Dynamic Seat Assignment

With Social Distancing

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Literature Review

Seat Planning with Social Distancing

- Seat planning on airplanes, classrooms, trains.

- Group seat assignment in amphitheaters, airplanes, theater.

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- Revenue management
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Problem Definition

Seat Planning with Social Distancing

- Group type $[M] = \{1, \ldots, M\}$
- Row $[N] = \{1, \ldots, N\}$
- s seats as the social distancing
- Let $n_i = i + s$ denote the new size of group type i for each $i \in [M]$.
- Let $L_j = S_j + s$ denote the length of row j for each $j \in [N]$, where S_j represents the number of seats in row j.

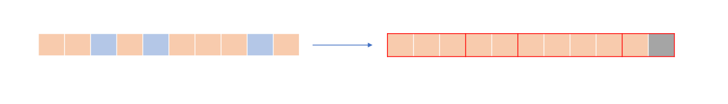


Figure: Problem Conversion with One Seat as Social Distancing

Dynamic Programming

Dynamic seat assignment can be characterized by DP:

$$V_t(\mathbf{L}) = E_i \left[\max_{k \in N: L_k \geq i+s} \{[V_{t-1}(\mathbf{L} - U_{ik}) + i], V_{t-1}(\mathbf{L})\} \right], \mathbf{L} \geq \mathbf{0}$$

$$V_{T+1}(\mathbf{L}) = 0,$$

- $\mathbf{L} = (L_1^r, L_2^r, \ldots, L_N^r)$, remaining capacity. L_j^r represents the number of remaining seats in row j.
- U_{ik} is a vector whose k-th element is n_i, with all other elements equal to 0.
- p_i: the probability of an arrival of group type i.

Some Definitions

- Pattern refers to the seat planning for each row.
- For each pattern k, α_k, β_k indicate the number of groups and the left seats, respectively.
- Denote by $\alpha_k + \beta_k - 1$ the loss for pattern k, $l(k)$. The loss represents the number of people lost compared to the situation without social distancing.
- Let I_1 be the set of patterns with the minimal loss. We call the patterns from I_1 are the largest. The patterns with zero left seat are called full patterns.
- Suppose there are n groups in a row, we use a descending form $P_k = (t_1, t_2, \ldots, t_n)$ to denote seat planning for pattern k, where t_h is the new group size, $h \in [n]$.

Example

- Suppose the social distancing is one seat and there are four types of groups. Then the new sizes of groups are $2, 3, 4, 5$, respectively.
- The length of one row is $L = 21$.
- Then these patterns, $(5, 5, 5, 5), (5, 4, 4, 4, 4), (5, 5, 5, 3, 3)$, belong to I_1.
- Pattern $(5, 5, 5, 5)$ is not full because there is one left seat.

Property

- Let $u = M + s$, then a largest pattern can be obtained greedily, i.e., select the maximal group size, u, as many as possible and the left space is occupied by the group with the corresponding size.
- Let $L = u \cdot q + r$. The loss of the largest pattern is $q - f(r)$, where $f(r) = 1$ if $r = 0$; $f(r) = 0$ if $r \neq 0$.
- For a seat layout, $\{S_1, S_2, \ldots, S_N\}$, the total loss is $\sum_j (\lfloor \frac{S_j+1}{u} \rfloor - f((S_j + 1) \mod u))$. The maximal number of people assigned is $\sum_j (S_j - \lfloor \frac{S_j+1}{u} \rfloor + f((S_j + 1) \mod u))$.

