Scenario-based Dynamic Seat Assignment

with Social Distancing

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Problem Definition

Seat Planning with Social Distancing

- Group type $[M] = \{1, \dots, M\}$
- Row $[N] = \{1, \dots, N\}$
- s seats as the social distancing
- Let $n_i = i + s$ denote the new size of group type i for each $i \in [M]$.
- Let $L_j = S_j + s$ denote the length of row j for each $j \in [N]$, where S_j represents the number of seats in row j.

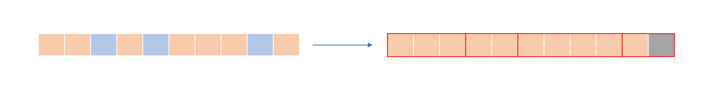

Figure: Problem Conversion

Dynamic Programming

Dynamic seat assignment can be characterized by DP:

$$V_t(\mathbf{L}) = E_i \left[\max_{k \in N: L_k \geq i+s} \{[V_{t-1}(\mathbf{L} - U_{ik}) + i], V_{t-1}(\mathbf{L})\} \right], \mathbf{L} \geq \mathbf{0}$$

$$V_{T+1}(\mathbf{L}) = 0,$$

- $\mathbf{L} = \{\}$, remaining capacity.
- U_{ik}
- p_i: the probability of an arrival of group type i.

Example

Suppose the social distancing is one seat, then the new sizes of groups are $2, 3, 4, 5$, respectively.
The length of one row is $L = 21$.
The demand is $[10, 12, 9, 8]_d$. Then these patterns, $(5, 5, 5, 5, 1), (5, 4, 4, 4, 4), (5, 5, 5, 3, 3)$, belong to I_1.
For pattern 1, $(5, 5, 5, 5, 1)$, $P_1 = \{5\}$, thus a group with a size smaller than 5 cannot be put in this pattern.

Properties

- α_k indicates the number of items for pattern k. β_k indicates the left space for maximal pattern k. Notice that the left space is the true loss.
- Denote $\alpha_k + \beta_k - 1$ as the loss for pattern k, $l(k)$. When $l(k)$ reaches minimum, the corresponding pattern k is the optimal solution for a single row.
- If the group sizes are consecutive integers starting from 2, $\{2, 3, \ldots, u\}$, then a greedy-based pattern is optimal, i.e., select the maximal group size,u, as many as possible and the left space is occupied by the group with the corresponding size. The loss is $k + 1$, where k is the number of times u selected. Let $S = u \cdot k + r$.

Property

- Let I_1 be the set of patterns with the minimal loss.
-
- For a seat layout, $\{S_1, S_2, \ldots, S_N\}$, the total loss is $\sum_j (\lfloor \frac{S_j+1}{u} \rfloor - f((S_j+1) \mod u))$. The maximal number of people assigned is $\sum_j (S_j - \lfloor \frac{S_j+1}{u} \rfloor + f((S_j+1) \mod u))$.

Scenario-based Stochastic Programming

Scenario-based Stochastic Programming

$$
\begin{aligned}
\max \quad & E_\omega \left[\sum_{i=1}^{M-1} (n_i - s)(\sum_{j=1}^{N} x_{ij} + y^+_{i+1,\omega} - y^+_{i\omega}) + (n_M - s)(\sum_{j=1}^{N} x_{Mj} - y^+_{M\omega}) \right] \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} - y^+_{i\omega} + y^+_{i+1,\omega} + y^-_{i\omega} = d_{i\omega}, \quad i \in [M-1], \omega \in \Omega \\
& \sum_{j=1}^{N} x_{ij} - y^+_{i\omega} + y^-_{i\omega} = d_{i\omega}, \quad i = M, \omega \in \Omega \\
& \sum_{i=1}^{M} n_i x_{ij} \leq L_j, j \in [N] \\
& y^+_{i\omega}, y^-_{i\omega} \in \mathbb{Z}_+, \quad i \in [M], \omega \in \Omega \\
& x_{ij} \in \mathbb{Z}_+, \quad i \in [M], j \in [N].
\end{aligned} \tag{1}
$$

Two-stage

$$
\begin{aligned}
\max \quad & c'\mathbf{x} + z(\mathbf{x}) \\
\text{s.t.} \quad & \mathbf{n}\mathbf{x} \leq \mathbf{L} \\
& \mathbf{x} \in \mathbb{Z}_{+}^{M \times N},
\end{aligned} \tag{2}
$$

where $z(\mathbf{x})$ is the recourse function defined as

$$
z(\mathbf{x}) := E(z_\omega(\mathbf{x})) = \sum_{\omega \in \Omega} p_\omega z_\omega(\mathbf{x}),
$$

and for each scenario $\omega \in \Omega$,

$$
\begin{aligned}
z_\omega(\mathbf{x}) := \max \quad & \mathbf{f}'\mathbf{y}_\omega \\
\text{s.t.} \quad & \mathbf{x}\mathbf{1} + \mathbf{V}\mathbf{y}_\omega = \mathbf{d}_\omega \\
& \mathbf{y}_\omega \geq 0.
\end{aligned} \tag{3}
$$

Solve the Second Stage Problem

$$
\begin{aligned}
\min \quad & {\alpha'}_{\omega}(\mathbf{d}_{\omega} - \mathbf{x1}) \\
\text{s.t.} \quad & {\alpha'}_{\omega}\mathbf{V} \geq \mathbf{f}'
\end{aligned}
\tag{4}
$$

Let $P = \{\alpha | \alpha' V \geq \mathbf{f}'\}$. The feasible region of problem (4), P, is bounded. In addition, all the extreme points of P are integral.

