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# Partitioning procedures for solving mixed-variables programming problems*,**

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## 1 Introduction

In this paper two slightly different procedures are presented for solving mixed-variables programming problems of the type

$$\max\{c^T x + f(y) \mid Ax + F(y) \leqq b, \quad x \in R_p, y \in S\}\ , \tag{1.1}$$

where $x \in R_p$ (the $p$-dimensional Euclidean space), $y \in R_q$, and $S$ is an arbitrary subset of $R_q$. Furthermore, $A$ is an $(m, p)$ matrix, $f(y)$ is a scalar function and $F(y)$ an $m$-component vector function both defined on $S$, and $b$ and $c$ are fixed vectors in $R_m$ and $R_p$ respectively.

An example is the mixed-integer programming problem in which certain variables may assume any value an a given interval, whereas others are restricted to integral values only. In this case S is a set of vectors in $R_q$ with integral-valued components. Various methods for solving this problem have been proposed by Beale [1], Gomory [9] and Land and Doig [11]. The use of integer variables, in particular for incorporating in the programming problem a choice from a set of alternative discrete decisions, has been discussed by Dantzig [4].

Other examples are those in which certain variables occur in a linear and others in a non-linear fashion in the formulation of the problem (see e.g. Griffith and Stewart [7]). In such cases $f(y)$ or some of the components of $F(y)$ are non-linear functions defined on a suitable subset $S$ of $R_q$.

Obviously, after an arbitrary partitioning of the variables into two mutually exclusive subsets, any linear programming problem can be considered as being of type (1.1). This may be advantageous if the structure of the problem indicates

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a natural partitioning of the variables. This happens, for instance, if the problem is actually a combination of a general linear programming and a transportation problem. Or, if the matrix shows a block structure, the blocks being linked only by some columns, to which also many other block structures can easily be reduced. A method of solution for linear programming problems efficiently utilizing such block structures, has been designed by Dantzig and Wolfe, [5].

The basic idea behind the procedures to be described in this report is a partitioning of the given problem (1.1) into two sub problems : a programming problem (which may be linear, non-linear, discrete, etc.) defined on $S$, and a linear programming problem defined in $R_p$. Then, in order to avoid the very laborious calculation of a complete set of constraints for the feasible region in the first problem, two multi-step procedures have been designed both leading, in a finite number of steps, to a set of constraints determining an optimum solution of problem (1.1). Each step involves the solution of a general programming problem. The two procedures differ only in the way the linear programming problem is solved.

Earlier versions of these procedures constitute part of the author's doctoral dissertation [2]. This paper, however, contains a more detailed description of the computational aspects.

## 2 Preliminaries

We assume the reader to be familiar with the theory of convex polyhedral sets and with the computational aspects of solving a linear programming problem by the simplex method; see e.g. Tucker [13], Goldman [8] and Gass [6].

Throughout this paper $u$, $v$ and $z$ denote vectors in $R_m$; $u_0$, $x_0$ and $z_0$ are scalars.

For typographical convenience the partitioned column vectors

$$\begin{pmatrix} x_0 \\ x \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ z_0 \end{pmatrix}, \quad \begin{pmatrix} x \\ z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_0 \\ u \end{pmatrix}$$

are written in the form $(x_0, x, y)$, $(x, y)$, $(x, z_0)$, $(x, z)$ and $(u_0, u)$, respectively. The letter $e$ will always stand for a vector of appropriate dimension with all components equal to one.

If A is the $(m, p)$ matrix and $c$ the vector in $R_p$ both occurring in the formulation of problem (1.1), we will define

(a) the convex polyhedral cone $C$ in $R_{m+1}$ by

$$C = \{(u_0, u) | A^T u - cu_0 \geqq 0, \ u \geqq 0, \ u_0 \geqq 0\}, \tag{2.1}$$

(b) the convex polyhedral cone $C_0$ in $R_m$ by

$$C_0 = \{u | A^T u \geqq 0, \ u \geq 0\}, \tag{2.2}$$

(c) the convex polyhedron $P$ (which may be empty) in $R_m$ by

$$P = \{u | A^T u \geqq c, \ u \geqq 0\}. \tag{2.3}$$

## 3 A partitioning theorem

Introducing a scalar variable $x_0$, we write problem (1.1) first in the equivalent form

$$\max\{x_0 | x_0 - c^T x - f(y) \leqq 0,\ Ax + F(y) \leqq b,\ x \geqq 0,\ y \in S\}, \tag{3.1}$$

i. e. $(\bar{x}_0, \bar{x}, \bar{y})$ is an optimum solution of problem (3.1) if and only if $\bar{x}_0 = c^T \bar{x} + f(\bar{y})$ and $(\bar{x}, \bar{y})$ is an optimum solution of problem (1.1).

