1. **Background**

A cooperative game consists of a set of players *N*, where any subset of players may form a coalition *s*, and *N* is called the grand coalition. There is a characteristic function defined from the set of coalitions to a set of payoffs. In the context of cost minimization, the characteristic function *c*(*s*) describes how much a coalition *s* would pay when the players in *s* work collaboratively. The key issue is to design a fair way to share the coalition cost, especially for the grand coalition *c*(*N*), among the players in the coalition. We may use a vector **x**={*xj*, } to denote a cost allocation where *xj* is the cost allocated to player *j*.

There are several key concepts regarding fair allocation among the players, where the most basic one is the core. The core of a cooperative game is the set of cost allocation vectors satisfying (1) and (2) for any coalition *s*. Here constraint (1) enforces the budget balance, and constraint (2) ensures that no coalition has a cost smaller than the sum of its members' cost shares. If the core is not empty, then under an allocation in the core, no coalition has incentive to leave the grand coalition. In such a case, we say the grand coalition is stable, which is also known as a balanced game in the literature. In practice, a stable grand coalition is often desired because it represents the best of the social efficiency.

It is well known that the core may be empty for many cooperative games. For such games, alternative concepts have been proposed for compromised solution, including the least core (e.g., Kern and Paulusma 2003) and *ɛ*-approximate core (e.g., Blaser and Ram 2008). These concepts can be used to achieve the global collaboration as long as the players internally agree to tolerate certain loss. For example, under the least core, each coalition *s* may accept a cost allocation bounded by *c*(*s*)+*z*, with *z* to be minimized; under the *ɛ*-approximate core, each coalition *s* may accept a cost allocation bounded by (1+*ɛ*)*c*(*s*), with *ɛ* to be minimized. Though not often explicitly mentioned, these solution concepts can be used by the players to cooperate and achieve the global optimum. In general, we refer to such concepts as schemes for stabilizing the grand coalition or stabilizing the cooperative game.

In this project we take a different perspective to stabilize a cooperative game, which exceeds the current scope that only considers the players within the game. We consider the case where there may exist an outside party who will step in to stabilize the grand coalition, for example, for the sake of social welfare. This outside party can be a government agency, the headquarters of a large enterprise, or even an industrial association. Despite its clear practical meaning, how to stabilize grand coalitions by an outside party has been largely overlooked in the literature, with a few work published only recently. The purpose of this project is to explore different approaches available for stabilizing grand coalitions by an outside party. To make the project concrete, we will mainly focus on cooperative games originating from sequencing/scheduling applications; but our methodology, major results, and new concepts will be applicable to other games, especially games with a background of operations research.

Under the framework of cooperative game theory, researchers have studied various models for the possibility of collaboration in sequencing and scheduling problems, e.g., early works such as Curiel et al. (1989, 1994, 2002), and more recent works such as Aydinliyim and Vairaktarakis (2010), Cai and Vairaktarakis (2012), Çiftçi et al. (2013), and Grundel et al. (2013). While such works highlight the importance of cooperation in sequencing and scheduling, there is not much work on stabilizing a grand coalition when the core is empty.

The PI has been working in the field of scheduling for more than ten years with 20+ papers published on scheduling problems. Some representative work includes Qi et al. (1999, 2002, 2004), Qi (2005, 2006, 2008, 2012), Ou et al. (2010), Wan and Qi (2010), Liu et al. (2012), and Lu et al. (2013). Recently, the PI has started research on cooperative games, with papers on recycling operations (Lu et al. 2014) and scheduling (Liu et al. 2014). The PI has the technical capability to carry out the proposed research. Regarding the preparation of this project, we already have some preliminary results such as Liu and Qi (2014a, 2014b), the details of which will be elaborated later.

1. **The Motivating Example**

In this proposal, we use one collaborative scheduling problem to facilitate discussion. In carrying out the project, we will consider studying different typical variations of the problems, for example, with other cost measures. Suppose that there are *n* agents, each agent *j* having a job to process with a processing time *pj*. Any agent can activate a machine by paying a setup cost *S*. The cost incurred to an agent includes two parts, one measured by the completion time of its job, and the other one being the machine setup cost. These two cost terms are normalized to be additive. To have a specific instance, we assume *n*=4, *p*1=2, *p*2=3, *p*3=4, *p*4=5, and *S*=9.5.

Under the framework of the cooperative game, we first need to calculate the characteristic functions for each coalition. From the research on scheduling, we know that the minimum cost to each coalition can be obtained by enumerating the number of activated machines where jobs will be alternatively scheduled on the activated machines in an SPT order (shortest processing time first). Hence we have the following values of the characteristic functions.

*c*( {1} ) = 11.5, *c*( {2} ) = 12.5, *c*( {3} ) = 13.5, *c*( {4} ) = 14.5

*c*( {1,2} ) = 16.5, *c*( {1,3} ) = 17.5, *c*( {1,4} ) = 18.5

*c*( {2,3} ) = 19.5, *c*( {2,4} ) = 20.5, *c*( {3,4} ) = 22.5

*c*( {1,2,3} ) = 25.5, *c*( {1,2,4} ) = 26.5, *c*( {1,3,4} ) = 28.5, *c*( {2,3,4} ) = 31.5

*c*( {1,2,3,4} ) = 38.

