An Optimal Algorithm for Online Multiple Knapsack

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─── **Abstract** ───

In the online multiple knapsack problem, an algorithm faces a stream of items, and each item has to be either rejected or stored irrevocably in one of n bins (knapsacks) of equal size. The gain of an algorithm is equal to the sum of sizes of accepted items and the goal is to maximize the total gain.

So far, for this natural problem, the best solution was the 0.5-competitive algorithm FIRSTFIT (the result holds for any $n \geq 2$). We present the first algorithm that beats this ratio, achieving the competitive ratio of $1/(1 + \ln(2)) - O(1/n) \approx 0.5906 - O(1/n)$. Our algorithm is deterministic and optimal up to lower-order terms, as the upper bound of $1/(1 + \ln(2))$ for randomized solutions was given previously by Cygan et al. [TOCS 2016].

1 Introduction

Knapsack problems have been studied in theoretical computer science for decades [13, 14]. In particular, in the *multiple knapsack* problem [2, 5, 6, 7, 10, 12, 18], items of given sizes and profits have to be stored in n bins (knapsacks), each of capacity 1. The goal is to find a subset of all items that maximizes the total profit and can be feasibly packed into bins without exceeding their capacities. We consider an *online scenario*, where an online algorithm is given a sequence of items of unknown length. When an item is presented to an algorithm, it has to either irrevocably reject the item or accept it to a chosen bin (which cannot be changed in the future). The actions of an online algorithm have to be made without the knowledge of future items.

Proportional case. In this paper, we focus on the most natural, *proportional variant* (sometimes called *uniform*), where item profits are equal to item sizes and the goal is to maximize the sum of profits of all accepted items.

The single-bin case ($n = 1$) has been fully resolved: no deterministic online algorithm can be competitive [16], and the best randomized algorithm RONE by Böckenhauer et al. [4]

achieves the optimal competitive ratio of 0.5.[1]

Less is known for multiple-bin case ($n \geq 2$). Cygan et al. [7] showed that the FIRSTFIT algorithm is 0.5-competitive and proved that no algorithm (even a randomized one) can achieve a competitive ratio greater than R, where

$$R = 1/(1 + \ln 2) \approx 0.5906.$$

Other variants.   Some authors focused on the variant, where the goal is to maximize the *maximum* profit over all bins, instead of the sum of the profits. For this objective, optimal competitive ratios are already known: 0.5-competitive deterministic algorithm was given by Böckenhauer et al. [4], and the upper bound of 0.5 holding even for randomized solutions were presented by Cygan et al. [7].

The multiple knapsack problem can be generalized in another direction: profits and sizes may be unrelated. However, already *the unit variant*, where the profit of each item is equal to 1, does not admit any competitive solutions (even randomized ones) [5].

These results together mean that the proportional case studied in this paper is the only variant, whose online complexity has not been fully resolved yet.

1.1   Our results

The main result of this paper is an $(R - O(1/n))$-competitive deterministic online algorithm for the proportional variant of the multiple knapsack problem. We give insights for our construction in Section 1.3 below and the definition of our algorithm later in Section 2. Given the upper bound of R for randomized solutions [7], our result is optimal up to lower-order terms also for the class of randomized solutions.

It is possible to show that for deterministic algorithms, the term $O(1/n)$ in the competitive ratio is inevitable: we show how the upper bound construction given in [7] can be tweaked and extended to show that the competitive ratio of any deterministic algorithm is at most $R - O(1/n)$.

1.2   Related work

Some previous papers focused on a *removable* scenario, where an accepted item can be removed afterwards from its bin [2, 7, 8, 10, 11]. Achievable competitive ratios are better than their non-removable counterparts; in particular, the proportional variant admits constant-competitive deterministic algorithms even for a single bin [10].

The online knapsack problem has been also considered in relaxed variants: with resource augmentation, where the bin capacities of an online algorithm are larger than those of the optimal offline one [11, 17], with a resource buffer [9], or in the variant where an algorithm may accept fractions of items [17].

The hardness of the variants with arbitrary profits and sizes as well as applications to online auctions motivated another strand of research focused on the so-called random-order model [1, 3, 15, 19]. There, the set of items is chosen adversarially, but the items are presented to an online algorithm in random order.

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[1] An online algorithm is called α-competitive if, for any input instance, its total profit is at least fraction α of the optimal (offline) solution. While many papers use the reciprocal of α as the competitive ratio, the current definition is more suited for accounting arguments in our proofs.

1.3 Algorithmic challenges and ideas

Our algorithm splits items into three categories: large (of size greater than $1/2$), medium (of size from the interval $[\phi, 1/2]$) and small (of size smaller than ϕ). We defer the actual definition of ϕ.

First, we explain what an online algorithm should do when it faces a stream of large items. Note that no two large items can fit together in a single bin. If an algorithm greedily collects all large items, then the adversary may give n items of size $1/2 + \epsilon$ (accepted by an online algorithm) followed by n items of size 1 (accepted by an optimal offline algorithm OPT), and the resulting competitive ratio is then 0.5. On the other hand, if an algorithm stops after accepting some number of large items, OPT may collect all of them.

Our RISING THRESHOLD ALGORITHM (RTA) balances these two strategies. It chooses a non-decreasing threshold function $\mathsf{f}\colon [0,1] \to [1/2, 1]$ and ensures that the size of the i-th accepted large item is at least $\mathsf{f}(i/n)$. While an actual definition of f is given later, to grasp a general idea, it is worth looking at its plot in Figure 1 (left). A natural adversarial strategy is to give large items meeting these thresholds, and once RTA fills k bins, present n items of sizes slightly smaller than the next threshold $\mathsf{f}((k+1)/n)$. These items will be rejected by RTA but can be accepted by OPT. Analyzing this strategy and ensuring that the ratio is at least R for any choice of k yields boundary conditions. Analyzing these conditions for n tending to infinity, we obtain a differential equation, whose solution is the function f used in our algorithm.

The actual difficulty, however, is posed by medium items. RTA never proactively rejects them and it keeps a subset of *marked* medium items in their own bins (one item per one bin), while it stacks the remaining, non-marked ones (places them together in the same bin, possibly combining items of similar sizes). This strategy allows RTA to combine a large item with marked medium items later. However, the amount of marked items has to be carefully managed as they do not contribute large gain alone. A typical approach would be to partition medium items into discrete sub-classes, control the number of items in each class, and analyze the gain on the basis of the *minimal* size item in a particular subclass. To achieve optimal competitive ratio, we however need a more fine-grained approach: we use a carefully crafted continuous function ξ to control the number of marked items larger than a given value. Analyzing all possible adversarial strategies gives boundary conditions for ξ. In particular, the value ϕ that separates medium items from small ones was chosen as the minimum value that ensures the existence of function ξ satisfying all boundary conditions.

Finally, we note that simply stacking small items in their own bins would not lead to the desired competitive ratio. Instead, RTA tries to stack them in a single bin, but whenever its load exceeds ϕ, RTA tries to merge them into a single medium item and verify whether such an item could be marked. This allows for combining them in critical cases with large items.

1.4 Preliminaries

We have n bins of capacity 1, numbered from 1 to n. An input is a stream of items from $(0,1]$, defined by their sizes. Upon seeing an item, an online algorithm has to either reject it or place it in an arbitrary bin without violating the bin's capacity. The *load* of a bin b, denoted $\mathsf{load}(b)$, is the sum of item sizes stored in bin b. We define the load of a set of items as the sum of their sizes and the total load as the load of all items collected by an algorithm. Additionally, for any $x \le 1/2$, we define $\mathsf{pile}(x) = \max\{2/3, 2x\}$. Note that if we put medium items of sizes at least x (till it is possible) into a bin b, then $\mathsf{load}(b) \ge \mathsf{pile}(x)$.

To simplify calculations, for any set Z of items, we define the *gain* of Z, denoted $\mathsf{g}(Z)$, as

their load divided by n; similarly, the total gain is the total load divided by n. Furthermore, we use $\mathsf{min}(Z)$ to denote the minimum size of an item in set Z. If Z is accepted by our online algorithm, $\mathsf{b}(Z)$ denotes the number of bins our algorithm uses to accommodate these items, divided by n. For any value $x \in [0, 1]$, $Z^{\geq x}$ is the set of all items from Z of size greater or equal x. Whenever we use terms $\mathsf{g}(Z)$, $\mathsf{min}(Z)$ or $\mathsf{b}(Z)$ for a set Z that varies during runtime, we mean these values for the set Z after an online algorithm terminates its execution.

For any input sequence σ and an algorithm A, we use $A(\sigma)$ to denote the total gain of A on sequence σ. We denote the optimal offline algorithm by Opt.

1.5   Neglecting lower-order terms

As our goal is to show the competitive ratio $R - O(1/n)$, we introduce a notation that allows to neglect terms of order $1/n$. We say that x is *approximately equal* to y (we write $x \approxeq y$) if $|x - y| = O(1/n)$. Furthermore, we say that x is *approximately greater* than y (we write $x \gtrsim y$) if $x \geq y$ or $x \approxeq y$; we define relation $\lesssim$ analogously. Each of these relations is transitive when composed a constant number of times.

In our analysis, we are dealing with Lipschitz continuous functions (their derivative is bounded by a universal constant). For such function h, (i) the relation $\approxeq$ is preserved after application of h, and (ii) an integral of h can be approximated by a sum, as stated in the following facts, used extensively in the paper.