To any point $(u_0, u) \in C$ we adjoin the region in $R_{q+1}$, defined by

$$\{(x_0, y) | u_0 x_0 + u^T F(y) - u_0 f(y) \leqq u^T b, y \in S\}. \tag{3.2}$$

$G$ will denote the intersection (which may be empty) of all these regions:

$$G = \bigcap_{(u_0,u)\in C} \{(x_0, y) | u_0 x_0 + u^T F(y) - u_0 f(y) \leqq u^T b, y \in S\}. \tag{3.3}$$

Since $C$ is a pointed convex polyhedral cone, it is the convex hull of finitely many extreme halflines. It follows that there are $H$ points $(u_0^h, u^h)$, $h = 1, \ldots, H$ in $C$ so that

$$G = \bigcup_{h \leqq H} \{(x_0, y) | u_0^h x_0 + (u^h)^T F(y) - u_0^h f(y) \leqq (u^h)^T b, y \in S\}. \tag{3.4}$$

**Theorem 3.1.** (Partitioning theorem for mixed-variables programming problems).

(1) Problem (1.1) is not feasible if and only if the programming problem

$$\max\{x_0 | (x_0, y) \in G\} \tag{3.5}$$

is not feasible, i.e if and only if the set $G$ is empty.

(2) Problem (1.1) is feasible without having an optimum solution, if and only if problem (3.5) is feasible without having an optimum solution.

(3) If $(\bar{x}, \bar{y})$ is an optimum solution of problem (1.1) and $\bar{x}_0 = c^T \bar{x} + f(\bar{y})$, then $(\bar{x}_0, \bar{y})$ is an optimum solution of problem (3.5) and $\bar{x}$ is an optimum solution of the linear programming problem

$$\max\{c^T x | A\, x \leqq b - F(\bar{y}),\ x \geqq 0\}. \tag{3.6}$$

(4) If $(\bar{x}_0, \bar{y})$ is an optimum solution of problem (3.5), then problem (3.6) is feasible and the optimum value of the objective function in this problem is equal to $\bar{x}_0 - f(\bar{y})$. If $\bar{x}$ is an optimum solution of problem (3.6), then $(\bar{x}, \bar{y})$ is an optimum solution of problem (1.1), with optimum value $\bar{x}_0$ for the objective function.

*Proof.* If $x_0^*$ is an arbitrary number and $y^*$ is an arbitrary point in $S$, it follows from the theorem of Farkas (see Tucker [13]) that the linear system

$$\begin{aligned} Ax &\leqq b - F(y^*) \\ -c^T x &\leqq -x_0^* + f(y^*), \quad x \geqq 0 \end{aligned}$$

is feasible if and only if

$$u_0 x_0^* + u^T F(y^*) - u_0 f(y^*) \leqq u^T b$$

for any point $(u_0, u) \in C$.

Hence if $(x_0^*, x^*, y^*)$ is a feasible solution of problem (3.1), $(x_0^*, y^*)$ is a feasible solution of problem (3.5). Conversely, if $(x_0^*, y^*)$ is a feasible solution of problem (3.5), there is a vector $x^* \in R_p$ so that $(x_0^*, x^*, y^*)$ is a feasible solution of problem (3.1). Since the problems (1.1) and (3.1) are equivalent, this proves items (1) and (2) of Theorem 3.1. Moreover it follows that if $(\bar{x}, \bar{y})$ is an optimum solution of problem (1.1) and $\bar{x}_0 = c^T \bar{x} + f(\bar{y})$, then $(\bar{x}_0, \bar{y})$ is an optimum solution of problem (3.5). Finally, if $(\bar{x}_0, \bar{y})$ is an optimum solution of problem (3.5), there is a vector $\bar{x} \in R_p$, so that $(\bar{x}_0, \bar{x}, \bar{y})$ is an optimum solution of problem (3.1). Then $\bar{x}_0 = c^T \bar{x} + f(\bar{y})$ and since $c^T x + f(\bar{y}) \leqq \bar{x}_0$ for any feasible solution $(x, \bar{y})$ of problem (1.1) ($\bar{y}$ fixed!) it follows that $\bar{x}$ is an optimum solution of problem (3.6). This completes the proof of Theorem 3.1.

The partitioning theorem does not require any further specification of the subset $S$ and of the functions $f(y)$ and $F(y)$ defined on $S$. In actual practice, however, $S$, $f(y)$ and $F(y)$ must have such properties that problem (3.5) can be solved by existing methods, in other words, we must be able to detect whether this problem is not feasible or feasible without having an optimum solution, or we must be able to find an optimum solution if one exists (for special cases, see Sect. 6). If these assumptions are satisfied, Theorem 3.1 asserts that problem (1.1) can be solved by a two-step procedure. The first step involves the solution of problem (3.5), leading to the conclusion that problem (1.1) is not feasible, or that it is feasible without having an optimum solution, or to the optimum value of the objective function in problem (1.1) and to an optimum vector $\bar{y}$ in $S$. In the latter case a second step is required for calculating an optimum vector $\bar{x}$ in $R_p$, which is obtained by solving the linear programming problem (3.6).

The solution of problem (3.5) must be considered in more detail. For, even if a procedure is available for solving problems of this type, a direct solution of problem (3.5) would require the calculation in advance of a complete set of constraints, determining the set $G$. According to (3.4) this could be done by calculating all extreme half lines of the convex polyhedral cone $C$, but this is practically impossible because of the enormous calculating effort required. However, since we are interested in an optimum solution of problem (3.5) rather than in the set $G$ itself, it would suffice to calculate only those constraints of $G$ which determine an optimum solution. In the next section we will derive an efficient procedure for calculating such constraints.

