Inverse optimization

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1 The Model

The original LP model is:

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & A'x \geq b \end{aligned}$$

The dual of the LP is:

$$\begin{aligned} \min \quad & by \\ \text{s.t.} \quad & A'^T y = c \end{aligned}$$

Among this, y is dual variable. Thus, the inverse optimization model can be expressed as the following:

$$\begin{aligned} \min |A' - A| & \\ \text{s.t.} \quad A'^T y = c & \quad (1) \\ cx^0 \geq by & \quad (2) \\ A'x^0 \geq b & \quad (3) \end{aligned}$$

We can split this into two situations: given the value or given the solution. If we give the solution, just as the model we showed. If we give the value, we don't need the (3) constraints. Expand the matrix to the specific elements. We can obtain the corresponding model:

$$
\begin{aligned}
\min \quad & \sum_i \sum_j (e_{ij} + f_{ij}) \\
& \sum_{i=1}^{m} (e_{ij} - fij + aij) y_i && = c_j \\
& \sum_{j=1}^{n} (e_{ij} - fij + aij) x_j^0 && \geq b_i \\
& \sum_{i=1}^{m} b_i y_i && \leq v_0 \\
& e_{ij} \geq 0 \qquad f_{ij} \geq 0
\end{aligned}
\tag{4}
$$

Add the lagrangian multiplier:

$$
T(e,f,y) = \min \sum_i \sum_j (e_{ij} + f_{ij}) + \sum_{j=1}^{n} \lambda_j g_j(e,f,y) + \sum_{i=1}^{m} \mu_i f_i(e,f,y)
$$

$$
g_j(e,f,y) = \sum_{i=1}^{m} (e_{ij} - f_{ij} + a_{ij}) y_i - c_j = 0 \tag{God}
$$

$$
f_i(e,f,y) = b_i - \sum_{j=1}^{n} (e_{ij} - f_{ij} + a_{ij}) x_j^0 \leq 0 \tag{5}
$$

$$
h(y) = \sum_{i=1}^{m} b_i y_i - v_0 \leq 0 \tag{6}
$$

$$
K(e,f) = e_{ij} f_{ij} = 0 \tag{7}
$$

$$
M(e) = -e_{ij} \leq 0 \tag{8}
$$

$$
N(f) = -f_{ij} \leq 0 \tag{9}
$$

The corresponding multipliers are $\lambda_j, \mu_i, \alpha, \beta_{ij}, m_{ij}, n_{ij}$.

Using the KKT constraints, we can obtain the following equations.

$$
\begin{aligned}
\frac{\partial T(e,f,y)}{\partial e_{ij}} &= 1 + \beta_{ij} f_{ij} - m_{ij} + \lambda_j y_i + \mu_i(-x_j^0) = 0 \\
\frac{\partial T(e,f,y)}{\partial f_{ij}} &= 1 + \beta_{ij} e_{ij} - n_{ij} - \lambda_j y_i + \mu_i x_j^0 = 0 \\
\frac{\partial T(e,f,y)}{\partial y_i} &= \sum_{j=1}^{n} (e_{ij} - f_{ij} + a_{ij})\lambda_j + \alpha b_i = 0 \\
g_j(e,f,y) &= 0 \\
\mu_i f_i &= 0 \quad \mu_i \geq 0 \\
K(e,f) &= e_{ij} f_{ij} = 0 \\
\alpha h(y) &= 0 \quad \alpha \geq 0 \\
m_{ij} M(e) &= 0 \quad m_{ij} \geq 0 \\
n_{ij} N(f) &= 0 \quad n_{ij} \geq 0
\end{aligned}
$$

We can solve these equations to obtain the solution $(e_{ij}^*, f_{ij}^*, y_i^*)$. And the e_{ij}^*, f_{ij}^* corresponds the optimal adjustment, the y_i^* corresponds the optimal lagrangian multipiers.

Consider the particularity of this problem. One possible method to solve the systems of nonlinear equations is