Dynamic Seat Assignment with Social Distancing

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Introduction

Social Distancing under pandemic

- Government issued the social distancing constraint.
- Theater has to follow that.

Literature Review

Seat Planning with Social Distancing

- Seat planning on airplanes, classrooms, trains.

- Group seat assignment in amphitheaters, airplanes, theater.

Dynamic Seat Assignment

- Dynamic multiple knapsack problem

- Revenue management with the feature of Assign-to-seat

Problem Definition

Seat Planning with Social Distancing

- Group type $\mathcal{M} = \{1, \ldots, M\}$.
- Row $\mathcal{N} = \{1, \ldots, N\}$.
- The social distancing: δ seat(s).
- $n_i = i + \delta$: the new size of group type i for each $i \in \mathcal{M}$.
- The number of seats in row j: $S_j, j \in \mathcal{N}$.
- $L_j = S_j + \delta$: the length of row j for each $j \in \mathcal{N}$.

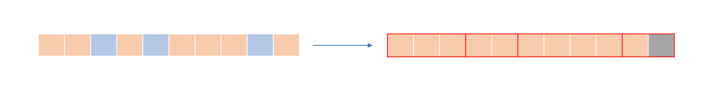


Figure: Problem Conversion with One Seat as Social Distancing

Basic Concepts

- Pattern refers to the seat planning for one row.
- A descending form $P_k = (t_1, t_2, \ldots, t_n)$ to denote pattern k, where t_h is the new group size, $h = 1, \ldots, n$.
- For each pattern k, α_k, β_k indicate the number of groups and the left seats, respectively.
- Loss for pattern k: $\alpha_k \delta + \beta_k - \delta$.
- I_1: the set of patterns with the minimal loss– the largest.
- Full patterns: $\beta_k = 0$.

- Example: $\delta = 1$, $M = 4$, $n_i = i + 1, i \in \mathcal{M}$, $L = 21$. Largest pattern: $(5,5,5,5), (5,4,4,4,4), (5,5,5,3,3)$. Full pattern: $(5,5,5,5)$.

Loss of The Largest Patterns

- A largest pattern can be obtained by the greedy way: select the maximal group size, n_M, as many times as possible, then $L = n_M \cdot q + r, 0 \leq r < n_M$, r is the number of empty seats.
- When $r > \delta$, these seats can be occupied by the group type $(r - \delta)$; when $r \leq \delta$, leave these seats empty.

* Loss of the largest patterns: $q\delta - \delta + f(r)$, where $f(r) = 0$ if $r > \delta$; $f(r) = r$ if $r \leq \delta$.
* For a seat layout, $\{S_1, S_2, \ldots, S_N\}$, the minimal total loss: $\sum_j (\lfloor \frac{S_j+\delta}{n_M} \rfloor - \delta + f((S_j + \delta) \mod n_M))$. The maximal number of people assigned: $\sum_j (S_j - \lfloor \frac{S_j+\delta}{n_M} \rfloor + \delta - f((S_j + \delta) \mod n_M))$.

Dynamic Seat Assignment Problem

- There is and only one group arrival each period. $T+1$ periods in total.
- The probability of an arrival of group type i: p_i. Value function: $V_t(\mathbf{L})$.
* Remaining capacity: $\mathbf{L} = (L_1^r, L_2^r, \ldots, L_N^r)$.
* The number of remaining seats in row j: L_j^r.

Dynamic seat assignment can be characterized by DP:

$$V_t(\mathbf{L}) = \mathbb{E}_{i\sim p}\left[\max_{\substack{j\in\mathcal{N}:\\ L_j \geqslant n_i}} \left\{V_{t+1}\left(\mathbf{L} - n_i \mathbf{e}_j^\intercal\right) + i, V_{t+1}(\mathbf{L})\right\}\right], V_{T+1}(\mathbf{L}) = 0.$$

- DP is computationally complex due to the curse of dimensionality.
- We give a seat planning by stochastic programming firstly, then apply stochastic planning policy to make the decision.