Scenario-based Stochastic Programming

Scenario-based Stochastic Programming

$$
\begin{aligned}
\max \quad & E_\omega \left[\sum_{i=1}^{M-1} (n_i - s)(\sum_{j=1}^{N} x_{ij} + y^+_{i+1,\omega} - y^+_{i\omega}) + (n_M - s)(\sum_{j=1}^{N} x_{Mj} - y^+_{M\omega}) \right] \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} - y^+_{i\omega} + y^+_{i+1,\omega} + y^-_{i\omega} = d_{i\omega}, \quad i \in [M-1], \omega \in \Omega \\
& \sum_{j=1}^{N} x_{ij} - y^+_{i\omega} + y^-_{i\omega} = d_{i\omega}, \quad i = M, \omega \in \Omega \\
& \sum_{i=1}^{M} n_i x_{ij} \leq L_j, j \in [N] \\
& y^+_{i\omega}, y^-_{i\omega} \in \mathbb{Z}_+, \quad i \in [M], \omega \in \Omega \\
& x_{ij} \in \mathbb{Z}_+, \quad i \in [M], j \in [N].
\end{aligned} \tag{1}
$$

Two-stage

$$\begin{aligned} \max \quad & c'\mathbf{x} + z(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{n}\mathbf{x} \leq \mathbf{L} \\ & \mathbf{x} \in \mathbb{Z}_+^{M\times N}, \end{aligned} \tag{2}$$

where $z(\mathbf{x})$ is the recourse function defined as

$$z(\mathbf{x}) := E(z_\omega(\mathbf{x})) = \sum_{\omega\in\Omega} p_\omega z_\omega(\mathbf{x}),$$

and for each scenario $\omega \in \Omega$,

$$\begin{aligned} z_\omega(\mathbf{x}) := \max \quad & \mathbf{f}'\mathbf{y}_\omega \\ \text{s.t.} \quad & \mathbf{x}\mathbf{1} + \mathbf{V}\mathbf{y}_\omega = \mathbf{d}_\omega \\ & \mathbf{y}_\omega \geq 0. \end{aligned} \tag{3}$$

Solve the Second Stage Problem

The dual of problem (3) is

$$\begin{aligned} \min \quad & \alpha'_{\omega}(\mathbf{d}_{\omega} - \mathbf{x1}) \\ \text{s.t.} \quad & \alpha'_{\omega}\mathbf{V} \geq \mathbf{f}' \end{aligned} \tag{4}$$

Let $P = \{\alpha | \alpha' V \geq \mathbf{f}'\}$. The feasible region of problem (4), P, is bounded. In addition, all the extreme points of P are integral.

Delayed Constraint Generation

LP of problem (1) can be obtained by solving following restricted benders master problem(RBMP):

$$\begin{aligned} \max \quad & c'x + \sum_{\omega \in \Omega} p_\omega z_\omega \\ \text{s.t.} \quad & \sum_{i=1}^{M} n_i x_{ij} \leq L_j, j \in [N] \\ & (\alpha^k)'(\mathbf{d}_\omega - \mathbf{x1}) \geq z_\omega, \alpha^k \in \mathcal{O}^t, \forall \omega \\ & \mathbf{x} \geq 0 \end{aligned} \tag{5}$$

Constraints will be generated from problem (4) until the value of RBMP converges.

Benders Decomposition Algorithm

Step 1. Solve LP 5 with all $\alpha_\omega^0 = \mathbf{0}$ for each scenario. Then, obtain the solution $(\mathbf{x}_0, \mathbf{z}^0)$.

Step 2. Set the upper bound $UB = c'\mathbf{x}_0 + \sum_{\omega\in\Omega} p_\omega z_\omega^0$.

Step 3. For x_0, we can obtain α_ω^1 and $z_\omega^{(0)}$ for each scenario, set the lower bound $LB = c'x_0 + \sum_{\omega\in\Omega} p_\omega z_\omega^{(0)}$

Step 4. For each ω, if $(\alpha_\omega^1)'(\mathbf{d}_\omega - \mathbf{x}_0\mathbf{1}) < z_\omega^0$, add one new constraint, $(\alpha_\omega^1)'(\mathbf{d}_\omega - \mathbf{x}\mathbf{1}) \geq z_\omega$, to RBMP.

Step 5. Solve the updated RBMP, obtain a new solution (x_1, z^1) and update UB.

Step 6. Repeat step 3 until $UB - LB < \epsilon$.(In our case, UB converges.)

