $t^k_L < t < t^k_{L+1}$, we have: $\delta^k_{\mathrm{OPAC}}(t) = \delta^k_{\mathrm{OPAC}}(t^k_L, t) + \sum_{0\le i\le L-1} \delta^k_{\mathrm{OPAC}}(t^k_i, t^k_{i+1})$. Lemma B.4 implies that if condition (†) and (‡) are satisfied up to time t, then

$$N^k_{j,\,\mathrm{PAC}}(s) = N^k_{j,\,\mathrm{OPAC}}(s) \qquad \text{and} \qquad \zeta^{j,k}_{\mathrm{PAC}}(s) = \zeta^{j,k}_{\mathrm{OPAC}}(s)$$

for all j and $0\le s\le t$. That is, if we apply either heuristic to the same realization of customer arrivals, the same outcome of the probabilistic allocation rule, and the same realization of capacity usage, then the sample-path realizations generated under PAC and the OPAC heuristics are identical as long as (†) and (‡) are satisfied. Now, the term $\tilde{\Delta}^k_{\mathrm{OPAC}}(s,t)$ can be interpreted as the size of error (deviation from expected capacity usage) accumulated during time interval $[s,t)$. The weighted sum $\tilde{\Delta}^k_{\mathrm{OPAC}}(t^k_{L(s)}, s) + \sum_{i=1}^{L(s)} \tilde{\Delta}^k_{\mathrm{OPAC}}(t^k_{i-1}, t^k_i)((1-s)/(1-t^k_i))$ can be roughly interpreted as the size of accumulated error under the OPAC heuristic during time interval $[0,s)$. Note that the cumulative error during time interval $[0,s)$ is *not* $\tilde{\Delta}^k_{\mathrm{OPAC}}(t^k_{L(s)}, s) + \sum_{i=1}^{L(s)} \tilde{\Delta}^k_{\mathrm{OPAC}}(t^k_{i-1}, t^k_i)$. Rather, it is a weighted sum of them. Consider, as an example, the term $\tilde{\Delta}^k_{\mathrm{OPAC}}(0, t^k_1)$. The weight of this term at time $s > t^k_1$ is equal to $(1-s)/(1-t^k_1)$. This is reminiscent of Lemma 4.1 where the same "corrective power" was demonstrated in the special case. In the general case too, OPAC (and equivalently PAC) attempts to correct the cumulative deviation at a uniform rate over the remaining horizon, resulting in the formula that we see.

Step 3. Define a hitting time τ^k as the minimum of 1 and the first time $t\in[0,1)$, when either condition (†) or (‡) is violated in the kth system. Note that, by definition, τ^k is a bounded stopping time. τ^k can be interpreted as the time beyond which we can no longer guarantee that the structure of the optimal solution to the perturbed DLP is not significantly changed. That is, the size of cumulative errors (perturbation) may have become sufficiently large that the formula in Appendix B.2 for the solution of the perturbed DLP no longer applies. Alternatively, τ^k can be interpreted as the time when total cumulative errors have became too big for the heuristic to get a good performance without immediate re-solving. We define hitting time τ^k as the first time when either condition (†) or (‡) is violated in the kth system and then compute a bound for $\mathbf{E}[\tau]$. This is done in Appendix C.2. We state the key lemma, which is the analog of Lemma 4.2 below.

Lemma C.1. *Suppose that $k > 2\xi_{\max}/Q$. Then, there exist constants ρ and $\hat{\rho}$ independent of k such that, for all choices of re-solving times $\Gamma^k = \{t^k_l\colon l=1,\ldots,M^k\}$, and for all $0 < v(k,t) \le \min\{1, 1/\xi_{\max}\}$, integrable in the second argument, we have*:

$$\mathbf{E}[1-\tau^k] \le \rho k^{-1} + \int_0^{1-\theta^k} \min\{1, \hat{\rho}F(k,t)\}\,dt,$$

where $\theta^k = 2\xi_{\max}/(kQ)$, $F(k,t) = \exp[kv(k,t)^2 G(k,t) - kv(k,t)(1-t)]$ and

$$G(k,t) = (t - t^k_{i-1}) + \sum_{j=1}^{i-1} (t^k_j - t^k_{j-1}) \frac{(1-t)^2}{(1-t^k_j)^2}, \qquad t^k_{i-1} \le t < t^k_i,\quad 0\le t\le 1.$$

Step 4. Consider yet another relaxation of PAC, named MPAC, which works as follows: we implement the same allocation rule as in PAC and keep re-solving the DLP at times $t^k_1, t^k_2, \ldots,$ until we hit τ^k. When we hit τ^k, we re-solve the DLP for the last time and implement the resulting allocation rule until the end of the activity horizon without paying attention to the capacity constraints. That is, MPAC is identical to PAC up to time τ^k. At time τ^k, while PAC may or may not re-solve the DLP, depending on whether $\tau^k\in\Gamma^k$, MPAC prescribes that we re-solve the DLP and cancel any scheduled re-solving times that come afterward. Another difference is that while PAC always meets the capacity constraints, MPAC only does so up until time τ^k. Beyond τ^k, MPAC operates as if we have infinite capacity resources. During $[\tau^k, 1)$, MPAC may present offers that cannot be satisfied in the real system because of capacity constraints. However, we count the resulting revenue anyway under MPAC.

Not surprisingly, this significant relaxation yields revenue that is the same as the LP upper bound. To be precise, we have:

Lemma C.2. *Suppose that $k > 2\xi_{\max}/Q$. Then, $V^k_{DLP} - \mathbf{E}[R^k_{MPAC}] = 0$.*

The proof of Lemma C.2 can be found in Appendix C.3.

Step 5. In the final step, we argue that even if we ignore the revenue collected after time τ^k by PAC, analogous to what we did in §4.7, we would still get a bound on the expected revenue loss that is tight enough to establish Theorem 5.3. The key lemma to complete the proof is stated below and proved in Appendix C.4.

LEMMA C.3. *Suppose that $k > 2\xi_{\max}/Q$. Then there exists a constant $\hat{\rho}$ independent of k such that*

$$V_{DLP}^k - \mathbf{E}[R_{PAC}^k] \le k\hat{\rho}\mathbf{E}[1-\tau^k].$$

This completes the proof of Theorem 5.3 using Lemma C.1 in Step 3.

6. Re-solving at endogenously determined times. In §5, we discussed the performance of PAC with multiple re-solvings when the choice of re-solving times is determined exogenously at time zero. A natural question that follows is, what if we pick re-solving times endogeneously? One would expect that using re-solve times that are endogenously determined in a clever way would do better than the exogenous case. Indeed, the underlying arguments in the proof of Theorem 5.3 can be used to prove the following theorem.

THEOREM 6.1. *For each M, there exist positive constants $\rho(M)$ and $\hat{\rho}(M)$ independent of k, and a set of stopping times $0 < \tau_1^k < \tau_2^k < \cdots < \tau_M^k < 1$, such that PAC with re-solving at these times has the following performance guarantee*:

$$V_{DLP}^k - \mathbf{E}[R_{PAC}^k] \le \rho(M) + \hat{\rho}(M)k^{(1/2^{M+1})}.$$

Using $M = 1$ in Theorem 6.1 tells us that a single re-solve is enough to reduce the order of expected revenue loss from $O(\sqrt{k})$ to $O(k^{1/4})$. This result is similar to that of Reiman and Wang [12]. The proof of Theorem 6.1 does contain the explicit construction of the stopping times. So re-solving at endogenously determined times is at least theoretically possible. The choice of whether to use an exogeneous or endogenous re-solving schedule might depend on prevailing constraints and objectives.

7. Future research. We have made quite a few assumptions in our model and restricted attention to a single class of heuristics that we have called PAC. This is sufficient to achieve our goal for this paper—to show that a simple heuristic, based on re-solving the certainty equivalent optimization problem and properly implemented, can have excellent performance in a very general class of problems. Moreover, we have quantified the trade-off between the re-solving frequency and the performance guarantee in a general way. Despite the limitations of our model, we believe that our results provide a justification for, or at least suggest the viability of, the use of re-solving based heuristics for other revenue management settings. (A final note regarding nondegeneracy of the LP. Mathematical generality is indeed limited by Assumption 2.1. Unfortunately, the techniques in this paper do not allow us to relax this assumption. We leave this challenging extension to future work, perhaps by others.) We intend to explore the following issues in future work.

(i) Nonhomogenous arrival processes and estimation error: In our model, all arrival rates are assumed to be known and constant. In reality neither assumption holds. Predictable nonhomogenity can be handled via suitable time-changes. The more interesting issue is handling parametric uncertainty, especially about arrival rates. Here we will have to re-estimate periodically in addition to re-solving. The key question is, what is the trade-off between the number of re-estimations and the frequency of re-solvings? Moreover, what do we lose if we keep re-solving an inaccurate model?

(ii) Other implementations of DLP-based re-solving heuristics: In this paper we have focused solely on PAC, which is implemented probabilistically. There are many other implementations that are used in practice such as bid-prices or booking limits. In a subsequent paper, we show that the benefit of frequent re-solves depends largely on how the DLP solution is implemented.

(iii) Decomposition-based decentralized control: While solving a large LP is tractable, it may still be preferable to solve smaller, decentralized problems. How to decompose the large centralized problem into smaller decentralized problems will also be the topic of future work.

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Appendix A. Proof of results in §2.

A.1. Proof of Theorem 2.1. Fix a policy $\pi \in \Pi$. Note that

$$\sum_{j=1}^{n}\sum_{u=1}^{N_{j,\pi}} A^{j,u} = \sum_{i=1}^{\infty}\sum_{j\in S_{q(i)}\cup\{0\}} A^{j,u(i,j)}\mathbf{1}\{\pi(q(i),t(i)) = j\},$$

where $t(i)$ is the time of the ith arrival, $q(i)$ is the corresponding customer type, $u(i,j)$ is the number of times offer j has been presented up to and including time $t(i)$, and $A^{j,l}$ is the lth copy (realization) of random vector A^j. $\pi(q(i), t(i))$ is the decision made under policy π at time $t(i)$, after observing customer type $q(i)$ *and* the whole history up to time $t(i)$. Since we limit our activity horizon to time interval $[0,1]$, without loss of generality, we can set $\pi(q(i), t(i)) = 0$ for all $t(i) > 1$. The decision $\pi(q(i), t(i)) = 0$ simply means that no offer is presented at time $t(i)$.