Seat Planning by Stochastic Programming

Algorithm Flow

- Transform deterministic equivalent form(DEF) to the two-stage stochastic programming. Problem (1) $\rightarrow$ Problem (2).
- Transform Problem (2) to restricted benders master problem.

Scenario-based Stochastic Programming

$$
\begin{aligned}
(DEF)\max \quad & E_\omega \left[\sum_{i=1}^{M-1} (n_i - \delta)(\sum_{j=1}^{N} x_{ij} + y_{i+1,\omega}^{+} - y_{i\omega}^{+}) + (n_M - \delta)(\sum_{j=1}^{N} x_{Mj} - y_{M\omega}^{+}) \right] \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} - y_{i\omega}^{+} + y_{i+1,\omega}^{+} + y_{i\omega}^{-} = d_{i\omega}, \quad i = 1, \ldots, M-1, \omega \in \Omega \\
& \sum_{j=1}^{N} x_{ij} - y_{i\omega}^{+} + y_{i\omega}^{-} = d_{i\omega}, \quad i = M, \omega \in \Omega \\
& \sum_{i=1}^{M} n_i x_{ij} \leq L_j, j \in \mathcal{N} \\
& y_{i\omega}^{+}, y_{i\omega}^{-} \in \mathbb{Z}_{+}, \quad i \in \mathcal{M}, \omega \in \Omega \\
& x_{ij} \in \mathbb{Z}_{+}, \quad i \in \mathcal{M}, j \in \mathcal{N}.
\end{aligned} \tag{1}
$$

For any i, ω, at most one of $y_{i\omega}^{+}$ and $y_{i\omega}^{-}$ can be positive.

Two-stage Stochastic Programming

$$
\begin{aligned}
\max \quad & \mathbf{c}^{\intercal}\mathbf{x} + z(\mathbf{x}) \\
\text{s.t.} \quad & \mathbf{n}\mathbf{x} \leq \mathbf{L} \\
& \mathbf{x} \in \mathbb{Z}_{+}^{M \times N},
\end{aligned} \tag{2}
$$

where $z(\mathbf{x})$ is the recourse function defined as

$$
z(\mathbf{x}) := E(z_{\omega}(\mathbf{x})) = \sum_{\omega \in \Omega} p_{\omega} z_{\omega}(\mathbf{x}),
$$

and for each scenario $\omega \in \Omega$,

$$
\begin{aligned}
z_{\omega}(\mathbf{x}) := \max \quad & \mathbf{f}^{\intercal}\mathbf{y}_{\omega} \\
\text{s.t.} \quad & \mathbf{x}\mathbf{1} + \mathbf{V}\mathbf{y}_{\omega} = \mathbf{d}_{\omega} \\
& \mathbf{y}_{\omega} \geq 0.
\end{aligned} \tag{3}
$$

Solve The Second Stage Problem

The dual of problem (3) is

$$\begin{aligned} \min \quad & \alpha_\omega^\intercal(\mathbf{d}_\omega - \mathbf{x}\mathbf{1}) \\ \text{s.t.} \quad & \alpha_\omega^\intercal \mathbf{V} \geq \mathbf{f}^\intercal \end{aligned} \tag{4}$$

Let $P = \{\alpha | \alpha^\intercal V \geq \mathbf{f}^\intercal\}$. The feasible region of problem (4), P, is bounded. In addition, all the extreme points of P are integral.

Delayed Constraint Generation

LP of problem (1) can be obtained by solving following restricted benders master problem(RBMP):

$$
\begin{aligned}
\max \quad & \mathbf{c}^\intercal x + \sum_{\omega \in \Omega} p_\omega z_\omega \\
\text{s.t.} \quad & \sum_{i=1}^{M} n_i x_{ij} \leq L_j, j \in \mathcal{N} \\
& (\alpha^k)^\intercal (\mathbf{d}_\omega - \mathbf{x1}) \geq z_\omega, \alpha^k \in \mathcal{O}^t, \forall \omega \\
& \mathbf{x} \geq 0
\end{aligned}
\tag{5}
$$

Constraints will be generated from problem (4) until the value of RBMP converges.

