Dynamic Seat Assignment with Social Distancing

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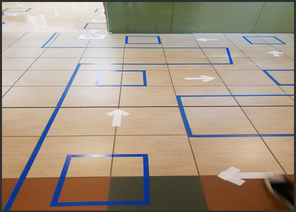
Social Distancing under Pandemic

- Social distancing measures.



Social Distancing under Pandemic

- Social distancing in seating areas.




Situations

- Deterministic Demand
  We know the specific demand for each group, then make the seat planning.
- Stochastic Demand
  We know the demand distribution, need to give a seat planning before the demand realization.
- Dynamic Demand
  1. We assign seats to each group after its realization.
  2. We accept or reject each group after its realization, but assign them later.

Literature Review

Seat Planning with Social Distancing

- Allocation of seats on airplanes (Ghorbani et.al 2020), classroom layout planning (Bortolete et al. 2022), seat planning in long-distancing trains (Haque & Hamid 2022).

- Social distancing can be enforced in different groups (Moore et al. 2021).

- Seating planning for known groups in amphitheaters (Haque & Hamid 2022), airplanes (Salari et al. 2022), theater (Blom et al. 2022).

Dynamic Seat Assignment

- Related to multiple knapsack problem (Pisinger et al. 1999) and dynamic knapsack problem (Kleywegt et al. 1998).
- Dynamic seat assignment on airplane (Hamdouch et al. 2011), train (Berge et al. 1993).
- Assign-to-seat: dynamic capacity control for selling high-speed train tickets (Zhu et al. 2023).

Deterministic Demand Situation

Seat Planning with Social Distancing

- Group type $\mathcal{M} = \{1, \ldots, M\}$.
- Row $\mathcal{N} = \{1, \ldots, N\}$.
- The social distancing: δ seat(s).
- $n_i = i + \delta$: the new size of group type i for each $i \in \mathcal{M}$.
- The number of seats in row j: $L_j^0, j \in \mathcal{N}$.
- $L_j = L_j^0 + \delta$: the length of row j for each $j \in \mathcal{N}$.

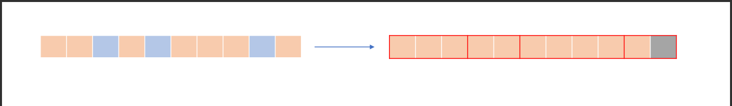


Figure: Problem Conversion with One Seat as Social Distancing

Basic Concepts

- $\boldsymbol{h} = (h_1, \ldots, h_M)$, pattern, the seat planning for each group type in one row. Let $L = q(M+\delta) + r$.
  - The number of people accommodated:
  $|\boldsymbol{h}| = \sum_{i=1}^{M} ih_i = qM + \max\{r - \delta, 0\}$.
  - Loss for pattern $\boldsymbol{h}$: $L - \delta - |\boldsymbol{h}| = q\delta - \delta + \min\{r, \delta\}$.
- Largest patterns: $|\boldsymbol{h}| \geq |\boldsymbol{h}'|$ for any $\boldsymbol{h}'$.
- Full patterns: $\sum_{i=1}^{M} n_i h_i = L$.

- Example:
  $\delta = 1$, $M = 4$, $n_1 = 2, n_2 = 3, n_3 = 4, n_4 = 5$, $L = 21$.
  Largest patterns: $(0,0,0,4), (0,0,4,1), (0,2,0,3)$.
  Largest may not be full: $(0,0,0,4)$.
  Full may not be largest: $(1,1,4,0)$.

Deterministic Formulation

To obtain an integral seat planning, we use the deterministic formulation.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{M} \sum_{j=1}^{N} (n_i - \delta) x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} \leq d_i, \quad i \in \mathcal{M}, \\
& \sum_{i=1}^{M} n_i x_{ij} \leq L_j, j \in \mathcal{N}, \\
& x_{ij} \in \mathbb{Z}_+, \quad i \in \mathcal{M}, j \in \mathcal{N}.
\end{aligned} \tag{1}
$$

Problem (1) can generate a feasible seat planning.

