Homework 1

2-5

(a) Write down a recursion for $f^{(n)}(x)$ in terms of $f^{(n-1)}(\cdot)$ and the parameters of the problem.

For any participant, with n more spins, he/she can choose to spin or stop. When choosing **stopping**, he/she will take away the current accumulation of x. We can obtain $f^{(n)}(x) = x$. While choosing **spinning**, it is also easy to know that

$f^{(n)}(x) = \sum_i f^{(n-1)}(x + r_i) \cdot p_i$, where $i = 1, \ldots, m$.

$f^{(n)}(x) = \max\{\sum_i f^{(n-1)}(x + r_i) \cdot p_i,\ x\}$

(b) Solve the problem numerically for the case in which $m = 3, p_i = 1/4$, for $i = 1, 2, 3, r_1 = 1, r_2 = 5, r_3 = 10$, and there are one or two more spins available.

At the beginning, the winning is 0, so must choose to spin.

$$f^{(2)}(0) = \begin{cases} f^{(1)}(1), \text{probability } 1/4 \\ f^{(1)}(5), \text{probability } 1/4 \\ f^{(1)}(10), \text{probability } 1/4 \\ 0 \quad \text{probability } 1/4 \end{cases}$$

$$f^{(1)}(x) = \begin{cases} x+1, \text{probability } 1/4 \\ x+5, \text{probability } 1/4 \\ x+10, \text{probability } 1/4 \\ 0 \quad \text{probability } 1/4 \\ x \quad \text{choose to stop} \end{cases}$$

So when spinning once, the excepted winnings are $1/4 \times (1 + 5 + 10) = 4$. And when spinning twice, $f^{(1)}(x) = 1/4 \times [(x+1) + (x+5) + (x+10)]$. $f^{(2)}(0) = 1/4 \times (f^{(1)}(1) + f^{(1)}(5) + f^{(1)}(10)) = 1/4 \times (4 + 3/4 + 4 + 15/4 + 4 + 30/4) = 6$

In particular, $f^{(0)}(x) = x, f^{(1)}(x) = \max\{3/4x + 4, x\}$. Let $3/4x + 4 = x$, $x = 16$. When $x > 16$, the participant will take away the winnings; when $x < 16$, the participant will spin again. Consequently, the expected total winnings are 16 for this case.

2-6

(a) Write down a recursion for $v(x)$ and use it to find an explicit expression for $v(x)$ in terms of $x, \Phi(x), I(x)$, and α, where $I(x) := E(X - x)^+ = \int_x^\infty (\xi - x)\phi(\xi)d\xi$.

$v(x,n) = v(x,n-1) \cdot \alpha \cdot \Phi(x) + \int_x^\infty \xi\phi(\xi)d\xi$

Let $V = \int_x^\infty \xi\phi(\xi)d\xi$.

$v(x,n) = v(x,1) \cdot [\alpha \cdot \Phi(x)]^{(n-1)} + V[1 + [\alpha \cdot \Phi(x)] + [\alpha \cdot \Phi(x)]^2 + \cdots + [\alpha \cdot \Phi(x)]^{n-2}]$

$v(x,1) = \int_x^\infty \xi\phi(\xi)d\xi = V$

Then we can obtain that $v(x) = V\frac{1-(\alpha\cdot\Phi(x))^n}{1-\alpha\cdot\Phi(x)}$, $\alpha \in (0,1)$. Because this is an infinite period, let $n \to \infty$, so $v(x) = \frac{V}{1-\alpha\cdot\Phi(x)}$.

$I(x) = \int_x^\infty (\xi - x)\phi(\xi)d\xi = \int_x^\infty \xi\phi(\xi)d\xi - x(1-\Phi(x))$

So $V = I(x) + x(1-\Phi(x))$, substitute it in the $v(x)$, we get $v(x) = \frac{I(x)+x(1-\Phi(x))}{1-\alpha\cdot\Phi(x)}$

(b) $\alpha = 0.99$ bids $\sim N(1000, 200)$. x = 1200. So $v(1200) = \frac{I(1200)+1200(1-\Phi(1200))}{1-\alpha\Phi(1200)}$, $I(1200) = 200I_N(1)$, $\Phi(1200) = \Phi_N(1) = 0.841$,$I_N(1) = \phi_N(1) - 1 + \Phi_N(1) = 0.083$.
By calculating, $v(1200) = 1238.9$

2-7

(a) Display Ω.

$$\begin{aligned}\Omega = \{&(3,3,1),(2,3,1),(1,3,1),(0,3,1),(3,0,1),\\&(2,0,1),(1,0,1),(0,0,1),(1,1,1),(2,2,1),\\&(3,3,2),(2,3,2),(1,3,2),(0,3,2),(3,0,2),\\&(2,0,2),(1,0,2),(0,0,2),(1,1,2),(2,2,2)\}\end{aligned}$$

Total: 20

(b)$S_0 = \{(3,3,1)\}$
$S_1 = \{(2,3,2),(2,2,2),(1,3,2)\}$
$S_2 = S_0 \cup \{(2,3,1)\}$
$S_3 = S_1 \cup \{(0,3,2)\}$
$S_4 = S_2 \cup \{(1,3,1)\}$

$S_5 = S_3 \cup \{(1,1,2)\}$
$S_6 = S_4 \cup \{(2,2,1)\}$
$S_7 = S_5 \cup \{(2,0,2)\}$
$S_8 = S_6 \cup \{(3,0,1)\}$
$S_9 = S_7 \cup \{(1,0,2)\}$
$S_{10} = S_8 \cup \{(2,0,1),(1,1,1)\}$
$S_{11} = S_9 \cup \{(0,0,2)\}$.

m = 11.