Benders Decomposition Algorithm

Step 1. Solve LP (5) with all $\alpha_\omega^0 = \mathbf{0}$ for each scenario. Then, obtain the solution $(\mathbf{x}_0, \mathbf{z}^0)$.

Step 2. Set the upper bound $UB = c'\mathbf{x}_0 + \sum_{\omega \in \Omega} p_\omega z_\omega^0$.

Step 3. For x_0, we can obtain α_ω^1 and $z_\omega^{(0)}$ for each scenario, set the lower bound $LB = c'x_0 + \sum_{\omega \in \Omega} p_\omega z_\omega^{(0)}$

Step 4. For each ω, if $(\alpha_\omega^1)'(\mathbf{d}_\omega - \mathbf{x}_0\mathbf{1}) < z_\omega^0$, add one new constraint, $(\alpha_\omega^1)'(\mathbf{d}_\omega - \mathbf{x}\mathbf{1}) \geq z_\omega$, to RBMP.

Step 5. Solve the updated RBMP, obtain a new solution (x_1, z^1) and update UB.

Step 6. Repeat step 3 until $UB - LB < \epsilon$.(In our case, UB converges.)

Deterministic Formulation

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{M} \sum_{j=1}^{N} (n_i - s) x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} \leq s_i, \quad i \in \mathcal{M}, \\
& \sum_{i=1}^{M} n_i x_{ij} \leq L_j, j \in \mathcal{N} \\
& x_{ij} \in \mathbb{Z}_+, \quad i \in \mathcal{M}, j \in \mathcal{N}.
\end{aligned} \tag{6}
$$

Substitute the first constraint with $\sum_{j=1}^{N} x_{ij} \geq s_i, i \in \mathcal{M}$, we can obtain the problem with lower bound supply.

Equivalent with Deterministic Model

When $|\Omega| = 1$ in problem (1), the stochastic programming will be

$$\begin{aligned}
\max \quad & \sum_{i=1}^{M} \sum_{j=1}^{N} (n_i - s) x_{ij} - \sum_{i=1}^{M} y_i^{+} \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} - y_i^{+} + y_{i+1}^{+} + y_i^{-} = d_i, \quad i = 1, \ldots, M-1, \\
& \sum_{j=1}^{N} x_{ij} - y_i^{+} + y_i^{-} = d_i, \quad i = M, \\
& \sum_{i=1}^{M} n_i x_{ij} \leq L_j, j \in \mathcal{N} \\
& y_i^{+}, y_i^{-} \in \mathbb{Z}_{+}, \quad i \in \mathcal{M} \\
& x_{ij} \in \mathbb{Z}_{+}, \quad i \in \mathcal{M}, j \in \mathcal{N}.
\end{aligned} \tag{7}$$

Obtain The Feasible Seat Planning

Step 1. Obtain the solution, $\mathbf{x}^*$, from stochatic linear programming by benders decomposition. Aggregate $\mathbf{x}^*$ to the number of each group type, $s_i^0 = \sum_j x_{ij}^*, i \in \mathbf{M}$.

Step 2. Solve problem (6) to obtain the optimal solution, $\mathbf{x}^1$. Aggregate $\mathbf{x}^1$ to the number of each group type, $s_i^1 = \sum_j x_{ij}^1, i \in \mathbf{M}$.

Step 3. Obtain the optimal solution, $\mathbf{x}^2$, from problem (20) with supply $\mathbf{s}^1$. Aggregate $\mathbf{x}^2$ to the number of each group type, $s_i^2 = \sum_j x_{ij}^2, i \in \mathbf{M}$.

Step 4. For each row, construct a full pattern.

Dynamic Seat Assignment for Each Group Arrival

Stochatic Planning Policy

Stochastic planning policy involves

- Group-type Control
  - Seat planning.
  - Small assigned to larger seats. $\rightarrow$ Find an arbitrary row to assign.
- Value of Acceptance and Rejection
  - Compare the value of stochastic programming when assigning in the row versus not assigning.