Since $\mathbf{E}[A^{j,u(i,j)}\mathbf{1}\{\pi(q(i),t(i)) = j\}] = \bar{A}^j\mathbf{E}[\mathbf{1}\{\pi(q(i),t(i)) = j\}]$, that is, once the decision has been made regarding which offer to present, the realization of the capacity consumption vector for that offer is independent of the decision, and we have

$$\mathbf{E}\left[\sum_{j=1}^{n}\sum_{u=1}^{N_{j,\pi}} A^{j,u}\right] = \sum_{j=1}^{n}\bar{A}^j\mathbf{E}[N_{j,\pi}] = \bar{A}\mathbf{E}[N_\pi],$$

where $\mathbf{E}[N_{j,\pi}] = \mathbf{E}[\sum_{i:\,t(i)\le 1}\mathbf{1}\{\pi(q(i),t(i)) = j\}]$. Similarly,

$$\mathbf{E}\left[\sum_{j=1}^{n}\sum_{u=1}^{N_{j,\pi}} P^j\right] = P\mathbf{E}[N_\pi] \qquad \text{and} \qquad \mathbf{E}\left[\sum_{j=1}^{n}\sum_{u=1}^{N_{j,\pi}} r_j(A^{j,u})\right] = \sum_{j=1}^{n}\bar{r}_j\mathbf{E}[N_{j,\pi}].$$

So, if π is a feasible policy under optimization (1), $x = \mathbf{E}[N_\pi]$ is a feasible solution to $\mathbf{DLP}[C,\lambda]$. We conclude that $V_{\text{opt}} \le V_{\text{DLP}}$. The inequality $\mathbf{E}[R_{\text{PAC}}] \le V_{\text{opt}}$ simply follows from the definition of V_{opt}. This completes the proof.

Appendix B. Preliminaries for the proofs. This section contains notation, a policy definition, and preliminary results that will be repeatedly used throughout the rest of the appendix.

B.1. Matrix decomposition. First, we define three sets of indices as follows: $J_\lambda = \{j\colon Y_j = \lambda_{q(j)}\}$, $J_y = \{j\colon 0 < Y_j < \lambda_{q(j)}\}$, and $J_0 = \{j\colon Y_j = 0\}$ (note: Y is the unique optimal solution to $\mathbf{DLP}[C,\lambda]$), corresponding to *full*, *fractional*, and *zero* components, respectively. Similarly, without loss of generality, we write $Y = [Y_\lambda; Y_y; Y_0]$. The subscripts λ, y, 0 refer to the indices of full, fractional, and zero components of optimal solution Y, respectively.

Now, we rearrange $\bar{A}$ and P as:

$$\bar{A} = \begin{bmatrix}\bar{A}_B\\ \bar{A}_N\end{bmatrix} = \begin{bmatrix}\bar{A}_{B,\lambda} & \bar{A}_{B,y} & \bar{A}_{B,0}\\ \bar{A}_{N,\lambda} & \bar{A}_{N,y} & \bar{A}_{N,0}\end{bmatrix},$$

$$P = \begin{bmatrix}P_B\\ P_N\end{bmatrix} = \begin{bmatrix}P_{B,\lambda} & P_{B,y} & P_{B,0}\\ P_{N,\lambda} & P_{N,y} & P_{N,0}\end{bmatrix},$$

and, similarly, the capacity and arrival rates vector, C and λ, can be rearranged as

$$C = \begin{bmatrix}C_B\\ C_N\end{bmatrix} \qquad \text{and} \qquad \lambda = \begin{bmatrix}\lambda_B\\ \lambda_N\end{bmatrix}.$$

In the above, B stands for "binding" and N stands for "nonbinding." Sub-matrix $\bar{A}_B$ corresponds to the rows of $\bar{A}$ whose constraints are binding (or active) at optimal solution Y, and submatrix $\bar{A}_N$ corresponds to the rows of $\bar{A}$ whose constraints are not binding (not active) at optimal solution Y (similarly, P_B and P_N). By abuse of notation, we also interpret C_B and C_N (similarly, λ_B and λ_N) as the components of the capacity vector (arrival rates vector) that correspond to the binding and nonbinding constraints. As before, the subscripts refer to the indices of full, fractional, and zero components of optimal solution Y.

By our discussion on the structure of matrix P in §2.1, without loss of generality, we assume that submatrix P_B can be further decomposed into

$$P_B = \begin{bmatrix}P_{B_1}\\ P_{B_2}\end{bmatrix} = \begin{bmatrix}P_{B_1,\lambda} & P_{B_1,y} & P_{B_1,0}\\ P_{B_2,\lambda} & P_{B_2,y} & P_{B_2,0}\end{bmatrix},$$

where $P_{B_1,\lambda}$ is a $|J_\lambda|$ by $|J_\lambda|$ identity matrix, and $P_{B_1,y}$, $P_{B_2,\lambda}$ are both zero matrices. Define an augmented matrix W as follows:

$$W = \begin{bmatrix}\bar{A}_{B,y}\\ P_{B_2,y}\end{bmatrix}.$$

We now make two observations:

OBSERVATION B.1. J_y is not empty.

PROOF. Suppose to the contrary that J_y is empty. So, Y does not have any fractional components—that is, for all j, either $Y_j = 0$ or $Y_j = \lambda_{q(j)}$. Since, by Assumption 2.1, Y is the unique optimal solution to the nondegenerate **DLP**$[C, \lambda]$, we have exactly n binding (active) constraints at Y (Bertsimas and Tsitsiklis [2]). These must include the rows of submatrix $P_{B_1,\lambda}$ and the rows of constraints $x \geq 0$ correspond to the zero components of Y. Added together, these already give a total of n constraints, which implies that none of the rows of $\bar{A}$ can be binding. So, $\bar{A}Y < C$. Now, suppose that submatrix P_{B_2} is a zero matrix. Then, Y is also an optimal solution to DLP (2). But $\bar{A}Y < C$ contradicts Assumption 2.2. So, P_{B_2} cannot be a zero matrix, which implies that Y is *not* an optimal solution to **DLP**$[C, \lambda]$ (because $\bar{A}Y < C$ and so we can increment the value of y_j for some $j \in J_0$). We conclude that J_y is not empty. □

OBSERVATION B.2. W is a square invertible matrix.

PROOF. There are really two things that we have to prove here: that W is a square matrix and that it is invertible. We first argue that W is a square matrix. From Observation B.1 we know that J_y is not empty. Let row(Q) denote the number of rows of matrix Q. We have: $\text{row}(\bar{A}_B) + \text{row}(P_B) = |J_\lambda| + |J_y|$ (because we have n active constraints at optimal solution Y and $|J_\lambda| + |J_y| + |J_0| = n$) and $\text{row}(P_{B_1}) = |J_\lambda|$. Since $\text{row}(P_B) = \text{row}(P_{B_1}) + \text{row}(P_{B_2})$, we must have $\text{row}(\bar{A}_B) + \text{row}(P_{B_2}) = |J_y|$. So, W has $|J_y|$ rows. But, we know that W has $|J_y|$ columns. We conclude that W is a square matrix. Note that W has to be invertible because if it were otherwise, the optimal solution to **DLP**$[C, \lambda]$ would not be unique. This completes the proof. □

B.2. Perturbation analysis. We study the solution of **DLP**$[C - \Delta, \lambda]$ for a sufficiently small perturbation vector $\Delta = [\Delta_B; \Delta_N]$ (the same decomposition as in $C = [C_B; C_N]$). Define Y_Δ as follows: $Y_{\Delta,j} = Y_j$ for all $j \in J_\lambda \cup J_0$ and

$$Y_{\Delta,y} = Y_y - W^{-1}\begin{bmatrix}\Delta_B \\ O\end{bmatrix} = Y_y - H\Delta_B,$$

where O is a zero vector and H is a submatrix of W^{-1}. Then, as long as the following feasibility conditions are satisfied:

$$\begin{bmatrix}\bar{A}_N \\ P_N\end{bmatrix} Y_\Delta < \begin{bmatrix}C_N - \Delta_N \\ \lambda_N\end{bmatrix} \qquad \text{and} \qquad Y_{\Delta,y} > 0,$$

Y_Δ is the unique optimal solution to **DLP**$[C - \Delta, \lambda]$. The proof of optimality follows from Karush-Kuhn-Tucker (KKT) conditions and the proof of uniqueness follows from Bertsimas and Tsitsiklis [2] (see Exercise 3.6). [There is no need to check the feasibility condition $P_B Y_\Delta \leq \lambda_B$ since this is immediately satisfied by construction of Y_Δ.] For future reference, we split the above feasibility conditions into three separate conditions:

Condition 1.

$$P_N Y_\Delta < \lambda_N.$$

Condition 2.

$$Y_{\Delta,y} > 0.$$

Condition 3.

$$\bar{A}_N Y_\Delta < C_N - \Delta_N.$$

REMARK. If we scale both C and λ by a factor of γ, the solution to **DLP**$[\gamma C - \Delta, \gamma\lambda]$ is given by $Y^\gamma_{\Delta,j} = \gamma Y_j$ for all $j \in J_\lambda \cup J_0$ and $Y^\gamma_{\Delta,y} = \gamma Y_y - H\Delta_B$.

B.3. Expression for $\hat{Y}^k(t)$ and some properties of OPAC. Without loss of generality, we assume that $\tilde{\Delta}^k_{\text{OPAC}}$ can be written as $[\tilde{\Delta}^k_{B,\text{OPAC}}, \tilde{\Delta}^k_{N,\text{OPAC}}]$, where the subscripts B and N correspond to the binding and nonbinding rows of $\bar{A}$, respectively (at optimal solution, $Y^k(0) = kY$). Define $L(t)$ to be the unique integer satisfying $t^k_{L(t)} \leq t < t^k_{L(t)+1}$ and define:

$$\bar{Y}^k_y(t) = (1-t)kY_y - H\left(\tilde{\Delta}^k_{B,\text{OPAC}}(t^k_{L(t)}, t) + \sum_{i=1}^{L(t)} \tilde{\Delta}^k_{B,\text{OPAC}}(t^k_{i-1}, t^k_i)\frac{1-t}{1-t^k_i}\right), \quad \text{for all } t \in [t^k_{L(t)}, t^k_{L(t)+1}). \tag{B1}$$

Now, we give the definition of $\hat{Y}^k(t)$:

$$\hat{Y}^k_j(t) = (1-t)kY_j, \quad \text{for all } t,\ j \in J_\lambda \cup J_0,$$

$$\hat{Y}^k_y(t) = \begin{cases} \bar{Y}^k_y(t), & \text{if } \bar{Y}^k_y(t) \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The following properties of OPAC are immediate consequences of the assumptions made and are useful in the sequel. Assuming $0 \leq t^k_L \leq s < t \leq t^k_{L+1} < 1$,

$$\mathbf{E}[N^k_{j,\,\text{OPAC}}(s,t) \mid \Im_s] = (t-s)k\lambda_{q(j)} p^k_{j,\,\text{OPAC}}(t^k_L) \qquad \text{and} \qquad \mathbf{E}[\zeta^{j,k}_{\text{OPAC}}(s,t) \mid \Im_s] = \bar{A}^j \mathbf{E}[N^k_{j,\,\text{OPAC}}(s,t) \mid \Im_s].$$