▶ **Fact 1.** *Fix any Lipschitz continuous function h and values $x \approxeq y$ from its domain. Then, $h(x) \approxeq h(y)$. Furthermore, if h is non-decreasing, then $x \lesssim y$ implies $h(x) \lesssim h(y)$ and $x \gtrsim y$ implies $h(x) \gtrsim h(y)$.*

▶ **Fact 2.** *For any Lipschitz continuous function h and integers a, b satisfying $1 \leq a \leq b \leq n$, it holds that $(1/n) \cdot \sum_{i=a+1}^{b} h(i/n) \approxeq \int_{a/n}^{b/n} h(x)\, \mathrm{d}x$.*

1.6   Roadmap of the proof

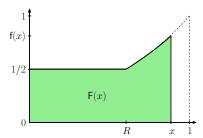We present our algorithm in Section 2. Its analysis consists of three main parts.

- In Section 3, we investigate the gain of Rta on large items and explain the choice of the threshold function f.
- In Section 4, we study properties of medium items, marking routine, function ξ and show how the marked items influence the gain on other non-large items.
- In Section 5, we study the impact of marked items on bins containing large items.

Each of these parts is concluded with a statement that, under certain conditions, Rta is $(R - O(1/n))$-competitive (cf. Lemma 5, Lemma 15, Lemma 20 and Lemma 21). In Section 6, we argue that these lemmas cover all possible outcomes. For succinctness, some technical claims have been moved to Section 7.

2   Rising Threshold Algorithm

We arrange items into three categories: small, medium and large. We say that an item is *large* if its size is in the range $(1/2, 1]$, *medium* if it is in the range $[\phi, 1/2]$, and otherwise it is *small*, where we define

$$\xi_{\mathrm{c}} = (1 + (2/3) \cdot \ln(4/3)) \cdot R - 2/3 \approx 0.0372 \quad \text{and} \tag{1}$$

$$\phi = (2/3) \cdot \xi_{\mathrm{c}} \, / \, (2/3 - R + \xi_{\mathrm{c}}) \approx 0.2191. \tag{2}$$
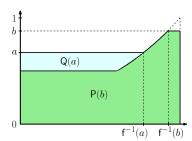
Figure 1 Left: function f and its integral F. The value of $F(x)$ roughly corresponds to our lower bound on the gain of RTA when it collects $n \cdot x$ large items. Right: functions P and Q used in estimating the gain in Section 4 and Section 5; note that their arguments are marked at Y axis.

We further arrange medium items into subcategories M_2, M_3 and M_4: a medium item belongs to M_i if its size is from range $(1/(i+1), 1/i]$. As we partition only medium items this way, M_4 contains items of sizes from $[\phi, 1/4]$. Note that at most i items of category M_i fit in a single bin.

At some times (defined precisely later) a group of small items of a total load from $[\phi, 2\phi)$ stored in a single bin may become *merged*, and from that point is treated as a single medium item. We ensure that such merging action does not violate invariants of our algorithm.

Our algorithm RTA applies labels to bins; the possible labels are E, A, S_*, M_S, M_2, M_3, M_4 and L_+. Each bin starts as an E-bin, and RTA can relabel it later. The label determines the content of a given bin:

- an E-bin is empty,
- an A-bin (an *auxiliary bin*) contains small items of a total load smaller than ϕ and at most one A-bin exists at any time,
- an S_*-bin contains one or multiple small items,
- an M_S-bin contains a single marked medium item,
- an M_i-bin contains one or more medium items of category M_i,
- an L_+-bin contains a single large item and possibly some other non-large ones.

For any label C, we define a corresponding set, also denoted C, containing all items stored in bins of label C. For instance, L_+ is a set containing all items stored in L_+-bins. Furthermore, we define L as the set of all large items (clearly $L \subseteq L_+$ and $b(L) = b(L_+)$) and the set $M_* = M_2 \uplus M_3 \uplus M_4$.

RTA processes a stream of items, and it operates until the stream ends or there are no more empty bins (even if an incoming item could fit in some partially filled bin). Upon the arrival of an item, RTA classifies it by its size and proceeds as described below.

Large items. Whenever a large item arrives, RTA compares its size with the threshold $f(b(L) + 1/n)$, and if the item is smaller, RTA rejects it. The function $f : [0, 1] \to [1/2, 1]$ is defined as

$$f(x) = \begin{cases} 1/2 & \text{if } x \leq R, \\ (2e)^{x-1} & \text{otherwise,} \end{cases} \tag{3}$$

and depicted in Figure 1 (left). If the item meets the threshold, RTA attempts to put it in an M_S-bin with sufficient space left (relabeling it to L_+), and if no such bin exists, RTA puts the item in any empty bin.

Medium items. We fix a continuous and decreasing function ξ that maps medium item sizes to $[0, \xi_c/\phi]$:

$$\xi(x) = \begin{cases} \xi_c/x & \text{if } x \in [\phi, 1/3], \\ 9\xi_c \cdot (1 - 2x) & \text{if } x \in (1/3, 1/2]. \end{cases} \tag{4}$$

We say that the subset Z of medium items is ξ-dominated if $|Z^{\geq x}|/n \leq \xi(x)$ for any $x \in Z$. Intuitively, it means that if we sort items of Z from largest to smallest, then all points $(i/n, x_i)$ are under or at the plot of ξ^{-1}, see Figure 3 (left).

RTA never proactively rejects medium items, i.e., it always accepts them if it has an empty bin. Some medium items become *marked* upon arrival; we denote the set of medium marked items by D. Large or small items are never marked. Marked medium items are never combined in a single bin with other marked medium items. At all times, RTA ensures that the set D is ξ-dominated. As no two items from D are stored in a single bin, this corresponds to the condition $b(D^{\geq x}) \leq \xi(x)$ for any $x \in D$. Each marked item is stored either in an M_S-bin (alone) or in an L_+-bin (together with a large item and possibly some other non-marked items). That is, $M_S \subseteq D \subseteq M_S \uplus L_+$.

Whenever a medium item arrives, RTA attempts to put it in an L_+-bin. If it does not fit there, RTA verifies whether marking it (including it in the set D) preserves ξ-domination of D. If so, RTA marks it and stores it in a separate M_S-bin. Otherwise, RTA *fails to mark* the item and the item is stored in an M_i-bin (where i depends on the item size): it is added to an existing bin whenever possible and a new M_i-bin is opened only when necessary.

We emphasize that if RTA puts a large item in an M_S-bin later (and relabel it to L_+), the sole medium item from this bin remains marked (i.e., in the set D). However, if a medium item fits in an L_+-bin at the time of its arrival, it avoids being marked, even though its inclusion might not violate ξ-dominance of the set D. Note also that M_* contains medium items RTA failed to mark.

Small items. RTA never proactively rejects any small item. Whenever a small item arrives, RTA attempts to put this item in an L_+-bin, in an S_*-bin, and in the A-bin, in this exact order. If the item does not fit in any of them (this is possible only if the A-bin does not exist), RTA places it in an empty bin and relabels this bin to A.

If RTA places the small item in an already existing A-bin and in effect its load reaches or exceeds ϕ, RTA attempts to *merge* all its items into a single medium marked item. If the resulting medium item can be marked and included in D without violating its ξ-dominance, RTA relabels the A-bin to M_S and treats its contents as a single marked medium item from now on. Otherwise, it simply changes the label of the A-bin to S_*.

3 Gain on large items

In this section, we analyze the gain of RTA on large items. To this end, we first calculate the integral of function f, denoted F (see Figure 1, left) and list its properties that can be verified by routine calculations.

$$\mathsf{F}(x) = \int_0^x \mathsf{f}(y)\, dy = \begin{cases} x/2 & \text{if } x \leq R, \\ R \cdot (2e)^{x-1} & \text{otherwise.} \end{cases} \tag{5}$$

▶ **Lemma 3.** *The following properties hold for function F.*
1. $f^{-1}(c) = 1 + R \cdot \ln c$ *and* $\mathsf{F}(f^{-1}(c)) = R \cdot c$ *for any* $c \in (1/2, 1]$.

2. $F(x)/f(x) = \min\{x, R\}$ *for any* $x \in [0, 1]$.
3. $(1/n) \cdot \sum_{i=1}^{\ell} f(i/n) \approx \int_0^{\ell/n} f(x)\, dx = F(\ell/n)$ *for any* $\ell \in \{0, \dots, n\}$.

Using Lemma 3, we may bound on the gain of RTA on large items L and use this bound to estimate its competitive ratio when it terminates with empty bins.

▶ **Lemma 4.** *It holds that* $g(L) \gtrsim F(b(L)) = F(b(L_+))$. *Moreover,* $g(L) \gtrsim F(b(L)) + g(L^{\geq x}) - x \cdot b(L^{\geq x})$ *for any* $x \geq f(b(L))$.

Proof. For the first part of the lemma, we sort large items from L in the order they were accepted by RTA. The size of the i-th large item is at least the threshold $f(i/n)$. Hence, by Lemma 3, $g(L) \geq (1/n) \cdot \sum_{i=1}^{|L|} f(i/n) \approx F(|L|/n) = F(b(L))$.

To show the second part, we fix any $x \geq f(b(L))$ and for each large item of size greater than x we reduce its size to x. The total gain of the removed parts is exactly $g(L^{\geq x}) - x \cdot b(L^{\geq x})$. The resulting large item sizes still satisfy acceptance thresholds, and thus the gain on the remaining part of L is approximately greater than $F(b(L))$. Summing up yields $g(L) \gtrsim F(b(L)) + g(L^{\geq x}) - x \cdot b(L^{\geq x})$. ◀

3.1 When RTA terminates with some empty bins

▶ **Lemma 5.** *If* RTA *terminates with some empty bins, then it is* $(R - O(1/n))$-*competitive.*

Proof. Fix an input sequence σ. As RTA terminates with empty bins, it manages to accept all medium and small items from σ. Furthermore, it accepts large items from σ according to the thresholds given by function f. Recall that f is non-decreasing: at the beginning it is equal to $1/2$ (RTA accepts any large item) and the acceptance threshold grows as RTA accepts more large items. Let $x = f(b(L) + 1/n)$ be the value of the acceptance threshold for large items when RTA terminates. We consider two cases.

