Online Multi-type Multiple Knapsack Problem

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Abstract

Keywords: multi-type multiple knapsack problem, revenue management, .

1 Introduction

We will address the online multi-type multiple knapsack problem.

We develop several policies for online MMKP.

The rest of this paper is structured as follows. We review the relevant literature in Section 2. Then we introduce the key concepts of seat planning with social distancing and formulate the . Section 3 presents the bid-price and resolving dynamic primal policies to assign seats for incoming requests. Section 4 presents the experimental results and provide insights gained from implementing social distancing. Conclusions are shown in Section 5.

2 Literature Review

Multiple Knapsack problem (Martello and Toth, 1990) is a practical problem that presents unique challenges in various applications. While existing literature has primarily focused on deriving bounds or competitive ratios for general multiple knapsack problems (Khuri et al., 1994; Ferreira et al., 1996; Pisinger, 1999; Chekuri and Khanna, 2005), our work distinguishes itself by analyzing the specific structure and properties of solutions to the MMKP problem.

While the dynamic stochastic knapsack problem (e.g., Kleywegt and Papastavrou (1998, 2001), Papastavrou et al. (1996)) has been extensively studied in the literature, these works primarily consider a single knapsack scenario where requests arrive sequentially and their resource requirements and rewards are unknown until they arrive. In contrast, the online MMKP problem extends this framework by incorporating multiple knapsacks, adding another layer of complexity to the decision-making process. Research on the dynamic or stochastic multiple knapsack problem is limited. Perry and Hartman (2009) employs multiple knapsacks to model multiple time periods for solving a multiperiod, single-resource capacity reservation problem. This essentially remains a dynamic knapsack problem but involves time-varying capacity. Tönissen et al. (2017) considers a two-stage stochastic multiple knapsack problem with a set of scenarios, wherein the capacity of the knapsacks may be subject to disturbances. This problem is similar to the SPSR problem in our work, where the number of items is stochastic.

Generally speaking, the online MMKP problem relates to the revenue management (RM) problem, which has been extensively studied in industries such as airlines, hotels, and car rentals, where perishable inventory must be allocated dynamically to maximize revenue (van Ryzin and Talluri, 2005). Network revenue management (NRM) extends traditional RM by considering multiple resources (e.g., flight legs, hotel nights) and interdependent demand (Williamson, 1992). The standard NRM problem is typically formulated as a dynamic programming (DP) model, where decisions involve accepting or rejecting requests based on their revenue contribution and remaining capacity (Talluri and van Ryzin, 1998). However, a significant challenge arises because the number of states grows exponentially with the problem size, rendering direct solutions computationally infeasible. To address this, various control policies have been proposed, such as bid-price (Adelman, 2007; Bertsimas and Popescu, 2003), booking limits (Gallego and van Ryzin, 1997), and dynamic programming decomposition (Talluri and van Ryzin, 2006; Liu and van Ryzin, 2008). These methods typically assume that demand arrives individually (e.g., one seat per booking). However, in our problem, customers often request multiple units simultaneously, requiring decisions that must be made on an all-or-none basis for each request. This requirement introduces significant complexity in managing group arrivals (Talluri and van Ryzin, 2006).

A notable study addressing group-like arrivals in revenue management examines hotel multi-day stays (Bitran and Mondschein, 1995; Goldman et al., 2002; Aydin and Birbil, 2018). While these works focus on customer classification and room-type allocation, they do not prioritize real-time assignment. The work of Zhu et al. (2023), which addresses the high-speed train ticket allocation and processes individual seat requests and implicitly accommodates group-like traits through multi-leg journeys (e.g., passengers retaining the same seat across connected segments).

3 Online MMKP

Consider N knapsacks, with each knapsack j containing $c_j \in \mathbb{Z}^+$ capacities, for $j \in \mathcal{N} := \{1, 2, \dots, N\}$. There are M distinct item types, where each item type i, $i \in \mathcal{M} := \{1, 2, \dots, M\}$, requiring $w_i \in \mathbb{Z}^+$ consecutive size in one capacity. The profit of each item type i is $r_i \in \mathbb{Z}^+$. Suppose that the profit-weight ratio is increasing monotone from 1 to M for type i. Requests arrive sequentially over time, and the seller must immediately decide whether to accept or reject each request upon arrival. If a request is accepted, the seller must also determine the specific knapsack to assign. Importantly, each item must be either fully accepted or entirely rejected; once the item is assigned to a knapsack, it cannot be altered.

To model this problem, we formulate it using dynamic programming approach in a discrete-time framework. Time is divided into T periods, indexed forward from 1 to T. We assume that in each period, at most one request arrives and the probability of an arrival for an item type i is denoted as λ_i^t, where $i \in \mathcal{M}$. The probabilities satisfy the constraint $\sum_{i=1}^{M} \lambda_i^t \leq 1$, indicating that the total probability of any item arriving in a single period does not exceed one. We introduce the probability $\lambda_0^t = 1 - \sum_{i=1}^{M} \lambda_i^t$ to represent the probability of no arrival in each period. To simplify the analysis, we assume that the arrivals of different item types are independent and the arrival probabilities remain constant over time. This assumption can be extended to consider dependent arrival probabilities over time if necessary.

The remaining capacity in each knapsack is represented by a vector $\mathbf{C} = (c_1, c_2, \ldots, c_N)$, where c_j denotes the capacity of knapsack j. Upon the arrival of an item type i at time t, the seller needs to make a decision denoted by $u_{i,j}^t$, where $u_{i,j}^t = 1$ indicates acceptance of type i in knapsack j during period t, while $u_{i,j}^t = 0$ signifies rejection of that type in knapsack j. The feasible decision set is defined as

$$U^t(\mathbf{L}) = \left\{ u_{i,j}^t \in \{0,1\}, \forall i \in \mathcal{M}, \forall j \in \mathcal{N} \,\middle|\, \sum_{j=1}^{N} u_{i,j}^t \leq 1, \forall i \in \mathcal{M}; w_i u_{i,j}^t \mathbf{e}_j \leq \mathbf{C}, \forall i \in \mathcal{M}, \forall j \in \mathcal{N} \right\}.$$

Here, $\mathbf{e}_j$ represents an N-dimensional unit column vector with the j-th element being 1, i.e., $\mathbf{e}_j = (\underbrace{0, \cdots, 0}_{j-1}, 1, \underbrace{0, \cdots, 0}_{N-j})$. The decision set $U^t(\mathbf{C})$ consists of all possible combinations of acceptance and rejection decisions for each type in each knapsack, subject to the constraints that at most one item of each type can be accepted in any knapsack, and the size of each accepted item must not exceed the remaining capacity of the knapsack.

Let $V^t(\mathbf{C})$ denote the maximum expected revenue earned by the optimal decision regarding item assignments at the beginning of period t, given the remaining capacity $\mathbf{C}$. Then, the dynamic programming formulation for this problem can be expressed as:

$$V^t(\mathbf{C}) = \max_{u_{i,j}^t \in U^t(\mathbf{C})} \left\{ \sum_{i=1}^{M} \lambda_i^t \Big(\sum_{j=1}^{N} r_i u_{i,j}^t + V^{t+1}(\mathbf{C} - w_i u_{i,j}^t \mathbf{e}_j) \Big) + \lambda_0^t V^{t+1}(\mathbf{C}) \right\} \tag{1}$$

with the boundary conditions $V^{T+1}(\mathbf{C}) = 0, \forall \mathbf{C}$, which implies that the revenue at the last period is 0 under any capacity. The initial capacity is denoted as $\mathbf{C}_0 = (C_1, C_2, \ldots, C_N)$. Our objective is to determine item assignments that maximize the total expected revenue during the horizon from period 1 to T, represented by $V^1(\mathbf{C}_0)$.

Solving the dynamic programming problem in equation (1) presents computational challenges due to the curse of dimensionality that arises from the large state space.

We propose our policy for assigning arriving requests. First, we employ the traditional bid-price control policy. Then, we present the bid-price control policy based on the patterns.

3.1 BPC Policy

Bid-price control is a classical approach discussed extensively in the literature on network revenue management. It involves setting bid prices for different types, which determine the eligibility of items to take the sizes. Bid-prices refer to the opportunity costs of taking one size of capacity. As usual, we estimate the bid price of one size by the shadow price of the capacity constraint corresponding to some knapsack. In this section, we will demonstrate the implementation of the bid-price control policy.

First, we give the formulation of the LP relaxation of the multi-type multiple knapsack problem (MMKP). Let x_{ij} represent the number of items of type i placed in knapsack j over T periods. The whole request of each item type during T periods is represented by the expected demand $d_i = \sum_{t=1}^{T} \lambda_i^t$.