Note that the second identity follows from the fact that $N^k_{j,\,\text{OPAC}}$ is independent of $A^{j,u}$. Thus, taking the expectation by first conditioning on $N^k_{j,\,\text{OPAC}}$ gives the result. Also, the amount of capacity used during time interval $[s,t)$ is given by

$$\begin{aligned} \sum_j \zeta^{j,k}_{\text{OPAC}}(s,t) &= \sum_j \bar{A}^j \mathbf{E}[N^k_{j,\,\text{OPAC}}(s,t) \mid \Im_s] + \sum_j \Delta^{j,k}_{\text{OPAC}}(s,t) \\ &= \bar{A}\mathbf{E}[N^k_{\text{OPAC}}(s,t) \mid \Im_s] + \tilde{\Delta}^k_{\text{OPAC}}(s,t). \end{aligned}$$

Lemma B.1. *Suppose that $0 \leq t^k_L \leq s < t \leq t^k_{L+1} < 1$. Then, there exists a constant Φ independent of k, j, s, t, L, and $\Im_s$ such that*

$$\mathbf{E}[|\tilde{\Delta}^k_{OPAC}(s,t)|^2 \mid \Im_s] \leq \Phi \mathbf{e} k(t-s),$$

where $\mathbf{e}$ *is a vector of ones.*

Proof. We first argue that $\mathbf{E}[|\Delta^{j,k}_{i,\,\text{OPAC}}(s,t)|^2 \mid \Im_s] \leq \Phi k(t-s)$ for some constants Φ independent of k, j, s, t, L, and $\Im_s$. To see this, one simply needs to note that, conditioned on $N^k_{j,\,\text{OPAC}}(s,t) = n_j$, and using the fact that $A^{j,u}$ is independent of $N^k_{j,\,\text{OPAC}}$, we have:

$$\mathbf{E}[\Delta^{j,k}_{i,\,\text{OPAC}}(s,t)^2 \mid N^k_{j,\,\text{OPAC}}(s,t) = n_j, \Im_s] = \operatorname{var}\left(\sum_{u=1}^{n_j} A^u_{ij}\right) = n_j \operatorname{var}(A_{ij}) \leq n_j \sigma^2_{\max},$$

where $\sigma^2_{\max} = \max_{i,j} \operatorname{var}(A_{ij})$. Since the above bound holds for all i, and since, conditioned on $\Im_s$, $N^k_{j,\,\text{OPAC}}(s,t)$ is a random variable with mean $k\lambda_{q(j)}(t-s)p^k_{j,\,\text{OPAC}}(t^k_L)$, we conclude that there exists a constant Φ independent of k, j, s, t, L, and $\Im_s$ such that $\mathbf{E}[|\Delta^{j,k}_{i,\,\text{OPAC}}(s,t)|^2 \mid \Im_s] \leq \Phi k(t-s)$ for all i.

Noting that

$$\begin{aligned} \mathbf{E}[|\tilde{\Delta}^k_{i,\,\text{OPAC}}(s,t)|^2 \mid \Im_s] &= \sum_{j=1}^n \mathbf{E}[|\Delta^{j,k}_{i,\,\text{OPAC}}(s,t)|^2 \mid \Im_s] + \sum_{1\leq j<j'\leq n} \mathbf{E}[\Delta^{j,k}_{i,\,\text{OPAC}}(s,t)\Delta^{j',k}_{i,\,\text{OPAC}}(s,t) \mid \Im_s] \\ &\leq \sum_{j=1}^n \mathbf{E}[|\Delta^{j,k}_{i,\,\text{OPAC}}(s,t)|^2 \mid \Im_s] + \sum_{1\leq j<j'\leq n} \mathbf{E}[\Delta^{j,k}_{i,\,\text{OPAC}}(s,t)^2 \mid \Im_s]^{1/2} \mathbf{E}[\Delta^{j',k}_{i,\,\text{OPAC}}(s,t)^2 \mid \Im_s]^{1/2} \\ &\leq \left(n + \frac{n(n-1)}{2}\right)\Phi k(t-s), \end{aligned}$$

completes the proof. □

Finally, we make two useful observations:

Lemma B.2. *There exists a constant Φ independent of k, j, and t such that*

$$\mathbf{E}[|\tilde{\Delta}^k_{OPAC}(t)|^2] \leq \Phi \mathbf{e} k t.$$

Proof. Let Φ be the constant stated in Lemma B.1 Suppose that $t^k_L \leq t < t^k_{L+1}$. By definition, $\tilde{\Delta}^k_{\text{OPAC}}(t) = \tilde{\Delta}^k_{\text{OPAC}}(t^k_L, t) + \sum_{0\leq i\leq L-1} \tilde{\Delta}^k_{\text{OPAC}}(t^k_i, t^k_{i+1})$. So $\tilde{\Delta}^k_{\text{OPAC}}(t)$ is the sum of $L+1$ terms. Because of the expectation of cross terms equal to zero (i.e., by conditioning on the term with earlier time), we have:

$$\mathbf{E}[|\tilde{\Delta}^k_{\text{OPAC}}(t)|^2] = \mathbf{E}[\tilde{\Delta}^k_{\text{OPAC}}(t^k_L, t)^2] + \sum_{0\leq i\leq L-1} \mathbf{E}[\tilde{\Delta}^k_{\text{OPAC}}(t^k_i, t^k_{i+1})^2].$$

Applying Lemma B.1 to each term and adding them up give the desired result. □

Lemma B.3. *For each fixed k, $\{\delta^k_{OPAC}(t)\}$ and $\{\tilde{\Delta}^k_{OPAC}(t)\}$ are Martingales with respect to filtration $\{\Im_t\}$.*

Proof. Follows directly from the definition of $\delta^k_{\text{OPAC}}(t)$ and $\tilde{\Delta}^k_{\text{OPAC}}(t)$. □

B.4. Relationship between PAC and OPAC. First, define Ψ to be the maximum absolute value of elements of matrix $P_{N,y}H$, H, $-\bar{A}_{N,y}H$, $\bar{A}_{B,y}$ and I. Also, define Q to be the minimum value of elements of vectors $\lambda_N - P_N Y$, $\bar{A}_{B,y}Y_y$, Y_y, $C_N - \bar{A}_N Y$. Note that the element of Q is positive. Let $\mathbf{\Psi}$ (similarly, $\mathbf{Q}$ and $\xi_{\mathbf{max}}$) be a matrix (vectors) of appropriate sizes whose components are all equal to Ψ (similarly Q and $\xi_{\max}$). We reproduce the statement of Lemma B.4 from the text for convenience here.

LEMMA B.4. *Fix a set of re-solving times* $\Gamma^k = \{t_1^k, \ldots, t_{M^k}^k\}$ *and fix a time* $t \in [0, 1)$. *Then, if the following conditions hold throughout time interval* $[0, t]$,

CONDITION †.

$$\Psi\left|\tilde{\Delta}^k_{OPAC}(t^k_{L(s)}, s) + \sum_{i=1}^{L(s)} \tilde{\Delta}^k_{OPAC}(t^k_{i-1}, t^k_i)\frac{1-s}{1-t^k_i}\right| < (1-s)k\mathbf{Q} - \xi_{\mathbf{max}},$$

CONDITION ‡.

$$\Psi|\tilde{\Delta}^k_{OPAC}(s)| < k\mathbf{Q} - \xi_{\mathbf{max}},$$

we have: (i) $C^k_{OPAC}(s) \ge \xi_{\mathbf{max}}$ *for all time* $s \in [0, t]$, (ii) *the optimal solution to* $\mathbf{DLP}[C^k_{OPAC}(t), (1-t)k\lambda]$ *is given by* $Y^k_{OPAC}(t) = \hat{Y}^k(t)$, *and* (iii) *total capacity usage under OPAC heuristic up to time* t *is equal to*

$$\sum_j \zeta^{j,k}_{OPAC}(t) = \bar{A}kYt + \tilde{\Delta}^k_{OPAC}(t^k_{L(t)}, t) + \sum_{i=1}^{L(t)}\left(I - \bar{A}_y\tilde{H}\frac{t-t^k_i}{1-t^k_i}\right)\tilde{\Delta}^k_{OPAC}(t^k_{i-1}, t^k_i),$$

where $\tilde{H} = [H, O]$, *with* O *being a zero matrix with an appropriate size.*

A remark before we proceed with the proof. Conditions (†) and (‡) are more than sufficient for PAC and OPAC to be identical as stated in the text. In fact, a more natural set of conditions, the one which we will use in the proof, is:

CONDITION 1.

$$P_N(1-s)kY - P_{N,y}H\left(\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_{L(s)}, s) + \sum_{i=1}^{L(s)}\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_{i-1}, t^k_i)\frac{1-s}{1-t^k_i}\right) < (1-s)k\lambda_N,$$

CONDITION 2.

$$(1-s)kY_y - H\left(\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_{L(s)}, s) + \sum_{i=1}^{L(s)}\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_{i-1}, t^k_i)\frac{1-s}{1-t^k_i}\right) > 0,$$

CONDITION 3.

$$\tilde{\Delta}^k_{N,\mathrm{OPAC}}(s) - \bar{A}_{N,y}H\tilde{\Delta}^k_{B,\mathrm{OPAC}}(s) < kC_N - \bar{A}_N kY.$$

Conditions 1–3 are related to the feasibility conditions for the perturbed LP discussed in Appendix B.2. We claim that if conditions (†) and (‡) are satisfied, then conditions 1–3 are also satisfied. To see why, note that conditions 1–3 can be written as:

CONDITION 1.

$$P_N(1-s)kY - P_{N,y}\tilde{H}\left(\tilde{\Delta}^k_{\mathrm{OPAC}}(t^k_{L(s)}, s) + \sum_{i=1}^{L(s)}\tilde{\Delta}^k_{\mathrm{OPAC}}(t^k_{i-1}, t^k_i)\frac{1-s}{1-t^k_i}\right) < (1-s)k\lambda_N,$$

CONDITION 2.

$$(1-s)kY_y - \tilde{H}\left(\tilde{\Delta}^k_{\mathrm{OPAC}}(t^k_{L(s)}, s) + \sum_{i=1}^{L(s)}\tilde{\Delta}^k_{\mathrm{OPAC}}(t^k_{i-1}, t^k_i)\frac{1-s}{1-t^k_i}\right) > 0,$$

CONDITION 3.

$$Z\tilde{\Delta}^k_{\mathrm{OPAC}}(s) < kC_N - \bar{A}_N kY,$$

where $Z = [-\bar{A}_{N,y}H, I]$ with I being an identity matrix with a proper size. It is now just a matter of routine algebra to check that conditions (†) and (‡) are indeed stronger than Conditions 1–3.

