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Performance of an LP-Based Control for Revenue Management with Unknown Demand Parameters

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We consider a standard network revenue management (RM) problem and study the performance of a linear program (LP)-based control, the *Probabilistic Allocation Control* (PAC), in the presence of unknown demand parameters. We show that frequent re-optimizations of PAC without re-estimation suffice to *shrink* the asymptotic impact of estimation error on revenue loss. If, in addition to re-optimizations, we also frequently re-estimate the parameters, we prove that the performance of PAC in the unknown parameters setting is almost as good as the performance of PAC in the known parameters setting. Our numerical experiments show that PAC yields a revenue improvement of order 0.5%–1.5% relative to LP-based Booking Limit and Bid Price in most cases. Given the small margin in RM industries, such as the airline industry (about 2%), this level of improvement can easily translate into a significant increase in profit.

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1. Introduction

We consider a standard quantity-based network revenue management (RM) problem with finite selling horizon, limited inventories, and exogenously predetermined prices. Despite two decades of research on RM, most existing RM literature still assumes that the seller knows exactly the true demand rate (Chiang et al. 2007). This is an obvious limitation for at least two reasons. First, in practice, the true demand rate is rarely known and the seller needs to estimate the unknown demand parameters from the available data. Second, it is not clear a priori whether the heuristics developed in the setting of known demand rate are sufficiently robust in the presence of estimation error. The purpose of this paper is to partially address these issues; in particular, we will focus on a linear program (LP)-based control and show that it is strongly robust in the presence of estimation error.

The complexity of the RM problem has led many researchers to find heuristics that are not only near optimal but also are computationally tractable. Our work is motivated by the tractability and strong performance of an LP-based control, called *Probabilistic Allocation Control* (PAC), in the known parameters setting, as recently shown by Reiman and Wang (2008) and Jasin and Kumar (2012). To elaborate, in the setting of independent Poisson demands with known rates, when the size of the problem as represented by k (i.e., total expected demands or total initial inventories) becomes large, it can be shown that expected revenue loss of PAC with respect to the optimal policy is $\Theta(\sqrt{k})$. Given that the size of demand variation under Poisson arrivals is also $\Theta(\sqrt{k})$, this result is not surprising; the more important question is whether there is an easy way to improve the performance of PAC. Reiman and Wang (2008) are the first to show that a *single* re-optimization of PAC at a proper hitting time reduces its expected revenue loss from $O(\sqrt{k})$ to $O(k^{1/4})$. This result was surprising and enlightening; it tells us that an LP-based control may not be naïve. (It was generally believed at the time that the primary drawback of *any* LP-based control is that it considers only the expected demand and ignores demand variation; see Talluri and van Ryzin 2004, pg. 94.) Motivated by the work of Reiman and Wang (2008), Jasin and Kumar (2012) later show that the expected revenue loss of PAC under *frequent* re-optimizations is actually $O(1)$, that is, the revenue loss of PAC is *independent* of the size of the problem. This result is very strong. It not only highlights the effectiveness of simple re-optimizations as a tactical solution for dealing with demand randomness but also further suggests the viability of PAC for practical implementation.

In this paper, we continue to study the performance of PAC in the setting of unknown demand parameters. We ask: What is the impact of re-optimizations on revenue loss if we only estimate the parameters once and never re-estimate them during the remaining selling horizon? (The answer to this question is important to isolate the benefit of re-optimizations apart from re-estimations, i.e., whether re-optimization still has a significant impact on performance in the presence of estimation error like it does in the known parameters setting.) What is the impact of joint re-optimizations and re-estimations on revenue loss? We show that while frequent re-optimizations of PAC without re-estimation suffice to reduce its revenue loss from $O(k^{3/4})$ to $O(\sqrt{k})$, joint re-optimizations and re-estimations further reduce its revenue loss to $O(\log^2 k)$. These show that PAC is *strongly* robust in the presence of estimation error. Apart from Reiman and Wang (2008) and Jasin and Kumar (2012), our work is most closely related to Ciocan and Farias (2012). In their paper, Ciocan and Farias analyze an RM model with stochastic rates and provide a bound for their proposed heuristic under a combined re-optimization and re-estimation scheme. Although our analysis can be essentially extended to their setting, for clarity of the analysis, we will simply assume stationary rates in this paper. Our work differs from their work in two aspects. First, while Ciocan and Farias use competitive ratio as their performance measure, our objective is to characterize a more precise asymptotic revenue loss bound. Second, we explicitly characterize the impact of different re-optimization and re-estimation frequencies on revenue loss. Thus our work complements their work in different dimensions. The remainder of this paper is organized as follows. In §2, we discuss the model and problem setting. In §§3–4, we analyze the performance of PAC. Section 5 reports results from numerical experiments. In §6, we conclude the paper. All proofs can be found in the electronic companion (available as supplemental material at http://dx.doi.org/10.1287/opre.2015.1390).