Then, the problem can be expressed as:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{M}\sum_{j=1}^{N} r_i x_{ij} && (2)\\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} \le d_i, \quad i \in \mathcal{M}, && (3)\\
& \sum_{i=1}^{M} w_i x_{ij} \le c_j, j \in \mathcal{N}, && (4)\\
& x_{ij} \ge 0, \quad i \in \mathcal{M}, j \in \mathcal{N}.
\end{aligned}
$$

The objective function (2) is to maximize the revenue. Constraint (3) ensures the total number of accommodated items does not exceed the number of requests for each type. Constraint (4) stipulates that the total size in each knapsack does not exceed its capacity.

The increasing nature of the ratio $\frac{r_i}{w_i}$ with respect to type i leads to preferential inclusion of larger items in the optimal fractional assignment. This intuitive property is illustrated in Proposition 1.

Proposition 1. *For the LP relaxation of the* MMKP *problem, there exists an index* $\tilde{i}$ *such that the optimal solutions satisfy the following conditions:* $x^*_{ij} = 0$ *for all* j, $i = 1, \ldots, \tilde{i} - 1$; $\sum_{j=1}^{N} x^*_{ij} = d_i$ *for* $i = \tilde{i} + 1, \ldots, M$; $\sum_{j=1}^{N} x^*_{ij} = \frac{L - \sum_{i=\tilde{i}+1}^{M} d_i w_i}{w_{\tilde{i}}}$ *for* $i = \tilde{i}$.

The dual of LP relaxation of the MMKP problem is:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{M} d_i z_i + \sum_{j=1}^{N} c_j \beta_j \\
\text{s.t.} \quad & z_i + \beta_j w_i \ge r_i, \quad i \in \mathcal{M}, j \in \mathcal{N} \\
& z_i \ge 0, i \in \mathcal{M}, \beta_j \ge 0, j \in \mathcal{N}.
\end{aligned} \tag{5}
$$

In (5), β_j can be interpreted as the bid-price for one size in knapsack j. A request is only accepted if the revenue it generates is no less than the sum of the bid prices of the sizes it uses. Thus, if $r_i - \beta_j w_i \ge 0$, meanwhile, the capacity allows, we will accept the item type i. And choose knapsack $j^* = \arg\max_j \{r_i - \beta_j w_i\}$ to allocate that item.

Lemma 1. *The optimal solution to problem* (5) *is given by* $z_1 = \ldots = z_{\tilde{i}} = 0$, $z_i = \frac{r_i w_{\tilde{i}} - r_{\tilde{i}} w_{\tilde{i}}}{w_{\tilde{i}}}$ *for* $i = \tilde{i} + 1, \ldots, M$ *and* $\beta_j = \frac{r_{\tilde{i}}}{w_{\tilde{i}}}$ *for all* j.

According to Lemma 1, the decision inequality becomes $r_i - \beta_j w_i = r_i - \frac{r_{\tilde{i}}}{w_{\tilde{i}}} w_i \ge 0$. This establishes the threshold policy: reject item type $i, i < \tilde{i}$ and accept item type $i, i \ge \tilde{i}$.

Algorithm 1: Bid-Price Control

for $t = 1, \dots, T$ **do**
- Observe a request of item type i;
- Solve problem (5) with $\boldsymbol{d}^{[t,T]}$ and $\mathbf{C}^t$;
- Obtain $\tilde{i}$ such that the aggregate optimal solution is $xe_{\tilde{i}} + \sum_{i=\tilde{i}+1}^{M} d_i^t e_i$;
- **if** $i \geq \tilde{i}$ *and* $\max_{j \in \mathcal{N}} c_j^t \geq w_i$ **then**
  - Set $k = \arg\min_{j \in \mathcal{N}} \{c_j^t | c_j^t \geq w_i\}$ and break ties;
  - Assign the item to knapsack k, let $c_k^{t+1} \leftarrow c_k^t - w_i$;
- **else**
  - Reject the request;
- **end**

end

Let z(BPC), z(DLP) denote the optimal value of (5) and the LP relaxation of the MMKP problem with expected demand, respectively. Then we have $z(\text{BPC}) = z(\text{DLP})$.

However, the BPC policy has two drawbacks. First, the capacity feasibility need to be checked when to accept a request. Second, when capacity permits, the policy treats all knapsacks as equally preferable, making no distinction among them.

Example 1. *Consider $M = 3$, $N = 4$, $w_1 = 3, w_2 = 4, w_3 = 5$, $r_1 = 4, r_2 = 6, r_3 = 8$, $\boldsymbol{C} = [7, 8, 8, 4]$, $T = 8$, stationary arrival probability: $\lambda_1 = \lambda_3 = \frac{1}{4}$, $\lambda_2 = \frac{1}{2}$. Then the expected demand for each type is $\boldsymbol{d} = (2, 4, 2)$.*

For the traditional bid-price control, for each j, $\beta_j^* = \frac{4}{3}$. $z(BPC) = \frac{124}{3}$. Two drawbacks: there is no difference among knapsacks. Even $r_3 - \beta_4^* * w_3 > 0$, it is infeasible for type 3 assigned in knapsack 4.

3.2 BPC Policy Based on Patterns (BPP)

To account for the differences in placing items across knapsacks, we propose an enhanced dynamic programming (DP) formulation. The key idea, as detailed next, is to represent knapsack configurations using patterns rather than merely tracking the residual capacities.

A feasible pattern $\boldsymbol{h} = [h_1, \dots, h_M]$ for knapsack j satisfies $\sum_{i=1}^{M} w_i h_i \leq c_j$. Suppose that $S(c_j)$ is the set of all feasible patterns for knapsack j.

Let $v^t(\boldsymbol{C})$ denote the maximal expected value-to-go at time t, given the remaining capacity $\boldsymbol{C}$. The enhanced dynamic programming formulation is as follows:

$$v^t(\boldsymbol{C}) \geq \mathbb{E}_{i \sim \lambda^t} \left[\begin{cases} \max \left\{ \max_{j: \boldsymbol{h} \in S(c_j), h_i \geq 1} \left\{ v^{t+1} \left(\boldsymbol{C} - e_j^T \cdot w_i \right) + r_i \right\}, v^{t+1}(\boldsymbol{C}) \right\}, & \exists j, \boldsymbol{h} \in S(c_j), h_i \geq 1, \\ v^{t+1}(\boldsymbol{C}) & \text{otherwise.} \end{cases} \right] \tag{6}$$

The DP formulation involves two layers of maximization when there exists at least one knapsack j satisfying $\boldsymbol{h} \in S(c_j), h_i \geq 1$. The inner maximization evaluates the optimal placement of item type i across all feasible knapsacks j where the pattern h is feasible for knapsack j (i.e., $\boldsymbol{h} \in S(c_j)$) and the item type i can be accommodated (i.e., $h_i \geq 1$). The outer maximization compares the value of accepting i (via the inner maximization) and rejecting i (retaining $v^{t+1}(\boldsymbol{C})$). If no such j exists, the request i is rejected.

For notational convenience, we define q_i as follows:

$$q_i = \begin{cases} \max_{j:h \in S(c_j), h_i \geq 1} \left\{ v^{t+1}\left(C - e_j^T w_i\right) + r_i \right\} & \text{if } \exists j \text{ satisfying } h \in S\left(c_j\right), h_i \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

We can solve the following program to compute $v^1(\boldsymbol{C})$ for any given capacity $\boldsymbol{C}$:

$$\begin{aligned} \min \quad & v^1(\boldsymbol{C}) \\ \text{s.t.} \quad & v^t(\boldsymbol{C}) \geq \mathbb{E}_{i \sim \lambda^t}\left[\max\left\{q_i, v^{t+1}(\boldsymbol{C})\right\}\right], \\ & v^{T+1}(\boldsymbol{C}) \geq 0. \end{aligned} \tag{7}$$

Solving (7) remains computationally prohibitive. Following from the ADP approach, we approximate $v^t(\boldsymbol{C})$ as

$$\hat{v}^t(\boldsymbol{C}) = \theta^t + \sum_{j=1}^{N} \max_{\boldsymbol{h} \in S(c_j)} \{\sum_{i=1}^{M} \beta_{ij}^{\dagger} h_i\}. \tag{8}$$

The term $\beta_{ij}^{\dagger}$ can be regarded as the approximated value for each type i in knapsack j. Unlike traditional linear approximations, our approach retains the linear term θ^t but introduces a nonlinear component for each knapsack j. Specifically, we maximize the linear combination $\sum_{i=1}^{M} \beta_{ij}^{\dagger} h_i$ over the feasible set $S(c_j)$.

