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Schedule-based transit assignment model with vehicle capacity and seat availability

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ABSTRACT

In this paper, we propose a new schedule-based equilibrium transit assignment model that differentiates the discomfort level experienced by sitting and standing passengers. The notion of seat allocation has not been considered explicitly and analytically in previous schedule-based frameworks. The model assumes that passengers use strategies when traveling from their origin to their destination. When loading a vehicle, standing on-board passengers continuing to the next station have priority to get available seats and waiting passengers are loaded on a First-Come-First-Serve (FCFS) principle. The stimulus of a standing passenger to sit increases with his/her remaining journey length and time already spent on-board. When a vehicle is full, passengers unable to board must wait for the next vehicle to arrive. The equilibrium conditions can be stated as a variational inequality involving a vector-valued function of expected strategy costs. To find a solution, we adopt the method of successive averages (MSA) that generates strategies during each iteration by solving a dynamic program. Numerical results are also reported to show the effects of our model on the travel strategies and departure time choices of passengers.

1. Introduction

System planners and transit operators are interested in the passengers' choice of transit services for determining the performance and revenue generated from the transit system. Particularly in the peak hours, passengers are often unevenly distributed over different transit vehicles and time periods due to various reasons (e.g. double headings for the case of buses) causing an inefficient utilization of the service capacity. Therefore, a transit assignment model, which takes into account the temporal distribution of demand and congestion over the transit network, is useful in estimating how passengers utilize a given transit system. This model will also serve as a tool for system planners and transit operators to plan and schedule the transit services for optimizing their objectives (e.g. minimizing the total delay or maximizing the revenue).

In the literature of transit assignment studies, models could be classified into two different categories: static (frequency-based) and dynamic (schedule-based) transit assignment models. Similar to the traditional static user equilibrium assignment on road networks, static transit assignment models consider a constant transit passenger demand such that the strategy, or hyperpath, costs of individual passengers are minimized (Spiess and Florian, 1989; De Cea and Fernandez, 1993; Lam et al., 1999). In static transit assignment models, as there is no time dimension, all model characteristics are averaged out over the modeling period (e.g. the morning peak hour). As the average values are adopted, static transit assignment models could not reveal the bottleneck induced congestion problem (Schmoecker et al., 2008) and are not able to properly evaluate the transit network under dramatically changing network conditions (e.g. passenger arrival rate and loading of

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Nomenclature

N	set of nodes (with $i,j,k \in N$)
A	set of arcs (with $a \in A$)
$D^g_{(q,r)}$	travel demand for OD pair (q,r) and group g
$[t^-_{(q,r)}(g), t^+_{(q,r)}(g)]$	desired arrival time for OD pair (q,r) and group g
c_{ij}	travel time for arc $a = (i,j)$
L	set of transit lines (with $l \in L$)
T	set of time intervals (with $t \in T$)
δ	duration of one time interval
i_t	time-expanded node
$I^-(i_t)$	set of predecessor nodes for $i_t : \{k_{(t-c_{ki})}\} \cup \{i_{t-1}\}$
$I^+(i_t)$	set of successor nodes for $i_t : \{i_{(t+c_{ij})}\} \cup \{i_{t+1}\}$
$S_{(q,r)}$	set of strategies for OD pair (q,r) (with $s \in S_{(q,r)}$)
m_l	number of runs for transit line l
n_l	number of transit nodes for line l (with $1 \leqslant n \leqslant n_l$)
$\{i_1(l), i_2(l), \ldots, i_{n_l}(l)\}$	route sequence associated to line l
$\left\{t_1^{i_n(l)}, t_2^{i_n(l)}, \ldots, t_{m_l}^{i_n(l)}\right\}$	departure times at transit node $i_n(l)$
u^t_{ij}	seat capacity for arc (i,j) at time t
$\tilde{u}^t_{ij}$	standing capacity for arc (i,j) at time t
v^t_{ij}	transit fare on arc (i,j) at time t
η^g_1	early arrival penalty (in monetary units) for group g
η^g_2	late arrival penalty (in monetary units) for group g
η^g_3	early departure penalty (in monetary units) for group g
η_4	crowding penalty (in monetary units)
η_5	stimulus parameter associated to passengers's time spent on-board
η_6	stimulus parameter associated to passenger's remaining journey length
γ_{travel}	value of time for traveling
γ_{wait}	value of time for waiting
$e^{t,g}_{qi}$	departure penalty cost for access arc $(q_t, i_{(t+c_{qi})})$ and group g
$e^{t,g}_{ir}$	late penalty cost for egress arc $(i_t, r_{(t+c_{ir})})$ and group g
$\hat{e}^t_{ij}$	crowding cost function on arc (i,j) at time t
$W^{t,sit}_i$	priority class of sitting passengers having continuance priority at i_t
$W^{t,std}_i$	priority class of standing passengers having continuance priority at i_t
$W^{t,\tau}_i$	non-priority class of passengers having reached node i at time τ
$E^{s,t}_i$	user-preference set for strategy s, node i and time t
$x^s_{(q,r,g)}$	number of passengers for OD pair (q,r) and group g assigned to strategy s
X	strategy assignment (SA) vector (with $x^s_{(q,r,g)}$ its components)
$f^{s,t}_{ij}$	number of sitting passengers using strategy s and traveling on arc (i,j) at time t
$\tilde{f}^{s,t}_{ij}$	number of standing passengers using strategy s and traveling on arc (i,j) at time t
$\pi^{s,t}_{ij}(X)$	success-to-sit probability: probability that a passenger using strategy s will succeed to sit on the vehicle serving arc (i,j) at time t
$\tilde{\pi}^{s,t}_{ij}(X)$	success-to-stand probability: probability that a passenger using strategy s will succeed to stand on the vehicle serving arc (i,j) at time t
$\hat{\pi}^{s,t}_{ij}(X)$	failure-to-board probability: probability that a passenger using strategy s will fail to board the vehicle serving arc (i,j) at time t
$z^{s,t}_i(X)$	number of passengers using strategy s who are at node i at time t
$\alpha^{s,t}_i(X)$	node arrival probability: probability that a passenger using strategy s will arrive at node i at time t
$C(X)$	vector of expected strategy costs (with $C^s_{(q,r,g)}$ its components)

transit services) during the period of analysis. Despite its weaknesses, static transit assignment models are commonly adopted for the strategic and long-term planning/evaluation. In order to precisely model the dynamic characteristics in transit networks, dynamic transit assignment models have gained their importance in the past two decades. For the majority of these models, timetables of the transit services are assumed to be sufficiently reliable, which in contrast to the use of average frequencies in static models. In the dynamic case, transit passengers are not only choosing their strategies/hyperpaths, but also their departure and arrival time for minimizing their generalized cost. In order to incorporate this time dependent choice, a time-dependent transit network should be adopted for the dynamic (schedule-based) transit assignment studies. Poon et al. (2004) suggested to classify the time-dependent transit network into: (a) diachronic graph representation (Nuzzolo et al., 2001); (b) dual graph representation (Moller-Pedersen, 1999); (c) forward star network formulation (Tong and Wong, 1998), and; (d) space-time formulation (Nguyen et al., 2001; Hamdouch and Lawphongpanich, 2008).

Over the past few decades, many studies have proposed models for dynamic transit assignment. Sumi et al. (1990) proposed a stochastic approach to model departure times and route choices of passengers on a mass transit system. Alfa and Chen (1995) developed a transit assignment model for forecasting the temporal demand distribution along a corridor under a random assumption of passenger boarding. Their model, however, neglected the congestion effect at the transit station, which affects the passengers' waiting time, and in-vehicle discomfort, which affects the passengers' preference on that transit line. Tong and Wong (1998), and Poon et al. (2004) proposed a dynamic user equilibrium model that considers the effect of congestion at transit stations on the time-dependent demand distribution and accounts for the First-Come-First-Serve (FCFS) principle when loading passengers. In Poon et al. (2004), the authors model the user equilibrium transit assignment problem as an optimization problem with an unaccountably infinite number of decision variables and later discretize the problem in order to find an approximate solution. Tian et al. (2007a) improved the model of Alfa and Chen (1995) by introducing the in-vehicle congestion through a bulk-queue model and analyzed theoretical properties of the equilibrium flows. However, their model and analysis are limited to transit networks with a single destination.

Optimal strategy approach is one of the commonly adopted formulations for transit assignment problems. The core idea for optimal strategy is that a traveler will select at each node of the network, a set of attractive lines that allows him/her to reach his/her destination at a minimum expected cost. Spiess and Florian (1989) were the first to propose a transit assignment model based on the optimal strategy approach. In their study, the optimal strategy approach is used to solve a linear, many-to-one and uncongested transit network. In Spiess and Florian (1989), as departure time is not considered in the strategy, their model is only applicable to static transit assignment. Wu et al. (1994) further extended the model in Spiess and Florian (1989) to accommodate asymmetric cost that models the waiting and in-vehicle cost as a function of the transit flow. Although the proposed algorithm could effectively solve the transit assignment problem, authors of that paper suggested that unless decomposition approach is used, considerable amount of computer storage is needed for large scale problems. Marcotte et al. (2004) formulated the strategic flow problem of a capacitated and static transit network into a variational inequality (VI) format. A combined Frank-Wolfe and projection algorithm is proposed by the authors and is proven to be an efficient and robust algorithm for solving the strategic flow problem. Different from the previous static models, Hamdouch and Lawphongpanich (2008) proposed a dynamic schedule-based transit assignment where the choice of strategy is an integral part of user behavior. In that study, passengers specify their travel strategy by providing, at each transit station and each point in time, an ordered list of transit lines they prefer to use to continue their journey. For a given passenger, the user-preference sets at each time-expanded (TE) node collectively yield a set of potential paths that depart from the passenger's origin at the same time and generally arrive at the destination at different times. Also, when loading a transit vehicle at a station, on-board passengers continuing to the next station remains on the vehicle and waiting passengers are loaded in a first-come-first-serve (FCFS) basis.

Among the transit assignment models developed in the literature, capacity constrained transit assignment have been widely considered (De Cea and Fernandez, 1993; Lam et al., 1999; Kurauchi et al., 2003; Hamdouch et al., 2004; Cepeda et al., 2006; Hamdouch and Lawphongpanich, 2008) for replicating the actual capacity constraint of transit vehicles. De Cea and Fernandez (1993) had indirectly incorporate the capacity constraint of transit vehicles in the waiting time of passengers at transit stops that leaded to the definition of effective frequency of transit services. In their study, the passenger waiting time will increase as the volume to capacity ratio of the transit services increases. However, as this setup will only increase the waiting time as the transit vehicle is more congested, it still allows the capacity to be exceeded. Different from the study of De Cea and Fernandez (1993), the capacity constraint are strictly enforced in the transit assignment models formulated in Kurauchi et al. (2003) and Hamdouch et al. (2004). In Kurauchi et al. (2003), the absorbing Markov chain is adopted to solve the capacity constrained transit assignment problem. The strict capacity constraint is incorporated through the consideration of failure-to-board probabilities, which is dependent on the residual capacity of the transit vehicles. Similar to Kurauchi et al. (2003), Hamdouch et al. (2004) had considered the boarding probability to incorporate the capacity constraint. Hamdouch and Lawphongpanich (2008) extended the model to the dynamic setting where the fail-to-board passengers are assigned to the waiting arc to wait for their next preferred transit services with residual capacities.

Despite the rich literature in capacity constrained transit assignment, few studies had drawn attention to the issue of differentiating between sitting and standing capacities. As mentioned in the second edition of the Transit Capacity and Quality of Service Manual (Kittelson & Associates et al., 2003), the seat capacity is one of the critical factors for assessing the transit quality of service. The treatment of seat allocation can allow transit planners to consider the allocation of space in vehicle to seat and standing passengers. This is an important issue in countries with high utilization of transit service (e.g. Hong Kong and Japan). Some passengers, particularly those with a long-distance journey, may prefer to wait for a transit service with more available seats or decide to arrive at stations earlier to increase their chances of getting a seat. Tian et al. (2007b) extended their previous model to differentiate the in-vehicle congestion effects of sitting and standing passengers in many-to-one transit networks. Schmoecker et al. (2009) proposed a static frequency-based assignment model that considers travelers probability of finding a seat in their perception of route choice and used Markov chain process to find an equilibrium solution. In their model, standing on-board passengers have priority to get available seats and waiting passengers are loaded on a random manner when they mingle on stations. Leurent (2010) incorporated the seat capacity into the hyperpath-based transit assignment model and considered different discomfort cost for the seating and standing passengers. In their model, priority rules are adopted to govern the probability of getting a seat and are used to assign the passengers to sit, or stand, along each of the transit segment. Leurent and Liu (2009) had further applied the model in Leurent (2010) to Paris for demonstrating its applicability in large-scale network. Apart from the aforementioned static models, Sumalee et al. (2009) formulated a dynamic transit assignment model with explicit consideration of stochastic seat allocation among the standing and boarding passengers, and proposed a solution algorithm based on Monte–Carlo simulation. In their seat allocation process, the probability of getting a seat depends on the time that passengers spent and is going to spend on the transit line.

The objective of the paper is to extend the schedule-based transit assignment model in Hamdouch and Lawphongpanich (2008) to differentiate the discomfort level experienced by the sitting and standing passengers. As compared to the model in Sumalee et al. (2009), we propose an analytical model that captures the stochastic nature of the standing and boarding passengers to get a seat. Each class of passengers, grouped by their remaining journey lengths and times already spent on-board, is assigned success-to-sit, success-to-stand and failure-to-board probabilities. These probabilities are computed by performing a dynamic network loading. When loading a vehicle, standing on-board passengers continuing to the next station have priority to get available seats and waiting passengers are loaded on a First-Come-First-Serve (FCFS) principle. The stimulus of a standing passenger to sit increases with his/her remaining journey length and time already spent on-board. When a vehicle is full, passengers unable to board must wait for the next vehicle to arrive.

For the remainder, Section 2 presents the notation, network representation and assumptions of the proposed model. Travel strategies and the dynamic loading process are described in Section 3. Section 4 shows the calculation of the expected cost associated with an optimal strategy. Section 5 formulates the transit assignment problem as a variational inequality and discusses the existence of an equilibrium solution. To illustrate the effects of our model on the strategy choices and departure times of passengers, Section 6 presents numerical results from two transit networks. Finally, Section 7 concludes the paper.

2. Notation and network representation

2.1. Network representation

Consider a route network that displays the entire transit system in a static and compact manner. Nodes in a route network consist of origins and destinations and transit stations (or station nodes) where a transit vehicle stops to load and unload passengers. To illustrate, Fig. 1 displays a system with two origin nodes q and o, two destination nodes r and y, and three transit lines l_1, l_2 and l_3. Nodes labeled a, b, c and d are station nodes. In this example, there are four walk arcs, two access arcs (q,a) and (o,b), and two egress arcs (d,r) and (c,y). The remaining arcs correspond to route segments of the three transit lines. As an example, Line 1 or l_1 begins its route at node a, travels to node b, then to node c, and finally terminates at node d. Thus, $\{a,b,c,d\}$ is the route sequence associated to line l_1.

In the route network, the number next to each arc is the "travel time" c_{ij}. For walk arcs, c_{ij} represents the time to walk from i to j. When (i,j) corresponds to a transit-line segment, c_{ij} denotes the difference between, e.g. the scheduled departure times at stations i and j (see the bottom of Fig. 2). In practice, this difference includes the travel time from i to j and the time for passengers to board and alight.

Associated with each transit line, there is a schedule that lists the times at which a transit vehicle must leave its starting station as well as the scheduled arrival or departure times at stations along its route throughout each day. One method for incorporating such temporal information is by using a time-expanded (TE) network (see, e.g. Hamdouch and Lawphongpanich, 2008).