2. Model

2.1. The Setting

We consider a discrete-time RM model with T periods. Without loss of generality, we assume that at most one request arrives during each period. (It is also possible to use a continuous-time model with Poisson arrivals and the results in this paper still hold.) There are n_r resources and n_p products. Demands across different periods are assumed to be independent and identically distributed. The price for product j is f_j and the arrival probability (demand rate) of a request for product j during period t is $\lambda_j \in (0, 1]$. Let λ_0 denote the probability of no arrival. By definition, we must have $\lambda_0 + \sum_{j=1}^{n_p} \lambda_j = 1$. Let C denote the vector of initial resources capacity (unless otherwise noted, all vectors are to be understood as column vectors). We assume that a single unit of product j requires $A_{ij} \geqslant 0$ units of resource i. Let A^j denote the jth column of A. Throughout this paper, we will use the following convention: time t is used to denote the end of period t (equivalently, the beginning of period $t+1$). In addition, interval $(s, t]$ refers to period $s+1$ up to and including period t.

For any control π, we define a new control π^R to be its relaxation in the sense that π^R implements the same control as π but ignores the capacity constraint. For example, if control π dictates that the seller accepts at most 100 requests for product 1 subject to available capacity during $(T/2, T]$, then π^R simply dictates the seller to accept at most 100 requests for product 1 during $(T/2, T]$. Below, we provide a list of useful notations (some of which will only appear in the proofs). For all $0 \leqslant s < t \leqslant T$, we have:

$\Im_s$ = the cumulative information up to time s,
$D(s, t)$ = total demands during $(s, t]$,
$\hat{\lambda}(s)$ = estimate of λ used at time s,
$\epsilon(s)$ = estimation error at time s $(=\hat{\lambda}(s) - \lambda)$,
$R_\pi(s, t)$ = total revenue earned under control π during $(s, t]$,
$N_\pi(s, t)$ = total accepted requests (sold products) under control π during $(s, t]$,
$\hat{N}_\pi(s, t)$ = estimate of total accepted requests under control π during $(s, t]$,
$\phi_{\pi, s}(s, t) = \mathbf{E}[\hat{N}_{\pi^R}(s, t) \mid \Im_s] - \mathbf{E}[N_{\pi^R}(s, t) \mid \Im_s]$.

Note that, with the exception of $\Im_s$ and $R_\pi(s, t)$, all the above notations are to be understood as vector notations. Moreover, we say that $\hat{N}_\pi(s, t)$ is an estimate of $N_\pi(\cdot, \cdot)$ in a distributional sense, i.e., $\mathbf{E}[\hat{N}_\pi(s, t) \mid \Im_s]$ is the expected number of sales under $\hat{\lambda}(s)$, whereas $\mathbf{E}[N_\pi(s, t) \mid \Im_s]$ is the expected number of sales under the true demand parameters λ. For notational brevity, whenever there is no confusion, we will suppress the dependency on π.

2.2. The Stochastic and Deterministic Formulations

The dynamics of the RM problem are as follows: Upon a customer's arrival and observing his request, the seller decides whether to sell the requested product. If the request is granted, capacity is consumed and revenue is earned; if the request is denied, capacity remains intact and no new revenue is earned. This is the standard model for the so-called quantity-based RM (see Talluri and van Ryzin 2004, chapter 3). The optimal control formulation of the RM problem is given by $V_{\text{opt}} = \sup_{\pi \in \Pi} \mathbf{E}[\sum_{j=1}^{n_p} f_j N_{\pi, j}(0, T)]$ s.t. $\sum_{j=1}^{n_p} N_{\pi, j}(0, T) A^j \leqslant C$ and $0 \leqslant N_\pi(0, t) \leqslant D(0, t)$ for all $t = 1, \ldots, T$, where all constraints must be satisfied almost surely (or with probability one) and Π is the set of admissible nonanticipating policies. The deterministic linear program (DLP) formulation of the RM problem is

$$\mathbf{DLP}[C, \lambda T]: \quad \max f'x \quad \text{s.t. } Ax \leqslant C \quad \text{and} \quad 0 \leqslant x \leqslant \lambda T.$$

For analytical tractability, we will make two assumptions on the network structure: (1) $\mathbf{DLP}[C, \lambda T]$ is nondegenerate and has a unique optimal solution x^* and (2) $A\lambda T \nleqslant C$. The first assumption on nondegeneracy has been used in Jasin and Kumar (2012) and is required for the proof; in particular, it allows us to explicitly characterize the solution of the perturbed linear program. The second assumption corresponds to the case where, on average, we do not have sufficient resources to satisfy the demands. This is quite natural and appropriate, especially during high seasons. Let V_{DLP} denote the optimal value of $\mathbf{DLP}[C, \lambda T]$. The relationship between the optimal stochastic formulation, the DLP, and any control $\pi \in \Pi$ is given by $\mathbf{E}[R_\pi(0, T)] \leqslant V_{\text{opt}} \leqslant V_{\text{DLP}}$. (See Jasin and Kumar 2012, for proof.) Hence, as the performance measure of any control $\pi \in \Pi$, we will use the expected revenue loss defined as $V_{\text{DLP}} - \mathbf{E}[R_\pi(0, T)]$.

