

Introduction to Optimization

Dis·count

School of Management
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Summary

- 1 Backpack problem
- 2 Playing Around With Our New Toy
- 3 Fourier's Physics Playground
 - Maxwell's Electrodynamics
 - Heisenberg's Uncertainty Principle

Backpack problem

Ordinary Differential Equations

$$\frac{d}{dx}y(x) + \frac{1}{CR}y(x) = 0$$

$$\frac{d^2}{dx^2}y(x) + \gamma \frac{d}{dx}y(x) + \omega_0^2 y(x) = f(x)$$

Ordinary Differential Equations

$$\frac{d}{dx}y(x) + \frac{1}{CR}y(x) = 0$$

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Livin' La Vida Loca

$$\frac{d^2}{dx^2}y(x) + \gamma \frac{d}{dx}y(x) + \omega_0^2 y(x) = f(x)$$

$$\Downarrow$$

$$\left[\frac{d^2}{dx^2} + \gamma \frac{d}{dx} + \omega_0^2 \right] y(x) = f(x)$$

$$\Downarrow$$

$$y(x) = \frac{f(x)}{\frac{d^2}{dx^2} + \gamma \frac{d}{dx} + \omega_0^2}$$

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Livin' La Vida Loca



Pandora's Box



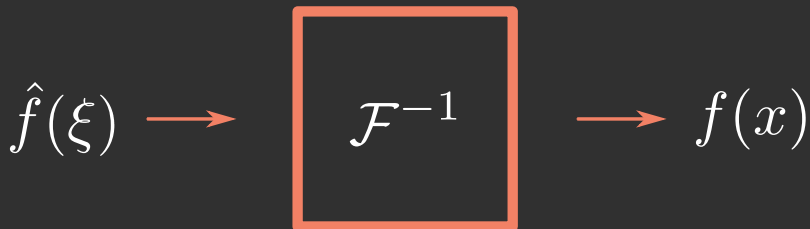
Pandora's Box

$$(f + \alpha g)(x) \longrightarrow \boxed{\mathcal{F}} \longrightarrow \hat{f}(\xi) + \alpha \hat{g}(\xi)$$

Pandora's Box

$$\frac{d}{dx} f(x) \longrightarrow \boxed{\mathcal{F}} \longrightarrow i\xi \hat{f}(\xi)$$

Pandora's Box



Pandora's Box

$$\left[\frac{d^2}{dx^2} + \gamma \frac{d}{dx} + \omega_0^2 \right] y(x) = f(x)$$



$$\mathcal{F}$$



$$[-\xi^2 + i\gamma\xi + \omega_0^2] \hat{y}(\xi) = \hat{f}(\xi)$$

Box Proposal

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ix\xi} dx$$

$$\mathcal{F}^{-1}[\hat{f}](x) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{ix\xi} d\xi$$

Box Proposal

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Quality Control

$$(\widehat{f + \alpha g})(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (f(x) + \alpha g(x)) e^{-ix\xi} dx$$

$$\Downarrow$$

$$(\widehat{f + \alpha g})(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ix\xi} dx + \frac{\alpha}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x) e^{-ix\xi} dx$$

$$\Downarrow$$

$$(\widehat{f + \alpha g})(\xi) = \hat{f}(\xi) + \alpha \hat{g}(\xi)$$

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$$(\widehat{f + \alpha g})(\xi) = \hat{f}(\xi) + \alpha \hat{g}(\xi)$$

Quality Control

$$\widehat{f}'(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f'(x) e^{-ix\xi} dx$$

$$\Downarrow$$

$$\widehat{f}'(\xi) = \left. \frac{f(x) e^{-ix\xi}}{\sqrt{2\pi}} \right|_{-\infty}^{+\infty} + i\xi \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ix\xi} dx$$

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$$\widehat{f}'(\xi) = i\xi \widehat{f}(\xi)$$

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$$\Downarrow$$

$$\widehat{f}'(\xi) = i\xi \widehat{f}(\xi)$$

Quality Control

The inverse does work for appropriate functions

and, sometimes, the Fourier Transform of a function is not in the same set as the original function, but let's forget about this since we do not know a decent theory of integration