## 4 A computational procedure for solving mixed-variables programming problems

In this section we assume that the set $S$ is closed and bounded, and that $f(y)$ and the components of $F(y)$ are continuous functions on a subset $\bar{S}$ of $R_q$ containing $S$. These assumptions are satisfied in most applications and they rule out complications caused by feasible programming problems which have no solution. It may happen that $S$ is not bounded explicitly in the formulation of problem (1.1). In that case we can add bounds for the components of the vector $y$ which are so large that either it is known beforehand that there is an optimum solution satisfying these bounds or that components of $y$ exceeding these bounds have no realistic interpretation.

**Lemma 4.1.** If problem (3.5) is feasible and $S$ is bounded, then $x_0$ has no upper bound on $G$ if and only if the polyhedron $P$ is empty.

*Proof.* By assumption, there is at least one point $(x_0^*, y^*) \in G$. However, if $P$ is empty, then $u_0 = 0$ for any point $(u_0, u) \in C$. Hence $G$ assumes the form

$$G = \bigcap_{u \in C_0} \{(x_0, y) | u^T F(y) \leqq u^T b,\, y \in S\}, \tag{4.1}$$

and it follows that $(x_0, y^*) \in G$ for any value of $x_0$.

If $P$ is not empty, there is at least one point $(1, \bar{u}) \in C$. Hence, for any feasible solution $(x_0, y)$ of problem (3.5) we have, by the assumptions imposed on the set $S$ and on the functions $F(y)$ and $f(y)$

$$x_0 \leqq \max_{y \in S} \{\bar{u}^T b - \bar{u}^T F(y) + f(y)\} < \infty. \tag{4.2}$$

Let $Q$ be a non-empty subset of $C$ and let the subset $G(Q)$ of $R_{q+1}$ be defined by

$$G(Q) = \bigcap_{(u_0, u) \in Q} \{(x_0, y) | u_0 x_0 + u^T F(y) - u_0 f(y) \leqq u^T b,\, y \in S\}. \tag{4.3}$$

We consider the programming problem

$$\max\{x_0 | (x_0, y) \in G(Q)\}. \tag{4.4}$$

If problem (4.4) is not feasible, then, since $G \subset G(Q)$, problem (3.5) is not feasible. On the other hand, if $(\bar{x}_0, \bar{y})$ is an optimum solution of problem (4.4) we have to answer the question whether $(\bar{x}_0, \bar{y})$ is also an optimum solution of problem (3.5) and, if not, how a “better” subset $Q$ of $C$ can be obtained.

**Lemma 4.2.** If $(\bar{x}_0, \bar{y})$ is an optimum solution of problem (4.4), it is also an optimum solution of problem (3.5) if and only if

$$\min\{(b - F(\bar{y}))^T u | u \in P\} = \bar{x}_0 - f(\bar{y}). \tag{4.5}$$

*Proof.* Since the maximum value of $x_0$ on the set $G(Q)$ is assumed to be finite, if follows from Lemma 4.1 that $Q$ contains at least one point $(u_0, u)$ for which $u_0 > 0$. Hence the polyhedron $P$ is not empty, i.e. the linear programming problem

$$\min\{(b - F(\bar{y}))^T u | u \in P\} \tag{4.6}$$

is feasible.

Now we observe first that an optimum solution $(\bar{x}_0, \bar{y})$ of problem (4.4) is also an optimum solution of problem (3.5) if and only if $(\bar{x}_0, \bar{y}) \in G$. The necessity of this condition is obvious. Moreover, since $Q \subset C$, we have

$$\max\{x_0 | (x_0, y) \in G(Q)\} \geqq \max\{x_0 | (x_0, y) \in G\}, \tag{4.7}$$

hence the condition is also sufficient.

By the definition of $G$, the point $(\bar{x}_0, \bar{y}) \in G$ if and only if

$$(b - F(\bar{y}))^T u + (-\bar{x}_0 + f(\bar{y}))u_0 \geqq 0 \tag{4.8}$$

for any point $(u_0, u) \in C$. This happens if and only if

$$(b - F(\bar{y}))^T u \geqq 0 \quad \text{for any } u \in C_0$$

and

$$(b - F(\bar{y}))^T u \geqq \bar{x}_0 - f(\bar{y}) \quad \text{for any } u \in P,$$

i.e. if and only if

$$\min\{(b - F(\bar{y}))^T u | u \in P\} \geqq \bar{x}_0 - f(\bar{y}). \tag{4.9}$$

By the duality theorem for linear programming problems, it follows that the linear programming problem

$$\max\{c^T x | Ax \leqq b - F(\bar{y}), x \geqq 0\} \tag{4.10}$$

has a finite optimum solution $\bar{x}$ for which

$$c^T \bar{x} = \min\{(b - F(\bar{y}))^T u | u \in P\}. \tag{4.11}$$

Since $(\bar{x}, \bar{y})$ is a feasible solution of problem (1.1) it follows from Theorem 3.1 and $G \subset G(Q)$ that

$$c^T \bar{x} + f(\bar{y}) \leqq \max\{x_0 | (x_0, y) \in G\} \leqq \max\{x_0 | (x_0, y) \in G(Q)\} = \bar{x}_0. \tag{4.12}$$

Finally, it follows from a combination of the relations (4.9), (4.11) and (4.12) that the inequality (4.9) can be replaced by the equality (4.5). This completes the proof of Lemma 4.2.

If the linear programming problem (4.6) has a finite optimum solution, at least one of the vertices of the polyhedron $P$ is contained in the set of optimum solutions.