Deterministic Formulation

When $|\Omega| = 1$ in problem (1), the stochastic programming will be

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{M}\sum_{j=1}^{N}(n_i - s)x_{ij} - \sum_{i=1}^{M} y_i^+ \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} - y_i^+ + y_{i+1}^+ + y_i^- = d_i, \quad i \in [M-1], \\
& \sum_{j=1}^{N} x_{ij} - y_i^+ + y_i^- = d_i, \quad i = M, \\
& \sum_{i=1}^{M} n_i x_{ij} \leq L_j, j \in [N] \\
& y_i^+, y_i^- \in \mathbb{Z}_+, \quad i \in [M] \\
& x_{ij} \in \mathbb{Z}_+, \quad i \in [M], j \in [N].
\end{aligned}
\tag{6}
$$

Deterministic Formulation

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{M} \sum_{j=1}^{N} (n_i - s) x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} \leq d_i, \quad i \in [M], \\
& \sum_{i=1}^{M} n_i x_{ij} \leq L_j, j \in [N] \\
& x_{ij} \in \mathbb{Z}_+, \quad i \in [M], j \in [N].
\end{aligned} \tag{7}
$$

Substitute the first constraint with $\sum_{j=1}^{N} x_{ij} \geq d_i, i \in [M]$, we can obtain the problem with lower bound demand.

Obtain the Feasible Seat Planning

Step 1. Obtain the solution, $\mathbf{x}^*$, from linear stochastic programming by benders decomposition.

Step 2. Aggregate the solution to the supply, $s_i^0 = \sum_j x_{ij}^*$.

Step 3. Obtain the optimal solution, $\mathbf{x}^1$, from problem (7) by setting the supply $\mathbf{s}^0$ as the upper bound.

Step 4. Aggregate the solution to the supply, $s_i^1 = \sum_j x_{ij}^1$.

Step 5. Obtain the optimal solution, $\mathbf{x}^2$, from problem (19) by setting the supply $\mathbf{s}^1$ as the lower bound.

Step 6. Aggregate the solution to the supply, $s_i^2 = \sum_j x_{ij}^2$, which is the feasible seat planning.

Dynamic Seat Assignment

Assign-to-seat Rules

- When the supply of one arriving group is enough, we will accept the group directly.
- When the supply of one arriving group is 0, the demand can be satisfied by only one larger-size supply.
- When one group is accepted to occupy the larger-size seats, the rest empty seat(s) can be reserved for future demand.

The difference of expected served people between acceptance and rejection on group i occupying $(j+s)$-size seats:
$d(i,j) = i + (j-i-1)P(D_{j-i-1} \geq x_{j-i-1}+1) - jP(D_j \geq x_j), j > i$.
Find $d(i,j^*) = \max_j d(i,j)$, if $d(i,j^*) > 0$, accept group type i in $j^* + s$-size seats; otherwise, reject it.

Dynamic Seat Assignment for Each Group Arrival

Step 1. Obtain the set of patterns, $\mathbf{P} = \{P_1, \ldots, P_N\}$, from the feasible seat planning algorithm. The corresponding aggregated supply is $\mathbf{X} = [x_1, \ldots, x_M]$.

Step 2. For the arrival group type i at period T', find the first $k \in [N]$ such that $i \in P_k$. Accept the group, update $P_k = P_k/(i)$ and $x_i = x_i - 1$. Go to step 4.

Step 3. If $i \notin P_k, \forall k \in [N]$, find $d(i, j^*)$. If $d(i, j^*) > 0$, find the first $k \in [N]$ such that $j^* \in P_k$. Accept group type i and update $P_k = P_k/(j^*)$, $x_{j^*} = x_{j^*} - 1$. Then update $x_{j-i-1} = x_{j-i-1} + 1$ and $P_k = P_k \cup (j^* - i - 1)$ when $j^* - i - 1 > 0$. If $d(i, j^*) \leq 0$, reject group type i.