Playing Around With Our New Toy

Fourier Transforming

$$f(t) = \cos(\omega_0 t) e^{-\pi t^2}$$

$$\hat{f}(\omega) = \frac{e^{-\frac{(\omega - \omega_0)^2}{4\pi}} + e^{-\frac{(\omega + \omega_0)^2}{4\pi}}}{2\sqrt{2\pi}}$$

$$\omega = 2\pi\nu$$

Fourier Transforming

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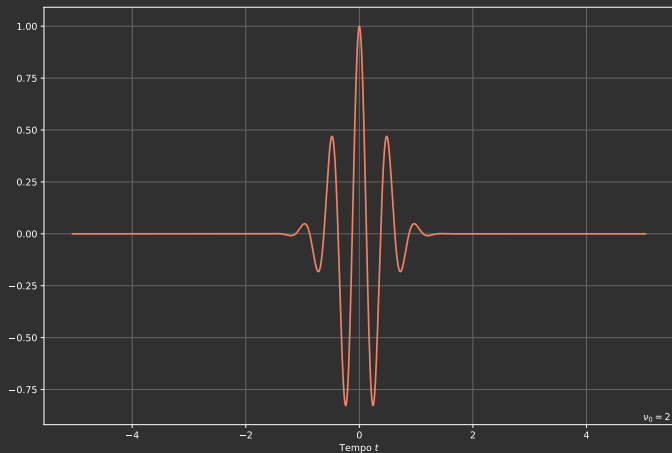
$$\omega = 2\pi\nu$$

Fourier Transforming

$$f(t) = \cos(2\pi\nu_0 t)e^{-\pi t^2}$$
$$\hat{f}(\nu) = \frac{e^{-\pi(\nu-\nu_0)^2} + e^{-\pi(\nu+\nu_0)^2}}{2\sqrt{2\pi}}$$

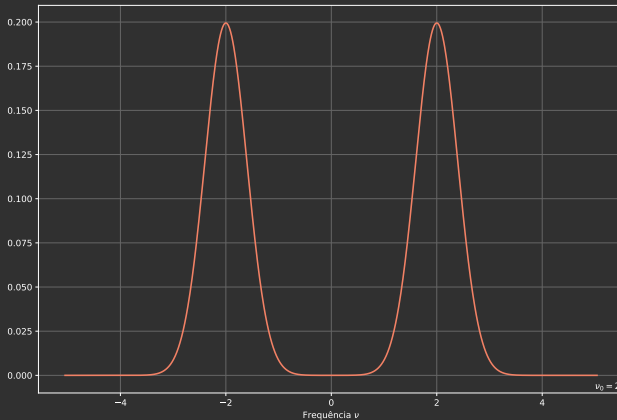
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Fourier Transforming

$$\hat{f}(\nu) = \frac{e^{-\pi(\nu-\nu_0)^2} + e^{-\pi(\nu+\nu_0)^2}}{2\sqrt{2\pi}}$$



A Harder Example

$$f(t) = e^{i\omega_0 t} = \cos(\omega_0 t) + i \sin(\omega_0 t)$$

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega_0 t} e^{-i\omega t} dt$$

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The Mathematical Moonwalk

$$f(t) = e^{i\omega_0 t}$$

$$e^{i\omega_0 t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{i\omega t} d\omega$$

$$\hat{f}(\omega) = \sqrt{2\pi} \delta(\omega - \omega_0)$$

The Mathematical Moonwalk

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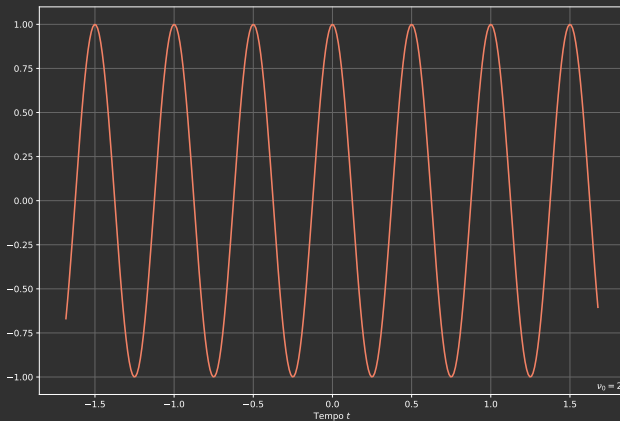
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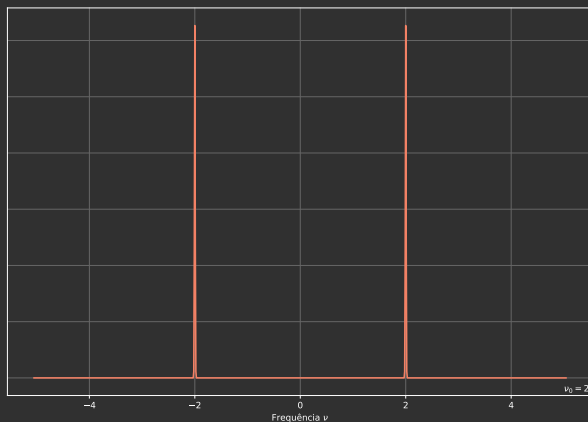
Cosines

$$f(t) = \cos(\omega_0 t) = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}$$



Cosines

$$\hat{f}(\omega) = \sqrt{\frac{\pi}{2}} (\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$$



Fourier's Physics Playground

Maxwell's Electrodynamics

In the beggining, God said:

$$\left\{ \begin{array}{l} \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{array} \right.$$

and there was light!

