Introduction to
Optimization Method

Dis·*count*

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Dynamic Programming

0-1 Knapsack Problem

The most common problem being solved is the 0-1 knapsack problem, which restricts the number x_i of copies of each kind of item to zero or one. Given a set of n items numbered from 1 up to n, each with a weight w_i and a value v_i, along with a maximum weight capacity W,

$$\max \sum_{i=1}^{n} v_i x_i$$

$$\text{s.t.} \sum_{i=1}^{n} w_i x_i \leq W \text{ and } x_i \in \{0, 1\}.$$

0-1 Knapsack Problem

Assume $w_1, w_2, \ldots, w_n, W$ are strictly positive integers. Define $m[i, w]$ to be the maximum value that can be attained with weight less than or equal to w using items up to i. We can define $m[i, w]$ recursively as follows:

$$m[0, w] = 0$$
$$m[i, w] = m[i-1, w] \text{ if } w_i > w$$
$$m[i, w] = \max(m[i-1, w], m[i-1, w-w_i] + v_i) \text{ if } w_i \leq w.$$

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0-1 Knapsack Problem

The solution can then be found by calculating $m[n, W]$. To do this efficiently, we can use a table to store previous computations. This solution will therefore run in $O(nW)$ time and $O(nW)$ space.

Weight Limit (i):	0	1	2	3	4	5	6	7	8	9	10	11
$w_1 = 1\ v_1 = 1$	0	1	1	1	1	1	1	1	1	1	1	1
$w_2 = 2\ v_2 = 6$	0	1	6	7	7	7	7	7	7	7	7	7
$w_3 = 5\ v_3 = 18$	0	1	6	7	7	18	19	24	25	25	25	25
$w_4 = 6\ v_4 = 22$	0	1	6	7	7	18	22	24	28	29	29	40
$w_5 = 7\ v_5 = 28$	0											

Optimal substructure

- Fibonacci sequence

$$fib(n) = fib(n-1) + fib(n-2)$$

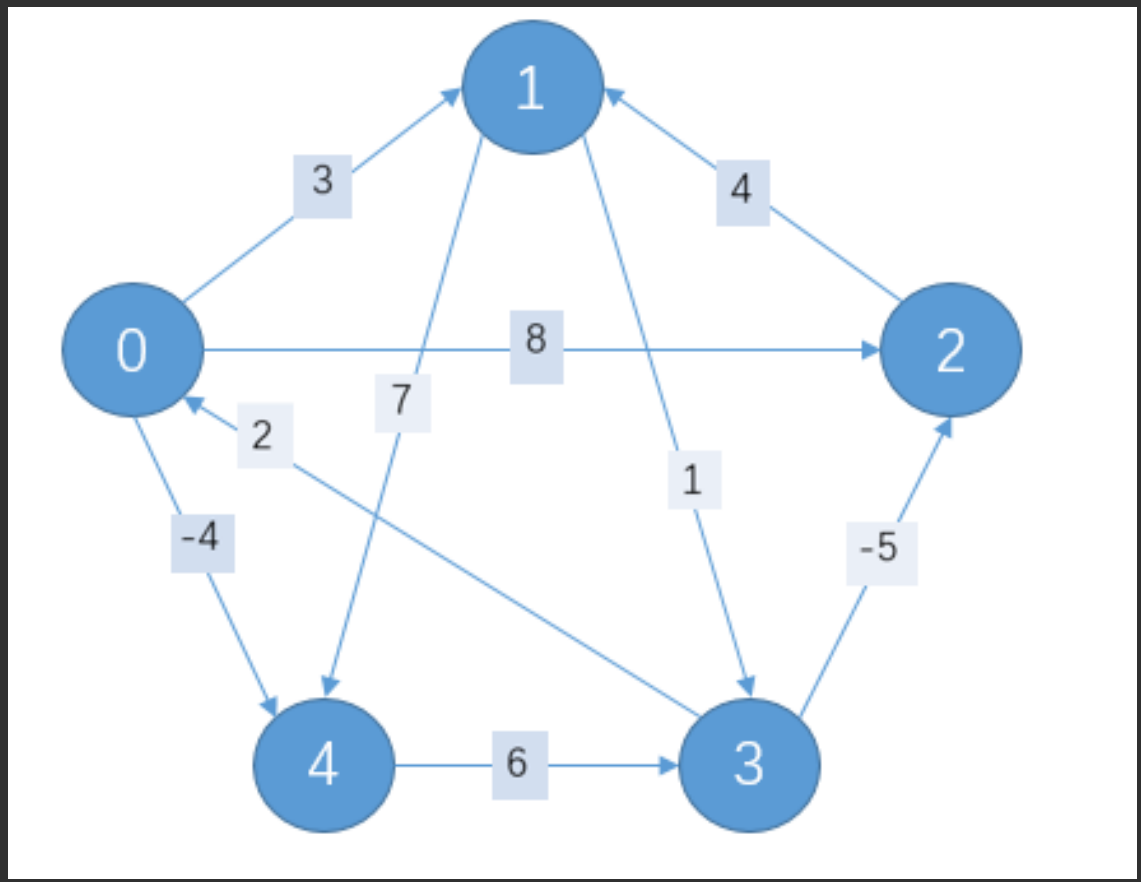
- Dijkstra's algorithm for the shortest path problem

$$d[y] = \min_x \{d[y], d[x] + w(x, y)\}$$

How to define the status and stage of problems is essential.