2.3. PAC and Asymptotic Setting

We now provide the definition of PAC. Let $\Gamma = \{t_n: n = 0, 1, 2, \ldots, N\}$ denote the set of re-optimization times ($t_n < t_{n+1}$ for all n). (Since we only focus on LP-based heuristic, these will be the set of times when we actually re-optimize the LP.) The initial time t_0 will be used to serve two purposes: $t_0 = 0$ is used to denote the beginning of the selling horizon and $t_0 > 0$ is used to denote the beginning of the actual selling periods (see §3). For example, a periodic schedule with $t_0 = 0$ and period $h > 0$ is given by $\Gamma = \{0, h, 2h, \ldots, Nh\}$, where N is the greatest integer satisfying $Nh \leqslant T$. For convenience, we will write $t_{N+1} = T$. Let $C(t)$ denote the vector of remaining resources capacity at time t and let $x(t)$ denote the optimal solution of the following LP:

$$\mathbf{DLP}[C(t), (T-t)\hat{\lambda}(t)]: \quad \max f'x \quad \text{s.t. } Ax \leqslant C(t)$$
$$\text{and} \quad 0 \leqslant x \leqslant (T-t)\hat{\lambda}(t).$$

Let $\Im_n = \Im_{t_n}$ and $\phi_n(\cdot) = \phi_{t_n}(\cdot)$. Given a schedule $\Gamma = \{t_n\}$, the definition of PAC is given below.

Definition 1 (Probabilistic Allocation Control). For each $n \geqslant 0$, do (1) at time t_n, solve $\mathbf{DLP}[C(t_n), (T-t_n)\hat{\lambda}(t_n)]$ and (2) during $(t_n, t_{n+1}]$, accept each incoming request for product j with probability $q_j(t_n) = x_j(t_n)/[(T-t_n)\hat{\lambda}_j(t_n)]$ subject to capacity.

(We use the convention $0/0 = 0$. So, $\hat{\lambda}(t_n) = 0$ implies $q_j(t_n) = 0$.) By definition, under the relaxed PAC, $\mathbf{E}[\hat{N}_j(t_n, t) \mid \Im_n] = q_j(t_n)(t-t_n)\hat{\lambda}_j(t_n)$. Since $\mathbf{E}[N_j(t_n, t) \mid \Im_n] = q_j(t_n)(t-t_n)\lambda_j$, we can bound $||\phi_n(t_n, t)||_\infty \leqslant (t-t_n)||\hat{\lambda}(t_n) - \lambda||_\infty = (t-t_n)||\epsilon(t_n)||_\infty$. This observation is critical for the analysis. Note that the above definition implicitly includes the possibility of re-estimation. If no re-estimation is scheduled at time t_n, we simply use the most recent estimate of λ; if re-estimation is scheduled, we first compute the new estimate and then re-optimize the LP using the new estimate.

Following the standard asymptotic setting in the literature, in this paper, we consider a sequence of problems increasing in size where the number of selling periods and the initial capacity vector are scaled by a factor of $k > 0$. To be precise, the kth problem is parameterized by $(T^k, C^k) = (kT, kC)$. (The superscript k will be used throughout this paper as a reference to the kth problem.) As discussed in Jasin and Kumar (2012), the scaling factor k can be interpreted as the relative size of the problem (i.e., large k corresponds to the setting of large demands and large inventories). The asymptotic notations $\Omega(\cdot)$, $\Theta(\cdot)$, $O(\cdot)$, and $o(\cdot)$ have their standard meaning and will be used regularly throughout the paper. For notational brevity, whenever there is no confusion, we will often suppress the superscript k on all notations.

3. The Impact of Re-Optimizations

In this section, we study the impact of re-optimization without re-estimation. We show that, even without re-estimation, frequent re-optimizations of PAC alone manage to *shrink* the asymptotic impact of estimation error on revenue loss. This tells us that PAC is strongly robust in the presence of estimation error. To characterize the benefit of re-optimizations apart from re-estimation, we first divide the selling horizon into two segments: the *initial learning* periods $(0, t_0]$, where we simply accept all incoming requests; and the *actual selling* periods $(t_0, T]$, where we apply PAC with re-optimizations. The complete algorithm for our scheme is given below.