We introduce conditions (†) and (‡) in the lemma only for simplicity; checking these two conditions are much easier than checking conditions 1–3. So, while in the actual proof of Lemma B.4 we will mostly work directly with conditions 1–3, in the later analysis we will only work with conditions (†) and (‡).

We are now ready to prove Lemma B.4.

Proof. We first show that conditions (†) and (‡) imply (ii) and (iii) (we will prove that (†) and (‡) imply (i) later). We proceed by induction. For the base case, we use t with $L(t)=0$. Fix $t\in[0,t_1^k)$. At the beginning of the activity horizon (time 0), under the OPAC heuristic, we compute $Y^k_{\text{OPAC}}(0)$ and get $Y^k_{\text{OPAC}}(0)=Y$. So, during time interval $[0,t)$, we are to present offer j to customer type $q(j)$ with probability $p^k_{j,\text{OPAC}}(0)=Y^k_{j,\text{OPAC}}(0)/k\lambda_{q(j)}$.

By definition of ζ and $\tilde{\Delta}$, total capacity usage during time interval $[0,t)$ is given by

$$\sum_j \zeta^{j,k}_{\text{OPAC}}(0,t)=\bar{A}kYt+\tilde{\Delta}^k_{\text{OPAC}}(0,t).$$

So, (iii) is trivially satisfied. It is now left for us to prove that the solution to $\mathbf{DLP}[C^k_{\text{OPAC}}(t),(1-t)k\lambda]$ is equal to $Y^k_{\text{OPAC}}(t)=\hat{Y}^k(t)$.

First, note that $\mathbf{DLP}[C^k_{\text{OPAC}}(t),(1-t)k\lambda]$ is given by

$$\begin{aligned} \text{maximize}\quad & \bar{r}'x\\ \text{s.t.}\quad & \bar{A}x\le kC-\bar{A}kYt-\tilde{\Delta}^k_{\text{OPAC}}(0,t),\\ & Px\le(1-t)k\lambda,\\ & x\ge 0. \end{aligned}\tag{$*$}$$

The first two constraints can be written as

$$\begin{bmatrix}\bar{A}_B\\ P_B\end{bmatrix}x\le\begin{bmatrix}kC_B-\bar{A}_BkYt-\tilde{\Delta}^k_{B,\text{OPAC}}(0,t)\\ (1-t)k\lambda_B\end{bmatrix}=\begin{bmatrix}(1-t)kC_B-\tilde{\Delta}^k_{B,\text{OPAC}}(0,t)\\ (1-t)k\lambda_B\end{bmatrix},\qquad\text{and}$$

$$\begin{bmatrix}\bar{A}_N\\ P_N\end{bmatrix}x\le\begin{bmatrix}kC_N-\bar{A}_NkYt-\tilde{\Delta}^k_{N,\text{OPAC}}(0,t)\\ (1-t)k\lambda_N\end{bmatrix}.$$

By our perturbation analysis (see Appendix B.2), it is not difficult to see that, as long as the following conditions (which are the original conditions 1–3) are satisfied at time t:

Condition 1.
$$P_N(1-t)kY-P_{N,y}H\tilde{\Delta}^k_{B,\text{OPAC}}(0,t)<(1-t)k\lambda_N,$$

Condition 2.
$$(1-t)kY_y-H\tilde{\Delta}^k_{B,\text{OPAC}}(0,t)>0,$$

Condition 3.
$$\tilde{\Delta}^k_N(0,t)-\bar{A}_{N,y}H\tilde{\Delta}^k_{B,\text{OPAC}}(0,t)<kC_N-\bar{A}_NkY,$$

then $x^*_j=(1-t)Y^k_{j,\text{OPAC}}(0)$ for all $j\in J_\lambda\cup J_0$, and $x^*_y=(1-t)kY_y-H\tilde{\Delta}^k_{B,\text{OPAC}}(0,t)$ is an optimal solution to ($*$). So, we have: $Y^k_{\text{OPAC}}(t)=x^*$. But, by definition of $\hat{Y}^k(\cdot)$, we also have $\hat{Y}^k(t)=x^*$. We conclude that $Y^k_{\text{OPAC}}(t)=\hat{Y}^k(t)$. This completes the proof for case $L(t)=0$.

Now, the induction step. Suppose that the statement of the lemma is true for all t with $L(t)\le L-1<M^k$. We will show that it is also true for $t\in[t^k_L,t^k_{L+1})$. By our induction hypothesis, the lemma is true up to time t^k_L. So, as scheduled, under the OPAC heuristic, we re-compute Y^k_{OPAC} at time t^k_L and get: $Y^k_{j,\text{OPAC}}(t^k_L)=(1-t^k_L)kY_j$ for all $j\in J_\lambda\cup J_0$, and

$$Y^k_{y,\text{OPAC}}(t^k_L)=(1-t^k_L)kY_y-H\left(\tilde{\Delta}^k_{B,\text{OPAC}}(t^k_{L-1},t^k_L)+\sum_{i=1}^{L-1}\tilde{\Delta}^k_{B,\text{OPAC}}(t^k_{i-1},t^k_i)\frac{1-t^k_L}{1-t^k_i}\right).$$

Under the OPAC heuristic, during time interval $[t^k_L,t)$, we are to present offer j to customer type $q(j)$ with probability $p^k_{j,\text{OPAC}}(t^k_L)=Y^k_{j,\text{OPAC}}(t^k_L)/[(1-t^k_L)k\lambda_{q(j)}]$. Suppose that conditions 1–3 are satisfied during time interval $[t^k_L,t)$; we need to check total capacity usage up to time t. By the induction hypothesis,

$$\sum_j\zeta^{j,k}_{\text{OPAC}}(0,t^k_L)=\bar{A}kYt^k_L+\tilde{\Delta}^k_{\text{OPAC}}(t^k_{L-1},t^k_L)+\sum_{i=1}^{L-1}\left(I-\bar{A}_y\tilde{H}\frac{t^k_L-t^k_i}{1-t^k_i}\right)\tilde{\Delta}^k_{\text{OPAC}}(t^k_{i-1},t^k_i).$$

By definition of $\zeta^{j,k}_{\mathrm{OPAC}}(\cdot,\cdot)$ (see Appendix B.3), total capacity usage during time interval $[t^k_L, t)$ using allocation probability $p^k_{j,\mathrm{OPAC}}(t^k_L)$ is given by

$$\sum_j \zeta^{j,k}_{\mathrm{OPAC}}(t^k_L, t) = \bar{A}(t-t^k_L)kY - \bar{A}_y\tilde{H}\left(\tilde{\Delta}^k_{\mathrm{OPAC}}(t^k_{L-1}, t^k_L) + \sum_{i=1}^{L-1}\tilde{\Delta}^k_{\mathrm{OPAC}}(t^k_{i-1}, t^k_i)\frac{1-t^k_L}{1-t^k_i}\right)\frac{t-t^k_L}{1-t^k_L} + \tilde{\Delta}^k_{\mathrm{OPAC}}(t^k_L, t).$$

Adding $\sum_j \zeta^{j,k}_{\mathrm{OPAC}}(0, t^k_L)$ and $\sum_j \zeta^{j,k}_{\mathrm{OPAC}}(t^k_L, t)$ together, we get the desired form, so (iii) is satisfied. It is now left for us to prove that the solution to $\mathbf{DLP}[C^k_{\mathrm{OPAC}}(t), (1-t)k\lambda]$ is given by $Y^k_{\mathrm{OPAC}}(t) = \hat{Y}^k(t)$. But, this follows from the observation that $\mathbf{DLP}[C^k_{\mathrm{OPAC}}(t), (1-t)k\lambda]$ is given by:

$$\begin{aligned} \text{maximize}\quad & \bar{r}'x \\ \text{s.t.}\quad & \bar{A}x \le kC - \sum_j \zeta^{j,k}_{\mathrm{OPAC}}(t), \\ & Px \le (1-t)k\lambda, \\ & x \ge 0. \end{aligned} \tag{$**$}$$

So, using the fact that $\bar{A}_{B,y}H = I$ (which follows from the definition of H) and $\bar{A}_B Y = C$, the first two constraints can be written as

$$\begin{bmatrix} \bar{A}_B \\ P_B \end{bmatrix} x \le \begin{bmatrix} (1-t)kC_B - \tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_L, t) - \sum_{i=1}^{L}\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_{i-1}, t^k_i)\dfrac{1-t}{1-t^k_i} \\ (1-t)k\lambda_B \end{bmatrix}, \qquad \text{and}$$

$$\begin{bmatrix} \bar{A}_N \\ P_N \end{bmatrix} x \le \begin{bmatrix} kC_N - \bar{A}_N kYt - \tilde{\Delta}^k_{N,\mathrm{OPAC}}(0, t) + \bar{A}_{N,y}H\sum_{i=1}^{L}\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_{i-1}, t^k_i)\dfrac{t-t^k_i}{1-t^k_i} \\ (1-t)k\lambda_N \end{bmatrix}.$$

It is routine to check that, as long as the following conditions (1–3) are satisfied at time t, i.e.,

Condition 1.

$$P_N(1-t)kY - P_{N,y}H\left(\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_L, t) + \sum_{i=1}^{L}\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_{i-1}, t^k_i)\frac{1-t}{1-t^k_i}\right) < (1-t)k\lambda_N,$$

Condition 2.

$$(1-t)kY_y - H\left(\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_L, t) + \sum_{i=1}^{L}\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_{i-1}, t^k_i)\frac{1-t}{1-t^k_i}\right) > 0,$$

Condition 3.

$$\tilde{\Delta}^k_{N,\mathrm{OPAC}}(t) - \bar{A}_{N,y}H\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t) < kC_N - \bar{A}_N kY,$$

then the optimal solution to $(**)$ is given by x^{**}, where $x^{**}_j = (1-t)kY_j$ for all $j \in J_\lambda \cup J_0$, and

$$x^{**}_y = (1-t)kY_y - H\left(\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_L, t) + \sum_{i=1}^{L}\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_{i-1}, t^k_i)\frac{1-t}{1-t^k_i}\right).$$

So, $Y^k_{\mathrm{OPAC}}(t) = x^{**}$. But, by definition, we also have $\hat{Y}^k(t) = x^{**}$. So, $Y^k_{\mathrm{OPAC}}(t) = \hat{Y}^k(t)$. This completes the induction.