It can be verified that the core is empty for this game. In other words, a proper way does not exist for the agents to share the grand coalition cost *c*({1,2,3,4})=38, and hence the grand coalition is not stable.

For any decision maker who is responsible for all four jobs, the global optimal schedule should have a total cost of *c*({1,2,3,4})=38. Now we consider an outside party who, without a direct control of the jobs, regards the global optimum as the best social welfare that should be achieved. We will study how this outside party can step in to stabilize the grand coalition. To this end, we introduce the following mechanisms that can be used by the outside party.

1. **Research Issues and Methodologies**

We will investigate four mechanisms for an outside party to stabilize cooperative games in this project. The first two mechanisms are based on two independent ideas, subsidization and taxation, both motivated by public policies in the general economics. Using subsidization to stabilize a cooperative game has been proposed in the literature recently, but there is no result on scheduling games; using taxation is a new idea proposed in this project. The more creative mechanism is the third one, to combine subsidization and taxation for providing additional flexibility. The fourth mechanism is a further integration with other existing mechanisms. For each mechanism, we will use an example to show the idea, roughly define the problem by a brief mathematical description, and present specific tasks to solve the problem.

**3.1 Mechanism 1: Subsidizing the game**.

In the above example, if the outside player can share a cost of 0.75 for the grand coalition, then the four agents need to share the remaining cost of 38-0.75=37.25, and the cost allocation **x**=(6, 8.75, 10.75, 11.75) will be acceptable for them because for any coalition *s*. In this way, the grand coalition is stabilized by a subsidy from the outside party. For the outside party, the question is how to compute the minimum required subsidy. Referring to the example, we see that any subsidy higher than 0.75 will be enough to stabilize the grand coalition. In general, we need to find the minimum required subsidy because this is the cost that has to be paid by the outside party.

The problem can be formulated as

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The difficulty of solving the problem comes from two sources: First, for given a solution, even its feasibility test needs to check *O*(2*n*) number of inequalities, , each for one coalition *s*. Second, calculating each *c*(*s*) may be NP-hard, though it is not the case for the motivating example.

In the literature, study on subsidy has started recently. Bachrach et al. (2009) refer to this concept as cost of stability, and derive bounds on it for several classes of games. Further results on this concept can be found in Resnick et al. (2009) and Meir et al. (2011). Roughly speaking, not much algorithmic work has been developed along this stream. Caprara and Letchford (2010) address an equivalent problem with the aim of calculating the maximum total cost that can be allocated to the players, where they propose a framework based on linear programming (LP) relaxation techniques.

*Recently, we have done some preliminary work on this topic.* In one working paper (Liu and Qi 2014a), we propose a generic framework to compute the maximum cost allocation based on Lagrangian Relaxation, and implement it on facility location games. We have presented the idea on INFORMS and POMS conferences, and obtained positive feedbacks. By design, the framework can be applied to a broad class of cooperative games if we have an effective Lagrangian relaxation for the underlying optimization problem, but the implementation on each specific game needs further development. In this project, we plan to take two tasks to study the mechanism of subsidization, especially for scheduling games.

**Task 1.1: Implementing the LP or Lagrangian Relaxation-based frameworks to scheduling games.** In both cases, we will first write the standard integer programming (IP) formulation for the optimal scheduling problem. To implement the LP-based approach proposed in Caprara and Letchford (2010), we need to further refine the IP formulation that includes only those so-called assignable constraints, which is the most challenging part. To implement the Lagrangian relaxation-based approach, the idea is to decompose the game into sub-games that will be tackled separately. The key is to find the best way of doing relaxation and decomposition. Extensive computational study will be conducted.

**Task 1.2: Designing combinatorial algorithms specific to scheduling games.** While the algorithms under the general frameworks may work, they may overlook some special structures in scheduling games. It is possible to have more effective algorithm if we study how to take advantage of these structures to design specialized algorithms. The possibility may come from the fact that in some scheduling games, e.g., the motivating example, *c*(*s*), the optimal schedule for each coalition for each coalition *s*, can be relatively easy to obtain, hence avoiding a major general difficulty.

**3.2 Mechanism 2: Taxing the resource.**

While providing subsidy can stabilize the grand coalition, the mechanism needs to be externally funded, which may require extra justification in some cases. We now propose another mechanism that is free of external funding. This mechanism applies to cases where the outside party has certain legal power or authority so that he can tax the resource to be used. By doing this, it increases the cost for non-cooperating players, thus enhancing the incentive for collaboration.

For example, in the scheduling game, if the outside party charges an extra cost of 6.5 for each activated machine, then the equivalent machine activation cost becomes 16, which leads to the global optimal schedule to use only one machine with a cost of 46. It can be verified that the game now becomes balanced, e.g., with a cost allocation (8, 11, 13, 14) in the core.

