Operations Research

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Dynamic Bid Prices in Revenue Management

Daniel Adelman,

OPERATIONS RESEARCH
Vol. 55, No. 4, July–August 2007, pp. 647–661
ISSN 0030-364X | EISSN 1526-5463 | 07 | 5504 | 0647

DOI 10.1287/opre.1060.0368

Dynamic Bid Prices in Revenue Management

Daniel Adelman
Graduate School of Business, University of Chicago, Chicago, Illinois 60637, dan.adelman@chicagogsb.edu

We formally derive the standard deterministic linear program (LP) for bid-price control by making an affine functional approximation to the optimal dynamic programming value function. This affine functional approximation gives rise to a new LP that yields tighter bounds than the standard LP. Whereas the standard LP computes static bid prices, our LP computes a time trajectory of bid prices. We show that there exist dynamic bid prices, optimal for the LP, that are individually monotone with respect to time. We provide a column generation procedure for solving the LP within a desired optimality tolerance, and present numerical results on computational and economic performance.

Subject classifications: revenue management, pricing: network; bid prices; dynamic programming/optimal control: applications, approximate.
Area of review: Manufacturing, Service, and Supply Chain Operations.
History: Received July 2005; revision received June 2006; accepted June 2006. Published online in *Articles in Advance* July 20, 2007.

1. Introduction

The notion of "bid-price controls" (Talluri and van Ryzin 1998, Simpson 1989, Williamson 1992) has been a powerful and influential solution concept in revenue management for more than a decade. Major airlines, for instance, have used bid-price control policies for deciding when to open and close customer fare classes for sale. More generally, they can be used in revenue management settings where the supply of resources is fixed and customer requests arrive over a finite time horizon to consume various resource configurations. The arriving requests must either be accepted or rejected, with the objective of maximizing expected profit over the time horizon. The basic idea of bid-price control is simple: Accept the request if the revenue earned exceeds the value of the resources consumed as measured by bid prices. Typically, the bid prices are computed as optimal dual prices, i.e., marginal resource values, of a simple deterministic linear program.

While the system under control is dynamic, to date there only exist models for computing static bid prices, which do not change as a function of time. The effect of dynamically changing prices is created by re-solving the static bid-price model through time as the system evolves. One purpose of this paper is to derive and explore a tractable model for computing a time trajectory of bid prices all at once within a single model. In implementation this model may still be re-solved over time, and this turns out to be a good strategy, but hopefully the prices will be more accurate and effective due to the fact that system dynamics must somehow be taken into account in computing them.

The second purpose of this paper is to further formalize the connection between bid-price control in revenue management and dynamic programming, beyond Talluri and van Ryzin (1998). Given the standard bid-price linear program, Talluri and van Ryzin (1998) heuristically interpret the embedded pricing mechanism as making a linear functional approximation to the dynamic programming value function. However, the linear program itself has never been derived directly from the dynamic program. Rather, its prices are heuristically interpreted ex post in terms of a value function approximation.

In this paper, we start with the dynamic program, and a priori make an affine functional approximation to the value function, i.e., based on a linear combination of affine basis functions. If r_i is the number of seats (resources) still available for flight (resource type) i in period t, then we approximate the value of state vector $\vec{r}$ by

$$v_t(\vec{r}) \approx \theta_t + \sum_i V_{t,i} r_i$$

for parameters $V_{t,i}$ and θ_t. From this, we derive the standard bid-price linear program. We do this by substituting this approximation into a linear programming formulation of the optimality equations, and then aggregating constraints in the dual. We thereby show that the intermediate linear program, which computes dynamic bid prices $V_{t,i}$, provides stronger bounds than the standard bid-price linear program. Furthermore, we may interpret the standard, static bid prices as approximating the dynamic ones.

This analysis also leads to a new, alternative proof that the standard linear program provides an upper bound on the optimal dynamic programming value function. Previous proofs were based on sample-path arguments, for example, see the discussion in Bertsimas and Popescu (2003). We provide numerical results showing that the standard bound can be as much as 50.4% larger than our new bound. This improved bound can be used by researchers to numerically obtain improved optimality guarantees for new heuristics.

This is the first paper to use the linear programming approach to approximate dynamic programming in the context of revenue management. This approach was first considered by Schweitzer and Seidmann (1985), and more recently by de Farias and Van Roy (2003). The idea is to approximate the value function with a linear combination of basis functions, and then substitute the approximation into the linear programming formulation of the optimality equations. To our knowledge, this approach has not been considered previously in the finite-horizon case, such as we have here.

We show that this approach has foundational significance in the context of bid-price controls in revenue management. Indeed, bid prices need not be associated with a linear functional approximation, as they are usually thought of. In general, they may be viewed as prices on basis-weighted flow-balance constraints that appear in a primal problem, for any arbitrary collection of basis functions used to approximate the value function. The standard bid prices are associated with linear basis functions.

We show how to get this approach to work effectively in the affine basis case. Hopefully, our progress will aid future researchers in exploring stronger functional forms for basis functions, or in applying these methods on other variations of revenue management problems. For instance, in the general approximation case, we establish that obtaining feasibility is trivial and that there is a simple optimality guarantee. We demonstrate the importance of structural properties satisfied by optimal bid-price trajectories. We prove that the optimal bid prices are individually nonincreasing over time, and by adding this as dual constraints, we obtain substantial computational speedup by as much as a factor of 85. It turns out that this is key to implementing the model on real-world industrial-sized instances. This structural result also permits us to interpret the approximate linear program as finding a time threshold policy that is optimal with respect to an approximate policy evaluation. We also report on an effective column generation algorithm for solving the model within any desired optimality tolerance. The economic performance of the resulting policy, with re-solving, beats standard bid-price control by as much as 21.4% in our numerical tests. This is substantial in the context of airline revenues, for instance.

Previous work has considered various other, but quite different, approximate dynamic programming approaches to revenue management. As mentioned above, Talluri and van Ryzin (1998) intepret various revenue management models in terms of approximating the value function. Bertsimas and Popescu (2003) consider using the exact value functions of math programming models, in particular, the standard bid-price linear program, to compute opportunity costs. Our model could in fact be used in their procedure, and therefore complements their work. Bertsimas and de Boer (2005) use simulation-based methods to estimate gradients useful in updating booking limits. The latter two references, the book (Talluri and van Ryzin 2004) and the review article (McGill and van Ryzin 1999), provide excellent, detailed reviews of the literature on other approaches to, and aspects of, the revenue management problem.

This paper is organized as follows. In §2, we provide some background, including a standard dynamic programming formulation and the standard bid-price control models. Then, in §3, we consider the general case of approximating the value function with a linear combination of basis functions. This leads to a primal model with an interesting structural form, as well as results regarding feasibility and optimality guarantees. The heart of the paper is §4, which makes the affine functional approximation, derives the standard bid-price linear program from it, provides structural properties, and gives a useful column generation algorithm. We provide numerical results in §5 on computational times, comparative bound strength, and policy performance.

2. Preliminaries

In this section, we provide the basic formulations and notation we will use throughout the paper. We first present a known formulation of the network revenue management problem as a Markov decision process. We then formulate the standard linear program used for bid-price control.

2.1. Markov Decision Process Formulation

Our formulation essentially follows Cooper and Homem-de-Mello (2003), and similar models have appeared in the literature (McGill and van Ryzin 1999). The model is a finite-horizon discrete-time Markov decision process with time units $t = 1, \ldots, \tau$, where τ is the last period in which sales can occur. The objective is to maximize the total expected revenue.

There are m resources and n product or customer classes. There is an m-vector of resources $\vec{c} \equiv (c_i)$, and an $(m \times n)$-matrix $A \equiv (a_{i,j})$. The entry $c_i > 0$ represents the integer amount of resource i available at the beginning of the time horizon ($t = 1$), and the entry $a_{i,j}$ represents the integer amount of resource i required by a class j customer. To ease notation, we reserve the symbols i, j, and t for resources, classes, and time, respectively, and we omit writing the index sets. For example, the notation $\forall\, i$ means $\forall\, i \in \{1, \ldots, m\}$ and $\sum_i$ means $\sum_{i \in \{1,\ldots,m\}}$.

In each period, at most one customer arrives. A class j customer arrives in period t with probability $p_{t,j}$, and with probability $1 - \sum_j p_{t,j}$ no customer arrives. When a customer of class j arrives, the controller must decide whether to accept or reject this customer. If the controller chooses to accept, she receives $f_j > 0$ in revenue, and resources are consumed according to the jth column of matrix A, denoted by A^j. (Our analysis permits the revenue to depend on time if we so wished, i.e., it could be given by $f_{t,j}$.) If there are not enough resources available to satisfy the request, then the request must be rejected and there is no reward. Even if enough resources are available, a request may be rejected if it is more profitable to reserve resources for potential future customers.

The state at the beginning of any period t is an m-vector of resources $\vec{r}$ that satisfies

$$\vec{r} \in \mathscr{R} \equiv \{\vec{r} \in \mathbb{Z}_+^m\colon r_i \in \{0, 1, \ldots, c_i\}\ \forall\, i\}.$$

In Period 1, we have $\vec{r} = \vec{c}$, and so for convenience we define

$$\mathscr{R}_t = \begin{cases} \{\vec{c}\} & \text{if } t = 1, \\ \mathscr{R} & \text{if } t = 2, \ldots, \tau. \end{cases}$$

Given an initial state $\vec{c}$ and the arrival probabilities $p_{t,j}$, not all states are reachable, and so the sets $\mathscr{R}_t$ are larger than needed. The controller chooses a vector $\vec{u} \in \{0, 1\}^n$, where $u_j = 1$ if the controller would accept a class j customer if one arrives this period, and $u_j = 0$ otherwise. Given that the system is in state $\vec{r}$, this vector must satisfy

$$\vec{u} \in \mathscr{U}_{\vec{r}} = \{\vec{u} \in \{0, 1\}^n\colon \vec{r} \geqslant A^j u_j\ \forall\, j\} \quad \forall\, \vec{r} \in \mathscr{R}$$

to ensure that there are enough resources available to satisfy a class j request if $u_j = 1$.

Given $\vec{r}$ and $\vec{u}$, with probability $p_{t,j}$ the reward for the current period will be $f_j u_j$ and the next state will be $\vec{r} - A^j u_j$. With probability $1 - \sum_j p_{t,j}$, the reward in the current period will be zero and the next state will be $\vec{r}$.

Let $v_t(\vec{r})$ denote the expected cost-to-go over periods $t, \ldots, \tau$ starting from state $\vec{r}$ at the beginning of period t. Define the terminal value $v_{\tau+1}(\vec{r}) \equiv 0\ \forall\, \vec{r}$, and assume that this holds throughout. The optimality equations are

$$v_t(\vec{r}) = \max_{\vec{u} \in \mathscr{U}_{\vec{r}}} \left\{ \sum_j p_{t,j}[f_j u_j + v_{t+1}(\vec{r} - A^j u_j)] + \left(1 - \sum_j p_{t,j}\right) v_{t+1}(\vec{r}) \right\} \quad \forall\, t,\ \vec{r} \in \mathscr{R}_t. \tag{1}$$

It is easy to show that an optimal policy chooses, in state $\vec{r}$ and period t,

$$u_{t,j}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \geqslant A^j,\ f_j \geqslant v_{t+1}(\vec{r}) - v_{t+1}(\vec{r} - A^j), \\ 0 & \text{otherwise}, \end{cases} \quad \forall\, t, j,\ \vec{r} \in \mathscr{R}_t, \tag{2}$$

where $f_j \geqslant v_{t+1}(\vec{r}) - v_{t+1}(\vec{r} - A^j)$ means that the revenue for a class j customer meets or exceeds the opportunity cost of the resources that would be consumed. Note that the control policy is *nested*: for any two classes j_1 and j_2 for the same itinerary $A^{j_1} = A^{j_2}$, if $f_{j_2} > f_{j_1}$ then the policy accepts j_2 if it accepts j_1.