Shortest path problem

Given a directed graph (V, A) with source node s, target node t, and cost w_{ij} for each edge (i, j) in A, consider the program with variables x_{ij}.



Shortest path problem

- Integer programming formulation:

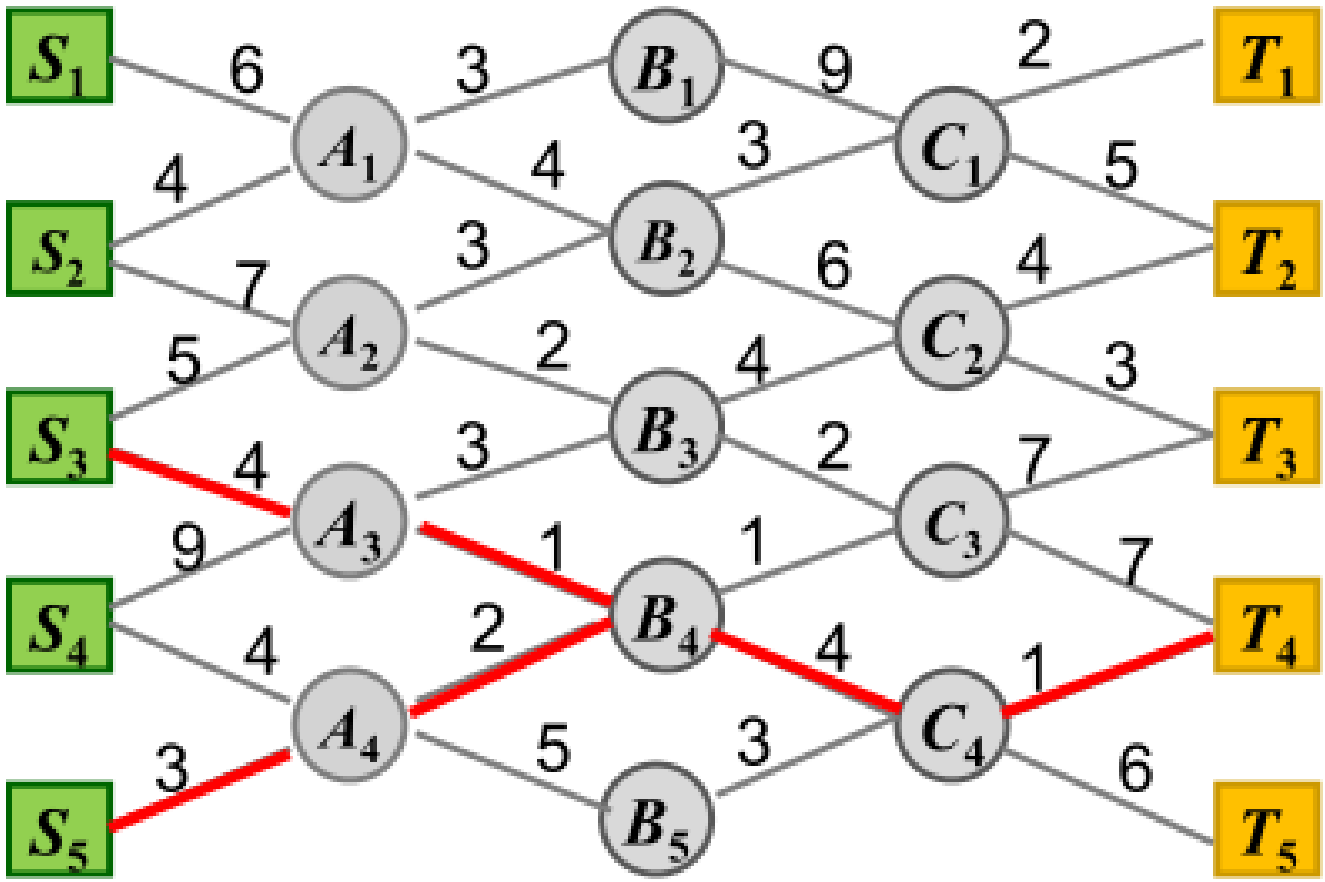
$$\min \sum_{(i,j)\in A} w_{ij}x_{ij}$$

$$\text{s.t.} \sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} 1, & \text{if } i = s; \\ -1, & \text{if } i = t; \\ 0, & \text{otherwise.} \end{cases}$$

$$x \in \{0, 1\} \text{ and for all } i.$$

Shortest path problem

- Find the shortest path from s to t.



