Electronic Companion to
Seating Management under Social Distancing

EC.1 Policies for Dynamic Seat Assignment

Relaxed Dynamic Programming Heuristic (RDPH)

According to the RDP formulation in (17), we can determine whether to accept or reject each request. For accepted requests, we must then decide on specific seat assignments. However, without a predefined seat plan, this assignment process lacks clear guidelines. To resolve this, we implement a modified Best Fit rule [1], assigning each group to the row with the minimal remaining capacity that can still accommodate it. An important prerequisite for this assignment is verifying seat availability. Specifically, if the group size exceeds the maximum remaining capacity across all rows, the request must be rejected.

This policy is stated in the following algorithm.

Algorithm EC.1: RDP Heuristic

Calculate $V^t(l)$ by (17), $\forall t = 2, \ldots, T; \forall l = 1, 2, \ldots, l^1 = \tilde{L}$;
for $t = 1, \ldots, T$ **do**
    Observe a request of group type i;
    if $\max_{j \in \mathcal{N}} L_j^t \geq n_i$ *and* $V^{t+1}(l^t) \leq V^{t+1}(l^t - n_i) + i$ **then**
        Set $k = \arg\min_{j \in \mathcal{N}} \{L_j^t | L_j^t \geq n_i\}$ and break ties;
        Assign the group to row k, let $L_k^{t+1} \leftarrow L_k^t - n_i$, $l^{t+1} \leftarrow l^t - n_i$;
    else
        Reject the group and let $L_k^{t+1} \leftarrow L_k^t$, $l^{t+1} \leftarrow l^t$;
    end
end

Bid-Price Control (BPC) Policy

Bid-price control is a classical approach discussed extensively in the literature on network revenue management. It involves setting bid prices for different group types, which determine the eligibility of groups to take the seats. Bid-prices refer to the opportunity costs of taking one seat. As usual, we estimate the bid price of a seat by the shadow price of the capacity constraint corresponding to some row. In this section, we will demonstrate the implementation of the bid-price control policy.

The dual of LP relaxation of the SPDR problem is:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{M} d_i z_i + \sum_{j=1}^{N} L_j \beta_j \\
\text{s.t.} \quad & z_i + \beta_j n_i \geq (n_i - \delta), \quad i \in \mathcal{M}, j \in \mathcal{N} \\
& z_i \geq 0, i \in \mathcal{M}, \beta_j \geq 0, j \in \mathcal{N}.
\end{aligned} \tag{1}
$$

In (1), β_j can be interpreted as the bid-price for a seat in row j. A request is only accepted if the revenue it generates is no less than the sum of the bid prices of the seats it uses. Thus, if $i - \beta_j n_i \geq 0$, we will accept the group type i. And choose $j^* = \arg\max_j \{i - \beta_j n_i\}$ as the row to allocate that group.

Lemma EC.1. The optimal solution to problem (1) is given by $z_1, \ldots, z_{\tilde{i}} = 0$, $z_i = \frac{\delta(n_i - n_{\tilde{i}})}{n_{\tilde{i}}}$ for $i = \tilde{i}+1, \ldots, M$ and $\beta_j = \frac{n_{\tilde{i}} - \delta}{n_{\tilde{i}}}$ for all j.

The bid-price decision can be expressed as $i - \beta_j n_i = i - \frac{n_{\tilde{i}} - \delta}{n_{\tilde{i}}} n_i = \frac{\delta(i - \tilde{i})}{n_{\tilde{i}}}$. When $i < \tilde{i}$, $i - \beta_j n_i < 0$. When $i \geq \tilde{i}$, $i - \beta_j n_i \geq 0$. This implies that group type i greater than or equal to $\tilde{i}$ will be accepted if the capacity allows. However, it should be noted that β_j does not vary with j, which means the bid-price control cannot determine the specific row to assign the group to. We maintain the same tie-breaking rule as in RDPH, assigning each group to the row with the minimal residual capacity while still satisfying the accommodation requirement.

The bid-price control policy based on the static model is stated below.