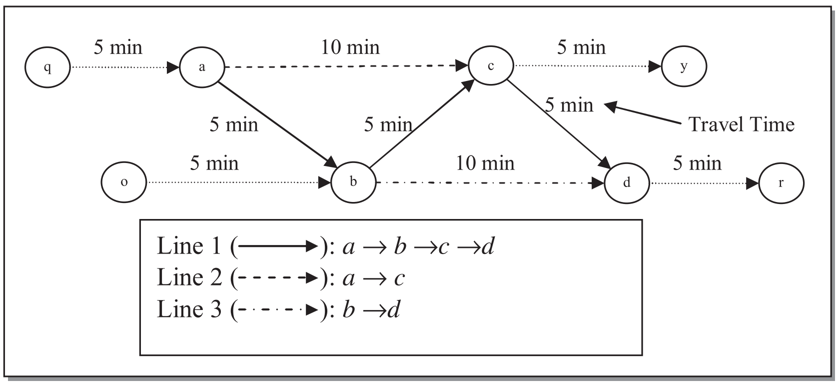
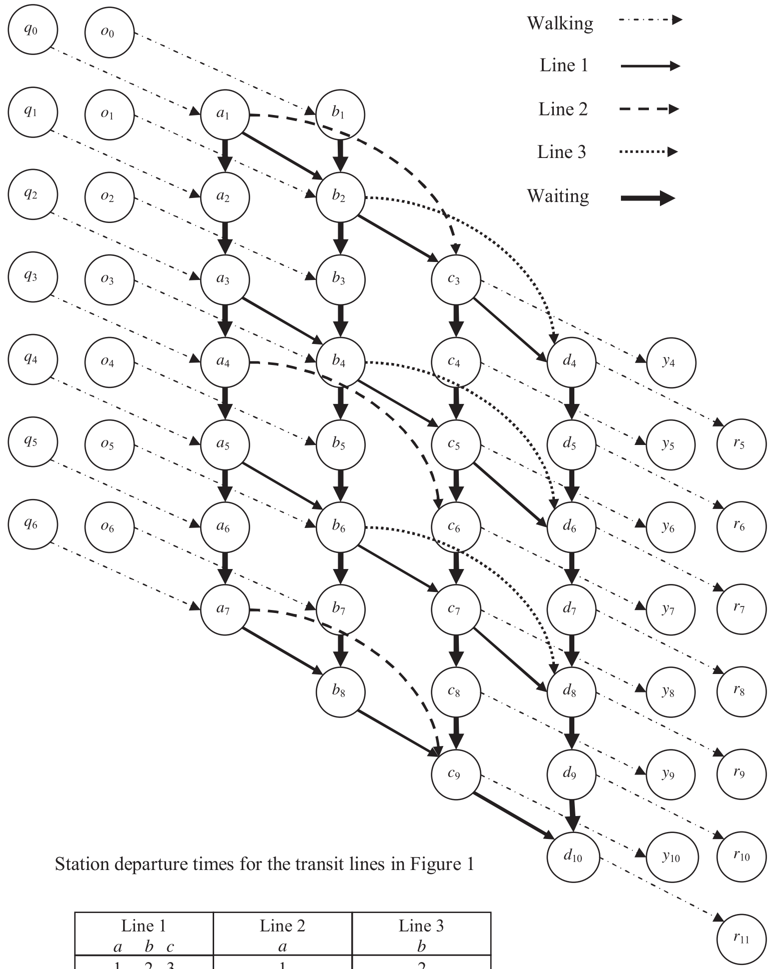


Fig. 1. A route network with three transit lines.



Station departure times for the transit lines in Figure 1

Line 1			Line 2	Line 3
a	b	c	a	b
1	2	3	1	2
3	4	5	4	4
5	6	7	7	6
7	8	9		

Fig. 2. The time-expanded network of the route network in Fig. 1.

In a TE network, we assume that all times (arrivals, departures, travel times, etc.) are in multiples of δ. To simplify our presentation, we ignore δ, thus a travel time of two units means $2 * \delta$ minutes. The value of δ depends on the network data and can be set to 1 min or even 0.5 in case there is an arc with travel time equal to, e.g. 9.5 min. In general, each node i in the route network is expanded into nodes of the form i_t, where $t \in T$, in the TE network. Similarly, an arc (i,j) in the route network is expanded into arcs of the form $(i_t, j_{(t+c_{ij})})$ where, c_{ij} denotes the travel time for arc (i,j) in the route network. As before, we refer to arcs of the form $(i_t, j_{(t+c_{ij})}), (q_t, j_{(t+c_{qj})})$, and $(i_t, r_{(t+c_{ir})})$ as "in-vehicle", "access", and "egress" arcs, respectively. The last two represent walking from an origin to a station and from another station to a destination. In addition, there are arcs of the form (i_t, i_{t+1}) that represent passengers having to wait at station i from time t to $(t+1)$. Thus, a route network, $G(N,A)$, is expanded into a TE network of the form $G(V,E)$ where $V = \{i_t | i \in N, t \in T\}$ and $E = \{(i_t, j_{(t+c_{ij})}) \mid (i,j) \in A,\ t = t_1^{i_n(l)}, t_2^{i_n(l)}, \ldots, t_{m_l}^{i_n(l)}$ for all lines $l \in L$ such as (i,j) is served by l and $i = i_n(l)\} \cup \{(i_t, i_{(t+1)}) | i \in N, t \in T\}$.

Fig. 2 displays a portion of the TE network associated with the route network in Fig. 1 given the scheduled departure times for the three transit lines at the bottom. In Fig. 2, $T = \{0,1,2,\ldots,11\}$, $\delta = 5$ min, line l_1 has four runs ($m_{l_1} = 4$) and lines l_2 and l_3 have 3 runs ($m_{l_2} = m_{l_3} = 3$). At node a, there are four departure times $\left(t_1^{a(l_1)} = 1, t_2^{a(l_1)} = 3, t_3^{a(l_1)} = 5 \text{ and } t_4^{a(l_1)} = 7\right)$ for transit line l_1, therefore arc (a,b) is expanded into four TE arcs (a_1,b_2), (a_3,b_4), (a_5,b_6) and (a_7,b_8). As an example of paths, $p = q_2 \to a_3 \to b_4 \to c_5 \to d_6 \to r_7$ corresponds to passengers leaving node q at time 2, arriving at station a at time 3, boarding Line 1 at time 3, arriving station d (via stations b and c) at time 6, and walking from there to reach r at time 7.

Theoretically, when δ is small, the route network has to be expanded by a large number of dummy arcs. However, in finding the equilibrium solution, it is not necessary to explicitly construct and maintain the TE network in computer memory. Time-expanded nodes and arcs can often be generated as needed using the route network, its parameters, and the scheduled departure times for all transit lines.

2.2. Model assumptions

Many assumptions are made within the model and are presented below.

(i) All transit services arrive on time. Timetable is sufficiently reliable and can be considered as deterministic.
(ii) Every passenger strongly prefers to sit when provided with an available seat. Any standing passenger will, therefore, attempt to get a seat at a station.
(iii) When loading a vehicle, standing on-board passengers continuing to the next station have priority to get available seats and waiting passengers are loaded on a First-Come-First-Serve (FCFS) principle.
(iv) The stimulus of a standing passenger to sit increases with his/her remaining journey length and time already spent on-board. For an on-board passenger, the longer his/her time already spent on board, the higher stimulus for him/her to get a seat. Moreover, passengers will experience high stimulus to get available seats if their remaining journeys are relatively long.
(v) Transit fares are collected based on arcs. If the fares are not directly proportional to the travel distance or time, one can construct a direct in-vehicle arc between each pair of connected nodes in the TE network. The trade-off is an increase number of arcs in the TE network (see, e.g. Lo et al. (2003) for more details).
(vi) All wait arcs have zero fares, zero penalties and infinite capacities.
(vii) All access and egress arcs have zero fares and infinite capacities. However, there are penalties associated with these arcs to, e.g. account for lost opportunities associated with early departure and arrivals outside the desired interval. Typically, these penalties are different for different groups because of their desired arrival intervals.
For egress arcs, i.e., arcs of the form $(i_t, r_{(t+c_{ir})})$, where c_{ir} is the time to walk from station node i to destination r, one form of such penalty is as follows:

$$e_{ir}^{t,g} = \eta_1^g \max\{0, t^-_{(q,r)}(g) - (t + c_{ir})\} + \eta_2^g \max\{0, (t + c_{ir}) - t^+_{(q,r)}(g)\}. \tag{1}$$

For access arcs, one such penalty is based on the latest possible time (denoted as $LT^g_{(q,r)}$) a passenger can depart from his or her destination and arrive within the desired interval. Nguyen et al. (2001) provide a procedure for computing $LT^g_{(q,r)}$by traversing the TE network in a reverse topological order. Given $LT^g_{(q,r)}$, the following is a departure penalty for access arc $(q_t, i_{(t+c_{qi})})$

$$e_{qi}^{t,g} = \eta_3^g \max\{0, LT^g_{(q,r)} - t\}. \tag{2}$$

(viii) All in-vehicle arcs have transit fares, seat and standing capacities. In addition, a standing passenger will experience an additional in-vehicle congestion discomfort whereas a sitting passenger will not experience any congestion discomfort. For example, such a discomfort function can be defined as follows:

$$\hat{e}_{ij}^t(f_{ij}^t) = 0 \tag{3}$$

$$\hat{e}_{ij}^t(\tilde{f}_{ij}^t) = \eta_4 \left(\frac{\tilde{f}_{ij}^t}{\tilde{u}_{ij}^t}\right)^2, \tag{4}$$

where $\tilde{u}_{ij}^t$ is the standing capacity of arc (i,j) at time t, and f_{ij}^t and $\tilde{f}_{ij}^t$ are the total number of sitting and standing passengers on arc (i,j) at time t.

3. Travel strategies and dynamic loading

In this section, we introduce the concept of travel strategies and show how to calculate the expected cost of a strategy using a dynamic loading procedure that computes arc flows and arc probabilities in the TE network.

3.1. Travel strategies

We assume that passengers use strategies when traveling. To specify a strategy (denoted as s), passengers must provide, at each node i_t, a preference set $E_i^{s,t} \subseteq I^+(i_t)$ of subsequent TE nodes at which they want to reach via a transit line, walking, or waiting at a station. Theoretically, the preference set may include an other index $\tau \leqslant t$ when passengers are loaded on an FCFS basis and τ denotes the time passengers arrive at node i (see Hamdouch and Lawphongpanich (2008) for more details). The order in which nodes are listed in $E_i^{s,t}$ gives the passengers' preference, i.e., the first node in the set is the most preferred and the last is the least. To each node in the preference set that can be reached via an in-vehicle arc, we associate an index representing the corresponding transit line. Adding this index will model the situation when there are overlapping transit lines serving the same pair of TE nodes. The user-preference set may be empty at some nodes. In particular, nodes irrelevant to the strategy, inaccessible from the origin, or unable to access the destination of an OD pair have empty user-preference sets.

For example, Table 1 displays two valid strategies s^1 and s^2 for OD pair (q,r), each starting at time 0 and 2 respectively (rows in which every node has an empty user-preference set are not shown in the table). For passengers using s^1, the order of nodes in the user-preference set at node a_1, i.e., $[b_2(l_1), c_3(l_2), a_2]$, indicates that the passenger prefers Line 1 over Line 2 and

Table 1
Two travel strategies for OD pair (q,r).

Node	Pref.	Node	Pref.	Node	Pref.	Node	Pref.	Node	Pref.
Strategy: s^1									
q_0	$[a_1]$	a_1	$[b_2(l_1), c_3(l_2), a_2]$	b_2	$[c_3(l_1), d_4(l_3), b_3]$	c_3	$[d_4(l_1), c_4]$	d_4	$[r_5]$
q_1	$\emptyset$	a_2	$[a_3]$	b_3	$[b_4]$	c_4	$[c_5]$	d_5	$\emptyset$
q_2	$\emptyset$	a_3	$[b_4(l_1), a_4]$	b_4	$[c_5(l_1), d_6(l_3), b_5]$	c_5	$[d_6(l_1), c_6]$	d_6	$[r_7]$
q_3	$\emptyset$	a_4	$[c_6(l_2), a_5]$	b_5	$[b_6]$	c_6	$[c_7]$	d_7	$\emptyset$
q_4	$\emptyset$	a_5	$[b_6(l_1), a_6]$	b_6	$[d_8(l_3), c_7(l_1), b_7]$	c_7	$[d_8(l_1), c_8]$	d_8	$[r_9]$
q_5	$\emptyset$	a_6	$[a_7]$	b_7	$[b_8]$	c_8	$[c_9]$	d_9	$\emptyset$
q_6	$\emptyset$	a_7	$[c_9(l_2)]$	b_8	$[c_9(l_1)]$	c_9	$[d_{10}(l_1)]$	d_{10}	$[r_{11}]$
Strategy: s^2									
q_2	$[a_3]$	a_3	$[b_4(l_1), a_4]$	b_4	$[c_5(l_1), d_6(l_3), b_5]$	c_5	$[d_6(l_1), c_6]$	d_6	$[r_7]$
q_3	$\emptyset$	a_4	$[a_5]$	b_5	$[b_6]$	c_6	$[c_7]$	d_7	$\emptyset$
q_4	$\emptyset$	a_5	$[b_6(l_1), a_6]$	b_6	$[d_8(l_3), c_7(l_1), b_7]$	c_7	$[d_8(l_1), c_8]$	d_8	$[r_9]$
q_5	$\emptyset$	a_6	$[a_7]$	b_7	$[b_8]$	c_8	$[c_9]$	d_9	$\emptyset$
q_6	$\emptyset$	a_7	$[b_8(l_1)]$	b_8	$[c_9(l_1)]$	c_9	$[d_{10}(l_1)]$	d_{10}	$[r_{11}]$

Line 2 over waiting. Fig. 3 displays the strategy subgraph (SS) induced by s^1. In the figure, there are several directed paths emanating from q_0 and reaching node r at four different times 5, 7, 9 and 11. Using this strategy, the arrival time at the destination depends on, e.g. whether passengers are able to board Line 1, Line 2 or have to wait at station a for the next vehicle. For the latter, a positive amount of flow on (a_1, a_2) indicates that vehicles for Lines 1 and 2 departing at time 1 are full.