It is well-known that, in this case, the simplex method leads to an optimum vertex $\bar{u}$ of $P$.

According to Lemma 4.2, if $(b - F(\bar{y}))^T \bar{u} = \bar{x}_0 - f(\bar{y})$, we have found an optimum solution $(\bar{x}_0, \bar{y})$ of problem (3.5). Furthermore, the simplex method provides us, at the same time, with an optimum solution $\bar{x}$ of the dual linear programming problem (4.10) and it follows from Theorem 3.1 that $(\bar{x}, \bar{y})$ is an optimum solution of problem (1.1) .

If

$$(b - F(\bar{y}))^T \bar{u} < \bar{x}_0 - f(\bar{y}), \tag{4.13}$$

the point $(1, \bar{u})$ of $C$ does not belong to $Q$. In this case we form a new subset $Q^*$ of $C$ by adding the point $(1, \bar{u})$ to $Q$.

If the linear programming problem (4.6) has no finite optimum solution, the simplex method leads to a vertex $\bar{u}$ of $P$ and to a direction vector $\bar{v}$ of one of the extreme halflines of $C_0$ so that the value of the objective function $(b - F(\bar{y}))^T u$ tends to infinity along the halfline

$$\{u | u = \bar{u} + \lambda \bar{v}, \lambda \geqq 0\}.$$

Moreover we have the inequality

$$(b - F(\bar{y}))^T \bar{v} < 0, \tag{4.14}$$

from which it follows that the point $(0, \bar{v})$ of $C$ does not belong to $Q$. In this case we form a new subset $Q^*$ of $C$ by adding the point $(0, \bar{v})$ to $Q$.

In any case, let $(x_0^*, y^*)$ be an optimum solution of the programming problem

$$\max\{x_0 | (x_0, y) \in G(Q^*)\}. \tag{4.15}$$

Then, in the first case we have

$$(b - F(y^*))^T \bar{u} \geqq x_0^* - f(y^*), \tag{4.16}$$

and in the second case

$$(b - F(y^*))^T \bar{v} \geqq 0. \tag{4.17}$$

From this in combination with (4.13) and (4.14) it follows that

$$(x_0^*, y^*) \neq (\bar{x}_0, \bar{y}). \tag{4.18}$$

Furthermore, since $Q^* \supset Q$, we have $G(Q^*) \subset G(Q)$, hence

$$x_0^* \leqq \bar{x}_0. \tag{4.19}$$

In case the linear programming problem (4.6) has no finite solution, it may be that the above-mentioned vertex $\bar{u}$ satisfies the inequality (4.13). Then, both the point $(1, \bar{u})$, and $(0, \bar{v})$ do not belong to $Q$ and the new subset $Q^*$ of $C$ may be obtained by adding both points to $Q$. It is also important to note that the constrained set $G(Q^*)$ is obtained from $G(Q)$ by adding the constraint

$$x_0 + \bar{u}^T F(y) - f(y) \leqq \bar{u}^T b$$

and/or the constraint

$$\bar{v}^T F(y) \leqq \bar{v}^T b$$

to the set of constraints determining this set $G(Q)$.

We are now prepared for the derivation of a finite multi-step procedure for solving mixed-variables programming problems of type (1.1).

**Procedure 4.1.** The procedure starts from a given finite subset $Q^0 \subset C$.

*Initial step.* If $u_0 > 0$ for at least one point $(u_0, u) \in Q^0$, go to the first part of the iterative step.

If $u_0 = 0$ for any point $(u_0, u) \in Q^0$, put $x_0^0 = +\infty$, take for $y^0$ an arbitrary point of $G(Q^0)$ and go to the second part of the iterative step.

If $G(Q^0)$ is empty, the procedure terminates: problem (1.1) is not feasible.

*Iterative step, first part.* If the $\nu$-th step has to be performed, solve the programming problem

$$\max\{x_0 | (x_0, y) \in G(Q^\nu)\}. \tag{4.20}$$

If problem (4.20) is not feasible, the procedure terminates: problem (1.1) is not feasible.

If $(x_0^\nu, y^\nu)$ is found to be an optimum solution of problem (4.20), go to the second part of the iterative step.

*Iterative step, second part.* Solve the linear programming problem

$$\min\{(b - F(y^\nu))^T u | A^T u \geqq c, \ u \geqq 0\}. \tag{4.21}$$

If problem (4.21) is not feasible, problem (1.1) is either not feasible, or it has no finite optimum solution. (This situation can only be encountered in the first iterative step!)

If problem (4.21) has a finite optimum solution $u^\nu$ and

$$(b - F(y^\nu))^T u^\nu = x_0^\nu - f(y^\nu), \tag{4.22}$$

the procedure terminates. In this case, if $x^\nu$ is the optimum solution for the dual problem of problem (4.21), then $(x^\nu, y^\nu)$ is an optimum solution of problem (1.1) and $x_0^\nu$ is the optimum value of the objective function in this problem. Then if

$$(b - F(y^\nu))^T u^\nu < x_0^\nu - f(y^\nu), \tag{4.23}$$

form the set

$$Q^{\nu+1} = Q^\nu \cup \{(1, u^\nu)\}, \tag{4.24}$$

replace the step counter $\nu$ by $\nu + 1$ and repeat the first part of the iterative step.