- $b(L) \leq R - 1/n$. The threshold used for each large item is at most $x \leq f(R) = 1/2$, i.e., RTA accepts all large items. Then, RTA accepts all items and is 1-competitive.
- $b(L) > R - 1/n$. Let N be the set of all non-large items accepted by RTA. By Lemma 4,

$$\text{RTA}(\sigma) = g(L) + g(N) \gtrsim F(b(L)) + g(L^{\geq x}) - x \cdot b(L^{\geq x}) + g(N)$$
$$\approx R \cdot x + g(L^{\geq x}) - x \cdot b(L^{\geq x}) + g(N).$$

where for the last relation we used $F(b(L)) \approx F(f^{-1}(x)) = R \cdot x$ (by Lemma 3).

As RTA takes all non-large items and all large items that are at least x, the input sequence σ contains items taken by RTA and possibly some large items smaller than x. Thus, the gain of OPT on large items is maximized when it takes $L^{\geq x}$ and fills the remaining $n - |L^{\geq x}|$ bins with large items from σ smaller than x. The total gain of OPT is thus at most

$$\text{OPT}(\sigma) \leq g(L^{\geq x}) + x \cdot (1 - b(L^{\geq x})) + g(N) = x + g(L^{\geq x}) - x \cdot b(L^{\geq x}) + g(N).$$

Comparing the bounds on gains of RTA and OPT and observing that the term $g(L^{\geq x}) - x \cdot b(L^{\geq x}) + g(N)$ is non-negative, yields $\text{RTA}(\sigma) \geq R \cdot \text{OPT}(\sigma) - O(1/n)$. As $\text{OPT}(\sigma) \geq g(L) = \Omega(1)$, we obtain $\text{RTA}(\sigma) \geq (R - O(1/n)) \cdot \text{OPT}(\sigma)$. ◀

As an immediate corollary, we observe that if σ contains large items only, then RTA is $(R - O(1/n))$-competitive: If it terminates with empty bins, then its competitive ratio follows by Lemma 5. Otherwise, it terminates with n large items, and hence, by Lemma 4, $\text{RTA}(\sigma) \gtrsim F(b(L)) = F(1) = R$. On the other hand, $\text{OPT}(\sigma) \leq 1$, and therefore the competitive ratio is at most $R - O(1/n)$ also in this case.
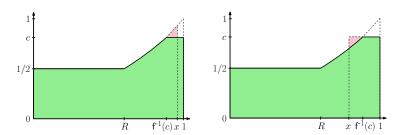
■ **Figure 2** A geometric interpretation of the second property of Lemma 6: using P to lower-bound the sum of $\mathsf{F}(x)$ and the rectangle $c \cdot (1 - x)$ for the case $\mathsf{f}^{-1}(c) \leq x$ (left) and $\mathsf{f}^{-1}(c) > x$ (right).

4   Gain on medium items

In the remaining part of the analysis, we make use of the following functions. For any $c \in (1/2, 1]$, let $\mathsf{P}(c) = \int_0^1 \min\{\mathsf{f}(y), c\} \, \mathrm{d}y$ and $\mathsf{Q}(c) = \int_0^1 \max\{c - \mathsf{f}(y), 0\} \, \mathrm{d}y$. Both functions are increasing and depicted in Figure 1 (right). As we show below (cf. the last property of Lemma 6), $\mathsf{P}(c)$ lower-bounds the gain of RTA in the case when its load on non-L_+ bins is at least c.

▶ **Lemma 6.** *Fix any $c \in (1/2, 1]$ and any $x \in [0, 1]$. It holds that*
1. $\mathsf{P}(c) = c - R \cdot c \cdot \ln(2c)$,
2. $\mathsf{Q}(c) = R \cdot c \cdot \ln(2c)$,
3. $\mathsf{P}(c) + \mathsf{Q}(c) = c$,
4. $\mathsf{F}(x) + c \cdot (1 - x) \geq \mathsf{P}(c)$.

Proof of Lemma 6. We fix any $c \in (1/2, 1]$ and any $x \in [0, 1]$. For the first property, observe that

$$\mathsf{P}(c) = \int_0^{\mathsf{f}^{-1}(c)} \mathsf{f}(y) \, \mathrm{d}y + \int_{\mathsf{f}^{-1}(c)}^1 c \, \mathrm{d}y = \mathsf{F}(\mathsf{f}^{-1}(c)) + c \cdot (1 - \mathsf{f}^{-1}(c)) = c - R \cdot c \cdot \ln(2c),$$

where for the last equality we used Lemma 3. Similarly, the second property follows as

$$\mathsf{Q}(c) = \int_0^{\mathsf{f}^{-1}(c)} c - \mathsf{f}(y) \, \mathrm{d}y = c \cdot \mathsf{f}^{-1}(c) - \mathsf{F}(\mathsf{f}^{-1}(c)) = R \cdot c \cdot \ln(2c).$$

The third relation, $\mathsf{P}(c) + \mathsf{Q}(c) = c$, follows immediately by the first two. Finally, for the last relation, we use

$$\mathsf{F}(x) + c \cdot (1 - x) = \int_0^x \mathsf{f}(y) \, \mathrm{d}y + \int_x^1 c \, \mathrm{d}y \geq \int_0^x \min\{\mathsf{f}(y), c\} \, \mathrm{d}x + \int_x^1 \min\{f(y), c\} \, \mathrm{d}y = \mathsf{P}(c).$$

See also Figure 2 for a geometric argument.   ◀

4.1   Boundary conditions on function ξ

We start with a shorthand notation. Let $\mathsf{T}(a, b) = (a + b - 1/2) \cdot (\xi(b) - \xi(a))$, where $a, b \in [\phi, 1/2]$ (so that the values of $\xi(a)$ and $\xi(b)$ are well defined).

Our choice of function ξ satisfies the conditions below. In fact, for our analysis to hold, function ξ could be replaced by any Lipschitz continuous and non-increasing function mapping $[\phi, 1/2]$ to $[0, 1]$ satisfying these properties.

▶ **Lemma 7.** *The following properties hold for function ξ:*
1. $x \cdot \xi(x)$ *is a non-increasing function of* $x \in [\phi, 1/2]$,
2. $P(\mathsf{pile}(x)) + x \cdot \xi(x) \geq R$ *for* $x \in [\phi, 1/2]$,
3. $P(1 - \phi) + 2\phi \cdot \xi(2\phi) \geq R$,
4. $2/3 - (2/3 - \phi) \cdot \xi(x) \geq R$ *for* $x \in [\phi, 1/3]$,
5. $P(1 - \phi) + Q(1 - x) + (x + \phi - 1) \cdot \xi(x) + \max\{T(x, y), 0\} \geq R$ *for* $x \in [1/3, 1/2]$ *and* $y \in [\phi, 2\phi]$,
6. $P(\mathsf{pile}(y)) + Q(1 - x) + (x - \mathsf{pile}(y)) \cdot \xi(x) + T(x, y) \geq R$ *for* $x \in [1/3, 1/2]$ *and* $y \in [\phi, x]$,
7. $P(2x) + Q(1 - x) - x \cdot \xi(x) \geq R$ *for* $x \in [1/3, 1/2]$.

4.2 Marked and tight items

We start with a simple bound on the gain of RTA on M_S-bins. Recall that these bins store single marked items.

▶ **Lemma 8.** *If* RTA *terminates with at least one M_S-bin, then* $\mathsf{g}(M_S) \geq \mathsf{min}(M_S) \cdot \mathsf{b}(M_S)$, *and* $\mathsf{b}(M_S) \leq \xi(\mathsf{min}(M_S))$.

Proof. The first condition follows trivially as each M_S-bin contains a single medium item of size at least $\mathsf{min}(M_S)$. For the second condition, note that $M_S \subseteq D$, and thus also $M_S \subseteq D^{\geq \mathsf{min}(M_S)}$. As D is ξ-dominated, $\mathsf{b}(M_S) \leq \mathsf{b}(D^{\geq \mathsf{min}(M_S)}) \leq \xi(\mathsf{min}(M_S))$. ◀

We now take a closer look at the marked items and their influence on the gain on other sets of items. We say that a medium marked item $x \in D$ is *tight* if it is on the verge of violating ξ-domination invariant.

▶ **Definition 9.** *An item $x \in D$ is tight if* $\mathsf{b}(D^{\geq x}) > \xi(x) - 1/n$.

If an item $x \in D$ is tight, then another item of size x or greater cannot be included in D without violating ξ-domination invariant. Figure 3 (left) illustrates this concept. As D can only grow, once an item becomes tight, it remains tight till the end. We emphasize that items smaller than x are not relevant for determining whether x is tight. If D contains a tight item, then $\mathsf{mt}(D)$ denotes the size of the minimum tight item in D. This important parameter influences the gain both on set D and also on stacking bins M_* and S_*.

▶ **Lemma 10.** *If D contains a tight item, then* $\mathsf{g}(D) \gtrsim \mathsf{mt}(D) \cdot \xi(\mathsf{mt}(D))$.

Proof. Fix a tight item $d \in D$ of size $\mathsf{mt}(D)$. By Definition 9, $\mathsf{b}(D^{\geq d}) > \xi(d) - 1/n$, and thus $\mathsf{g}(D) \geq \mathsf{g}(D^{\geq d}) \geq d \cdot \mathsf{b}(D^{\geq d}) \gtrsim d \cdot \xi(d)$. ◀

4.3 Impact of tight items on stacking bins

By Property 1 of Lemma 7, $x \cdot \xi(x)$ is a non-increasing function of x. Therefore, the smaller $\mathsf{mt}(D)$ is, the larger is the lower bound on $\mathsf{g}(D)$ guaranteed by Lemma 10. Now we argue that the larger $\mathsf{mt}(D)$ is, the better is the gain on stacking bins M_* and S_*.