Algorithm EC.2: Bid-Price Control

for $t = 1, \ldots, T$ **do**
- Observe a request of group type i;
- Solve the LP relaxation of the SPDR problem with $\boldsymbol{d}^t = (T - t) \cdot \boldsymbol{p}$ and $\mathbf{L}^t$;
- Obtain $\tilde{i}$ such that the aggregate optimal solution is $x e_{\tilde{i}} + \sum_{i=\tilde{i}+1}^{M} d_i^t e_i$;
- **if** $i \geq \tilde{i}$ *and* $\max_{j \in \mathcal{N}} L_j^t \geq n_i$ **then**
  - Set $k = \arg\min_{j \in \mathcal{N}} \{L_j^t | L_j^t \geq n_i\}$ and break ties;
  - Assign the group to row k, let $L_k^{t+1} \leftarrow L_k^t - n_i$;
- **else**
  - Reject the group;
- **end**

end

Booking-Limit Control (BLC) Policy

The booking-limit control policy involves setting a maximum number of reservations that can be accepted for each request. By controlling the booking-limits, revenue managers can effectively manage demand and allocate inventory to maximize revenue. In this policy, we solve the SPDR problem with the expected demand. Then for every type of requests, we only allocate a fixed amount according to the static solution and reject all other exceeding requests.

Algorithm EC.3: Booking-Limit Control

for $t = 1, \ldots, T$ **do**
    Observe a request of group type i;
    Solve the SPDR problem with $\boldsymbol{d}^t = (T - t) \cdot \boldsymbol{p}$ and $\mathbf{L}^t$;
    Obtain the seat plan, $\boldsymbol{H}^t$;
    if $X_i > 0$ **then**
        Set $k = \arg\min_{j \in \mathcal{N}} \{L_j^t - \sum_i n_i H_{ij}^t | H_{ij}^t > 0\}$;
        Break ties arbitrarily;
        Assign the group to row k, let $L_k^{t+1} \leftarrow L_k^t - n_i$, $H_{ik}^{t+1} \leftarrow H_{ik}^t - 1$;
    else
        Reject the group;
    end
end

EC.2 Proofs

Proof of Proposition 1

First, we regard this problem as a special case of the Multiple Knapsack Problem (MKP), then we consider the LP relaxation of this problem. Treat the groups as the items, the rows as the knapsacks. There are M types of items, the total number of which is $K = \sum_{i=1}^{M} d_i$, each item k has a profit p_k and weight w_k. Sort these items according to profit-to-weight ratios $\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \ldots \geq \frac{p_K}{w_K}$. Let the break item b be given by $b = \min\{j : \sum_{k=1}^{j} w_k \geq \tilde{L}\}$, where $\tilde{L} = \sum_{j=1}^{N} L_j$ is the total size of all knapsacks. For the LP relaxation of (2), the Dantzig upper bound [2] is given by $u_{\text{MKP}} = \sum_{j=1}^{b-1} p_j + \left(\tilde{L} - \sum_{j=1}^{b-1} w_j\right) \frac{p_b}{w_b}$. The corresponding optimal solution is to accept the whole items from 1 to $b-1$ and fractional $(\tilde{L} - \sum_{j=1}^{b-1} w_j)$ item b. Suppose the item b belong to type $\tilde{i}$, then for $i < \tilde{i}$, $x^*_{ij} = 0$; for $i > \tilde{i}$, $x^*_{ij} = d_i$; for $i = \tilde{i}$, $\sum_j x^*_{ij} = (\tilde{L} - \sum_{i=\tilde{i}+1}^{M} d_i n_i)/n_{\tilde{i}}$. ■

Proof of Proposition 2

First, we construct a feasible pattern with the size of $qM + \max\{r - \delta, 0\}$, then we prove this pattern is largest. Let $L = n_M \cdot q + r$, where q represents the number of times n_M is selected (the quotient), and r represents the remainder, indicating the number of remaining seats. It holds that $0 \leq r < n_M$. The number of people accommodated in the pattern $\boldsymbol{h}_g$ is given by $|\boldsymbol{h}_g| = qM + \max\{r-\delta, 0\}$. To establish the optimality of $|\boldsymbol{h}_g|$ as the largest number of people accommodated given the constraints of L, δ, and M, we can employ a proof by contradiction.