The expected cost of a strategy s depends directly on the success-to-sit probabilities $\pi_{ij}^{s,t}(X)$, the success-to-stand probabilities $\tilde{\pi}_{ij}^{s,t}(X)$ and the failure-to board probabilities $\hat{\pi}_{ij}^{s,t}(X)$ associated with vehicles serving arcs (i,j) at time t. The procedure for determining these probabilities involves *loading*the TE network according to a given strategy assignment vector X and is an extension to the one in Hamdouch and Lawphongpanich (2008). The loading process computes arc flows, $f_{ij}^{s,t}$ and $\tilde{f}_{ij}^{s,t}$, and arc probabilities, $\pi_{ij}^{s,t}(X)$, $\tilde{\pi}_{ij}^{s,t}(X)$ and $\hat{\pi}_{ij}^{s,t}(X)$, by processing TE nodes one at a time and in a topological and chronological (T&C) order, i.e., a node with no predecessor and the smallest time index is processed first. In loading the TE network, we should ensure that, at each node i_t, the summation of probabilities associated to outgoing TE arcs should be equal to 1:

$$\sum_{j_{(t+c_{ij})} \in I^+(i_t)} (\pi_{ij}^{s,t}(X) + \tilde{\pi}_{ij}^{s,t}(X)) + (\hat{\pi}_{ij}^{s,t}(X)) = 1, \quad \forall i_t \in V. \tag{5}$$

3.2. Flow priority and FCFS rule

Consider processing node i at time t. Because of the ordering in which the nodes are processed, the amounts of flows on arcs terminating at i_t are known. These flows represent the number of passengers using various strategies arriving at station i at time t on various transit lines as well as those who has been waiting. Let $L_i^t \subset L$ be the set of lines that traverse node i at time t. For each line $l \in L_i^t$, node i can be viewed as $i = i_n(l)$, $(1 \leqslant n \leqslant n_l)$. For each line l such that $1 < n < n_l$ (i.e $i_n(l)$ is neither the starting nor the ending node of line l), if $t_n = t - c_{i_{n-1}(l)i}$ denotes the time that transit vehicle l leave node $i_{n-1}(l)$, then, for each strategy s such that the first choice in the user preference set $E_i^{s,t}(1) = \{i_{n+1}(l)\}$, the number of sitting passengers using strategy s on arc $(i_{n-1}(l), i)$ have priority to get a seat on arc $(i, i_{n+1}(l))$. Therefore, the first priority class, $W_i^{t,sit}$, consists of all sitting passengers that have continuance priority at node i_t:

$$W_i^{t,sit} = \cup_s \left\{ z_i^{s,t,sit}, E_i^{s,t}(1) = \{i_{n+1}(l)\} \right\}, \tag{6}$$

where $z_i^{s,t,sit}$ denotes the number of sitting passengers using strategy s who have continuance priority at node i_t:

$$z_i^{s,t,sit} = \sum_{l \in L_i} \pi_{i_{n-1}(l)i}^{s,t_n} z_{i_{n-1}(l)}^{s,t_n} \tag{7}$$

The second priority class $W_i^{t,std}$ consists of all standing passengers who have continuance priority at node i_t. The rule to load these standing passengers is based on the stimulus to sit that increases with the passenger's remaining journey length and time already spent on-board. For each strategy s such that $E_i^{s,t}(1) = \{i_{n+1}(l)\}$ let, $t_1^{l,s}$ and $t_2^{l,s}$, denote the boarding time on and alighting time from transit vehicle l of passengers using strategy s. The boarding and alighting times can be computed from the first choice of the user preference sets as follows:

$$t_1^{l,s} = t' \text{ such that } E_{j'}^{s,t'}(1) = i_{n'}(l)(1 \leqslant n' \leqslant n_l) \text{ and } E_k^{s,t'-c_{kj'}}(1) \neq i_{n'-1}(l)\left(k_{t'-c_{kj'}} \in I^-(j'_{t'})\right) \tag{8}$$

$$t_2^{l,s} = t'' \text{ such that } E_{j''}^{s,t''}(1) = i_{n''}(l)(1 \leqslant n'' \leqslant n_l) \text{ and } E_k^{s,t''+c_{kj''}}(1) \neq i_{n''+1}(l)\left(k_{t''+c_{kj''}} \in I^+(j''_{t''})\right) \tag{9}$$

Thus, the time the passenger has already spent standing in the vehicle and the remaining journey length at arc $(i, i_{n+1}(l))$ can be calculated as: $t_{ii_{n+1}(l)}^{1,s} = t - t_1^{l,s}$ and $t_{ii_{n+1}(l)}^{2,s} = t_2^{l,s} - t$. As in Sumalee et al. (2009), the stimulus to get a seat can be measured as:

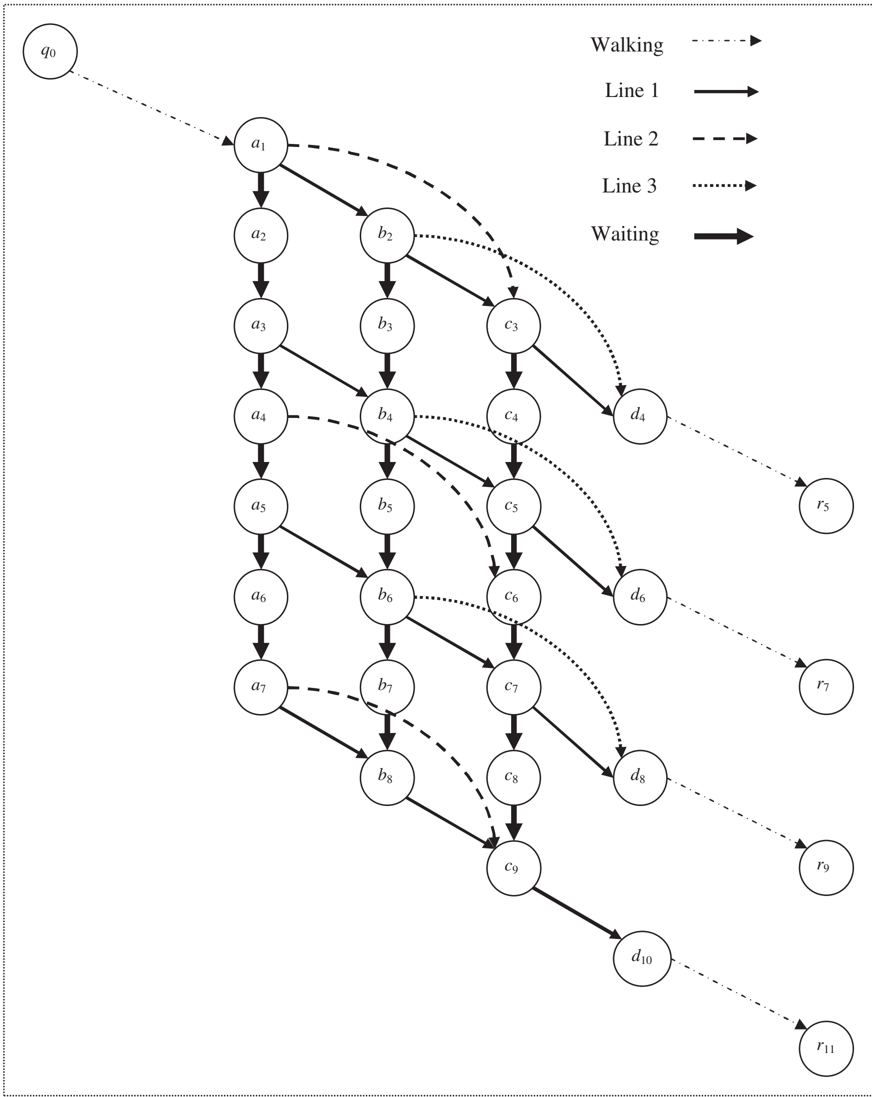


Fig. 3. The strategy subgraph induced by strategy s^1.

$$p^{s,t}_{ii^{n+1}_l} = \sqrt{\eta_5\left(t^{1,s}_{ii_{n+1}(l)}\right)^2 + \eta_6\left(t^{2,s}_{ii_{n+1}(l)}\right)^2}, \tag{10}$$

where η_5 and η_6 are stimulus parameters in which $\eta_5 + \eta_6 = 1$, $\eta_5 \geqslant 0$, $\eta_6 \geqslant 0$. The number of standing passengers using strategy s (with stimulus value $p^{s,t}_{ii^{n+1}_l}$) who have continuance priority at node i_t is equal to:

$$z^{s,t,std}_i = \sum_{l \in L_i} \tilde{\pi}^{s,t_n}_{i_{n-1}(l)i} z^{s,t_n}_{i_{n-1}(l)}, \tag{11}$$

and $W^{t,std}_i$ is defined as:

$$W^{t,std}_i = \cup_s \left\{ z^{s,t,std}_i, E^{s,t}_i(1) = \{i_{n+1}(l)\} \right\}. \tag{12}$$

The last class of passengers consists of all passengers arriving at node i at time t on various transit lines and who want to transfer to other transit lines as well as those who has been waiting at node i_t. To enforce the FCFS rule, we classify these passengers according to their arrival time at node i. We denote by $z^{s,t,\tau}_i$ the number of passengers using strategy s at node i_t and having reached node i at time $\tau \leqslant t$ and group all flow into a class $W^{t,\tau}_i$ restricted to passengers having reached node i at time τ:

$$W^{t,\tau}_i = \cup_s \{ z^{s,t,\tau}_i, E^{s,t}_i \neq \emptyset \}. \tag{13}$$

The flow $z_i^{s,t,\tau}$ is computed according to the recursion:

$$z_i^{s,t,\tau} = \begin{cases} z_i^{s,t-1} \sum_{j_{\hat{t}} \in I^+(i_{t-1})} \hat{\pi}_{ij}^{s,t-1} & \text{if } \tau \leqslant t-1 (\hat{t} = t-1+c_{ij}) \\ \sum_{k_{t_c} \in I^-(i_t)} \left(\pi_{ki}^{s,t_c} + \tilde{\pi}_{ki}^{s,t_c}\right) z_k^{s,t_c} & \text{if } \tau = t (t_c = t - c_{ki}) \text{ and} \\ & (k,i) \in l \text{ and } \left(i, E_i^{s,t,\tau}(1)\right) \in l' \neq l, \end{cases} \tag{14}$$

where the first term denotes passengers who reach node i before time $\tau - 1$ and the second term denotes those who reach node i at time t.

3.3. loading process

Consider processing node i at time t. The loading process starts by loading passengers belonging to the first priority class $W_i^{t,sit}$. For each line $l \in L_i^t$ and for each strategy s such that $E_i^{s,t}(1) = \{i_{n+1}(l)\}$, the number of sitting passengers using strategy s on arc $(i_{n-1}(l), i)$, $f_{i_{n-1}(l)i}^{s,t_n}$, have priority to get a seat on arc $(i, i_{n+1}(l))$ and are added to $f_{ii_{n+1}(l)}^{s,t}$. Then, the sitting capacities of all arcs $(i, i_{n+1}(l))$ are updated and the process ends for class $W_i^{t,sit}$.

The process will then focus on the second priority class $W_i^{t,std}$. If the sitting capacity of arc $(i, i_{n+1}(l))$ is exhausted, then for each strategy s such that $E_i^{s,t}(1) = \{i_{n+1}(l)\}$, the number of passengers using strategy s on arc $(i_{n-1}(l), i)$, $\tilde{f}_{i_{n-1}(l),i}^{s,t_n}$, will remain standing and are added to $\tilde{f}_{ii_{n+1}(l)}^{s,t}$. Otherwise, standing passengers, $\tilde{f}_{i_{n-1}(l),i}^{s,t_n}$ will attempt to get a seat on arc $(i, i_{n+1}(l))$. The rule to load these standing passengers is based on the stimulus to sit that increases with the passenger's remaining journey length and time already spent on-board. The number of standing passengers using strategy s who can get a seat on arc $(i, i_{n+1}(l))$ is equal to:

$$pr_{i_{n-1}(l),i}^{s,t_n} = \min\left\{ \frac{p_{ii_l^{n+1}}^{s,t} \tilde{f}_{i_{n-1}(l),i}^{s,t_n}}{\sum_{s' \neq s} p_{ii_l^{n+1}}^{s',t} \tilde{f}_{i_{n-1}(l),i}^{s,t_n}} u_{ii_{n+1}(l)}^t, \tilde{f}_{i_{n-1}(l),i}^{s,t_n} \right\} \tag{15}$$

If $\frac{p_{ii_l^{n+1}}^{s,t} \tilde{f}_{i_{n-1}(l),i}^{s,t_n}}{\sum_{s' \neq s} p_{ii_l^{n+1}}^{s',t} \tilde{f}_{i_{n-1}(l),i}^{s,t_n}} u_{ii_{n+1}(l)}^t \geqslant \tilde{f}_{i_{n-1}(l),i}^{s,t_n}$, then all standing passengers using strategy s will get a seat and are added to $f_{ii_{n+1}(l)}^{s,t}$. Otherwise a proportion $pr_{i_{n-1}(l),i}^{s,t_n}$ will get a seat and are added to $f_{ii_{n+1}(l)}^{s,t}$ and the remaining passengers $\tilde{f}_{i_{n-1}(l),i}^{s,t_n} - pr_{i_{n-1}(l),i}^{s,t_n}$ will keep standing and are added to $\tilde{f}_{ii_{n+1}(l)}^{s,t}$.

After loading all on-board passengers who want to continue their journey in the same transit vehicle, the process will load, in the FCFS order, the remaining passengers belonging to the classes $W_i^{t,\tau} (\tau \leqslant t)$ who, according to their strategy s, prefer to access arc (i,j) at time t, i.e., load and add to $f_{ij}^{s,t}$ those who arrive earlier before those who arrive later until the remaining capacity of the arc is exhausted $\left(u_{ij}^t + \tilde{u}_{ij}^t = 0\right)$. Those who cannot sit will stand on arc (i,j). Because all these passengers will board in-vehicle arc (i,j) at time t, the stimulus to get a seat will depend only on the alighting time from transit line l corresponding to arc (i,j). Passengers who cannot be loaded must use the wait arc (i_t, i_{t+1}).

Once all arcs emanating from i_t are loaded, the arc probabilities are computed as follows:

$$\pi_{ij}^{s,t} = \frac{f_{ij}^{s,t}}{z_j^{s,t}} \tag{16}$$

$$\tilde{\pi}_{ij}^{s,t} = \frac{\tilde{f}_{ij}^{s,t}}{z_j^{s,t}} \tag{17}$$

$$\hat{\pi}_{ij}^{s,t} = \frac{z_j^{s,t} - f_{ij}^{s,t} - \tilde{f}_{ij}^{s,t}}{z_j^{s,t}}. \tag{18}$$

The loading procedure will be explained in detail on the example in Fig. 4 where we focus on the loading process at nodes o_3, a_3, b_4 and c_5. The parameters used in the stimulus function are set as $\eta_5 = \eta_6 = 0.5$. At node o_3, 15 passengers using strategy s^2 are loaded onto access arc (o_3, b_4). Thus, $f_{ob}^{s^2,3} = 15$, $\pi_{ob}^{s^2,3} = 1$ and $\tilde{\pi}_{ob}^{s^2,3} = 0$ (for access arcs, walking plays the role of sitting and standing is not applicable). At node a_3, 10 passengers using strategy s^1 and 5 passengers using strategy s^3, want to board line 1 and access arc (a,b) at time 3. The stimulus to get a seat depends on the alighting time from line 1, which is time 6 for strategy s^1 and time 5 for strategy s^3. Therefore, the stimulus values are: $p_{ab}^{s^1,3} = \sqrt{0.5(6-3)^2} = 2.12$ and $p_{ab}^{s^3,3} = \sqrt{0.5(5-3)^2} = 1.41$. Using these two stimulus measures, the number of passengers using strategies s^1 and s^3 who will get a seat on arc (a,b) are equal to:

$$f_{ab}^{s^1,3} = \frac{\frac{2.12}{3.53}(10)}{\frac{2.12}{3.53}(10) + \frac{1.41}{3.53}(5)}(10) = 0.75 \cdot 10 = 7.5$$

$$f_{ab}^{s^3,3} = \frac{\frac{1.41}{3.53}(5)}{\frac{2.12}{3.53}(10) + \frac{1.41}{3.53}(5)}(10) = 0.25 \cdot 10 = 2.5,$$

and the number of passengers using strategies s^1 and s^3 who will stand on arc (a,b) are equal to:

$$\tilde{f}_{ab}^{s^1,3} = 10 - 7.5 = 2.5$$
$$\tilde{f}_{ab}^{s^3,3} = 5 - 2.5 = 2.5$$

When the process undergoes node b_4, there are four classes of passengers: $W_b^{4,sit} = \{z_b^{s^1,4,sit} = 7.5, z_b^{s^3,4,sit} = 2.5\}$, $W_b^{4,std} = \{z_b^{s^1,4,std} = 2.5, z_b^{s^3,4,std} = 2.5\}$, $W_b^{4,2} = \{z_b^{s^1,4,2} = 5, z_b^{s^3,4,2} = 3\}$ and $W_b^{4,4} = \{z_b^{s^2,4,4} = 15\}$. The process starts by loading passengers belonging to $W_b^{4,sit}$ where 7.5 passengers using strategy s^1 and 2.5 passengers using s^3 will get a seat on arc (b,c) and the residual seating capacity of arc (b,c) becomes exhausted. After that, the process will load passengers belonging to the class $W_b^{4,std}$ where 2.5 passengers using strategy s^1 and 2.5 passengers using s^3 will be standing on arc (b,c) and the residual capacity of arc (b,c) becomes 5. The next step is to load passengers belonging to the class $W_b^{4,2}$. There are five passengers using s^1 and three passengers using s^3 who want to access arc (b,c). On average, 3.13 passengers using s^1 and 1.87 passengers using s^3 will be standing on arc (b,c). The remaining 1.13 passengers using s^3 will use the waiting arc (b_4,b_5) and the remaining 1.87 passengers using s^1 will board line 3 and get a seat on arc (b,d). The last step is to load passengers belonging to $W_b^{4,4}$. Among the 15 passengers using strategy s^2 and belonging to this class, 3.13 passengers will get a seat and five passengers will be standing on arc (b,d); the remaining 6.87 will use the waiting arc (b_4,b_5).