Shortest path problem

$$Stage1 \quad F(C_l) = \min_m \{C_l T_m\}$$

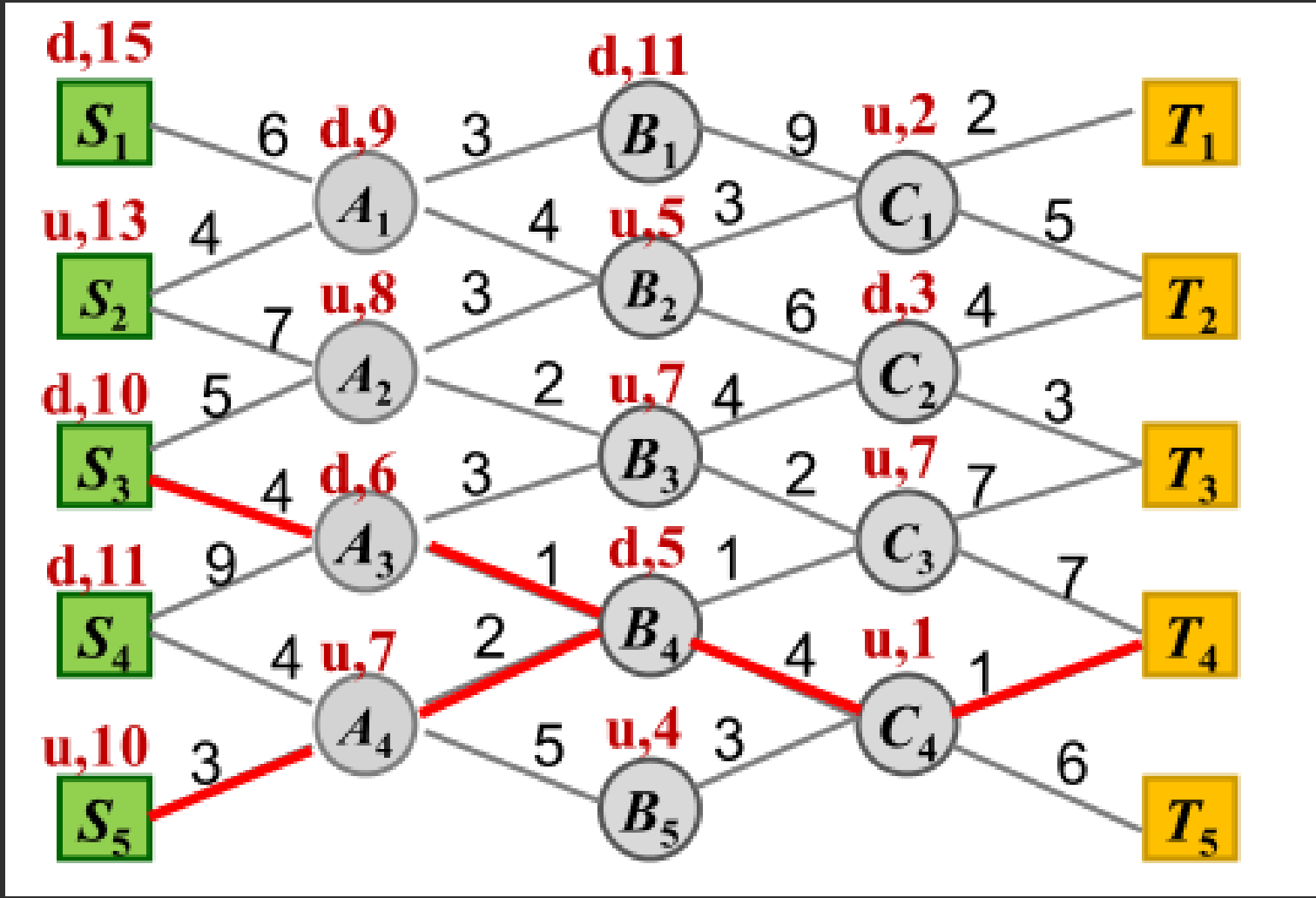
$$Stage2 \quad F(B_k) = \min_l \{B_k C_l + F(C_l)\}$$

$$Stage3 \quad F(A_j) = \min_k \{A_j B_k + F(B_k)\}$$

$$Stage4 \quad F(S_i) = \min_j \{S_i A_j + F(A_j)\}$$

Shortest path problem

- Find the shortest path from s to t.



Integer & Linear Programming

Integer Programming

- Shortest path problem

$$\min \sum_{(i,j)\in A} w_{ij}x_{ij}$$

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- Maximum flow problem
- Assignment problem

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Totally Unimodular Matrix

- Every entry in A is 0, $+1$, or -1;
- Every column of A contains at most two non-zero (i.e., $+1$ or -1) entries;
- If two non-zero entries in a column of A have the same sign, then the row of one is in B, and the other in C;
- If two non-zero entries in a column of A have opposite signs, then the rows of both are in B, or both in C.

TU Matrix

- Totally unimodular matrices are extremely important in polyhedral combinatorics and combinatorial optimization since they give a quick way to verify that a linear program is integral (has an integral optimum, when any optimum exists).
- Specifically, if A is TU and b is integral, then linear programs of forms like $\{\min cx \mid Ax \geq b, x \geq 0\}$ or $\{\max cx \mid Ax \leq b\}$ have integral optima, for any c. Hence if A is totally unimodular and b is integral, every extreme point of the feasible region (e.g. $\{x \mid Ax \geq b\}$) is integral and thus the feasible region is an integral polyhedron.

Another Perspective

Recall the simplex method for linear programming.

$$Bx = b$$
$$x^* = (B^{-1}b, 0)$$

How to obtain the inverse of B?