In principle, the optimal value function at the initial state $\vec{c}$ can be computed by the linear program

$$\textbf{(D0)} \quad \min_{v(\cdot)} v_1(\vec{c}),$$

$$v_t(\vec{r}) \geqslant \sum_j p_{t,j}[f_j u_j + v_{t+1}(\vec{r} - A^j u_j)] + \left(1 - \sum_j p_{t,j}\right) v_{t+1}(\vec{r}) \quad \forall\, t, \vec{r} \in \mathscr{R}_t, \vec{u} \in \mathscr{U}_{\vec{r}}, \tag{3}$$

with decision variables $v_t(\vec{r})\ \forall\, t, \vec{r}$. This can be shown from complementary slackness. It is apparently nontraditional to consider linear programming for stochastic dynamic programs in the finite-horizon case. However, we can view this dynamic program as a "positive dynamic program" (Puterman 1994) with the time index included in the state space, and therefore general results for that model, including ones dealing with linear programming, hold here.

For notational convenience, we write the inequality above as

$$v_t(\vec{r}) \geqslant \sum_j p_{t,j} f_j u_j + \mathsf{E}[v_{t+1}(\vec{R}_{t+1}) \mid \vec{r}, \vec{u}],$$

where $\vec{R}^{t+1} \in \mathscr{R}_{t+1}$ is the random vector representing the resource state at the beginning of period $t+1$. Throughout the paper, the above conditional expectation relies only on the one-step transition probabilities, which involve the demand probabilities $p_{t,j}$ as shown above.

The following result is elementary. It shows that any feasible solution to **(D0)** provides an upper bound on the expected profit-to-go for an optimal policy from every state and time period. Later, we will consider feasible solutions that arise from special functional forms for $v_t(\cdot)$.

PROPOSITION 1. *Suppose that $v_t(\cdot)$ solves the optimality Equations* (1), *and $\hat{v}_t(\cdot)$ is any feasible solution to* **(D0)**. *Then,*

$$\hat{v}_t(\vec{r}) \geqslant v_t(\vec{r}) \quad \forall\, t,\ \vec{r} \in \mathscr{R}_t.$$

PROOF. We show this by induction. For all $\vec{r} \in \mathscr{R}_\tau$, $\vec{u} \in \mathscr{U}_{\vec{r}}$, dual feasibility (3) implies

$$\hat{v}_\tau(\vec{r}) \geqslant \sum_j p_{\tau,j} f_j u_j.$$

Hence,

$$\hat{v}_\tau(\vec{r}) \geqslant \max_{\vec{u} \in \mathscr{U}_{\vec{r}}} \sum_j p_{\tau,j} f_j u_j = v_\tau(\vec{r}),$$

where the equality is the optimality Equation (1) for period $t = \tau$. Now suppose the result is true for $t+1$. Then, for all $\vec{r} \in \mathscr{R}_t$, $\vec{u} \in \mathscr{U}_{\vec{r}}$, by dual feasibility we have

$$\hat{v}_t(\vec{r}) \geqslant \sum_j p_{t,j} f_j u_j + \mathsf{E}[\hat{v}_{t+1}(\vec{R}_{t+1}) \mid \vec{r}, \vec{u}]$$

$$\geqslant \sum_j p_{t,j} f_j u_j + \mathsf{E}[v_{t+1}(\vec{R}_{t+1}) \mid \vec{r}, \vec{u}].$$

In particular,

$$\hat{v}_t(\vec{r}) \geqslant \max_{\vec{u}\in\mathcal{U}_{\vec{r}}} \sum_j p_{t,j} f_j u_j + \mathsf{E}[v_{t+1}(\vec{R}_{t+1}) \mid \vec{r}, \vec{u}] = v_t(\vec{r}),$$

where the last equality is the optimality equation for period t. □

2.2. Standard LP for Bid-Price Control

In general, (1) and **(D0)** are intractable because of the high-dimensional state space, known as *Bellman's curse of dimensionality*. Consequently, the standard approach to revenue management solves a much simpler linear program that ignores time dynamics. Let Y_j denote the expected number of units of class j we plan to satisfy over the finite horizon. Then, the standard linear program for bid-price control is

$$\textbf{(LP)} \quad z_{\mathrm{LP}} = \max_Y \sum_j f_j Y_j, \tag{4}$$

$$\sum_j a_{i,j} Y_j \leqslant c_i \quad \forall\, i, \tag{5}$$

$$0 \leqslant Y_j \leqslant \sum_t p_{t,j} \quad \forall\, j. \tag{6}$$

The dual of **(LP)** is

$$\min_{V,\mu} \sum_i V_i c_i + \sum_j \mu_j \left(\sum_t p_{t,j}\right),$$

$$\sum_i a_{i,j} V_i + \mu_j \geqslant f_j \quad \forall j,$$

$$V, \mu \geqslant 0,$$

where V_i are dual prices on (5) and μ_j are dual prices on the right-hand inequality of (6). Let V_i^* and μ_j^* denote optimal dual prices. The idea of bid-price control is to accept a class j arrival if $f_j \geqslant \sum_i a_{ij} V_i^*$, and reject it otherwise. In practice, **(LP)** is re-solved frequently. Williamson (1992) shows that this policy performs quite well, and in fact this has been a widely used approach to revenue management in industry.

3. Generalized Functional Approximations

We now lay down a general framework for approximately solving **(D0)**. Consider a set of basis functions $\phi_k(\vec{r})$ for all k in some set that indexes them, and approximate

$$v_t(\vec{r}) \approx \hat{v}_t(\vec{r}) = \sum_k V_{t,k} \phi_k(\vec{r}) \quad \forall\, t,\ \vec{r} \in \mathcal{R}_t, \tag{7}$$

where $V_{t,k}$ is a parameter that weights the kth basis function at time t. Note that if we specify a separate basis function for every possible state $\vec{r}$, then we recover the exact model **(D0)**.

The general idea is this: By substituting the approximation (7) into **(D0)**, we restrict it into an optimization problem over only the parameters $V_{t,k}$. If there are relatively few parameters and there is enough structure in the ϕ_k functions, then hopefully the restricted optimization problem will be considerably easier to solve than **(D0)**. From Proposition 1, the resulting approximate value function provides an upper bound on the optimal value function for every state. The approximate linear program in fact finds the lowest upper bound on $v_1(\vec{c})$ of the form (7). Futhermore, we obtain a policy whose performance can be compared with the upper bound using simulation. It sets

$$u_{t,j}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \geqslant A^j, \\ & f_j \geqslant \sum_k V_{t+1,k}(\phi_k(\vec{r}) - \phi_k(\vec{r} - A^j)), \\ 0 & \text{otherwise}, \end{cases} \qquad \forall\, t, j,\ \vec{r} \in \mathcal{R}_t. \tag{8}$$

This is computationally easy to implement provided the given set of basis functions is not too large and the basis functions are easy to evaluate.

Note that this policy does not fit the definition of "bid-price controls" given in Talluri and van Ryzin (1998), yet it is still based on (dual) prices. Indeed, substituting (7) into **(D0)** yields an optimization problem over the parameters $V_{t,k}$,

$$(\mathbf{D}_\phi) \quad \min_V \sum_k V_{1,k} \phi_k(\vec{c}), \tag{9}$$

$$\sum_k (V_{t,k} \phi_k(\vec{r}) - V_{t+1,k} \mathsf{E}[\phi_k(\vec{R}^{t+1}) \mid \vec{r}, \vec{u}]) \geqslant \sum_j p_{t,j} f_j u_j \quad \forall\, t,\ \vec{r} \in \mathcal{R}_t,\ \vec{u} \in \mathcal{U}_{\vec{r}}. \tag{10}$$

Note that

$$\mathsf{E}[\phi_k(\vec{R}^{t+1}) \mid \vec{r}, \vec{u}] = \sum_j p_{t,j} \phi_k(\vec{r} - A^j u_j) + \left(1 - \sum_j p_{t,j}\right) \phi_k(\vec{r}).$$

Its dual is

$$(\mathbf{P}_\phi) \quad Z_\phi = \max_{X \geqslant 0} \sum_{t,\, \vec{r} \in \mathcal{R}_t,\, \vec{u} \in \mathcal{U}_{\vec{r}}} \left(\sum_j p_{t,j} f_j u_j\right) X_{t,\vec{r},\vec{u}}, \tag{11}$$

$$\sum_{\vec{r} \in \mathcal{R}_t,\, \vec{u} \in \mathcal{U}_{\vec{r}}} \phi_k(\vec{r}) X_{t,\vec{r},\vec{u}} = \begin{cases} \phi_k(\vec{c}) & \text{if } t = 1, \\ \displaystyle\sum_{\vec{r} \in \mathcal{R}_{t-1},\, \vec{u} \in \mathcal{U}_{\vec{r}}} \mathsf{E}[\phi_k(\vec{R}^t) \mid \vec{r}, \vec{u}] X_{t-1,\vec{r},\vec{u}} & \forall\, t = 2, \ldots, \tau \end{cases} \quad \forall\, k,\ t. \tag{12}$$

The constraints (12) are basis-weighted flow-balance constraints, whereby flow balance is maintained around each basis function $\phi_k(\cdot)$ separately. The $V_{t,k}$ are dual prices on these constraints, and may be viewed as generalized bid prices.

The problem $(\mathbf{P}_\phi)$ has relatively few constraints if not many basis functions are used but many variables. Thus, it may be well-suited for solution via column generation, provided the subproblems can be solved efficiently. Denote the reduced profit of $X_{t,\vec{r},\vec{u}}$ by

$$\pi_{t,\vec{r},\vec{u}} = \sum_j p_{t,j} f_j u_j - \sum_k (V_{t,k}\phi_k(\vec{r}) - V_{t+1,k}\,\mathsf{E}[\phi_k(\vec{R}^{t+1}) \mid \vec{r}, \vec{u}]).$$

Given an initial set of prices $V_{t,k}$, to generate a new column for $(\mathbf{P}_\phi)$, or prove that none exist with positive reduced profit, we solve

$$\max_{t,\,\vec{r}\in\mathcal{R}_t,\,\vec{u}\in\mathcal{U}_{\vec{r}}} \pi_{t,\vec{r},\vec{u}}.$$

Later we consider a special case in which this subproblem becomes a linear integer program for each t.

Despite the enormous number of variables in $(\mathbf{P}_\phi)$, obtaining an initial feasible solution to begin a column generation procedure is trivial. There exists one supported by only τ decision variables, corresponding with the "offer nothing" policy.