Finally, at node c_5, all passengers using s^3 will alight from line *1* to take the egress arc (c,y). Therefore, only passengers using strategy s^1 will continue on line *1* and will access arc (c,d). Among these passengers, 7.5 passengers who are already seated will have priority to get a seat on arc (c,d) and the residual seating capacity of arc (c,d) becomes 2.5. Among the remaining 5.63 passengers, 2.5 passengers will get a seat and 3.13 passengers will be standing on arc (c,d). The loading process at nodes o_3, a_3, b_4 and c_5 is summarized in Table 2.

3.4. Expected strategy cost

Similar to the arc probabilities, $\alpha_i^{s,t}(X)$ is a node arrival probability or the probability that a passenger using strategy s will reach node i at time t. These node arrival probabilities are useful when computing the expected cost of a strategy and can be calculated recursively after computing arc probabilities when processing node i_t:

$$\alpha_i^{s,t}(X) = \begin{cases} 0 & \text{if node } i_t \text{ is not in the strategy} \\ & \quad \text{subgraph of strategy } s \\ 1 & \text{if } i \text{ is an origin and } t \text{ is a} \\ & \quad \text{starting time of strategy } s \\ \sum_{k_{\tilde{t}} \in I^-(i_t)} \alpha_k^{s,\tilde{t}}(X)\pi_{ki}^{s,\tilde{t}}(X) & \text{if node } i_t \text{ is in the strategy} \\ \quad + \sum_{k_{\tilde{t}} \in I^-(i_t)} \alpha_k^{s,\tilde{t}}(X)\tilde{\pi}_{ki}^{s,\tilde{t}}(X) & \quad \text{subgraph of strategy } s, \\ \quad + \alpha_i^{s,(t-1)}(X)\sum_{j_{\hat{t}} \in I^+(i_{(t-1)})} \hat{\pi}_{ij}^{s,(t-1)}(X) & \quad \tilde{t} = t - c_{ki} \text{ and } \hat{t} = t - 1 + c_{ij} \end{cases} \tag{19}$$

Note that the proportion of passengers using strategy s who reach node i_t, $\alpha_i^{s,t}(X)$, is the sum of the proportion of sitting passengers on ingoing arcs $(k_{\tilde{t}}, i_t)$, $\sum_{k_{\tilde{t}} \in I^-(i_t)} \alpha_k^{s,\tilde{t}}(X)\pi_{ki}^{s,\tilde{t}}(X)$, the proportion of standing passengers on arcs $(k_{\tilde{t}}, i_t)$, $\sum_{k_{\tilde{t}} \in I^-(i_t)} \alpha_k^{s,\tilde{t}}(X)\tilde{\pi}_{ki}^{s,\tilde{t}}(X)$, and the proportion of passengers who failed to board the vehicle serving arc (i,j) at time $t-1$, $\alpha_i^{s,(t-1)}(X)\sum_{j_{\hat{t}} \in I^+(i_{(t-1)})} \hat{\pi}_{ij}^{s,(t-1)}(X)$.

Using the arc probabilities $\pi_{ij}^{s,t}(X)$, $\tilde{\pi}_{ij}^{s,t}(X)$ and $\hat{\pi}_{ij}^{s,t}(X)$ and the node arrival probabilities $\alpha_i^{s,t}(X)$, the expected strategy cost can be expressed in terms of arcs in the strategy subgraph as follows:

$$\begin{aligned} C_{(q,r,g)}^s(X) = {} & \sum_{(i,j)\in A}\sum_{t\in T}(\gamma c_{ij} + v_{ij}^t + e_{ij}^{t,g})\alpha_i^{s,t}(X)\pi_{ij}^{s,t}(X) + \sum_{(i,j)\in A}\sum_{t\in T}(\gamma_{travel}c_{ij} + v_{ij}^t + e_{ij}^{t,g} + \hat{e}_{ij}^t(\tilde{f}_{ij}^t))\alpha_i^{s,t}(X)\tilde{\pi}_{ij}^{s,t}(X) \\ & + \sum_{(i,j)\in A}\sum_{t\in T}\gamma_{wait}\alpha_i^{s,t}(X)\hat{\pi}_{ij}^{s,t}(X), \end{aligned}$$

where we assume that node arrival and arc probabilities associated with events occurring prior to the start time of a strategy s are irrelevant and assumed to be zero. In the above expression, the first summation represents the cost associated with sitting passengers that includes the penalty, $e_{ij}^{t,g}$, for arriving outside the desired interval or departing too early. The second summation represents the cost associated with standing passengers that includes the penalty $e_{ij}^{t,g}$ as well as the penalty, $\hat{e}_{ij}^t(\tilde{f}_{ij}^t)$, for being in a crowded vehicle. Finally, the last summation represents the cost associated with waiting.

4. Computation of an optimal strategy

In finding a strategic equilibrium solution, we need to compute, for each triplet (q,r,g), an optimal strategy $s_{(q,r,g)}^*$ with the least expected cost given (or in response to) the current strategy assignment X:

$$C_{(q,r,g)}^{s^*}(X) = \min_{s\in S_{(q,r)}} C_{(q,r,g)}^s(X). \tag{21}$$

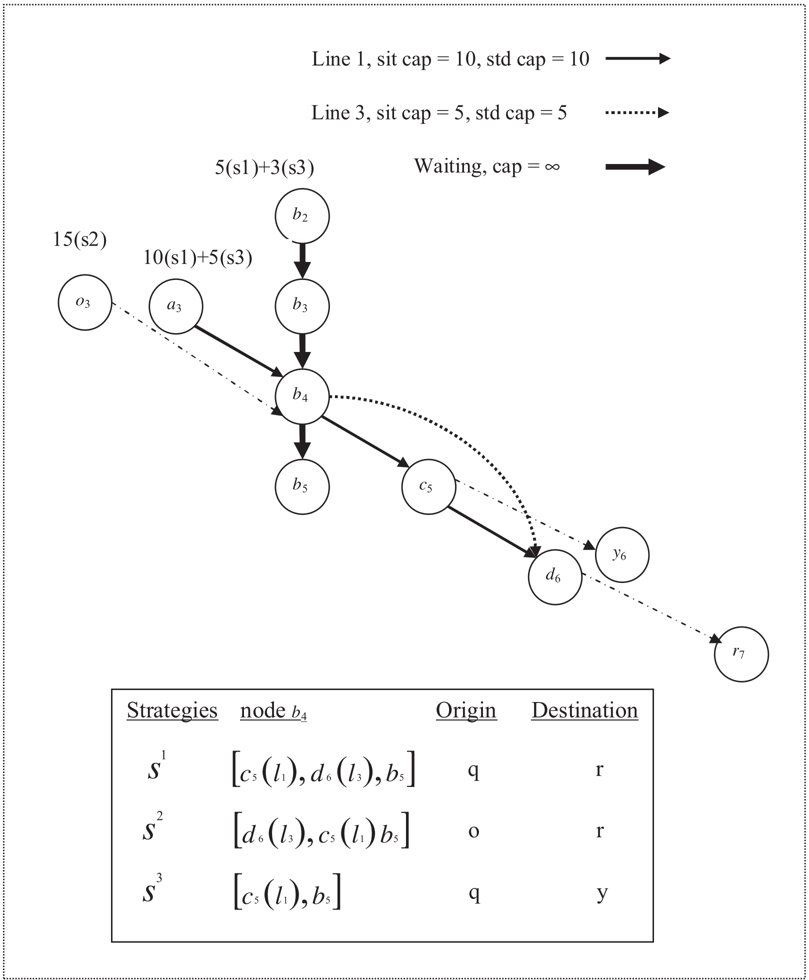


Fig. 4. An example of dynamic loading.

Table 2
Dynamic loading process at nodes o_3, a_3, b_4 and c_5.

	(o_3,a_3)	(a_3,b_4)	(b_4,c_5)	(b_4,d_6)	(c_5,d_6)
$f_{ij}^{s,t}$	$f_{ob}^{s^2,3} = 15$	$f_{ab}^{s^1,3} = 7.5$ $f_{ab}^{s^3,3} = 2.5$	$f_{bc}^{s^1,4} = 7.5$ $f_{bc}^{s^3,4} = 2.5$	$f_{bd}^{s^1,4} = 1.87$ $f_{bd}^{s^2,4} = 3.13$	$f_{cd}^{s^1,5} = 10$
$\tilde{f}_{ij}^{s,t}$	$\tilde{f}_{ob}^{s^2,3} = 0$	$\tilde{f}_{ab}^{s^1,3} = 2.5$ $\tilde{f}_{ab}^{s^3,3} = 2.5$	$\tilde{f}_{bc}^{s^1,4} = 5.63$ $\tilde{f}_{bc}^{s^3,4} = 4.37$	$\tilde{f}_{bd}^{s^1,4} = 0$ $\tilde{f}_{bd}^{s^2,4} = 5$	$\tilde{f}_{cd}^{s^1,5} = 3.13$
$\pi_{ij}^{s,t}$	$\pi_{ob}^{s^2,3} = 1$	$\pi_{ab}^{s^1,3} = \frac{7.5}{10}$ $\pi_{ab}^{s^3,3} = \frac{2.5}{5}$	$\pi_{bc}^{s^1,4} = \frac{7.5}{15}$ $\pi_{bc}^{s^3,4} = \frac{2.5}{8}$	$\pi_{bd}^{s^1,4} = \frac{1.87}{15}$ $\pi_{bd}^{s^2,4} = \frac{3.13}{15}$	$\pi_{cd}^{s^1,5} = \frac{10}{13.13}$
$\tilde{\pi}_{ij}^{s,t}$	$\tilde{\pi}_{ob}^{s^2,3} = 0$	$\tilde{\pi}_{ab}^{s^1,3} = \frac{2.5}{10}$ $\tilde{\pi}_{ab}^{s^3,3} = \frac{2.5}{5}$	$\tilde{\pi}_{bc}^{s^1,4} = \frac{5.63}{15}$ $\tilde{\pi}_{bc}^{s^3,4} = \frac{4.37}{8}$	$\tilde{\pi}_{bd}^{s^1,4} = 0$ $\tilde{\pi}_{bd}^{s^2,4} = \frac{5}{15}$	$\tilde{\pi}_{cd}^{s^1,5} = \frac{3.13}{13.13}$
$\hat{\pi}_{ij}^{s,t}$	$\hat{\pi}_{ob}^{s^2,3} = 0$	$\hat{\pi}_{ab}^{s^1,3} = 0$ $\hat{\pi}_{ab}^{s^3,3} = 0$	$\hat{\pi}_{bc}^{s^1,4} = 0$ $\hat{\pi}_{bc}^{s^3,4} = \frac{1.13}{8}$	$\hat{\pi}_{bd}^{s^1,4} = 0$ $\hat{\pi}_{bd}^{s^2,4} = \frac{6.87}{15}$	$\hat{\pi}_{cd}^{s^1,5} = 0$

The construction of the optimal strategy s^* to the above problem is achieved by a procedure that plays the role of the shortest path algorithm in standard schedule-based transit assignment. It is based on dynamic programming and uses the information (strategic flows in the classes $W_i^{t,sit}$, $W_i^{t,std}$ and $W_i^{t,\tau}(\tau \leqslant t)$) generated by the loading process.

Since the computation of the expected cost $C^{s^*}_{(q,r,g)}(X)$ involves the access probabilities associated with the optimal (unknown) strategy being constructed, these probabilities have to be computed, in a reverse T&C order. The resulting procedure resembles the dynamic loading process described in Section 3 with the small difference that the flow corresponding to the optimal strategy being computed is set to zero. While this *micro-loading* phase (loading of zero or virtual flow) has been implemented in the case without seat availability consideration, new challenges arise when seat and standing capacities are taken into account. Indeed, since loading is performed in reverse T&C order, one might be unaware of the sitting/standing status of the virtual flow at loading time. To make up for this, we analyze all situations:

(i) The virtual (zero) flow is a sitting virtual passenger and the micro-loading is performed over the set $W_i^{t,sit} \cup \{s^*\}$ yielding the expected cost $w_i^{s^*,t,sit}$ for reaching destination r from node $i(=i_n(l))$ at time t.
(ii) The virtual (zero) flow is a standing virtual passenger and the micro-loading is performed over the sets $W_i^{t,sit}$ and $W_i^{t,std} \cup \{s^*\}$ yielding the expected cost $w_i^{s^*,t,std,n'}$ for reaching destination r from node $i(=i_n(l))$ at time t and boarding line l at node $i_{n'}(l)(1 \leqslant n' \leqslant n)$.
(iii) The virtual (zero) flow will arrive at node i at time $\tau(\tau = 1,2,\ldots,t)$ and will try to board transit line l at node $i = i_n(l)$. The micro-loading will then be performed over the sets $W_i^{t,sit}, W_i^{t,std}$ and $W_i^{t,\tau} \cup \{s^*\}$ yielding the expected cost $w_i^{s^*,t,\tau}$ for reaching destination r from node $i(=i_n(l))$ at time t.

The user-preference set $E_i^{s^*,t}$ and the expected costs $w_i^{s^*,t,sit}, w_i^{s^*,t,std,n'}$ and $w_i^{s^*,t,\tau}$ are computed by scanning TE nodes in reverse T&C order and applying Bellman's generalized recursion.

Ending Conditions:

For the current destination r **associated to** s^*,

(i) Set $E_r^{s^*,t} = \emptyset, \forall t \in T$.
(ii) Set $w_r^{s^*,t,std} = 0, \forall t \in T$.
(iii) Set $w_r^{s^*,t,std,n'} = 0, \forall t \in T$, and $1 \leqslant n' \leqslant n$.
(iv) Set $w_r^{s^*,t,\tau} = 0, \forall t \in T$, and $\tau = 1, \ldots, t$.

For the destination $\hat{r} \neq r$ **not covered by** s^*,

(i) Set $E_{\hat{r}}^{s^*,t} = \emptyset, \forall t \in T$.
(ii) Set $w_{\hat{r}}^{s^*,t,std} == \infty, \forall t \in T$.
(iii) Set $w_{\hat{r}}^{s^*,t,std,n'} = \infty, \forall t \in T$, and $1 \leqslant n' \leqslant n$.
(iv) Set $w_{\hat{r}}^{s^*,t,\tau} = \infty, \forall t \in T$, and $\tau = 1, \ldots, t$.

Recursions at node $i_t(i \neq q)$:

To compute the user-preference set $E_i^{s^*,t}$ and the expected costs at node i_t, we must calculate the arc probabilities $\pi_{ij}^{s^*,t}$, $\tilde{\pi}_{ij}^{s^*,t}$ and $\hat{\pi}_{ij}^{s^*,t}$. As mentioned before, to make up for the unawareness of the sitting/standing status at the current time t, we consider three cases.

