Inverse optimization

Dec 2019

1 The Model

The original LP model is:

$$
\begin{aligned}
\min \quad & cx \\
\text{s.t.} \quad & A'x \geq b
\end{aligned}
$$

The dual of the LP is:

$$
\begin{aligned}
\min \quad & by \\
\text{s.t.} \quad & A'^T y = c
\end{aligned}
$$

Among this, y is dual variable. Thus, the inverse optimization model can be expressed as the following:

$$
\begin{aligned}
\min \|A' - A\|_1 & \\
\text{s.t.} \quad A'^T y = c & \qquad (1.1) \\
cx^0 \geq by & \qquad (1.2) \\
A'x^0 \geq b & \qquad (1.3)
\end{aligned}
$$

We can split this into two situations: given the value or given the solution. If we give the solution, just as the model we showed. If we give the value, we don't need the (1.3) constraints. Expand the matrix to the specific elements. We can obtain the corresponding model:

$$
\begin{aligned}
\min \quad & \sum_i \sum_j (e_{ij} + f_{ij}) \\
\text{s.t.} \quad & \sum_{i=1}^{m} (e_{ij} - fij + aij) y_i && = c_j \\
& \sum_{j=1}^{n} (e_{ij} - fij + aij) x_j^0 && \geq b_i \\
& \sum_{i=1}^{m} b_i y_i && \leq v_0 \\
& e_{ij} \geq 0 \qquad f_{ij} \geq 0
\end{aligned} \tag{1.4}
$$

Notice that there are $(2mn + m)$ variables but only $(m + n + 1)$ constraints.

The theoretical method is to use the KKT conditions which is usually used to solve the constrained nonlinear programming. Add the lagrangian multiplier:

$$
T(e, f, y) = \min \sum_i \sum_j (e_{ij} + f_{ij}) + \sum_{j=1}^{n} \lambda_j g_j(e, f, y) + \sum_{i=1}^{m} \mu_i f_i(e, f, y)
$$

$$
\text{s.t.} \quad g_j(e, f, y) = \sum_{i=1}^{m} (e_{ij} - f_{ij} + a_{ij}) y_i - c_j = 0 \tag{God}
$$

$$
f_i(e, f, y) = b_i - \sum_{j=1}^{n} (e_{ij} - f_{ij} + a_{ij}) x_j^0 \leq 0 \tag{1.5}
$$

$$
h(y) = \sum_{i=1}^{m} b_i y_i - v_0 \leq 0 \tag{1.6}
$$

$$
K(e, f) = e_{ij} f_{ij} = 0 \tag{1.7}
$$

$$
M(e) = -e_{ij} \leq 0 \tag{1.8}
$$

$$
N(f) = -f_{ij} \leq 0 \tag{1.9}
$$

The corresponding multipliers are $\lambda_j, \mu_i, \alpha, \beta_{ij}, m_{ij}, n_{ij}$.

Using the KKT constraints, we can obtain the following equations.

$$
\begin{aligned}
\frac{\partial T(e,f,y)}{\partial e_{ij}} &= 1+\beta_{ij}f_{ij}-m_{ij}+\lambda_j y_i+\mu_i(-x_j^0)=0\\
\frac{\partial T(e,f,y)}{\partial f_{ij}} &= 1+\beta_{ij}e_{ij}-n_{ij}-\lambda_j y_i+\mu_i x_j^0=0\\
\frac{\partial T(e,f,y)}{\partial y_i} &= \sum_{j=1}^{n}(e_{ij}-f_{ij}+a_{ij})\lambda_j+\alpha b_i=0\\
g_j(e,f,y) &= 0\\
\mu_i f_i &= 0 \quad \mu_i \geq 0\\
K(e,f) &= e_{ij}f_{ij}=0\\
\alpha h(y) &= 0 \quad \alpha \geq 0\\
m_{ij}M(e) &= 0 \quad m_{ij}\geq 0\\
n_{ij}N(f) &= 0 \quad n_{ij}\geq 0
\end{aligned}
$$

If we can solve these equations, we can obtain the solution $(e_{ij}^*, f_{ij}^*, y_i^*)$. And the e_{ij}^*, f_{ij}^* corresponds the optimal adjustment, the y_i^* corresponds the optimal lagrangian multipiers. However, the system of nonlinear equations doesnot seem to be solved in this case.

Consider the particularity of this problem.

Case 1. All the $f_{ij}=0, e_{ij}>0$

The original model is transformed as:

$$
\begin{aligned}
& G=\min \quad \sum_i\sum_j e_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{m}(e_{ij}+aij)y_i \qquad = c_j\\
& \sum_{j=1}^{n}(e_{ij}+aij)x_j^0 \qquad \geq b_i\\
& \sum_{i=1}^{m} b_i y_i \qquad \leq v_0\\
& e_{ij}\geq 0
\end{aligned}
\tag{1.10}
$$

If all $a_{ij}>0$, we can further transform this model as a geometric programming(GP), because the functions are all posynomialfunction. Let $z_i=\ln x_i \Rightarrow x_i=e^{z_i}$, and $e_{ij}=x_k,\ k=1,\dots,i\times j,\ y_i=x_k, k=i\times j+1,\dots,i\times j+m$

$f(x)=\sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}}\cdots x_n^{a_{nk}}=\sum_{k=1}^{K} e^{a_k^T y+b_k}$

Then, the model can be written as:

Because the optimal solution satisfies the

And the system is changed as: $h(y) = 0$ and

$$\begin{aligned}
1 - m_{ij} + \lambda_j y_i + \mu_i(-x_j^0) &= 0 \\
1 + \beta_{ij} e_{ij} - n_{ij} - \lambda_j y_i + \mu_i x_j^0 &= 0 \\
\sum_{j=1}^{n} (e_{ij} + a_{ij}) \lambda_j + \alpha b_i &= 0 \\
\sum_{i=1}^{m} (e_{ij} + a_{ij}) y_i - c_j &= 0 \\
b_i - \sum_{j=1}^{n} (e_{ij} + a_{ij}) x_j^0 &\leq 0 \\
\mu_i f_i = 0 \quad \mu_i &\geq 0 \\
\sum_{i=1}^{m} b_i y_i = v_0 \quad \alpha &\geq 0 \\
m_{ij} e_{ij} = 0 \quad m_{ij} &\geq 0 \\
n_{ij} &\geq 0
\end{aligned}$$

Let all $m_{ij} = 0$

Another information is that if x_j^0 satisfies some equation $f_i = 0, i \in I$. For other $i \in \{1, \ldots, n\} \backslash I, f_i \leq 0$ and $\mu_i = 0$.