Cramer's rule:

$$B^{-1} = B^*/\det(B)$$

Simplex Method

- Feasible region(Convex polytope)
- Basic feasible solution(Extreme point)
- Basic variables(Identity matrix)
- Entering variable selection
- Leaving variable selection
- Pivot operation
- Reduced costs

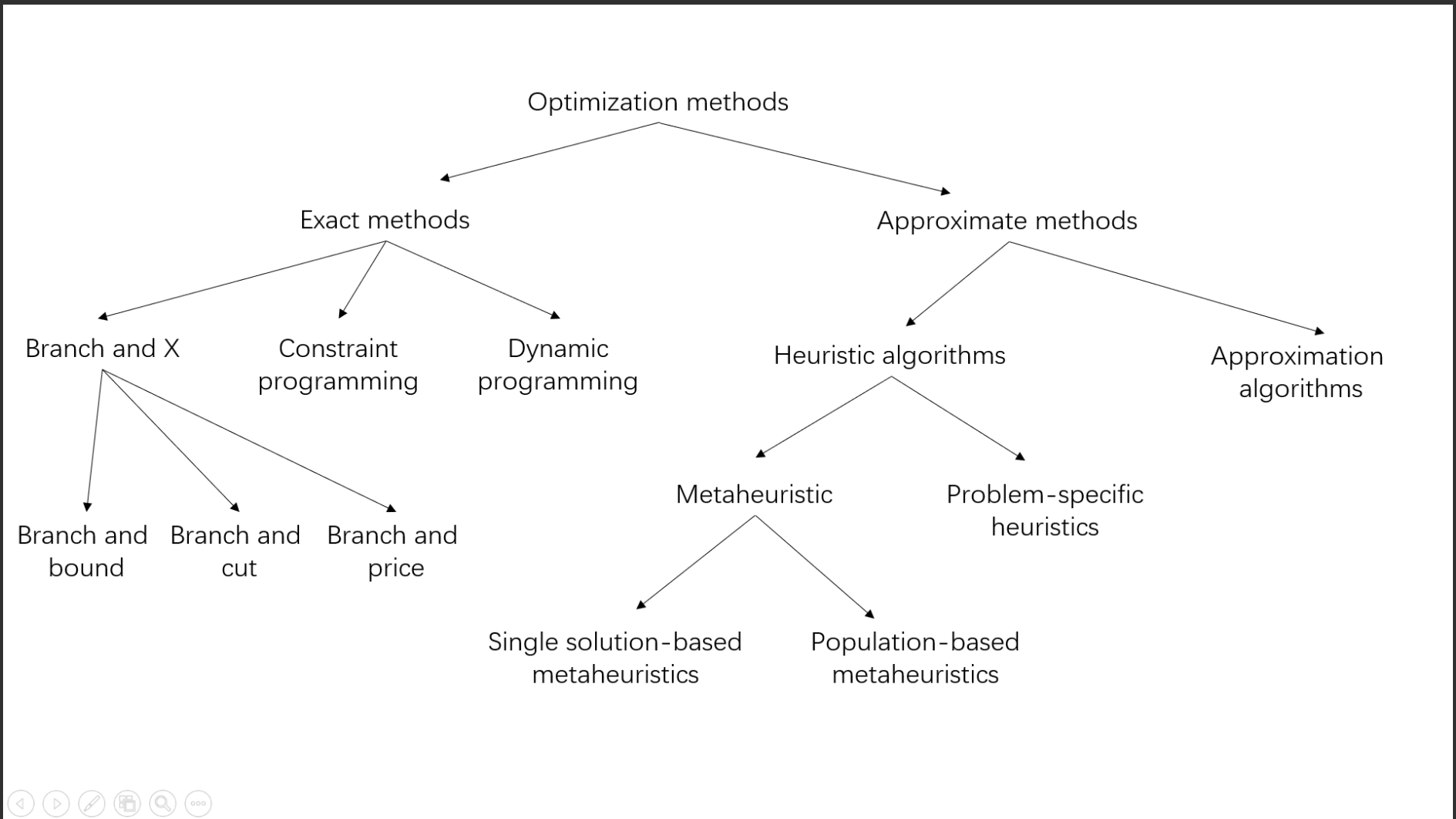
Another Perspective

The simplex method is an iteration process.

$$x' = x - \theta\lambda$$

- λ: Gradient direction(As small as possible)
  Entering variable selection
- θ: Step length(As long as possible)
  Leaving variable selection

