Authors' Reply
Lagrangian Heuristic for Simultaneous Subsidization and Penalization: Implementations on Rooted Travelling Salesman Games

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We would like to thank the Associate Editor for processing our submission efficiently and providing guidance for the revision, and also thank the Referee for the detailed comments on the paper. We have carefully studied all the comments and addressed them in our manuscript.

Please find below our point-by-point reply to the Referee. To facilitate reading, the original comments are in *italics*.

Reply to Referee 1

We appreciate your detailed comments and clarification of your question. We have carefully studied the technical concerns you raised, and revised our manuscript accordingly.

The main issues raised in your report are addressed as follows:

Question 1. *However, my question focused on the LP (2) is as follows: Is it true that in LP (2), one might replace constraint $\beta(V) = c(V) - w$ by $\beta(V) \geq c(V) - w$, such that for any optimal solution to LP (2) this constraint will be binding?*

$$\beta(S) \leq c_l(S) + z \leq c(S) + z$$

$$\beta(V) \geq c_u(V) + z \geq c(V) + z$$

Note: According to your context, there may be some typos in your comments, and they are now revised in red.

For example, it will become obvious that LP with constraints

$$\beta(S) \leq c_l(S) + z \leq c(S) + z$$

$$\beta(V) \geq c_u(V) + z \geq c(V) + z$$

will be the proper restriction of LP (2). Moreover, replacing $c_l(S)$ and $c_u(V)$ by $c_u(S)$ and $c_l(V)$ $\beta(S) \leq c_u(S) + z$ $\beta(V) \geq c_l(V) + z$

Reply 1. Thanks for your question. With regard to the conversion from the equality to the inequality, the constraint will be binding for LP(2). Both methods are correct for this After careful consideration, we decided to change to The formation will be easy to understand.

With regard to the conversion from the equality to the inequality, we have added more detailed explanations on page 9. For the second part of this question, we are sorry that the original term "restricted LP" on page 7 is misunderstanding, and it is now replaced with term "variant LP" to describe the relationships between LPs (2) and (4).

To be specific, when $c(S)$ and $c(V)$, in LP (2), are respectively replaced by $c_l(S)$ and $c_u(V)$ in LP (4).

Constraint $\beta(S) \leq c_l(S) + z \leq c(S) + z$ for any S is a restriction of the original one, while $\beta(V) \geq c_u(V) + z \geq c(V) + z$ is not. Therefore, LP(4) can be viewed as a variant, rather than a restriction, of LP(2). Under the joint effects of the replacement of both constraints, we can find that LP(4) indeed can provide an upper bound for LP(2), as we have shown in Theorem 1.

This conversion doesn't hold because the constraint $\beta(V) = c(V) - \omega$ equals two constraints: $\beta(V) \geq c(V) - \omega$ and $\beta(V) \leq c(V) - \omega$ When we change $\beta(V) = c(V) - \omega$ to $\beta(V) \geq c(V) - \omega$, which means we discard this constraint, $\beta(V) \leq c(V) - \omega$. Thus, we will obtain a lower bound of $z(\omega)$ by relaxing the original constraint. Denote this lower bound as $z_l(\omega)$. By Changing $\beta(V) \geq c(V) - \omega$ to $\beta(V) \geq c_u(V) - \omega$, we can obtain an upper bound of $z_l(\omega)$, but that cannot ensure this way will obtain an upper bound of

$z(\omega)$. Therefore, in order to obtain an upper bound of $z(\omega)$, the conversion order on constraint (4) is important, we cannot change it.

Regarding the lower bound, we relax $\beta(V) = c(V) - \omega$ to $\beta(V) \geq c_l(V) - \omega$, This part holds as you said. I hope this clarification can anwser your questions.

Question 2. *Other responses have not appeared clear to me either. For example, When I asked about upper bound on c(S) I am now referred to page 7, where it is stated that Lagrangian relaxation is used to find both bounds. How can Lagrangian relaxation be used to find both bounds? In any case, I have not seen any discussion about possible ways to obtain the upper bound $c_u(V)$ and have not noticed the exact choice of the upper bound employed in the computational study.*

Reply 2. Thanks a lot for your suggestion. In the first half of this manuscript, we are trying to introduce a general framework of computing feasible subsidy-penalty pairs under which the grand coalition is stabilized. Therefore, the upper bound $c_u(V)$ in LP(4) could be any value larger than $c(V)$. It could be either named by some authority, or computed by some well known heuristic algorithm (such as the linear relaxation based heuristics). In the second half of this manuscript, when the Lagrangian relaxation technique is applied for approximation, $c_u(V)$ is obtained by a Lagrangian relaxation based heuristic for the sake of consistency and generality. We have added this clarification on page 7.