3-4

(a) Write down a recursion for $f^n(x|S)$.

$$f^{(n)}(x|S) = \begin{cases} \sum_i f^{(n-1)}(x + r_i|S) \cdot p_i, \ x < S. \\ x \quad , \ x \geq S. \end{cases}$$

(b) Being indifferent means $\sum_i f^{(n-1)}(S + r_i|S) \cdot p_i = S = f^{(n)}(S|S) \Rightarrow \sum_i^m (S^* + r_i)p_i = S^* \Rightarrow S^* = \frac{\mu}{p}$.

3-5

(a) Write a recursion for f_n in terms of f_{n-1}.

$f_n = f_{n-1} \cdot \alpha \cdot \Phi(\alpha f_{n-1}) + \int_{\alpha f_{n-1}}^{\infty} \xi \phi(\xi) d\xi = I(\alpha f_{n-1}) + \alpha f_{n-1}$

(b) Calculate f_n.

$I_N(-0.05) = \phi_N(-0.05) + 0.05 \times (1 - \Phi_N(-0.05) = 0.398 - 1 + 0.48 = 0.4244$

$I(990) = 200 I_N(-0.05) = 200 \times (0.4244) = 84.88$

$f_2 = \alpha\mu + I(\alpha\mu) = 0.99 * 1000 + 84.88 = 1074.88$

$f_3 = \alpha f_2 + I(\alpha f_2) = 1115.92$

$f_4 = \alpha f_3 + I(\alpha f_3) = 1142.87$

(c) Let $f_n = f_{n-1}$, we can obtain that $f_n = \alpha f_n \Phi(\alpha f_n) + I(\alpha f_n) + \alpha f_n (1 - \Phi(\alpha f_n))$. Obviously, we know that f_n is the solution of $x = \frac{I(\alpha x)}{1-\alpha}$.

3-6

(a) Because there is a discount factor which is less than 1, when you wait it for a long period, your profit will decrease as time went by. So, it is obvious that we don't need to wait for all bids to be received.

(b)

$$f_n(x) = \begin{cases} \alpha E(f_{n-1}(X) - f_{n-1}(x))^+ + f_{n-1}(x), \text{not accept this period.} \\ I(x) + x \quad\quad, \text{accept.} \end{cases}$$

$$f_n(x) = \max\{\alpha(E(f_{n-1}(X) - f_{n-1}(x))^+ + f_{n-1}(x)), I(x) + x\}$$

4-4

The original formulation is $v(x, S) = \begin{cases} c & \text{if } x = 0 \\ px + qv(x-1, S) & \text{if } 1 \leq x \leq S \\ v(S, S) & \text{if } S < x \end{cases}$,

for this case, we can continue drive until we find an unoccupied space. So define a function $F(x), f(x, i)$, where $x < 0$.

$F(x) = pf(x, 0) + qf(x, 1)$ and

$$f(x, i) = \begin{cases} -x & \text{if } i = 0 \\ F(x-1) & \text{if } i = 1 \end{cases}$$

$i = 0$ means the space isn't occupied, only choose to park here. Now, we can obtain $F(x) = p(-x) + qF(x-1) = p(\sum_{i=-x}^{n-x-1} iq^{i+x}) = \frac{q[1+(n-1)q^n - nq^{n-1}]}{(1-q)^2} - x\frac{1-q^n}{1-q}, x < 0$, let $n = \infty, F(-1) = \frac{q}{p^2} + \frac{1}{p}$.

Let $v(0,S) = F(-1)$, we can obtain the formulation for this question:

$$v(x,S) = \begin{cases} v(0,S) = \frac{q}{p^2} + \frac{1}{p} & \text{if } x = 0 \\ px + qv(x-1,S) & \text{if } 1 \le x \le S \\ v(S,S) & \text{if } S < x \end{cases}$$

Consequently, $v(S,S) = p\sum_{i=0}^{S} q^i(S-i) + q^S\frac{1}{(1-q)^2} = S - \frac{q(1-q^S)}{p} + q^S(\frac{q}{p^2} + \frac{1}{p})$.

4-5

$$f^{(n)}(x|S) = \begin{cases} \sum_i f^{(n-1)}(x + r_i|S) \cdot p_i, \; x < S. \\ x \quad , \; x \ge S. \end{cases}$$

Define $G^n(x) = -x + \sum_i^m f^{(n-1)}(x + r_i)p_i$.

$$G^{(n)}(x) = \begin{cases} -x + \sum_i (G^{(n-1)}(x) + x) \cdot p_i, \; x < S. \\ -x + \sum_i p_i \cdot x \quad , \; x \ge S. \end{cases}$$

Use mathematical induction to prove $G^{(n)}(x)$ is decreasing in x for each n.

1. $G^{'(i+1)}(x) = -1 + \sum_i^m p_i < 0$.

If $f^i(x|S) = 0$, then $G^{'(i+1)}(x) = -1 < 0$.

2. Suppose $G^{'(n)}(x) < 0$, then prove $G^{'(n+1)}(x) < 0$.

Omitted.