▶ **Lemma 11.** *Assume* RTA *failed to mark a medium item y. Then, a tight item exists and* $\mathsf{mt}(D) \lesssim y$.

Proof. Let $D_{\mathrm{ext}} = D \cup \{y\}$. By the lemma assumption, D_{ext} is not ξ-dominated, i.e., there exists an item $x \in D_{\mathrm{ext}}$ such that $\mathsf{b}(D_{\mathrm{ext}}^{\geq x}) > \xi(x)$. Note that $x \leq y$, as otherwise we would have $D^{\geq x} = D_{\mathrm{ext}}^{\geq x}$, and thus $\mathsf{b}(D^{\geq x}) > \xi(x)$, which would contradict ξ-domination of D.
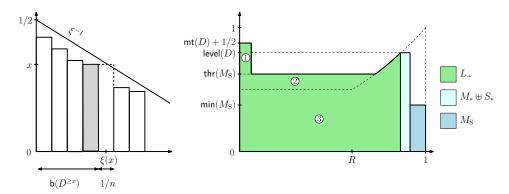
■ **Figure 3** Left: set D of marked items with a tight (gray) item of size x. As x is tight, insertion of another item of size x (with a dashed border) would violate ξ-domination of D. Right: items collected by RTA, when it terminates without empty bins and with M_S-bins. Gain on L_+-sets is split into three parts, where the first part corresponds to the gain of T^* (cf. Lemma 17). The minimum guaranteed load in M_S-bins is given by Lemma 8 and in the bins of $M_* \uplus S_*$ by Lemma 13.

Let $d \geq x$ be the minimum size of an item from $D^{\geq x}$. Then, $D^{\geq d} = D^{\geq x}$, and thus

$$\mathsf{b}(D^{\geq d}) = \mathsf{b}(D^{\geq x}) = \mathsf{b}(D^{\geq x}_{\text{ext}}) - 1/n > \xi(x) - 1/n \geq \xi(d) - 1/n, \tag{6}$$

where in the last inequality we used monotonicity of ξ. By (6), d is tight. On the other hand, ξ-domination of D implies that $\mathsf{b}(D^{\geq d}) \leq \xi(d)$. This, combined with (6), yields $\xi(d) \approx \xi(x)$, and thus $d \approx x \leq y$. Note that d remains tight till the end of the execution. This concludes the lemma, as the minimum tight item, $\mathsf{mt}(D)$, can be only smaller than d. ◀

▶ **Lemma 12.** *If* RTA *finishes*
- *with at least one M_*-bin, then $\mathsf{mt}(D)$ is defined and $\mathsf{mt}(D) \lesssim \mathsf{min}(M_*)$;*
- *with at least one S_*-bin, then $\mathsf{mt}(D)$ is defined and $\mathsf{mt}(D) \lesssim 2\phi$.*

Proof. For the first part of the lemma, fix a medium item from M_* of size $\mathsf{min}(M_*)$. By the definition of RTA, it failed to mark this item. Hence, by Lemma 11, $\mathsf{mt}(D)$ is defined and $\mathsf{mt}(D) \lesssim \mathsf{min}(M_*)$.

Assume now that RTA finishes with at least one S_*-bin. When the first such S_*-bin was created, RTA placed a small item $s < \phi$ in the already existing A-bin of load $r < \phi$, and the merge action failed, because RTA failed to mark the resulting item of size $s + r$. Thus, again by Lemma 11, $\mathsf{mt}(D)$ is defined and $\mathsf{mt}(D) \lesssim s + r < 2\phi$. ◀

To estimate the gain on M_*-bins and S_*-bins, we define

$$\mathsf{level}_* = \begin{cases} \mathsf{min}\{\mathsf{pile}(\mathsf{mt}(D)), 1 - \phi\} & \text{if } D \text{ contains a tight item and } S_* \neq \emptyset, \\ \mathsf{pile}(\mathsf{mt}(D)) & \text{if } D \text{ contains a tight item and } S_* = \emptyset, \\ 1 & \text{if } D \text{ does not contain any tight item.} \end{cases} \tag{7}$$

▶ **Lemma 13.** *It holds that $\mathsf{g}(M_* \uplus S_*) \gtrsim \mathsf{level}_* \cdot \mathsf{b}(M_* \uplus S_*)$.*

Proof. If D does not contain a tight item, then, by Lemma 12, both M_* and S_* are empty, and the lemma follows trivially. Thus, in the following we assume that D contains a tight item and we take a closer look at the contents of S_*-bins and M_*-bins.

Assume that M_* is non-empty. RTA creates a new M_i-bin (for $i \in \{2, 3, 4\}$) if the incoming medium item of category M_i (of size $(1/(i+1), 1/i]$) does not fit in any of the existing M_i

bins. Hence, each M_i-bin (except at most one) has exactly i items, and therefore its load is greater than $i/(i+1) \geq 2/3$ and is also at least $i \cdot \min(M_i) \geq 2 \cdot \min(M_*)$. Thus,

$$\mathsf{g}(M_*) \gtrsim \max\{2/3, 2 \cdot \min(M_*)\} \cdot \mathsf{b}(M_*) = \mathsf{pile}(\min(M_*)) \cdot \mathsf{b}(M_*) \gtrsim \mathsf{pile}(\mathsf{mt}(D)) \cdot \mathsf{b}(M_*), \quad (8)$$

where the second inequality follows by Lemma 12 and by monotonicity of function pile. Note that (8) holds trivially also when there are no M_*-bins.

If there are no S_*-bins, then, $\mathsf{g}(M_* \uplus S_*) = \mathsf{g}(M_*) \gtrsim \mathsf{pile}(\mathsf{mt}(D)) \cdot \mathsf{b}(M_*) = \mathsf{level}_* \cdot \mathsf{b}(M_* \uplus S_*)$, and the lemma follows.

If there are some S_*-bins, recall that RTA creates a new S_*-bin only if the considered small item does not fit in any existing S_*-bin. Thus, the load of each S_*-bin (except at most one) is at least $1 - \phi$, and therefore $\mathsf{g}(S_*) \gtrsim (1 - \phi) \cdot \mathsf{b}(S_*)$. Combining this with (8) implies $\mathsf{g}(M_* \uplus S_*) \gtrsim \mathsf{pile}(\mathsf{mt}(D)) \cdot \mathsf{b}(M_*) + (1 - \phi) \cdot \mathsf{b}(S_*) \geq \min\{\mathsf{pile}(\mathsf{mt}(D)), 1 - \phi\} \cdot \mathsf{b}(M_* \uplus S_*)$. ◄

4.4 When RTA terminates without empty bins and without M_S-bins

Using tight items, we may analyze the case when RTA terminates without empty bins and without M_S-bins, and show that in such case its gain is approximately greater than R. As the gain of OPT is at most 1, this yields the desired competitive ratio.

▶ **Lemma 14.** *If* RTA *terminates without empty bins, then* $\mathsf{b}(L) + \mathsf{b}(M_S) + \mathsf{b}(M_*) + \mathsf{b}(S_*) \approx 1$.

Proof. There is at most one A-bin. The remaining bins (at least $n - 1$ many) are of classes L_+, M_S, M_* or S_*, and thus $\mathsf{b}(L) + \mathsf{b}(M_S) + \mathsf{b}(M_*) + \mathsf{b}(S_*) \approx 1$. ◄

▶ **Lemma 15.** *If on input σ,* RTA *terminates without empty bins and without M_S-bins, then* $\mathrm{RTA}(\sigma) \gtrsim R$.

Proof. We analyze the gain of RTA on three disjoint sets: L, D and $M_* \uplus S_*$.

$$\begin{aligned}
\mathrm{RTA}(\sigma) &\geq \mathsf{g}(L) + \mathsf{g}(M_* \uplus S_*) + \mathsf{g}(D) \\
&\gtrsim \mathsf{F}(\mathsf{b}(L)) + \mathsf{level}_* \cdot (1 - \mathsf{b}(L)) + \mathsf{g}(D) && \text{(by L. 4, L. 13 and L. 14)} \\
&\gtrsim \mathsf{P}(\mathsf{level}_*) + \mathsf{g}(D) && \text{(by L. 6)}
\end{aligned}$$

If D does not contain a tight item, then $\mathsf{level}_* = 1$, and thus $\mathrm{RTA}(\sigma) \gtrsim \mathsf{P}(1) = R$.

If D contains a tight item, then by Lemma 10, $\mathsf{g}(D) \geq \mathsf{mt}(D) \cdot \xi(\mathsf{mt}(D))$, and therefore $\mathrm{RTA}(\sigma) \gtrsim \mathsf{P}(\mathsf{level}_*) + \mathsf{mt}(D) \cdot \xi(\mathsf{mt}(D))$. We consider two cases.