Our approximation extends classical linear ADP by incorporating resource-specific nonlinear terms through constrained maximization over feasible allocations. While similar separable corrections appear in resource allocation ADP (e.g., Powell, 2007), our explicit use of $\max_{\boldsymbol{h} \in S(c_j)}$ captures local constraints more directly.

Substituting (8) into (7), we have:

$$\theta^t - \theta^{t+1} = \hat{v}^t(\boldsymbol{C}) - \hat{v}^{t+1}(\boldsymbol{C}) \geq \sum_i \lambda_i^t \max\left\{q_i - v^{t+1}(\boldsymbol{C}), 0\right\} \tag{9}$$

For the case where there exists a knapsack j satisfying both conditions: $\boldsymbol{h}^* \in \arg\max_{\boldsymbol{h} \in S(c_j)} \sum_i \beta_{ij}^{\dagger} h_i$ and $h_i^* \geq 1$, we establish the value difference:

$$
\begin{aligned}
&v^{t+1}(\boldsymbol{C} - e_j^T w_i) - v^{t+1}(\boldsymbol{C}) \\
=& \max_{\boldsymbol{h}\in S(c_j - w_i)} \{\sum_i \beta_{ij}^{\dagger} h_i\} - \max_{\boldsymbol{h}\in S(c_j)} \{\sum_i \beta_{ij}^{\dagger} h_i\} \\
=& -\beta_{ij}^{\dagger} \leq 0
\end{aligned}
$$

The acceptance threshold is then defined as:

$$
\alpha_i = \max\left\{\max_j \left\{r_i - \beta_{ij}^{\dagger}\right\}, 0\right\},
$$

with $\alpha_i = 0$ when no qualifying knapsack j exists.

Let $\gamma_j = \max_{\boldsymbol{h}\in S(c_j)}\{\sum_i \beta_{ij}^{\dagger} h_i\}$. This yields:

$$
\begin{aligned}
\theta^1 &= \sum_{t=1}^{T} (\theta^t - \theta^{t+1}) \geq \sum_t \sum_i \alpha_i \lambda_i^t = \sum_i d_i \alpha_i \\
\hat{v}^1(\boldsymbol{C}) &= \sum_i d_i \alpha_i + \sum_j \gamma_j
\end{aligned}
$$

Since $\hat{v}^1(\boldsymbol{C})$ constitutes a feasible solution to (7), we have $\hat{v}^1(\boldsymbol{C}) \geq v^1(\boldsymbol{C}) = V^{DP}$.

If all j exist for $\boldsymbol{h}^* \in \arg\max_{\boldsymbol{h}\in S(c_j)} \sum_i \beta_{ij}^{\dagger} h_i, h_i^* \geq 1$, the corresponding bid-price problem can be expressed as:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{M} \alpha_i d_i + \sum_{j=1}^{N} \gamma_j \\
\text{s.t.} \quad & \alpha_i + \beta_{ij}^{\dagger} \geq r_i, \quad \forall i, j, \\
& \sum_{i=1}^{M} \beta_{ij}^{\dagger} h_i \leq \gamma_j, \quad \forall j, \boldsymbol{h} \in S(c_j), \\
& \alpha_i \geq 0, \forall i, \quad \beta_{ij}^{\dagger} \geq 0, \forall i, j \\
& \gamma_j \geq 0, \quad \forall j.
\end{aligned}
\tag{10}
$$

α_i represents marginal revenue for type i. $\beta_{ij}^{\dagger}$ represents the cost for type i assigned in knapsack j. γ_j represents the capacity cost associated with knapsack j.

When no knapsack j exists satisfying both $\boldsymbol{h}^* \in \arg\max_{\boldsymbol{h}\in S(c_j)} \sum_i \beta_{ij}^{\dagger} h_i$ and $h_i^* \geq 1$, the first set of constraints for (i, j) should be removed from the formulation (10). Although these constraints appear difficult to enforce in advance, the following lemma reveals that in practice we can avoid imposing additional restrictions. Simply solving problem (10) will inherently satisfy all required conditions.

Lemma 2. *Define $\mathcal{J} = \{j \in \mathcal{N} \mid r_i - \beta_{ij}^{\dagger *} > 0\}$. Then there exists a $j_0 \in \mathcal{J}$ such that:*

$$
\boldsymbol{h}^* \in \arg\max_{\boldsymbol{h}\in S(c_{j_0})} \sum_i \beta_{ij_0}^{\dagger *} h_i, h_i^* \geq 1.
$$

This lemma guarantees that when such a j_0 exists, the first set of constraints for i becomes active; otherwise, it remains inactive. Consequently, we have $z(\text{BPP}) = \hat{v}^1(\boldsymbol{C})$.

Then, the control policy becomes as follow. If $\alpha_i > 0$, accept the type i. Find knapsack $k = \arg\max_{j\in\mathcal{N}}\{r_i - \beta^{\dagger}_{ij}\}$, if multiple maximizers exist, assign the type i to a knapsack j satisfying:

$$\boldsymbol{h}^* \in \arg\max_{\boldsymbol{h}\in S(c_j)} \sum_i \beta^{\dagger*}_{ij} h_i, h^*_i \geq 1.$$

If $\alpha_i = 0$, check whether there exists a knapsack j such that: $\boldsymbol{h}^* \in \arg\max_{\boldsymbol{h}\in S(c_j)} \sum_i \beta^{\dagger*}_{ij} h_i, h^*_i \geq 1$. If no such j exists, reject the type i; otherwise, accept the type i and assign it to knapsack j.

Let $z(\text{BPC})$ and $z(\text{BPP})$ denote the expected optimal value of (5) and (10), respectively.

Lemma 3. *For the optimal β^*_j in (5), there exist optimal $\beta^{\dagger*}_{ij}$ in (10) satisfying $\beta^{\dagger*}_{ij} \leq w_i \beta^*_j$ for all i. Furthermore, $z(BPC) \geq z(BPP)$.*

Both bid-price approaches give upper bounds on the value function at any state, meanwhile it follows that BPP provides a tighter approximation to the value function more accurately than BPC does.

Under the approximation (8), the BPP policy operates as follows:

Algorithm 2: Bid-Price Control Based on Patterns

for $t = 1, \ldots, T$ **do**
- Observe a request of item type i;
- Solve problem (10) with $\boldsymbol{d}^{[t,T]}$ and $\mathbf{L}^t$;
- **if** $r_i - \beta^{\dagger}_{ij} > 0$ **then**
  - Set $k = \arg\max_{j\in\mathcal{N}}\{r_i - \beta^{\dagger}_{ij}\}$;
  - If multiple ks exist, assign the type i to a knapsack j satisfying:

    $$\boldsymbol{h}^* \in \arg\max_{\boldsymbol{h}\in S(c_j)} \sum_i \beta^{\dagger*}_{ij} h_i, h^*_i \geq 1.$$

    Let $c^{t+1}_j \leftarrow c^t_j - w_i$;
- **else**
  - **if** *There exists j such that* $\boldsymbol{h}^* \in \arg\max_{\boldsymbol{h}\in S(c_j)} \sum_i \beta^{\dagger}_{ij} h_i, h^*_i \geq 1$ **then**
    - Assign the item in knapsack j, let $c^{t+1}_j \leftarrow c^t_j - w_i$;
  - **else**
    - Reject the request;
  - **end**
- **end**

end

Unlike the traditional BPC, the BPP does not require explicit feasibility checks on capacity. However, it still needs to verify whether the optimal pattern contains the given request. This introduces a new challenge, as bid-price policies are inherently derived from a dual formulation, which may inherently omit key information preserved in the primal problem.

Example 2. *Continue with the above example:*

For the BPP, $\boldsymbol{\beta}_1^* = [4,4,4,6]$*,* $\boldsymbol{\beta}_2^* = [6,6,6,6]$*,* $\boldsymbol{\beta}_3^* = [10,8,8,8]$*.* $z(BPP) = 40$*.* $\alpha = [0,0,0]$*,* $\gamma = [10,12,12,6]$*.*

$r_1 - \boldsymbol{\beta}_1^* = [0,0,0,-2]$*,* $r_2 - \boldsymbol{\beta}_2^* = [0,0,0,0]$*,* $r_3 - \boldsymbol{\beta}_3^* = [-2,0,0,0]$*.*

Drawback: need to check whether j *exists for* $\boldsymbol{h}^* \in \arg\max_{\boldsymbol{h}\in S(c_{j_0})} \sum_i \beta_{ij_0}^{\dagger *} h_i, h_i^* \geq 1$*.*

For example, $r_3 - \boldsymbol{\beta}_3^* = [-2,0,0,0]$ *indicates type 3 cannot be assigned in knapsack 1. While the generated patterns for knapsack 4 are* $[0,1,0],[1,0,0]$*, which do not contain* h_3*, type 3 can only be assigned to knapsack 2 or 3.*

While this verification is cumbersome under pure bid-price frameworks, the primal problem offers a more direct way to guarantee the existence of such a request-pattern match. To address this gap, we propose a dynamic primal formulation.