Strictly speaking, Lagrangian relaxation is used to obtain a lower bound. But we can obtain an upper bound during the calculation of Lagrangian relaxation. Therefore, the upper bound is obtained by a Lagrangian relaxation based heuristic.

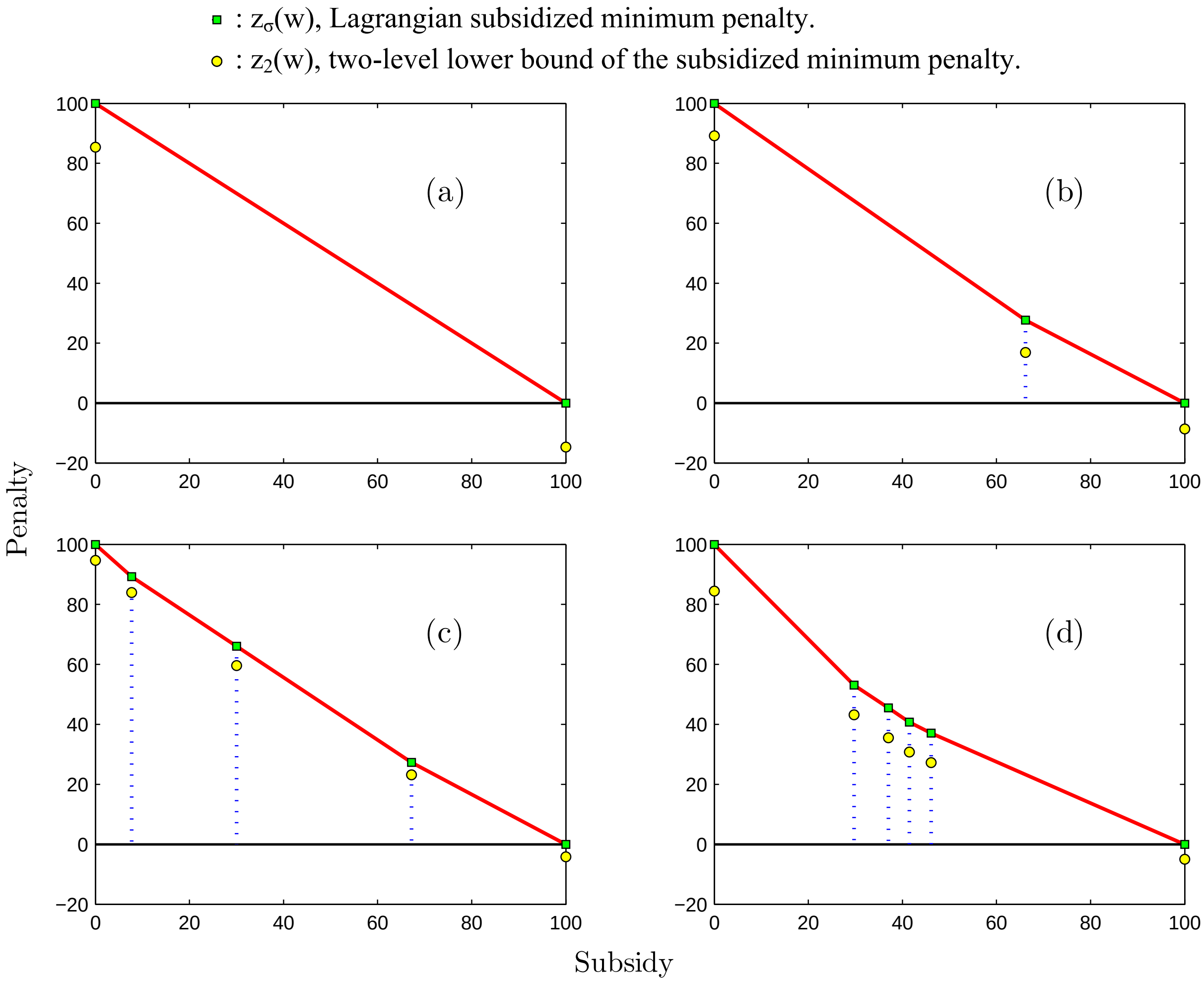
Question 3. *The new clarification appeared on page 17 is still confusing. For example, 'the curve in subfigure (a) represents that the respective lines passing the two squared points have the same slope' – I am not sure I understand how different lines can pass through two points, so the entire paragraph is confusing. Further on, I understand that the figure presents results for 4 different games. Why does a) instance obtain zero penalty-subsidy combinations, b) – 1, c) – 3, d) – 4 combinations, what is the logic behind this?*