- If $\mathsf{level}_* \geq \mathsf{pile}(\mathsf{mt}(D))$, then $\mathrm{RTA}(\sigma) \gtrsim \mathsf{P}(\mathsf{pile}(\mathsf{mt}(D))) + \mathsf{mt}(D) \cdot \xi(\mathsf{mt}(D)) \geq R$, where the last inequality follows by Property 2 of Lemma 7.
- The opposite case, $\mathsf{level}_* < \mathsf{pile}(\mathsf{mt}(D))$, is possible only if S_*-bins exist and $\mathsf{level}_* = 1 - \phi$. By Lemma 12, the existence of S_*-bins implies $\mathsf{mt}(D) \lesssim 2\phi$. As the function $x \cdot \xi(x)$ is non-increasing (cf. Property 1 of Lemma 7), $\mathrm{RTA}(\sigma) \gtrsim \mathsf{P}(1 - \phi) + 2\phi \cdot \xi(2\phi) \geq R$. The last inequality follows by Property 3 of Lemma 7. ◄

5 Gain on large items revisited

In this section, we assume that RTA terminates without empty bins and with at least one M_S-bin. Recall that Lemma 4 allows us to estimate $\mathsf{g}(L_+)$ by calculating the gain on *large* items alone. Now we show how to improve this bound by taking into account non-large items in L_+. First, we leverage the fact that if a (marked) medium item is in M_S, then RTA must have failed to combine it with a large item, and we obtain a better lower bound on the size

of each large item. Second, we show that in some cases marked medium items must be in L_+ which increases its load. If an M_S-bin exists, we define

$$\mathsf{T}^* = \begin{cases} \mathsf{T}(\min(M_S), \mathsf{mt}(D)) & \text{if } \mathsf{mt}(D) \text{ is defined and } \min(M_S) > \max\{\mathsf{mt}(D), 1/3\} \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathsf{thr}(M_S) = \min\{1 - \min(M_S), 1/2 + \phi\}.$$

Note that T^* is always non-negative. In particular, $\mathsf{T}(\min(M_S), \mathsf{mt}(D)) = (\min(M_S) + \mathsf{mt}(D) - 1/2) \cdot (\xi(\mathsf{mt}(D)) - \xi(\min(M_S)) \geq 0$ because $\min(M_S) + \mathsf{mt}(D) \geq 1/2$ for $\min(M_S) > 1/3$ and $\mathsf{mt}(D) \geq \phi$.

▶ **Lemma 16.** *Assume* RTA *terminates with at least one M_S-bin. Then, the load of any L_+-bin is at least $\mathsf{thr}(M_S)$.*

Proof. Consider any L_+-bin b and let y be the large item contained in this bin. If b contains an additional medium item, then its load is greater than $1/2 + \phi$, and the lemma follows. Hence, in the following, we assume that y was not combined with a medium item in a single bin. As RTA finishes with an M_S-bin, we fix a medium item x of size $\min(M_S)$. We consider three cases.

- Item x arrived (or was created by merging some small items) before the arrival of y. RTA did not place y in the M_S-bin containing x, because $y + x > 1$. Thus, $\mathsf{load}(b) \geq y \geq 1 - x = 1 - \min(M_S)$.
- Item x arrived after the arrival of y. (Some small items might be placed together with y prior to the arrival of x.) As RTA placed x in a separate bin, it did not fit in b, i.e., the load of b at the time of the arrival of x was greater than $1 - x = 1 - \min(M_S)$.
- Item x was created by merging small items after the arrival of y. Let $s < x$ be the small item that caused the creation of x. RTA placed s in A-bin, because s did not fit in b, i.e., the load of b at that time was greater than $1 - s > 1 - x = 1 - \min(M_S)$. ◀

▶ **Lemma 17.** *Assume that* RTA *terminates with at least one M_S-bin. Then, $\mathsf{g}(L_+) \gtrsim \mathsf{T}^* + \int_0^{b(L)} \max\{\mathsf{f}(y), \mathsf{thr}(M_S)\}\, \mathrm{d}y$.*

Proof. We sort accepted large items by their arrival time and denote the bin containing the i-th large item by b_i. The bin b_i contains a large item of size at least $\mathsf{f}(i/n)$ because of the acceptance threshold, and its load is at least $\mathsf{thr}(M_S)$ by Lemma 16, i.e., $\mathsf{load}(b_i) \geq \max\{\mathsf{f}(i/n), \mathsf{thr}(M_S)\}$.

We now show how to decrease the load in L_+-bins, so that the remaining load in bin b_i remains at least $\max\{\mathsf{f}(i/n), \mathsf{thr}(M_S)\}$ and the change in the total gain is approximately equal to T^*. This claim is trivial for $\mathsf{T}^* = 0$, so we assume $\mathsf{T}^* > 0$. This is possible only if a tight item exists, $\min(M_S) > \mathsf{mt}(D)$ and $\min(M_S) > 1/3$. As $\min(M_S) > \mathsf{mt}(D)$, every marked medium item of size from the interval $[\mathsf{mt}(D), \min(M_S))$ is in (a separate) L_+-bin; let $\tilde{L}$ be the set of these bins. As $M_S \subseteq D^{\geq \mathsf{mt}(D)}$, $n \cdot \mathsf{b}(\tilde{L}) = |D^{\geq \mathsf{mt}(D)} \setminus M_S| = |D^{\geq \mathsf{mt}(D)}| - |M_S|$. Using the tightness of $\mathsf{mt}(D)$ and Lemma 8, $\mathsf{b}(\tilde{L}) = \mathsf{b}(D^{\geq \mathsf{mt}(D)}) - \mathsf{b}(M_S) \gtrsim \xi(\mathsf{mt}(D)) - \xi(\min(M_S))$. From each bin of $\tilde{L}$ we remove a load of $\mathsf{mt}(D) + \min(M_S) - 1/2$. The induced change in the total gain is then approximately equal to $\mathsf{b}(\tilde{L}) \cdot (\mathsf{mt}(D) + \min(M_S) - 1/2) = \mathsf{T}^*$.

We now analyze the load of bin b_i after the removal. The original load of bin b_i was at least $\mathsf{f}(i/n) + \mathsf{mt}(D)$, and after removal it is at least $\mathsf{f}(i/n) + \min(M_S) - 1/2$. This amount is at least $\mathsf{f}(i/n)$ (as $\min(M_S) \leq 1/2$) and at least $1 - \min(M_S) \geq \mathsf{thr}(M_S)$ (as $\mathsf{f}(i/n) \geq 1/2$). Hence, the remaining load of b_i is at least $\max\{\mathsf{f}(i/n), \mathsf{thr}(M_S)\}$.

Thus, $g(L_+) - T^* \geq \frac{1}{n} \sum_{i=1}^{n \cdot b(L)} \max\{f(i/n), thr(M_S)\} \approx \int_0^{b(L)} \max\{f(y), thr(M_S)\} \, dy$. Particular subsets of L_+ are depicted in Figure 3 (right); the removed part of gain T^* is depicted as (1). ◄

▶ **Lemma 18.** *Assume that* RTA *terminates with at least one M_S-bin. Then, $g(L_+) + \eta \cdot (1 - b(L)) \gtrsim P(\eta) + Q(\min\{thr(M_S), \eta\}) + T^*$ for any $\eta \in (1/2, 1]$.*

Proof. We fix any $\eta \in (1/2, 1]$ and define $h = \min\{thr(M_S), \eta\}$. By Lemma 17, $g(L_+) - T^* + \eta \cdot (1 - b(L)) \gtrsim \int_0^{b(L)} \max\{f(y), h\} \, dy + \int_{b(L)}^1 \eta \, dy$. We denote this lower bound by $A(b(L))$ and we analyze it as a function of $b(L)$. When $b(L) < f^{-1}(h)$, then using $h \leq \eta$ we obtain $A(b(L)) = \int_0^{b(L)} h \, dy + \int_{b(L)}^1 \eta \, dy \geq \int_0^{f^{-1}(h)} h \, dy + \int_{f^{-1}(h)}^1 \eta \, dy = A(f^{-1}(h))$. Therefore, we need to lower-bound the value of $A(b(L))$ only for $b(L) \geq f^{-1}(h)$. In such case,

$$A(b(L)) = \int_0^{b(L)} \max\{h, f(y)\} \, dy + \int_{b(L)}^1 \eta \, dy$$
$$= \int_0^{b(L)} f(y) \, dy + \int_{b(L)}^1 \eta \, dy + \int_0^{b(L)} \max\{h - f(y), 0\} \, dy$$
$$\geq \int_0^1 \min\{f(y), \eta\} + \int_0^1 \max\{h - f(y), 0\} \, dy = P(\eta) + Q(h). \quad ◄$$

5.1 When RTA terminates without empty bins and with some M_S-bins

The following lemma combines our bounds on gains on L_+, M_*, S_* and M_S.

▶ **Lemma 19.** *Assume that* RTA *run on input σ terminates without empty bins and with at least one M_S-bin. Then, for any $\eta \in (1/2, level_*]$,*

$$\text{RTA}(\sigma) \gtrsim P(\eta) + Q(\min\{\eta, thr(M_S)\}) + T^* + (min(M_S) - \eta) \cdot \xi(min(M_S)).$$

Proof. By the lemma assumptions,

$$\begin{aligned}
\text{RTA}(\sigma) &\geq g(L_+) + g(M_* \uplus S_*) + g(M_S) \\
&\gtrsim g(L_+) + \eta \cdot b(M_* \uplus S_*) + min(M_S) \cdot b(M_S) && \text{(by L. 13 and L. 8)} \\
&\approx g(L_+) + \eta \cdot (1 - b(L_+)) + (min(M_S) - \eta) \cdot b(M_S). && \text{(by L. 14)}
\end{aligned}$$

Applying the guarantee of Lemma 18 to $g(L_+) + \eta \cdot (1 - b(L_+))$ concludes the proof. ◄

▶ **Lemma 20.** *Assume that* RTA *run on input σ terminates without empty bins and with at least one M_S-bin. If $min(M_S) \leq 1/3$, then $\text{RTA}(\sigma) \gtrsim R$.*

Proof. As $level_* \geq 2/3$, we may apply Lemma 19 with $\eta = 2/3$. Note that $thr(M_S) \geq 2/3$ for $min(M_S) \leq 1/3$. Then,

$$\begin{aligned}
\text{RTA}(\sigma) &\gtrsim P(2/3) + Q(2/3) + (min(M_S) - 2/3) \cdot \xi(min(M_S)) \\
&= 2/3 + (\phi - 2/3) \cdot \xi(min(M_S)) \geq R. && \text{(by L. 6)}
\end{aligned}$$

The last inequality follows by Property 4 of Lemma 7. ◄

▶ **Lemma 21.** *Assume that* RTA *run on input σ terminates without empty bins and with at least one M_S-bin. If $min(M_S) > 1/3$, then $\text{RTA}(\sigma) \gtrsim R$.*

Proof. As $\min(M) > 1/3$, $\mathsf{thr}(M_S) < 1 - \min(M_S) \geq 2/3$. Lemma 19 applied with any $\eta \in [2/3, \mathsf{level}_*]$ yields

$$\mathrm{RTA}(\sigma) \gtrsim \mathsf{P}(\eta) + \mathsf{Q}(1 - \min(M_S)) + \mathsf{T}^* + (\min(M_S) - \eta) \cdot \xi(\min(M_S)). \tag{9}$$

First, we assume that D has no tight items. Then, $\mathsf{level}_* = 1$, and we may use (9) with $\eta = 2 \cdot \min(M_S)$ obtaining $\mathrm{RTA}(\sigma) \geq \mathsf{P}(2 \cdot \min(M_S)) + \mathsf{Q}(1 - \min(M_S)) - \min(M_S) \cdot \xi(\min(M_S)) \geq R$, where the last inequality follows by Property 7 of Lemma 7.