3.3 Dynamic Primal Based on Patterns

Let $y_{j\boldsymbol{h}}$ denote the proportion of pattern $\boldsymbol{h}$ used in knapsack j. The primal problem can be formulated as:

$$\begin{aligned} \max \quad & \sum_{i=1}^{M}\sum_{j=1}^{N} r_i x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} \leq d_i, \quad i \in \mathcal{M}, \\ & x_{ij} \leq \sum_{\boldsymbol{h}\in S(c_j)} h_i y_{j\boldsymbol{h}}, \quad i \in \mathcal{M}, j \in \mathcal{N}, \\ & \sum_{\boldsymbol{h}\in S(c_j)} y_{j\boldsymbol{h}} \leq 1, \quad j \in \mathcal{N}. \end{aligned} \tag{11}$$

The first set of constraints demonstrate that for each item type i, the sum of assigned items and unassigned items equals the total demand. The second set of constraints shows that the number of items of type i assigned in knapsack j is not larger than the sum of h_i (the count of type i items in pattern $\boldsymbol{h}$) weighted by the pattern proportions $y_{i\boldsymbol{h}}$. The total proportion of patterns uesd in knapsack j cannot exceed 1.

Lemma 4. *The optimal solution* x_{ij}^* *to* (11) *satisfies* $x_{ij}^* > 0$ *if and only if there exists a knapsack* j *such that*

$$\boldsymbol{h}^* \in \arg\max_{\boldsymbol{h}\in S(c_j)} \sum_i \beta_{ij}^{\dagger} h_i, h_i^* \geq 1.$$

In contrast to Lemma 2, this lemma demonstrates the equivalence between the condition $x_{ij}^* > 0$ and the existance of a knapsack j satisfying $\boldsymbol{h}^* \in \arg\max_{\boldsymbol{h}\in S(c_j)} \sum_i \beta_{ij}^{\dagger} h_i, h_i^* \geq 1$. This equivalence eliminate the need for explicit existance verification.

Lemma 5. $z(DLP) \geq V^{HO}$ *results from the concave property.*

Consider the standard linear program: $\phi(\boldsymbol{d}) = \{\max c^T \boldsymbol{x} : A\boldsymbol{x} \leq \boldsymbol{d}, \boldsymbol{x} \geq \boldsymbol{0}\}$. Suppose that $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are two demand vectors, the optimal solution is $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. For any $\lambda \in [0,1]$, $\boldsymbol{d}_\lambda = \lambda \boldsymbol{d}_1 + (1-\lambda)\boldsymbol{d}_2$.

Let $\boldsymbol{x}_\lambda = \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2$, then $A\boldsymbol{x}_\lambda = A(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) \leq \lambda \boldsymbol{d}_1 + (1-\lambda)\boldsymbol{d}_2 = \boldsymbol{d}_\lambda$. Thus, $\boldsymbol{x}_\lambda$ is a feasible solution for $\boldsymbol{d}_\lambda$. Then, $\phi(\boldsymbol{d}_\lambda) \geq \boldsymbol{c}^T\boldsymbol{x}_\lambda = \lambda \boldsymbol{c}^T\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{c}^T\boldsymbol{x}_2 = \lambda\phi(\boldsymbol{d}_1) + (1-\lambda)\phi(\boldsymbol{d}_2)$, which indicates $\phi(\boldsymbol{d})$ is concave. Let $\phi(\boldsymbol{d})$ indicate the optimal value of the linear relaxation of the SPDR problem. Substitute $\boldsymbol{x}$ with $y_{j\boldsymbol{h}}$ and view $y_{j\boldsymbol{h}}$ as the decision variables, then the concave property still holds for (11). $V^{\text{HO}} = E[\phi(\boldsymbol{d})] \leq \phi(E[\boldsymbol{d}]) = z(\text{DLP})$.

3.3.1 Solve the dynamic primal

The pattern $\boldsymbol{h}$ is efficient for knapsack j if and only if, for some $(\alpha_1, \ldots, \alpha_M, \gamma_j)$ (except that $\alpha_i = r_i, \forall i$), $\boldsymbol{h}$ is the optimal solution to

$$\max_{\boldsymbol{h}} \sum_{i=1}^{M} (r_i - \alpha_i) h_i - \gamma_j$$

To generate all efficient patterns, we need to solve the subproblem for each knapsack j:

$$\begin{aligned}
\max \quad & \sum_{i=1}^{M} (r_i - \alpha_i) h_i - \gamma_j \\
\text{s.t.} \quad & \sum_{i=1}^{M} w_i h_i \leq c_j, \\
& h_i \in \mathbb{N}, \quad i \in \mathcal{M}.
\end{aligned} \tag{12}$$

If the optimal value of (12) is larger than 0, the primal (11) reaches the optimal. Otherwise, a new pattern can be generated.

One important fact is that only efficient sets are used in the solution to (11). Specifically,

Lemma 6. *If $y^*_{j\boldsymbol{h}} > 0$ is the optimal solution to (11), then $\boldsymbol{h}$ is an efficient pattern.*

A pattern $\boldsymbol{h}$ is dominant if there is no distinct pattern $\boldsymbol{h}'$ where every component of $\boldsymbol{h}'$ is greater than or equal to the corresponding component of $\boldsymbol{h}$. The efficient pattern is a dominating pattern. (If $\alpha_i = r_i$, (11) reaches the optimal and no pattern will be generated.)

The relation between the capacity and the demand shows the different structure of the optimal solution.

Lemma 7. *When $\sum_{i=1}^{M} d_i w_i < \sum_{j=1}^{N} c_j$, we have $\gamma^*_j = 0, \forall j$, $\beta^{\dagger *}_{ij} = 0, \forall i, j$ and $\alpha^*_i = r_i, \forall i$. There exists at least one knapsack j such that $\sum_{\boldsymbol{h} \in S(c_j)} y^*_{j\boldsymbol{h}} < 1$.*

When $\sum_{i=1}^{M} d_i w_i \geq \sum_{j=1}^{N} c_j$, we have $\sum_{\boldsymbol{h} \in S(c_j)} y^*_{j\boldsymbol{h}} = 1, \forall j$.

Algorithm 3: Dynamic Primal

for $t = 1, \dots, T$ **do**
  Observe a request of type i;
  if $c_j^t = w_i, \exists j$ **then**
    Assign the item to knapsack j;
    continue
  end
  Solve problem (11) with $\boldsymbol{d}^{[t,T]}$;
  Obtain an optimal solution x_{ij} ;
  if $\max_j\{x_{ij}\} > 0$ **then**
    Set $k = \arg\max_j\{x_{ij}\}$ and break ties;
    Assign the item to knapsack k, let $c_k^{t+1} \leftarrow c_k^t - w_i$;
  else
    Reject the request;
  end
end

Meanwhile, it guarantees feasible placement. Once a request is accepted, the policy ensures it can be assigned to a suitable knapsack without additional feasibility checks.