Reply 3. *It represents four situations,*

we can obtain the slope at any given penalty-subsidy point and the subsidy when given subsidy.

we use the breakpoint to construct the SPF, what we doing is to obtain the breakpoints. For the figure a)

Thanks for your suggestions. We are sorry for not making ourselves clear: Figure 2 presents four representations of the curves of Lagrangian SPFs. The x axis and the y axis respectively represent the amount of subsidy and the amount of penalty in all 4 subfigures. We have added the definitions of squared points and round points into the legend of the figure.



In addition, the meaning of curves in subfigures (a) and (c) of Figure 2 is explained on page 17. And the meaning of all curves in Figure 2 is shown below. The curve in subfigure (a) represents that the respective lines passing the two squared points have the same slope, thus there is no breakpoint between them. The curve in subfigure (b) represents that the two lines passing the two squared endpoints meets a point, besides, the value of $z_\theta(\omega)$ *at x-coordinate value of the point is just equal to the value of y-coordinate of the point which indicates that this point is a breakpoint, which is the middle squared one. The curve in subfigure (c) represents that the two lines passing the two squared endpoints intersects a point under the third squared point. (Until here , we have not got the third squared point, but we know they have the same x-coordinate value.) Then with x-coordinate value of this intersection point, the third squared point can be obtained by Algorithm 1 in our manuscript. With the IPC algorithm proposed by Liu et al. (2018), we can generate a line passing the third squared point. Finally, this line intersects the two lines we mentioned at the beginning at the second and fourth squared points, respectively. As for the curve in subfigure (d), the only difference between the curve in (c) and the curve in (d) is that the curve in (d) has one more line intersection process than that in (c) at the right half part of the subfigure.*

Question 4. *I am still unclear why you introduce the symmetric TSP problems with the full set of binary variables x_{ij}, ij instead of $x_{ij}, i < j$. This does not seem to be necessary in relation to coalitions S, please clarify.*

Reply 4. *Thanks a lot for your comment.*

What we replied in the first round is aimed at why not using the x_e, which cannot reflect the players.

To define the TSP game, using $x_{ij}, i < j$ is OK, but the reasons why we did not adopt $x_{ij}, i < j$ are as follows:

1. x_{ij}, ij will be helpful to derive (17), (but $i < j$ cannot). we can change the subscript($i/toj, j/toi$) by using symmetry.

2. On page And 1971 conclude the relation between the symmetric TSP game and spanning tree, which help to solve the problem. If we define TSP game with x_{ij}, ij, we can continue to finish our calculation with the existing literature.

However, in the context of cooperative game theory (e.g., TSP game), since we have to indicate the existence of some player in a coalition (e.g., $\gamma_j^S = 1$ indicates that player j is in coalition S), it would be useful and neat to introduce variable x_{ij} for better descriptions.

Reply to Minor Comments. *Thank you very much for pointing out the typos of our manuscript in Minor Comments. We have made corresponding revisions.*

1. page 10, lines 19-21, s_1 and s_2 should probably be corrected to S_2 and S_2

2. page 4, line 36: linear programming problems

3. page 5, line 5: program

4. page 12, line 17, please add space after min

5. page 13, line 43, why do you need the notation 't' in $c_t(S)$?

-t shorted for TSP, for the difference of the common c(s)

6. page 14, line 10-11, could you please clarify the phrase 'constraints (14) and (15) are redundant due to (12)'?

-constraint (12) means

7. page 15, line 41, 55: s.t.