To complete the proof of the lemma, we will now argue that indeed, conditions (†) and (‡) also imply (i). To do this, suppose that conditions 1–3 are satisfied by time $t \in [t^k_L, t^k_{L+1})$. By our induction analysis above, we know that the remaining capacity at time t is given by

$$\begin{bmatrix} C^k_{B,\mathrm{OPAC}}(t) \\ C^k_{N,\mathrm{OPAC}}(t) \end{bmatrix} = \begin{bmatrix} (1-t)kC_B - \tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_L, t) - \sum_{i=1}^{L}\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_{i-1}, t^k_i)\dfrac{1-t}{1-t^k_i} \\ kC_N - \bar{A}_N kYt - \tilde{\Delta}^k_{N,\mathrm{OPAC}}(0, t) + \bar{A}_{N,y}H\sum_{i=1}^{L}\tilde{\Delta}^k_{B,\mathrm{OPAC}}(t^k_{i-1}, t^k_i)\dfrac{t-t^k_i}{1-t^k_i} \end{bmatrix}.$$

It is important to note though that conditions 1–3 already imply $C^k_{B,\,\mathrm{OPAC}}(t) = \bar{A}_B Y^k_{\mathrm{OPAC}}(t) \geq 0$ and $C^k_{N,\,\mathrm{OPAC}}(t) \geq \bar{A}_N Y^k_{\mathrm{OPAC}}(t) \geq 0$. To have (i), stronger conditions than that of 1–3 are required. By definition of $\mathbf{Q}$ and Ψ, and using the fact that $C_B = \bar{A}_B Y > \bar{A}_{B,\,y} Y_y$, Condition (†) implies

$$C^k_{B,\,\mathrm{OPAC}}(t) \geq k\bar{A}_{B,\,y}(1-t)Y_y - \tilde{\Delta}^k_{B,\,\mathrm{OPAC}}(t^k_L, t) - \sum_{i=1}^{L} \tilde{\Delta}^k_{B,\,\mathrm{OPAC}}(t^k_{i-1}, t^k_i)\frac{1-t}{1-t^k_i} \geq \xi_{\mathbf{max}}.$$

Also, condition (‡) implies $C^k_{N,\,\mathrm{OPAC}}(t) - \bar{A}_N Y^k_{\mathrm{OPAC}}(t) = kC_N - \bar{A}_N kY - Z\tilde{\Delta}^k_{\mathrm{OPAC}}(t) \geq \xi_{\mathbf{max}}$, where $Z = [-\bar{A}_{N,\,y}H, I]$. So, $C^k_{N,\,\mathrm{OPAC}}(t) \geq \xi_{\mathbf{max}}$. This completes the proof. $\square$

Appendix C. Completing the proofs of results in §5.

C.1. Completing the proof of Theorem 5.3. The lemmas that remain to be proved in the outline of §5.3 are reproduced below for convenience.

Recall the definition of τ^k as the minimum of 1 and the first time $t \in [0, 1)$, when either condition (†) or condition (‡) (see Appendix B.4) is violated. The following lemma will be proved in Appendix C.2.

Lemma C.1. *Suppose that $k > 2\xi_{\max}/Q$. Then, there exist constants ρ and $\hat{\rho}$ independent of k such that, for all choices of re-solving times $\Gamma^k = \{t^k_l\colon l = 1, \ldots, M^k\}$, and for all $0 < v(k,t) \leq \min\{1, 1/\xi_{\max}\}$, integrable in the second argument, we have*:

$$\mathbf{E}[1-\tau^k] \leq \rho k^{-1} + \int_0^{1-\theta^k} \min\{1, \hat{\rho}F(k,t)\}\,dt, \tag{C1}$$

where $\theta^k = 2\xi_{\max}/(kQ)$, $F(k,t) = \exp[kv(k,t)^2 G(k,t) - kv(k,t)(1-t)]$ and

$$G(k,t) = (t - t^k_{i-1}) + \sum_{j=1}^{i-1}(t^k_j - t^k_{j-1})\frac{(1-t)^2}{(1-t^k_j)^2}, \quad t^k_{i-1} \leq t < t^k_i, \quad 0 \leq t \leq 1.$$

Recall the definition of MPAC, where we implement the same allocation rule as in PAC and keep re-solving the DLP at times $t^k_1, t^k_2, \ldots,$ until we hit τ^k. When we hit τ^k, we re-solve the DLP for the last time and implement the resulting allocation rule until the end of the activity horizon without paying attention to the capacity constraints. The following lemma will be proved in Appendix C.3.

Lemma C.2. *Suppose that $k > 2\xi_{\max}/Q$. Then, $V^k_{DLP} - \mathbf{E}[R^k_{MPAC}] = 0$.*

The following lemma is a consequence of Lemma C.2 and will be proved in Appendix C.4.

Lemma C.3. *Suppose that $k > 2\xi_{\max}/Q$. Then, there exists a constant $\hat{\rho}$ independent of k such that*

$$V^k_{DLP} - \mathbf{E}[R^k_{PAC}] \leq k\hat{\rho}\mathbf{E}[1-\tau^k],$$

where τ^k is as defined in Lemma C.1.

Finally, putting together Lemma C.3 with Lemma C.1, we complete the proof of Theorem 5.3. Note that the case, $1 \leq k < 2\xi_{\max}/Q$, can be easily incorporated into Theorem 5.3 by noting that $\max_{1\leq k<2\xi_{\max}/Q} V^k_{\mathrm{DLP}}$ is bounded by a constant independent of k.

C.2. Proof of Lemma C.1. The following result will be useful:

Lemma C.4. *Suppose that during time interval $[s,t)$ we apply OPAC with allocation probability $\{p^k_j\}$. Also, let A^u_{ij}, $u = 1, 2, \ldots,$ denote an i.i.d. copy of A_{ij}. Then, as long as $|r\xi_{\max}| \leq 1$, we have*:

$$\begin{aligned}\mathbf{E}[\exp(r\tilde{\Delta}^k_{i,\,OPAC}(s,t))|\{p^k_j\}] &= \mathbf{E}\left[\exp\left(r\sum_{j=1}^{n}\sum_{u=1}^{N^k_{j,\,OPAC}(s,t)} A^u_{ij} - r\sum_{j=1}^{n} k\lambda_{q(j)}(t-s)p^k_j\bar{A}_{ij}\right)\right]\\ &\leq \prod_q \exp\left(k\lambda_q(t-s)r^2\sum_{j\in S_q} p^k_j\mathbf{E}[A^2_{ij}]\right) \leq \exp(k\varphi(t-s)r^2),\end{aligned}$$

for some constant φ independent of i.

PROOF. We proceed in several steps.

Step 1. Conditioned on $\{N^k_{j,\text{OPAC}}(s,t)\}$, we have:

$$\begin{aligned}
&\mathbf{E}\left[\exp\left(r\sum_{j=1}^{n}\sum_{u=1}^{N^k_{j,\text{OPAC}}(s,t)} A^u_{ij} - r\sum_{j=1}^{n} k\lambda_{q(j)}(t-s)p^k_j\bar{A}_{ij}\right)\,\middle|\, N^k_{j,\text{OPAC}}(s,t),\ j=1,\dots,n\right]\\
&=\prod_{j=1}^{n}\mathbf{E}\left[\exp\left(r\sum_{u=1}^{N^k_{j,\text{OPAC}}(s,t)} A^u_{ij} - rk\lambda_{q(j)}(t-s)p^k_j\bar{A}_{ij}\right)\,\middle|\, N^k_{j,\text{OPAC}}(s,t),\ j=1,\dots,n\right]\\
&=\prod_{j=1}^{n}\{\mathbf{E}[\exp(rA_{ij})]^{N^k_{j,\text{OPAC}}(s,t)}.\exp(-rk\lambda_{q(j)}(t-s)p^k_j\bar{A}_{ij})\}\\
&=\prod_{j=1}^{n}(\hat{Z}_{ij}(r))^{N^k_{j,\text{OPAC}}(s,t)}\prod_{j=1}^{n}\exp(-rk\lambda_{q(j)}(t-s)p^k_j\bar{A}_{ij}),
\end{aligned}$$

where $\hat{Z}_{ij}(r)=\mathbf{E}[\exp(rA_{ij})]$ for all i and j. Note that the first and second equalities follow from the definition of the OPAC heuristic (see Appendix B.3), which implies that, conditioned on the realization of $N^k_{j,\text{OPAC}}(s,t)$, capacity consumption A^u_{ij}, $u=1,2,\dots,$ are i.i.d. random variables.

Step 2. Note that

$$\mathbf{E}\left[\prod_{j=1}^{n}(\hat{Z}_{ij}(r))^{N^k_{j,\text{OPAC}}(s,t)}\right]=\prod_{q=1}^{N}\mathbf{E}\left[\prod_{j\in S_q}(\hat{Z}_{ij}(r))^{N^k_{j,\text{OPAC}}(s,t)}\right].$$

Now, for each q, conditioning on $\Lambda^k_q(s,t)=\Lambda$, we have:

$$\begin{aligned}
&\mathbf{E}\left[\prod_{j\in S_q}(\hat{Z}_{ij}(r))^{N^k_{j,\text{OPAC}}(s,t)}\,\middle|\,\Lambda^k_q(s,t)=\Lambda\right]\\
&=\sum_{\{n_j\ge 0:\ \sum_{j\in S_q}n_j\le\Lambda\}}\left\{\prod_{j\in S_q}(\hat{Z}_{ij}(r))^{n_j}\frac{\Lambda!}{n_0!\prod_{j\in S_q}n_j!}\left(1-\sum_{j\in S_q}p^k_j\right)^{n_0}\prod_{j\in S_q}(p^k_j)^{n_j}\right\}\\
&=\left(1-\sum_{j\in S_q}p^k_j+\sum_{j\in S_q}p^k_j\hat{Z}_{ij}(r)\right)^{\Lambda},\quad\text{where } n_0=\Lambda-\sum_{j\in S_q}n_j.
\end{aligned}$$

So, using the moment-generating function for Poisson random variables:

$$\begin{aligned}
\mathbf{E}\left[\prod_{j\in S_q}(\hat{Z}_{ij}(r))^{N^k_{j,\text{OPAC}}(s,t)}\right]&=\mathbf{E}\left[\left(1-\sum_{j\in S_q}p^k_j+\sum_{j\in S_q}p^k_j\hat{Z}_{ij}(r)\right)^{\Lambda^k_q(s,t)}\right]\\
&=\mathbf{E}\left[\exp\left(\Lambda^k_q(s,t)\log\left(1-\sum_{j\in S_q}p^k_j+\sum_{j\in S_q}p^k_j\hat{Z}_{ij}(r)\right)\right)\right]\\
&=\exp\Big(k\lambda_q(t-s)\sum_{j\in S_q}p^k_j(\hat{Z}_{ij}(r)-1)\Big).
\end{aligned}$$

Step 3. Putting everything together, we get:

$$\mathbf{E}\left[\exp\left(r\sum_{j=1}^{n}\sum_{u=1}^{N^k_{j,\text{OPAC}}(s,t)}A^u_{ij}-r\sum_{j=1}^{n}k\lambda_{q(j)}(t-s)p^k_j\bar{A}_{ij}\right)\right]=\prod_{q=1}^{N}\exp\left(k\lambda_q(t-s)\sum_{j\in S_q}p^k_j(\hat{Z}_{ij}(r)-1-r\bar{A}_{ij})\right).$$

The term $\hat{Z}_{ij}(r)-1-r\bar{A}_{ij}$ can be written as $\mathbf{E}[\exp(rA_{ij})-1-rA_{ij}]$. Since $e^x-1-x\le x^2$ for $|x|\le 1$, as long as $|r\xi_{\max}|\le 1$, which implies $|rA_{ij}|\le 1$, we have $\exp(rA_{ij})-1-rA_{ij}\le(rA_{ij})^2$. This completes the proof. □

We now proceed to prove Lemma C.1.

