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Scenario-based Stochastic Programming

$$V_t(\mathbf{L}) = E_i\left[\max_{k\in N:L_k\geq i+s}\{[V_{t-1}(\mathbf{L}-U_{ik})+i], V_{t-1}(\mathbf{L})\}\right], \mathbf{L}\geq \mathbf{0}$$

$$V_{T+1}(\mathbf{L}) = 0,$$

- $\mathbf{L}$, remaining capacity.
- n_i.
- p_i: the probability of an arrival of group type i.

Scenario-based Stochastic Programming

$$
\begin{aligned}
\max \quad & E_\omega\left[\sum_{i=1}^{M-1}(n_i - s)(\sum_{j=1}^{N} x_{ij} + y^+_{i+1,\omega} - y^+_{i\omega}) + (n_M - s)(\sum_{j=1}^{N} x_{Mj} - y^+_{M\omega})\right] \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} - y^+_{i\omega} + y^+_{i+1,\omega} + y^-_{i\omega} = d_{i\omega}, \quad i \in [M-1], \omega \in \Omega \\
& \sum_{j=1}^{N} x_{ij} - y^+_{i\omega} + y^-_{i\omega} = d_{i\omega}, \quad i = M, \omega \in \Omega \\
& \sum_{i=1}^{M} n_i x_{ij} \le L_j, j \in [N] \\
& y^+_{i\omega}, y^-_{i\omega} \in \mathbb{Z}_+, \quad i \in [M], \omega \in \Omega \\
& x_{ij} \in \mathbb{Z}_+, \quad i \in [M], j \in [N].
\end{aligned}
\tag{1}
$$

Two-stage

$$\begin{aligned} \max \quad & c'\mathbf{x} + z(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{n}\mathbf{x} \le \mathbf{L} \\ & \mathbf{x} \in \mathbb{Z}_+^{M \times N}, \end{aligned} \tag{2}$$

where $z(\mathbf{x})$ is the recourse function defined as

$$z(\mathbf{x}) := E(z_\omega(\mathbf{x})) = \sum_{\omega \in \Omega} p_\omega z_\omega(\mathbf{x}),$$

and for each scenario $\omega \in \Omega$,

$$\begin{aligned} z_\omega(\mathbf{x}) := \max \quad & \mathbf{f}'\mathbf{y}_\omega \\ \text{s.t.} \quad & \mathbf{x}\mathbf{1} + \mathbf{V}\mathbf{y}_\omega = \mathbf{d}_\omega \\ & \mathbf{y}_\omega \ge 0. \end{aligned} \tag{3}$$

Solve the Second Stage Problem

$$\begin{aligned} \min \quad & \alpha'_\omega(\mathbf{d}_\omega - \mathbf{x1}) \\ \text{s.t.} \quad & \alpha'_\omega \mathbf{V} \geq \mathbf{f}' \end{aligned} \tag{4}$$

Let $P = \{\alpha | \alpha' V \geq \mathbf{f}'\}$. The feasible region of problem (4), P, is bounded. In addition, all the extreme points of P are integral.

Restricted Benders Master Problem

$$
\begin{aligned}
\max \quad & c'x + \sum_{\omega \in \Omega} p_\omega z_\omega \\
\text{s.t.} \quad & \sum_{i=1}^{M} n_i x_{ij} \le L_j, j \in [N] \\
& (\alpha^k)'(\mathbf{d}_\omega - \mathbf{x1}) \ge z_\omega, \alpha^k \in \mathcal{O}^t, \forall \omega \\
& \mathbf{x} \ge 0
\end{aligned} \tag{5}
$$

Deterministic Model

Formulation

When $|\Omega| = 1$ in problem (1), the stochastic programming will be

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{M}\sum_{j=1}^{N}(n_i - s)x_{ij} - \sum_{i=1}^{M} y_i^+ \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} - y_i^+ + y_{i+1}^+ + y_i^- = d_i, \quad i \in [M-1], \\
& \sum_{j=1}^{N} x_{ij} - y_i^+ + y_i^- = d_i, \quad i = M, \\
& \sum_{i=1}^{M} n_i x_{ij} \le L_j, j \in [N] \\
& y_i^+, y_i^- \in \mathbb{Z}_+, \quad i \in [M] \\
& x_{ij} \in \mathbb{Z}_+, \quad i \in [M], j \in [N].
\end{aligned}
\tag{6}
$$

Formulation

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{M} \sum_{j=1}^{N} (n_i - s) x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} \le d_i, \quad i \in [M], \\
& \sum_{i=1}^{M} n_i x_{ij} \le L_j, j \in [N] \\
& x_{ij} \in \mathbb{Z}_+, \quad i \in [M], j \in [N].
\end{aligned} \tag{7}
$$

Analysis

Branch and Price

- Solve restricted master problem with the initial columns.
- Use pricing problem to generate columns.
- When column generation terminates, is the solution integral?
  Yes, update lower bound.
  No, update upper bound. And fathom node or branch to create two nodes.
- Select the next node until all nodes are explored.