In the LP relaxation of problem (1), there exists an index v such that the optimal solutions satisfy the following conditions:

- For $i = 1, \ldots, v-1$, $x^*_{ij} = 0$ for all rows, indicating that no group type i are assigned to any rows before index v.
- For $i = v+1, \ldots, M$, the optimal solution assigns $\sum_j x^*_{ij} = d_i$ group type i to meet the demand for group type i.
- For $i = v$, the optimal solution assigns $\sum_j x^*_{ij} = \frac{L - \sum_{i=v+1}^{M} d_i n_i}{n_v}$ group type v to the rows. This quantity is determined by the available supply, which is calculated as the remaining seats after accommodating the demands for group types $v+1$ to M, divided by the size of group type v, denoted as n_v.

Generate The Full or Largest Pattern

Given a specific pattern, we can convert it into a largest or full pattern while ensuring that the original group type requirements are met. When multiple full patterns are possible, our objective is to generate the pattern with minimal loss.
Mathematically, for any pattern $\boldsymbol{h} = (h_1, \ldots, h_N)$, we seek to find a pattern $\boldsymbol{h}' = (h_1', \ldots, h_N')$ that satisfies the following programming.

$$\begin{aligned} \max \quad & |\boldsymbol{h}'| \\ \text{s.t.} \quad & h_1' \geq h_1 \\ & h_1' + h_2' \geq h_1 + h_2 \\ & \ldots \\ & h_1' + \ldots + h_N' \geq h_1 + \ldots + h_N. \end{aligned} \tag{2}$$

Stochastic Demand Situation

Method Flow

We aim to obtain a seat planning before the demand realization.

- The formulation of scenario-based stochastic programming(SSP).
- Reformulate SSP to the benders master problem(BMP) and subproblem.
- The optimal solution can be obtained by solving BMP iteratively.
- To avoid solving IP directly, we consider the linear relaxation form.
- Obtain integral seat planning composed of full or largest patterns by deterministic model.

Scenario-based Stochastic Programming

$$
\begin{aligned}
(SSP)\max\quad & E_\omega\left[\sum_{i=1}^{M-1}(n_i-\delta)(\sum_{j=1}^{N}x_{ij}+y_{i+1,\omega}^{+}-y_{i\omega}^{+})+(n_M-\delta)(\sum_{j=1}^{N}x_{Mj}-y_{M\omega}^{+})\right]\\
\text{s.t.}\quad & \sum_{j=1}^{N}x_{ij}-y_{i\omega}^{+}+y_{i+1,\omega}^{+}+y_{i\omega}^{-}=d_{i\omega},\quad i=1,\ldots,M-1,\omega\in\Omega\\
& \sum_{j=1}^{N}x_{ij}-y_{i\omega}^{+}+y_{i\omega}^{-}=d_{i\omega},\quad i=M,\omega\in\Omega\\
& \sum_{i=1}^{M}n_i x_{ij}\leq L_j, j\in\mathcal{N}\\
& y_{i\omega}^{+},y_{i\omega}^{-}\in\mathbb{Z}_+,\quad i\in\mathcal{M},\omega\in\Omega\\
& x_{ij}\in\mathbb{Z}_+,\quad i\in\mathcal{M},j\in\mathcal{N}.
\end{aligned}
\tag{3}
$$

Reformulation

$$
\begin{aligned}
\max \quad & \mathbf{c}^\intercal \mathbf{x} + z(\mathbf{x}) \\
\text{s.t.} \quad & \mathbf{n}\mathbf{x} \leq \mathbf{L} \\
& \mathbf{x} \in \mathbb{Z}_+^{M \times N},
\end{aligned} \tag{4}
$$

where $z(\mathbf{x})$ is defined as

$$
z(\mathbf{x}) := E(z_\omega(\mathbf{x})) = \sum_{\omega \in \Omega} p_\omega z_\omega(\mathbf{x}),
$$

and for each scenario $\omega \in \Omega$,

$$
\begin{aligned}
z_\omega(\mathbf{x}) := \max \quad & \mathbf{f}^\intercal \mathbf{y} \\
\text{s.t.} \quad & \mathbf{x}\mathbf{1} + \mathbf{V}\mathbf{y} = \mathbf{d}_\omega \\
& \mathbf{y} \geq 0.
\end{aligned} \tag{5}
$$