Step 1. Since τ^k is a nonnegative random variable,

$$\mathbf{E}[\tau^k]=\int_0^1 P(\tau^k\ge t)\,dt=1-\int_0^1 P(\tau^k<t)\,dt.$$

We have: $\mathbf{E}[1-\tau^k]=\int_0^1 P(\tau^k<t)\,dt=\int_0^{1-\theta^k}P(\tau^k<t)\,dt+\int_{1-\theta^k}^1 P(\tau^k<t)\,dt\le\int_0^{1-\theta^k}P(\tau^k<t)\,dt+\theta^k$. So, we need to find an upper bound for the term $P(\tau^k<t)$ for $t\in[0,1-\theta^k]$.

Step 2. Note that τ^k can be written as the minimum of hitting time τ_1^k and τ_2^k, where τ_1^k is defined as the minimum of 1 and the first time $t \in [0, 1)$ such that condition (†) is violated and τ_2^k is defined as the minimum of 1 and the first time $t \in [0, 1)$ such that condition (‡) is violated.

Now, $P(\tau^k < t) \le P(\tau_1^k < t) + P(\tau_2^k < t)$. The second probability is easy to compute: from Lemma B.2, we know that $\mathbf{E}[\tilde{\Delta}_{i,\,\mathrm{OPAC}}^k(t)^2] \le \Phi k t$ for all $i = 1, \ldots, m$, and, from Lemma B.3, we know that $\{\tilde{\Delta}_{i,\,\mathrm{OPAC}}^k(t)\}$ is a Martingale with respect to filtration $\{\Im_t\}$, so by Doob's inequality (see, for example: Williams [16]) we have

$$\begin{aligned}
P(\tau_2^k < t) &= P(\Psi|\tilde{\Delta}_{i,\,\mathrm{OPAC}}^k(s)| \ge kQ - \xi_{\max} \text{ for some } i, \text{ for some } s \in [0, t)) \\
&\le P\left(\Psi \sup_{0 \le s \le t} |\tilde{\Delta}_{i,\,\mathrm{OPAC}}^k(s)| \ge kQ - \xi_{\max} \text{ for some } i\right) \\
&\le \sum_{i=1}^m P\left(\Psi \sup_{0 \le s \le t} |\tilde{\Delta}_{i,\,\mathrm{OPAC}}^k(s)| \ge kQ - \xi_{\max}\right) \\
&\le \frac{m\Psi^2 \Phi k t}{(kQ - \xi_{\max})^2} \le \frac{4m\Psi^2\Phi t}{Q^2 k},
\end{aligned}$$

where the last inequality follows from $kQ > 2\xi_{\max}$. So, $\int_0^{1-\theta^k} P(\tau_2^k < t)\,dt \le 4m\Phi\Psi^2 Q^{-2} k^{-1}$. We will now focus our attention on $P(\tau_1^k < t)$.

Step 3. Fix a choice of re-solving times $t_1^k, t_2^k, \ldots$, and define $E^k(t)$ as follows:

$$E_i^k(t) = \tilde{\Delta}_{i,\,\mathrm{OPAC}}^k(t_L^k, t) + \sum_{j=1}^{L} \frac{1-t}{1-t_j^k} \tilde{\Delta}_{i,\,\mathrm{OPAC}}^k(t_{j-1}^k, t_j^k), \quad t_L^k \le t < t_{L+1}^k.$$

Fix $t \in [0, 1)$. Now, define $X^k(s)$ and $\hat{X}^k(s)$ for $s \in [0, t]$ as follows:

$$X^k(s) = \frac{E^k(s)}{1-s} = \frac{\tilde{\Delta}_{i,\,\mathrm{OPAC}}^k(t_L^k, s)}{1-s} + \sum_{j=1}^{L} \frac{\tilde{\Delta}_{i,\,\mathrm{OPAC}}^k(t_{j-1}^k, t_j^k)}{1-t_j^k}, \quad \text{for all } s \in [t_L^k, t_{L+1}^k),$$

$$\hat{X}^k(s) = \begin{cases} \dfrac{\tilde{\Delta}_{i,\,\mathrm{OPAC}}^k(t_L^k, s)}{1-t_{L+1}^k} + \displaystyle\sum_{j=1}^{L} \dfrac{\tilde{\Delta}_{i,\,\mathrm{OPAC}}^k(t_{j-1}^k, t_j^k)}{1-t_j^k} & \text{for all } s \in [t_L^k, t_{L+1}^k), \ \ L < L(t), \\[2ex] \dfrac{\tilde{\Delta}_{i,\,\mathrm{OPAC}}^k(t_{L(t)}^k, s)}{1-t} + \displaystyle\sum_{j=1}^{L(t)} \dfrac{\tilde{\Delta}_{i,\,\mathrm{OPAC}}^k(t_{j-1}^k, t_j^k)}{1-t_j^k} & \text{for all } s \in [t_{L(t)}^k, t]. \end{cases}$$

Since $t \le 1 - \theta^k$ implies $kQ(1-t) - \xi_{\max} \ge kQ(1-t)/2$, for all $v(k, t) > 0$, we have:

$$\begin{aligned}
P(\tau_1^k < t) &= P(\Psi|E_i^k(s)| \ge kQ(1-s)/2 \text{ for some } i, \text{ for some } s \in [0, t)) \\
&= P(\Psi|X_i^k(s)| \ge kQ/2 \text{ for some } i, \text{ for some } s \in [0, t)) \\
&\le P\left(\Psi \sup_{0 \le s \le t} |X_i^k(s)| \ge kQ/2 \text{ for some } i\right) \\
&\le \sum_i P\left(\Psi \sup_{0 \le s \le t} |X_i^k(s)| \ge kQ/2\right) \\
&\le \sum_i P\left(\Psi \sup_{0 \le s \le t} |\hat{X}_i^k(s)| \ge kQ/2\right) \\
&\le \sum_i P\left(\Psi \sup_{0 \le s \le t} v(k, t)(1-t)|\hat{X}_i^k(s)| \ge v(k, t) kQ(1-t)/2\right) \\
&= \sum_i P\left(\Psi \sup_{0 \le s \le t} \exp[v(k, t)(1-t)|\hat{X}_i^k(s)|] \ge \exp[v(k, t) kQ(1-t)/2]\right).
\end{aligned}$$

The third inequality follows because, by construction, $|X_i^k(s)| \ge kQ/2$ always implies $|\hat{X}_i^k(s)| \ge kQ/2$.

Step 4. We are interested in process $\hat{X}^k(s)$ because, for a fixed t, $\{\hat{X}^k(s)\}_{0\le s\le t}$ is a Martingale with respect to filtration $\{\Im_s\}$, which implies that $\{\exp[v(k,t)(1-t)|\hat{X}_i^k(s)|]\}_{0\le s\le t}$ is a sub-Martingale (i.e., for all $v(k,t)>0$) with respect to the same filtration. So, for each j, we can apply Doob's sub-Martingale inequality and get

$$\begin{aligned}P\bigg(\Psi\sup_{0\le s\le t}\exp[v(k,t)(1-t)|\hat{X}_i^k(s)|]\ge\exp[v(k,t)kQ(1-t)/2]\bigg)\\ \le\min\bigg\{1,\frac{\mathbf{E}(\Psi\exp[v(k,t)(1-t)|\hat{X}_i^k(t)|])}{\exp[v(k,t)kQ(1-t)/2]}\bigg\}\\ =\min\bigg\{1,\frac{\Psi\mathbf{E}(\exp[v(k,t)|E_i^k(t)|])}{\exp[v(k,t)kQ(1-t)/2]}\bigg\}.\end{aligned}$$

Step 5. We will now bound the term $\mathbf{E}(\exp[v(k,t)|E_i^k(t)|])$. First, note that

$$\exp[v(k,t)|E_i^k(t)|]\le\exp[v(k,t)E_i^k(t)]+\exp[-v(k,t)E_i^k(t)].$$

Now, we claim that there exists a constant φ independent of i such that $\mathbf{E}(\exp[v(k,t)E_i^k(t)])\le\exp(k\varphi v(k,t)^2G(k,t))$ and $\mathbf{E}(\exp[-v(k,t)E_i^k(t)])\le\exp(k\varphi v(k,t)^2G(k,t))$, where G is as defined in the statement of Lemma C.1.

To prove the above claim, note that $\mathbf{E}(\exp[v(k,t)E_i^k(t)])$ can be written as:

$$\mathbf{E}\bigg(\exp[v(k,t)\tilde{\Delta}^k_{i,\mathrm{OPAC}}(t^k_{L(t)},t)]\prod_{j=1}^{L(t)}\exp\bigg[v(k,t)\tilde{\Delta}^k_{i,\mathrm{OPAC}}(t^k_{j-1},t^k_j)\frac{1-t}{1-t^k_j}\bigg]\bigg).$$

Using the fact that $\tilde{\Delta}^k_{i,\mathrm{OPAC}}(t^k_{L(t)},t)$ is independent of the other terms given history up to time $t^k_{L(t)}$, by conditioning on $\Im_{t^k_{L(t)}}$ and applying Lemma C.4, we have

$$\mathbf{E}(\exp[v(k,t)E_i^k(t)])\le\exp(k\varphi v(k,t)^2(t-t^k_{L(t)}))\mathbf{E}\bigg(\prod_{j=1}^{L(t)}\exp\bigg[v(k,t)\tilde{\Delta}^k_{i,\mathrm{OPAC}}(t^k_{j-1},t^k_j)\frac{1-t}{1-t^k_j}\bigg]\bigg).$$

Similarly, $\tilde{\Delta}^k_{i,\mathrm{OPAC}}(t^k_{L(t)-1},t^k_{L(t)})$ is independent of the rest of the other terms given history up to time $t^k_{L(t)-1}$. So, again, by conditioning on $\Im_{t^k_{L(t)-1}}$ and applying Lemma C.4, we get

$$\begin{aligned}&\mathbf{E}\bigg(\prod_{j=1}^{L(t)}\exp\bigg[v(k,t)\tilde{\Delta}^k_{i,\mathrm{OPAC}}(t^k_{j-1},t^k_j)\frac{1-t}{1-t^k_j}\bigg]\bigg)\\ &\quad\le\exp\bigg(k\varphi v(k,t)^2(t^k_{L(t)}-t^k_{L(t)-1})\frac{(1-t)^2}{(1-t^k_{L(t)})^2}\bigg)\mathbf{E}\bigg(\prod_{j=1}^{L(t)-1}\exp\bigg[v(k,t)\tilde{\Delta}^k_{i,\mathrm{OPAC}}(t^k_{j-1},t^k_j)\frac{1-t}{1-t^k_j}\bigg]\bigg).\end{aligned}$$

Proceeding in the same manner until we get to the last term gives the desired result. The bound for $\mathbf{E}(\exp[-v(k,t)E_i^k(t)])$ can be proved using the same argument. This completes the proof.