Example 3. *For the primal,* $\boldsymbol{x}_1^* = [1,1,0,0]$*,* $\boldsymbol{x}_2^* = [1,0,2,1]$*,* $\boldsymbol{x}_3^* = [0,1,0,0]$*. It indicates that type 1 can be assigned in knapsacks 1 or 2, type 2 cannot be assigned in knapsack 2, type 3 can only be assigned in knapsack 2.*

It may contain multiple efficient patterns for one knapsack. In this example, there is exactly one efficient pattern for each knapsack: $[1,1,0]$*,* $[1,0,1]$*,* $[0,2,0]$*,* $[0,1,0]$*. It shows that knapsack 1 can assign type 1 and 2, so on and so forth.*

Using x_{ij} *to make the decision is straightforward and easy to implement.*

Example 4. *Consider* $M = 3$*,* $N = 4$*,* $w_1 = 3, w_2 = 4, w_3 = 5$*,* $r_1 = 4, r_2 = 6, r_3 = 8$*,* $\boldsymbol{C} = [7,8,8,4]$*,* $T = 8$*, stationary arrival probability,* $\lambda_1 = \lambda_3 = \frac{1}{4}$*,* $\lambda_2 = \frac{1}{2}$*. Then the expected demand for each type is* $\boldsymbol{d} = (2,4,2)$*.*

For the traditional bid-price control, for each j*,* $\beta_j^* = \frac{4}{3}$*.* $z(BPC) = \frac{124}{3}$*. Two drawbacks: there is no difference among knapsacks. Even* $r_3 - \beta_4^* * w_3 > 0$*, it is infeasible for type 3 assigned in knapsack 4.*

For the BPP, $\boldsymbol{\beta}_1^* = [4,4,4,6]$*,* $\boldsymbol{\beta}_2^* = [6,6,6,6]$*,* $\boldsymbol{\beta}_3^* = [10,8,8,8]$*.* $z(BPP) = 40$*.* $\alpha = [0,0,0]$*,* $\gamma = [10,12,12,6]$*.*

$r_1 - \boldsymbol{\beta}_1^* = [0,0,0,-2]$*,* $r_2 - \boldsymbol{\beta}_2^* = [0,0,0,0]$*,* $r_3 - \boldsymbol{\beta}_3^* = [-2,0,0,0]$*.*

Drawback: need to check whether j *exists for* $\boldsymbol{h}^* \in \arg\max_{\boldsymbol{h} \in S(c_{j_0})} \sum_i \beta_{ij_0}^{\dagger *} h_i, h_i^* \geq 1$*.*

For example, $r_3 - \boldsymbol{\beta}_3^* = [-2,0,0,0]$ *indicates type 3 cannot be assigned in knapsack 1. While the generated patterns for knapsack 4 are* $[0,1,0], [1,0,0]$*, which do not contain* h_3*, type 3 can only be assigned to knapsack 2 or 3.*

For the primal, $\boldsymbol{x}_1^* = [1,1,0,0]$*,* $\boldsymbol{x}_2^* = [1,0,2,1]$*,* $\boldsymbol{x}_3^* = [0,1,0,0]$*. It indicates that type 1 can be* *assigned in knapsacks 1 or 2, type 2 cannot be assigned in knapsack 2, type 3 can only be assigned in knapsack 2.*

It may contain multiple efficient patterns for one knapsack. In this example, there is exactly one efficient pattern for each knapsack: $[1, 1, 0]$, $[1, 0, 1]$, $[0, 2, 0]$, $[0, 1, 0]$. *It shows that knapsack 1 can assign type 1 and 2, so on and so forth.*

Using x_{ij} *to make the decision is straightforward and easy to implement.*

Asymptotic loss:

Lemma 8. *Loss:* $V_\theta^{HO} - V_\theta^{BPC} = O(\sqrt{\theta})$.

Lemma 9. *Loss:* $V_\theta^{HO} - V_\theta^{DPP} = O(1)$.

Let $T_i = sup\{t \leq T : \lambda_i^t > 0\}$.

Resource allocation Applications: Cloud computing, Ad placement, Production scheduling, Energy management, Network bandwidth allocation

3.4 Static BLC Policy

Booking limit control policy:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{M}\sum_{j=1}^{N} r_i x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} \le d_i, \quad i \in \mathcal{M}, \\
& \sum_{i=1}^{M} w_i x_{ij} \le L_j, j \in \mathcal{N},
\end{aligned} \tag{13}
$$

Let $d_i^* = \sum_j x_{ij}^*$, x_{ij}^* is an integral optimal solution to (13) with $d_i = \sum_t p_i^t$ (Expected demand).

Let d_i indicate the number of type i during time T. $d_i = \sum_t \mathbf{1}_{i_t=i}$. Let $val(I;\{d_i\})$ denote the optimal objective value of (13).

$V^{BL}(I) = E_{\{d_i\}}[\sum_i (n_i - \delta)\min\{d_i^*, d_i\}]$, $V^{OPT}(I) = E_{\{d_i\}}[val(I;\{d_i\})] \le val(I;\{E[d_i]\})$.

$val(I;\{d_i\})$ is concave in d_i.

$$
\begin{aligned}
& V^{OPT}(I) - V^{BL}(I) \\
\le & val(I;\{E[d_i]\}) - V^{BL}(I) \\
= & val(I;\{E[d_i]\}) - val(I;\{\lfloor E[d_i] \rfloor\}) + val(I;\{\lfloor E[d_i] \rfloor\}) - E_{\{d_i\}}[\sum_i (n_i - \delta)\min\{d_i^*, d_i\}] \\
\le & \sum_i (n_i - \delta) + N\sum_i i + E_{\{d_i\}}[\sum_i (n_i - \delta)(d_i^* - \min\{d_i^*, d_i\})] \\
= & \sum_i (n_i - \delta) + N\sum_i i + E_{\{d_i\}}[\sum_i \frac{1}{2}(n_i - \delta)(d_i^* - d_i + |d_i^* - d_i|)] \\
\overset{(a)}{\le} & \sum_i (n_i - \delta) + N\sum_i i + \frac{1}{2}\sum_i (n_i - \delta)(d_i^* - E[d_i] + |d_i^* - E[d_i]| + \sqrt{\mathrm{Var}[d_i]}) \\
\le & \sum_i (n_i - \delta) + N\sum_i i + \frac{1}{2}\sum_i (n_i - \delta)\sqrt{\mathrm{Var}[d_i]} \\
\le & \sum_i (n_i - \delta) + N\sum_i i + \frac{1}{2}\sum_i (n_i - \delta)\sqrt{Tp_i(1-p_i)} = O(\sqrt{T})
\end{aligned}
$$

Thus, $\lim_{T\to\infty}(V^{OPT}(I) - V^{BL}(I))/T \to 0$.

$val(I;\{E[d_i]\}) - val(I;\{\lfloor E[d_i] \rfloor\}) \le val(I;\{\lceil E[d_i] \rceil\}) - val(I;\{\lfloor E[d_i] \rfloor\}) = \sum_i (n_i - \delta)$

$LP - IP \le \sum_i \sum_j (n_i - \delta)(x_{ij}^* - \lfloor x_{ij}^* \rfloor) \le N\sum_i i \Rightarrow val(I;\{\lfloor E[d_i] \rfloor\}) \le IP + N\sum_i i$.

$IP = \sum_i \sum_j (n_i - \delta)x_{ij}^* = \sum_i (n_i - \delta)d_i^*$

(a) results from the following inequalities: $|d_i^* - d_i| = |(d_i^* - E[d_i]) + (E[d_i] - d_i)| \le |d_i^* - E[d_i]| + |d_i - E[d_i]|$. Take the expectation, we have $E[|d_i^* - d_i|] \le |d_i^* - E[d_i]| + E[|d_i - E[d_i]|]$. $E[|d_i - E[d_i]|] \le \sqrt{\mathrm{Var}[d_i]}$(Since $E[|X|] \le \sqrt{E[X^2]}$). $d_i^* \le E[d_i]$.

Surrogate relaxation (0-1 single):

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{M} r_i x_i && (14)\\
\text{s.t.} \quad & x_i \le d_i, \quad i \in \mathcal{M}, && (15)\\
& \sum_{i=1}^{M} n_i x_i \le L. && (16)
\end{aligned}
$$

LP optimal solution: $[0, \dots, 0, X_{\tilde{i}}, d_{\tilde{i}+1}, \dots, d_M]$, $X_{\tilde{i}} = \frac{L - \sum_{i=\tilde{i}+1}^{M} d_i w_i}{n_{\tilde{i}}}$.

One feasible IP optimal solution: $[0, \dots, 0, \lfloor X_{\tilde{i}} \rfloor, d_{\tilde{i}+1}, \dots, d_M]$.

$LP - IP \le \tilde{i}(X_{\tilde{i}} - \lfloor X_{\tilde{i}} \rfloor)$

$$
\begin{aligned}
& V^{OPT}(I) - V^{BL}(I) \\
\le & val(I; \{E[d_i]\}) - V^{BL}(I) \\
= & val(I; \{E[d_i]\}) - val(I; \{\lfloor E[d_i] \rfloor\}) + val(I; \{\lfloor E[d_i] \rfloor\}) - E_{\{d_i\}}[\sum_i (n_i - \delta) \min\{d_i^*, d_i\}] \\
\le & \sum_i (n_i - \delta) + \tilde{i}(X_{\tilde{i}} - \lfloor X_{\tilde{i}} \rfloor) + E_{\{d_i\}}[\sum_i (n_i - \delta)(d_i^* - \min\{d_i^*, d_i\})] \\
\le & \sum_i (n_i - \delta) + \tilde{i}(X_{\tilde{i}} - \lfloor X_{\tilde{i}} \rfloor) + \frac{1}{2} \sum_i (n_i - \delta) \sqrt{T p_i (1 - p_i)}
\end{aligned}
$$

$E[\text{loss}] = V^{\text{off}} - V_{\pi}^{on} \ge V^{\text{opt}} - V_{\pi}^{on}$

One sample path. d^r realization of M types.