Solution to Subproblem

Problem (3) is easy to solve with a given $\mathbf{x}$ which can be seen by the dual problem:

$$\begin{aligned} \min \quad & \alpha_\omega^\intercal(\mathbf{d}_\omega - \mathbf{x}\mathbf{1}) \\ \text{s.t.} \quad & \alpha_\omega^\intercal \mathbf{V} \geq \mathbf{f}^\intercal \end{aligned} \tag{6}$$

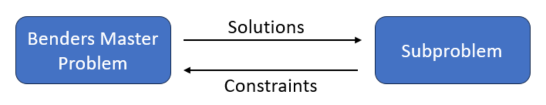
- The feasible region of problem (6), $P = \{\alpha | \alpha^\intercal V \geq \mathbf{f}^\intercal\}$, is bounded. In addition, all the extreme points of P are integral.
- The optimal solution to this problem can be obtained directly according to the complementary slackness property.

Benders Decomposition Procedure

Let z_ω be the lower bound of problem (6), SSP can be obtained by solving following restricted benders master problem:

$$
\begin{aligned}
\max \quad & \mathbf{c}^\intercal \mathbf{x} + \sum_{\omega \in \Omega} p_\omega z_\omega \\
\text{s.t.} \quad & \mathbf{n}\mathbf{x} \leq \mathbf{L} \\
& (\alpha^k)^\intercal (\mathbf{d}_\omega - \mathbf{x}\mathbf{1}) \geq z_\omega, \alpha^k \in \mathcal{O}, \forall \omega \\
& \mathbf{x} \in \mathbb{Z}_+
\end{aligned} \tag{7}
$$

Constraints will be generated from problem (6) until an optimal solution is found.



To avoid solving IP directly, we consider the linear relaxation of Problem (7).

Obtain Seat Planning Composed of Full or Largest Patterns

Step 1. Obtain the solution, $\mathbf{x}^*$, by benders decomposition.
Aggregate $\mathbf{x}^*$ to the number of each group type,
$s_i^0 = \sum_j x_{ij}^*, i \in \mathbf{M}$.

Step 2. Solve problem (1) to obtain the optimal solution, $\mathbf{x}^1$.
Aggregate $\mathbf{x}^1$ to the number of each group type,
$s_i^1 = \sum_j x_{ij}^1, i \in \mathbf{M}$.

Step 3. For each row, construct a full or largest pattern.

Construction

There exists an optimal solution to the stochastic programming problem such that the patterns associated with this optimal solution are composed of the full or largest patterns under any given scenarios.
Algorithm to obtain the solution of problem (2).

Running time of Benders Decomposition and IP

# of scenarios	demands	running time of IP(s)	Benders (s)	# of rows	# of groups	# of seats
1000	(150, 350)	5.1	0.13	30	8	(21, 50)
5000		28.73	0.47	30	8	
10000		66.81	0.91	30	8	
50000		925.17	4.3	30	8	
1000	(1000, 2000)	5.88	0.29	200	8	(21, 50)
5000		30.0	0.62	200	8	
10000		64.41	1.09	200	8	
50000		365.57	4.56	200	8	
1000	(150, 250)	17.15	0.18	30	16	(41, 60)
5000		105.2	0.67	30	16	
10000		260.88	1.28	30	16	
50000		3873.16	6.18	30	16	

Verify the performance

To evaluate the effectiveness of the initial seat planning, we employ a dynamic situation where decisions are made based on real-time feedback. In this context, we utilize a fixed seat planning as the foundation, and then make informed decisions accordingly.

For any given seat planning, we can make the decision (by using group-type control).

Customers have more freedom to choose their seats with fixed seat planning.

