Authors' Reply
Lagrangian Heuristic for Simultaneous Subsidization and Penalization: Implementations on Rooted Travelling Salesman Games

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We would like to thank the Associate Editor and the Referees for the encouraging and detailed comments on the paper. We have carefully studied all the comments and addressed them in our manuscript.

Please find below our point-by-point reply to each of the Referees. To facilitate reading, the original comments are in *italics*.

Reply to Associate Editor

Thank you very much for processing our submission efficiently and providing guidance for the revision.

Reply to Referee 1

We appreciate your detailed comments made directly on the manuscript. We have carefully studied the technical concerts you raised. Thank you for your comments which are very helpful to enrich our manuscript, especially for pointing out the upper bound $c_u(V)$ we did not specify in our manuscript. We have adopted your suggestions and revised our paper accordingly. The main issues raised in your report are addressed as follows:

Question 1. *Constraint # 2 in problem (4). It appears from the next page that the equality can be replaced to $\geq$ inequality in this problem formulation. I would like the authors to expand the discussion of reasons for that being the case as the brief explanation on page 9 does not seem sufficient. Moreover, it looks like a good idea to do it right after introduction of problem (4), since in this case it becomes obvious that (4) is a restriction of (2) and therefore $z_r(w)$ is an upper bound on z(w):*

$\beta(s) \leq c_l(s) + z \leq c(s) + z$

$\beta(V) \geq c_u(U) + z \geq c(s) + z$

Reply 1. Thanks for mentioning that, we have added a more detailed explanation on page 9. Then we will explain that LP(4) is not a restriction of LP(2). Maybe the term 'restricted LP' in page 7 caused the misunderstanding and we now use 'variant' to replace 'restricted'.

First of all, this formula $\beta(s) \leq c_l(s) + z \leq c(s) + z$ you wrote is correct, which indicates that $\beta(s) \leq c_l(s) + z$ in LP(4) is a restriction of $\beta(s) \leq c(s) + z$ in LP(2). However, we think there are some mistakes in the formula $\beta(V) \geq c_u(U) + z \geq c(s) + z$ you wrote. The first one is, U should be a written error. The second one is, $\beta(V)$ has no direct relation with z. In fact, according to the cooperative game theory, coalitional stability constraints refer to all subcoalitions $S \in \mathbb{S} \setminus \{V\}$ except the grand coalition V. Thus, the second constraint in (4), $\beta(V) = c_u(V) - \omega$ i.e., the budget balance constraint is not a restriction of $\beta(V) = c(V) - \omega$ in (2). Based on the above analysis, LP(4) is not a restriction of LP(2), which can also explain why we cannot get a relaxation of LP(2) if we switch the places of $c_l(s)$ and $c_u(s)$ you mentioned in Question 10.

Question 2. *When approximate problem (4) is introduced, it is not mentioned which upper bound $c_u(V)$ is used. Probably a discussion of upper bound possibilities will be good here.*

Reply 2. Thanks a lot for your suggestion. We indeed forgot to mention the upper bound. In fact, all the upper and lower bounds of $c(S)$ are obtained by the Lagrangian relaxation method to keep the consistency and continuity. We add this clarification in page 7.

Question 3. *Proof of Theorem 1. This is probably a general question but it also directly affects the proof of Theorem 1. The vector of cost allocation beta is defined to be in R^v. I would like to see some discussion on why some individual cost allocation is allowed to be negative (implying a gain for a player, I guess) and how that is possible with existence of grand coalition. My concern is that beta components in the proof of Theorem 1 can become negative, but, again, I am not sure I understand the meaning of* *negative cost allocation.*

Reply 3. Thanks for asking that. The beta components in the proof indeed can be negative as you said, that is because the vector of cost allocation which is defined to be in R^v in the proof is mathematically valid. When the cost assigned to some player is negative, it can be understood as a gain for the player as you guessed. In cooperative game theory, cost allocation does not mean cost assigned to every player in the cooperative game must to be positive. As long as a cost allocation satisfies the budget balance and coalitional stability constraints we mentioned in the paper, it is meaningful and can make the grand coalition stable(In other words, the grand coalition can exist), even if some components of it are negative.