If the value of the objective function in problem (4.21) tends to infinity along the halfline

$$\{u | u = u^\nu + \lambda v^\nu, \lambda \geqq 0\},$$

$u^\nu$ being a vertex of $P$ and $v^\nu$ the direction of an extreme halfline of $C_0$, while

$$(b - f(y^\nu))^T u^\nu \geqq x_0^\nu - f(y^\nu), \tag{4.25}$$

form the set

$$Q^{\nu+1} = Q^\nu \cup \{(0, v^\nu)\}. \tag{4.26}$$

However, if (4.25) is not satisfied, i.e. if

$$(b - F(y^\nu))^T u^\nu < x_0^\nu - f(y^\nu), \tag{4.27}$$

form the set

$$Q^{\nu+1} = Q^\nu \cup \{(1, u^\nu), (0, v^\nu)\}. \tag{4.28}$$

In either case replace the step counter $\nu$ by $\nu + 1$ and repeat the first part of the iterative step.

This procedure terminates, within a finite number of steps, either with the conclusion that problem (1.1) is not feasible, or that this problem is feasible without a finite optimum solution, or because an optimum solution of problem (1.1) has been obtained.

This procedure is finite, since at each step where it does not terminate the preceding subset $Q^\nu$ is extended by the direction vector of at least one extreme halfline of the polyhedral cone $C$, which does not belong already to $Q^\nu$. Hence, within a finite number of steps either the procedure would terminate or a complete set of constraints determining the set $G$ would have been obtained and by Theorem 3.1 the procedure would stop after the next step.

The termination rules are justified by:

(1) $G(Q^\nu) \subset G$ in combination with Theorem (3.1), item (1): problem (1.1) is not feasible.
(2) Lemma (4.1) and Theorem (3.1), item (2): problem (1.1) has no finite optimum solution.
(3) Lemma (4.2) and Theorem (3.1), item (4): optimum solution for problem (1.1).

Since

$$G(Q^\nu) \supset G(Q^{\nu+1}) \supset G,$$

the sequence $\{x_0^\nu\}$ is non-decreasing and

$$\max\{x_0 | (x_0, y) \in G\} \leqq x_0^\nu \quad \text{for any} \quad \nu \geqq 0. \tag{4.29}$$

If problem (4.21) has an optimum solution $u^\nu$, then its dual problem

$$\max\{c^T x | Ax \leqq b - F(y^\nu), x \geqq 0\} \tag{4.30}$$

has an optimum solution $x^\nu$, while

$$(b - F(y^\nu))^T u^\nu = c^T x^\nu. \tag{4.31}$$

Since $(x^\nu, y^\nu)$ is a feasible solution of problem (1.1), it follows from Theorem (3.1), item (3) that

$$(b - F(y^\nu))^T u^\nu + f(y^\nu) \leqq \max\{x_0 | (x_0, y) \in G\}. \tag{4.32}$$

Hence, at the end of each step, we have upper and lower bounds for the maximum value of $x_0$ on the set $G$, or what is the same, for the maximum value of the objective function in problem (1.1)

$$\max_{k \leqq \nu}\{(b - F(y^k))^T u^k + f(y^k)\} \leqq \max\{x_0 | (x_0, y) \in G\} \leqq x_0^\nu. \tag{4.33}$$

Here, $(b - F(y^k))^T u^k = -\infty$ if problem (4.21) in the $k$-th iterative step has no finite optimum solution; otherwise it is the optimum value of the objective function in this problem.

The determination of an initial set $Q^0$ will depend much on the actual problem to be solved. In any case one may start from the set $Q^0$ containing only the origin of $R_{m+1}$, which always belongs to the cone $C$. The procedure then starts with the second part of the iterative step from an arbitrary point $y^0 \in S$, while $x_0^0$ is put equal to $+\infty$.

## 5 An alternative version of Procedure 4.1

In actual practice it is often more convenient to solve the dual problem

$$\max\{c^T x | Ax \leqq b - F(y^\nu),\ x \geqq 0\} \tag{5.1}$$

of problem (4.21), rather than this problem itself. In this section an efficient way is described of obtaining all information for the performance of Procedure 4.1 by solving (5.1) instead of problem (4.21)

First we observe that problem (5.1) may not be feasible since problem (4.21) may have an infinite optimum solution. In order to avoid infeasibility, we replace the convex polyhedron $P$ by the bounded convex polyhedron

$$P(M) = \{u | A^T u \geqq c, e^T u \leqq M, u \geqq 0\}, \tag{5.2}$$

the number $M$ being so large that all vertices of the polyhedron $P$ (if not empty) are contained in the region

$$\{u | e^T u \geqq M, u \geqq 0\}. \tag{5.3}$$

Problem (5.1) is then replaced by the problem

$$\max\{c^T x - Mz_0 | Ax - z_0 e \leqq b - F(y^\nu),\ x_0 \geqq 0,\ z_0 \geqq 0\}, \tag{5.4}$$

which is always feasible.

If $M$ is sufficiently large (see below), then:

(1) Problem (4.21) is not feasible if and only if problem (5.4) has an infinite optimum solution.