For the outside party, the problem is how to compute the minimum required tax level. Referring to the example, we can numerically find that any tax level higher than 6.5 will be enough to stabilize the grand coalition. In general, we need to find the minimum tax level because a high tax level will unavoidably cause dissatisfaction to the players. Using *c*(*s*; *T* ) to denote the minimum cost for coalition *s* under tax level *T*, we have the following formulation.

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*We emphasize that taxing on resource to stabilize cooperative games is a new concept*. Recently Zick et al. (2013) study the issue of taxation and stability for cooperative games, but in that work taxation actually refers to the least core, another related concept that we will discuss later. We formally state our task for the mechanism of taxing on resources as follows.

**Task 2.1 Designing algorithms to compute the minimum tax level.** This is a new idea that has no prior study in the literature. However, we anticipate that our study on subsidization may shed some light on solving the taxation problem, because these two different approaches actually play some complementary roles in stabilizing a game.

**3.3 Mechanism 3: Combining taxation and subsidization.**

While both taxation and subsidization can be used to stabilize the grand coalition, each has its drawbacks. In particular, although taxation does not require any external funding, which resolves the difficulty in providing subsidization, it is at the cost of players’ dissatisfaction caused by tax payment. Even worse, it may cause some new inefficiency to the system. In the above example, the taxation actually changes the resulted schedule, from using two machines in the original global optimal schedule, to using one machine in the “forced” collaborative schedule. Such a change is inconsistent with the real purpose of collaboration because it is less efficient than the ideal two-machine schedule.

The above discussion raises the question whether it is possible to stabilize a grand coalition not changed from the original global schedule, without any externally-funded subsidy. To have an affirmative answer, we propose a new mechanism referred to as subsidization funded by taxation. Using the example, we can explain this idea as follows. When the external party charges the machine tax at 0.5, which makes the machine activation cost at 10, the global optimal schedule is still to use two machines with a total cost at 39. At the same time, if the third party subsidizes the grand coalition by 1, then the four agents only need to share a cost of 38, and a cost allocation (6, 9, 11, 12) would be acceptable to the agents.

If we calculate the cash flow for the outside party, we find that he has an exact breakeven. While he collects a tax at 0.5×2=1, he also pays a subsidy of 1. In doing this, essentially the grand coalition is stabilization by the agents themselves. Mathematically, we use *m*(*N*; *T*) to denote the number of machines used by the grand coalition under the tax level *T*, then the budget balance constraint is changed to In addition, we may also add constraint *m*(*N*; *T*) = *m*(*N*; 0) to ensure that the induced stable schedule is the same as the centralized global optimal schedule. We have the following tasks to formalize and study the new mechanism motivated by the above example.

**Task 3.1: We need to explore the generality of the above phenomenon.** Is it a coincidence just for this example, or does a subsidy perfectly funded by taxation always exist? From intuition and a few numerical examples, we conjecture that it may exist. However, the existence is not straightforward because both *m*(*N*;*T*) and *c*(*N*;*T*) are discontinuous of *T*. Related to this question, we also will investigate the uniqueness of the tax level if it does exist, and design algorithms to compute it.

**Task 3.2: We need to understand the general pattern that characterizes the tradeoff between taxation and subsidization**. Generally speaking, the tax income, , even if not high enough, can be used to fund subsidy, and reduce *y*, the external funding required for subsidization, i.e.,

We need to conduct research to formally look into the tradeoffs between *T* and *y*. This will help the outside party understand the efficiency of using external funding with respect to lowering the dissatisfaction caused by taxation.

**3.4 Extensions to the integration with existing concepts.**

There are several existing concepts, such as the least core, that can be helpful to stabilize a grand coalition. The least core problem is to seek a value *z* and an allocation **x** such as

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For example, Schulz and Uhan 2010, 2013) study the problem of calculating the least core for a class of cooperative games including games originating from scheduling problems. In the definition of the least core, the least core value, *z*, refers to the maximum loss that a coalition has to tolerate in a grand coalition. This can be regarded as an internal effort among the players to stabilize the game, which is different from our approach that assumes an outside party.

We point out that the two streams, internal efforts and outside actions, can merge as an integrated tool for a broader range of flexibility. For example, *in our preliminary study (Liu and Qi, 2014b), we show how adding subsidy to a cooperative game can reduce the least core value, or reduce the negative goodwill caused to players due to the loss to be tolerated*.

Following this direction, we have a number of issues to address. For example, we can study the impact of taxing resources on the least core value. Note that this is different from the case of providing subsidy. Both taxation and loss tolerance will bring negative goodwill to players.

We do not to explicitly list the tasks in this part because of space limit. Meanwhile, we will keep our options open for other extensions that we may be able to realize in the future.

1. **Research Plan**

This project includes four major steps, each on one type of schemes. The project will take 36 months. In months 1 to 6, we will work on subsidization based on our preliminary research; in months 7 to 15, we will work on taxation; in months 16 to 24, we will work on combining subsidization and taxation; and in the last 12 months, we will work on integration with traditional concepts, and also explore other extensions if time allows.

We plan to make the following deliverables. Every year we will make a presentation at international conferences for comments and feedbacks. We plan to submit papers for journal review starting from the second year. We expect that the project will generate three to four papers published in good journals.