$V_t(l) = \sum_{i=\hat{i}+1}^{M} r_i d_i^r + r_{\hat{i}}(l - \sum_{i=\hat{i}+1}^{M} d_i^r)$

Let $V^{\text{OPT}}(I)$ denote the expected value under offline optimal policy (relaxed) during T periods for instance I (capacity, probability distribution).

The revenue loss between the static deterministic heuristic and the optimal is bounded by $C\sqrt{T}$.

Let γ_i, γ_i^0 denote the number of type i accepted and rejected by some heuristic policy, respectively.

$$
\begin{aligned}
\text{OPT}(L, \hat{d}, \gamma): \quad \max \quad & \sum_{i=1}^{M} r_i x_i \\
\text{s.t.} \quad & x_i^0 + x_i = \hat{d}_i, \quad i \in \mathcal{M}, \\
& x_i \ge \gamma_i, \quad i \in \mathcal{M}, \\
& x_i^0 \ge \gamma_i^0, \quad i \in \mathcal{M}, \\
& \sum_{i=1}^{M} w_i x_i \le L.
\end{aligned}
$$

Heuristic policy: At time t, solve problem (14) with $d_i = d_i^t = (T - t) * p_i$, $L = L^t$. When $x_i \ge 1$ for the request of type i, accept the request.

$d^{[1,T]}$ is the demand realization during $[1,T]$. $\gamma^{[1,t)}$ represents the number of requests rejected and accpeted by some heuristic policy during $[1,t)$.

$OPT(L, d^{[1,T]}, \gamma^{[1,t+1)})$ can be interpreted as the total reward obtained under a virtual policy where we first follow the heuristic policy during $[1,t+1)$ and then from time $t+1$ we follow the optimal solution assuming that we know the future demands.

For one sample path of the requests, the revenue loss can be decomposed into T increments.

$$\begin{aligned}
&OPT(C, d^{[1,T]}, 0) - OPT(C, d^{[1,T]}, \gamma^{[1,T]}) \\
=&\sum_{t=1}^{T}[OPT(C, d^{[1,T]}, \gamma^{[1,t)}) - OPT(C, d^{[1,T]}, \gamma^{[1,t+1)})]
\end{aligned}$$

Let $c_j^t = c_j - \sum_{i=1}^{M} w_i \gamma_{ij}^{[1,t)}$.

The expected revenue loss can be upper bounded:

$$\begin{aligned}
&E[OPT(C, d^{[1,T]}, 0) - OPT(C, d^{[1,T]}, \gamma^{[1,T]})] \\
\leq& l \sum_{t=1}^{T} P(OPT(C, d^{[1,T]}, \gamma^{[1,t)}) - OPT(C, d^{[1,T]}, \gamma^{[1,t+1)}) > 0) \\
=&(n_M - \delta) \sum_{t=1}^{T} P(OPT(L^t, d^{[t,T]}, 0) - OPT(L^t, d^{[t,T]}, \gamma^{[t,t+1)}) > 0) \\
\leq&(n_M - \delta) \sum_{t=1}^{T} P(x_{it}^{*,t} < 1) \\
=&(n_M - \delta) \sum_{t=T_0}^{T} P(x_{it}^{*,t} < 1) \\
\leq&(n_M - \delta) \max_i \{\frac{1}{p_i}\}
\end{aligned}$$

Lemma 10. $OPT(L^1, \hat{d} + d^{[1,t_2)}, \gamma^{[1,t_2)}) = \sum_i (n_i - \delta)\gamma_i^{[1,t_1)} + OPT(L^t, \hat{d} + d^{[t_1,t_2)}, \gamma^{[t_1,t_2)})$

For any optimal solution x^* of $OPT(L^t, \hat{d} + d^{[t_1,t_2)}, \gamma^{[t_1,t_2)})$, $x^* + \gamma^{[1,t_1)}$ is a feasible solution of $OPT(L^1, \hat{d} + d^{[1,t_2)}, \gamma^{[1,t_2)})$. For any optimal solution x^* of $OPT(L^1, \hat{d} + d^{[1,t_2)}, \gamma^{[1,t_2)})$, $x^* - \gamma^{[1,t_1)}$ is a feasible solution of $OPT(L^t, \hat{d} + d^{[t_1,t_2)}, \gamma^{[t_1,t_2)})$ because $x^* - \gamma^{[1,t_1)} \geq \gamma^{[1,t_2)} - \gamma^{[1,t_1)} = \gamma^{[t_1,t_2)}$.

The first inequality results from $E[A] \leq r_M E[\mathbf{1}_{A>0}] = r_M P(A > 0)$.

The first equation follows from Lemma. (Let $t_1 = t_2 = t$, $\hat{d} = d^{[t,T]}$; let $t_1 = t, t_2 = t+1$, $\hat{d} = d^{[t+1,T]}$).

The second equation is as follows. If $x_{it}^{*,t} \geq 1$, then $x^{*,t}$ is still feasible for $OPT(L^t, d^{[t,T]}, \gamma^{[t,t+1)})$. (Because the optimal policy)

$x_{it}^{*,t}$ is the optimal solution for OPT$(L^t, d^{[t,T]}, 0)$ at time t.

Let $T - T_0 = \max_i\{\frac{1}{p_i}\}$

For N rows,

$$
\begin{aligned}
OPT(\boldsymbol{L}, \hat{d}, \gamma): \quad \max \quad & \sum_{i=1}^{M}\sum_{j=1}^{N} r_i x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} + x_{i0} = \hat{d}_i, \quad i \in \mathcal{M}, \\
& \sum_{j=1}^{N} x_{ij} \geq \gamma_i, \quad i \in \mathcal{M}, \\
& x_{i0} \geq \gamma_i^0, \quad i \in \mathcal{M}, \\
& \sum_{i=1}^{M} w_i x_{ij} \leq c_j, \quad j \in \mathcal{N}.
\end{aligned}
$$

4 Computational Experiments

We conduct several experiments, including analyzing the performances of different policies, evaluating the impact of implementing social distancing, and comparing the performance under varied constraints. In the experiments, we set the following parameters.

The default parameters in the experiments are as follows, the size of social distancing $\delta = 1$, the maximum group size $M = 4$, the number of rows $N = 10$, and the size of each row $L_j = 21$, for all $j \in \mathcal{N}$. We simulate the arrival of exactly one group in each period, i.e., $p_0 = 0$. Each experiment result is the average of 100 instances. In each instance, the number of scenarios in SBSP is $|\Omega| = 1000$.

To assess the performances of different policies across varying demand levels, we conduct experiments spanning a range of 60 to 100 periods. The arrival probabilities for each group, denoted as $[p_1, p_2, p_3, p_4]$, are as follows for our analysis: $D_1 : [0.18, 0.7, 0.06, 0.06]$, $D_2 : [0.2, 0.8, 0, 0]$, $D_3 : [0.34, 0.51, 0.07, 0.08]$ and $D_4 : [0.12, 0.5, 0.13, 0.25]$. The first two probability distributions, D_1 and D_2, are tested in Blom et al. (2022), where D_1 represents the statistical distribution of group sizes observed in historical data. In contrast, D_2 aligns with our constrained situation by capping the maximum group size at two. The other two distributions, D_3 and D_4, are derived from real-world movie data representing two distinct types of films, which we collected for this study. The specific procedure is detailed in Appendix. We use D_4 as the default probability distribution in the other experiments.

4.1 Performance Evaluation of Seat-Plan-Based Assignment

We evaluate the performance of our proposed policies. To do this, we compare it with three other policies: the traditional bid-price control (BPC), bid-price control based on patterns, first-come-first-served. For each policy, we compute the performance relative to the optimal policy derived from solving the deterministic model with perfect information regarding all requests. Table 1 summarizes the ratio of assigned individuals under each policy compared to those assigned under the optimal policy for different demand probability distributions and time periods.

Our results demonstrate that the policy outperforms the BPC and FCFS policies. The and BPC policies can make the initial decision to accept or reject a request, but lack the capability to optimize

Table 1: Performances

Distribution	T	BPC (%)	BPP (%)	Primal (%)	FCFS (%)
	60				
	70				
D_1	80				
	90				
	100				
	60				
	70				
D_2	80				
	90				
	100				
	60				
	70				
D_3	80				
	90				
	100				
	60	98.04		98.96	
	70	97.15		98.82	
D_4	80	96.91		98.54	
	90	96.93		98.41	
	100	97.43		99.01	

seat assignments. The policy strictly adheres to predetermined booking limits; even if the supply of one type is exhausted, it does not utilize seats planned for other types to accept the request, leading to its limited effectiveness.