C.3. Proof of Lemma C.2. As usual, let $N^k_{j,\mathrm{MPAC}}(s,t)$ denote the number of times offer j is presented during time interval $[s,t)$, and let $\zeta^{j,k}_{\mathrm{MPAC}}(s,t)$ denote its corresponding total capacity usage vector. By abuse of notation, we will use $N^k_{j,\mathrm{MPAC}}(t)$ to denote the number of times offer j is presented up to time t, and, similarly, $\zeta^{j,k}_{\mathrm{MPAC}}(t)$, the total capacity usage by offer j up to time t.

By our previous remarks, MPAC and OPAC heuristics are identical up to time τ^k. So, under the same sample-path realization, we have:

$$N^k_{j,\mathrm{OPAC}}(0,s)=N^k_{j,\mathrm{MPAC}}(0,s)\qquad\text{and}\qquad\zeta^{j,k}_{\mathrm{OPAC}}(0,s)=\zeta^{j,k}_{\mathrm{MPAC}}(0,s)\quad\text{for all }0\le s\le\tau^k.$$

We now proceed in several steps:

Step 1. Suppose that $t_L^k < \tau^k \le t_{L+1}^k$. Since MPAC is equivalent to OPAC up to time τ^k, we have:

$$N_{\mathrm{MPAC}}^k(0, t_1^k) = N_{\mathrm{OPAC}}^k(0, t_1^k) = kYt_1^k + \delta_{\mathrm{OPAC}}^k(0, t_1^k),$$

$$\begin{aligned} N_{\mathrm{MPAC}}^k(t_i^k, t_{i+1}^k) &= N_{\mathrm{OPAC}}^k(t_i^k, t_{i+1}^k) \\ &= (t_{i+1}^k - t_i^k)kY - \begin{bmatrix} O_\lambda \\ H\sum_{j=0}^{i}\tilde{\Delta}_{B,\mathrm{OPAC}}^k(t_j^k, t_{j+1}^k)\dfrac{t_{i+1}^k - t_i^k}{1 - t_{j+1}^k} \\ O_0 \end{bmatrix} + \delta_{\mathrm{OPAC}}^k(t_i^k, t_{i+1}^k), \quad \text{for } i = 1, \ldots, L-1, \end{aligned}$$

$$\begin{aligned} N_{\mathrm{MPAC}}^k(t_L^k, \tau^k) &= N_{\mathrm{OPAC}}^k(t_L^k, \tau^k) \\ &= (\tau^k - t_L^k)kY - \begin{bmatrix} O_\lambda \\ H\sum_{j=0}^{L-1}\tilde{\Delta}_{B,\mathrm{OPAC}}^k(t_j^k, t_{j+1}^k)\dfrac{\tau^k - t_L^k}{1 - t_{j+1}^k} \\ O_0 \end{bmatrix} + \delta_{\mathrm{OPAC}}^k(t_L^k, \tau^k). \end{aligned}$$

Also, by definition of the MPAC heuristic, during time interval $[\tau^k, 1)$, we have:

$$N_{\mathrm{MPAC}}^k(\tau^k, 1) = (1 - \tau^k)kY - \begin{bmatrix} O_\lambda \\ H\left(\tilde{\Delta}_{B,\mathrm{OPAC}}^k(t_L^k, \tau^k) + \sum_{j=0}^{L-1}\tilde{\Delta}_{B,\mathrm{OPAC}}^k(t_j^k, t_{j+1}^k)\dfrac{1 - \tau^k}{1 - t_{j+1}^k}\right) \\ O_0 \end{bmatrix} + \delta_{\mathrm{MPAC}}^k(\tau^k, 1),$$

where $\delta_{j,\mathrm{MPAC}}^k(\tau^k, 1) = N_{j,\mathrm{MPAC}}^k(\tau^k, 1) - \mathbf{E}[N_{j,\mathrm{MPAC}}^k(\tau^k, 1) \mid \Im_{\tau^k}]$. Note: by definition of the MPAC heuristic, $N_{j,\mathrm{MPAC}}^k(\tau^k, 1)$ has a marginal binomial distribution,

$$N_{j,\mathrm{MPAC}}^k(\tau^k, 1) = \mathrm{binomial}(\Lambda_{q(j)}^k(\tau^k, 1), p_{j,\mathrm{MPAC}}^k(\tau^k)),$$

where the offer probability $p_{j,\mathrm{MPAC}}^k(\tau^k)$ is equal to $Y_{j,\mathrm{MPAC}}^k(\tau^k)/[(1-\tau^k)k\lambda_{q(j)}]$, and $Y_{\mathrm{MPAC}}^k(\tau^k)$ is the optimal solution to $\mathbf{DLP}[C_{\mathrm{MPAC}}^k(\tau^k), (1-\tau^k)k\lambda]$.

Adding the expressions for $N_{\mathrm{MPAC}}^k(0, \tau^k)$ and $N_{\mathrm{MPAC}}^k(\tau^k, 1)$, we have

$$N_{\mathrm{MPAC}}^k(1) = kY - \begin{bmatrix} O_\lambda \\ H\tilde{\Delta}_{B,\mathrm{OPAC}}^k(\tau^k) \\ O_0 \end{bmatrix} + \delta_{\mathrm{OPAC}}^k(\tau^k) + \delta_{\mathrm{MPAC}}^k(\tau^k, 1).$$

Step 2. From Lemma B.3, we know that both $\{\delta_{\mathrm{OPAC}}^k(t)\}_t$ and $\{\tilde{\Delta}_{B,\mathrm{OPAC}}^k(t)\}_t$ are Martingales with respect to filtration $\{\Im_t\}_t$. Since, by definition, τ^k is a bounded stopping time, we can apply an optional stopping theorem (see, for example: Williams [16]) and get: $\mathbf{E}[\delta_{\mathrm{OPAC}}^k(\tau^k)] = \mathbf{E}[\delta_{\mathrm{OPAC}}^k(0)] = 0$. Similarly, $\mathbf{E}[\tilde{\Delta}_{B,\mathrm{OPAC}}^k(\tau^k)] = \mathbf{E}[\tilde{\Delta}_{B,\mathrm{OPAC}}^k(0)] = 0$. Together with the fact that $\mathbf{E}[\delta_{\mathrm{MPAC}}^k(\tau^k, 1)] = \mathbf{E}[\mathbf{E}[\delta_{\mathrm{MPAC}}^k(\tau^k, 1) \mid \tau^k]] = 0$, from Step 1, we get $\mathbf{E}[N_{\mathrm{MPAC}}^k(1)] = kY$. Finally, by the same argument as in the proof of Theorem 2.1,

$$\mathbf{E}[R_{\mathrm{MPAC}}^k(0, 1)] = \mathbf{E}\left[\sum_{j=1}^{n}\sum_{u=1}^{N_{j,\mathrm{MPAC}}^k(1)} r_j(A^{j,u})\right] = \sum_{j=1}^{n}\bar{r}_j\mathbf{E}[N_{j,\mathrm{MPAC}}^k(1)] = \sum_{j=1}^{n}\bar{r}_jkY_j = V_{DLP}^k.$$

This completes the proof of Lemma C.2.

C.4. Proof of Lemma C.3. We have:

$$\begin{aligned} V^k_{\mathrm{DLP}} - \mathbf{E}[R^k_{\mathrm{PAC}}(0,1)] &= V^k_{\mathrm{DLP}} - \mathbf{E}[R^k_{\mathrm{MPAC}}(0,1)] + \mathbf{E}[R^k_{\mathrm{MPAC}}(0,1)] - \mathbf{E}[R^k_{\mathrm{PAC}}(0,1)] \\ &= \mathbf{E}[R^k_{\mathrm{MPAC}}(0,1)] - \mathbf{E}[R^k_{\mathrm{PAC}}(0,1)] \\ &= \mathbf{E}[R^k_{\mathrm{MPAC}}(\tau^k,1)] - \mathbf{E}[R^k_{\mathrm{PAC}}(\tau^k,1)] \\ &\le \mathbf{E}[R^k_{\mathrm{MPAC}}(\tau^k,1)] \\ &\le k\hat{\rho}\mathbf{E}[1-\tau^k] \quad \text{for some constant } \hat{\rho}. \end{aligned}$$

The first equality follows from Lemma C.2 and the second equality follows from $R^k_{\mathrm{MPAC}}(0,\tau^k) = R^k_{\mathrm{PAC}}(0,\tau^k)$. The last inequality can be seen by imagining presenting all products to all incoming customers during time interval $[\tau^k, 1)$ and assuming that the customer always purchases the products, which we can do since we ignore capacity constraint. Since, for any given τ^k, the expected number of customers arriving during time interval $[\tau^k, 1)$ is bounded by $\hat{\rho}k(1-\tau^k)$ for some constants $\hat{\rho}$ independent of k (in fact, we can take $\hat{\rho} = \sum_q \lambda_q$), the result follows.

C.5. Proof of Theorem 5.1. Fix $0 < h \le 1$. For $t < 1$, by integral comparison, we have

$$\begin{aligned} \frac{t - t^k_{i-1}}{(1-t)^2} + \sum_{j=1}^{i-1} \frac{t^k_j - t^k_{j-1}}{(1-t^k_j)^2} &\le \frac{h}{(1-t)^2} + \sum_{j=1}^{i-1} \frac{h}{(1-jh)^2} \\ &\le \frac{h}{(1-t)^2} + \frac{h}{(1-(i-1)h)^2} + \frac{1}{1-(i-1)h} \\ &\le \frac{2h}{(1-t)^2} + \frac{1}{1-t}, \end{aligned}$$

for all i and $(i-1)h \le t < \min(ih, 1)$. So, $G(k,t) \le 2h + 1 - t$ (and this is true for all $t \in [0,1]$). Let $y = \min(1-2h, 1-1/k)$. We divide our analysis into three cases:

Case 1. $2h > 1/k$ and $kh > \xi^2_{\max}$.