(2) If $(x^\nu, z_0^\nu)$ is an optimum solution found for problem (5.4), and $z_0^\nu = 0$, then $(x^\nu, y^\nu)$ is a feasible solution of problem (1.1). For the optimum solution $u^\nu(M)$ found at the same time for the dual problem

$$\min\{(b - F(y^\nu))^T u | A^T u \geqq c,\ e^T u \leqq M,\ u \geqq 0\} \tag{5.5}$$

we have the relation

$$(b - F(y^\nu))^T u^\nu(M) = c^T x^\nu. \tag{5.6}$$

Since $u^\nu(M)$ is also a feasible solution of problem (4.21), it follows from (5.6) and the duality theorem for linear programming problems that it is also an optimum solution of problem (4.21). Then, applying Lemma 4.2 and Theorem 3.1 we find that $(x^\nu, y^\nu)$ is an optimum solution of problem (1.1) if and only if

$$c^T x^\nu + f(y^\nu) = x_0^\nu. \tag{5.7}$$

If $z_0^\nu = 0$, but relation (5.6) is not satisfied, or if $z_0^\nu > 0$, we consider the optimum solution $u^\nu(M)$ in more detail.

It is well-known that the components of $u^\nu(M)$ are equal to the components of the “$d$-row” (known also as the “$z_j - c_j$-row", see Gass [6]) in the optimum simplex tableau, corresponding to the initial slack variables. It follows from the definition of the “$d$-row” that it is a linear form in $M$, i.e.

$$d^\nu(M) = d^{1,\nu} + Md^{2,\nu} \tag{5.8}$$

with $d^{2,\nu} \geqq 0$ (otherwise the number $M$ would be too small, see below).

It follows that $u^\nu$ is also a linear form in $M$:

$$u^\nu(M) = u^{1,\nu} + Mu^{2,\nu} \tag{5.9}$$

The vectors $d^{1,\nu}$ and $d^{2,\nu}$, hence also the vectors $u^{1,\nu}$ and $u^{2,\nu}$ are obtained by replacing the objective function $c^T x - Mz_0$ in the optimum simplex tableau by $c^T x$ and $-z_0$, respectively and recalculating then the “$d$-row". We will refer to $d^{1,\nu}$ and $d^{2,\nu}$ as to the “$M$-components” of the $d$-row; similarly $u^{1,\nu}$ and $u^{2,\nu}$ are the “$M$-components” of the optimum solution of the dual problem. For any given optimum simplex tableau the $M$-components are independent of $M$.

By virtue of the construction of the polyhedron $P(M)$, any vertex of $P$ is a vertex of $P(M)$. Furthermore, as proved by Goldman ([8], Corollary 1A) we have that any vertex of $P(M)$ is of the form

$$u(M) = \bar{v} + \lambda \bar{u}, \tag{5.10}$$

where $\bar{u}$ is a vertex of $P$, $\bar{v}$ is either zero or it is the direction vector of an extreme halfline of $C_0$ and $\lambda$ is some non-negative number. Conversely, if $\bar{v}$ is such a direction vector, there is at least one vertex $\bar{u}$ of $P$ so that

$$u(M) = \bar{u} + \frac{M - e^T \bar{u}}{e^T \bar{v}} \bar{v} \tag{5.11}$$

is a vertex of $P(M)$.

Hence, the vertex (5.9) of $P(M)$ can be written in the form

$$u^\nu(M) = u^\nu + \lambda^\nu v^\nu \tag{5.12}$$

where $u^\nu$ is a vertex of $P$ and $v^\nu$ is the direction of an extreme halfline of $C_0$. The only problem is to calculate $u^\nu$ and $v^\nu$ from the $M$-components $u^{1,\nu}$ and $u^{2,\nu}$ of $u^\nu(M)$.

Obviously

$$v^\nu = u^{2,\nu}. \tag{5.13}$$

If

$$u^{2,\nu} = 0, \text{ then } u^\nu = u^{1,\nu}, \tag{5.14}$$

and, if $u^{2,\nu} \neq 0$, $u^\nu$ is that point on the halfline

$$\{u | u = u^{1,\nu} + M u^{2,\nu}, M \geqq 0\} \tag{5.15}$$

which also belongs to $P$ and corresponds to the smallest value of $M$ for which this happens.

It is a well-known property of the "$d$-row" of the optimum simplex tableau of problem (5.4) that this is the minimum value $M_{\min}$ of $M$ for which $d^\nu(M) = d^{1,\nu} + M d^{2,\nu} \geqq 0$. Hence

$$M_{\min} = \max_j \left\{ -\frac{d_j^{1,\nu}}{d_j^{2,\nu}} \,\middle|\, d_j^{2,\nu} > 0 \right\}. \tag{5.16}$$

By the duality theorem we have also the relation

$$(b - F(y^\nu))^T u^\nu = c^T x^\nu - M_{\min} z_0^\nu.$$

Now we get the following modification of Procedure 4.1.

**Procedure 5.1.** This procedure starts again from a given finite subset $Q^0 \subset C$. Moreover a suitable value of M must be known (see below).

*Initial step.* The same as in Procedure 4.1.

*Iterative step, first part.* The same as in Procedure 4.1.