Assuming the existence of a pattern $\boldsymbol{h}$ such that $|\boldsymbol{h}| > |\boldsymbol{h}_g|$, we can derive the following inequalities:

$$
\begin{aligned}
& \sum_i (n_i - \delta) h_i > qM + \max\{r - \delta, 0\} \\
\Rightarrow\ & L \geq \sum_i n_i h_i > \sum_i \delta h_i + qM + \max\{r - \delta, 0\} \\
\Rightarrow\ & q(M + \delta) + r > \sum_i \delta h_i + qM + \max\{r - \delta, 0\} \\
\Rightarrow\ & q\delta + r > \sum_i \delta h_i + \max\{r - \delta, 0\}
\end{aligned}
$$

(i) When $r > \delta$, the inequality becomes $q+1 > \sum_i h_i$. It should be noted that h_i represents the number of group type i in the pattern. Since $\sum_i h_i \leq q$, the maximum number of people that can be accommodated is $qM < qM + r - \delta$.

(ii) When $r \leq \delta$, we have the inequality $q\delta + \delta \geq q\delta + r > \sum_i \delta h_i$. Similarly, we obtain $q + 1 > \sum_i h_i$. Thus, the maximum number of people that can be accommodated is qM, which is not greater than $|\boldsymbol{h}_g|$.

Therefore, $\boldsymbol{h}$ cannot exist. The maximum number of people that can be accommodated in the largest pattern is $qM + \max\{r - \delta, 0\}$.

Additionally, any largest pattern $\boldsymbol{h}$ under M remains feasible for $(M+1)$, implying that $\phi(M, L^0, \delta) \leq \phi(M+1, L^0, \delta)$. Similarly, any largest pattern $\boldsymbol{h}$ under L is a feasible pattern under $L+1$, thus, $\phi(M, L^0, \delta) \leq \phi(M, L^0 + 1, \delta)$. Any largest pattern $\boldsymbol{h}$ under $\delta + 1$ is a feasible pattern under δ, thus, $\phi(M, L^0, \delta + 1) \leq \phi(M, L^0, \delta)$. So, $\phi(M, L^0, \delta)$ is non-decreasing with M and L^0, and non-increasing with δ, respectively.

■

Proof of Proposition 3

First of all, we demonstrate the feasibility of problem (4). Given the feasible seat plan $\boldsymbol{H}$ and $\tilde{d}_i = \sum_{j=1}^{N} H_{ij}$, let $\hat{x}_{ij} = H_{ij}, i \in \mathcal{M}, j \in \mathcal{N}$, then $\{\hat{x}_{ij}\}$ satisfies the first set of constraints. Because $\boldsymbol{H}$ is feasible, $\{\hat{x}_{ij}\}$ satisfies the second set of constraints and integer constraints. Thus, problem (4) always has a feasible solution.

Suppose there exists at least one pattern $\boldsymbol{h}$ is neither full nor largest in the optimal seat plan obtained from problem (4). Let $\beta = L - \sum_i n_i h_i$, and denote the smallest group type in pattern $\boldsymbol{h}$ by k. If $\beta \geq n_1$, we can assign at least n_1 seats to a new group to increase the objective value. Thus, we consider the situation when $\beta < n_1$. If $k = M$, then this pattern is largest. When $k < M$, let $h_k^1 = h_k - 1$ and $h_j^1 = h_j + 1$, where $j = \min\{M, \beta + k\}$. In this way, the constraints will still be satisfied but the objective value will increase when the pattern $\boldsymbol{h}$ changes. Therefore, by contradiction, problem (4) always generate a seat plan composed of full or largest patterns. ■

Proof of Proposition 4

Suppose that H is the seat plan associated with the optimal solution to SBSP, but there exists a pattern that is neither full nor the largest. The corresponding excess of supply is $\mathbf{y}^+$. According to Proposition 3, H' can be obtained from H. The seat plan, H', is composed of full or largest patterns and satisfies all constraints of SBSP. The corresponding excess of supply is $\mathbf{y}^{'+}$.

Then we will demonstrate that for each scenario ω, the objective function of SBSP, given by $\sum_{j=1}^{N} \sum_{i=1}^{M} i \cdot x_{ij} - \sum_{i=1}^{M} y_{i\omega}^+$, does not decrease when transitioning from H to $H^{'}$.