PAC with learning and re-optimization

1. During $(0, t_0]$, do:
   —Accept all incoming requests subject to the available capacity C.
2. At time t_0, do:
   —Compute $\hat{\lambda}(t_0) = D(0, t_0)/t_0$ and solve $\mathbf{DLP}[C(t_0), (T-t_0)\hat{\lambda}(t_0)]$.
   —Apply PAC during $(t_0, t_1]$ with $q_j(t_0) = x_j(t_0)/[(T-t_0)\hat{\lambda}_j(t_0)]$.
3. For each $n > 0$, do:
   —Solve $\mathbf{DLP}[C(t_n), (T-t_n)\hat{\lambda}(t_0)]$.
   —Apply PAC during $(t_n, t_{n+1}]$ with $q_j(t_n) = x_j(t_n)/[(T-t_n)\hat{\lambda}_j(t_0)]$.

We state our first result below.

Theorem 1. *Suppose that we use $t_0^k = \lceil \sqrt{T^k} \rceil$ and periodic re-optimization with period h^k. There exists a constant $M > 0$ independent of $k \geqslant 1$ and $h^k \geqslant 1$ such that*

$$V_{DLP}^k - \mathbf{E}[R^k(0, T^k)] \leqslant M\left[\sqrt{k} + \frac{h^k}{k^{1/4}}\right].$$

Some comments are in order. If we never re-optimize the LP (i.e., we use $h^k = T^k - t_0^k$), then the revenue loss of PAC is $O(k^{3/4})$. This is expected since the size of estimation error at time t_0^k is about $1/\sqrt{t_0^k} = 1/k^{1/4}$. (Thus the total revenue loss during $\Theta(k)$ periods is about $k \cdot 1/k^{1/4} = k^{3/4}$.) If, on the other hand, we re-optimize the LP $k^{1/4}$ times throughout the remaining selling horizon using $h^k \approx k^{3/4}$, the revenue loss of PAC is *shrunk* to $O(\sqrt{k})$. This is rather surprising: the fact that mere re-optimizations *without* re-estimations could have *shrank* the asymptotic impact of estimation error on revenue loss is neither natural nor obvious. First, this result is not natural as it does not hold just for any LP-based control. Using similar arguments shown in Jasin and Kumar (2013), it can be analytically shown that re-optimizing an LP-based control called *Booking Limit* does not shrink the asymptotic impact of estimation error on revenue loss, and this is true regardless of the variation of booking limit being used. Moreover, our numerical results in §5 suggest that re-optimizing another LP-based control called *Additive Bid Price* also does not shrink the asymptotic impact of estimation error on revenue loss. This tells us that the shrinking result of Theorem 1 owes its explanation to the particular manner in which the LP solution is translated into an admission control (see Remark 2). Second, this result is not obvious because it cannot be directly inferred from the results in Jasin and Kumar (2012). To be precise, while Jasin and Kumar show that frequent re-optimizations of PAC diminish the impact of *demand randomness* on revenue loss (up to $O(1)$), their analysis assumes that the seller knows the true demand parameters. In contrast, here we use $\hat{\lambda}(0) = \lambda + \epsilon(0)$ instead of λ; thus, in addition to having to deal with the errors from demand randomness, we also have to deal with estimation errors, which creates *systematic biases* in optimization. (In the presence of estimation errors, we are essentially re-optimizing an LP with wrong parameters. This results in a biased optimal solution everytime the LP is re-optimized.) It is not clear a priori whether frequent re-optimizations of PAC alone without re-estimation suffice to diminish the impact of estimation errors on revenue loss, or whether these errors will have an inevitable adverse impact on revenue. Fortunately, we show that frequent re-optimizations of PAC manage to also shrink the asymptotic impact of estimation errors on revenue loss. In fact, we can decouple the impact of demand randomness and estimation errors on revenue loss. We state this result more formally below. Define $\Phi^k = \mathbf{E}[||\epsilon^k(0)||_\infty^2]^{1/2}$. (This can be interpreted as the size of estimation error at time 0.) Theorem 1 can be seen as a corollary of the following proposition.

Proposition 1. *Suppose that we are given an estimate $\hat{\lambda}(0)$ at time 0 and never re-estimate the parameters. Under periodic re-optimization with period h^k, there exists a constant $M > 0$ independent of $k \geqslant 1$, $h^k \geqslant 1$, and Φ^k such that $V_{\mathrm{DLP}}^k - \mathbf{E}[R^k(0, T^k)] \leqslant M[\sqrt{h^k} + \Phi^k h^k + (\Phi^k)^2 k]$.*

(Mathematically, the result of Proposition 1 still holds even if we use $C^k + o(k)$ as our initial capacity instead of exactly C^k.) Note that the term $\sqrt{h^k}$ in Proposition 1 is similar to the bound for periodic PAC under the known parameters setting in Jasin and Kumar (2012). This tells us that the impact of estimation errors on revenue loss is essentially captured by the sum $\Phi^k h^k + (\Phi^k)^2 k$. If we rarely re-optimize the LP, the impact is about $\Phi^k k$; if we frequently re-optimize the LP, the impact is only about $(\Phi^k)^2 k$, which is smaller than $\Phi^k k$ because $\Phi^k \leqslant 1$.