Proposition 2. *A feasible solution to* $(\mathbf{P}_\phi)$ *is*

$$\hat{X}_{t,\vec{r},\vec{u}} = \begin{cases} 1 & \textit{if } \vec{r}=\vec{c},\ \vec{u}=\vec{0}, \\ 0 & \textit{otherwise}, \end{cases} \quad \forall\, t,\ \vec{r}\in\mathcal{R}_t,\ \vec{u}\in\mathcal{U}_{\vec{r}}.$$

Proof. For all t, we have $\vec{c}\in\mathcal{R}_t$ and $\vec{u}=\vec{0}\in\mathcal{U}_{\vec{c}}$. For all t and k, the left-hand side of (12) is

$$\sum_{\vec{r}\in\mathcal{R}_t,\,\vec{u}\in\mathcal{U}_{\vec{r}}} \phi_k(\vec{r})\hat{X}_{t,\vec{r},\vec{u}} = \phi_k(\vec{c})\hat{X}_{t,\vec{c},\vec{0}} = \phi_k(\vec{c}).$$

Likewise, for all $t>1$, the right-hand side is

$$\sum_{\vec{r}\in\mathcal{R}_{t-1},\,\vec{u}\in\mathcal{U}_{\vec{r}}} \mathsf{E}[\phi_k(\vec{R}^t) \mid \vec{r},\vec{u}]\hat{X}_{t-1,\vec{r},\vec{u}} = \phi_k(\vec{c})\hat{X}_{t-1,\vec{c},\vec{0}} = \phi_k(\vec{c}). \quad \square$$

The next result gives an upper bound on the optimality gap between an optimal solution and a given feasible solution. This kind of result is relatively standard, but we specialize it to our functional approximation setting. It is useful because it provides a stopping criterion for a column generation procedure. This result in fact holds for any basic feasible solution X, but we present it in the form in which we will use it, i.e., in terms of columns. Define

$$\pi_t^* = \max_{\vec{r}\in\mathcal{R}_t,\,\vec{u}\in\mathcal{U}_{\vec{r}}} \pi_{t,\vec{r},\vec{u}} \quad \forall\, t$$

to be the maximum reduced profit for period t under V.

Proposition 3. *Suppose that* $\{\phi_k\}_{\forall k}$ *is chosen so that* $\phi_0(\vec{r})=1\ \forall\,\vec{r}$. *Consider the restricted version of* $(\mathbf{P}_\phi)$ *containing only decision variables* $X_{t,\vec{r},\vec{u}}$, *whose indices are in a subset* $\mathcal{C}$ *of all possible indices. Let* $(\tilde{X},\tilde{V})$ *denote the corresponding optimal primal-dual pair for this restricted problem, and let* $\tilde{\pi}_t^*$ *be* π_t^* *computed with respect to* $\tilde{V}$. *Let* $Z_{\mathcal{C}}$ *denote its optimal objective value. Then,*

$$Z_\phi \leqslant Z_{\mathcal{C}} + \sum_{t=1}^{\tau} \tilde{\pi}_t^*.$$

Proof. For $k=0$, (12) becomes

$$\sum_{\vec{r}\in\mathcal{R}_t,\,\vec{u}\in\mathcal{U}_{\vec{r}}} X_{t,\vec{r},\vec{u}} = \begin{cases} 1 & \text{if } t=1, \\ \displaystyle\sum_{\vec{r}\in\mathcal{R}_{t-1},\,\vec{u}\in\mathcal{U}_{\vec{r}}} X_{t-1,\vec{r},\vec{u}} & \forall\, t=2,\ldots,\tau, \end{cases}$$

which implies

$$\sum_{\vec{r}\in\mathcal{R}_t,\,\vec{u}\in\mathcal{U}_{\vec{r}}} X_{t,\vec{r},\vec{u}} = 1 \quad \forall\, t. \tag{13}$$

Consider any X feasible to $(\mathbf{P}_\phi)$, and any numbers $V_{t,k}\ \forall\, t, k$. Multiply both sides of (12) by $V_{t,k}$ for each t,k, and add the resulting equations together with

$$Z(X) = \sum_{t,\vec{r},\vec{u}} \left(\sum_j p_{t,j} f_j u_j\right) X_{t,\vec{r},\vec{u}}.$$

We obtain

$$Z(X) - \sum_k V_{1,k}\phi_k(\vec{c}) = \sum_{t,\vec{r},\vec{u}} \pi_{t,\vec{r},\vec{u}} X_{t,\vec{r},\vec{u}} \leqslant \sum_{t,\vec{r},\vec{u}} \pi_t^* X_{t,\vec{r},\vec{u}} = \sum_t \pi_t^* \left(\sum_{\vec{r},\vec{u}} X_{t,\vec{r},\vec{u}}\right) = \sum_t \pi_t^*,$$

where the last equality follows from (13), and $\pi_{t,\vec{r},\vec{u}}$ is the reduced profit of $X_{t,\vec{r},\vec{u}}$. Because this relation is true for all feasible solutions X, it is true in particular for an optimal solution X^* to $(\mathbf{P}_\phi)$, having objective value $Z(X^*)=Z_\phi$. Furthermore, from strong duality applied to the restricted problem, we have

$$\sum_k \tilde{V}_{1,k}\phi_k(\vec{c}) = Z(\tilde{X}) = Z_{\mathcal{C}}.$$

Hence, we obtain

$$Z_\phi \leqslant Z_{\mathcal{C}} + \sum_{t=1}^{\tau} \tilde{\pi}_t^*,$$

which is the desired result. $\square$

4. Affine Functional Approximation

In this section, we study a specific form of functional approximation (7) with affine basis functions. We give the resulting primal-dual formulations, and then establish their relationship with the standard LP for bid-price control. We then give some results on structural properties satisfied by optimal solutions. While they are interesting in their own right, we will see later that these structural properties also have a dramatic impact on computational times for solving larger-scale problem instances. Lastly, we discuss a column generation algorithm.

4.1. Formulation

Consider the affine functional approximation

$$v_t(\vec{r}) \approx \theta_t + \sum_i V_{t,i} r_i \quad \forall\, t, \vec{r}, \tag{14}$$

where $V_{t,i}$ estimates the marginal value of a unit of resource i in period t, and θ_t is a constant offset. We implicitly assume $\theta_{\tau+1} = 0$ and $V_{\tau+1,i} = 0\ \forall\, i$. This approximation has the form (7), with componentwise basis functions $\phi_i(\vec{r}) = r_i\ \forall\, i$ and the constant basis function $\phi_0(\vec{r}) = 1$.

Under (14), the control policy (2) becomes

$$u_{t,j}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \geqslant A^j,\ f_j \geqslant \sum_i a_{i,j} V_{t+1,i}, \\ 0 & \text{otherwise}, \end{cases} \qquad \forall\, t, j,\ \vec{r} \in \mathscr{R}_t, \tag{15}$$

which has the well-known bid-price control structure (Talluri and van Ryzin 1998).

The linear program $(\mathbf{D}_\phi)$, specialized to this form, becomes

$$(\mathbf{D1}) \quad \min_{\theta, V} \theta_1 + \sum_i V_{1,i} c_i, \tag{16}$$

$$\theta_t - \theta_{t+1} + \sum_i (V_{t,i} r_i - V_{t+1,i}\, \mathsf{E}[R_i^{t+1} \mid \vec{r}, \vec{u}]) \geqslant \sum_j p_{t,j} f_j u_j \quad \forall\, t,\ \vec{r} \in \mathscr{R}_t,\ \vec{u} \in \mathscr{U}_{\vec{r}}. \tag{17}$$

By expanding the expectation, we can rewrite (17) as

$$\theta_t - \theta_{t+1} + \sum_i \left(V_{t,i} r_i - V_{t+1,i} \left(r_i - \sum_j p_{t,j} a_{i,j} u_j \right) \right) \geqslant \sum_j p_{t,j} f_j u_j \quad \forall\, t,\ \vec{r} \in \mathscr{R}_t,\ \vec{u} \in \mathscr{U}_{\vec{r}}.$$

The dual of $(\mathbf{D1})$ is

$$(\mathbf{P1}) \quad z_{\mathrm{P1}} = \max_X \sum_{t,\, \vec{r} \in \mathscr{R}_t,\, \vec{u} \in \mathscr{U}_{\vec{r}}} \left(\sum_j p_{t,j} f_j u_j \right) X_{t,\vec{r},\vec{u}}, \tag{18}$$

$$\sum_{\vec{r} \in \mathscr{R}_t,\, \vec{u} \in \mathscr{U}_{\vec{r}}} r_i X_{t,\vec{r},\vec{u}} = \begin{cases} c_i & \text{if } t = 1, \\ \displaystyle\sum_{\vec{r} \in \mathscr{R}_{t-1},\, \vec{u} \in \mathscr{U}_{\vec{r}}} \left(r_i - \sum_j p_{t-1,j} a_{i,j} u_j \right) \cdot X_{t-1,\vec{r},\vec{u}} & \forall\, t = 2, \ldots, \tau, \end{cases} \quad \forall\, i, t, \tag{19}$$

$$\sum_{\vec{r} \in \mathscr{R}_t,\, \vec{u} \in \mathscr{U}_{\vec{r}}} X_{t,\vec{r},\vec{u}} = \begin{cases} 1 & \text{if } t = 1, \\ \displaystyle\sum_{\vec{r} \in \mathscr{R}_{t-1},\, \vec{u} \in \mathscr{U}_{\vec{r}}} X_{t-1,\vec{r},\vec{u}} & \forall\, t = 2, \ldots, \tau, \end{cases} \tag{20}$$

$$X \geqslant 0.$$

Constraints (19) say that the expected mass of resources having type i flowing into time t must equal that flowing out. The first part of (19) for $t = 1$ can be eliminated by noting $\mathscr{R}_1 = \{\vec{c}\}$ and substituting (20) for $t = 1$ into the left-hand side. Furthermore, (20) can be replaced simply by

$$\sum_{\vec{r} \in \mathscr{R}_t,\, \vec{u} \in \mathscr{U}_{\vec{r}}} X_{t,\vec{r},\vec{u}} = 1 \quad \forall\, t. \tag{21}$$

This allows us to interpret the decision variables $X_{t,\vec{r},\vec{u}}$ as state-action probabilities. (Note, however, that the X solution does not, in general, yield an implementable randomized control policy. The support of X does not specify an action for every attainable state.) The objective function (18) maximizes the approximate total expected revenue over the finite time horizon.