General Optimization Methods



General Optimization Methods
Exact Methods

Exact Methods

- Branch and X
  1. Branch and bound
  2. Branch and cut
  3. Branch and price

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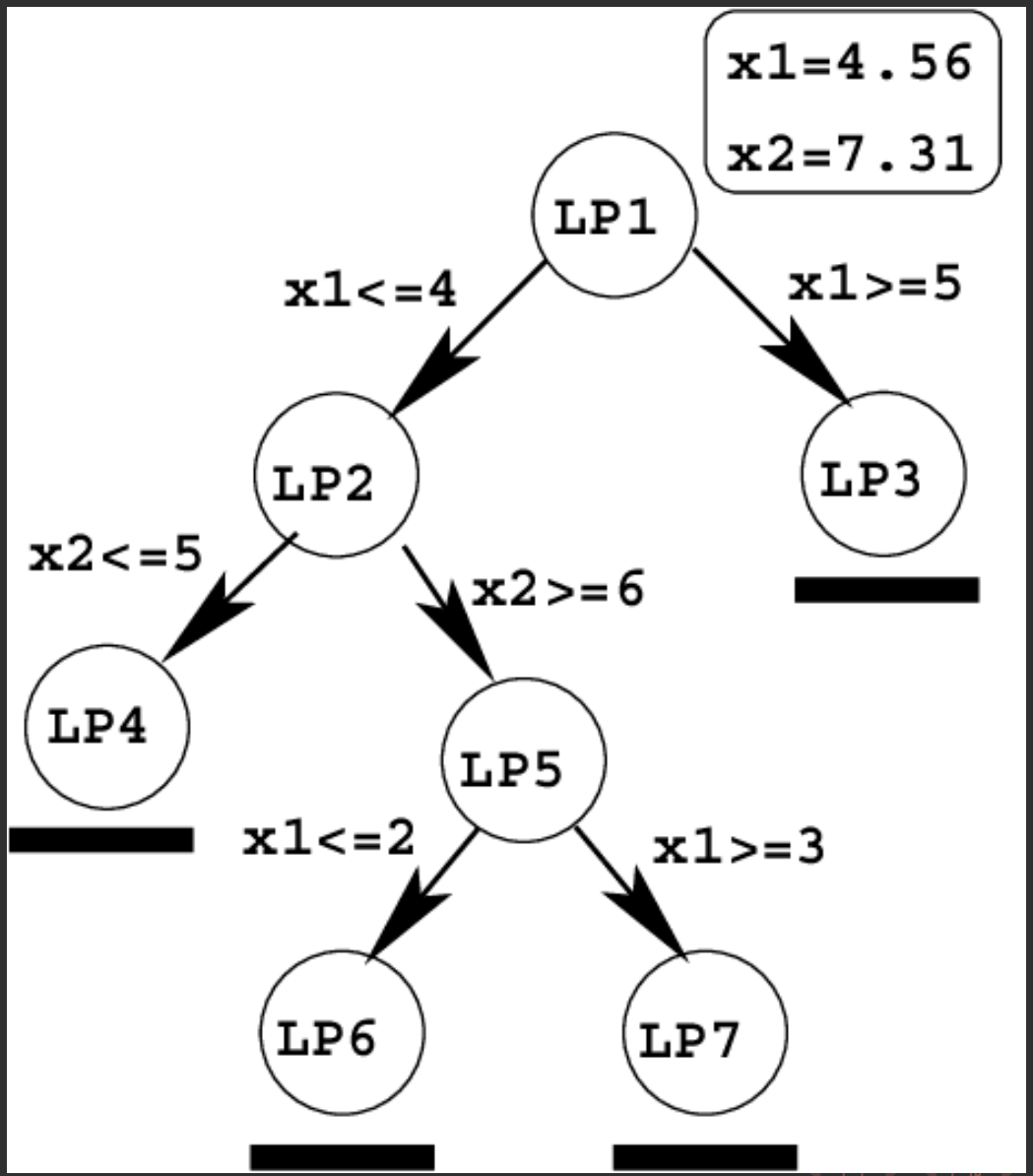
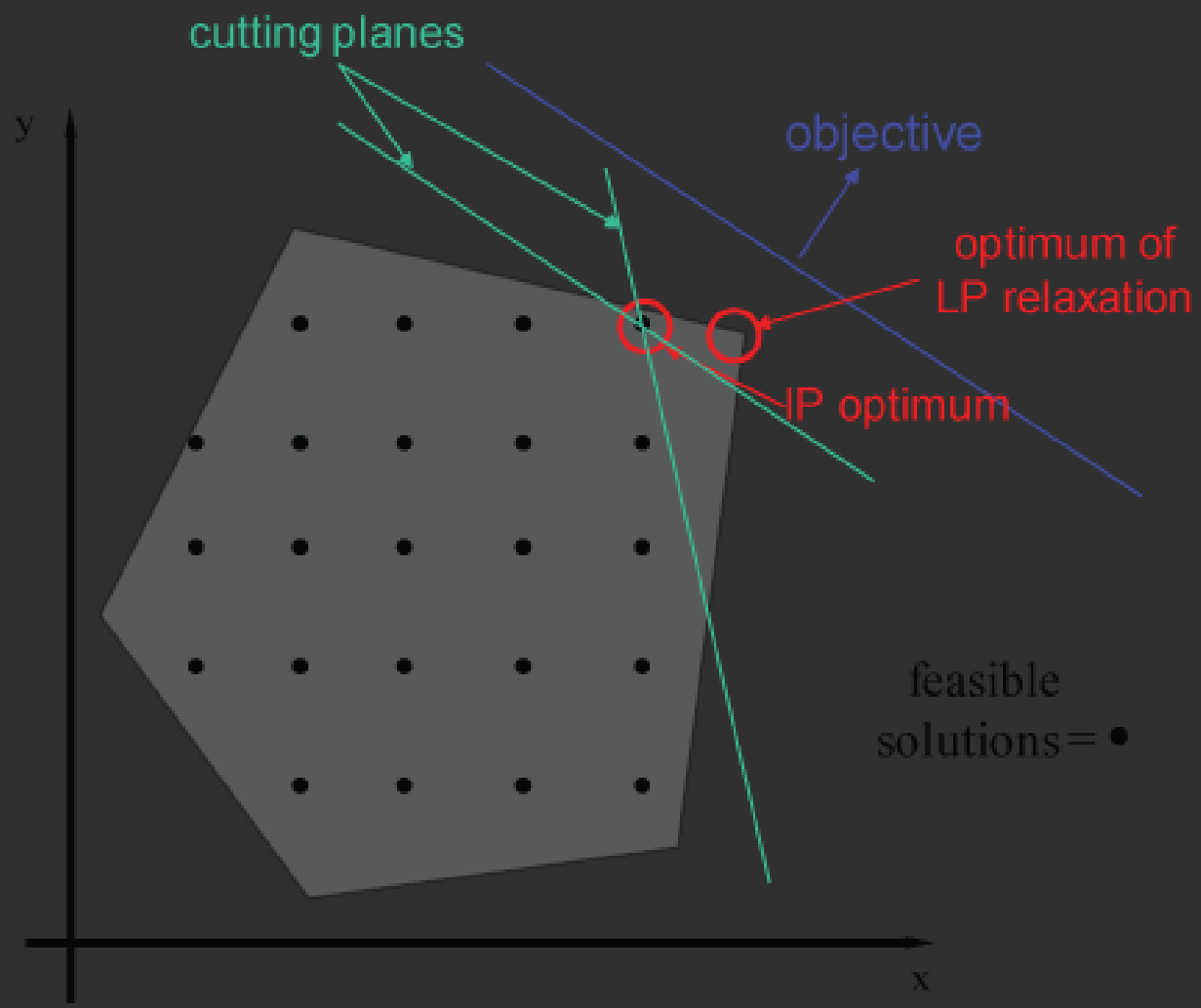
Branch and bound