Step 4. If $T' \leq T$, move to next period, set $T' = T' + 1$, go to step 2. Otherwise, terminate this algorithm.

Dynamic Seat Assignment after All Group Arrivals

Relax all rows to one row with the same capacity by $L = \sum_{j=1}^{N} L_j$.
Deterministic problem is:
$\{\max \sum_{i=1}^{M}(n_i - s)x_i : x_i \leq d_i, i \in [M], \sum_{i=1}^{M} n_i x_i \leq L, x_i \in \mathbb{Z}_+\}$.
The optimal solution can be easily obtained.
The gap between the relaxed deterministic problem and original deterministic problem is zero for most cases.
Thus, we use DP to make the decision.

$$V_t(L) = E_i[\max\{[V_{t-1}(L - n_i) + i], V_{t-1}(L)\}], L \geq 0$$

$$V_{T+1}(x) = 0, \forall x$$

Results

Running time of Benders Decomposition and IP

# of scenarios	demands	running time of IP(s)	Benders (s)	# of rows	# of group
1000	(150, 350)	5.1	0.13	30	8
5000		28.73	0.47	30	8
10000		66.81	0.91	30	8
50000		925.17	4.3	30	8
1000	(1000, 2000)	5.88	0.29	200	8
5000		30.0	0.62	200	8
10000		64.41	1.09	200	8
50000		365.57	4.56	200	8
1000	(150, 250)	17.15	0.18	30	16
5000		105.2	0.67	30	16
10000		260.88	1.28	30	16
50000		3873.16	6.18	30	16

Feasible Seat Planning versus IP Solution

# samples	T	probabilities	# rows	people served by FSP	IP
1000	45	[0.4,0.4,0.1,0.1]	8	85.30	85.3
1000	50	[0.4,0.4,0.1,0.1]	8	97.32	97.32
1000	55	[0.4,0.4,0.1,0.1]	8	102.40	102.40
1000	60	[0.4,0.4,0.1,0.1]	8	106.70	NA
1000	65	[0.4,0.4,0.1,0.1]	8	108.84	108.84
1000	35	[0.25,0.25,0.25,0.25]	8	87.16	87.08
1000	40	[0.25,0.25,0.25,0.25]	8	101.32	101.24
1000	45	[0.25,0.25,0.25,0.25]	8	110.62	110.52
1000	50	[0.25,0.25,0.25,0.25]	8	115.46	NA
1000	55	[0.25,0.25,0.25,0.25]	8	117.06	117.26
5000	300	[0.25,0.25,0.25,0.25]	30	749.76	749.76
5000	350	[0.25,0.25,0.25,0.25]	30	866.02	866.42
5000	400	[0.25,0.25,0.25,0.25]	30	889.02	889.44
5000	450	[0.25,0.25,0.25,0.25]	30	916.16	916.66

Each entry of people served is the average of 50 instances. IP will spend more than 2 hours in some instances, as ‘NA’ showed in the table.

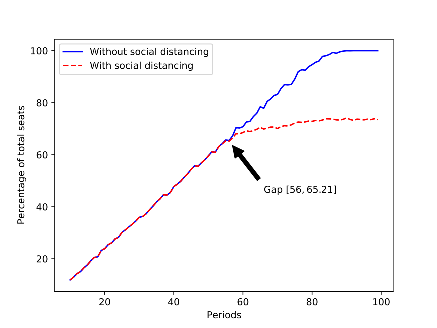
Results of Different Policies

T	probabilities	DSA(%)	FCFS(%)	DP-based(%)	FCFS-based(%)
60	[0.45, 0.05, 0.05, 0.45]	99.26	97.59	99.45	99.33
60	[0.35, 0.05, 0.35, 0.25]	99.46	98.06	99.66	99.47
60	[0.25, 0.05, 0.65, 0.05]	98.45	97.49	99.24	98.83
60	[0.35, 0.15, 0.15, 0.35]	99.10	97.94	99.41	99.12
60	[0.25, 0.15, 0.45, 0.15]	99.48	98.45	99.62	99.46
80	[0.45, 0.05, 0.05, 0.45]	96.26	92.43	98.93	93.68
80	[0.35, 0.05, 0.35, 0.25]	96.37	92.13	98.06	93.80
80	[0.25, 0.05, 0.65, 0.05]	98.32	94.30	99.08	95.15
80	[0.35, 0.15, 0.15, 0.35]	96.24	92.49	99.18	93.54
80	[0.25, 0.15, 0.45, 0.15]	96.68	92.84	99.36	93.87

