Authors' Reply
Lagrangian Heuristic for Simultaneous Subsidization and Penalization: Implementations on Rooted Travelling Salesman Games

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We would like to thank the Associate Editor and the reviewers for the encouraging and detailed comments on the paper. We have carefully studied all the comments and addressed them in our manuscript. The paper has therefore been substantially revised. Our revision follows the guidance that "***" and "***" To this end, we have made the following major changes.

First, regarding the proofs of the structural properties of the penalty-subsidy tradeoff function, we now have a much simpler proof which was suggested by the Associate Editor, and for which we are very grateful.

Second, regarding constructing the tradeoff function over the entire effective domain, we add a new approximation algorithm with a guaranteed error bound.

Third, regarding computing the tradeoff between penalty and subsidy at a specific point, we remove the Lagrangian relaxation heuristic method, and design two new solution approaches that are more theoretically sound, and that can find the exact solutions for some games of which $c(s)$ are solvable.

Fourth, regarding the demonstration games, we replace the TSP game with machine scheduling games. Indeed, we obtain some interesting properties by studying the special structures of these games.

There is another change mainly for the convenience of presentation. For the tradeoff function, we now define it as $\omega(z)$ instead of $z(\omega)$. Because of the one-to-one mapping between penalty z and subsidy ω, all main results still hold.

Please find below our point-by-point reply to the Associate Editor and each of the reviewers. To facilitate reading, the original comments are in *italics*.

Reply to Associate Editor

Thank you very much for processing our submission efficiently and providing guidance for the revision. We are especially grateful for the simple proofs that you suggested. The issues raised in your report are addressed as follows:

Question 1. *When trying to address one issue raised by one of the referees about the proof of Lemma 2 and Lemma 3, I came up with simple proofs for several key results in Section 2 and Section 3 by using linear programming duality. I find the proofs are quite straightforward. I include the derivations below.*

Reply 1. Thank you so much for providing simpler proofs. We have adopted your proofs.

Question 2. *On the other hand, I also agree with one of the referees that the discussion on TSP game, and more generally the case when computing* $c(s)$ *itself is hard, diverges the contribution of the paper. (And the contributions rely on known techniques for such games.) I believe the revision should focus on strengthening the contributions of the first part of the paper.*

Reply 2. We fully agree with the comment, and have focused more on theoretical results instead of solving a particular game by heuristics. Please refer to the major changes (2)-(4) for details.

Question 3. *If this were my own paper, I would try to identify interesting games for which one could utilize the structure of the cost function* c *to derive more, perhaps deeper, theoretical results about* $z(\omega)$.

Reply 3. Thank you for your suggestion. We believe that your suggestion will lead to many interesting studies. In the revision we have studied parallel machine scheduling games to demonstrate the applicability of the proposed model, algorithms, and solution approaches. As a result of studying the special structure of these games, we are indeed able to derive some deeper results. For example, as shown in Section 5.1, we can compute the exact value of $\omega(z)$ by a polynomial-time solvable linear program under any penalty z; moreover, we are able to obtain an upper bound for the number of breakpoints on function $\omega(z)$ (Theorem 7).