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and there was light!

Too hard, let's try something different

$$\begin{cases} \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} = \nabla \times \mathbf{A} \end{cases}$$

Wave Equations

$$\begin{cases} \nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0} \\ \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \end{cases}$$

All Wave Equations In One

$$\nabla^2 \psi(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = -g(\mathbf{r}, t)$$

Fourier's Opinion

$$\hat{g}(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\mathbf{r}, t) e^{-i\omega t} dt$$

$$g(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{g}(\mathbf{r}, \omega) e^{i\omega t} d\omega$$

Fourier's Opinion

$$\hat{\psi}(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(\mathbf{r}, t) e^{-i\omega t} dt$$

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Fourier's Opinion

$$\nabla^2 \psi(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = -g(\mathbf{r}, t)$$

$$\nabla^2 \hat{\psi}(\mathbf{r}, \omega) + \frac{\omega^2}{c^2} \hat{\psi}(\mathbf{r}, \omega) = -\hat{g}(\mathbf{r}, \omega)$$

Green Function

$$L\phi(\mathbf{r}) = -s(\mathbf{r})$$

$$LG(\mathbf{r} - \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$$

$$\phi(\mathbf{r}) = \int G(\mathbf{r} - \mathbf{r}') s(\mathbf{r}') d\tau'$$

$$L\phi(\mathbf{r}) = \int LG(\mathbf{r} - \mathbf{r}') s(\mathbf{r}') d\tau' = - \int \delta(\mathbf{r} - \mathbf{r}') s(\mathbf{r}') d\tau' = -s(\mathbf{r})$$

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One At a Time

$$\nabla^2 \hat{\psi}(\mathbf{r}, \omega) + \frac{\omega^2}{c^2} \hat{\psi}(\mathbf{r}, \omega) = -\hat{g}(\mathbf{r}, \omega)$$

$$\nabla^2 G(\mathbf{r} - \mathbf{r}') + \frac{\omega^2}{c^2} G(\mathbf{r} - \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$$

One At a Time

$$\nabla^2 \hat{\psi}(\mathbf{r}, \omega) + \frac{\omega^2}{c^2} \hat{\psi}(\mathbf{r}, \omega) = -\hat{g}(\mathbf{r}, \omega)$$

$$\nabla^2 G(\mathbf{r} - \mathbf{r}') + \frac{\omega^2}{c^2} G(\mathbf{r} - \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$$

Solution for $\mathbf{r} - \mathbf{r}' \neq 0$

$$\frac{1}{r} \frac{d^2(rG)}{dr^2} + k^2 G = 0$$

$$G(r) = \frac{A}{r} e^{\pm ikr}$$

Solution for $\mathbf{r} - \mathbf{r}' \neq 0$

$$\frac{1}{r} \frac{d^2(rG)}{dr^2} + k^2 G = 0$$

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Recovering 0 Psychological Trauma

$$\nabla^2 G(\mathbf{r} - \mathbf{r}') + \frac{\omega^2}{c^2} G(\mathbf{r} - \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$$

$$A \int \nabla^2 \frac{1}{r} d\tau' + 4\pi A \frac{\omega^2}{c^2} \int \frac{r^2}{r} dr = - \int \delta(\mathbf{r} - \mathbf{r}') d\tau'$$

$$-4\pi A = -1$$

Recovering 0 Psychological Trauma

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$$-4\pi A = -1$$

Back To Our Problem

$$\hat{\psi}(\mathbf{r}, \omega) = \int G(z) \hat{g}(\mathbf{r}', \omega) d\tau'$$

$$G(z) = \frac{1}{4\pi z} e^{\pm i k z}$$

$$\hat{\psi}(\mathbf{r}, \omega) = \frac{1}{4\pi} \int \frac{\hat{g}(\mathbf{r}', \omega) e^{\pm i k z}}{z} d\tau'$$

Back To Our Problem

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Actually Solving Our Problem

$$\psi(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{\psi}(\mathbf{r}, \omega) e^{i\omega t} d\omega$$

$$\psi(\mathbf{r}, t) = \frac{1}{4\pi\sqrt{2\pi}} \iint \frac{\hat{g}(\mathbf{r}', \omega) e^{i\omega t \pm i\omega \frac{z}{c}}}{z} d\omega d\tau'$$