4.2. Relationship with the Standard LP for Bid-Price Control

To derive **(LP)** from **(P1)**, define

$$Y_j \equiv \sum_{t,\, \vec{r} \in \mathscr{R}_t,\, \vec{u} \in \mathscr{U}_{\vec{r}}} (p_{t,j} u_j) X_{t,\vec{r},\vec{u}} \quad \forall\, j. \tag{22}$$

The objective function (4) follows immediately from (18). Now fix i, and sum (19) over time t to obtain

$$\sum_{t,\, \vec{r} \in \mathscr{R}_t,\, \vec{u} \in \mathscr{U}_{\vec{r}}} r_i X_{t,\vec{r},\vec{u}} = c_i + \sum_{t=2,\ldots,\tau,\ \vec{r} \in \mathscr{R}_{t-1},\, \vec{u} \in \mathscr{U}_{\vec{r}}} \left(r_i - \sum_j p_{t-1,j} a_{i,j} u_j \right) X_{t-1,\vec{r},\vec{u}}.$$

Rearranging and canceling terms yields

$$c_i = \sum_j a_{i,j} \sum_{t=1,\ldots,\tau-1,\, \vec{r} \in \mathscr{R}_t,\, \vec{u} \in \mathscr{U}_{\vec{r}}} (p_{t,j} u_j) X_{t,\vec{r},\vec{u}} + \sum_{\vec{r} \in \mathscr{R}_\tau,\, \vec{u} \in \mathscr{U}_{\vec{r}}} r_i X_{\tau,\vec{r},\vec{u}}. \tag{23}$$

Next, we bound the second term on the right-hand side. We first show that for all i, t, $\vec{r} \in \mathcal{R}_t$, $\vec{u} \in \mathcal{U}_{\vec{r}}$,

$$r_i \geqslant \sum_j a_{i,j} p_{t,j} u_j.$$

By definition of $\mathcal{U}_{\vec{r}}$, we have $r_i \geqslant a_{i,j} u_j \ \forall i, j$. Multiply both sides by $p_{t,j}$ for any j to obtain

$$r_i p_{t,j} \geqslant a_{i,j} p_{t,j} u_j.$$

Now sum over j to obtain

$$r_i \geqslant r_i \sum_j p_{t,j} \geqslant \sum_j a_{i,j} p_{t,j} u_j.$$

Hence, (23) implies

$$\begin{aligned} c_i &\geqslant \sum_j a_{i,j} \sum_{t,\, \vec{r} \in \mathcal{R}_t,\, \vec{u} \in \mathcal{U}_{\vec{r}}} (p_{t,j} u_j) X_{t,\vec{r},\vec{u}} \\ &= \sum_j a_{i,j} Y_j, \end{aligned}$$

which yields (5).

To derive the inequalities on the right-hand side of (6), start with (22) to obtain

$$\begin{aligned} Y_j &= \sum_t p_{t,j} \sum_{\vec{r} \in \mathcal{R}_t,\, \vec{u} \in \mathcal{U}_{\vec{r}}} u_j X_{t,\vec{r},\vec{u}} \\ &\leqslant \sum_t p_{t,j} \sum_{\vec{r} \in \mathcal{R}_t,\, \vec{u} \in \mathcal{U}_{\vec{r}}} X_{t,\vec{r},\vec{u}} = \sum_t p_{t,j} \end{aligned} \tag{24}$$

from (21).

Together with Proposition 1, this proves the following theorem.

Theorem 1. *Any feasible solution to* **(P1)** *yields a feasible solution to* **(LP)** *having the same objective value. Hence,* $z_{LP} \geqslant z_{P1} \geqslant v_1(\vec{c})$.

It is well known that z_{LP} is asymptotically optimal, i.e., converges to $v_1(\vec{c})$, as the demand, capacity, and time horizon τ scale linearly (e.g., see Cooper 2002). It follows, then, that z_{P1} is also asymptotically optimal. Also, although the bound obtained from **(P1)** is stronger than that obtained from **(LP)**, this does not theoretically guarantee that the associated policy (15) is better than standard bid-price control based on **(LP)**.

The following example shows that we can have $z_{\text{LP}} > z_{\text{P1}}$.

Example 1. Suppose that there is only one resource and one class, i.e., $m = n = 1$. Hence, we drop the indices i and j. Set $\tau = 2$, $c = 1$, $f = 1$, $a = 1$, and $p_t = 0.5$ for $t = 1, 2$. Without loss of optimality, we can eliminate X variables that set $u = 0$ when $r = 1$. Then, the linear program **(P1)** reduces to

$$\begin{aligned} &\max 0.5(X_{1,1,1} + X_{2,1,1}), \\ &\quad X_{1,1,1} = 1, \\ &\quad X_{2,1,1} = (1 - 0.5) X_{1,1,1}, \\ &\quad X_{2,0,0} + X_{2,1,1} = X_{1,1,1}, \\ &\quad X \geqslant 0, \end{aligned}$$

which trivially has optimal objective value equal to 0.75. In contrast, **(LP)** becomes

$$\begin{aligned} &\max Y, \\ &\quad 0 \leqslant Y \leqslant 1, \end{aligned}$$

with optimal objective value equal to one. The difference arises because, whereas **(LP)** aggregates over all sample paths according to (22), **(P1)** at least partially accounts for sample-path dynamics. As an aside, note that in this case the linear program **(P1)** is exact, meaning that it solves the original optimality Equations (1).

Comparing the objective function (16) with that of the dual of **(LP)** suggests that optimal dual prices from **(LP)** are estimates of the optimal functional approximation (14) computed by **(D1)**, i.e.,

$$V_i^* \approx V_{1,i}^* \quad \forall i,$$

$$\sum_j \mu_j^* \left(\sum_t p_{t,j} \right) \approx \theta_1^*.$$

This formalizes the heuristic interpretation of Talluri and van Ryzin (1998) that

$$V_1(\vec{r}) - V_1(\vec{r} - A^j) \approx \sum_i V_{1,i}^* a_{ij}.$$

Note, however, that (15) indicates the prices from time period 2, rather than 1, are relevant for the decision in time period 1.

In the appendix, we show that the above analysis can be carried out with a lower-fidelity, quasi-static value function approximation. The resource prices do not depend on time, as they do here, and so the upper bound is weaker.

4.3. Structural Properties

Given a feasible solution X to **(P1)**, define the first time resource i is used by

$$t_i^* = \operatorname{argmin}\{t \in \{1, \ldots, \tau\} \colon \exists j, \vec{r} \in \mathcal{R}_t, \vec{u} \in \mathcal{U}_{\vec{r}} \text{ with } X_{t,\vec{r},\vec{u}} > 0, u_j = 1, a_{i,j} > 0\} \quad \forall i. \tag{25}$$

It is not necessarily the case that $t_i^* = 1$ in an optimal solution X^*. For example, resource i's consumption may simultaneously require, under the matrix A, the unprofitable consumption of other resources.

Lemma 1. *Fix i. Assume that t_i^* exists (which implies $c_i > 0$) and $p_{t,j} > 0 \ \forall t, j$. Then,*
(1) *$\forall t > t_i^*$, $\exists X_{t,\vec{r},\vec{u}} > 0$ such that $r_i < c_i$, and*
(2) *$\forall t \leqslant t_i^*$, $X_{t,\vec{r},\vec{u}} > 0$ only if $r_i = c_i$.*

Proof. We start by showing the first part. Assume that $t_i^* < \tau$. Let $\vec{r'}$ and $\vec{u'}$ be such that $X_{t_i^*,\vec{r'},\vec{u'}} > 0$ and there exists an associated j' as in the definition of t_i^*. Observe that by definition of t_i^*, we have the strict inequality

$$r_i' - \sum_j p_{t_i^*, j} a_{i,j} u_j' < r_i' \leqslant c_i.$$

For period $t_i^* + 1$, (19) therefore reads

$$\begin{aligned}\sum_{\vec{r},\vec{u}} r_i X_{t_i^*+1,\vec{r},\vec{u}} &= \sum_{\vec{r},\vec{u}} \left(r_i - \sum_j p_{t_i^*,j} a_{i,j} u_j\right) X_{t_i^*,\vec{r},\vec{u}} \\ &\leqslant \left(r_i' - \sum_j p_{t_i^*,j} a_{i,j} u_j'\right) X_{t_i^*,\vec{r}',\vec{u}'} \\ &\quad + \sum_{\vec{r}\neq\vec{r}',\vec{u}\neq\vec{u}'} c_i X_{t_i^*,\vec{r},\vec{u}} < c_i\end{aligned}$$

because of (20) and $X_{t_i^*,\vec{r}',\vec{u}'} > 0$. If $r_i = c_i$ for all $\vec{r}$, $\vec{u}$ such that $X_{t_i^*+1,\vec{r},\vec{u}} > 0$, then by (20), the left-hand side equals c_i, a contradiction. The desired result for all $t > t_i^*$ follows by applying the same argument recursively.

The second claim is true for $t = 1$ by definition of $\mathscr{R}_1$. Now consider any $1 < t \leqslant t_i^*$, and suppose that the statement is true for $t-1$. Then, (19) reads

$$\sum_{\vec{r},\vec{u}} r_i X_{t,\vec{r},\vec{u}} = \sum_{\vec{r},\vec{u}} r_i X_{t-1,\vec{r},\vec{u}} = c_i \sum_{\vec{r},\vec{u}} X_{t-1,\vec{r},\vec{u}} = c_i$$

because of (20). Suppose that $r_i' < c_i$ for some $\vec{r}'$, $\vec{u}'$ such that $X_{t,\vec{r}',\vec{u}'} > 0$. Then, the left-hand side becomes

$$\begin{aligned}\sum_{\vec{r},\vec{u}} r_i X_{t,\vec{r},\vec{u}} &= \sum_{\vec{r}\neq\vec{r}',\vec{u}\neq\vec{u}'} r_i X_{t,\vec{r},\vec{u}} + r_i' X_{t,\vec{r}',\vec{u}'} \\ &< c_i \sum_{\vec{r}\neq\vec{r}',\vec{u}\neq\vec{u}'} X_{t,\vec{r},\vec{u}} + c_i X_{t,\vec{r}',\vec{u}'} = c_i,\end{aligned}$$

which is a contradiction. The desired result follows by induction. □

Theorem 2. *Assume that $c_i > 0\ \forall i$ and $p_{t,j} > 0\ \forall t, j$. There exists an optimal dual solution (θ^*, V^*) of* **(D1)** *and a set of indices $\{\tilde{t}_i^*\}_{\forall i}$ such that*

$$\theta_t^* \geqslant \theta_{t+1}^* \quad \forall t, \tag{26}$$

$$V_{t,i}^* = V_{t+1,i}^* \quad \forall i,\ t = 1, \ldots, \tilde{t}_i^* - 1, \tag{27}$$

$$V_{t,i}^* \geqslant V_{t+1,i}^* \quad \forall i,\ t = \tilde{t}_i^*, \ldots, \tau, \tag{28}$$

$$V^*, \theta^* \geqslant 0. \tag{29}$$

Proof. Suppose that t_i^*, as defined in (25), exists. Consider $\tilde{t}_i^* = t_i^*$. We later consider the case where t_i^* does not exist. We first focus attention on (27) and (28), which we prove in three steps:

(1) $V_{t,i}^* \geqslant V_{t+1,i}^*\ \forall t > t_i^*$.
(2) $V_{t,i}^* \leqslant V_{t+1,i}^*\ \forall t < t_i^*$.
(3) If $V_{t,i}^* < V_{t+1,i}^*$ for some $t \leqslant t_i^*$, then we can raise $V_{t,i}^*$ to $V_{t+1,i}^*$ without loss of optimality.

Finally, we prove the remaining claims.