Remark 1 (Exploration vs. Exploitation). Is it possible to use a different t_0^k and guarantee a smaller revenue loss than $O(\sqrt{k})$? Suppose that we use $t_0^k = \tau^k$ with $\tau^k/k \to 0$ as $k \to \infty$. Since the size of estimation error at time τ^k is about $1/\sqrt{\tau^k}$, by Proposition 1, the total revenue loss with $h^k = 1$ during $(\tau^k, T^k]$ is $O(1 + 1/\sqrt{\tau^k} + k/\tau^k) = O(1 + k/\tau^k)$. Thus the combined revenue loss during $(0, T^k]$ is about $\tau^k + k/\tau^k$. This loss is minimized when $\tau^k \approx \sqrt{k}$.

Remark 2 (PAC vs. Booking Limit and Additive Bid Price). As discussed in Jasin and Kumar (2013), the strong performance of PAC in the known parameters setting is due to the fact that PAC uses a *rate* interpretation of the LP solution; this allows PAC to do *continuous* correction as the LP is being re-optimized. In contrast, booking limit and bid price use a *cumulative* interpretation of the LP solution. Since small errors are relatively negligible in cumulative sense, booking limit and bid price do not actually correct for these errors until their cumulative magnitude is sufficiently large, i.e., the correction happens too late. In our setting, this downside is further exacerbated by the sensitivity of LP solution with respect to estimation error.

4. The Impact of Joint Re-Optimizations and Re-Estimations

In this section, we want to know whether re-estimations can significantly reduce the revenue loss bound in Theorem 1. We again divide the selling horizon into two segments: the initial learning periods $(0, t_0]$, where we simply accept all incoming requests, and the actual selling periods $(t_0, T]$, where we apply PAC with re-optimizations and re-estimations. Let $\Gamma_o = \{t_0, t_0 + h_o, t_0 + 2h_o, \ldots\} = \{t_n\}$ and $\Gamma_e = \{t_0, t_0 + h_e, t_0 + 2h_e, \ldots\}$ denote the re-optimization and re-estimation schedule, respectively. Let $z = h_e/h_o$. For simplicity, we assume that z and $(T - t_0)/h_e$ are integers. The combined learning and joint re-optimizations and re-estimations algorithm is given below.

PAC with learning, re-optimization, and re-estimation

1. During $(0, t_0]$, do:
   —Accept all incoming requests subject to the available capacity C.
2. At time t_0, do:
   —Compute $\hat{\lambda}(t_0) = D(0, t_0)/t_0$ and solve $\mathbf{DLP}[C(t_0), (T - t_0)\hat{\lambda}(t_0)]$.
   —Apply PAC during $(t_0, t_1]$ with $q_j(t_0) = x_j(t_0)/[(T - t_0)\hat{\lambda}_j(t_0)]$.
3. For each $i > 0$, do:
   a. At time t_{iz}, do:
      —Compute $\hat{\lambda}(t_{iz}) = D(0, t_{iz})/t_{iz}$ and solve $\mathbf{DLP}[C(t_{iz}), (T - t_{iz})\hat{\lambda}(t_{iz})]$.
      —Apply PAC during $(t_{iz}, t_{iz+1}]$ with $q_j(t_{iz}) = x_j(t_{iz})/[(T - t_{iz})\hat{\lambda}_j(t_{iz})]$.
   b. At time $t \in \{t_{iz+1}, \ldots, t_{(i+1)z-1}\}$, do:
      —Solve $\mathbf{DLP}[C(t), (T - t)\hat{\lambda}(t_{iz})]$.
      —Apply PAC during $(t, t + h_o]$ with $q_j(t) = x_j(t)/[(T - t)\hat{\lambda}_j(t_{iz})]$.

Let $J_0 = \{j\colon x_j^* = 0\}$ denote the set of *zero* products. (Note that x^* is the optimal solution of $\mathbf{DLP}[C, \lambda T]$. Thus the set J_0 can be viewed as the set of products deemed most unprofitable by the LP.) We state a theorem.