(i) With continuance priority and sitting:

To consider this case, we should have at least one transit line $l \in L_i^t$ such that $i = i_n(l)$ and $1 < n < n_l$. In this case, the virtual passenger using strategy s^* is added to the first priority class $W_i^{t,sit}$ and has continuance priority to sit on arc (i,j^1) where j^1 is the first element of the set $E_i^{s^*,t}$. Node $j^1 = E_i^{s^*,t}(1)$ is determined as:

$$j^1 = \arg\min_{l \in L_i: 1<n<n_l} \left\{ \varphi^{s^*,t}_{i_{n+1}(l)} \right\}, \tag{22}$$

where

$$\varphi_j^{s^*,t} = \gamma c_{ij} + v_{ij}^t + e_{ij}^{t,g} + \begin{cases} w_j^{s^*,t_c,sit}, & \text{if } (i,j) \text{ and } (j,k^1) \text{ belong} \\ & \text{to same transit line}, \\ w_j^{s^*,t_c,t_c}, & \text{otherwise}, \end{cases} \tag{23}$$

$t_c = t + c_{ij}, k^1$ is the first element in the user-preference set $E_j^{s^*,t_c}$ and $E_j^{s^*,t_c}, w_j^{s^*,t_c,sit}$ and $w_j^{s^*,t_c,t_c}$ are computed from the previous recursion.

After determining the first element of $E_i^{s,t}$, the expected cost, $w_i^{s^*,t,sit}$ is calculated as:

$$w_i^{s^*,t,sit} = \pi_{ij^1}^{s^*,t} \varphi_{j^1}^{s^*,t} = \varphi_{j^1}^{s^*,t}. \tag{24}$$

(ii) With continuance priority and standing: As in the first case, we should have at least one transit line $l \in L_i^t$ such that $i = i_n(l)$ and $1 < n < n_l$. In this case, the virtual passenger using strategy s^* is added to the second class $W_i^{t,std}$. Passengers belonging to this class have continuance priority to access arc (i,j^1) where j^1 is the first element of the set $E_i^{s^*,t}$ and is computed as before. Because passengers belonging to the first class $W_i^{s,sit}$ are loaded first, the virtual standing passenger using s^* could find a seat or keep standing depending on the available sitting capacity of arc (i,j^1). The computation of the expected cost for reaching destination r from node i_t depends on the stimulus to sit that requires the knowledge of the boarding time on and alighting time from transit vehicle l of passengers using strategy s^*. Because TE nodes are scanned in reverse T&C order, we can identify the alighting time from previous recursions using Eq. (9). However, we cannot know in advance the boarding time from transit vehicle l. At node $i = i_n(l)$, the virtual passenger using s^* can board line l at node $i_{n'}(l)$ where $n' = 1, \ldots, n$. At the current recursion, we compute the expected cost for each value of n' and calculate the real boarding time in subsequent recursions using Eq. (8) (see the example in Fig. 5). For each $n' = 1, \ldots, n-1$, we compute the stimulus value, $p_{ij^1}^{s^*,t}$, associated to $z_i^{s^*,t}$ and the expected cost:

$$w_i^{s^*,t,std,n'} = \pi_{ij^1}^{s^*,t}\varphi_{j^1}^{s^*,t} + \tilde{\pi}_{ij^1}^{s^*,t}\tilde{\varphi}_{j^1}^{s^*,t,n'}, \tag{25}$$

where

$$\tilde{\varphi}_{j^1}^{s^*,t,n'} = \gamma c_{ij^1} + v_{ij^1}^t + e_{ij^1}^{t,g} + \hat{e}_{ij^1}^t(\tilde{f}_{ij^1}^t) + \begin{cases} w_{j^1}^{s^*,t_c,std,n'}, & \text{if } (i,j^1) \text{ and } (j^1,k^1) \text{ belong} \\ & \quad \text{to same transit line,} \\ w_{j^1}^{s^*,t_c,t_c}, & \text{otherwise,} \end{cases} \tag{26}$$

$t_c = t + c_{ij^1}$ and k^1 is the first element in the user-preference set $E_{j^1}^{s^*,t_c}$.

(iii) Without continuance priority:
In this case, the virtual passenger using strategy s^* can arrive at node $i = i_n(l)$ at time τ, where $\tau = 1, \ldots, t$.
For each $\tau = 1, \ldots, t$, we load passengers successively over $W_i^{t,sit}$, $W_i^{t,std}$, $W_i^{t,1}, \ldots, W_i^{t,\tau-1}$ and the virtual passenger, $z_i^{s^*,t}$, is added to the class $W_i^{t,\tau}$. Then, the expected cost, $w_i^{s^*,t,\tau}$ is computed using the recursion

$$w_i^{s^*,t,\tau} = \sum_{j_{t_c}\in E_i^{s^*,t}} \pi_{ij}^{s^*,t}\varphi_j^{s^*,t} + \tilde{\pi}_{ij}^{s^*,t}\tilde{\varphi}_j^{s^*,t,n} + \hat{\pi}_{ij}^{s^*,t}\varphi_i^{s^*,t,\tau}, \tag{27}$$

where $t_c = t + c_{ij},\ \varphi_i^{s^*,t,\tau} = \gamma + w_i^{s,t+1,\tau}$ and the optimal preference set $E_i^{s^*,t}$ is the solution of the following combinatorial problem:

$$E_i^{s^*,t} = \arg\min_{E_i^{s,t}\subseteq I^+\{i_t\}}\left\{\sum_{j_{t_c}\in E_i^{s,t}} \pi_{ij}^{s,t}\varphi_j^{s,t} + \tilde{\pi}_{ij}^{s,t}\tilde{\varphi}_j^{s,t,n} + \hat{\pi}_{ij}^{s,t}\varphi_i^{s,t,\tau}\right\} \tag{28}$$

Determining the minimum cost strategy for (q,r,g):

(i) For each $t \in T$, compute

$$j_{t_c^*}^* = \arg\min_{j_{t_c}\in I^+(q_t)}\left\{\gamma c_{qj} + p_{qj}^{t,g} + w_j^{s^*,t_c,t_c}\right\}, \tag{29}$$

where $t_c^* = t + c_{qj^*}$ and $t_c = t + c_{qj}$ Then, set $E_q^{s^*,t} = \left\{j_{t_c^*}^*\right\}$ and

$$w_q^{s^*,t,t} = \gamma c_{qj^*} + e_{qj^*}^{t,g} + w_j^{s^*,t_c^*,t_c^*}. \tag{30}$$

(ii) Compute $t^* = \arg\min_t\{w_q^{s^*,t,t}\}$, the optimal starting time.

(iii) The expected cost for the optimal $s^*$is determined as $C_{(q,r,g)}^{s^*} = w_q^{s^*,t^*,t^*}$.

Note that one application of Bellman's generalized recursion provides optimal strategies from any node to the destination r.

We close this section with an illustration of one iteration of computing an optimal strategy. Let us consider the example depicted in Fig. 5, where we focus on computing the optimal preference set and the expected costs at node b_4 for the origin-destination (q,r) (to simplify, we suppose the existence of one group g). We assume that the discomfort parameter η_4 is set to 0.5, the parameter γ equals 0.5 and the transit fares for arcs (a_3,b_4), (b_4,c_5), (c_5,d_6) and (b_4,d_4) are 0.25, 0.5, 0.75 and 0.5 respectively. From the loading example illustrated in Section 3, we known the existence of four classes of passengers at node b_4: $W_b^{4,sit} = \{z_b^{s^1,4,sit} = 7.5, z_b^{s^3,4,sit} = 2.5\}$, $W_b^{4,std} = \{z_b^{s^1,4,std} = 2.5, z_b^{s^3,4,std} = 2.5\}$, $W_b^{4,2} = \{z_b^{s^1,4,2} = 5, z_b^{s^3,4,2} = 3\}$ and $W_b^{4,4} = \{z_b^{s^2,4,4} = 15\}$. As explained before, we consider three cases. In the first case, we suppose that the virtual passenger using the optimal strategy s^* is added to the first priority class $W_b^{4,sit}$. The first element of the optimal preference set $E_b^{s^*,4}$ is determined as:

$$\begin{aligned}\arg\min_{c,d}\left\{\varphi_c^{s^*,4},\varphi_d^{s^*,4}\right\} &= \arg\min_{c,d}\left\{\gamma c_{bc} + v_{bc}^4 + w_c^{s^*,5,sit}, \gamma c_{bd} + v_{bd}^4 + w_d^{s^*,6,6}\right\} = \arg\min_{c,d}\{0.5+0.5+1.5, 1+0.5+1.5\} \\ &= \{c(l_1)\}.\end{aligned}$$

Then, the virtual passenger using strategy s^* has continuance priority to sit on arc (b_4,c_5) of line 1 and the expected cost $w_b^{s^*,4,sit}$ is calculated as:

$$w_b^{s^*,4,sit} = \pi_{bc}^{s^*,4}\varphi_c^{s^*,4} = \varphi_c^{s^*,4} = 2.5$$

In the second case, we suppose that the virtual passenger using the optimal strategy s^* is added to the second priority class $W_b^{4,std}$. The alighting time from line 1 is time 6 and the boarding time on line 1 for the virtual passenger can be time 3 or time

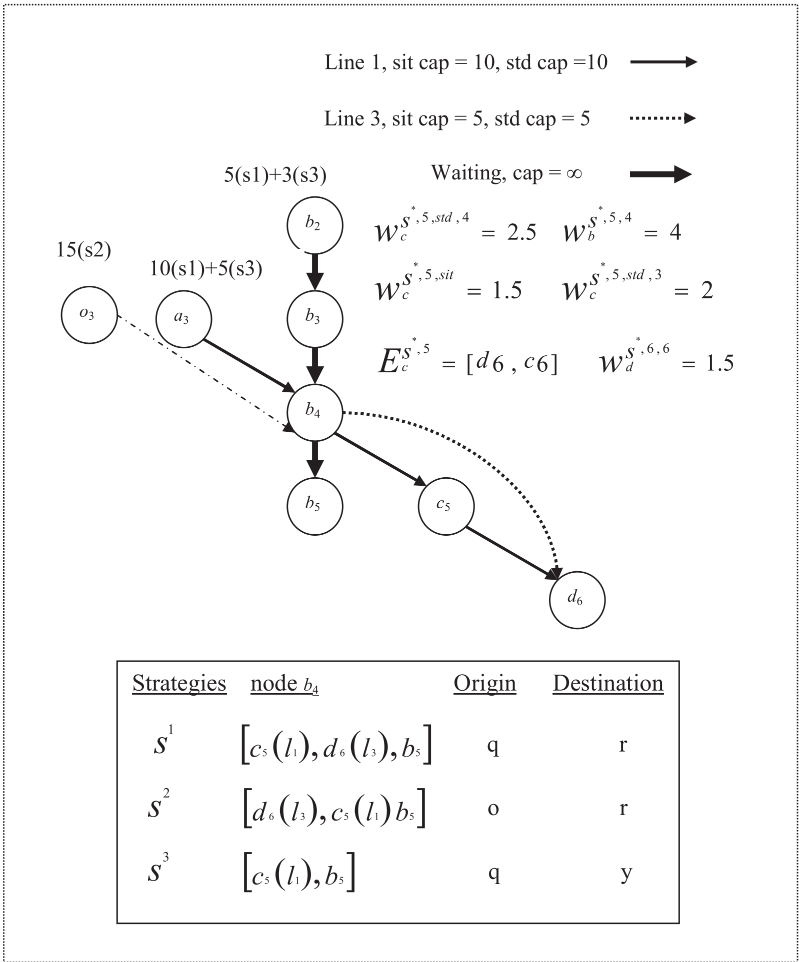


Fig. 5. Computing an optimal strategy at node b_4.

4. As mentioned before, we compute the expected cost for each value and the real boarding time would be known when loading node a_3. Because passengers belonging to the first class $W_b^{4,std}$ are loaded first, the seating capacity of the arc (b,c) becomes zero and therefore $\pi_{bc}^{s^*,4} = 0$ and the expected costs $w_b^{s^*,4,std,3}$ and $w_b^{s^*,4,std,4}$ are computed as:

$$
\begin{aligned}
w_b^{s^*,4,std,3} &= \pi_{bc}^{s^*,4}\varphi_c^{s^*,4} + \tilde{\pi}_{bc}^{s^*,4}\tilde{\varphi}_c^{s^*,4,3} \\
&= \tilde{\varphi}_c^{s^*,4,3} \\
&= \gamma c_{bc} + v_{bc}^4 + \eta_4\left(\frac{\tilde{f}_{bc}^4}{\tilde{u}_{bc}^4}\right)^2 + w_c^{s^*,5,std,3} \\
&= 0.5 + 0.5 + 0.5(1)^2 + 2 = 3.5 \\
w_b^{s^*,4,std,4} &= \tilde{\varphi}_c^{s^*,4,4} \\
&= \gamma c_{bc} + v_{bc}^4 + \eta_4\left(\frac{\tilde{f}_{bc}^4}{\tilde{u}_{bc}^4}\right)^2 + w_c^{s^*,5,std,4} \\
&= 0.5 + 0.5 + 0.5(1)^2 + 2.5 = 4
\end{aligned}
$$

In the last case, the virtual passenger using strategy s^* can arrive at node b at time τ, where $\tau = 2, 4$. It is easy to check that the optimal preference set $E_b^{s^*,4} = [c_5(l_1), d_6(l_3), b_5]$. For $\tau = 2$, the passenger using the optimal strategy s^* is added to the class $W_b^{4,2} = \{z_b^{s^1,4,2} = 5, z_b^{s^3,4,2} = 3\}$ and will be loaded in a similar way as the five passengers using strategy s^1. Therefore, the expected cost $w_b^{s^*,4,2}$ is calculated as:

$$w_b^{s^*,4,2} = \sum_{j \in E_b^{s^*,4}} \pi_{bj}^{s^*,4} \varphi_j^{s^*,4} + \tilde{\pi}_{bj}^{s^*,4} \tilde{\varphi}_j^{s^*,4,4} + \hat{\pi}_{bj}^{s^*,4} \varphi_b^{s^*,4,2} = \tilde{\pi}_{bc}^{s^*,4} \tilde{\varphi}_c^{s^*,4,4} + \pi_{bd}^{s^*,4} \varphi_d^{s^*,4}$$
$$= \frac{3.17}{5}(0.5 + 0.5 + 0.5(1)^2 + 2.5) + \frac{1.87}{5}(1 + 0.5 + 1.5) = 3.658$$

Finally, for $\tau = 4$, the passenger using the optimal strategy s^* is added to the class $W_b^{4,4} = \{z_b^{s^2,4,4} = 15\}$ and will be loaded in a similar way as the 15 passengers using strategy s^2. Then, the expected cost $w_b^{s^*,4,4}$ is computed as:

$$w_b^{s^*,4,4} = \sum_{j \in E_b^{s^*,4}} \pi_{bj}^{s^*,4} \varphi_j^{s^*,4} + \tilde{\pi}_{bj}^{s^*,4} \tilde{\varphi}_j^{s^*,4,4} + \hat{\pi}_{bj}^{s^*,4} \varphi_b^{s^*,4,4} = \pi_{bd}^{s^*,4} \varphi_d^{s^*,4} + \tilde{\pi}_{bd}^{s^*,4} \tilde{\varphi}_d^{s^*,4,4} + \hat{\pi}_{bj}^{s^*,4} \varphi_b^{s^*,4,4}$$
$$= \frac{3.13}{15}(1 + 0.5 + 1.5) + \frac{5}{15}(1 + 0.5 + 0.5(1)^2 + 1.5) + \frac{6.87}{15}(0.5 + 4) = 3.853$$

5. User equilibrium

A strategic assignment vector X^* is in a user equilibrium if no passenger has any incentive to change his or her strategy based on expected strategy costs. X^* is in a user equilibrium if and only if X^* solves the following variational inequality (denoted as $\mathrm{VI}[C(X), \mathcal{X}]$):

$$C(X^*)^T (X - X^*) \geqslant 0, \quad \forall X \in \mathcal{X}, \tag{31}$$

where $C(X)$ is a vector of expected strategy costs associated with X and $\mathcal{X}$ is the set of all feasible SA vectors:

$$\mathcal{X} = \left\{ X : \sum_{s \in S_{(q,r)}} x_{(q,r,g)}^s = D_{(q,r)}^g, \forall (q,r,g) \right\}. \tag{32}$$

This is an extension of the classical concept of a Wardrop equilibrium to capacitated networks, where we substitute for an explicit cost function, an implicit expected strategy cost function by a dynamic loading procedure that ensures the fulfillment of sitting and total line capacities $\left(f_{ij}^t \leqslant u_{ij}^t \text{ and } f_{ij}^t + \tilde{f}_{ij}^t \leqslant u_{ij}^t + \tilde{u}_{ij}^t\right)$.