Feasible Seat Planning versus IP Solution

# samples	T	probabilities	# rows	people served by FSP	IP
1000	45	[0.4,0.4,0.1,0.1]	8	85.30	85.3
1000	50	[0.4,0.4,0.1,0.1]	8	97.32	97.32
1000	55	[0.4,0.4,0.1,0.1]	8	102.40	102.40
1000	60	[0.4,0.4,0.1,0.1]	8	106.70	NA
1000	65	[0.4,0.4,0.1,0.1]	8	108.84	108.84
1000	35	[0.25,0.25,0.25,0.25]	8	87.16	87.08
1000	40	[0.25,0.25,0.25,0.25]	8	101.32	101.24
1000	45	[0.25,0.25,0.25,0.25]	8	110.62	110.52
1000	50	[0.25,0.25,0.25,0.25]	8	115.46	NA
1000	55	[0.25,0.25,0.25,0.25]	8	117.06	117.26
5000	300	[0.25,0.25,0.25,0.25]	30	749.76	749.76
5000	350	[0.25,0.25,0.25,0.25]	30	866.02	866.42
5000	400	[0.25,0.25,0.25,0.25]	30	889.02	889.44
5000	450	[0.25,0.25,0.25,0.25]	30	916.16	916.66

Each entry of people served is the average of 50 instances. IP will spend more than 2 hours in some instances, as 'NA' showed in the table.

Find the optimal solution

We aim to find the optimal integral solution from the optimal fractional solution. Suppose the aggregated supply is $(X_1, \ldots, X_M)$, let $f(\boldsymbol{X}; \Omega)$ indicate the value with the scenario set, Ω.

$$\begin{aligned} \max \quad & f(\boldsymbol{X'}; \Omega) \\ \text{s.t.} \quad & X_i - \delta \leq X_i' \leq X_i + \delta, i = 1, \ldots, M \\ & \text{for all feasible } \boldsymbol{X'} \end{aligned}$$

Use another formulation when the length of every row is the same?

$$\begin{aligned} \max \quad & c'\mathbf{x} + f'\mathbf{y} \\ \text{s.t.} \quad & \mathbf{T}\mathbf{x} + \mathbf{V}_\omega \mathbf{y}_\omega = \mathbf{D}_\omega, \omega \in \Omega \\ & \mathbf{1}\mathbf{x} = N \\ & \mathbf{x}, \mathbf{y} \geq \mathbf{0} \end{aligned}$$

Example

Dynamic Demand Situation

Dynamic Seat Assignment Problem

- There is one and only one group arrival at each period, $t = 1, \ldots, T+1$.
- The probability of an arrival of group type i: p_i.
- $\mathbf{L} = (l_1, l_2, \ldots, l_N)$, where $l_j = 0, \ldots, L_j, j \in \mathcal{N}$: Remaining capacity.
- $u_{i,j}^t$: Decision. Assign group type i to row j at period t, $u_{i,j}^t = 1$.
- $U^t(\mathbf{L}) = \{u_{i,j}^t \in \{0,1\}, \forall i,j \mid \sum_{j=1}^N u_{i,j}^t \leq 1, \forall i, n_i u_{i,j}^t \mathbf{e}_j^\top \leq \mathbf{L}, \forall i,j\}$.
- $\mathbf{e}_j^\top$: Unit row vector with j-th element being 1.
- $V^t(\mathbf{L})$: Value function at period t, given remaining capacity, $\mathbf{L}$.

$$V^t(\mathbf{L}) = \max_{u_{i,j}^t \in U^t(\mathbf{L})} \left\{ \sum_{i=1}^M p_i (\sum_{j=1}^N i u_{i,j}^t + V^{t+1}(\mathbf{L} - \sum_{j=1}^N n_i u_{i,j}^t \mathbf{e}_j)) + p_0 V^{t+1}(\mathbf{L}) \right\}$$

Method Overview

We make the decision based on the flexible seat planning.

- Obtain seat planning composed of full or largest patterns.
  - Linear seat planning from stochastic programming
  - Integral seat planning from deterministic model
  - Construct largest or full patterns.
- Dynamic seat assignment
  - Determine the group type
  - Decision on assigning the group to a specific row

Policies

We have the following policies when we make the instant allocation.

