Homework 1

Li Zikang

2-5

(a) Write donw a recursion for $f^{(n)}(x)$ in terms of $f^{(n-1)}(\cdot)$ and the parameters of the problem.

For any participant, with n more spins, he/she can choose to spin or stop. When choosing **stopping**, he/she will take away the current accumulation of x. We can obtain $f^{(n)}(x) = x$. While choosing **spinning**, it is also easy to know that

$f^{(n)}(x) = \sum_i f^{(n-1)}(x + r_i) \cdot p_i$, where $i = 1, \ldots, m$.

$f^{(n)}(x) = \max\{\sum_i f^{(n-1)}(x + r_i) \cdot p_i,\ x\}$

(b) Solve the problem numerically for the case in which $m = 3, p_i = 1/4$, for $i = 1, 2, 3, r_1 = 1, r_2 = 5, r_3 = 10$, and there are one or two more spins available.

At the beginning, the winning is 0, so must choose to spin.

$$f^{(2)}(0) = \begin{cases} f^{(1)}(1), \text{probability } 1/4 \\ f^{(1)}(5), \text{probability } 1/4 \\ f^{(1)}(10), \text{probability } 1/4 \\ 0 \quad \text{probability } 1/4 \end{cases}$$

$$f^{(1)}(x) = \begin{cases} x+1, \text{probability } 1/4 \\ x+5, \text{probability } 1/4 \\ x+10, \text{probability } 1/4 \\ 0 \quad \text{probability } 1/4 \\ x \quad \text{choose to stop} \end{cases}$$

So when spinning once, the excepted winnings are $1/4 \times (1 + 5 + 10) = 4$. And when spinning twice, $f^{(1)}(x) = 1/4 \times [(x+1) + (x+5) + (x+10)]$. $f^{(2)}(0) = 1/4 \times (f^{(1)}(1) + f^{(1)}(5) + f^{(1)}(10)) = 1/4 \times (4 + 3/4 + 4 + 15/4 + 4 + 30/4) = 6$

In particular, $f^{(0)}(x) = x, f^{(1)}(x) = \max\{3/4x + 4, x\}$. Let $3/4x + 4 = x$, $x = 16$. When $x > 16$, the participant will take away the winnings; when $x < 16$, the participant will spin again. Consequently, the expected total winnings are 16 for this case.

2-6

(a) Write down a recursion for $v(x)$ and use it to find an explicit expression for $v(x)$ in terms of $x, \Phi(x), I(x)$, and α, where $I(x) := E(X - x)^+ = \int_x^\infty (\xi - x)\phi(\xi)d\xi$.

$v(x,n) = v(x,n-1) \cdot \alpha \cdot \Phi(x) + \int_x^\infty \xi\phi(\xi)d\xi$

Let $V = \int_x^\infty \xi\phi(\xi)d\xi$.

$v(x,n) = v(x,1) \cdot [\alpha \cdot \Phi(x)]^{(n-1)} + V[1 + [\alpha \cdot \Phi(x)] + [\alpha \cdot \Phi(x)]^2 + \cdots + [\alpha \cdot \Phi(x)]^{n-2}]$

$v(x,1) = \int_x^\infty \xi\phi(\xi)d\xi = V$

Then we can obtain that $v(x) = V\frac{1-(\alpha\cdot\Phi(x))^n}{1-\alpha\cdot\Phi(x)}$, $\alpha \in (0,1)$. Because this is an infinite period, let $n \to \infty$, so $v(x) = \frac{V}{1-\alpha\cdot\Phi(x)}$.

$I(x) = \int_x^\infty (\xi - x)\phi(\xi)d\xi = \int_x^\infty \xi\phi(\xi)d\xi - x(1-\Phi(x))$

So $V = I(x) + x(1-\Phi(x))$, substitute it in the $v(x)$, we get $v(x) = \frac{I(x)+x(1-\Phi(x))}{1-\alpha\cdot\Phi(x)}$

(b) $\alpha = 0.99$ bids $\sim N(1000, 200)$. x = 1200. So $v(1200) = \frac{I(1200)+1200(1-\Phi(1200))}{1-\alpha\Phi(1200)}$, $I(1200) = 200 I_N(1)$, $\Phi(1200) = \Phi_N(1) = 0.841$,$I_N(1) = \phi_N(1) - 1 + \Phi_N(1) = 0.083$.
By calculating, $v(1200) = 1238.9$

2-7

(a) Display Ω.

$\Omega = \{(3,3,1),(2,3,2),(2,2,2),(1,3,2),(2,3,1),(0,3,2),(1,3,1),$ $(1,1,2),(2,2,1),(2,0,2),(3,0,1),(1,0,2),(2,0,1),(1,1,1),(0,0,2)\}$

(b) $S_0 = \{(3,3,1)\}$, $S_1 = \{(2,3,2),(2,2,2),(1,3,2)\}$,$S_2 = \{(2,3,1)\}$, $S_3 = \{(0,3,2)\}$, $S_4 = \{(1,3,1)\}$, $S_5 = \{(1,1,2)\}$, $S_6 = \{(2,2,1)\}$, $S_7 = \{(2,0,2)\}$, $S_8 = \{(3,0,1)\}$, $S_9 = \{(1,0,2)\}$, $S_{10} = \{(2,0,1),(1,1,1)\}$, $S_{11} = \{(0,0,2)\}$.

m = 11.

3-4

(a) Write down a recursion for $f^n(x|S)$.

$f^{(n)}(x|S) = \sum_i f^{(n-1)}(x + r_i|S) \cdot p_i$.

$$f^{(n)}(x|S) = \begin{cases} \sum_i f^{(n-1)}(x + r_i|S) \cdot p_i, & \text{spin or } x < S. \\ x & \text{, stop.} \end{cases}$$

(b) Being indifferent means $\sum_i f^{(n-1)}(S + r_i|S) \cdot p_i = S = f^{(n)}(S|S) \Rightarrow \sum_i^m (S^* + r_i)p_i = S^* \Rightarrow S^* = \frac{\mu}{p}$.

3-5

(a) Write a recursion for f_n in terms of f_{n-1}

$f_n(x) = f_{n-1}(x) \cdot \alpha + I(\alpha x)$

(b) Calculate f_n. $f_2 =$

3-6

To be honest, I'm confused by these questions about different ways of bidding.

4-4

4-5

Define $G^n(x) = -x + \sum_i^m f^{n-1}(x + r_i)p_i$.

The derivative of $G^n(x)$ equals $(-1 + \sum_i^m f'^{n-1}(x + r_i)p_i)$, because $f^{n-1}(x + r_i)$ is a linear function of x, and the coefficient of x is 1. So $\sum_i^m f'^{n-1}(x + r_i)p_i = \sum_i^m p_i = 1 - p < 1$. So the derivative of $G^n(x)$ is less than 0. Therefore, $G^n(x) < G^n(x + 1)$. According to Exercise 3.4, the decision rule we decided is indeed optimal.