Let $\Delta y_{M\omega}^+ = y_{M\omega}^{'+} - y_{M\omega}^+$, $\Delta \sum_{j=1}^{N} x_{Mj} = \sum_{j=1}^{N} x_{Mj}^{'} - \sum_{j=1}^{N} x_{Mj}$. According to (5), when i changes from M to 1, we obtain the following inequalities.

$$
\begin{aligned}
\Delta y^+_{M\omega} &\geq \Delta \sum_{j=1}^{N} x_{Mj} \\
\Delta y^+_{M-1,\omega} &\geq \Delta y^+_{M\omega} + \Delta \sum_{j=1}^{N} x_{M-1,j} \geq \Delta \sum_{j=1}^{N} (x_{Mj} + x_{M-1,j}) \\
\vdots \cdots &\geq \cdots \vdots \\
\Delta y^+_{1,\omega} &\geq \Delta \sum_{j=1}^{N} \sum_{i=1}^{M} x_{i,j}
\end{aligned}
$$

Since the objective function does not decrease, $H^{'}$ represents the optimal solution to SBSP and is composed of full or largest patterns. ■

Proof of Lemma 1

Note that $\mathbf{f}^\intercal = [-\mathbf{1},\ \mathbf{0}]$ and $\mathbf{V} = [\mathbf{W},\ \mathbf{I}]$. Based on this, we can derive the following inequalities: $\boldsymbol{\alpha}^\intercal \mathbf{W} \geq -\mathbf{1}$ and $\boldsymbol{\alpha}^\intercal \mathbf{I} \geq \mathbf{0}$. According to the expression of $\mathbf{W}$ and $\mathbf{I}$, we can deduce that $0 \leq \alpha_i \leq \alpha_{i-1} + 1$ for $i \in \mathcal{M}$ by letting $\alpha_0 = 0$. These inequalities indicate that the feasible region is nonempty and bounded. For $i \in \mathcal{M}$, α_i is only bounded by $\alpha_{i-1} + 1$ and 0, thus, all extreme points within the feasible region are integral. ■

Proof of Proposition 5

According to the complementary slackness property, we can obtain the following equations

$$
\begin{aligned}
&\alpha_i(d_{i0} - d_{i\omega} - y^+_{i\omega} + y^+_{i+1,\omega} + y^-_{i\omega}) = 0, i = 1, \ldots, M-1 \\
&\alpha_i(d_{i0} - d_{i\omega} - y^+_{i\omega} + y^-_{i\omega}) = 0, i = M \\
&y^+_{i\omega}(\alpha_i - \alpha_{i-1} - 1) = 0, i = 1, \ldots, M \\
&y^-_{i\omega}\alpha_i = 0, i = 1, \ldots, M.
\end{aligned}
$$

When $y^-_{i\omega} > 0$, we have $\alpha_i = 0$. When $y^+_{i\omega} > 0$, we have $\alpha_i = \alpha_{i-1} + 1$. When $y^+_{i\omega} = y^-_{i\omega} = 0$, let $\Delta d = d_\omega - d_0$,

- if $i = M$, $\Delta d_M = 0$, the value of objective function associated with α_M is always 0, thus we have $0 \leq \alpha_M \leq \alpha_{M-1} + 1$;
- if $i < M$, we have $y^+_{i+1,\omega} = \Delta d_i \geq 0$.
  - If $y^+_{i+1,\omega} > 0$, the objective function associated with α_i is $\alpha_i \Delta d_i = \alpha_i y^+_{i+1,\omega}$, thus to minimize the objective value, we have $\alpha_i = 0$.
  - If $y^+_{i+1,\omega} = 0$, we have $0 \leq \alpha_i \leq \alpha_{i-1} + 1$.