If the cost function C were continuous, the existence of an equilibrium solution would follow directly from classical fixed point theorems, such as Brouwer's or Kakutani's. However, a technical difficulty arises where a zero strategic flow want to access an arc with zero residual capacity. Similar to Hamdouch and Lawphongpanich (2008), the existence of an equilibrium flow follows from the lower semi-continuity of the cost function C on the set $\mathcal{X}$. The proof will not be repeated here.

If all strategies are available, then algorithms in the literature (see e.g. Facchinei and Pang, 2003) can be used to find a solution to $\mathrm{VI}[C(X), \mathcal{X}]$. However, it is time consuming to generate these strategies a priori, especially for large transit systems. As in Hamdouch and Lawphongpanich (2008), we use the method of successive averages (MSA) that generates strategies one at time by solving a dynamic program. Since the cost function C may fail to be monotone, this method is heuristic in our context. The proposed algorithm first assumes that the TE network is not loaded with passengers (i.e. $z_i^{s,t} = 0$ for all nodes within the TE network). With the empty TE network, the corresponding optimal strategy for each OD pair ($s^*[0]$) could be found by the method described in Section 4 and is set to be the initial strategy set $S^{[0]}$ for network loading. Also, the initial strategic flow $X^{[0]}$ is set to be the travel demand of the corresponding OD pair. Then, the strategic flow $X^{[0]}$ is loaded using the algorithm described in Sections 3.3 and 3.4 for getting the corresponding flow of passengers within the TE network, $z_i^{s,t}(X^{[\alpha]})$, and the expected cost of the strategies, $C(X^{[\alpha]})$. Based on the current flow of passengers, an updated optimal strategy $s^*[\alpha]$ could be found by the method described in Section 4 and the strategic assignment vector for this step, $Y^{[\alpha]}$ with $y_{(q,r,g)}^{s^*[\alpha]} = D_{(q,r)}^g$ and $y_{(q,r,g)}^s = 0, \forall s \neq s^*[\alpha]$, could be setup. With the current strategic assignment vector, the convergence of the algorithm is checked by the following relative gap function (see e.g. Marcotte et al., 2004):

$$g(x) = \frac{C(X^{[\alpha]})^T (X^{[\alpha]} - Y^{[\alpha]})}{C(X^{[\alpha]})^T X^{[\alpha]}} \tag{33}$$

If the above gap function is less than some predetermined tolerance, the algorithm stops with $X^{[\alpha]}$ and $S^{[\alpha]}$ as the optimal solutions. Otherwise, $X^{[\alpha]}$ and $S^{[\alpha]}$ will be updated for the next MSA step by the following equation:

$$S^{[\alpha+1]} = S^{[\alpha]} \cup s^*[\alpha] \tag{34}$$
$$X^{[\alpha+1]} = \frac{1}{(\alpha+1)}(\alpha X^{[\alpha]} + Y^{[\alpha]}), \quad \alpha = 1, 2, \ldots \tag{35}$$

The updated strategy set and strategic flow will be input to the dynamic loading process for getting the updated flow of passengers. A flowchart of the proposed algorithm is shown in Fig. 6.

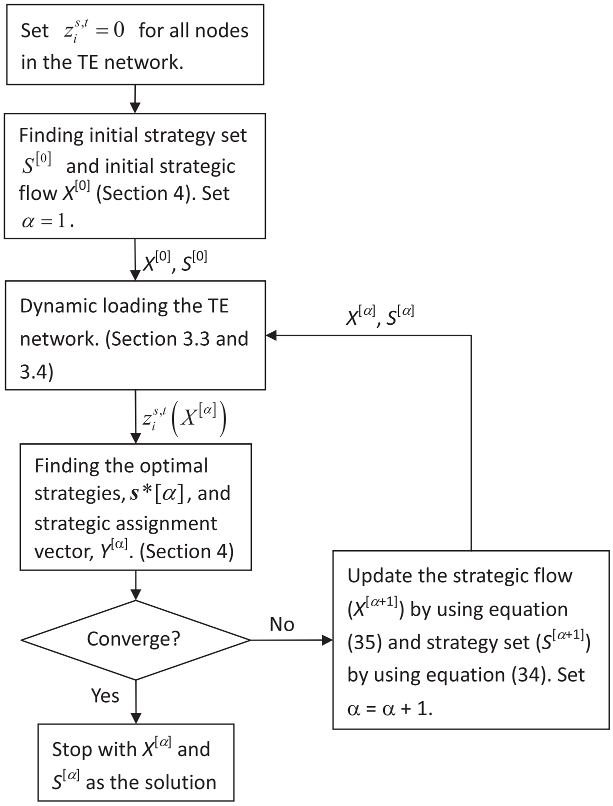


Fig. 6. Flowchart of the proposed solution algorithm.

6. Numerical results

To illustrate our approach, two different test networks: single line network and multiple line network, are setup for demonstrating the significance of the proposed seat model in affecting strategy choices and departure times of passengers.

6.1. Single line network

In order to demonstrate the key features of considering seat availability in the schedule-based transit assignment model, a simple network with a single many-to-one transit line, which shown in Fig. 7, is adopted. Based on this setup, there will be no transit line choice and therefore passengers will only focus on the departure time choice. This example has been studied by Alfa and Chen (1995) and Tian et al. (2007) for a transit system with single destination and by Sumalee et al. (2009) for transit assignment with simulation-based seat allocation.

In this test, we consider a single group of passenger (g_1) traveling in the morning peak period between 7:00 and 9:00 AM ([0,23]) with desired arrival interval of [18,19]. The total passenger demands from the three origins (nodes 1, 2, and 3) to the single destination (node 4) are 450 passengers, i.e., $D^1_{(1,4)} = D^2_{(1,4)} = D^3_{(1,4)} = 450$. The headway of the transit vehicles is assumed to be given and fixed at 10 min and the fare is 0.50 for all TE arcs. The seat and standing capacity $\left(u^t_{ij} \text{ and } \tilde{u}^t_{ij}\right)$ of each of the arc (i,j) and time t are taken as 80 and 120 respectively. Furthermore, penalties for early arrival (η^1_1), late arrival (η^1_2) and early departure (η^1_3) are taken as 0.25, 0.50 and 0.10 respectively for this group of passengers. Also the parameters for the standing discomfort function (η_4) is taken as 10 while the weights in the stimulus function (η_5 and η_6) are taken as 0.5. Lastly, the value of time for waiting (γ_{wait}) and traveling (γ_{travel}) for this group of passenger is taken as 0.75 and 0.50 respectively and the passengers board the vehicles at a FCFS manner. With all the above parameters, we solve the proposed model with the convergence result shown in Fig. 7. This example takes 350 MSA steps to achieve a relative gap less than 0.01.

In order to demonstrate the usefulness of considering seat availability in realistic modeling of passenger flow, the above example is also setup and solved for the case without such consideration (Hamdouch and Lawphongpanich, 2008). The departure and arrival time distribution for these two models are shown in Figs. 8 and 9 respectively. In Figs. 8 and 9, the labels in x-axis mark the beginning time of the corresponding period. In Fig. 8a, it could be seen that for the case that does not consider seat availability, the departure times of different OD pairs are evenly distributed within the period 7:00 AM to 8:50 AM. Among the passengers from different OD pairs, passengers for OD pair 1–4 tend to depart earlier (from 7:00 AM) as

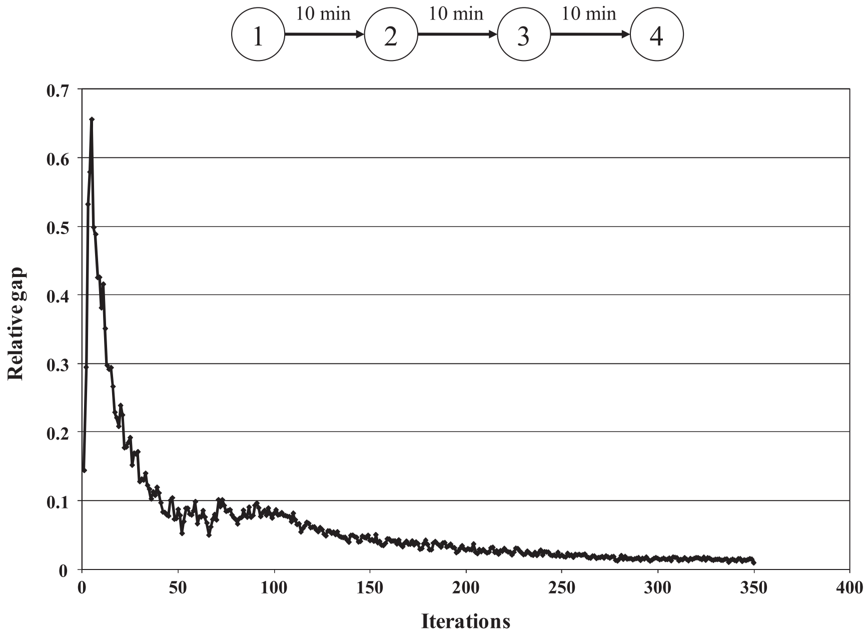


Fig. 7. Single line network and convergence result.

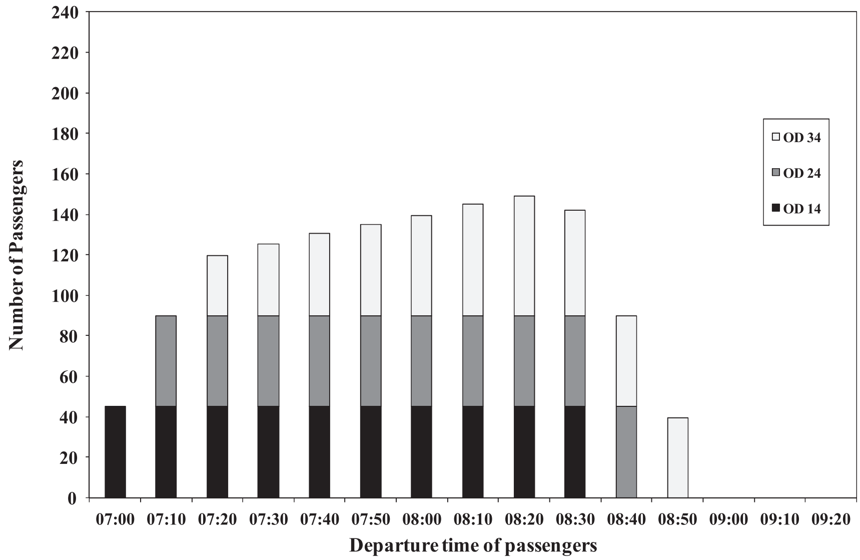
they are furthest away from the destination. For the case that considers seat availability (Fig. 8b), the distribution of departure time is completely different. In this case, the majority of passengers from node 1 are concentrated to depart between 7:30 AM to 8:10 AM, instead of 7:00 AM to 8:50 AM for the previous case. However, for the passengers from node 2 and 3, they choose to depart either earlier or later than the passengers from node 1.

The difference in the distribution of departure time discussed in the previous paragraph is due to the introduction of the seat availability concept in the proposed model. In this study, as passengers with a seat will not experience any in-vehicle discomfort cost ($\hat{e}_{ij}^{t}$), passengers will then change their departure time to trade-off this discomfort cost with their arrival and departure penalties for minimizing the expected cost of their trip. For passengers from node 1, as they are at the first station of the transit line, they only have to compete the seats among themselves. As a result, they have the highest chance to get a seat as compared to the passengers from the downstream stations. With this high chance to get a seat, departure time choice of passengers from node 1 will be more focused on fulfilling their desired arrival time with the consideration of the departure and arrival penalties. Thus, passengers from node 1 will have the least departure time penalty (depart from 40 min before the latest possible time of departure ($LT_{(1,4)}^{1}$)) and arrival time penalties (arrive from 30 min before to 10 min after the desire arrival interval, Fig. 9b) as compared to the other passengers.

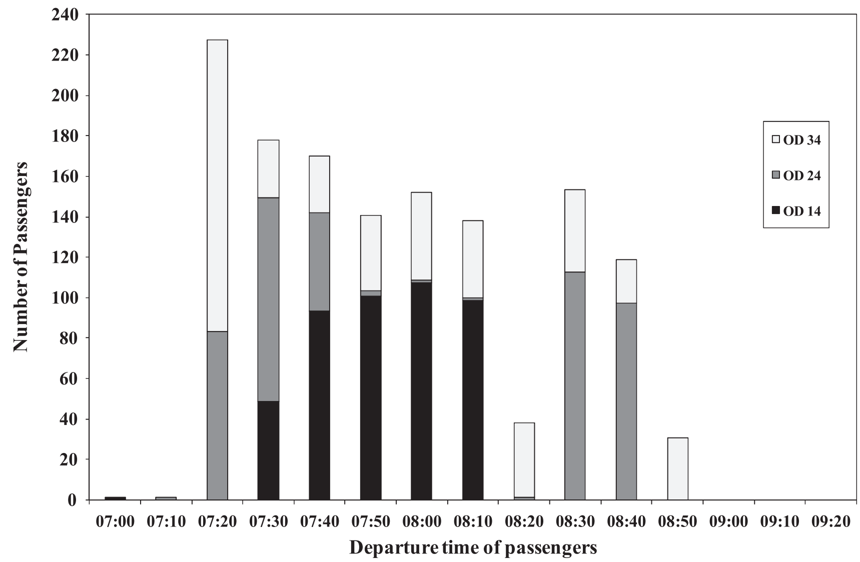
For the passengers from node 2 and 3, as they do not have the priority to get a seat, due to the trade-off discussed in the previous paragraph, they tend to depart much earlier, or later, in the less congested period. In the case for which seat availability is not considered (Fig. 8a), as the passengers in a vehicle will share the same level of in-vehicle congestion regardless of their boarding sequence, the departure time choice of passengers tend to even out the congestion and gives a evenly distributed departure time. Comparing Fig. 8a and b for the period 7:50 AM to 8:30 AM, almost all the passengers from node 2 have shifted to depart in earlier, or later, periods while similar number of passengers from node 3 have a departure time in this period. The significant shifting of departure time for node 2 passengers is mainly due to their boarding location. As they are at the second station of the transit line, they could effectively change their departure time to cope with the departure time distribution of the node 1 passengers for minimizing their in-vehicle congestion cost. But for the passengers from node 3, as they are at the third station (i.e. last station with boarding) of the transit line, there is very low chance for them to get a seat after the loadings from node 1 and 2. Thus, unless for the uncongested period (7:20 AM to 7:30 AM in Fig. 8b), there will be not much difference in their departure time distribution evaluated with or without the consideration of seat availability.

Fig. 9a and b are respectively the arrival time distribution resulted from the departure time distribution shown in Fig. 8a and b. These figures share a similar pattern as in Fig. 8 because no congestion effect is considered in the network. Due to the higher penalty for the late arrival as compared to the early arrival, node 1 passengers, for example, tend to arrive up to 30 min before the desired arrival interval but only up to 10 min after the desired interval.