Step 1. Assume that $t_i^* < \tau$. Consider an optimal primal-dual solution X^*, (V^*, θ^*). From Lemma 1, there exists an $X_{t,\vec{r},\vec{u}}^* > 0$ for which $r_i < c_i$. Therefore, complementary slackness implies that

$$\begin{aligned}0 = \theta_{t+1}^* - \theta_t^* + \sum_j p_{t,j}\left[f_j - \sum_i a_{i,j} V_{t+1,i}^*\right] u_j \\ - \sum_i (V_{t,i}^* - V_{t+1,i}^*) r_i.\end{aligned} \tag{30}$$

Consider a new $\vec{r}'$, equal to $\vec{r}$ except that $r_i' = r_i + 1 \leqslant c_i$. Note that $\vec{u}$ is still feasible, i.e., $\vec{u} \in \mathscr{U}_{\vec{r}'}$. By dual feasibility (17),

$$\begin{aligned}0 \geqslant \theta_{t+1}^* - \theta_t^* + \sum_j p_{t,j}\left[f_j - \sum_i a_{i,j} V_{t+1,i}^*\right] u_j \\ - \sum_{i'\neq i} (V_{t,i'}^* - V_{t+1,i'}^*) r_{i'} - (V_{t,i}^* - V_{t+1,i}^*)(r_i + 1),\end{aligned}$$

which yields the desired result, i.e., (28) for $t > t_i^*$, when combined with (30).

Step 2. We now show that $V_{t,i}^* \leqslant V_{t+1,i}^*$ for all $t < t_i^*$. Equation (20) implies that there exists an $X_{t,\vec{r},\vec{u}}^* > 0$ for some $\vec{r}$ and $\vec{u}$, and from Lemma 1 it has the property that $r_i = c_i$. By complementary slackness, we then have

$$\begin{aligned}0 = \theta_{t+1}^* - \theta_t^* + \sum_j p_{t,j}\left[f_j - \sum_i a_{i,j} V_{t+1,i}^*\right] u_j \\ - \sum_{i'\neq i} (V_{t,i'}^* - V_{t+1,i'}^*) r_{i'} - (V_{t,i}^* - V_{t+1,i}^*) c_i.\end{aligned} \tag{31}$$

Now consider a new $\vec{r}'$ equal to $\vec{r}$ except $r_i' = c_i - 1 \geqslant 0$. Because, by definition of t_i^*, $u_j = 0$ for all j such that $a_{i,j} > 0$, we know that $\vec{u}$ is still feasible, i.e., $\vec{u} \in \mathscr{U}_{\vec{r}'}$. By dual feasibility (19), we have

$$\begin{aligned}0 \geqslant \theta_{t+1}^* - \theta_t^* + \sum_j p_{t,j}\left[f_j - \sum_i a_{i,j} V_{t+1,i}^*\right] u_j \\ - \sum_{i'\neq i} (V_{t,i'}^* - V_{t+1,i'}^*) r_{i'} - (V_{t,i}^* - V_{t+1,i}^*)(c_i - 1).\end{aligned}$$

Combining the last two displays yields $V_{t,i}^* \leqslant V_{t+1,i}^*$.

Step 3. Given an optimal primal-dual solution X^*, (θ^*, V^*), consider the largest $t \leqslant t_i^*$ such that $V_{t,i}^* < V_{t+1,i}^*$. Note that if $V_{t_i^*,i}^* \geqslant V_{t_i^*+1,i}^*$, then (28) is satisfied for $t = t_i^*$. We will alter the optimal dual solution to produce another solution that achieves (27) (or (28) if $t = t_i^*$), yet is still optimal because it preserves dual feasibility and complementary slackness. Denote the modified solution by (θ', V'), and define it to be the same as (θ^*, V^*) except

$$V_{t,i}' = V_{t+1,i}^*,$$

$$\theta_t' = \theta_t^* + (V_{t,i}^* - V_{t+1,i}^*) c_i.$$

We first check the period t inequalities (17) for dual feasibility. Indeed, for any $\vec{r}$ and $\vec{u}$, the reduced profit is

$$\begin{aligned}&\theta^*_{t+1}-\theta'_t+\sum_j p_{t,j}\left[f_j-\sum_i a_{i,j}V^*_{t+1,i}\right]u_j\\&\quad-\sum_{i'\neq i}(V^*_{t,i'}-V^*_{t+1,i'})r_{i'}-(V'_{t,i}-V^*_{t+1,i})r_i\\&=\theta^*_{t+1}-\theta^*_t-(V^*_{t,i}-V^*_{t+1,i})c_i+\sum_j p_{t,j}\left[f_j-\sum_i a_{i,j}V^*_{t+1,i}\right]u_j\\&\quad-\sum_{i'\neq i}(V^*_{t,i'}-V^*_{t+1,i'})r_{i'}\leqslant 0.\end{aligned} \tag{32}$$

The inequality follows because the second expression is the reduced profit, under the original dual feasible solution (θ^*, V^*), for a (potentially) modified $\vec{r}$ having $r_i = c_i$ and $\vec{u}$ unchanged (yet still feasible). In particular, consider any $\vec{r}$, $\vec{u}$, for which $X^*_{t,\vec{r},\vec{u}} > 0$. Because $t \leqslant t^*_i$, from the argument above $r_i = c_i$ and (31) holds under (θ^*, V^*). Note that the expression in (32) is the same as (31), i.e., the reduced profit is the same under (θ', V') as under (θ^*, V^*), and so complementary slackness is preserved.

Next, we check dual feasibility and complementary slackness for the period $t-1$ inequalities (17), assuming that $t > 1$. For any $\vec{r}$, $\vec{u}$, (17) reads

$$\begin{aligned}&\theta^*_t-\theta^*_{t-1}+\sum_j p_{t-1,j}\left[f_j-\sum_i a_{i,j}V^*_{t,i}\right]u_j\\&\quad-\sum_{i'\neq i}(V^*_{t-1,i'}-V^*_{t,i'})r_{i'}-(V^*_{t-1,i}-V^*_{t,i})r_i\leqslant 0.\end{aligned}$$

Under (θ', V'), the reduced profit becomes

$$\begin{aligned}&\theta^*_t+(V^*_{t,i}-V^*_{t+1,i})c_i-\theta^*_{t-1}\\&\quad+\sum_j p_{t-1,j}\left[f_j-\sum_{i'\neq i}a_{i',j}V^*_{t,i'}-a_{i,j}V^*_{t+1,i}\right]u_j\\&\quad-\sum_{i'\neq i}(V^*_{t-1,i'}-V^*_{t,i'})r_{i'}-(V^*_{t-1,i}-V^*_{t+1,i})r_i.\end{aligned}$$

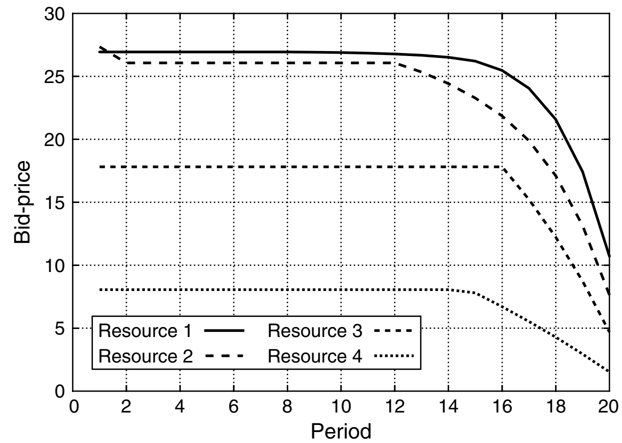
Subtracting the first reduced profit from the second yields

$$(V^*_{t,i}-V^*_{t+1,i})\left[(c_i-r_i)+\sum_j p_{t-1,j}a_{i,j}u_j\right]\leqslant 0;$$

hence, dual feasibility is preserved. Furthermore, for any $X^*_{t-1,\vec{r},\vec{u}} > 0$, we know that $r_i = c_i$ and $u_j = 0$ for all j such that $a_{i,j} > 0$. Therefore, the last inequality changes to equality, i.e., complementary slackness is preserved.

The modified variables $V'_{t,i}$ and θ'_t appear nowhere else besides where considered above, and so overall dual feasibility and complementary slackness is preserved. Now apply the same argument successively for each smaller t. Because $V^*_{t,i} \leqslant V^*_{t+1,i}$ (from Step 2 above), the procedure achieves (27).

Figure 1. Example dynamic bid-price trajectories.



Final details. If no t^*_i exists, then Step 2 of the argument above shows that $V^*_{t,i} \leqslant V^*_{t+1,i}$ for all $1 \leqslant t \leqslant \tau$. The variable modification procedure in Step 3 can still be applied to achieve $V^*_{t+1,i} = V^*_{t,i}$ for all t. Because $V^*_{\tau+1,i} = 0$, it follows that $V^*_{t,i} = 0$ for all t, hence, (27) holds for $\tilde{t}^*_i = \tau$.

The nonnegativity of $V^*_{t,i}$ in (29) follows from (28) and $V^*_{\tau+1,i} = 0$. The time monotonicity of θ^*_t, given in (26), follows directly from dual feasibility (17), because $\vec{r} = 0$ and $\vec{u} = 0$ is a feasible state-action pair. The nonnegativity of θ in (29) follows from (26) and the fact that $\theta^*_{\tau+1} = 0$. □

The following corollary follows directly from the time monotonicity of $V^*_{t,i}$ and $V^*_{\tau+1,i} = 0$.

Corollary 1. *There exists an optimal dual solution* (θ^*, V^*) *of* **(D1)** *and a set of* time thresholds $\{\sigma^*_j\}_{\forall j}$ *such that for each class* j,

$$f_j < \sum_i a_{i,j}V^*_{t+1,i} \quad \forall t = 1, \ldots, \sigma^*_j - 1,$$

$$f_j \geqslant \sum_i a_{i,j}V^*_{t+1,i} \quad \forall t = \sigma^*_j, \ldots, \tau.$$

The corresponding control policy (15) *is therefore*

$$u_{t,j}(\vec{r}) = \begin{cases}0 & \text{if } t < \sigma^*_j,\\ 1 & \text{if } t \geqslant \sigma^*_j \text{ and } \vec{r} \geqslant A^j,\end{cases} \qquad \forall t, j, \ \vec{r} \in \mathcal{R}_t.$$

Figure 2. Column generation algorithm for solving **(P1)** to within an optimality tolerance of Ω.

Algorithm Column generation

Set $\mathscr{C} = \{(t, \vec{c}, \vec{0})\ \forall t\}$, solve the restricted problem **(P1**($\mathscr{C}$)**)**, and set $\pi^*_t = \infty$ for all t.
while $\sum_t \pi^*_t > \Omega Z_{\mathscr{C}}$ do
  for all $t \in \{1, \ldots, \tau\}$
    compute $\pi^*_t = \max_{\vec{r},\vec{u}} \pi_{t,\vec{r},\vec{u}}$
    select an $(\vec{r}_t, \vec{u}_t) \in \arg\max_{\vec{r},\vec{u}} \pi_{t,\vec{r},\vec{u}}$
    update $\mathscr{C} \leftarrow \mathscr{C} \cup \{(t, \vec{r}_t, \vec{u}_t)\}$.
  solve **(P1**($\mathscr{C}$)**)**

Table 1. Dimensions of test instances.