Properties

- When column generation gives an integer solution at the first time, then we obtain the optimal solution.
- α_k indicates the number of items for pattern k. β_k indicates the left space for maximal pattern k. Notice that the left space is the true loss.
- Denote $(\alpha_k + \beta_k)$ as the loss for pattern k, $l(k)$. When $l(k)$ reaches minimum, the corresponding pattern k is the optimal solution for a single line.
- If the group sizes are consecutive integers starting from 2, $\{2, 3, \ldots, u\}$, then a greedy-based pattern is optimal, i.e., select the maximal group size,u, as many as possible and the left space is occupied by the group with the corresponding size. The loss is $k + 1$, where k is the number of times u selected. Let $S = u \cdot k + r$, when $r > 0$, we will have at least $\lfloor \frac{r+u}{2} \rfloor - r + 1$ optimal patterns with the same loss. When $r = 0$, we have only one optimal pattern.

- Let I_1 be the set of patterns with the minimal loss. And I_i be the set of patterns with i-th minimal loss.
- When supply is less than demand, we must select I_1.
- When $N \leq \max_{k \in I_1} \min_i \{\lfloor \frac{d_i}{b_i^k} \rfloor\}$, select the corresponding pattern from I_1 and it is the optimal solution. N is the number of lines, $i = 1, 2, \ldots, m$, d_m is the demand of the largest size, b_m^k is the number of group m placed in pattern k.

Integral Decomposition

- Denote by R^n_+ the nonnegative orthant of Euclidean n-space. A polyhedron $P \subset R^n_+$ is called lower comprehensive if $0 \leq y \leq x \in P$ implies $y \in P$.
- Let M be the matrix whose rows are the maximal integral points of P.
- An integral vector $x \in P$ is a maximal integral point of P if there is no integral vector $y \in P$ with $x \neq y \geq x$.
- Let kP be defined as $\{kx | x \in P\}$. The integral decomposition property holds for P if, for each integer $k \geq 1$ and each integral vector $x \in kP$, there exist integral vectors $x^i \in P, 1 \leq i \leq k$, for which $x = \sum_{i=1}^{k} x^i$.
- The knapsack polyhedron, denoted by P, is the convex hull of the set $\{x \in Z^n_+ | ax \leq b\}$.

Integer Round Up for CSP

- Cutting stock problem has the integer round-up property if and only if P has the integral decomposition property.
- Any two-dimension CSP has the IRU property.
- The numerical experiment shows that the form of $\{2, 3, \ldots, n\}$ has the IRU property.
- But we can check this property finitely. From S. BAUM and L.E. TROTTER.

Integer Round Up for CSP

- After the column generation gives the LP solution(fractional), calculate the supply quantity. If the number is integral, keep it. If the supply is not integral, we can construct an integer vector which can provide the largest integral profit.
- Construction: Calculate the space taken by fractional supply(the value must be integer), increase the corresponding supply having the same space, delete the fractional part.
- Now we have an integral supply vector. If this case has IRU property, there exists an integral decomposition.
- Minus the integral part of the patterns from the supply. Check if the rest can be placed in several lines.

$$\begin{aligned} \text{min} \quad & \sum y \\ \text{s.t.} \quad & yM \geq w \\ & y \geq 0 \text{ and integer} \end{aligned}$$

$$\begin{aligned} \text{max} \quad & cw \\ \text{s.t.} \quad & w \in \{Nx | x \in P\} \\ & w \leq w_0 \end{aligned}$$

When P has the integral decomposition property, there exist N integer vectors decomposing every w. Now, the question is how to find the maximal w.

Lower Bound for Demand

$$
\begin{aligned}
\text{max} \quad & \sum_{k=1}^{K} \left(\sum_{i=1}^{m} (s_i - 1) t_i^k \right) x_k \\
\text{s.t.} \quad & \sum_{k=1}^{K} x_k \leq N \\
& \sum_{k=1}^{K} t_i^k x_k \leq g_i, \quad i = 1, \ldots, m \\
& \sum_{k=1}^{K} t_i^k x_k \geq p_i, \quad i = 1, \ldots, m \\
& x_k \geq 0, \quad k = 1, \ldots, K
\end{aligned}
$$

- Use the similar method to solve.

Sub-problem

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{m} \left[(s_i - 1) - \lambda_i - \mu_i\right] y_i - \tau \\
\text{s.t.} \quad & \sum_{i=1}^{m} s_i y_i \leq S \\
& y_i \geq 0, \text{integer} \quad \text{for } i = 1, \ldots, m.
\end{aligned}
$$

- Use column generation to generate a new pattern until all reduced costs are negative.

IP Formulation

$$
\begin{aligned}
\text{max} \quad & \sum_{j=1}^{m}\sum_{i=1}^{n}(s_i - 1)x_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{n} s_i x_{ij} \le S, \quad j = 1, \dots, m \\
& p_i \le \sum_{j=1}^{m} x_{ij} \le g_i, \quad i = 1, \dots, n \\
& x_{ij} \ge 0 \text{ and integer}, \quad i = 1, \dots, n, j = 1, \dots, m
\end{aligned}
$$

- m indicates the number of lines.
- x_{ij} indicates the number of group type i placed in each line j.

Question

1. Feasibility. At first, use cutting stock model to check if the following constraints provide a feasible solution. If the minimal number of lines we need is less than the given number of lines, these constraints are feasible.

$$\sum_{i=1}^{n} s_i x_{ij} \leq S, \quad j = 1, \ldots, m$$

$$p_i \leq \sum_{j=1}^{m} x_{ij}, \quad i = 1, \ldots, n$$

The End