*Iterative step, second part.* Solve the linear programming problem

$$\max\{c^T x - Mz_0 | Ax - z_0 e \leqq b - F(y^\nu),\ x \geq 0,\ z_0 \geq 0\}. \tag{5.17}$$

If problem (5.17) has an infinite optimum solution, problem (1.1) is either not feasible or it has no finite solution. (This situation can only be encountered during the first iterative step.) If problem (5.17) has a finite optimum solution $(x^\nu, z_0^\nu)$ then if $z_0^\nu = 0$ and $c^T x^\nu + f(y^\nu) = x_0^\nu$, $(x^\nu, y^\nu)$ is an optimum solution of problem (1.1) with $x_0^\nu$ equal to the optimum value of the objective function; if $z_0^\nu = 0$ but $c^T x^\nu + f(y^\nu) < x_0^\nu$, or if $z_0^\nu > 0$, determine the $M$-components $d^{1,\nu}$ and $d^{2,\nu}$ of the "$d$-row" in the optimum simplex tableau and the $M$-components $u^{1,\nu}$ and $u^{2,\nu}$ of the optimum solution of the dual problem.

If

$$u^{2,\nu} = 0, \text{ put } u^\nu = u^{1,\nu} \textit{ and } v^\nu = 0 \tag{5.18}$$

If

$$u^{2,\nu} \neq 0, \text{ calculate}$$

$$M_{\min} = \max_j \left\{ -\frac{d_j^{1,\nu}}{d_j^{2,\nu}} \,\middle|\, d_j^{2,\nu} > 0 \right\} \tag{5.19}$$

and put

$$u^\nu = u^{1,\nu} + M_{\min} u^{2,\nu}, \tag{5.20}$$

$$v^\nu = u^{2,\nu}. \tag{5.21}$$

Then, if $c^T x^\nu - M_{\min} z_0^\nu < x_0^\nu - f(y^\nu)$, form the set

$$Q^{\nu+1} = Q^\nu \cup \{(1, u^\nu), (0, v^\nu)\} \tag{5.22}$$

and if $c^T x^\nu - M_{\min} z_0^\nu \geqq x_0^\nu - f(y^\nu)$, form the set

$$Q^{\nu+1} = Q^\nu \cup \{(0, v^\nu)\}. \tag{5.23}$$

Finally, replace the step counter $\nu$ by $\nu+1$ and repeat the first part of the iterative step.

This procedure terminates in a finite number of steps, with the conclusion that problem (1.1) is not feasible or that it is feasible without a finite optimum solution, or because an optimum solution of problem (1.1) has been obtained.

The inequalities (4.33), expressing upper and lower bounds for the ultimate optimum value of $x_0$, now assume the form:

$$\max_{k \leq \nu}\{c^T x^k + f(y^k) | z_0^k = 0\} \leqq \max\{x_0 | (x_0,\ y) \in G\} \leqq x_0^\nu. \tag{5.24}$$

It has been assumed that a suitable value of $M$, i.e. a value of $M$ so large that all vertices of $P$ are contained in the region

$$\{u|e^T u \leqq M,\ u \geqq 0\},$$

is known in advance. Such a value certainly exists, but need not to be known in actual applications. In any case one can start the second part of the iterative step with $M = +\infty$ which actually means that this part is done in two phases. In the first phase one maximizes the objective function $-z_0$. Then, in the second phase, the objective function $c^T x$ is maximized under the side conditions that $-z_0$ retains.the maximum value it reached during the first phase.

The procedure may be expected to be more efficient, however, if $M$ is not too large. One can start with any positive value of $M$. If, for fixed $M$, an optimum solution of problem (5.4) is obtained, but some components of $d^{2,\nu}$ are still negative, the value of $M$ must be increased until at least one of the corresponding components of $d^\nu$ becomes negative. Then the simplex calculations are continued with this new value of $M$. If one attains an infinite solution, the components of $d^{2,\nu}$, corresponding to columns with no positive elements in the actual simplex tableau, must be checked. If all these components are positive, the value of $M$ must be increased until the corresponding components of $d^\nu(M)$ become positive and then the simplex calculations are continued. If at least one of these components of $d^{2,\nu}$ is negative, or equal to zero but the corresponding component of $d^{1,\nu}$ is negative, an increase in the value of $M$ does not change this situation and the conclusion that problem (4.21) is not feasible is justified. In this way, in a finite number of simplex iterations a sufficiently large value of $M$ is obtained.

An important modification of Procedure 5.1 is obtained if we replace the upper bound inequality $e^T u \leqq M$ in (5.2) by the vector inequality $u \leqq Me$. Then problem (5.4) assumes the form

$$\max\{c^T x - Me^T z | Ax - z \leqq b - F(y^\nu),\ x \geqq 0, z \geqq 0\}, \tag{5.25}$$

i.e. the single variable $z_0$ in (5.4) is replaced by the vector $z$. The justification of this modification is slightly more complicated than for Procedure 5.1 but easily accomplished. From a computational point of view problem (5.25) is somewhat more flexible than problem (5.4).

In many applications the system of inequalities

$$Ax + F(y) \leqq b \tag{5.26}$$

assumes the form

$$\begin{aligned} A_1 x &\leqq b^1 \\ A_2 x + F_2(y) &\leqq b^2, \end{aligned} \tag{5.27}$$

i.e. the vector $y$ does not occur explicitly in some of the constraints determining the feasible region in problem (1.1). In this case problem (5.25) may be replaced conveniently by

$$\max\{c^T x - Me^T z | A_1 x \leqq b^1,\ A_2 x - z \leqq b^2 - F_2(y^\nu),\ x \geqq 0,\ z \geqq 0\}. \quad (5.28)$$

This means that an auxiliary variable $z_i$ has to be introduced only if the vector $y$ occurs explicitly in the corresponding inequality of the system (5.26).