The performance of , BPP, and BPC policies basically follows a pattern where it initially declines and then gradually improves as T increases. When T is small, the demand of requests is generally low, allowing these policies to achieve relatively optimal performance. However, as T increases, it becomes more challenging for these policies to consistently achieve a perfect allocation, resulting in a decrease in performance. Nevertheless, as T continues to grow, these policies tend to accept larger groups, thereby narrowing the gap between their performance and the optimal value. Consequently, their performances improve. In contrast, the BLC policy shows improved performance as T increases because it reduces the number of unoccupied seats reserved for the largest groups.

The performance of the policies can vary with different probabilities. For the different probability distributions listed, the policy performs more stably and consistently for the same demand. In contrast, the performances of the other policies fluctuate more significantly.

4.2 Impact of Social Distancing

To examine the impact of of implementing social distancing, we introduce three key terms, the threshold of request-volume, the threshold of occupancy rate, and the maximum achievable occupancy rate as follows.

The *threshold of request-volume*, q^{th}, is defined as

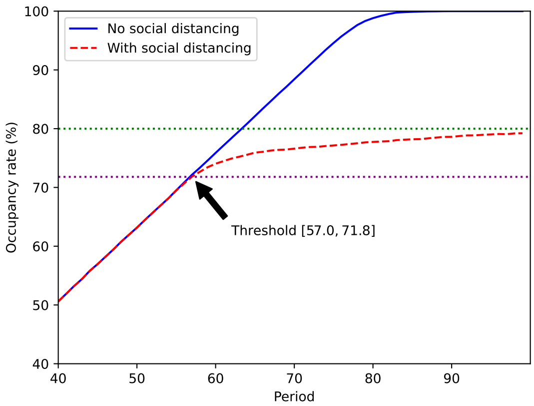
$$q^{\text{th}} = (1 - p_0) \cdot \max \left\{ T \,\middle|\, E^0(T) - E(T) < 1 \right\},$$

where $E(T)$ and $E^0(T)$ denotes the average number of assigned individuals by SPBA with social distancing level δ and without social distancing, respectively. Here, the maximization is performed over T while keeping all other parameters constant. Intuitively, the threshold of request-volume represents the maximum number of requests that can be accommodated while keeping the average loss below one.

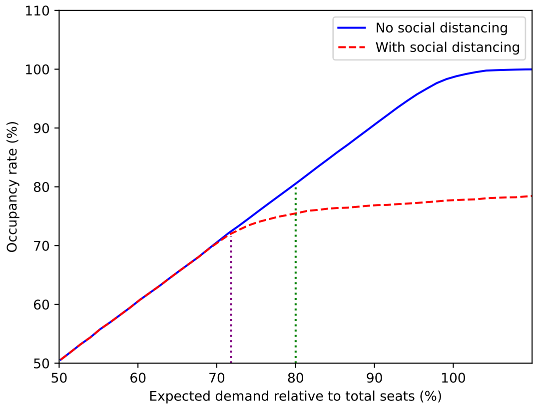
The occupancy rate corresponding to the threshold of request-volume is referred to as the *threshold of occupancy rate*, ρ^{tr}. This rate represents the maximum occupancy rate when the difference in the number of assigned individuals remains unaffected by the social distancing requirement.

The *maximum achievable occupancy rate* is attained when each row of a given layout is the largest pattern, denoted by ρ^{ac}, which is $\frac{\sum_{j=1}^{N} \phi(M, L_j^0, \delta)}{\sum_{j=1}^{N} L_j^0}$ as introduced in Section **??**.

We examine the impact of social distancing when implementing SPBA under varying levels of demand. Specifically, we calculate $E(T)$ with $\delta = 1$ and $E^0(T)$ across different values of T. The demand levels are varied by adjusting the parameter T from 40 to 100 in increments of 1. The results are visualized in Figure 1, which illustrates the occupancy rate under different demand levels.



(a)



(b)

Figure 1: Impact of social distancing

Figure 1(a) illustrates the occupancy rate over period, revealing three key metrics for policy evaluation: (1) the threshold of request-volume, $q^{\mathrm{th}} = 57$, (2) the threshold of occupancy rate, $\rho^{\mathrm{th}} = 71.8\%$, and (3) the maximum achievable occupancy rate $\rho^{\mathrm{ac}} = 80\%$. These metrics collectively serve as crucial indicators for assessing the effectiveness of the government policy.

Figure 1(b) presents the occupancy rate as a function of expected demand. The key distinction between Figure 1(a) and Figure 1(b) lies in their respective x-axes. From a demand perspective, we observe two distinct points: for expected demand below 71.8%, social distancing requirements impose no significant reduction in the number of assigned individuals; above this 71.8% threshold, the performance gap between scenarios with and without social distancing becomes increasingly pronounced.

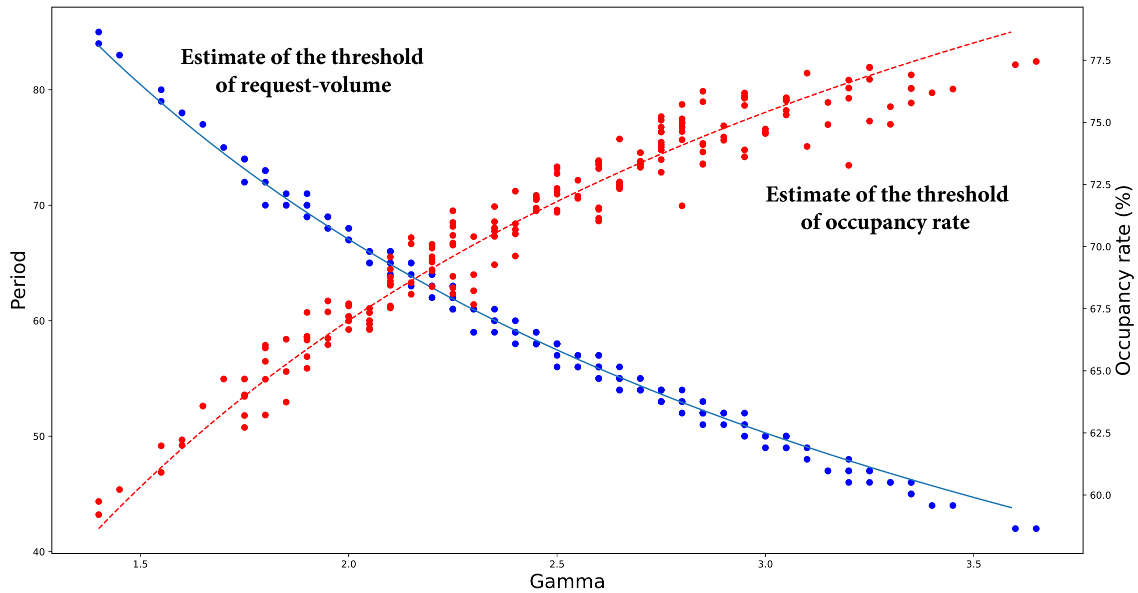
Given the conceptual similarity between period-based and demand-based occupancy rate analyses, we focus on these three metrics to concisely represent the core findings. Complete results and detailed interpretations are presented in Section 4.4, which examines performance under various policy constraints.

4.3 Robustness under Demand Distributions

As demonstrated earlier, SPBA maintains stable performance across four distinct probability distributions. To thoroughly assess its distributional robustness, we now examine its behavior over a wide range of probability distributions. For standardized comparison, we employ two key metrics: the threshold of request-volume q^{th} and the threshold of occupancy rate ρ^{th} (noting that the maximum achievable occupancy rate ρ^{ac} is distribution-invariant). Robustness is quantified by analyzing how these thresholds vary with the mean group size across different distributions. Finally, we provide estimates of q^{th} and ρ^{th} as functions of the mean group size.

The mean group size is $\gamma = \sum_{i=1}^{M} ip_i$. We analyze the relationship between q^{th}, ρ^{th} and γ using 200 random probability distributions ($M = 4$, $p_0 = 0$). Figure 2 shows q^{th} and ρ^{th} versus γ, along with their estimates: $\tilde{q}^{\text{th}} = \frac{c_1(1-p_0)\tilde{L}}{\gamma+\delta}$ and $\tilde{\rho}^{\text{th}} = \frac{c_2\gamma}{\gamma+\delta}\frac{(1-p_0)\tilde{L}}{\tilde{L}-N\delta}$, where c_1 and c_2 are discount factors, $\tilde{L}, \delta, N$ are defined in the problem setup, and $p_0 = 0$. Using the Ordinary Least Squares regression, we obtain perfect fits ($R^2 = 1.000$) with $c_1 = 0.9578$ and $c_2 = 0.9576$. The derivation and further details are provided in the Appendix.