For demonstrating the effect of passenger group, departure/arrival time penalties and seat stimulus on the proposed schedule-based transit assignment model, three different numerical examples for the single line network are proposed and compared with the original setup (base case). The first example aims at demonstrating the necessary of considering different passenger groups and its effect on the dynamic transit assignment. In this example, the 450 passengers for OD pair 1–4 in the base case are separated into two groups: g_1 and g_2. The demand for passenger group g_1 and g_2 are 300 and



(a) Without seat availability consideration

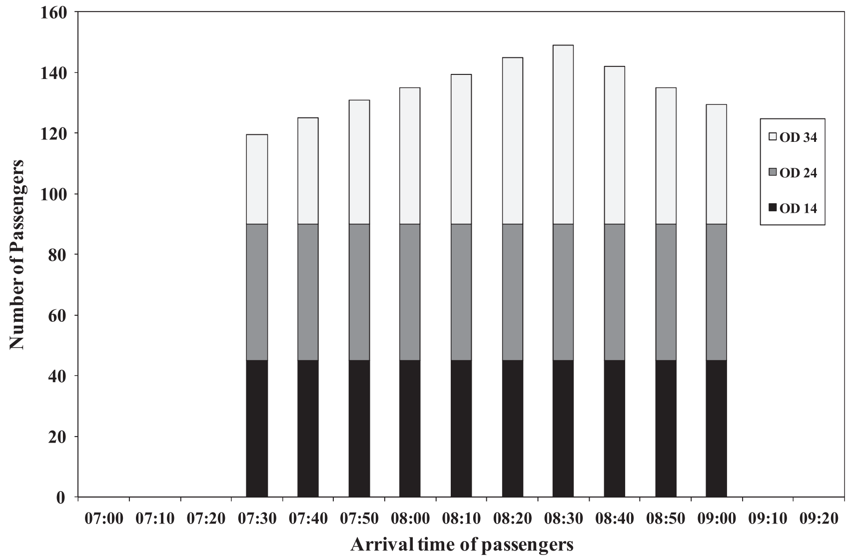


(b) With seat availability consideration

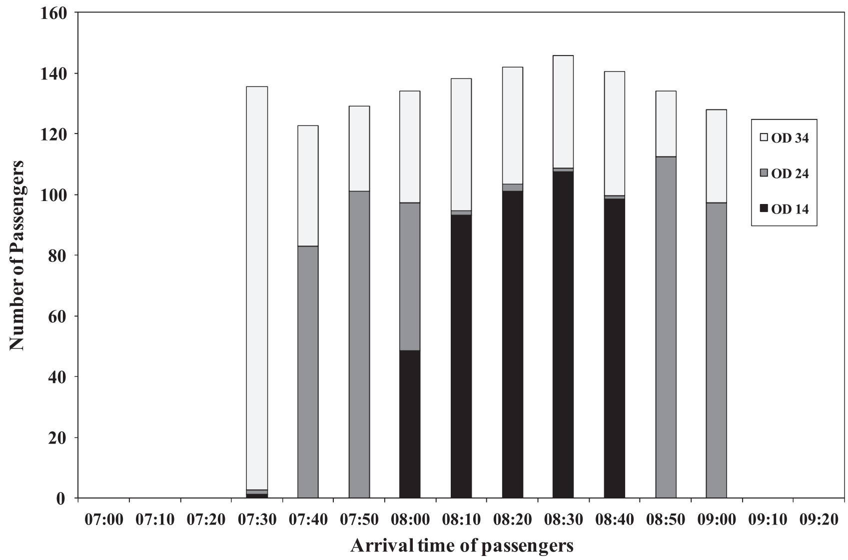
Fig. 8. Comparison of departure time distribution: single line network.

150 respectively. Both passenger groups share the same characteristics as the passengers from OD pair 1–4 in the base case except the desired arrival time of g_2 passengers which change to [12, 13]. After solving this example, the arrival time for different group of passengers are shown in Fig. 10. By comparing Fig. 10 to Fig. 9b, it could be seen that some of the passengers of OD pair 1–4 in the base case (Fig. 9b) have shifted to arrive in an earlier period 7:50 AM to 8:10 AM (Fig. 10). These shifted passengers are the passengers from group g_2, who change their travel strategy (departure time), to satisfy their desired arrival time of 8:00 AM. Consider the transit vehicle that reach node 4 by 8:10 AM, it is less congested as most of the passengers from OD pair 1–4 are shifted to arrive (depart) in an early period. As a result, passengers from OD pair 2–4, who arrive to their destination between 7:50 AM and 8:00 AM in the base case, are shifted to use this less congested transit vehicle and have an arrival time (8:10 AM) that is closer to their desired one (8:30 AM).

The second example aims at demonstrating the effect of departure and arrival penalties on the dynamic transit assignment model proposed in this paper. In this example, all the characteristics of the passengers from different OD pairs are the same as the base case except that departure/arrival penalties $\{\eta_1^1, \eta_2^1, \eta_3^1\}$ are reduced from {0.25,0.5,0.1} to {0.04,0.04,0.025}. After solving this example, the arrival time for passengers from different OD pairs are shown in Fig. 11. By comparing Fig. 11 to Fig. 9b, is could be seen that passengers from OD pair 1–4 tend to arrive later (8:50 AM $\sim$ 9:10



(a) Without seat availability consideration



(b) With seat availability consideration

Fig. 9. Comparison of arrival time distribution: single line network.

AM). It is because, as the late arrival penalty is reduced, passengers from OD 1–4 are more willing to depart later in a less congested period for getting a seat. For the passengers from OD pair 2–4, it could be seen that passengers tend to arrive earlier (7:30 AM ~ 7:40 AM). It is because, as the early departure η_3^1 and early arrival η_1^1 penalties are reduced, they tend to depart earlier (7:10 AM ~ 7:20 AM) to avoid the congestion from the passengers of OD pair 1–4.

The third example aims at demonstrating the effect of seat stimulus on the dynamic transit assignment model proposed in this paper. In this example, we consider a single group of passenger (g_1) traveling in the morning peak period between 8:00 AM and 9:30 AM. A single line network with 6 stations (Fig. 12) is adopted for this example and there are 3 OD pairs from the three origins (nodes 1, 2, and 3) to the three destinations (node 4, 5 and 6). The total passengers for each OD pairs are $D^1_{(1,4)} = 25$ and $D^2_{(2,5)} = D^3_{(3,6)} = 50$. The desired arrival time interval for passengers from OD pair 1–4, 2–5 and 3–6 are 8:50 AM ~ 9:00 AM, 9:00 AM ~ 9:10 AM, and 9:10 AM ~ 9:20 AM respectively. The headway of the transit vehicles is assumed to be given and fixed at 10 min and the fare is 0.10 for all TE arcs. The penalties for early arrival (η_1^1), late arrival (η_2^1), early departure (η_3^1) and standing discomfort (η_4^1) are taken as 1.2, 1.8, 0.6 and 1.5 respectively. The seat and standing capacity $\left(u^t_{ij}\right)$ and $\left(\tilde{u}^t_{ij}\right)$ of each of the arcs (i,j) at time t are respectively taken as 25 and 75. Lastly, the value of time for waiting (γ_{wait})

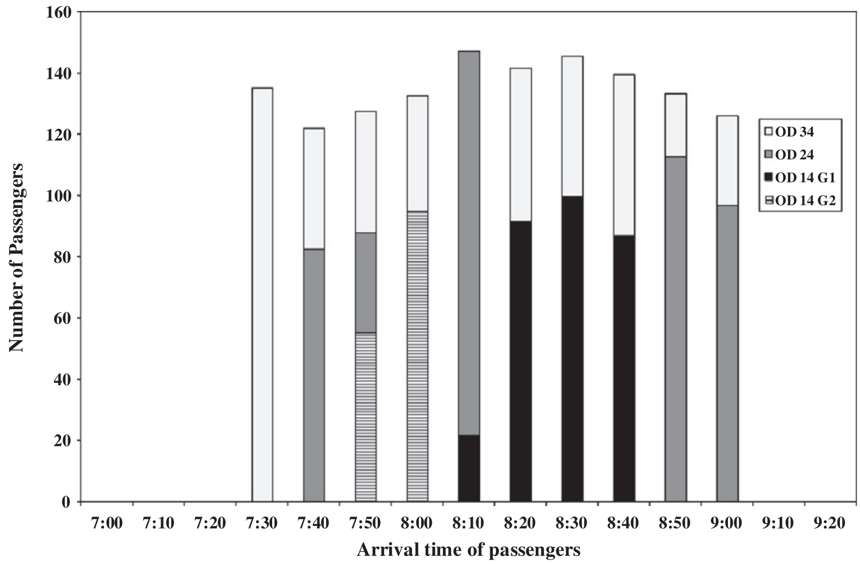


Fig. 10. Arrival time distribution: effect of passenger groups.

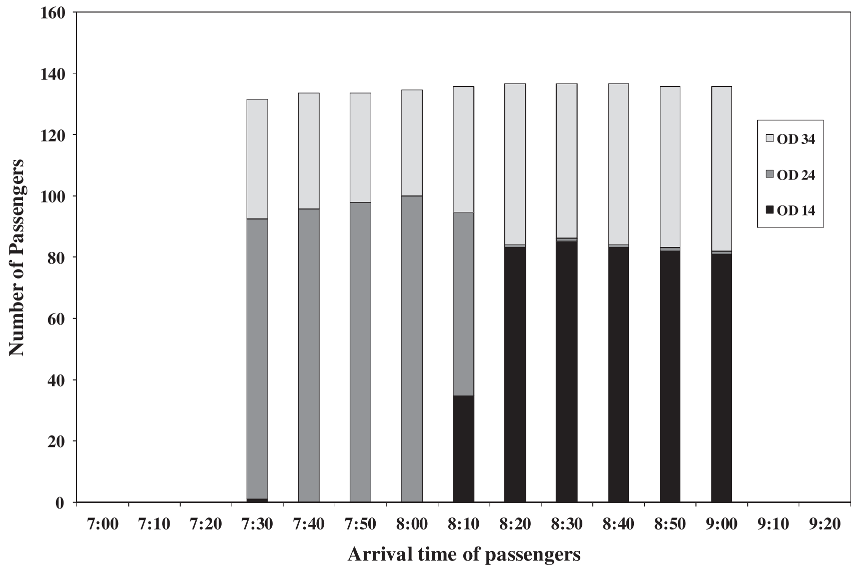


Fig. 11. Arrival time distribution: effect of departure/arrival time penalties.

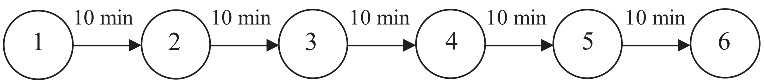


Fig. 12. Single line network: effect of seat stimulus.

and traveling (γ_{travel}) is taken as 0.2 and 0.1 respectively. With all the above parameters, three difference scenarios are setup and solved: (1) passengers with a longer remaining journey length have a higher stimulus to get a seat ($\eta_5 = 0.1$ and $\eta_6 = 0.9$); (2) all standing passengers have the same stimulus to get a seat $\left(p^{s,t}_{ii_l^{n+1}} = 1\,\forall i,l,n,s,t\right)$, and; (3) passengers spent a longer time on-board has a higher stimulus to get a seat ($\eta_5 = 0.9$ and $\eta_6 = 0.1$). Fig. 13a, b and c respectively show the departure time of OD pair 1–4, 2–5 and 3–6 for these three scenarios.

Considering the desired arrival time intervals and travel times between stations, the desired departure time interval for passengers from OD pair 1–4, 2–5 and 3–6 are 8:20 AM $\sim$ 8:30 AM, 8:30 AM $\sim$ 8:40 AM, and 8:40 AM $\sim$ 8:50 AM respectively. For the passengers from OD pair 1–4, as they will board the vehicle at the most upstream station (node 1), they could

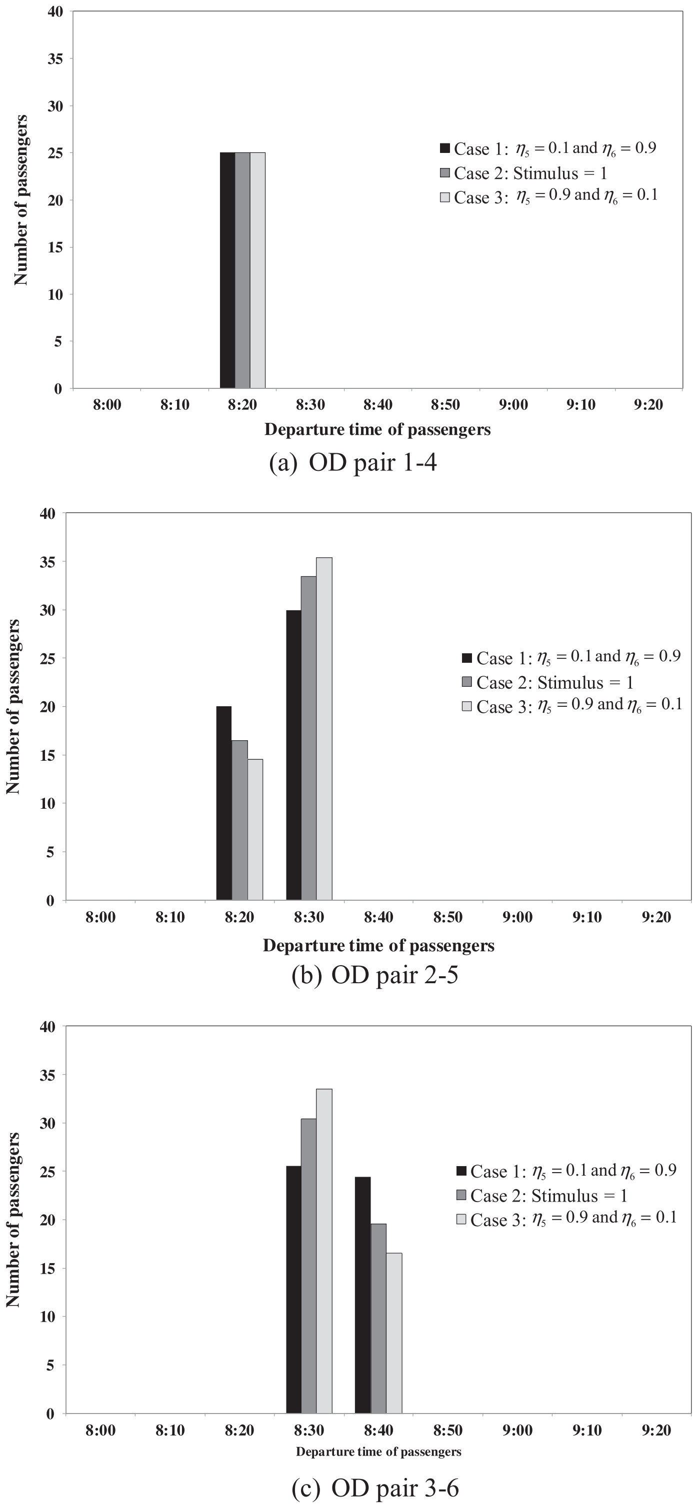


Fig. 13. Comparison of departure times for different seating stimuli.

always get a seat to avoid the crowding penalty. Thus, for the three tested scenarios, these passengers have scheduled their departure time between 8:20 AM and 8:30 AM to avoid the departure/arrival penalties (Fig. 13a). For passengers from OD pair 2–5 and 3–6, to depart in their desired departure time interval, they will be in the same vehicle with the passengers

Table 3
Iterates of MSA: multiple line network.

Iteration	Number of utilized strategies	Relative gap
1	8	0.16638
2	12	0.25521
5	12	0.12466
10	12	0.05583
20	12	0.06401
30	11	0.01586
50	10	0.00762
100	10	0.00174
150	9	0.00286
200	9	0.00076
213	9	0.00048

Table 4
Strategy utilization: multiple line network with seat availability.