- Dynamic seat assignment
- Bid-price control
- Dynamic programming based heuristic
- Booking limit control
- First come first served

Dynamic Seat Assignment(DSA)

- Determine the group type
  - Seat planning from stochastic programming.
  - When there is no small group, decide which group type to be assigned.

$$d^t(i,j) = \underbrace{i + (j-i-\delta)P(D_{j-i-\delta} \geq x_{j-i-\delta}+1; T-t)}_{\text{acceptance}} - \underbrace{jP(D_j \geq x_j; T-t)}_{\text{rejection}}.$$

For all $j > i$, find the maximum value denoted as $d^t(i, j^*)$.
If $d^t(i, j^*) > 0$, we will place the group of i in $(j^* + \delta)$-size seats. Otherwise, reject the group.

Dynamic Seat Assignment(DSA)

Decision on assigning the group to a specific row

- Break Tie for Determining A Specific Row
- Decision on Assigning The Group

- Value of Acceptance(VoA): approximation of $V_t(\mathbf{L} - n_i \mathbf{e}_j^\top) + i$. (Find a pattern containing group type j^*)
- Value of Rejection(VoR): approximation of $V_t(\mathbf{L})$.
- If VoA is no less than VoR, accept group type i, otherwise, reject it.

Regenerate the seat planning

- When $X_M = 0$
- When comparing VoA and VoR

Bid-price Control

The dual problem of LP relaxation of problem (1) is:

$$\begin{aligned} \min \quad & \sum_{i=1}^{M} d_i z_i + \sum_{j=1}^{N} L_j \beta_j \\ \text{s.t.} \quad & z_i + \beta_j n_i \geq (n_i - \delta), \quad i \in \mathcal{M}, j \in \mathcal{N} \\ & z_i \geq 0, i \in \mathcal{M}, \beta_j \geq 0, j \in \mathcal{N}. \end{aligned} \tag{8}$$

There exists h such that the aggregate optimal solution to relaxation of problem (1) takes the form $xe_h + \sum_{i=h+1}^{M} d_i e_i$, $x = (L - \sum_{i=h+1}^{M} d_i n_i)/n_h$.

Dynamic Programming Based Heuristic

- Relax all rows to one row with the same capacity by $L = \sum_{j=1}^{N} L_j$.
  - Deterministic problem: $\{\max \sum_{i=1}^{M} (n_i - \delta) x_i : x_i \leq d_i, i \in \mathcal{M}, \sum_{i=1}^{M} n_i x_i \leq L, x_i \in \mathbb{Z}_+\}$.
- Decision: u. If we accept a request in period t, $u(t) = 1$; otherwise, $u(t) = 0$.
  - DP with one row can be expressed as:

$$V_t(L) = \mathbb{E}_{i \sim p}[\max_{u \in \{0,1\}} \{[V_{t+1}(L - n_i u) + iu]\}], L \geq 0$$

$$V_{T+1}(x) = 0, \forall x.$$

- After accepting one group, assign it in some row arbitrarily when the capacity of the row allows.

Booking limit Control

Basic idea: for every type of requests, we only allocate a fixed amount according to the static solution and reject all other exceeding requests.

1. Observe the arrival group type i.
2. Solve problem (1) using the expected demand.
3. Obtain the optimal solution, x^*_{ij} and the aggregate optimal solution, $\mathbf{X}$.
4. If $X_i > 0$, accept the arrival and assign the group to row k where $x_{ik} > 0$, update $\mathbf{L}^{t+1} = \mathbf{L}^t - n_i \mathbf{e}_k^\top$; otherwise, reject it, let $\mathbf{L}^{t+1} = \mathbf{L}^t$.

Make A Later Allocation

This setting is particularly applicable to larger venues, such as stadiums, where an immediate decision is made when a group arrives, but the actual allocation of seats for that group is deferred to a later time.