Cutting Plane Method

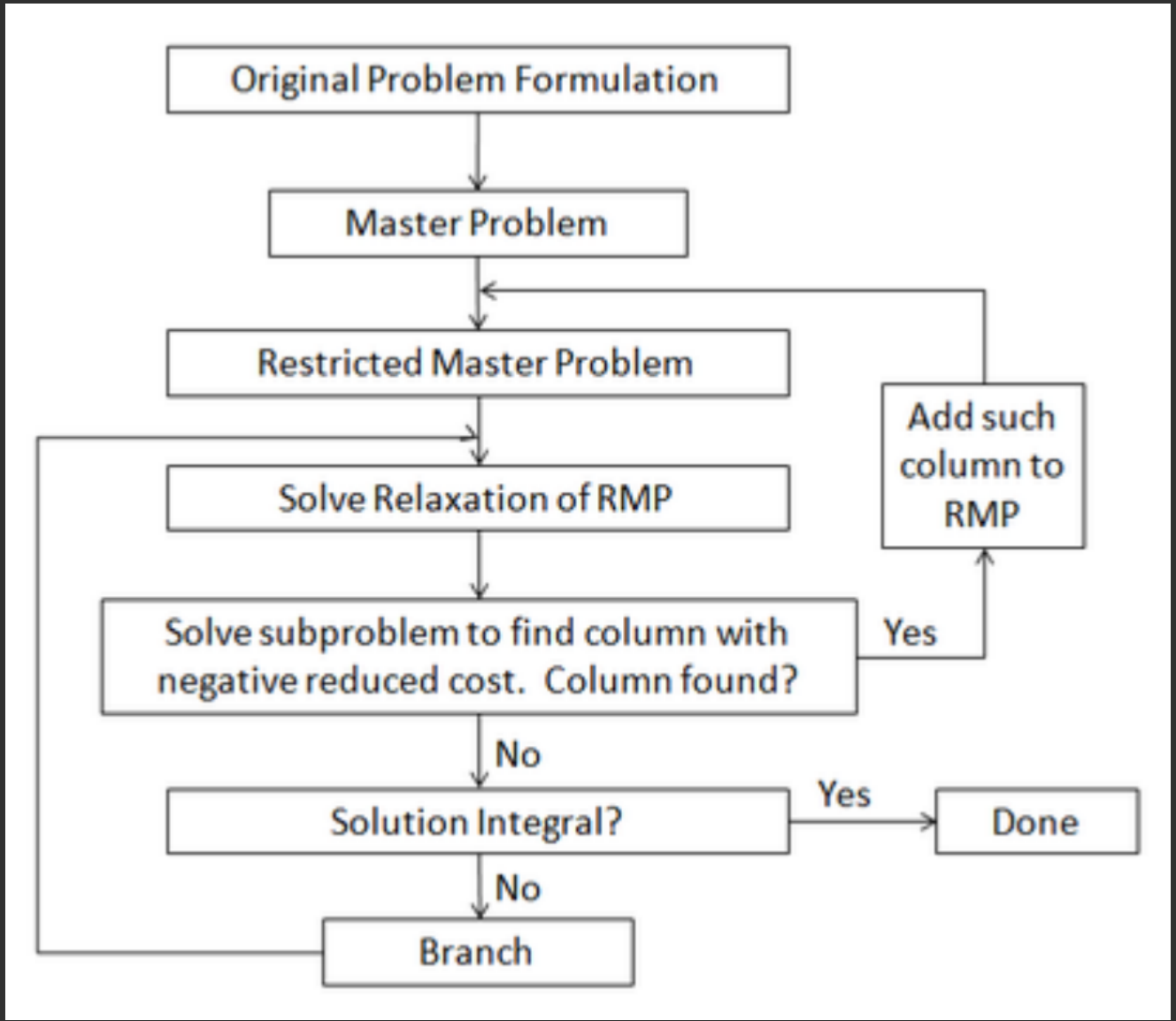


Cutting Planes

Gomory's Cut

- Using the simplex method: $x_i + \sum \bar{a}_{i,j} x_j = \bar{b}_i$
  where x_i are the basic variables and the x_j's are the nonbasic variables.
- Rewrite this equation: integer parts(left) and the fractional parts(right):
  $x_i + \sum \lfloor \bar{a}_{i,j} \rfloor x_j - \lfloor \bar{b}_i \rfloor = \bar{b}_i - \lfloor \bar{b}_i \rfloor - \sum (\bar{a}_{i,j} - \lfloor \bar{a}_{i,j} \rfloor) x_j.$
- Right side is less than 1 and the left side is an integer, therefore the inequality: $\bar{b}_i - \lfloor \bar{b}_i \rfloor - \sum (\bar{a}_{i,j} - \lfloor \bar{a}_{i,j} \rfloor) x_j \leq 0$ must hold for any integer point in the feasible region.
- The inequality above is a cut with the desired properties. Introducing a new slack variable x_k for this inequality, a new constraint is added to the linear program, namely
  $x_k + \sum (\lfloor \bar{a}_{i,j} \rfloor - \bar{a}_{i,j}) x_j = \lfloor \bar{b}_i \rfloor - \bar{b}_i$, $x_k \geq 0$, x_k an integer.

Branch and price



Exact Methods

- Branch and X
- Dynamic programming
- Constraint programming
- Enumeration method

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General Optimization Methods
Approximate Methods

Approximate Methods

- Heuristic algorithms
  1. Metaheuristic
     - Single solution-based metaheuristics
     - Population-based metaheuristics
  2. Problem-specific heuristics
- Approximate algorithms

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Single solution-based metaheuristics

- Simulated Annealing Algorithm
- Tabu Search
- Variable Neighborhood Search
- Adaptive Large Neighborhood Search