Second, we assume that D contains a tight item and we consider three cases.

- $\mathsf{level}_* < \mathsf{pile}(\mathsf{mt}(D))$. This relation is possible only when $\mathsf{level}_* = 1 - \phi$ and RTA terminates with at least one S_*-bin. In this case, Lemma 12 implies that $\mathsf{mt}(D) \leq 2\phi$. We apply (9) with $\eta = 1 - \phi$ obtaining $\mathrm{RTA}(\sigma) \geq \mathsf{P}(1 - \phi) + \mathsf{Q}(1 - \min(M_S)) + \mathsf{T}^* + (\min(M_S) + \phi - 1) \cdot \xi(\min(M_S))$, which is at least R by Property 5 of Lemma 7.
- $\mathsf{level}_* \geq \mathsf{pile}(\mathsf{mt}(D))$ and $\min(M_S) \leq \mathsf{mt}(D)$. Using monotonicity of pile, $\mathsf{level}_* \geq \mathsf{pile}(\min(M_S)) = 2 \cdot \min(M_S)$. Applying (9) with $\eta = 2 \cdot \min(M_S)$ yields $\mathrm{RTA}(\sigma) \gtrsim R$ by Property 7 of Lemma 7.
- $\mathsf{level}_* \geq \mathsf{pile}(\mathsf{mt}(D))$ and $\min(M) > \mathsf{mt}(D)$. In this case, $\mathsf{T}^* = \mathsf{T}(\min(M_S), \mathsf{mt}(D))$. Applying (9) with $\eta = \mathsf{pile}(\mathsf{mt}(D))$ yields $\mathrm{RTA}(\sigma) \gtrsim \mathsf{P}(\mathsf{pile}(\mathsf{mt}(D))) + \mathsf{Q}(1 - \min(M_S)) + \mathsf{T}(\min(M_S), \mathsf{mt}(D)) + (\min(M_S) - \mathsf{pile}(\mathsf{mt}(D)) \cdot \xi(\min(M_S))$, which is at least R by Property 6 of Lemma 7. ◄

6 Competitive ratio of RTA

▶ **Theorem 22.** *The competitive ratio of* RTA *for the multiple knapsack problem is at least* $R - O(1/n)$.

Proof. Fix an input σ. If $\mathrm{RTA}(\sigma)$ terminates with some empty bins, then its competitive ratio follows by Lemma 5.

Hence, below we assume that RTA terminates without empty bins. We presented three lemmas that cover all possible cases: there are no M_S-bins (Lemma 15), there are M_S-bins and $\min(M_S) \leq 1/3$ (Lemma 20), and there are M_S-bins and $\min(M_S) > 1/3$ (Lemma 21). In all these cases, we proved $\mathrm{RTA}(\sigma) \gtrsim R$. As $\mathrm{OPT}(\sigma) \leq 1$, the theorem follows. ◄

7 Proof of Lemma 7

We start with technical helper claims.

▶ **Fact 23.** *The functions below are the derivatives of functions* P, Q *and* ξ, *respectively.*

$$P'(x) = -R \cdot \ln x$$
$$Q'(x) = 1 + R \cdot \ln x$$
$$\xi'(x) = \begin{cases} -\xi_c/x^2 & \text{if } x < 1/3, \\ -18\xi_c & \text{if } x > 1/3 \end{cases}$$

▶ **Lemma 24.** $\mathsf{T}(x, y)$ *is a non-increasing function of* y *in the interval* $[\phi, 1/2]$.

Proof. Recall that $\mathsf{T}(x, y) = (x + y - 1/2) \cdot (\xi(y) - \xi(x))$ is defined for $x, y \in [\phi, 1/2]$. As the function $\mathsf{T}(x, y)$ is continuous and differentiable everywhere except $y = 1/3$, it suffices to show that its partial derivative $\partial_y \mathsf{T}(x, y)$ is non-positive (except $y = 1/3$). We have

$$\partial_y \mathsf{T}(x, y) = \xi(y) - \xi(x) + (x + y - 1/2) \cdot \xi'(y) \leq \xi(y) + y \cdot \xi'(y),$$

where for the inequality we used $\xi(x) \geq 0$ and $(x - 1/2) \cdot \xi'(y) \geq 0$.

If $y \leq 1/3$, then $\partial_y \mathsf{T}(x,y) \leq \xi_{\mathrm{c}}/y + y \cdot (-\xi_{\mathrm{c}}/y^2) = 0$. If $y > 1/3$, then $\partial_y \mathsf{T}(x,y) \leq 9\xi_{\mathrm{c}} \cdot (1 - 2y) - 18\xi_{\mathrm{c}} \cdot y = 9\xi_{\mathrm{c}} \cdot (1 - 4y) < 0$, which concludes the proof. ◄

▶ **Lemma 25.** $\mathsf{P}(2/3) = R - \xi_{\mathrm{c}}$ and for any $y \in [2/3, 1]$ it holds that $\mathsf{P}(y) \geq R + 3\xi_{\mathrm{c}} \cdot y - 3\xi_{\mathrm{c}}$.

Proof. For the first part of the lemma, observe that by the definition of ξ_{c} (see (1)),

$$\mathsf{P}(2/3) + \xi_{\mathrm{c}} = 2/3 - R \cdot (2/3) \cdot \ln(4/3) + (1 + (2/3) \cdot \ln(4/3)) \cdot R - 2/3 = R.$$

To show the second relation, note that $\mathsf{P}(2/3) = R - \xi_{\mathrm{c}}$ and $\mathsf{P}(1) = R$. Let $h(y) = R + 3\xi_{\mathrm{c}} \cdot y - 3\xi_{\mathrm{c}}$ be the linear function that coincides with $\mathsf{P}(y)$ for $y = 2/3$ and $y = 1$. As the function P is concave on its whole domain (its second derivative $\mathsf{P}''(y) = -R/y$ is negative), we have $\mathsf{P}(y) \geq h(y)$ for any $y \in [2/3, 1]$. ◄

Proof of Lemma 7. Note that $\xi(1/3) = 3 \cdot \xi_{\mathrm{c}}$ and $\xi(1/2) = 0$, For each property, we define an appropriate function G_i that we analyze; for all properties except the first one, we show that the function value is at least R for an appropriate range of arguments.

Property 1. Let $G_1(x) = x \cdot \xi(x)$. Then $G_1(x) = \xi_{\mathrm{c}}$ for $x \in [\phi, 1/3]$, and for $x \in [1/3, 1/2]$ it holds that $G_1(x) = 9\xi_{\mathrm{c}} \cdot x \cdot (1 - 2x)$, i.e., the function $G_1(x)$ is decreasing. Hence, $G_1(x)$ is non-increasing in the whole domain $[\phi, 1/2]$.

Property 2. Let $G_2(x) = \mathsf{P}(\mathsf{pile}(x)) + x \cdot \xi(x)$; we want to show that $G_2(x) \geq R$ for any $x \in [\phi, 1/2]$. For $x \in [\phi, 1/3]$, it holds that $G_2(x) = \mathsf{P}(2/3) + \xi_{\mathrm{c}} = R$ (by Lemma 25). For $x \in [1/3, 1/2]$, $G_2(x) = \mathsf{P}(2x) + x \cdot \xi(x)$. Using Lemma 25, we obtain $G_2(x) \geq R + 6\xi_{\mathrm{c}} \cdot x - 3\xi_{\mathrm{c}} + 9\xi_{\mathrm{c}} \cdot (x - 2x^2) = R - 18\xi_{\mathrm{c}} \cdot (x - 1/2) \cdot (x - 1/3) \geq R$. The last inequality follows as for any $x \in [1/3, 1/2]$ the term $(x - 1/2) \cdot (x - 1/3)$ is non-positive.

Property 3. Let $G_3 = \mathsf{P}(1 - \phi) + 2\phi \cdot \xi(2\phi)$. It can be verified numerically that $G_3 > 0.593 > R$.

Property 4. Let $G_4(x) = 2/3 - (2/3 - \phi) \cdot \xi(x)$. As the function $\xi(x)$ is decreasing, for any $x \in [\phi, 1/3]$ it holds that $G_4(x) \geq G_4(\phi) = 2/3 - (2/3) \cdot \xi_{\mathrm{c}}/\phi + \xi_{\mathrm{c}}$. Substituting the definition of ϕ (see (2)), we obtain $G_4(x) \geq G_4(\phi) \geq 2/3 - 2/3 + R - \xi_{\mathrm{c}} + \xi_{\mathrm{c}} = R$.

