Solution to Assignment 3

1. Integer Programming:

Method 1:

$$
\begin{aligned}
\min \quad & \sum_{i,j} D_{ij}x_{ij} + \sum_{j,k} C_{jk}y_{jk} \\
\text{s.t.} \quad & \sum_{i=1}^{n}\sum_{j=i}^{n} x_{ij} \le m && (1)\\
& \sum_{i=1}^{j} x_{ij} = \sum_{k=1,k\neq j}^{n} y_{jk}, \quad \forall j, && (2)\\
& \sum_{j=i}^{n} x_{ij} = \sum_{k=1,k\neq i}^{n} y_{ki}, \quad \forall i, && (3)\\
& u_j + y_{jk} \le u_k + (1-y_{jk})M, 2 \le k \neq j \le n && (4)\\
& u_i \le u_j + (1-x_{ij})M, 1 \le i \le j \le n && (5)\\
& \sum_{i=1}^{j} x_{ij} = \sum_{k=j+1}^{n} x_{j+1,k}, j = 1, \ldots, n-1 && (6)\\
& \sum_{j=1}^{n} x_{1j} = 1, && (7)\\
& \sum_{j=2}^{n} y_{j1} = 1, && (8)\\
& x_{ij} = \{0,1\}, \quad 1 \le i \le j \le n, && (9)\\
& y_{jk} = \{0,1\}, \quad 1 \le j \neq k \le n. && (10)
\end{aligned}
$$

u_i indicates the order of node i. $D_{ij} = C_{i,i+1} + \ldots, C_{j-1,j}$. $x_{ij} = 1$ indicates one segment starts from node i, ends at node j. $y_{jk} = 1$ indicates ending node j of one segment connects starting node k of another segment.

Constraint 1 represents the number of segment should be no larger than m.

Constraint 2 represents the last node of each segment should connect other nodes.

Constraint 3 represents the first node of each segment should connect other nodes.

Constraints 4 and 5 are sub-tour elimination constraints.

Constraint 6 represents the last node of one segment should connect the first node of another segment.

Constraints 7 and 8 represent node 1 is the first node of some segment.

Method 2:

The classical TSP-MTZ formulation:

$$
\begin{aligned}
\min \sum_{i=1}^{n} \sum_{j\neq i, j=1}^{n} c_{ij} x_{ij} \\
\text{s.t.} \sum_{i=1, i\neq j}^{n} x_{ij} = 1, j = 1, \ldots, n; \\
\sum_{j=1, j\neq i}^{n} x_{ij} = 1, i = 1, \ldots, n; \\
u_i - u_j + (n-1)x_{ij} \leq n-2, \quad 2 \leq i \neq j \leq n; \\
1 \leq u_i \leq n-1, \quad 2 \leq i \leq n; \\
x_{ij} \in \{0,1\}, i, j = 1, \ldots, n
\end{aligned}
$$

u_i indicates the order of node i. The meanings of all constraints are the same as those of TSP-MTZ formulation.

Notice that the nodes in each segment will be in order, i.e., $x_{i,i+1} = 1$, where node i and node $i+1$ are in the same segment.

For the break points of different segments, we have $x_{i,i+1} = 0$, where node i and node $i+1$ belong to two different segments.

Then we need to describe how to confine the number of segments.

Let $w_i = 1$ indicate node i is a break point(last node of one segment), $w_{20} = 1$.

Then we have $w_i \geq 1 - x_{i,i+1}$ and the number of segments constraint $\sum_i w_i \leq m$.

The whole programming will be:

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$$
\begin{aligned}
\min \sum_{i=1}^{n} \sum_{j\neq i, j=1}^{n} c_{ij}x_{ij} & \\
\text{s.t.} \sum_{i=1, i\neq j}^{n} & x_{ij} = 1, j = 1, \ldots, n; \\
& \sum_{j=1, j\neq i}^{n} x_{ij} = 1, i = 1, \ldots, n; \\
& u_i - u_j + (n-1)x_{ij} \leq n-2, \quad 2 \leq i \neq j \leq n; \\
& w_i \geq 1 - x_{i,i+1}; i = 1, \ldots, n-1; w_n = 1; \\
& \sum_{i=1}^{n} w_i \leq m; \\
& w_i \in \{0,1\}, i = 1, \ldots, n \\
& 1 \leq u_i \leq n-1, \quad 2 \leq i \leq n; \\
& x_{ij} \in \{0,1\}, i, j = 1, \ldots, n
\end{aligned}
$$

The solution is $[1, 2, 3, 4, \|7, 8, 9, \|19, 20, \|5, 6, \|14, 15, 16, 17, 18, \|10, 11, 12, 13]$. The corresponding cost is 155.

2. Dynamic Programming:

$$F(j) = \min_t (f(j,t) + F(t+1)), 0 \leq t - j \leq k-1, t \leq n.$$

$F(j)$ is the minimal cost of the segment TSP with $(n-j+1)$ nodes from node j to node n.
Thus, the optimal value for this problem is $F(1)$.
The boundary conditions are: $F(n) = C_{1,n}, F(n+1) = 0$.
$f(j,t)$ represents the minimal cost from node j to node $t+1$.
Specifically, $f(j,t) = C_{j,t+1}$ when $t = j$, and $f(n,n) = C_{1,n}$.
Then suppose $t \geq j+1$, let $\pi(j,t)$ be a permutation from node j to node t with first node j and last node t. $\Pi(j,t)$ be the set of all possible permutations from node j to node t. Denote by $C_{\pi(j,t)}$ the total cost of all adjacent nodes in the permutation $\pi(j,t)$.
Thus, $f(j,t) = \min_{\pi(j,t+1)\in\Pi(j,t+1)}\{C_{\pi(j,t+1)}\}$.
Add one dummy node $n+1$ satisfying $C_{i,n+1} = C_{1,i}, i = 2, \ldots, n$.
Best solution: $(1|2|3|4, 7, 5, 6|8|9|10|11|12|13, 15, 14|16|17|18|19|20)$.
Optimal value: $F(1) = 203$. Original cost: $\sum_{i=1}^{n} C_{i,i+1} + C_{1,n} = 217$.