No. of nonhub locations (L)	No. of legs (m)	No. of classes (n)
2	4	12
5	10	60
10	20	220
20	40	840

One might then interpret **(P1)** as searching over policies having the above form, but evaluating them only approximately. In the single resource-type problem ($m = 1$), it is well known that a time threshold policy is optimal. However, the time thresholds depend on the current state, whereas here they only depend on the initial state $\vec{c}$. Bertsimas and Popescu (2003) argue that the time threshold property does not hold in the network case. This reinforces how **(P1)** is still an approximation. Note that **(LP)** exhibits no notion of time thresholds, as it is too coarse.

Figure 1 shows dynamic bid-price trajectories for each of four resources in a problem instance with 20 periods. These were obtained by solving **(D1)** to optimality. The trajectories are monotone, without loss of optimality as stated by Theorem 2. Intuitively, the marginal value of a resource decreases over time because the opportunities to use it are fewer. In fact, after the horizon, $V_{\tau+1,i} = 0$ for all i. Note that the curves exhibit nonlinearity toward the end of the horizon. In the beginning of the horizon, the curves tend to be flat until an inflection point is reached. Theorem 2 shows that sometime at or before that point is reached, the resource is first used (i.e., at time t_i^*).

4.4. Column Generation

In passing from **(D0)** to **(D1)**, we have reduced the number of decision variables from

$$\sum_t |\mathscr{R}_t| = 1 + (\tau - 1) \cdot \prod_{i=1}^{m} (c_i + 1),$$

which is exponential in m, down to $\tau \cdot (m+1)$, a polynomial in m. However, there is still an exponential number of constraints. Hence, the primal program **(P1)** is potentially suited for solution via column generation.

To solve **(P1)** using column generation, first price out an initial primal feasible solution. This is supplied by Proposition 2. Denoting the resulting prices by V, θ, now solve

$$\max_{t,\, \vec{r} \in \mathscr{R}_t,\, \vec{u} \in \mathscr{U}_{\vec{r}}} \pi_{t,\vec{r},\vec{u}} = \max_{t,\, \vec{r} \in \mathscr{R}_t,\, \vec{u} \in \mathscr{U}_{\vec{r}}} \sum_j p_{t,j} \left[f_j - \sum_i a_{i,j} V_{t+1,i} \right] u_j - \sum_i (V_{t,i} - V_{t+1,i}) r_i - \theta_t + \theta_{t+1},$$

which maximizes the reduced profit from (17). If the objective value is greater than zero, then we add the column to the existing set of columns for **(P1)**; otherwise, we have attained optimality. For fixed $t > 1$, this is equivalent to solving the linear integer program

$$\max_{\vec{u},\vec{r}} \sum_j p_{t,j} \left[f_j - \sum_i a_{i,j} V_{t+1,i} \right] u_j - \sum_i (V_{t,i} - V_{t+1,i}) r_i - \theta_t + \theta_{t+1}, \tag{33}$$

$$a_{i,j} u_j \leqslant r_i \quad \forall\, i, j, \tag{34}$$

$$u_j \in \{0, 1\} \quad \forall\, j, \tag{35}$$

$$r_i \in \{0, \ldots, c_i\} \quad \forall\, i. \tag{36}$$

Note that if $a_{i,j}$ are integral, then without loss of optimality, we can relax the integrality of r_i. This is because fixing $\vec{u}$, in an extreme point solution either $r_i = \max_j \{a_{i,j} u_j\}$ or $r_i = c_i$. Furthermore, fixing $\vec{r}$, a $\vec{u}$ solution is given by (15). We believe that because of this structure, we find empirically that CPLEX can efficiently solve these subproblems to optimality.

We present the full algorithm in Figure 2, where **(P1($\mathscr{C}$))** denotes the restricted version of **(P1)** with columns coming only from $\mathscr{C}$, and $Z_{\mathscr{C}}$ denotes the corresponding optimal objective value. This is a standard algorithm, except that we enter a batch of columns at once, potentially one column for each time period. We found this to be much more effective than adding one column at a time.

As a consequence of Proposition 3, to ensure that the objective value of the current solution $\tilde{X}$ based on columns $\mathscr{C}$ is within Ω percent of an optimal solution, i.e.,

Table 2. Capacity and load statistics for large test instances.

	No. of nonhub locations (L)							
	2		5		10		20	
τ	Capacity per leg	Load factor	Capacity per leg	Load factor	Capacity per leg	Load factor	Capacity per leg	Load factor
20	3	1.68	1	2.66	1	1.45	1	0.76
50	7	1.79	4	1.66	2	1.81	2	0.95
100	15	1.68	8	1.66	4	1.81	2	1.90
200	31	1.62	16	1.66	9	1.61	4	1.90
500	79	1.59	42	1.58	22	1.65	12	1.59
1,000	159	1.58	84	1.58	45	1.61	24	1.59

Table 3. CPU seconds to solve **(P1)** with and without monotonicity constraints.

	No. of nonhub locations (L)							
	2		5		10		20	
τ	Mono	w/o Mono	Mono	w/o Mono	Mono	w/o Mono	Mono	w/o Mono
20	0.08	0.18	1.26	6.04	16.49	34.05	252.82	962.69
50	0.16	0.42	1.28	201.09	21.24	117.65	123.99	3,791.49
100	0.29	0.89	3.41	23.03	33.20	218.68	650.28	9,414.89
200	0.72	2.46	8.76	236.10	84.58	499.02	1,184.89	17,300.80
500	2.36	19.40	23.66	3,166.94	380.11	1,877.90	4,361.81	58,925.90
1,000	4.63	136.51	72.26	1,440.47	1,183.51	4,737.61	11,682.10	>84,000

$Z_\phi/Z_{\mathscr{C}} \leqslant 1+\Omega$, it suffices to ensure that

$$\frac{\sum_t \tilde{\pi}_t^*}{Z_{\mathscr{C}}} \leqslant \Omega.$$

This is merely sufficient; for the solution $\widetilde{X}$, the ratio $Z_\phi/Z_{\mathscr{C}}$ in fact may be closer to one than $1+\Omega$. We employ this as a stopping criterion for the algorithm.

We also considered more complicated versions of this basic algorithm. In one version, whenever a $(t, \vec{r}, \vec{u})$ was found for which $X_{t,\vec{r},\vec{u}}$ has positive reduced profit, we added this index to $\mathscr{C}$ for all t' for which $X_{t',\vec{r},\vec{u}}$ has positive reduced profit. This avoided having to solve as many integer programming subproblems, but this benefit was outweighed by having to solve more linear programs containing more columns. It was better to add columns with maximum reduced profit.

5. Numerical Results

We conducted numerical experiments to study three questions:

(1) How long does it take to solve instances of practical size?

(2) How much stronger is the upper bound given by **(P1)** as compared with **(LP)**?

(3) How well does the approximate policy perform in simulations, as compared with the upper bound given by **(P1)** and standard bid-price control?

Table 4. CPU seconds to solve **(P1)** with monotonicity constraints for the same instances except for half the load factor.

	No. of nonhub locations (L)			
τ	2	5	10	20
20	0.03	0.56	2.55	2.98
50	0.08	0.56	5.73	5.79
100	0.14	1.14	7.85	195.9
200	0.52	2.19	10.9	285.08
500	1.46	8.27	28.01	112.77
1,000	3.24	14.79	67.48	373.92

In our first set of experiments, we randomly generated 24 problem instances of varying complexity, having time horizons τ in the set $\{20, 50, 100, 200, 500, 1{,}000\}$. We considered an airline servicing L locations out of a single hub, where $L \in \{2, 5, 10, 20\}$. This is an important, basic network structure of revenue management problems found in the airline industry. Each location $l \in 1, \ldots, L$ is associated with two flight legs, i.e., to and from the hub, each departing at the end of the time horizon τ. Hence, there are $m = 2L$ resources. There are m single-leg itineraries and $L \cdot (L-1)$ two-leg itineraries. For each itinerary, we designate a high-fare class and a low-fare class, so that the total number of classes is $n = 2(m + L \cdot (L-1))$. Table 1 gives the resulting dimensions of our test instances. The dimensions of the largest instances are of real-world size.

We generated the revenue for each low-fare class from a discrete uniform distribution on the interval $[15, 49]$, and set the high fare for the same itinerary equal to five times that of the low fare. For simplicity, we considered stationary demand with the probability 0.2 for having no customer arrival in a period, and the other probabilities generated randomly. For each itinerary, we split the demand so that 25% was for the high-fare class, and 75% was for the low-fare class. For each instance, we set the initial capacity (seats), c, to be the same for each leg. Table 2 displays the capacity per leg and load factor for each instance. The load factor is defined as the expected demand for seat legs divided by the supply, i.e.,

$$\text{load factor} = \sum_{j,i,t} p_{t,j} a_{i,j} \Big/ cm.$$

Table 5. Approximate relative difference in upper bounds, **(LP)**/**(P1)**.

	No. of nonhub locations (L)			
τ	2	5	10	20
20	1.081	1.485	1.484	1.404
50	1.036	1.075	1.164	1.156
100	1.024	1.044	1.082	1.178
200	1.006	1.033	1.045	1.091
500	1.003	1.028	1.032	1.043
1,000	1.001	1.012	1.023	1.035

Table 6. Bound and policy results for stationary demand.

L	τ	Capacity per leg	Load factor	**(P1)** Bound	**(LP)** Bound	**BPC fixed** Mean (Std. err.)	**DBPC fixed** Mean (Std. err.)	**BPC** Mean (Std. err.)	**DBPC** Mean (Std. err.)
3	20	2	1.95	683.57	776.69	503.34 (17.15)	561.13 (21.55)	477.55 (17.50)	567.78 (20.54)
	50	6	1.62	1,987.84	2,065.04	1,510.57 (29.43)	1,661.67 (30.11)	1,592.68 (30.82)	1,759.91 (33.76)
	100	12	1.62	4,052.97	4,130.08	3,206.07 (40.21)	3,392.78 (56.83)	3,395.68 (44.25)	3,730.04 (53.87)
	200	24	1.62	8,179.11	8,260.16	6,560.23 (51.95)	6,956.47 (79.33)	7,018.59 (63.86)	7,683.04 (70.09)
	500	61	1.59	20,690.90	20,768.40	16,945.30 (93.15)	18,265.00 (100.23)	18,280.30 (118.10)	19,793.50 (132.23)
5	20	1	2.66	510.13	752.54	428.87 (17.29)	481.40 (17.82)	403.46 (16.13)	486.92 (17.86)
	50	4	1.66	2,072.87	2,202.42	1,686.61 (34.08)	1,816.97 (38.89)	1,657.13 (31.75)	1,874.34 (36.70)
	100	8	1.66	4,275.85	4,404.84	3,475.36 (50.13)	3,725.09 (52.24)	3,463.80 (49.51)	3,905.97 (50.49)
	200	16	1.66	8,677.93	8,809.68	7,277.38 (70.99)	7,624.10 (81.76)	7,309.92 (69.24)	8,109.55 (73.00)
	500	42	1.58	22,204.90	22,340.40	18,857.30 (119.21)	20,587.60 (132.26)	19,362.60 (123.96)	21,189.10 (125.73)

In Table 3, we report the CPU seconds required to solve **(P1)** guaranteed to within $\Omega = 5\%$, which is a practical tolerance given the data uncertainty encountered in the real world. We ran these instances on an Intel Xeon CPU running at 3.6 GHz with 3 GB of memory, using CPLEX 9.1. Adding monotonicity constraints to **(D1)** had a dramatic impact on the solution times, and in fact makes solution practical on large-scale instances. The improvement was on average by a factor of (at least) 21.32 times, with a range between 2.06 and 157.10 times. For the largest instance, with $L = 20$ and $\tau = 1{,}000$, the model solved in 3.25 hours with monotonicity constraints. Without them the model did not solve after one day, and so we terminated the algorithm.