OD	Strategies									Total demand	Expected cost
	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9		
(q,y)	15.07							9.93		25	2.20
(q,r)		2.55		9.30					8.15	20	3.45
(o,y)					8.46		1.54			10	1.79
(o,r)			13.11			1.89				15	2.39
Sit ratio	1.76	0.78	1.00	0.76	0.28	1.00	All*	All*	25.93		

* For this strategy, there is no in-vehicle standing nor waiting time incurred by passengers.

from OD pair 1–4 that broad the vehicle in 8:20 AM $\sim$ 8:30 AM. As the passengers from OD pair 1–4 has occupied all the seats, passengers from OD pair 2–5 and 3–6 have to stand inside the vehicle. After the passengers from OD pair 1–4 alight at node 4, the standing passengers, which are passengers from OD pair 2–5 departed in 8:30 AM $\sim$ 8:40 AM and passengers from OD pair 3–6 departed in 8:40 AM $\sim$ 8:50 AM, will compete for the seats based on their stimulus (Eq. (10)).

Considering the departure time 8:30 AM $\sim$ 8:40 AM in Fig. 13b, it could be seen that the number of passengers from OD pair 2–5 tends to increase as the weight of remaining journey length (time spent on-board) in the stimulus function is decreased (increased) from Scenario 1 to Scenario 3. On the contrary, there are fewer passengers from OD pair 3–6 to depart in 8:40 AM $\sim$ 8:50 AM as it moves from Scenario 1 to Scenario 3 (Fig. 13c). It is because, as more weight is put on the time spent on-board (i.e. less weight is put on remaining journey length), passengers from OD pair 2–5 have a higher probability to get a seat when they compete with the passengers from OD pair 3–6 at node 4. As a result, passengers from OD pair 2–5, who originally depart in 8:20 AM $\sim$ 8:30 AM, will shift to their desired departure time interval (8:30 AM $\sim$ 8:40 AM) to reduce their early departure/arrival penalty. For passengers from OD pair 3–6, as it is difficult for them to get a seat, they will shift to an earlier departure time (8:30 AM $\sim$ 8:40 AM) for getting a seat in a less congested vehicle.

6.2. Multiple line network

By analyzing a single line network, the example in the previous subsection demonstrated the impacts, in terms of departure time choices, of considering seat availability in formulating the schedule-based transit assignment. In order to further demonstrate such impact in a more realistic multiple-transit-line network, a transit network with multiple transit lines, which shown in Fig. 1, is considered.

In this example, we have four groups of passengers traveling within time period $[0,11]$ between four origin-destination pairs: (q,y), (q,r), (o,y) and (o,r). The desired arrival time interval for passengers ending at y and r are taken as $[5,7]$ and $[6,8]$ respectively. The total passenger demands for these 4 OD pairs are defined as $D^1_{(q,y)} = 25$, $D^2_{(q,r)} = 20$, $D^3_{(o,y)} = 10$ and $D^4_{(o,r)} = 15$. The headway of the transit vehicles are shown in Fig. 2 and the transit fares for arcs (a,b), (b,c), (c,d), (a,c) and (b,d) are 0.25, 0.50, 0.75, 1.00 and 0.5 respectively for all time t. The seat and standing capacities $\left(u^t_{ij} \text{ and } \tilde{u}^t_{ij}\right)$ are the same for each of the lines and are equal to 10, 15 and 5 for Line 1, 2 and 3 respectively. Furthermore, penalties for early arrival (η^g_1), late arrival (η^g_2) and early departure (η^g_3) are taken as 0.4, 0.4 and 0.2 respectively for all groups, while the parameters for the standing discomfort function (η_4) is taken as 0.8. The weights in the stimulus function and the value of time for waiting and traveling is this example are the same as in the previous example. Similar to the first example, passengers board the vehicles at a FCFS manner.

Table 3 lists the relative gaps and the number of utilized strategies at some selected iterates of MSA. The computational time for solving this multiple line network is 176 s on PC with Intel Core 2 Quad processor at 2.83 GHz and 4 GB RAM. From

Table 5
Strategy utilization: multiple line network without seat availability.

OD	Strategies								Total demand	Expected cost
	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8		
(q,y)				20.90				4.10	25	2.67
(q,r)	11.90				8.10				20	3.80
(o,y)		4.00					6.00		10	1.63
(o,r)			10.80			4.20			15	2.76

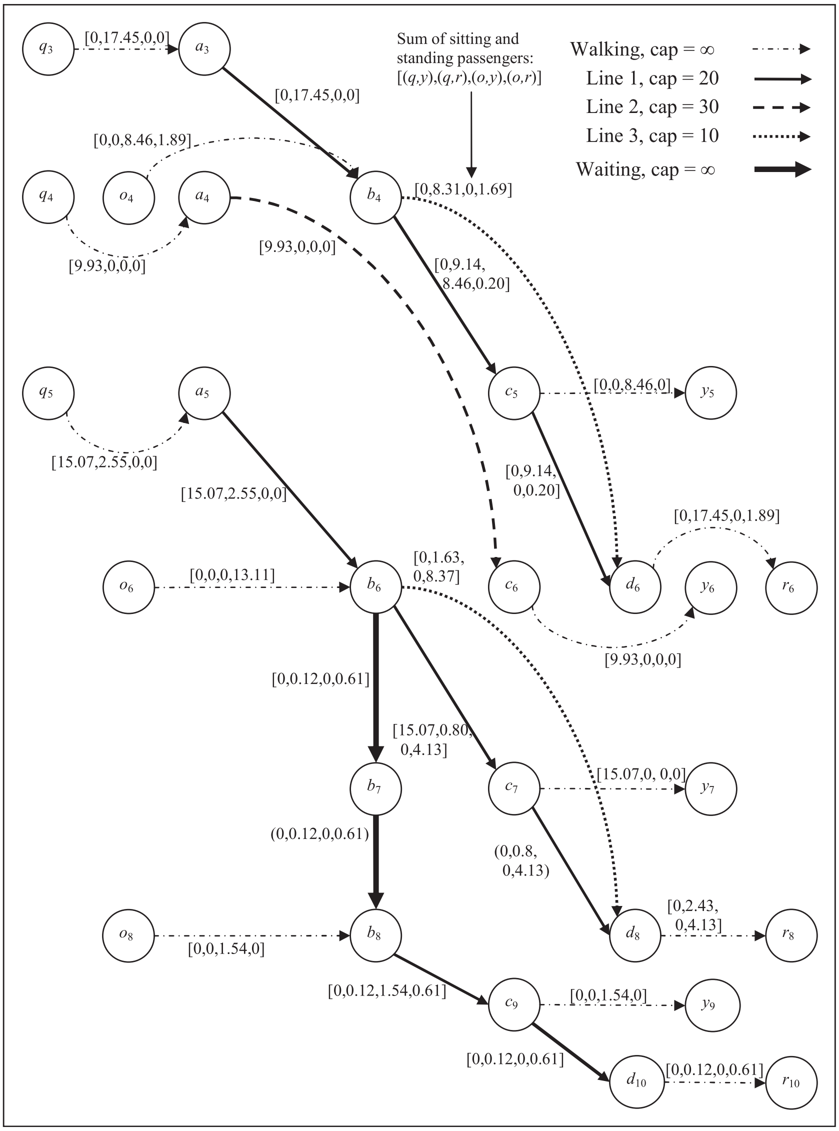


Fig. 14. Equilibrium arc flow of each passenger group: multiple line network.

the relative gap column of Table 3, it could be seen that the algorithm converges with a relative gap less than 0.0005, or 0.05%, at the 213th MSA step. From the column showing the number of utilized strategies, it could be seen that the algorithm is generating new strategies into the strategy set (increase from 8 to 12 during the first 20 iterations). The strategy set is updated by removing the unused strategies (decrease from 12 to 9). The number of utilized strategies is kept stable at 9 and the details of these nine strategies are shown in Table 4.

Table 4 shows the distribution of demand among the utilized strategies for each of the OD pairs at user equilibrium. In this example, each OD pair will have at least 2 utilized strategies with the same, and minimum, expected strategy cost (C^s).

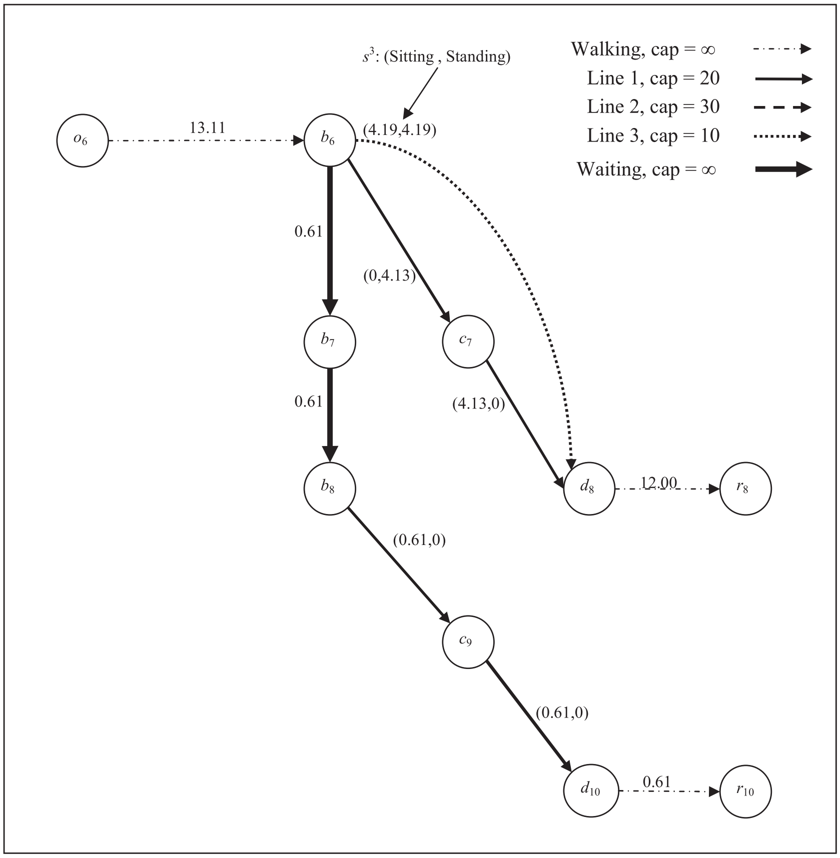


Fig. 15. Flow of strategy s_3: multiple line network.

The last row of Table 4 gives the ratio of the average sitting time to non-sitting time, which includes the standing time in transit vehicles and the waiting time at the stations, for each of the passengers using the corresponding strategy. The larger this ratio stands for the larger portion of average sitting time in that strategy. By comparing this sit ratio to the distribution of passengers among the same OD pair, no obvious relations could be found, as passengers' choice of strategy depends also on several other quantities such as departure and arrival penalties. Considering the strategies with high sit ratio (i.e. s_7, s_8 and s_9) they are either: departing at early time period (time period 3 for strategy s_9), departing at late time period (time period 8 for strategy s_7), or take only one line with boarding at the first station (strategy s_8). Table 5 shows the distribution of demand among the utilized strategies for the same setup but without the consideration of seat availability (Hamdouch and Lawphongpanich, 2008). By comparing Table 5 to Table 4, it could be seen that the consideration of seat availability not only results in a different expected cost, but also generates a different strategy set with different distribution of demand.

The equilibrium arc flows resulting from the utilization of the above nine strategies are shown in Fig. 14. The amount of flows on access and egress arcs, i.e. arcs between (q,a), (o,b), (c,y) and (d,r), determines the departure and arrival distributions. Note that in this example, the access and egress times are assumed to be zero, thus, all the access and egress arcs have the same time period at their both ends.

In Fig. 14, the square blanket besides each arc denotes the total passenger flow, sum of sitting and standing, for each OD pair on that arc. By tracing these values on Fig. 14, the flows and choices of passengers could be found. For example, 17.45 passengers, who head for the destination r, depart from node q at time period 3 and board transit Line 1 at node a_3. After arriving at node b_4, 8.31 passengers alighted Line 1 and switch to Line 3, while the remaining 9.14 passengers keep on traveling on Line 1 to node d. These two groups of passengers will then arrive at node d_6 and get to their destination r at time period 6. As this figure includes only the total passenger flow for the ease of display, no information on the number of sitting and standing passengers are shown. In order to demonstrate the ability of the proposed model in modeling sitting and standing passengers, the flow of strategy s_3 in Table 4 is plotted in Fig. 15.

In Fig. 15, the parenthesis besides each arc denotes the number of sitting and standing passengers following strategy s_3 on that arc. Considering arc (b_6,c_7) and (c_7,d_8) on transit Line 1, it could be seen that 4.13 passengers have got a seat after the transit vehicle passes node c_7. At node c_7, 15.07 passengers, who board the transit vehicle at node a_5, had alighted the vehicle at this point. As these 15.07 passengers have board the line at its first station (a), 10 of them will get a seat, while the remaining will stand. As they all alighted at c_7, 10 seats are free. Thus, the remaining 4.93 passengers in the vehicle, including the 4.13 passenger from the strategy s_3, will get a seat.

7. Conclusion

In this paper, we propose a new schedule-based transit assignment model in which passengers adopt strategies to travel from their origin to their destination. While this strategy concept has been successfully used in previous transit assignment studies, the new proposed model captures explicitly the stochastic nature of the standing and boarding passengers to get a seat. No such analytical schedule-based model has been developed in the literature to differentiate the discomfort level experienced by the sitting and standing passengers. When loading passengers on a first-come-first-serve basis, the model takes into account the sitting and standing capacities explicitly. The equilibrium conditions for this schedule-based transit assignment problem are stated as a variational inequality involving a vector-valued function of expected strategy costs. To find an equilibrium solution, we adopt the method of successive averages in which optimal strategy of each iteration is generated by solving a dynamic program. Using two different transit networks, we show empirically that the method converges to a user equilibrium solution and demonstrate the effects of the proposed seat model on passengers' decision making.

Compared to the other transit assignment models considering seat availability, the proposed model has the following merits: First, the introduction of analytical seat allocation model, instead of using Monte–Carlo simulation, could better describe the effect of stimulus on passengers' chance of getting a seat. Second, as compared to the static model in Schmoecker et al. (2009), the use of dynamic schedule-based model and the consideration of FIFO loading at transit stations help to better realize/model the tradeoff between seating comfort and early-departure/waiting-time penalty. Despite these merits, the proposed model needs the following improvements to be applied to real transit networks, such as Hong Kong and Japan. Firstly, as the variables in the proposed model are indexed by transit station, time period, strategy and transit line, the size of the matrices (computer storage) will growth exponentially as the TE network becomes larger or more transit lines are modeled. In such cases, additional effort should be made on the effective allocation of computer storage as most of the matrices are sparse matrices. Secondly, computational time is the other crucial issue that transit assignment models have to face as they are implemented for real transit networks. Compared to the other transit assignment models, the proposed model is more suitable to adopt parallel computing for reducing computational time. It is because the loading process (Section 3.3) and the computation of optimal strategy (Section 4) are performed on a node basis. Thus, for each of these processes, they could be started simultaneously from different nodes given that the specific criteria (T&C order for loading and reverse T&C order for computing optimal strategy) are satisfied. Lastly, walking arcs should not only between transit stops and origins/destinations, but also should be included in between transit stops. This kind of walking arcs will account for the extra penalty that passengers incurred as they transfer (e.g. walking from the bus stop at the road surface level to the underground railway platform). In the proposed model, apart from the set of inputs that could be directly adopted (e.g. transit fare and schedule), parameters, such as penalties and value of time, should be estimated by using stated-preference (SP) survey before the actual implementation of the model. Future research of this study will include the extension to different transit fare structures, the consideration of stochastic strategy choice behavior of transit passengers, and the development of efficient solution algorithms for large scale implementation.

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