Problem (5.28) may be not feasible because the system of inequalities $A_1 x \leqq b^1,\ x \geqq 0$ may be not feasible. This will be detected however during the first step of procedure (5.1). The procedure can then be terminated, since this means that the original problem (1.1) is not feasible.

## 6 Applications

The crucial point in the application of the Procedures 4.1 and 5.1 to the solution of actual problems is the existence of efficient procedures for solving the programming problem (4.20). We will consider in this section several special cases where this requirement is fulfilled.

(a) If $S = R_q$ (or a convex polyhedron in $R_q$), $F(y) = By$, $B$ being an $(m, q)$ matrix and $f(y) = r^T y, r \in R_q$, problem (4.20) becomes a linear programming problem which can be solved by the simplex method. Since in each step new constraints are added to the feasible region of problem (4.20) in the preceding step, the dual simplex procedure seems to be most suitable. Another possibility is to apply the primal simplex method to its dual problem. The Procedures 4.1 and 5.1 are now special versions of the well-known decomposition procedure for linear programming problems, developed by Dantzig and Wolfe [5].

(b) If $S = R_q$ and $f(y)$ and the components of $F(y)$ are convex and differentiable functions defined on $S$, problem (4.20) becomes a convex programming problem that can be solved by well-known methods e.g. by Kelley's cutting plane technique [10], by Rosen's gradient projection [12] or by Zoutendijk's methods of feasible directions [14].

(c) If $S$ is the set of all vectors in $R_q$ with non-negative integral-valued components, $F(y) = By,\ B$ being an $(m, q)$ matrix and $f(y) = r^T y,\ r \in R_q$, problem (1.1) is the well-known mixed-integer linear programming problem. Problem (4.20) now becomes an integer programming problem of a special type. Since the feasible region in problem (4.20) in the $(\nu + 1)$-th step is obtained by adding cutting planes to the feasible region in this problem in the $\nu$-th step the cutting plane technique of Gomory [9] seems to be very suitable for solving the integer sub-problems.

Of particular importance for applications is the case where $S$ constitutes the set of vertices of the unit cube, i.e. where the components of the vector $y$ may

**Table 1.** Successive upper and lower bounds for the ultimate optimum value of $x_0$

| Cycle | Upper bound for $x_0$ | Lower bound for $x_0$ | Cycle | Upper bound for $x_0$ | Lower bound for $x_0$ |
|---|---|---|---|---|---|
| 0 | $+\infty$ | $-\infty$ | 6 | 183.50 | 176.19 |
| 1 | 233.37 | 152.58 | 7 | 182.95 | 179.45 |
| 2 | 191.37 | 152.58 | 8 | 180.53 | 179.45 |
| 3 | 190.45 | 176.19 | 9 | 180.47 | 179.45 |
| 4 | 185.75 | 176.19 | 10 | 180.25 | 179.45 |
| 5 | 184.15 | 176.19 | 11 | 179.75 | 179.75 |

assume the values zero and one only. For these problems a slight modification of the combinatorial procedure for solving pure “zero-one” problems, developed by Benders, Catchpole and Kuiken [3] can be applied for solving problem (4.20). The computational effort for solving pure “zero-one” problems in this way depends exponentially on the number of integer variables involved, so that this procedure is only of limited use. From a number of experiments on a Ferranti Mark I* computer, it has been found that the calculating time is within reasonable bounds provided the number of “zero-one” variables does not exceed 30 to 40, for present day computing equipment available. This requirement is satisfied, however, in many applications.

When evaluating the Procedures 4.1 and 5.1 for practical purposes one is interested also in the number of steps required for reaching an optimum solution and in the calculating time for each separate step. Since useful theoretical estimates do not yet exist, experiments with actual problems are necessary for getting pertinent information.

Our experimental work is mainly restricted to the mixed-integer linear programming problem with all integer variables of the “zero-one” type. In the small number of test cases considered up to now we have used Procedure 5.1, with problem (5.47) replaced by the more flexible problem (5.25). The integer sub-problems have been solved exclusively by the above mentioned combinatorial procedure.

A typical test case involved 29 “continuous” variables, 27 integer variables and 34 linear constraints. The total number of steps required for reaching an optimum solution was eleven. Table 1 shows the upper and lower bounds for the ultimate optimum value of $x_0$ obtained in each step. In all our experiments these bounds were very instructive for estimating the “rate of convergence”.

The number of steps required for reaching an optimum solution was encouragingly small in all test cases. The calculating time per step was mainly used for solving the linear programming sub-problem.

A reduction of the total number of steps and hence of the total calculating time may be obtained by adding more than one or two new constraints per step to the integer sub-problem (4.20). Procedure 4.1 [or application of the dual simplex method for solving the problem (5.17) or (5.25) in Procedure 5.1] seems to be most suitable for doing this since, when solving the linear programming problem (4.21), at the end of each simplex iteration a basic feasible solution is available corresponding to an extreme half line of the polyhedral cone $C$. Hence at the end of each simplex iteration a constraint for the region $G$ can be calculated. An efficient way of doing this, while avoiding the addition of redundant constraints for determining the set $G$ as much as possible, has not yet been worked out completely.

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