Table 4 shows CPU times, with monotonicity constraints, when the load factor is cut in half by doubling the capacity c per leg. On average, the speedup was by a factor of 11.14, with a range between 1.38 and 84.84. There was a marked increase in speedup as the size of the network increased. This speedup can be explained by the fact that as the load factor decreases, the policy that always accepts tends to be closer to being optimal. The algorithm quickly generates the corresponding columns, and there is effectively a complexity reduction because other combinations of accept/reject decisions are less important.

In Table 5, we report the (approximate) relative difference in upper bounds between the standard linear program **(LP)** and our **(P1)**, i.e., $z_{\mathrm{LP}}/z_{\mathrm{P1}}$. The numbers reported in the table correspond with **(P1)** solved to an optimality tolerance of $\Omega = 5\%$, which does not strictly give an upper bound. However, when we solved **(P1)** to within $\Omega = 0.5\%$, the improvement in the objective value of **(P1)** was on average only about 1%, and this was achieved with an increase in CPU time by a factor of about five, on average. Therefore, the differences reported in the table are representative; they are also consistent with similar results that follow.

The gap between the bounds can be quite substantial, up to 48.5% on these instances. Two trends are noticeable. As the number of locations increases—i.e., the network becomes more complex—the gap in the bounds increases. Also, as the time horizon increases, the gap in the bounds decreases. On these instances, the capacity and demand scale up linearly as the time horizon increases. As a consequence of the asymptotic result in Cooper (2002), **(LP)** becomes asymptotically optimal, and hence so does **(P1)** due to Theorem 1.

We next considered policy performance. It has been shown by Williamson (1992) that static bid-price control, using **(LP)**, performs quite well when the bid prices are reoptimized frequently. This motivates the following policies, where the acronym DBPC stands for dynamic bid-price control.

- **DBPC fixed**: Solve **(P1)**–**(D1)** once. Given a set of fixed dynamic bid prices, use the policy given by (15).
- **DBPC**: Solve **(P1)**–**(D1)** multiple times spread evenly over the time horizon τ. Between solution epochs, use the policy given by (15).

Table 7. Bound and policy results for hi-lo demand.

L	τ	Capacity per leg	Load factor	**(P1)** Bound	**(LP)** Bound	**BPC fixed** Mean (Std. err.)	**DBPC fixed** Mean (Std. err.)	**BPC** Mean (Std. err.)	**DBPC** Mean (Std. err.)
3	20	1	2.52	312.54	460.47	209.10 (11.29)	263.03 (15.21)	218.86 (12.26)	265.37 (14.67)
	50	3	1.70	965.97	1,065.52	672.78 (19.42)	796.61 (30.61)	718.95 (22.54)	843.53 (27.57)
	100	5	1.74	1,704.30	1,803.52	1,246.27 (25.30)	1,447.50 (41.96)	1,333.62 (27.35)	1,556.42 (36.27)
	200	9	1.66	3,053.65	3,146.57	2,306.44 (38.59)	2,623.57 (53.85)	2,520.26 (44.52)	2,810.35 (50.54)
	500	19	1.63	6,480.50	6,575.19	4,961.15 (56.90)	5,533.88 (71.44)	5,459.09 (62.14)	5,915.06 (68.92)
5	20	1	1.72	374.73	563.77	292.70 (13.96)	333.58 (15.59)	287.19 (14.99)	331.61 (15.76)
	50	2	1.74	971.23	1,137.36	733.27 (22.42)	798.72 (26.92)	691.03 (22.78)	814.90 (25.41)
	100	3	1.98	1,683.69	1,862.10	1,201.05 (30.65)	1,410.97 (37.25)	1,193.96 (31.30)	1,449.24 (34.71)
	200	6	1.70	3,207.23	3,358.20	2,452.19 (42.34)	2,710.30 (44.88)	2,429.71 (42.97)	2,830.33 (46.58)
	500	13	1.63	6,919.78	7,068.77	5,465.29 (57.96)	5,905.84 (65.99)	5,701.03 (63.79)	6,312.69 (69.05)

Table 8. Bound and policy results for lo-hi demand.

L	τ	Capacity per leg	Load factor	**(P1)** Bound	**(LP)** Bound	**BPC fixed** Mean (Std. err.)	**DBPC fixed** Mean (Std. err.)	**BPC** Mean (Std. err.)	**DBPC** Mean (Std. err.)
3	20	1	2.52	326.17	460.47	259.49 (13.52)	298.61 (15.10)	271.52 (13.88)	296.40 (15.22)
	50	3	1.70	989.61	1,065.52	768.86 (23.46)	817.28 (29.15)	754.46 (24.01)	843.06 (26.79)
	100	5	1.74	1,732.14	1,803.52	1,389.86 (28.94)	1,496.75 (37.14)	1,366.83 (32.25)	1,558.59 (34.96)
	200	9	1.66	3,081.15	3,146.57	2,571.99 (41.88)	2,671.87 (53.63)	2,568.12 (39.70)	2,849.84 (48.54)
	500	19	1.63	6,509.29	6,575.19	5,414.64 (51.39)	5,847.40 (73.90)	5,507.38 (57.25)	6,020.40 (62.95)
5	20	1	1.72	374.73	563.77	323.61 (15.58)	348.72 (18.13)	298.25 (14.88)	347.16 (18.15)
	50	2	1.74	971.23	1,137.36	801.41 (23.59)	832.80 (26.57)	725.22 (24.56)	830.21 (26.64)
	100	3	1.98	1,683.69	1,862.10	1,299.61 (28.37)	1,362.37 (34.40)	1,221.30 (28.38)	1,396.68 (33.91)
	200	6	1.70	3,207.23	3,358.20	2,673.54 (35.13)	2,800.45 (44.67)	2,537.58 (36.62)	2,932.21 (39.22)
	500	13	1.63	6,919.78	7,068.77	5,947.20 (61.09)	6,209.08 (67.22)	5,700.24 (62.40)	6,474.42 (66.74)

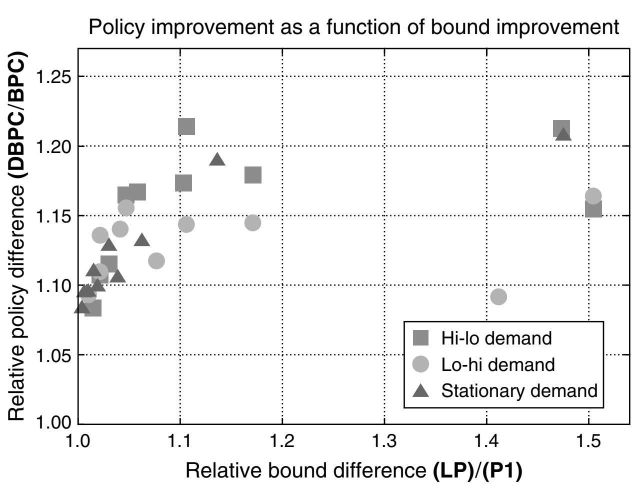
- **BPC fixed**: Solve **(LP)** once. Given a fixed set of static bid prices, use the policy given in §2.2.
- **BPC**: Solve **(LP)** multiple times spread evenly over the time horizon τ. Between solution epochs, use the policy given in §2.2.

Unless specified otherwise, we re-solve **(P1)–(D1)** and **(LP)** five times over the time horizon.

We noticed a small improvement in the DBPC policies when we used a simple single-period lookahead instead of (15). Let $\hat{v}$ denote the affine approximation (14), and $\hat{u}$ denote policy (15) under $\hat{v}$. Instead of using approximation (14) in policy (2), we instead approximate $v_{t+1}(\vec{r})$ and $v_{t+1}(\vec{r} - A^j)$ by looking ahead to period $t+1$, applying policy $\hat{u}$ in that period but evaluating the subsequent state in period $t+2$ using the affine approximation $\hat{v}$. Thus, we use the approximation

$$\tilde{v}_{t+1}(\vec{r}) = \sum_{j'} p_{t+1,\,j'}[f_{j'} + \hat{v}_{t+2}(\vec{r} - A^{j'})]\hat{u}_{t+1,\,j'}(\vec{r}) + \left(1 - \sum_{j'} p_{t+1,\,j'}\right)\hat{v}_{t+2}(\vec{r}) \quad \forall\, t \leqslant \tau - 2,\, \vec{r}.$$

Figure 3. Relationship between relative bound improvement and policy improvement.



We also noticed a small improvement in the DBPC policies when we required the revenue f_j to be strictly greater than the opportunity cost as measured by $\tilde{v}_{t+1}(\vec{r}) - \tilde{v}_{t+1}(\vec{r} - A^{j'})$.

In this set of experiments, we randomly generated three sets of instances having the same characteristics as the original set, except $L \in \{3, 5\}$ and $\tau \in \{20, 50, 100, 200, 500\}$. The first set has stationary demand, while the second and third sets have nonstationary *hi-lo* demand and *lo-hi* demand, respectively. Following Bertsimas and Popescu (2003), under hi-lo demand the arrival probabilities increase over time t for high-fare classes according to $p_{t,\,j} = p_j / \log_a(a + (\tau - t)\rho)$, where p_j is the "base" stationary arrival probability and a and ρ are parameters. The arrival probabilities decrease over time t for low-fare classes according to $p_{t,\,j} = p_j / \log_a(a + (t-1)\rho)$. Lo-hi demand is the reverse, with the arrival probabilities for high-fare classes decreasing over time, and the arrival probabilities for low-fare classes increasing over time. For our experiments, we used $a = 5$ and $\rho = 1$. We simulated each instance 100 times for each policy, using the same sequence of customer demands across different policies. We also solved **(P1)** with an optimality tolerance of $\Omega = 0.5\%$.

The results are shown in Tables 6, 7, and 8 with standard errors reported in parentheses. Figure 3 summarizes these results by plotting the relationship between the difference in the bounds and the difference in policy performance. The improvement of the **DBPC** policy over the **BPC** policy increases as the quality of the bound **(P1)** relative to **(LP)** improves. Furthermore, the differences are substantial: the **(LP)** bound can be up to 50.4% worse than the

Table 9. Relative policy improvement from re-solving every period.

L	τ	**BPC**	**DBPC**
3	20	1.053	1.018
	50	1.036	1.012
	100	1.054	1.016
5	20	0.999	0.999
	50	1.067	1.013
	100	1.088	1.022

Table 10. Bound and policy results for hi-lo demand with decreasing load factor across instances.