Policies:

- Dynamic programming based heuristic

Numerical Results

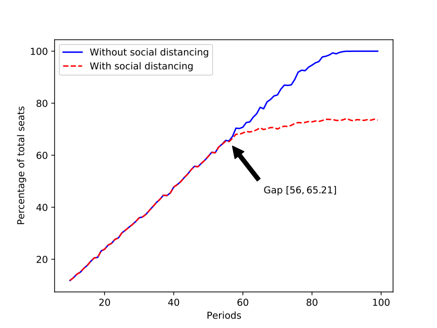
Performances of Different Policies

T	Probabilities	SPP(%)	DP1(%)	Bid(%)	Booking(%)	FCFS(%)
60	[0.25, 0.25, 0.25, 0.25]	99.12	98.42	98.38	96.74	98.17
70	[0.25, 0.25, 0.25, 0.25]	98.34	96.87	96.24	97.18	94.75
80	[0.25, 0.25, 0.25, 0.25]	98.61	95.69	96.02	98.00	93.18
90	[0.25, 0.25, 0.25, 0.25]	99.10	96.05	96.41	98.31	92.48
100	[0.25, 0.25, 0.25, 0.25]	99.58	95.09	96.88	98.70	92.54
60	[0.25, 0.35, 0.05, 0.35]	98.94	98.26	98.25	96.74	98.62
70	[0.25, 0.35, 0.05, 0.35]	98.05	96.62	96.06	96.90	93.96
80	[0.25, 0.35, 0.05, 0.35]	98.37	96.01	95.89	97.75	92.88
90	[0.25, 0.35, 0.05, 0.35]	99.01	96.77	96.62	98.42	92.46
100	[0.25, 0.35, 0.05, 0.35]	99.23	97.04	97.14	98.67	92.00
60	[0.15, 0.25, 0.55, 0.05]	99.14	98.72	98.74	96.61	98.07
70	[0.15, 0.25, 0.55, 0.05]	99.30	96.38	96.90	97.88	96.25
80	[0.15, 0.25, 0.55, 0.05]	99.59	97.75	97.87	98.55	95.81
90	[0.15, 0.25, 0.55, 0.05]	99.53	98.45	98.69	98.81	95.50
100	[0.15, 0.25, 0.55, 0.05]	99.47	98.62	98.94	98.90	95.25

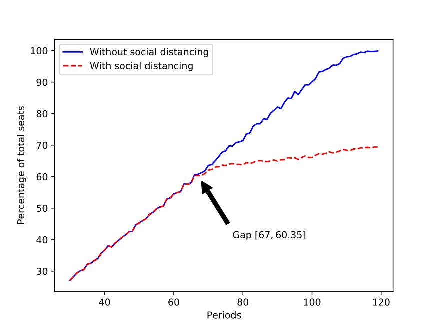
SPP has better performance than other policies under different demands.

Impact of Social Distancing as Demand Increases

$\gamma = p_1 * 1 + p_2 * 2 + p_3 * 3 + p_4 * 4$: the expected number of people at each period.



(a) When $\gamma = 2.5$



(b) When $\gamma = 1.9$

The gap point represents the first period where the number of people without social distancing is larger than that with social distancing and the gap percentage is the corresponding percentage of total seats.

Estimation of Gap Point

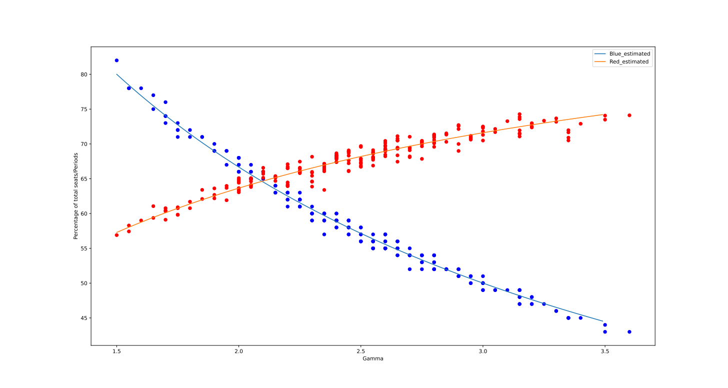


Figure: Gap points under 200 probabilities

Blue points: period of the gap point. Red points: occupancy rate of the gap point. Gap points can be estimated.

Conclusion

Conclusion

- We address the problem of dynamic seat assignment with social distancing.
- Our approach, stochastic planning policy, provides a comprehensive solution for optimizing seat assignments while ensuring the safety of customers under dynamic situation.
- We can estimate the occupancy rate when applying SPP according to γ.

The End