Delayed Constraint Generation

Restricted Benders Master Problem

$$
\begin{aligned}
\max \quad & c'x + \sum_{\omega\in\Omega} p_\omega z_\omega \\
\text{s.t.} \quad & \sum_{i=1}^{M} n_i x_{ij} \le L_j, j \in [N] \\
& (\alpha^k)'(\mathbf{d}_\omega - \mathbf{x1}) \ge z_\omega, \alpha^k \in \mathcal{O}^t, \forall\omega \\
& \mathbf{x} \ge 0
\end{aligned} \tag{5}
$$

Benders Decomposition Algorithm

Step 1. Solve LP ? with all $\alpha_\omega^0 = \mathbf{0}$ for each scenario. Then, obtain the solution $(\mathbf{x}_0, \mathbf{z}^0)$.

Step 2. Set the upper bound $UB = c'\mathbf{x}_0 + \sum_{\omega\in\Omega} p_\omega z_\omega^0$.

Step 3. For x_0, we can obtain α_ω^1 and $z_\omega^{(0)}$ for each scenario, set the lower bound $LB = c'x_0 + \sum_{\omega\in\Omega} p_\omega z_\omega^{(0)}$

Step 4. For each ω, if $(\alpha_\omega^1)'(\mathbf{d}_\omega - \mathbf{x}_0\mathbf{1}) < z_\omega^0$, add one new constraint, $(\alpha_\omega^1)'(\mathbf{d}_\omega - \mathbf{x}\mathbf{1}) \geq z_\omega$, to RBMP.

Step 5. Solve the updated RBMP, obtain a new solution (x_1, z^1) and update UB.

Step 6. Repeat step 3 until $UB - LB < \epsilon$.(In our case, UB converges.)

Deterministic Formulation

When $|\Omega| = 1$ in problem (1), the stochastic programming will be

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{M}\sum_{j=1}^{N}(n_i - s)x_{ij} - \sum_{i=1}^{M} y_i^+ \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} - y_i^+ + y_{i+1}^+ + y_i^- = d_i, \quad i \in [M-1], \\
& \sum_{j=1}^{N} x_{ij} - y_i^+ + y_i^- = d_i, \quad i = M, \\
& \sum_{i=1}^{M} n_i x_{ij} \le L_j, j \in [N] \\
& y_i^+, y_i^- \in \mathbb{Z}_+, \quad i \in [M] \\
& x_{ij} \in \mathbb{Z}_+, \quad i \in [M], j \in [N].
\end{aligned}
\tag{6}
$$

Formulation

$$\begin{aligned}
\max \quad & \sum_{i=1}^{M} \sum_{j=1}^{N} (n_i - s) x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} \leq d_i, \quad i \in [M], \\
& \sum_{i=1}^{M} n_i x_{ij} \leq L_j, j \in [N] \\
& x_{ij} \in \mathbb{Z}_+, \quad i \in [M], j \in [N].
\end{aligned} \tag{7}$$

Analysis

Obtain the Feasible Seat Planning

Step 1. Obtain the solution, $\mathbf{x}^*$, from stochatic linear programming by benders decomposition.

Step 2. Aggregate the solution to the supply, $s_i^0 = \sum_j x_{ij}^*$.

Step 3. Obtain the optimal solution, $\mathbf{x}^1$, from problem (7) by setting the supply $\mathbf{s}^0$ as the upper bound.

Step 4. Aggregate the solution to the supply, $s_i^1 = \sum_j x_{ij}^1$.

Step 5. Obtain the optimal solution, $\mathbf{x}^2$, from problem ?? by setting the supply $\mathbf{s}^1$ as the lower bound.

Step 6. Aggregate the solution to the supply, $s_i^2 = \sum_j x_{ij}^2$, which is the feasible seat planning.

Dynamic Seat Assignment

Assign-to-seat Rules

- When the supply of one arriving group is enough, we will accept the group directly.
- When the supply of one arriving group is 0, the demand can be satisfied by only one larger-size supply.
- When one group is accepted to occupy the larger-size seats, the rest empty seat(s) can be reserved for future demand.

$$d(i,j) = i + (j-i-1)P(D_{j-i-1} \geq x_{j-i-1}+1) - jP(D_j \geq x_j), j > i$$

Dynamic Seat Assignment for Each Group Arrival

Step 1. Obtain the set of patterns, $\mathbf{P} = \{P_1, \ldots, P_N\}$, from the feasible seat planning algorithm. The corresponding aggregated supply is $\mathbf{X} = [x_1, \ldots, x_M]$.

Step 2. For the arrival group type i at period T', find the first $k \in [N]$ such that $i \in P_k$. Accept the group, update $P_k = P_k/(i)$ and $x_i = x_i - 1$. Go to step 4.

Step 3. If $i \notin P_k, \forall k \in [N]$, find $d(i, j^*)$. If $d(i, j^*) > 0$, find the first $k \in [N]$ such that $j^* \in P_k$. Accept group type i and update $P_k = P_k/(j^*)$, $x_{j^*} = x_{j^*} - 1$. Then update $x_{j-i-1} = x_{j-i-1} + 1$ and $P_k = P_k \cup (j^* - i - 1)$ when $j^* - i - 1 > 0$. If $d(i, j^*) \leq 0$, reject group type i.

Step 4. If $T' \leq T$, move to next period, set $T' = T' + 1$, go to step 2. Otherwise, terminate this algorithm.

Dynamic Seat Assignment after All Group Arrivals

$$V_t(L) = E_i[\max\{[V_{t-1}(L - n_i) + i], V_{t-1}(L)\}]$$

Results

Running time of Benders Decomposition and IP

Feasible Seat Planning versus IP Solution

Results of Different Policies

Result of Different Demands

Results of the Number of Arriving People per Period

The End