Population-based heuristics

- Genetic Algorithm
- Ant Colony Optimization
- Partical Swarm Optimization

Combinatorial Optimization

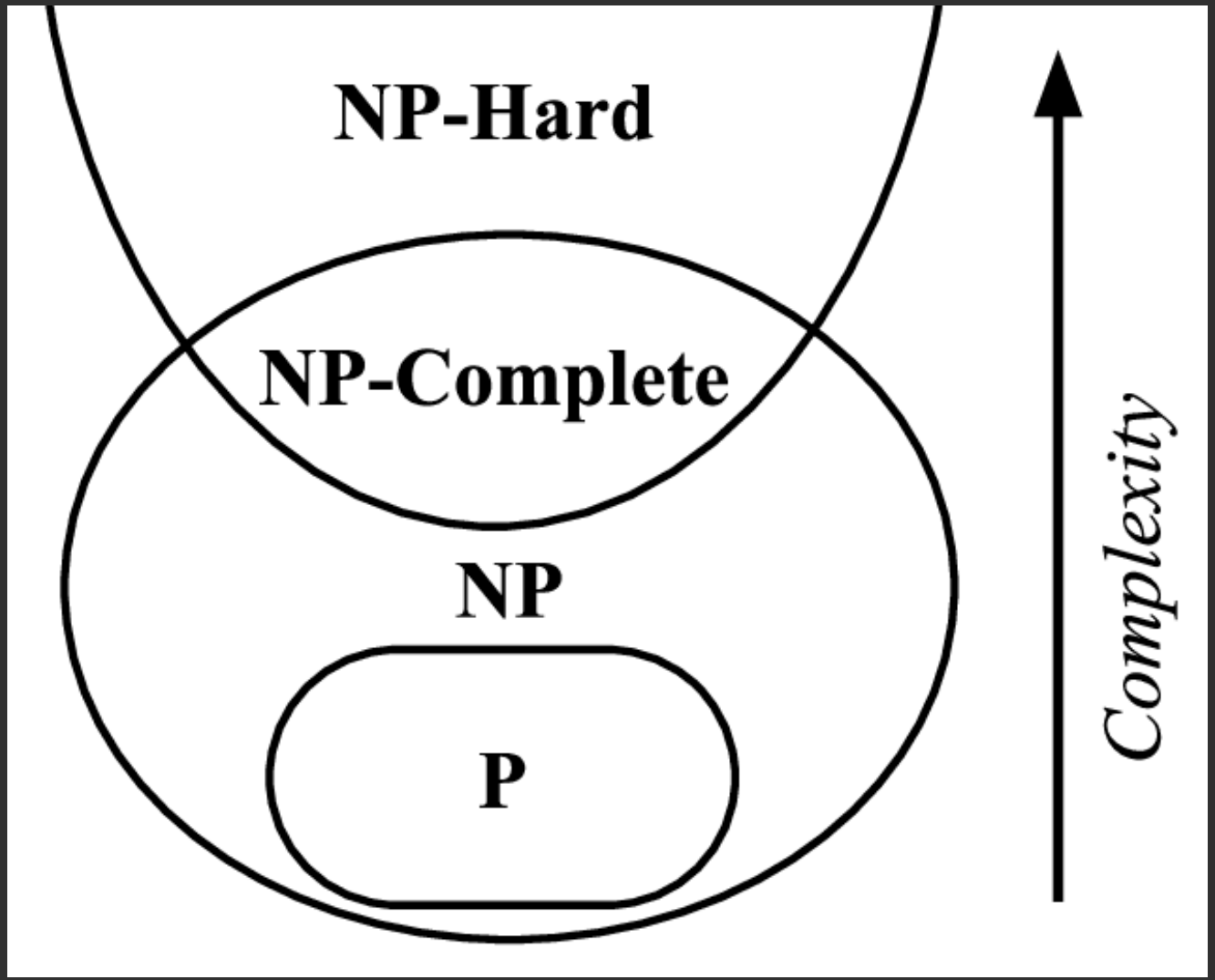
Classical problems

- Knapsack problem
- Minimum spanning tree
- Traveling salesman problem
- Set cover problem
- Matching problem
- Vehicle routing problem
- Facility Location Problem
- Production Scheduling Problem

NP-complete

- P(PTIME)
  It contains all decision problems that can be solved by a deterministic Turing machine using a polynomial amount of computation time, or polynomial time.
- NP(Nondeterministic Polynomial time)
  Class of computational decision problems for which a given yes-solution can be verified as a solution in polynomial time by a deterministic Turing machine.
- NP-complete
  Class of decision problems which contains the hardest problems in NP. Each NP-complete problem has to be in NP.
- NP-hard
  A problem H is NP-hard when every problem L in NP can be reduced in polynomial time to H. Class of problems which are at least as hard as the hardest problems in NP.

NP-complete



How to solve Combinatorial optimization problem

- Branch and bound
- Cutting plane
- Column generation
- .....

Summary

Finally, we should master how to settle a problem in that way, so that we will have more ideas and try different methods.
In the process of practice,they can guarantee to find the global solution, but in solving some typical practical problems, the price is too high, or it is easy to fall into local optimum. Because it's hard to accelerate the algorithm that can guarantee to find the optimal solution, that is, for most practical problems, it is difficult to find polynomial algorithm, because most of them are NP hard problems, then the rest selection is to design an algorithm that can jump out of local optimum.

General Optimization problem

Convex Optimization

- Convex Optimization
- Non-Convex Optimization

Convex Optimization

- Convex set: A set S is convex if for all members $x, y \in S$ and all $\theta \in [0, 1]$, we have that $\theta x + (1 - \theta)y \in S$.
- Convex function: For all $\theta \in [0, 1]$ and all x, y in S, the following condition holds: $f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$.
- Convex optimization:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, i = 1, 2, ..., m \\ & h_j(x) = 0, j = 1, 2, ..., n \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the optimization variable, the function $f : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, are convex, and $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, p$, are affine.

Standard Problems

- Linear Programming(LP)

$$\begin{aligned} \min \quad & c^T x + d \\ s.t. \quad & G(x) \preceq h \\ & A(x) = b \end{aligned}$$

- Quadratic Programming(QP)

$$\begin{aligned} \min \quad & \frac{1}{2} x^T P x + c^T x + d \\ s.t. \quad & G(x) \preceq h \\ & A(x) = b \end{aligned}$$

- Semidefinite Programming(SDP)

$$\begin{aligned} \min \quad & tr(CX) \\ s.t. \quad & tr(A_i X) = b_i, i = 1, 2, ...p \\ & X \succeq 0 \end{aligned}$$

Other problems

- Least squares
- Support Vector Machine(SVM)
- Quadratically Contrained Quadratic program(QCQP)
- Second-Order Cone Program(SOCP)
- Geometric Programming(GP)
- Conic Optimization

Property

- Local $=$ globle
- Non $\rightarrow$ Convex
- Many method to solve and wide application.

Methods

- Gradient Describe the basic iteration process.$x_{k+1} = x_k + \alpha_k d_k$ $\triangledown f(x_k)^T d_k < 0$ $f(x_{k+1}) = f(x_k + \alpha_k d_k) < f(x_k)$
- Sub-gradient(for can't derivative)
- Interior point

Gradient

Two essential elements: step and gradient. All kinds of methods are based on the fundamental.

- Random
- Batch
- .....

convergence condition

KKT

- Interior Methods
- Penelty
- Dual
- Lagrange
- Augmented Lagrangian
- Alternating Direction Method of Multipliers(ADMM)

Non-Convex

convert to Convex

heuristics The important thing is how to jump out of the local optimization. Change the parameter by experience.

Random optimization

Methods

Algorithmic idea

Summary

The author is extremely thankful to Prof. Liu for the short, yet wonderful, conversations about this seminar.

The End