Theorem 2. *Suppose that we use $t_0^k = \lceil \log^2 T^k \rceil$ and periodic re-optimization and re-estimation. There exists $M > 0$*

independent of $k \geqslant 1$, $h_o^k \geqslant 1$, *and* $h_e^k \leqslant (T^k - t_0^k)/2$ *such that*

$$V_{DLP}^k - \mathbf{E}[R^k(0,T^k)] \leqslant M\left[\sqrt{h_o^k} + \frac{(h_e^k)^2}{k}\right] \quad (\textit{if } J_0 = \varnothing) \quad \textit{and}$$

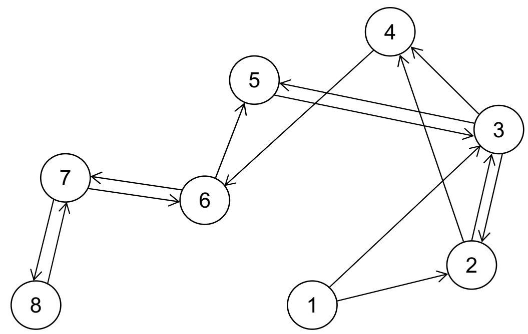
$$V_{DLP}^k - \mathbf{E}[R^k(0,T^k)] \leqslant M\left[\log^2 k + \sqrt{h_o^k} + \frac{(h_e^k)^2}{k}\right] \qquad (\textit{if } J_0 \neq \varnothing).$$

The proof of the second part of Theorem 2 proceeds by decomposing the total expected revenue loss into three parts: (1) the revenue loss incurred during the initial learning periods, which is $O(\log^2 k)$; (2) the revenue loss due to demand randomness, which is $O(\sqrt{h_o^k})$ (cf. Theorem 5.1 in Jasin and Kumar 2012); and (3) the revenue loss due to estimation errors, which is $O(1 + (h_e^k)^2/k)$. Two comments are in order. First, setting $h_e^k \approx \sqrt{k}$ effectively eliminates the impact of estimation error on revenue loss. Therefore, only $\Theta(\sqrt{k})$ re-estimations are needed under the periodic schedule. (This is in contrast to the $\Theta(k)$ re-optimizations needed to effectively turn off the term $\sqrt{h_o^k}$.) Second, if $J_0 = \varnothing$, we recover the original $O(1)$ revenue loss of PAC under the known parameters setting (Jasin and Kumar 2012). If, however, $J_0 \neq \varnothing$, there is a learning cost of magnitude $O(\log^2 k)$. This cost is incurred during the initial learning periods because we sell the "unprofitable" zero products. Is it possible to reduce the initial learning cost by cleverly modifying the schedule during the first $\Theta(\log^2 k)$ periods? One obvious approach is to use shorter initial learning periods with $t_0^k = \lceil \log^{1+\alpha} T^k \rceil$ for some $\alpha \in (0,1)$. It can be shown that this yields a reduced learning cost of order $\log^{1+\alpha} k$. Can we do better than $O(\log k)$? Perhaps another scheme that guarantees a stronger performance bound than $O(\log k)$ exists. However, this is beyond the scope of this paper.

Remark 3 (Less Intensive Schedule). Theorem 2 assumes the use of periodic schedule for re-optimizations and re-estimations. Is it possible to guarantee the same performance with a less intensive schedule? Consider the midppoint schedule introduced in Jasin and Kumar (2012), i.e., $t_1^k = \lceil (T^k + t_0^k)/2 \rceil$ and $t_{n+1}^k = \lceil (T^k + t_n^k)/2 \rceil$ for $n \geqslant 1$. Suppose that we re-estimate the parameters at times $\{t_0^k, t_0^k + 2, \ldots, t_0^k + 2^{M^k}\}$, where $M^k = \lfloor \log_2(T^k - t_0^k) \rfloor - 3$, and re-optimize the LP at times $\{t_0^k, t_0^k + 2, \ldots, t_0^k + 2^{M^k}, t_1^k, t_2^k, \ldots\}$. (Thus we only re-estimate and re-optimize about $\log_2 k$ times.) It can be shown using similar arguments as in the proof of Theorem 2 that the $O(\log^2 k)$ bound still holds.

Remark 4 (Using Capacity Rationing). Instead of accepting *all* requests during the initial learning periods in Theorem 2, it is possible to first ration the amount of capacity that can be used during the initial learning periods. For example, if we allocate $t_0 C/T$ to $(0, t_0]$ and the remaining $(T - t_0)C/T$ to $(t_0, T]$ (i.e., at time t_0, we solve a DLP with capacity $(T - t_0)C/T$), it can be shown using similar arguments as in the proof of Theorem 2 that the revenue loss incurred during $(0, t_0^k]$ is $O(\log^2 k)$ and the revenue loss incurred during $(t_0^k, T^k]$ is $O(\sqrt{h_o^k} + (h_e^k)^2/k)$.

Figure 1. A hub model with 8 cities and 14 connecting flights.