Question 4. *Proof of Remark 1. I am not able to see "the point-wise maximum of a finite set of straight lines (hyperplanes, you mean?)" in defining the $z_r(w)$ (19). Please clarity.*

Reply 4. Thanks for asking that, it is indeed not easy to understand. $z_r(\omega)$ in (19) is only related to ω meanwhile $c_u(V), c_l(s)$ can be calculated and have no relation with ω. By deriving $z_r(\omega)$ with respect to ω, we can obtain the slope is $\max_\rho -\rho_v$, which is the maximum of the slopes of a finite set of straight lines. We have added this clarification in the Proof of Remark 1.

Question 5. *In the description of the Algorithm 1, how is the initial restricted coalition set is constructed? For example, it is not clear what the first step means means if no initial set is defined. And also, I am not able to see how the initial values of Lagrangian coefficients lambda is constructed and how they change (if they do) during the algorithm. It would be great to add some clarifications.*

Reply 5. Thanks for your question. For the column genaration technique, the initial restricted coalition set can be selected arbitrarily. Commonly we select the set which we think can help to accelate computing speed. In this case, the restricted coalition set is $S' = \{\{1\}, \{2\}, \ldots, \{v\}\}$, it can also be other forms. And about the part of sub-gradient method, we have added the detailed clarification in page 9.

Question 6. *Thanks for mentioning that. The description of the TSP game is somewhat different from one can find in the literature. For example in Tamir (1989) the game was defined on an uncomplete graph, while here the game is defined on the complete graph. This affects all the models presented on page 13 and further. Probably both versions of the game exist, but I would like to understand why the descriptions are different.*

Reply 6. In Tamir(1989), the author also mentioned that in Potters et al. (1992) the game is equivalently defined on the complete graph. He defined the game on an uncomplete graph for the convenience's sake. In the TSP game we mentioned, use the complete graph can help us present our model clearly and conveniently. And we add this literature, Potters et al. (1992), to the Introduction.

Question 7. *Constraint (15): I am not sure why would one keep (15) in such an aggregated format when it is possible to disaggregate it to $x_{ij} \leq \gamma_i$ $x_{ij} \leq \gamma_j$ In terms of LP relaxation disaggregation gives a tighter bound and probably will lead to the improvement to Lagrangian relaxation just as well.*

Reply 7. Thanks for your suggestion. The disaggregation indeed gives a better bound when using LP relaxation. However, in the Lagrangian relaxation we do not change the integer property of the formulation, i.e., x_{ij} still has to be binary. Considering that the aggregated and disaggregated formats are fully equivalent, so the disaggregated format will not lead to the improvement to Lagrangian relaxation. For better understanding, we changed the aggregated format to the disaggregated one you mentioned.

Question 8. *The symmetry of TSP game. I would like to attract the attention of the authors to the Dantzig Fulkerson Johnson (1954) paper, which originated the development of the TSP theory. Most importantly, the authors there also considered a symmetric TSP problem. If the problem is symmetric, one does not need as many binary variables as was introduced by authors. For example, on page 13 x_{ij} exists together with x_{ji} but direction of the travel is not important for symmetric problem therefore it is sufficient to introduce x_e for e being an edge or x_{ij} for $i < j$ only. This is how the TSP problem was introduced in DFJ and this is something that can simplify many notations in this paper. Also, it will be probably a good idea to get rid of x_{ii} variables on page 13 and other optimization problems.*

Reply 8. Thanks for mentioning that. When it comes to a symmetric TSP problem, it is more convenient to introduce x_e or $x_{ij}, i < j$. However, the player is an important concept in the cooperative game. We have to express every single node in the TSP game. If we only use x_e or $x_{ij}, i < j$ to express the edge in the TSP game, the notation γ_j^S used in the rooted TSP game cannot be expressed. We have to introduce other notations to express every single node, which is obviously troublesome. Thus, we use the expression $x_{ij}, i, j \in V$ in our manuscript.