Property 5. Let $G_5(x, y) = \tilde{G}_5(x) + \max\{\mathsf{T}(x, y), 0\}$, where $\tilde{G}_5(x) = \mathsf{P}(1 - \phi) + \mathsf{Q}(1 - x) + (x + \phi - 1) \cdot \xi(x)$. We want to show that $G_5(x, y) \geq R$ for any $x \in [1/3, 1/2]$ and $y \in [\phi, 2\phi]$.

- If $x \in [1/3, 2\phi]$, then already $\tilde{G}_5(x) \geq R$. To show this relation, we estimate its derivative in the interval $[1/3, 2\phi]$:

$$\begin{aligned}
\tilde{G}_5'(x) &= -1 - R \cdot \ln(1 - x) + (x + \phi - 1) \cdot \xi'(x) + \xi(x) \\
&= -1 - R \cdot \ln(1 - x) - 36\xi_{\mathrm{c}} \cdot x - 18\xi_{\mathrm{c}} \cdot \phi + 27\xi_{\mathrm{c}} \\
&\leq -1 - R \cdot \ln(1 - 2\phi) - 36\xi_{\mathrm{c}} \cdot (1/3) - 18\xi_{\mathrm{c}} \cdot \phi + 27\xi_{\mathrm{c}} < -0.2479 < 0.
\end{aligned}$$

  Hence, $\tilde{G}_5$ is decreasing in the interval $[1/3, 2\phi]$, and thus for any $x \in [1/3, 2\phi]$ it holds that $G_5(x, y) \geq \tilde{G}_5(x) \geq \tilde{G}_5(2\phi) > 0.5997 > R$.

- If $x \in (2\phi, 1/2]$, then $\mathsf{T}(x, y) \geq 0$. As $\mathsf{T}(x, y)$ is a non-increasing function of y in the interval $[\phi, 2\phi]$ (by Lemma 24), it holds that $\mathsf{T}(x, y) \geq \mathsf{T}(x, 2\phi)$. Therefore,

$$G_5(x, y) \geq G_5(x, 2\phi) = \mathsf{P}(1 - \phi) + \mathsf{Q}(1 - x) - (\phi + 1/2) \cdot \xi(x) + (x + 2\phi - 1/2) \cdot \xi(2\phi).$$

  We now estimate its partial derivative for $x \in [2\phi, 1/2]$:

$$\begin{aligned}
\partial_x G_5(x, 2\phi) &= -1 - R \cdot \ln(1 - x) - (\phi + 1/2) \cdot \xi'(x) + \xi(2\phi) \\
&\leq -1 - R \cdot \ln(1/2) + 18\xi_{\mathrm{c}} \cdot (\phi + 1/2) + 9\xi_{\mathrm{c}} \cdot (1 - 4\phi) < -0.0673 < 0.
\end{aligned}$$

  Therefore $G_5(x, 2\phi)$ is decreasing as a function of x in the interval $[2\phi, 1/2]$. Thus, for the considered range of arguments, $G_5(x, y) \geq G_5(x, 2\phi) \geq G_5(1/2, 2\phi) > 0.5934 > R$.

Property 6. Let

$$G_6(x,y) = \mathsf{P}(\mathsf{pile}(y)) + \mathsf{Q}(1-x) + (x - \mathsf{pile}(y)) \cdot \xi(x) + \mathsf{T}(x,y)$$

Fix any pair $(x,y) \in [1/3, 1/2] \times [\phi, 1/2]$, such that $y \leq x$. We prove that $G_6(x,y) \geq R$, by showing each of the following inequalities

$$G_6(x,y) \geq G_6(x,y') \geq \tilde{G}_6(x,y') \geq \tilde{G}_6(1/2, y'') \geq R, \tag{10}$$

where $y' = \max\{y, 1/3\}$ and $y'' = y' + 1/2 - x$. The function $\tilde{G}_6$ is a lower bound on function G_6 created by using Lemma 25 (and defined formally later).

The first inequality of (10) is trivial for $y \geq 1/3$, hence we assume that $y < 1/3$. In such case, using the definition of function pile, we get $G_6(x,y) = \mathsf{P}(2/3) + \mathsf{Q}(1-x) + (x-2/3) \cdot \xi(x) + \mathsf{T}(x,y)$. As $\mathsf{T}(x,y)$ is a decreasing function of y, we obtain that $\mathsf{T}(x,y) \geq \mathsf{T}(x,y')$ and thus also $G_6(x,y) \geq G_6(x,y')$.

To show the second inequality of (10), we first simplify $G_6(x,y)$ using that $y \geq 1/3$ and the definition of pile:

$$G_6(x,y) = \mathsf{P}(2y) + \mathsf{Q}(1-x) + (x-2y) \cdot \xi(x) + (x+y-1/2) \cdot (\xi(y) - \xi(x))$$

Now, using Lemma 25, we have

$$G_6(x,y) \geq R + \mathsf{Q}(1-x) + 6\xi_c \cdot y - 3\xi_c + (1/2 - 3y) \cdot \xi(x) + (x+y-1/2) \cdot \xi(y).$$

We denote the estimate above by $\tilde{G}_6(x,y)$ and we inspect its two partial derivatives.

$$\partial_x \tilde{G}_6(x,y) = -1 - R \cdot \ln(1-x) + (1/2 - 3y) \cdot \xi'(x) + \xi(y)$$
$$\partial_y \tilde{G}_6(x,y) = 6\xi_c - 3 \cdot \xi(x) + \xi(y) + (x+y-1/2) \cdot \xi'(y)$$

The directional derivative along the vector $\mathbf{v} = (1,1)$ is then equal to

$$\begin{aligned}
\partial_{\mathbf{v}} \tilde{G}_6(x,y) &= \partial_x \tilde{G}_6(x,y) + \partial_y \tilde{G}_6(x,y) \\
&= -1 - R \cdot \ln(1-x) + 6\xi_c - 3 \cdot \xi(x) + 2 \cdot \xi(y) - 18\xi_c \cdot (x - 2y) \\
&= -1 - R \cdot \ln(1-x) + 36\xi_c \cdot x - 3\xi_c \\
&\leq -1 - R \cdot \ln(1/2) + 36\xi_c \cdot (1/2) - 3\xi_c < -0.0322 < 0.
\end{aligned}$$

This means that if we take any point $(x,y') \in [1/3, 1/2] \times [1/3, 1/2]$, where $y' \leq x$, and move along vector $\mathbf{v}$, to point $(1/2, y'') = (1/2, y + 1/2 - x)$, the value of the function $\tilde{G}_6$ can only decrease. This concludes the proof of the third inequality of (10).

To show the final inequality of (10), we fix any $y'' \in [1/3, 1/2]$. Then $\tilde{G}_6(1/2, y'') = R + 6\xi_c \cdot y - 3\xi_c + y'' \cdot \xi(y'') = R - 18\xi_c \cdot (y'' - 1/2) \cdot (y'' - 1/3) \geq R$, where the last inequality holds as $(y'' - 1/2) \cdot (y'' - 1/3)$ is non-positive.

Property 7. Let $G_7(x) = \mathsf{P}(2x) + \mathsf{Q}(1-x) - x \cdot \xi(x)$. For $x \in [1/3, 1/2]$, it holds that $G_7(x) = G_6(x,x)$. Hence $G_7(x) \geq R$ for $x \in [1/3, 1/2]$ follows by Property 6.   ◄

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◼ **Algorithm 1** RTA pseudocode for handling an item x

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if x is large **then**
    if $x \geq \mathsf{f}(\mathsf{b}(L) + 1/n)$ **then**
        if M_S-bin b with x space left exists **then**
            put x in b $\triangleright\ M_\mathrm{S} \to L_+$
        else
            put x in an empty bin $\triangleright\ E \to L_+$
    else
        reject x

if x is medium **then**
    Let $i \in \{2, 3, 4\}$ be such that $x \in [1/(i+1), 1/i)$
    if L_+-bin b with x space left exists **then**
        put x in b $\triangleright\ L_+ \to L_+$
    else if $D \cup \{x\}$ is ξ-dominated **then**
        $D \leftarrow D \cup \{x\}$ $\triangleright\ \text{mark } x$
        put x in an empty bin $\triangleright\ E \to M_\mathrm{S}$
    else if M_i-bin b with x space left exists **then**
        put x in b $\triangleright\ M_i \to M_i$
    else
        put x in an empty bin $\triangleright\ E \to M_i$

if x is small **then**
    if L_+-bin b with x space left exists **then**
        put x in b $\triangleright\ L_+ \to L_+$
    else if an S_*-bin b with x space left exists **then**
        put x in b $\triangleright\ S_* \to S_*$
    else if an A-bin b exists **then**
        put x in b $\triangleright\ A \to A$
        if $\mathsf{load}(b) \geq \phi$ **then**
            if $D \cup \{\mathsf{load}(b)\}$ is ξ-dominated **then**
                merge all items from b into medium item y of size $\mathsf{load}(b)$
                $D \leftarrow D \cup \{y\}$ $\triangleright\ \text{mark } y,\ A \to M_\mathrm{S}$
            else
                relabel b to S_* $\triangleright\ A \to S_*$
        else
            put x in an empty bin $\triangleright\ E \to A$

if there are no empty bins left **then**
    terminate

A   Pseudocode of RTA

The pseudocode of our algorithm RTA for handling a single incoming item is presented in Algorithm 1. The right margin shows label transitions of affected bins.

Upper bound

To show the optimality of RTA, we prove that the competitive ratio of any deterministic algorithm for the multiple knapsack problem is at most $R - O(1/n)$. This shows that the term $O(1/n)$ occurring in the competitive ratio of our deterministic algorithm RTA is inevitable.