5. Numerical Experiments

The purpose of this section is to illustrate the theoretical result in §4 as well as to compare the robustness of PAC with two LP-based controls, *Partition Booking Limit Control* (PBLC) and *Additive Bid Price Control* (ABPC). (The definition of these controls can be found in Jasin and Kumar 2013.) Our simulation setting is motivated by an example from airline industry discussed in Gallego and van Ryzin (1997). We use a hub-and-spoke model with 8 cities, 14 connecting flights, and 41 itineraries (see Figure 1 and Table 1). The normalized capacity for each flight can be seen in Table 2 and the normalized length of the selling horizon is 1 (i.e., $T = 1$). Per our definition, the actual capacity size and the actual length of selling horizon in the problem with scaling factor k are given by kC and kT, respectively. For example, in our setting, $k = 2{,}000$ corresponds to the problem with 2,000 periods, total expected demand for each itinerary ranging from 20 to 80, and total capacity for each flight ranging from 60 to 240. We run four different experiments. In experiment 1, we use the combined scheme discussed in Theorem 2 with $t_0^k = \lceil \log^2 k \rceil$ and $h_o^k = h_e^k = 1$ for each control. Experiment 2 uses the same scheme as in experiment 1, only with $\Theta(\sqrt{k})$ instead of $\Theta(k)$ re-estimations. In experiment 3, we use $\Theta(\sqrt{k})$ joint re-optimizations and re-estimations. Finally, in experiment 4, we use the less intensive schedule discussed in Remark 3.

We report the average revenue loss (with respect to V_{DLP}) out of 100 Monte Carlo simulations together with its standard deviation in Table 3. In addition, we also report the percentage reduction in revenue loss and percentage improvement in revenue by PAC relative to PBLC and ABPC in the electronic companion. Some comments are in order. First, PAC shows a generally better performance than PBLC and ABPC in all four experiments. (Despite their weaker asymptotic performance guarantee, PBLC and

Table 1. The value of demand rate and deterministic price for each itinerary.

Itinerary	Route	Arrival prob.	Price ($)	Itinerary	Route	Arrival prob.	Price ($)
1	1–2	0.01	800	22	3–4	0.04	450
2	1–2	0.01	850	23	3–5	0.04	400
3	1–2–4	0.01	1,400	24	3–4–6	0.04	600
4	1–3–5	0.01	1,200	25	4–6	0.01	450
5	1–2	0.02	600	26	4–6	0.04	200
6	1–3	0.04	650	27	5–3–2	0.01	1,200
7	1–2–4	0.04	1,000	28	5–3	0.01	800
8	1–3–5	0.04	900	29	5–3–4	0.01	1,250
9	2–3	0.01	800	30	5–3–2	0.04	800
10	2–4	0.01	825	31	5–3	0.04	500
11	2–3–5	0.01	1,000	32	5–3–4	0.04	800
12	2–4–6	0.01	1,000	33	6–5	0.04	800
13	2–3	0.04	500	34	6–7	0.01	800
14	2–4	0.04	550	35	6–7	0.04	500
15	2–3–5	0.04	800	36	7–6	0.01	800
16	2–4–6	0.04	825	37	7–6	0.04	500
17	3–2	0.01	600	38	7–8	0.01	700
18	3–4	0.01	650	39	7–8	0.04	400
19	3–5	0.01	625	40	8–7	0.01	700
20	3–4–6	0.01	800	41	8–7	0.04	400
21	3–2	0.02	425				

Table 2. Normalized capacity for each flight.

Flight/route	1–2	1–3	2–3	2–4	3–2	3–4	3–5	4–6	5–3	6–5	6–7	7–6	7–8	8–7
Capacity	0.06	0.08	0.08	0.12	0.06	0.12	0.12	0.12	0.12	0.03	0.04	0.04	0.04	0.04

ABPC perform reasonably well in the first two experiments.) Given the theoretical results in §3, this observation is not too surprising. Moreover, the results of experiment 4 also confirm the prediction of Remark 3 that it is possible to use a less intensive schedule and still obtain an excellent revenue performance. We want to stress that although the percentage revenue improvement of PAC relative to either PBLC or ABPC is only about 0.5%–1.5% in most cases, in many large industries, this can easily translate into a significant increase in profit margin. Indeed, as reported by the International Air Transport Association IATA (2013), the average net margins for airline industry is typically very small (only about 2%). Second, in comparison to PAC, PBLC and ABPC appear to be less robust with respect to estimation error. In particular, to guarantee a strong revenue performance, frequent re-optimizations of either PBLC or ABPC coupled with sufficiently frequent re-estimations appear to be necessary. This can be seen clearly by comparing the results of experiments 1–3. Although the combination of less frequent re-estimations with frequent re-optimizations in experiment 2 worsen by only a little bit, the performance of PBLC and ABPC in experiment 1, the combination of less frequent re-estimations with less frequent re-optimizations in experiment 3 worsen the performance of PBLC and ABPC by significantly more. In contrast, the performance of PAC in experiment 3 is only slightly worse than its performance in experiment 1. Similar outcome can also be observed by comparing the results of experiments 1 and 4. Although the performance of PAC in both experiments are not very different, the performance of PBLC and ABPC in experiment 4 are obviously much worse than their performance in experiment 1. All these observations reinforce our earlier discussions in §4 that not all heuristics are equally robust in the presence of estimation error; in particular, whereas some such as PBLC and ABPC are reasonably robust in the sense that they do not blow up the impact of estimation error on revenue loss, others such as PAC is very robust because it actually shrinks the impact of estimation error on revenue loss.