Bid-price Control

The dual problem of linear relaxation of problem (6) is:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{M} d_i z_i + \sum_{j=1}^{N} L_j \beta_j \\
\text{s.t.} \quad & z_i + \beta_j n_i \geq (n_i - \delta), \quad i \in \mathcal{M}, j \in \mathcal{N} \\
& z_i \geq 0, i \in \mathcal{M}, \beta_j \geq 0, j \in \mathcal{N}.
\end{aligned}
\tag{8}
$$

There exists h such that the aggregate optimal solution to relaxation of problem (6) takes the form $xe_h + \sum_{i=h+1}^{M} d_i e_i$, $x = (L - \sum_{i=h+1}^{M} d_i n_i)/n_h$.
The bid-price control policy will make the decision to accept group type i, where i is greater than or equal to h, if the capacity allows.

Dynamic Programming Base-heuristic

Relax all rows to one row with the same capacity by $L = \sum_{j=1}^{N} L_j$.
Deterministic problem is:
$\{\max \sum_{i=1}^{M} (n_i - \delta) x_i : x_i \leq d_i, i \in \mathcal{M}, \sum_{i=1}^{M} n_i x_i \leq L, x_i \in \mathbb{Z}_+\}$.
Let u denote the decision, where $u(t) = 1$ if we accept a request in period t, $u(t) = 0$ otherwise, the DP with one row can be expressed as:

$$V_t(L) = \mathbb{E}_{i \sim p}[\max_{u \in \{0,1\}} \{[V_{t+1}(L - n_i u) + iu]\}], L \geq 0$$

$$V_{T+1}(x) = 0, \forall x.$$

After accepting one group, assign it in some row arbitrarily when the capacity of the row allows.

Numerical Results

Running time of Benders Decomposition and IP

# of scenarios	demands	running time of IP(s)	Benders (s)	# of rows	# of groups	# of seats
1000	(150, 350)	5.1	0.13	30	8	(21, 50)
5000		28.73	0.47	30	8	
10000		66.81	0.91	30	8	
50000		925.17	4.3	30	8	
1000	(1000, 2000)	5.88	0.29	200	8	(21, 50)
5000		30.0	0.62	200	8	
10000		64.41	1.09	200	8	
50000		365.57	4.56	200	8	
1000	(150, 250)	17.15	0.18	30	16	(41, 60)
5000		105.2	0.67	30	16	
10000		260.88	1.28	30	16	
50000		3873.16	6.18	30	16	

Feasible Seat Planning versus IP Solution

# samples	T	probabilities	# rows	people served by FSP	IP
1000	45	[0.4,0.4,0.1,0.1]	8	85.30	85.3
1000	50	[0.4,0.4,0.1,0.1]	8	97.32	97.32
1000	55	[0.4,0.4,0.1,0.1]	8	102.40	102.40
1000	60	[0.4,0.4,0.1,0.1]	8	106.70	NA
1000	65	[0.4,0.4,0.1,0.1]	8	108.84	108.84
1000	35	[0.25,0.25,0.25,0.25]	8	87.16	87.08
1000	40	[0.25,0.25,0.25,0.25]	8	101.32	101.24
1000	45	[0.25,0.25,0.25,0.25]	8	110.62	110.52
1000	50	[0.25,0.25,0.25,0.25]	8	115.46	NA
1000	55	[0.25,0.25,0.25,0.25]	8	117.06	117.26
5000	300	[0.25,0.25,0.25,0.25]	30	749.76	749.76
5000	350	[0.25,0.25,0.25,0.25]	30	866.02	866.42
5000	400	[0.25,0.25,0.25,0.25]	30	889.02	889.44
5000	450	[0.25,0.25,0.25,0.25]	30	916.16	916.66

Each entry of people served is the average of 50 instances. IP will spend more than 2 hours in some instances, as ‘NA’ showed in the table.