Figure 2: The estimates of q^{th} and ρ^{th}



For fixed values of all other parameters, the estimated threshold of request-volume $\tilde{q}^{\text{th}}$ decreases as the mean group size γ increases, while the estimated threshold of occupancy rate $\tilde{\rho}^{\text{th}}$ increases with γ. This behavior aligns with the trends observed in Figure 2.

4.4 Performance Evaluation under Varied Constraints

To comprehensively evaluate the effectiveness of government policy strictness and venue-specific characteristics, we assess the performances of four critical factors: government-mandated maximum allowable occupancy rates, variations in maximum group sizes, different physical distances ($\delta \in \{1, 2\}$) and alternative seat layouts.

4.4.1 Government-Mandated Maximum Allowable Occupancy Rates

Government policies may impose a maximum allowable occupancy rate (ρ^{al}) to enforce stricter public health measures. The effectiveness of this policy depends on its relationship to two key metrics, threshold of occupancy rate ρ^{th} and maximum achievable occupancy rate ρ^{ac}.

When $\rho^{\text{al}} < \rho^{\text{th}}$, only the occupancy rate requirement is binding. Once the occupancy rate reaches ρ^{al}, all subsequent requests are rejected. When $\rho^{\text{th}} \leq \rho^{\text{al}} < \rho^{\text{ac}}$, both maximum allowable occupancy rate and social distancing requirements jointly govern seat assignments. Again, once ρ^{al} is reached, further requests are rejected. When $\rho^{\text{al}} \geq \rho^{\text{ac}}$, the maximum allowable occupancy rate constraint becomes redundant because the occupancy rate will never exceed ρ^{al} under social distancing.

The conclusion can be summarized in Table 2.

Table 2: Effectiveness of requirements

	Social distancing requirement	Requirement of ρ^{al}
$\rho^{\text{al}} < \rho^{\text{th}}$	Ineffective	Effective
$\rho^{\text{th}} \leq \rho^{\text{al}} < \rho^{\text{ac}}$	Effective	Effective
$\rho^{\text{al}} \geq \rho^{\text{ac}}$	Effective	Ineffective

4.4.2 Different Maximum Group Sizes and Physical Distances

When M is restricted at 3, given the probability distribution [0.12, 0.5, 0.13, 0.25], we discard the fourth component and normalize the remaining three components to generate a new probability distribution: [0.16, 0.67, 0.17]. Similarly, when $M = 2$, the probability distribution is [0.19, 0.81]. We also consider the impact of different distances. We present the corresponding threshold of request-volume, the threshold of occupancy rate and the maximum achievable occupancy rate in the table below.

Table 3: Impact of Ms and δs

M	δ	q^{th}	ρ^{th}	ρ^{ac}
2	1	74	66.8 %	70.0 %
2	2	54	48.8 %	50.0 %
3	1	68	68.3 %	75.0 %
3	2	53	53.1 %	60.0 %
4	1	57	71.8 %	80.0 %
4	2	47	59.2 %	70.0 %

For the fixed physical distance δ, an increase in the maximum group size M generally leads to a larger mean group size γ (since $\gamma = \sum_{i=1}^{M} ip_i$). This in turn causes the threshold of occupancy rate ρ^{th} to increase while reducing the threshold of request-volume $\tilde{q}^{\text{th}}$, as evidenced by their respective estimates. The maximum achievable occupancy rate ρ^{ac} increases because the size of the largest pattern $\Phi(M, L_j^0, \delta)$ defined in Proposition **??**, is monotonically increasing in M.

For the fixed maximum group size M, the experimental results show that when the physical distance δ increases from 1 to 2 seats, both the threshold of occupancy rate q^{th} and threshold of request-volume ρ^{th} decrease, which is consistent with our estimates. Assuming constant discount factors c_1 and c_2, both estimated thresholds q^{th} and ρ^{th} would decrease monotonically with increasing δ, provided the total capacity satisfies $\sum_{j=1}^{N} L_j^0 \geq N\gamma$ (See Appendix). This condition is generally satisfied in most practical seating layouts, further validating our experimental observations. The maximum achievable occupancy rate, ρ^{ac}, decreases since $\Phi(M, L_j^0, \delta)$ is monotonically decreasing in δ.

4.4.3 Alternative Seat Layouts

We experiment with several realistic seat layouts selected from a theater seat plan website, choosing five layouts labeled A, B, C, D, and E. Layouts A, D, and E are approximately rectangular, Layout C is a standard rectangular layout, and Layout B is irregular. In these layouts, wheelchair seats and management seats are excluded, while seats with sufficient space for an aisle are treated as new rows. The specific layouts are detailed in Appendix.

The occupancy rate over demand follows the typical pattern of Figure 1. The threshold of request-volume, the threshold of occupancy rate and the maximum achievable occupancy rate are also given in the following table. The maximum achievable occupancy rate can be calculated from Proposition **??**.

Table 4: Impact of the layouts

Layout	q^{th}	ρ^{th}	ρ^{ac}
A	36	72.3 %	82.4 %
B	38	75.8 %	84.1 %
C	32	72.8 %	80.0 %
D	43	74.1 %	83.6 %
E	102	72.4 %	81.7 %

While the venue layouts may differ in shapes (rectangular or otherwise) and row configurations (varying lengths), the maximum achievable occupancy rate ρ^{ac} do not shows the deterministic variation pattern.

Consistent with our estimates, the threshold of request-volume q^{th} follows $\tilde{q}^{\mathrm{th}} = \frac{c_1(1-p_0)\tilde{L}}{\gamma+\delta}$, exhibiting a positive correlation with total size of the layout $\tilde{L}$.

While the estimate shows that $\tilde{\rho}^{\mathrm{th}} = \frac{c_2\gamma}{\gamma+\delta}\frac{(1-p_0)\tilde{L}}{\tilde{L}-N\delta}$ should decrease with increasing $\tilde{L}$, in practice this variation becomes negligible as the ratio $\frac{\tilde{L}}{\tilde{L}-N\delta}$ remains nearly constant for typical venue sizes. This explains the observed lack of deterministic variation in ρ^{th} across layouts with different seating configurations.

5 Conclusion

We study the seating management problem under social distancing requirements. Specifically, we first consider the seat planning with deterministic requests problem. To utilize all seats, we introduce the full and largest patterns. Subsequently, we investigate the seat planning with stochastic requests problem. To tackle this problem, we propose a scenario-based stochastic programming model. Then, we utilize the Benders decomposition method to efficiently obtain a seat plan, which serves as a reference for dynamic seat assignment. Last but not least, to address the seat assignment with dynamic requests, we introduce the SPBA policy by integrating the relaxed dynamic programming and the group-type control allocation.

We conduct several numerical experiments to investigate various aspects of our approach. First, we compare SPBA with three benchmark policies: BPC, BLC, and RDPH. Our proposed policy demonstrates superior and more consistent performance relative to these benchmarks. All policies are assessed against the optimal policy derived from a deterministic model with perfect foresight of request arrivals.

Building upon our policies, we further evaluate the impact of implementing social distancing. By introducing the concept of the threshold of request-volume to characterize situations under which social distancing begins to cause loss to an event, our experiments show that the threshold of request-volume depends mainly on the mean of the group size. This leads us to estimate the threshold of request-volume by the mean of the group size.

Our models and analyses are developed for the social distancing requirement on the physical distance and group size, where we can determine a threshold of occupancy rate for any given event in a venue, and a maximum achievable occupancy rate for all events. Sometimes the government may impose a maximum allowable occupancy rate to tighten the social distancing requirement. This maximum allowable rate is effective for an event if it is lower than the threshold of occupancy rate of the event. Furthermore, the maximum allowable rate becomes redundant if it is higher than the maximum achievable rate for all events. These qualitative insights are stable concerning the tightness of the policy as well as the specific characteristics of various venues.

Future research can be pursued in several directions. First, when seating requests are predetermined, a scattered seat assignment approach can be explored to maximize the distance between adjacent groups when sufficient seating is available. Second, more flexible scenarios could be considered, such as allowing individuals to select seats based on their preferences. Third, research could also investigate scenarios where individuals arrive and leave at different times, adding an additional layer of complexity to the problem.

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