6. Closing Remarks

In this paper, we have studied the performance of PAC in the setting of unknown demand parameters. We show that PAC is strongly robust in the presence of estimation error. Moreover, our numerical results also show that PAC yields a revenue improvement of order 0.5%–1.5% relative to LP-based Booking Limit and Bid Price in most cases. Given the typical small margin in RM industries such as the airline industry (about 2%), this can easily translate into a significant increase in profit. Three things are worth mentioning at the conclusion of this paper. First, although the discussions in this paper focus only on a particular LP-based control, more sophisticated heuristics have been studied in the literature, such as the dynamic Bid Price developed in Zhang and Adelman (2009). Due to the popularity of Bid Price heuristic in practice, it is useful to know whether

Table 3. Expected revenue loss under different controls with varying k.

	k (10^3)	PAC			PBLC			ABPC		
		Loss	St dev	% loss	Loss	St dev	% loss	Loss	St dev	% loss
Exp 1	1	7,434	623	1.39	15,207	558	2.84	8,110	563	1.75
(k, k)	2	10,930	785	1.02	21,701	711	2.03	12,051	780	1.13
	4	11,957	1,064	0.56	23,651	951	1.10	13,547	959	0.63
	6	11,246	1,125	0.35	27,242	996	0.85	15,122	1,089	0.47
	8	13,081	1,492	0.31	26,767	1,256	0.62	20,438	1,297	0.48
	10	13,491	1,586	0.25	28,943	1,384	0.54	23,254	1,657	0.43
	20	13,960	2,354	0.13	31,698	1,866	0.30	33,479	2,562	0.31
Exp 2	1	8,493	483	1.59	15,651	858	2.92	10,345	584	1.93
$(k, \sqrt{k})$	2	10,031	742	0.94	24,405	720	2.28	11,685	740	1.09
	4	12,429	1,094	0.58	23,709	977	1.11	14,259	1,002	0.67
	6	11,596	1,209	0.36	27,441	1,051	0.85	17,407	854	0.54
	8	11,728	1,354	0.27	27,170	1,297	0.63	19,157	1,259	0.45
	10	14,750	1,634	0.28	29,796	1,341	0.56	25,398	1,182	0.47
	20	14,135	1,908	0.13	37,482	1,946	0.35	36,677	1,923	0.34
Exp 3	1	15,292	571	2.86	16,322	561	3.05	15,337	592	2.86
$(\sqrt{k}, \sqrt{k})$	2	19,670	764	1.84	22,752	730	2.12	22,096	631	2.06
	4	22,231	1,122	1.04	34,283	974	1.60	34,537	969	1.61
	6	29,877	1,399	0.93	44,601	1,304	1.39	44,676	1,277	1.39
	8	25,668	1,555	0.60	52,498	1,505	1.23	53,606	1,496	1.25
	10	26,967	1,737	0.50	62,009	1,656	1.16	63,095	1,657	1.18
	20	32,355	2,415	0.30	98,509	2,347	0.92	99,795	2,376	0.93
Exp 4	1	15,420	676	2.88	18,282	605	3.41	16,978	456	3.17
$(\log k, \log k)$	2	18,589	926	1.74	23,730	633	2.22	23,160	662	2.16
	4	23,141	1,220	1.08	40,520	910	1.89	40,630	928	1.90
	6	27,546	1,559	0.86	52,300	1,147	1.63	53,500	1,195	1.67
	8	29,404	1,802	0.69	78,360	1,408	1.83	77,760	1,351	1.82
	10	31,143	1,648	0.58	87,750	1,777	1.64	86,720	1,577	1.62
	20	33,711	2,609	0.31	163,100	3,794	1.52	166,510	2,878	1.55
	30	34,965	3,156	0.22	307,480	2,954	1.91	309,420	3,076	1.93

Note. The pair under the experiment label indicates the order of the number of re-optimizations and re-estimations, respectively.

these heuristics are more robust than LP-based Bid Price, or even PAC. Second, our schemes assume that the seller has no knowledge about the true demand parameters when starting the selling season. Although this is convenient for the analysis, in practice, the seller is likely to have some information about the parameters. A relevant research question would be how to integrate the seller's prior knowledge with an estimation scheme that uses the newly observed data. Third, throughout the paper, we have assumed that demand rates are stationary. Again, in reality, this is rarely the case. An important and impactful research question is: How can we extend the results in this paper to the setting where demand rates change gradually over time.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/opre.2015.1390.

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