L	τ	Capacity per leg	Load factor	**(P1)** Bound	**(LP)** Bound	**BPC** Mean (Std. err.)	**DBPC** Mean (Std. err.)
3	20	2	1.26	498.93	586.11	342.86 (16.29)	384.28 (18.06)
	50	6	0.85	1,268.93	1,287.93	1,068.17 (29.92)	1,130.42 (31.36)
	100	12	0.72	2,214.10	2,218.28	2,070.93 (43.58)	2,130.19 (42.93)
	200	24	0.62	3,810.55	3,810.55	3,756.89 (61.00)	3,790.21 (60.46)
	500	61	0.51	7,920.97	7,920.97	7,701.83 (82.73)	7,704.72 (82.54)
5	20	1	1.72	374.73	563.77	287.19 (14.99)	331.61 (15.76)
	50	4	0.87	1,295.13	1,351.88	1,059.23 (28.50)	1,126.13 (30.09)
	100	8	0.74	2,317.61	2,328.16	2,072.87 (42.09)	2,163.22 (43.80)
	200	16	0.64	3,999.29	3,999.29	3,819.94 (56.24)	3,902.94 (54.38)
	500	42	0.51	8,311.58	8,313.30	8,224.35 (97.15)	8,228.00 (97.59)

(P1) bound, and the policy **DBPC** can perform up to 21.4% better than **BPC**. There appears to be no obvious dependency on the demand characteristics—i.e., stationary, hi-lo, or lo-hi—although the policy improvements appear slightly weaker for lo-hi demand.

Table 9 considers the effect of more frequent re-solving when demand is stationary. It reports the mean policy performance from re-solving every period, divided by the mean policy performance from Table 6 for re-solving five times over the time horizon. Re-solving leads to more improvement for **BPC** than for **DBPC**. On the other hand, as shown in Tables 6, 7, and 8, re-solving for **BPC** may actually degrade the policy (a well-known effect), whereas re-solving consistently improves **DBPC**. We suspect this is because the model **(P1)** is more accurate than **(LP)**, and therefore its recommendations are more on target with outcomes.

We also tested how the policies are affected by the load factor. We considered the same instances as for the stationary demand case in Table 6, except the stationary arrival probabilities are used as base probabilities in the models given above for hi-lo and lo-hi demand. As shown in Tables 10 and 11, this gives a load factor that decreases as τ increases. For hi-lo demand (the lo-hi demand case is similar), in Figure 4 we plot how the policy and bound differences change as a function of load factor. As the load factor decreases, the advantage of **(P1)** over **(LP)** decreases because the problem simply becomes easier: Accepting every customer becomes optimal. On the other hand, as demand becomes greater than supply, i.e., the load factor increases, the problem becomes harder because now some demand must be turned away. Hence, using **(P1)** and the corresponding policy **DBPC** yields greater improvements precisely when they are needed the most. As shown earlier, this increased performance comes with a computational price.

In our experiments, we also noticed that increasing the ratio of high fare to low fare improves the performance of **DBPC**. Although unrealistic, in one instance with a high-to-low fare ratio of 20 to 1, we found the simulated mean revenue gap between **BPC** and **DBPC** to be 43%.

Appendix

We can, in fact, derive the standard linear program **(LP)** using the following quasistatic affine functional approximation to the value function

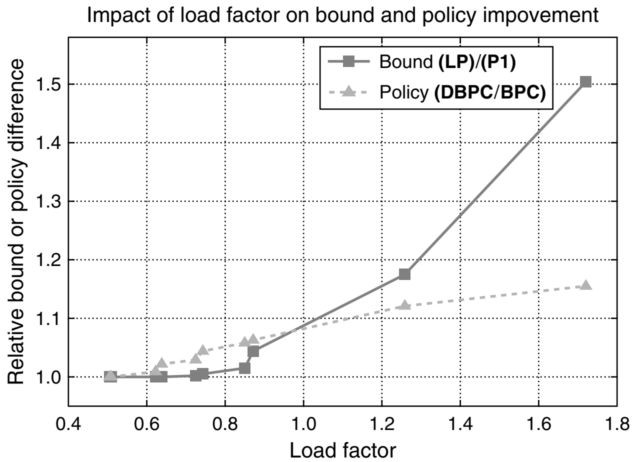
$$v_t(\vec{r}) \approx \theta_t + \sum_i V_i r_i \quad \forall\, t,\ \vec{r} \in \mathcal{R}_t. \tag{A1}$$

This is a simpler form than (14), which has parameters $V_{t,i}$ depending on the time index t, whereas here the parameters V_i do not depend on t. We can therefore interpret the optimal dual prices V_i^* of **(LP)** as approximating the

Table 11. Bound and policy results for lo-hi demand with decreasing load factor across instances.

L	τ	Capacity per leg	Load factor	**(P1)** Bound	**(LP)** Bound	**BPC** Mean (Std. err.)	**DBPC** Mean (Std. err.)
3	20	2	1.26	514.64	586.11	400.21 (19.25)	439.10 (20.91)
	50	6	0.85	1,271.94	1,287.93	1,092.10 (32.49)	1,132.24 (33.20)
	100	12	0.72	2,216.17	2,218.28	2,050.07 (39.96)	2,091.76 (40.46)
	200	24	0.62	3,810.55	3,810.55	3,704.92 (57.82)	3,732.64 (57.39)
	500	61	0.51	7,920.97	7,920.97	7,733.36 (84.17)	7,733.36 (84.17)
5	20	1	1.72	394.56	563.77	298.25 (14.88)	347.16 (18.15)
	50	4	0.87	1,307.89	1,351.88	1,064.69 (30.85)	1,149.21 (33.91)
	100	8	0.74	2,320.23	2,328.16	1,993.58 (43.96)	2,088.39 (44.34)
	200	16	0.64	3,999.29	3,999.29	3,850.33 (47.07)	3,921.71 (47.31)
	500	42	0.51	8,313.30	8,313.30	8,243.18 (84.76)	8,244.62 (84.76)

Figure 4. Impact of load on relative policy and bound improvement over standard methods for instances with hi-lo demand.



V_i^* coming from the following linear program. Namely, by substituting (A1) into **(D0)**, we obtain

$$\min_{V,\theta} \sum_i V_i c_i + \theta_1,$$

$$\theta_t - \theta_{t+1} \geqslant \sum_j p_{t,j}\left(f_j - \sum_i a_{i,j} V_i\right) u_j \qquad \forall\, t < \tau,\ \vec{r} \in \mathscr{R}_t,\ \vec{u} \in \mathscr{U}_{\vec{r}},$$

$$\sum_i V_i r_i + \theta_\tau \geqslant \sum_j p_{t,j} f_j u_j \quad \forall\, \vec{r} \in \mathscr{R}_\tau,\ \vec{u} \in \mathscr{U}_{\vec{r}}.$$

Its dual is

$$z_{\text{quasi}} = \max_{X \geqslant 0} \sum_{t,\, \vec{r} \in \mathscr{R}_t,\, \vec{u} \in \mathscr{U}_{\vec{r}}} \left(\sum_j p_{t,j} f_j u_j\right) X_{t,\vec{r},\vec{u}},$$

$$\sum_{t,\, \vec{r} \in \mathscr{R}_t,\, \vec{u} \in \mathscr{U}_{\vec{r}}:\, t<\tau} \left(\sum_j p_{t,j} a_{i,j} u_j\right) X_{t,\vec{r},\vec{u}} + \sum_{\vec{r} \in \mathscr{R}_\tau,\, \vec{u} \in \mathscr{U}_{\vec{r}}} r_i X_{\tau,\vec{r},\vec{u}} = c_i \quad \forall\, i, \tag{A2}$$

$$\sum_{\vec{r} \in \mathscr{R}_t,\, \vec{u} \in \mathscr{U}_{\vec{r}}} X_{t,\vec{r},\vec{u}} = \begin{cases} 1 & \text{if } t = 1, \\ \displaystyle\sum_{\vec{r} \in \mathscr{R}_{t-1},\vec{u} \in \mathscr{U}_{\vec{r}}} X_{t-1,\vec{r},\vec{u}} & \\ \quad \text{if } t \in \{2, \dots, \tau\}\ \forall\, t. \end{cases} \tag{A3}$$

Now define Y_j as before in (22). Constraints (A3) are equivalent to (21), and therefore (24) still holds, deriving (6). Constraints (A2) are the same as (23), and therefore the same argument after (23) holds to derive (5). The remaining details follow as before.

The same example from §4.2 shows that there can be a gap between z_{quasi} and z_{LP}. Furthermore, the above dual program restricts **(D1)** by adding the constraints

$$V_{t,i} = V_{t+1,i} \quad \forall\, t < \tau, i.$$

Because the objective senses minimize, from Theorem 1 and Proposition 1, we have

$$z_{\text{LP}} \geqslant z_{\text{quasi}} \geqslant z_{\text{P1}} \geqslant v_1(\vec{c}).$$

In general, one may consider functional approximations to the value function having the hybrid form

$$v_t(\vec{r}) \approx \sum_k V_{t,k} \phi_k(\vec{r}) + \sum_l W_l \zeta_l(\vec{r}) \quad \forall\, \vec{r} \in \mathscr{R}_t.$$

Here, the approximation consists of the usual dynamic component, i.e., the first sum, as in (7). It also includes a static component in the second sum, where the parameters W_l do not depend on time and $\zeta_l(\cdot)$ is the lth basis function. The quasistatic approximation (A1) is a special case. It should be computationally easier to handle higher-fidelity basis functions for the static component than for the dynamic component because fewer parameters need to be optimized over.

Acknowledgments

This work was presented in a series of seminars at the University of British Columbia, Division of Operations and Logistics in the Sauder School of Business in June 2005. The author gratefully acknowledges the constructive feedback provided by the seminar participants. He especially thanks Marty Puterman and Maurice Queyranne. The author, as usual, assumes responsibility for any errors. He is grateful for financial support from the Graduate School of Business, University of Chicago.

References

Bertsimas, D., S. de Boer. 2005. Simulation-based booking limits for airline revenue management. *Oper. Res.* **53** 90–106.

Bertsimas, D., I. Popescu. 2003. Revenue management in a dynamic network environment. *Transportation Sci.* **37** 257–277.

Cooper, W. L. 2002. Asymptotic behavior of an allocation policy for revenue management. *Oper. Res.* **50** 720–727.

Cooper, W. L., T. Homem-de-Mello. 2003. A class of hybrid methods for revenue management. Working Paper 03-015, Northwestern University, Evanston, IL.

de Farias, D. P., B. Van Roy. 2003. The linear programming approach to approximate dynamic programming. *Oper. Res.* **51** 850–865.

McGill, J., G. J. van Ryzin. 1999. Revenue management: Research overview and prospects. *Transportation Sci.* **33** 233–256.

Puterman, M. L. 1994. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. Wiley, New York.

Schweitzer, P. J., A. Seidmann. 1985. Generalized polynomial approximations in Markovian decision processes. *J. Math. Anal. Appl.* **110** 568–582.

Simpson, R. W. 1989. Using network flow techniques to find shadow prices for market and seat inventory control. Technical report, MIT Flight Transportation Laboratory Memorandum M89-1, Massachusetts Institute of Technology, Cambridge, MA.

Talluri, K. T., G. J. van Ryzin. 1998. An analysis of bid-price controls for network revenue management. *Management Sci.* **44** 1577–1593.

Talluri, K. T., G. J. van Ryzin. 2004. *The Theory and Practice of Revenue Management*. Kluwer Academic Publishers, Norwell, MA.

Williamson, E. L. 1992. Airline network seat control. Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge, MA.