Note that, in this case, $y = 1-2h$. We split the limits of integration in the integral in Theorem 5.3 into two: from 0 to $\min(1-2h, 1-1/k)$ and the other from $\min(1-2h, 1-1/k)$ to 1. For the first integral, use $v(k,t) = 1/\sqrt{k(1-t)}$, and for the second integral, use $v(k,t) = 1/\sqrt{kh}$. We have:

$$\int_0^y F(k,t)\,dt \le \int_0^y \exp(2 - \sqrt{k(1-t)})\,dt \le e^2 \int_0^y \frac{4!}{k^2(1-t)^2}\,dt \le \frac{e^2 4!}{k}, \qquad \text{and} \tag{C2}$$

$$\int_y^1 F(k,t)\,dt \le \int_y^1 \exp\left(4 - \sqrt{\frac{k}{h}}(1-t)\right) dt \le e^4 \sqrt{\frac{h}{k}}. \tag{C3}$$

The first inequality in (C2) follows because $G(k,t) \le 2(1-t)$ for all $t \in [0, y]$, and the second inequality follows from Taylor's expansion, which yields

$$\exp(-\sqrt{k(1-t)}) \le \frac{4!}{k^2(1-t)^2}.$$

The inequality in (C3) follows because $G(k,t) \le 4h$ for all $t \in (y, 1]$.

Case 2. $2h \le 1/k$ and $kh > \xi^2_{\max}$.

In this case, $y = 1 - 1/k$. Again, we split the limits of integration in the integral in Theorem 5.3 into two: from 0 to $\min(1-2h, 1-1/k)$ and from $\min(1-2h, 1-1/k)$ to 1. We assume that $\xi^2_{\max} \le 1/2$; otherwise this case does not apply. For the first integral, we use $v(k,t) = 1/\sqrt{k(1-t)}$, and for the second integral, we use $v(k,t) = 1/\sqrt{kh}$. Note that we still have $G(k,t) \le 2(1-t)$ for all $t \in [0,y]$. So, nothing changes with the first integral. As for the second integral, we have $G(k,t) \le 2/k$ for all $t \in (y, 1]$. So,

$$\int_y^1 F(k,t)\,dt \le \int_y^1 \exp\left(\frac{2}{kh} - \sqrt{\frac{k}{h}}(1-t)\right) dt \le e^{2/(\xi^2_{\max})} \sqrt{\frac{h}{k}}.$$

Case 3. $kh \le \xi^2_{\max}$.

Let $v(k,t) = r = \min(1/2, 1/\xi_{\max})$. We have:

$$\int_0^1 F(k,t)\,dt = \int_0^1 \exp(2khr^2 - k(r - r^2)(1-t))\,dt \le \frac{e^{2\xi^2_{\max} r^2}}{k(r-r^2)}.$$

Combining the three cases above, we have completed the proof.

C.6. Proof of Theorem 5.2. For $1-2^{-(i-1)} \le t < 1-2^{-i}$, $0 \le t < 1$, we have:

$$\frac{t-t^k_{i-1}}{(1-t)^2} + \sum_{j=1}^{i-1} \frac{t^k_j - t^k_{j-1}}{(1-t^k_j)^2} \le \frac{t-(1-2^{-(i-1)})}{(1-t)^2} + \sum_{j=1}^{i-1} 2^j$$
$$\le \frac{2^{-i}}{(1-t)^2} + 2^i \le 2^i + 2^i \le \frac{4}{1-t}.$$

So, $G(k,t) \le 4(1-t)$ (and this is true for all $t \in [0,1]$). Now, we divide our analysis into two cases: for $k \le \xi^2_{\max}$ and $k > \xi^2_{\max}$. For the first case, the integral in Theorem 5.3 is bounded by 1. So, $k\int_0^1 \min\{1, \rho' F(k,t)\}\,dt \le \xi^2_{\max}$. For the second case, we split the limits of integration in the integral in Theorem 5.3 into two: from 0 to $y = 1-\xi^2_{\max}/k$ and from y to 1. For the first integral, use $v(k,t) = 1/\sqrt{k(1-t)}$, and for the second integral, use $v(k,t) = 1/\xi_{\max}$. We have:

$$\int_0^y F(k,t)\,dt \le \int_0^y \exp(4-\sqrt{k(1-t)})\,dt \le e^4 \int_0^y \frac{4!}{k^2(1-t)^2}\,dt \le \frac{e^4 4!}{\xi^2_{\max} k}, \qquad \text{and}$$
$$\int_y^1 \min\{1, \rho' F(k,t)\}\,dt \le 1-y = \frac{\xi^2_{\max}}{k}.$$

This completes the proof.

Remark. It is also possible to bound the term $\exp(-\sqrt{k(1-t)})$ by $2!/[k(1-t)]$. Running the integration with this bound gives an $O(\log(k))$ expected revenue loss bound, which may be more useful for problems with moderate k because the resulting constants are smaller.

C.7. Proof of Corollary 5.1. For $t < 1$, note that

$$\frac{t-t^k_{i-1}}{(1-t)^2} + \sum_{j=1}^{i-1} \frac{t^k_j - t^k_{j-1}}{(1-t^k_j)^2} \le \frac{1}{(1-t)^2},$$

for all i and $t^k_{i-1} \le t < t^k_i$. So, $G(k,t) \le 1$ (and this is true for all $t \in [0,1]$). Now, using $v(k,t) = \min(k^{-1/2}, 1/\xi_{\max})$, for $k \ge \xi^2_{\max}$, we have:

$$\int_0^1 \min\{1, \rho' F(k,t)\}\,dt \le e\rho' \int_0^1 \exp(-\sqrt{k}(1-t))\,dt \le \frac{e\rho'}{\sqrt{k}}.$$

For $k < \xi^2_{\max}$, the above integral is bounded by the constant $\rho'/\xi_{\max}$. So, $V^k_{\text{opt}} - \mathbf{E}[R^k_{\text{PAC}}] \le \rho + \hat{\rho}\sqrt{k}$ for some positive constants ρ and $\hat{\rho}$. This completes the proof.

Appendix D. Proof of results in §6.

D.1. Proof of Theorem 6.1. Below we only give the outline of the proof:

(i) Fix M and let $(r_l)_{l=1}^M$ be a sequence of increasing numbers, $0 < r_l < 1$ for all l. We define τ^k_l, $l = 1, \dots, M$ as follows:

- Define τ^k_1 as the minimum of 1 and the first time $t \in [0, 1-1/k)$ such that at least one of the following two conditions fails to hold:

$$\Psi|\tilde{\Delta}^k_{\text{OPAC}}(0,t)| < r_1 k\mathbf{Q}(1-t) - r_1\xi_{\max} \qquad \text{and} \qquad \Psi|\tilde{\Delta}^k_{\text{OPAC}}(0,t)| < r_1 k\mathbf{Q} - r_1\xi_{\max}.$$

Remark. We use the same Ψ as defined in Appendix B.4.

- Define: $\tau^k_0 = 0$. For $l = 2, \dots, M$, define τ^k_l recursively as the minimum of 1 and the first time $t \in [\tau^k_{l-1}, 1-1/k)$ such that at least one the following two conditions fails to hold:

$$\Psi\left|\tilde{\Delta}^k_{\text{OPAC}}(\tau^k_{l-1}, t) + \sum_{j=0}^{l-2} \tilde{\Delta}^k_{\text{OPAC}}(\tau^k_j, \tau^k_{j+1}) \frac{1-t}{1-\tau^k_{j+1}}\right| < r_l k\mathbf{Q}(1-t') - r_l\xi_{\max} \qquad \text{and}$$
$$\Psi|\tilde{\Delta}^k_{\text{OPAC}}(t)| < r_l k\mathbf{Q} - r_l\xi_{\max}.$$

- The following can be proved:

$$\mathbf{E}[1-\tau_1^k] = O(k^{-1/2}) \tag{D1}$$

$$\mathbf{E}[1-\tau_l^k] = O\Big(r_{l-1}^{-1} k^{-1/2} \sqrt{\mathbf{E}[1-\tau_{l-1}^k]}\Big) \quad \text{for all } l = 2, \ldots, N. \tag{D2}$$

(C3) is a special case of Lemma C.1 (see Appendix C) when Γ^k is an empty set and (D1) can be proved recursively, or by induction.

(ii) This is an analog of Step 3 in the proof of Theorem 5.3. But there is a difference: instead of throwing away all the revenue earned during time interval $[\tau_N^k, 1)$, we take them into account by way of providing a tighter bound. That is, we have:

$$\begin{aligned} V_{\text{opt}}^k - \mathbf{E}[R_{\text{PAC}}^k(0,1)] &\le \mathbf{E}[R_{\text{MPAC}}^k(\tau_N^k, 1)] - \mathbf{E}[R_{\text{PAC}}^k(\tau_M^k, 1)] \\ &\le \hat{\rho}\mathbf{E}[\sqrt{k(1-\tau_M^k)}] \quad \text{for some constants } \hat{\rho} \text{ independent of } k. \end{aligned}$$

To see why the second inequality is true, note that, conditioned on the history up to time τ_N^k, $\mathbf{E}[R_{\text{MPAC}}^k(\tau_N^k, 1)]$ is the optimal value of $\mathbf{DLP}[C_{\text{MPAC}}^k(\tau_1^k), k\lambda(1-\tau_1^k)]$, which, by sample path argument, can be shown to equal $\mathbf{DLP}[C_{\text{PAC}}^k(\tau_N^k), k\lambda(1-\tau_N^k)]$. So, the term $\mathbf{E}[R_{\text{MPAC}}^k(\tau_1^k, 1)] - \mathbf{E}[R_{\text{PAC}}^k(\tau_1^k, 1)]$ can be bounded using the same argument as in the standard result without re-solving (i.e., we assume that the problem starts at time τ_1^k and apply the PAC heuristic until time 1 without re-solving (see Reiman and Wang [12]).

(iii) Finally, applying Jensen's inequality to the bound in the previous step and putting everything together gives the desired result. This completes the proof of Theorem 6.1.

Note that the choice of $(r_l)_{l=1}^M$ above is arbitrary. So, one may want to choose a sequence of $(r_l)_{l=1}^N$ that minimizes the resulting upper bound.

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