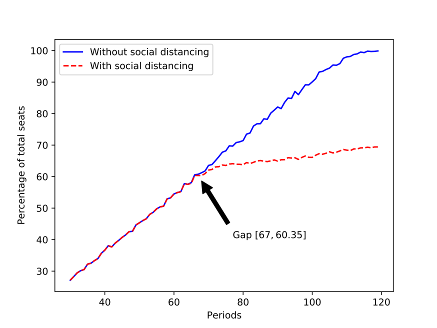
We compare the performance of different policies to the optimal value. Specifically, we consider two policies for seat assignment after all group arrivals: DSA and FCFS. In addition, we evaluate two policies for seat assignment for each group arrival: one based on dynamic programming (DP) and the other based on first-come, first-served (FCFS) scheduling.

Result of Different Demands

Let $c = p_1 * 1 + p_2 * 2 + p_3 * 3 + p_4 * 4$ denote the number of people each period.



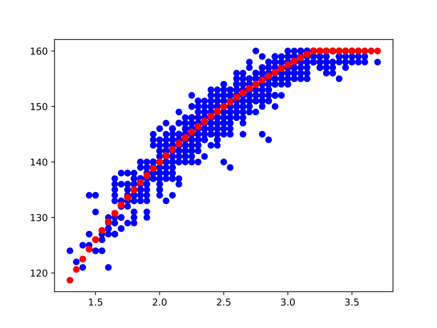
(a) When $c = 2.5$



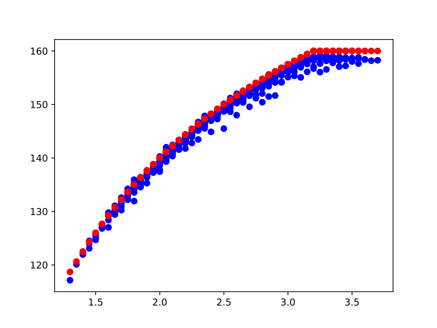
(b) When $c = 1.9$

The gap point represents the first period where the number of people without social distancing is larger than that with social distancing and the gap percentage is the corresponding percentage of total seats.

Results of the Number of Arriving People per Period



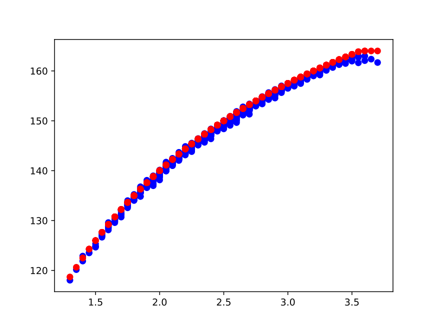
(c) One instance for each probability combination



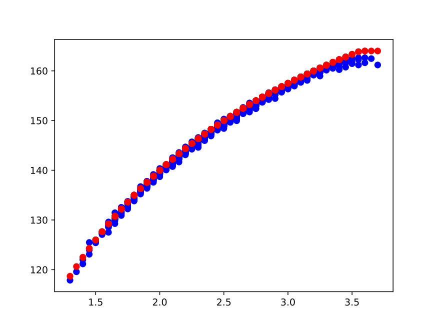
(d) Average of 50 instances for each probability combination

Figure: The number of people served versus c

Results of Different Seat Layouts



(a) Average of 50 instances for step-size seat layout



(b) Average of 50 instances for random seat layout

Figure: The number of people served versus c

The End