$\blacksquare$

Proof of Lemma EC.1

According to the Proposition 1, the aggregate optimal solution to LP relaxation of problem (2) takes the form $xe_{\tilde{i}} + \sum_{i=\tilde{i}+1}^{M} d_i e_i$, then according to the complementary slackness property, we know that $z_1, \dots, z_{\tilde{i}} = 0$. This implies that $\beta_j \geq \frac{n_i - \delta}{n_i}$ for $i = 1, \dots, \tilde{i}$. Since $\frac{n_i - \delta}{n_i}$ increases with i, we have $\beta_j \geq \frac{n_{\tilde{i}} - \delta}{n_{\tilde{i}}}$. Consequently, we obtain $z_i \geq n_i - \delta - n_i \frac{n_{\tilde{i}} - \delta}{n_{\tilde{i}}} = \frac{\delta(n_i - n_{\tilde{i}})}{n_{\tilde{i}}}$ for $i = h+1, \dots, M$.

Given that **d** and **L** are both no less than zero, the minimum value will be attained when $\beta_j = \frac{n_{\tilde{i}} - \delta}{n_{\tilde{i}}}$ for all j, and $z_i = \frac{\delta(n_i - n_{\tilde{i}})}{n_{\tilde{i}}}$ for $i = \tilde{i}+1, \dots, M$. $\blacksquare$

Proof of Corollary 1

According to Proposition 2, $\phi(M, L^0, \delta)$ is non-decreasing with M and L^0, and non-increasing with δ, respectively. Consequently, when M increases while $\boldsymbol{L}$ remains unchanged, the maximum achievable occupancy rate does not decrease; when δ increases while M and $\boldsymbol{L}^0$ remain unchanged, the maximum achievable occupancy rate does not increase; when $\boldsymbol{L}^0$ increases while M and δ remain unchanged, the maximum achievable occupancy rate may either increase or decrease.

$\blacksquare$

EC.3 Probabilities Estimation and Realistic Seat Layouts

EC.3.1 Probabilities Estimation

We select Movie A (representing the suspense genre) and Movie B (representing the family fun genre) as target movies to analyze group information and their corresponding probability distributions, denoted as $D3$ and $D4$, respectively.

We make the screenshots about the ticket seat plans from a Hong Kong cinema website at different time intervals. When tickets were sold in advance of the movie screening, the seats were typically scattered. Therefore, we treated consecutive seats as belonging to the same group, while excluding cases where the number of consecutive seats exceeds four.

We counted the frequency of different group types in the seat plans to derive their probability distributions. For Movie A, the frequencies for the four group types are 112, 460, 121, and 226, with a total of 919 observations. For Movie B, the frequencies are 116, 178, 23, and 28, with a total of 345 observations. We keep two decimal places, then obtain the probability:

$p_1^A = 0.12$, $p_2^A = 0.50$, $p_3^A = 0.13$, $p_4^A = 0.25$ and $p_1^B = 0.34$, $p_2^B = 0.52$, $p_3^B = 0.07$, $p_4^B = 0.08$.

Using the normal distribution approximation method (with a 95% confidence interval), the confidence intervals for the probabilities of each group type for Movie A is presented as follows: $CI_1^A = 0.122 \pm 0.011$, $CI_2^A = 0.501 \pm 0.016$, $CI_3^A = 0.132 \pm 0.011$, $CI_4^A = 0.246 \pm 0.014$

Similarly, the confidence intervals for the probabilities of each group type for Movie B are:

$CI_1^B = 0.336 \pm 0.025$, $CI_2^B = 0.516 \pm 0.027$, $CI_3^B = 0.067 \pm 0.013$, $CI_4^B = 0.081 \pm 0.015$