Question 9. *I am a bit confused by Figure 2. Does it represent 4 different games? What exactly is on y axis and x axis, Penalty and Subsidy on y and x axis for all 4 figures? I suggest to add definitions of squared points and round points to the legend of the figure. Why is it that one figure gets two points and another gets 6 and they are evaluated at different levels of subsidy?*

Reply 9. Thanks for your suggestions. X axis and y axis represent penalty and subsidy respectively for all 4 subfigures. We have added the definitions of squared points and round points to the legend of the figure and the meaning of curves (a) and (c) in Figure 2 is explained in page 17. The meaning of all curves in Figure 2 are shown below. (a) represents that the slopes at the two points are equal, thus there is no breakpoint between them. (b) represents that the two lines passing the two points at the beginning and the end meets a point meanwhile the value of $z_\theta(\omega)$ at x-coordinate value of the point equals the value of y-coordinate of the point which indicates that this point is a breakpoint. (c) represents that the two lines passing the two squared points at the beginning and the end intersects a point under the third squared points. Then with x-coordinate value of the point, the third squared points can be obtained by Algorithm 1 in our manuscript. With the IPC algorithm proposed by Liu et al. (2018), we can generate the line passing the third squared point. Finally, this line intersects the two lines passing the two points at the beginning and the end at the second and fourth squared points, respectively. The difference of (c) and (d) is that (d) has one more line intersection process than (c) at right half part.

Question 10. *In regards to results of experiments in Table 3, what exactly was used and the upper bound $\phi(N)$? and finally a general remark on the lower bound construction z(w): – if we consider problem (4) as a restriction of (2), then it is also possible to create a relaxation of the problem in the similar was as problem (4) was constructed.*

Reply 10. Thanks for your question. We forgot to mention the upper bound as we said in Reply 2. The $\phi(N)$ is the upper bound $c_u(V)$ obtained by the Lagrangian relaxation method. As we mentioned in Reply 1. LP(4) is not a restriction of LP(2). So we know what you concerned, we tried this before but it won't work. Thus we design new methods to obtain the upper and lower bound.

Minor Comments Reply. Thank you very much for pointing out these minor mistakes. We have made the corresponding revisions accroding to the Minor comments except the fourth one.

Based on what we demonstrated in Reply 1, LP(4) is not a restriction of LP(2). And what we mean right here is eliminating some constraints of LP(2) to construct a lower bound of the subsidized minimum penalty. Thus, it should be LP(2) in Page 10.

Reply to Referee 2

Thank you very much for your appreciation. We have corrected all the typos and grammatical errors that you pointed out.

Question 1. *Notation. The paper does not conform with the standard notation in TU-games. Usually, coalitions are referred with capital letters and its cardinal in lower case letters, i.e.* $S \subset N$ *and* $|S| = s$*. This paper uses a different notation with* s *for coalitions and then some inconsistencies appears when referring to* v *in some places. I would suggest to adapt the notation to the standard to ease the readability of potential readers.*

Reply 1. Thanks a lot for your suggestion. We have changed the notation to the standard form all through the revised manuscript.

Question 2. *Some confusion appears, here and there, when referring to LP or MIP. For instance, in page 4 line 3, it is mentioned that (2) is a combinatorial optimization problem. However, (2) is a LP since in its description* c *is given and thus all constraints and variables are linear and continuous. The same confusion can be found at other places of the paper. Please clarify!*

Reply 2. We have changed our expression with only using LP problem to avoid the confusion.

Question 3. *Page 4 line -13: The authors must be more precise. The Lagrangean bound is more accurate than the linear relaxation whenever the problem does not fulfill the integrality property.*

Reply 3. Thank you for pointing this out. We have added this statement.

Question 4. *The statement of Theorem 1 should be modified since the value of the LP is one of the many possible upper bounds not the only one as stated there.*

Reply 4. This should be "an". Thanks for pointing this out.

Question 5. *Remark 1. The meaning of* v *is unclear. One should guess that it refers to* $|V|$ *but this has to be made explicit.*

Reply 5. We have pointed this out in the revised manuscript in page 8.

Question 6. *Page 8, line -12. Note that (5) is not an LP but an ILP.*

Reply 6. Thank you for pointing this out. This should be MIP.

Question 7. *To better illustrate the proposed methodology, it would be advisable to apply it not only to the rooted traveling salesman problem. I would suggest to add another class of combinatorial games, for instance location games, to the computational study.*

Reply 7. How to add a computation?