Our adversarial strategy is parameterized with a non-decreasing sequence of $n + 1$ numbers $s(i)$, such that $1/2 < s(1) \le s(2) \le s(3) \le \ldots \le s(n) \le s(n + 1) = 1$ and proceeds in at most $n + 1$ phases numbered from 1. In phase i, the adversary issues n items, all of size $s(i)$, and stops immediately once DET accepts one of these items (places it into a bin). If all n items of a phase are rejected by DET, the adversary finishes the input sequence.

Let j be the number of phases where DET accepts an item. As the size of each item from the input is strictly greater than $1/2$, at most one item fits in a single bin, and thus $j \le n$. Then, total load of DET is equal to $\text{DET}(j) = \sum_{i=1}^{j} s(i)$. Note that in the final phase $j + 1$, n items of size $s(j + 1)$ are presented (but rejected by DET). OPT may accept all of them obtaining the total load $n \cdot s(j + 1)$. By comparing these two gains and as DET may choose the value of j, we get that the competitive ratio of DET is at most

$$U := \max_{0 \le j \le n} \frac{\text{DET}(j)/n}{\text{OPT}(j)/n} = \max_{0 \le j \le n} \frac{(1/n) \cdot \sum_{i=1}^{j} s(i)}{s(j + 1)}. \tag{11}$$

It remains to choose an appropriate sequence s and compute U using (11). As a warm-up, we show a simple construction leading to a weaker upper bound of R, which we later refine to obtain the bound of $R - O(1/n)$. We note that the bound of R given by Cygan et al. [7] uses concepts similar to ours, although in their construction, the adversary tries to control the number of bins that store particular amounts of items, while our approach is more fine-grained, and the adversary controls the contents of particular bins.

For our both bounds we use the following sequence s with the function f defined in (3), parameter α that we define later and infinitesimally small but positive ϵ.

$$s(i) = \begin{cases} f\left(\frac{i-1}{n} + \alpha\right) + \epsilon & \text{if } i \in \{1, \ldots, n\}, \\ 1 & \text{if } i = n + 1. \end{cases} \tag{12}$$

The role of ϵ is to ensure that $s(i) > 1/2$ for all i. We will neglect the load caused by ϵ in our proofs below; such approach could be formalized by taking $\epsilon = c/n^2$ with a constant c tending to zero.

B.1   Upper bound of R

▶ **Lemma 26.** *The competitive ratio of any deterministic algorithm* DET *for the multiple knapsack problem is at most R.*

Proof. The adversary uses the strategy described above with sequence s defined by (12) parameterized with $\alpha = 0$. We fix any $j \in \{0, \ldots, n\}$ and observe that $s(j + 1) \ge f(j/n)$ (for $j = n$ we use $f(1) = 1$). By (11), (5) and Lemma 3, the competitive ratio of DET is upper-bounded by

$$U = \frac{\sum_{i=1}^{j} s(i)/n}{s(j + 1)} = \frac{\frac{1}{n} \cdot \sum_{i=1}^{j} f\left(\frac{i-1}{n}\right)}{s(j + 1)} \le \frac{\int_0^{j/n} f(x)\,dx}{s(j + 1)} \le \frac{F(j/n)}{f(j/n)} \le R. \qquad \blacktriangleleft$$

B.2   Upper bound of R - O(1/n)

By choosing parameter $\alpha = 1/(8n)$, we may improve the upper bound by a term of $\Theta(1/n)$. As in the proof of Lemma 26, we approximate the sum of $s(i)$'s by an appropriate integral, but this time we also bound the error due to such approximation.

▶ **Lemma 27.** *For the sequence s defined in* (12) *parameterized with $\alpha = 1/(8n)$, and for any $j \in \{0, \ldots, n\}$, it holds that $\sum_{i=1}^{j} s(i)/n \leq \mathsf{F}(j/n)$. Moreover, for $n \geq 3$, it holds that $\sum_{i=1}^{n} s(i)/n \leq \mathsf{F}(1) - 1/(52n)$.*

Proof. Note that $\sum_{i=1}^{j} s(i)/n = \sum_{i=0}^{j-1} s(i+1)/n = \sum_{i=0}^{j-1} \frac{1}{n} \cdot f(i/n + \alpha)$. On the other hand, $F(j/n) = \int_{0}^{j/n} f(x)\, dx = \sum_{i=0}^{j-1} \int_{0}^{1/n} \mathsf{f}(i/n + x)\, dx$. Therefore, $F(j/n) - \sum_{i=1}^{j} s(i)/n = \sum_{i=0}^{j-1} D_i$, where

$$D_i = \int_{0}^{1/n} \mathsf{f}(i/n + x) - \mathsf{f}(i/n + \alpha)\, dx. \tag{13}$$

Now we estimate D_i. We consider two cases.

1. $i/n + \alpha < R$. Then, $f(i/n + \alpha) = 1/2$ and $\mathsf{f}(i/n + x) \geq 1/2$ for any $x \in [0, 1/n]$. Substituting these bounds to (13) yields $D_i \geq 0$.

2. $i/n + \alpha \geq R$. In this case, $\mathsf{f}(i/n + \alpha) = (2\mathrm{e})^{i/n + \alpha - 1}$ and $\mathsf{f}(i/n + x) \geq (2\mathrm{e})^{i/n + x - 1}$ for any $x \in [0, 1/n]$. Thus,

$$D_i \geq \int_{0}^{1/n} (2\mathrm{e})^{i/n + x - 1} - (2\mathrm{e})^{i/n + \alpha - 1}\, dx = (2\mathrm{e})^{i/n - 1} \cdot \int_{0}^{1/n} (2\mathrm{e})^{x} - (2\mathrm{e})^{\alpha}\, dx.$$

By the Taylor expansion of function $(2\mathrm{e})^{x}$, for any $\beta \in [0, R]$ it holds that $1 + \beta/R \leq (2\mathrm{e})^{\beta} \leq 1 + 2 \cdot \beta/R$. Thus,

$$D_i \geq (2\mathrm{e})^{i/n - 1} \cdot \int_{0}^{1/n} \frac{x}{R} - \frac{2\alpha}{R}\, dx \geq (2\mathrm{e})^{i/n - 1} \cdot \left(\frac{1/n^2}{2R} - \frac{2\alpha/n}{R} \right)$$
$$\geq \frac{1}{2\mathrm{e}} \cdot \frac{1}{4R \cdot n^2} > \frac{1}{13 \cdot n^2}.$$

As $D_i \geq 0$ in either case, we immediately get the first part of the lemma. To show the second part, we set $j = n$ and observe that all integers $i \in \{\lceil n \cdot R \rceil, \ldots, j-1\}$ are considered in the second case above. Therefore $F(1) - \sum_{i=1}^{n} s(i)/n = \sum_{i=0}^{n-1} D_i \geq (n - \lceil n \cdot R \rceil)/(13 \cdot n^2)$. For $n \geq 3$, it holds that $n - \lceil n \cdot R \rceil \geq n/4$, which concludes the proof.   ◀

▶ **Theorem 28.** *The competitive ratio of any deterministic algorithm* DET *for the multiple knapsack problem is at most $R - 1/(52n)$.*

Proof. The adversary uses the strategy described above with sequence s defined by (12) parameterized with $\alpha = 1/(8n)$. We compute the upper bound U on the competitive ratio of DET using (11).

We separately consider the case of small value of n. If $n = 1$, then $s(1) = 1/2$ and $s(2) = 1$, and therefore $U = 1/2$. If $n = 2$, then $s(1) = s(2) = 1/2$ and $s(3) = 1$, and thus again $U = 1/2$. In these cases, $U = 1/2 < R - 1/(52n)$, and the theorem follows.

Below, we assume that $n \geq 3$. We fix any $j \in \{0, \ldots, n\}$ and let $y = j/n$. We consider two cases.

1. $j < n$. Then, by Lemma 27, $\sum_{i=1}^{j} s(i)/n \leq \mathsf{F}(y)$. Additionally, $s(j+1) = \mathsf{f}(y + \alpha) + \epsilon > \mathsf{f}(y + \alpha)$. Using Lemma 3, we obtain

$$U = \frac{(1/n) \cdot \sum_{i=1}^{j} s(i)}{s(j+1)} < \frac{\mathsf{F}(y)}{\mathsf{f}(y + \alpha)} = \frac{\mathsf{F}(y)}{\mathsf{f}(y)} \cdot \frac{\mathsf{f}(y)}{\mathsf{f}(y + \alpha)} = \min\{y, R\} \cdot \frac{\mathsf{f}(y)}{\mathsf{f}(y + \alpha)}. \tag{14}$$

It suffices to show that $U \leq R - \alpha/2 = R - 1/(16n)$.

a. If $y < R - \alpha/2$, then $U < y \cdot 1 < R - \alpha/2$.

b. If $y \geq R - \alpha/2$, then we upper-bound the fraction $\mathsf{f}(y) / \mathsf{f}(y + \alpha)$. As $y + \alpha/2 \geq R$,

$$\frac{\mathsf{f}(y)}{\mathsf{f}(y + \alpha)} \leq \frac{\mathsf{f}(y + \alpha/2)}{\mathsf{f}(y + \alpha)} = \frac{(2\mathrm{e})^{y+\alpha/2-1}}{(2\mathrm{e})^{y+\alpha-1}} = (2\mathrm{e})^{-\alpha/2} \leq 1 - \frac{\alpha}{2R}. \tag{15}$$

In the last inequality, we used the Taylor expansion of the function $(2\mathrm{e})^x$. Combining (14) with (15) yields $U \leq R - \alpha/2$ as desired.

2. $j = n$. By Lemma 27, $\sum_{i=1}^{n} s(i)/n \leq R - 1/(52n)$ and $s(n+1) = 1$. Therefore, in this case, $U \leq R - 1/(52n)$. ◄