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Actually Solving Our Problem

$$\psi(\mathbf{r}, t) = \frac{1}{4\pi\sqrt{2\pi}} \iint \frac{\widehat{g}(\mathbf{r}', \omega) e^{i\omega(t \pm \frac{z}{c})}}{z} d\omega d\tau'$$

$$\psi(\mathbf{r}, t) = \frac{1}{4\pi} \int \frac{g(\mathbf{r}', t \pm \frac{z}{c})}{z} d\tau'$$

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$$\psi(\mathbf{r}, t) = \frac{1}{4\pi} \int \frac{g(\mathbf{r}', t - \frac{z}{c})}{z} d\tau'$$

Back at Maxwell's

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t - \frac{r}{c})}{r} d\tau'$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t - \frac{r}{c})}{r} d\tau'$$

One Last Step

$$\begin{cases} \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} = \nabla \times \mathbf{A} \end{cases}$$

Jefimenko Equations

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\mathbf{z}}}{z^2} [\rho] + \frac{\hat{\mathbf{z}}}{cz} \left[\frac{\partial \rho}{\partial t} \right] - \frac{1}{c^2 z} \left[\frac{\partial \mathbf{J}}{\partial t} \right] d\tau'$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left(\frac{1}{z^2} [\mathbf{J}] + \frac{1}{cz} \left[\frac{\partial \mathbf{J}}{\partial t} \right] \right) \times \hat{\mathbf{z}} d\tau'$$

Fourier's Physics Playground

Heisenberg's Uncertainty Principle

Position and Momentum

$$\psi(x) = \langle x | \psi \rangle = \int \langle x | k \rangle \langle k | \psi \rangle dk = \frac{1}{\sqrt{2\pi}} \int e^{ikx} \psi(k) dk$$

$$\psi(k) = \langle k | \psi \rangle = \int \langle k | x \rangle \langle x | \psi \rangle dx = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \psi(x) dx$$

Position and Momentum

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Position and Momentum (but weirder)

$$\begin{cases} X |\psi\rangle = x\psi(x) \\ K |\psi\rangle = -i\frac{\partial\psi}{\partial x} \end{cases}$$

$$\begin{cases} X |\psi\rangle = -i\frac{\partial\psi}{\partial k} \\ K |\psi\rangle = k\psi(k) \end{cases}$$

Position and Momentum (but weirder)

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Fourier Diplomacy

$$|x\rangle \xleftrightarrow[\mathcal{F}^{-1}]{\mathcal{F}} |k\rangle$$

Fourier Uncertainty

1 $\psi(x)$: what is x ?

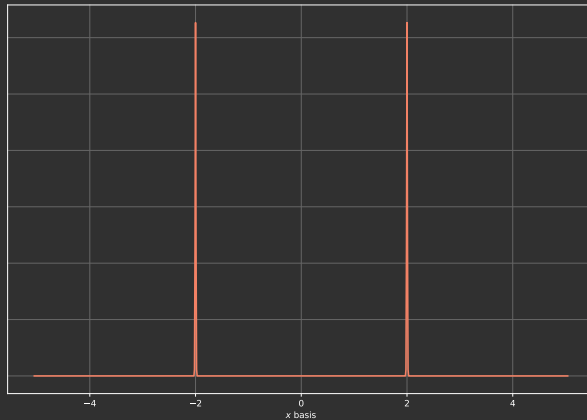
2 $\psi(k)$: what is k ?

Fourier Uncertainty

- 1 $\psi(x)$: what is x ?
- 2 $\psi(k)$: what is k ?

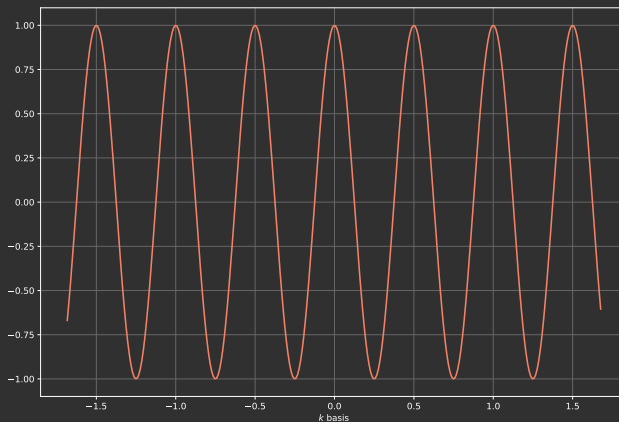
Definite Position

$$\psi(x) = \sqrt{\frac{\pi}{2}} (\delta(x - x_0) + \delta(x + x_0))$$



Undefinite Momentum

$$\psi(k) = \cos(x_0 k)$$



Uncertainty Relation

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

The uncertainty relation is a consequence of the general fact that anything narrow in one space is wide in the transform space and vice versa. So if you are a 45 kg weakling and are taunted by a 270 kg bully, just ask him to step into momentum space!

Ramamurti Shankar

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