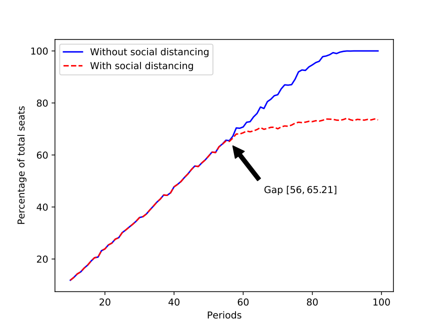
Performances of Different Policies to Optimal

T	probabilities	Sto(%)	DP1(%)	Bid-price(%)	FCFS(%)
60	[0.25, 0.25, 0.25, 0.25]	99.12	98.42	98.38	98.17
70	[0.25, 0.25, 0.25, 0.25]	98.34	96.87	96.24	94.75
80	[0.25, 0.25, 0.25, 0.25]	98.61	95.69	96.02	93.18
90	[0.25, 0.25, 0.25, 0.25]	99.10	96.05	96.41	92.48
100	[0.25, 0.25, 0.25, 0.25]	99.58	95.09	96.88	92.54
60	[0.25, 0.35, 0.05, 0.35]	98.94	98.26	98.25	98.62
70	[0.25, 0.35, 0.05, 0.35]	98.05	96.62	96.06	93.96
80	[0.25, 0.35, 0.05, 0.35]	98.37	96.01	95.89	92.88
90	[0.25, 0.35, 0.05, 0.35]	99.01	96.77	96.62	92.46
100	[0.25, 0.35, 0.05, 0.35]	99.23	97.04	97.14	92.00
60	[0.15, 0.25, 0.55, 0.05]	99.14	98.72	98.74	98.07
70	[0.15, 0.25, 0.55, 0.05]	99.30	96.38	96.90	96.25
80	[0.15, 0.25, 0.55, 0.05]	99.59	97.75	97.87	95.81
90	[0.15, 0.25, 0.55, 0.05]	99.53	98.45	98.69	95.50
100	[0.15, 0.25, 0.55, 0.05]	99.47	98.62	98.94	95.25

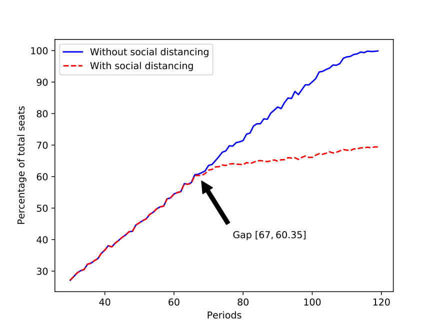
We compare the performance of different policies to the optimal value. Specifically, we evaluate four policies for seat assignment for each group arrival: stochastic planning policy, bid-price control, dynamic programming base-heuristic and first-come, first-served (FCFS).

Impact of Social Distance as Demands Increase

Let $\gamma = p_1 * 1 + p_2 * 2 + p_3 * 3 + p_4 * 4$ denote the number of people each period.



(a) When $\gamma = 2.5$



(b) When $\gamma = 1.9$

The gap point represents the first period where the number of people without social distancing is larger than that with social distancing and the gap percentage is the corresponding percentage of total seats.

When Supply and Demand Are Close

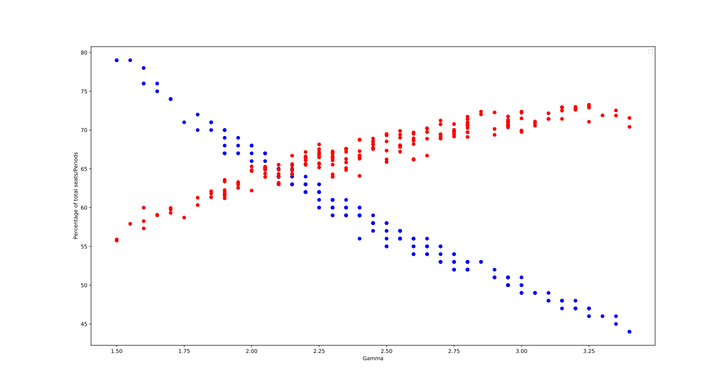


Figure: Gap points under 200 probabilities

Blue points: the first period that shows the difference.
Red points: percentage of total seats at the corresponding period.

The End