EC.3.2 Realistic Seat Layouts

Figure EC.1: Layout A

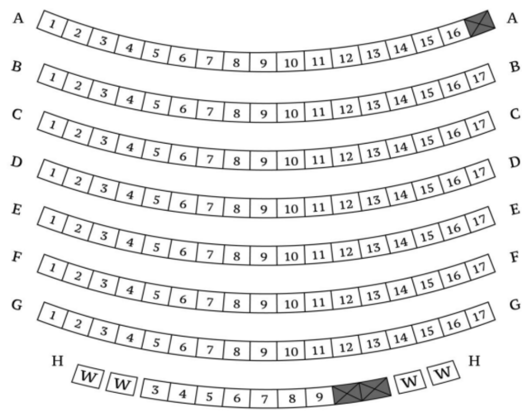


Figure EC.2: Layout B

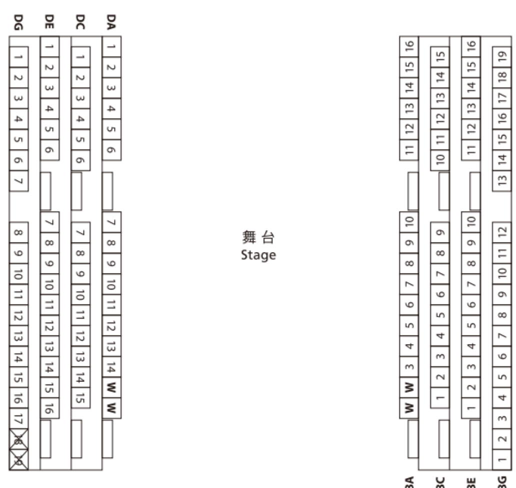


Figure EC.3: Layout C

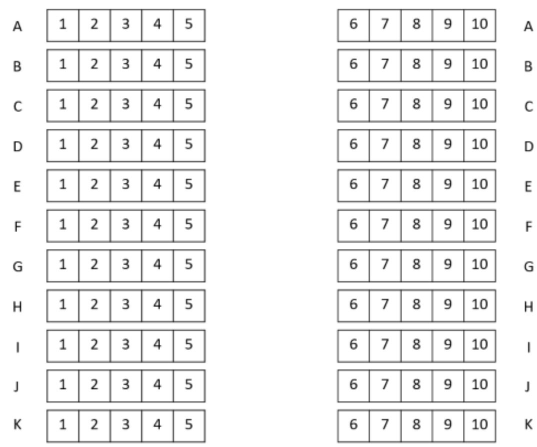


Figure EC.4: Layout D

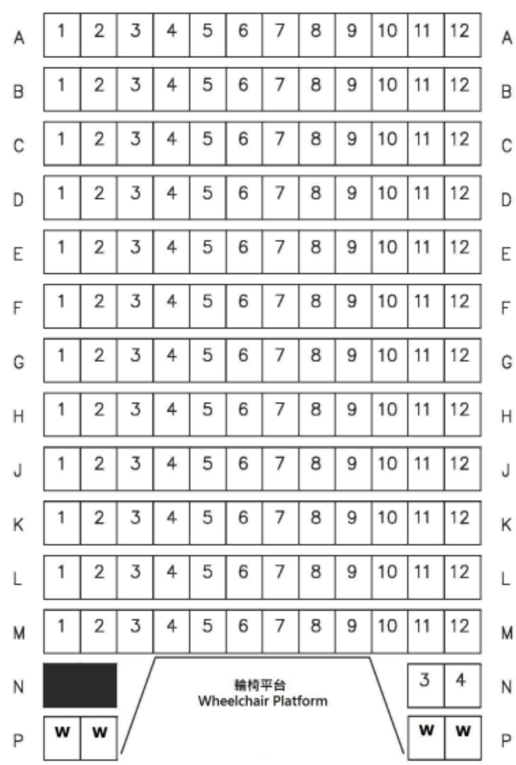
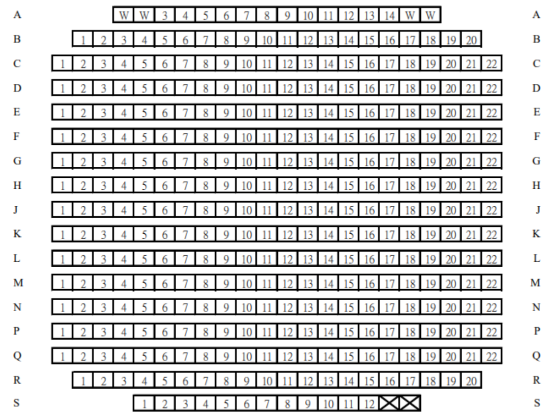


Figure EC.5: Layout E



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[1] D. S. Johnson, "Fast algorithms for bin packing," *Journal of Computer and System Sciences*, vol. 8, no. 3, pp. 272–314, 1974.

[2] G. B. Dantzig, "Discrete-variable extremum problems," *Operations Research*, vol. 5, no. 2, pp. 266–288, 1957.