Two-dimensional eccentric elliptic electromagnetic cloaks

Do-Hoon Kwon[a)] and Douglas H. Werner[b)]
Department of Electrical Engineering, The Pennsylvania State University, University Park, Pennsylvania 16802, USA

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A spatial transformation technique is applied to an important class of two-dimensional electromagnetic cloaks that do not possess rotational symmetry around the longitudinal axis. These cloaks are based on an eccentric elliptic annular geometry, which represents a generalization of previously published cloaking configurations. The required material properties of the cloak are derived in terms of the geometrical parameters and the coordinates in the transformed system. The validity of the cloak parameters is verified by a full-wave finite-element simulation of plane wave scattering from uncloaked and cloaked perfectly conducting cylinders with elliptic cross section.
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Recently, design recipes for electromagnetic cloaking devices have been presented in terms of spatial transformations.[1–3] An electromagnetic cloak bends and guides incoming waves smoothly around the cloaked region, so that the fields after having emerged from the cloak are the same as if the incident waves had just passed through free space. These cloaks require material parameters that are both inhomogeneous and anisotropic. The cloaking principle was experimentally demonstrated in the microwave regime[4] using a design approach based on low-index electromagnetic metamaterials.[5]

At optical wavelengths, a nonmagnetic cloak design has been proposed that incorporates metallic nanowire inclusions to achieve the required low-index bulk electric metamaterial properties.[6] It was predicted that the reflection at the cloak boundary associated with the original nonmagnetic design could be dramatically reduced by introducing nonlinear coordinate transformations.[7] The effect of a circular annular cloak was numerically investigated for the cases where a source is located in close proximity to the cloak or inside the cloak itself.[8] Toward practical cloak realizations in the optical regime, the use of layered media with alternating homogeneous isotropic dielectrics was suggested[9] to avoid anisotropic subwavelength inclusions. The sensitivity of the perfect cylindrical cloaks to perturbations of the material parameters was analyzed in Ref. 10.

The same transformation technique was also applied to obtain a concentric circular annular medium that is capable of rotating fields.[11] Application of the transformation approach was extended to investigate two-dimensional (2D) acoustic cloaks composed of anisotropic mass media.[12] It has also led to a theoretical demonstration of a circular concentrator,[13] an optical device that concentrates the incident field into its central core without producing any scattering.

Cloak material parameters have been reported and investigated mostly for spherical and circular cylindrical shell geometries[1,4,6,14] primarily due to the simplicity of analysis for structures that possess radial and axial symmetries. In addition to the concentrator, Ref. 13 presents a square electromagnetic cloak, a novel design that features not only flat surfaces but also sharp corners. All of the cloak designs reported to date have inner and outer boundaries with well-defined centers that are colocated.

This letter presents a generalization of concentric cloak designs to eccentric cloaks by introducing a 2D configuration that has an eccentric elliptic annular shape. Nonorthogonal coordinate systems are introduced and the associated transformations are presented. Full-wave simulation results are provided for verification. The generalization introduced here represents an important step toward the realization of arbitrarily shaped and unsymmetrical cloaks.

For an eccentric elliptic annular cloak, we first define a coordinate system (q_1, q_2, q_3) in terms of the Cartesian coordinates (x, y, z) via the relationships

$$q_1 = \sqrt{\left(\frac{x - q_1 x_c}{a}\right)^2 + \left(\frac{y - q_1 y_c}{b}\right)^2}, \tag{1}$$

$$q_2 = \tan^{-1}\frac{(y - q_1 y_c)/b}{(x - q_1 x_c)/a}, \tag{2}$$

$$q_3 = z. \tag{3}$$

The resulting (q_1, q_2, q_3) system is illustrated in Fig. 1(a). The constant-q_1 contours in the x-y plane represent a family of ellipses with a constant axial ratio b/a. Note that the center position of the ellipse changes linearly with q_1 according to $(x, y) = (q_1 x_c, q_1 y_c)$. The constant-$q_2$ contours are a collection of radial lines. This system is very different from the conventional elliptic coordinate system.[15] Moreover, (q_1, q_2, q_3) does not form an orthogonal coordinate system.

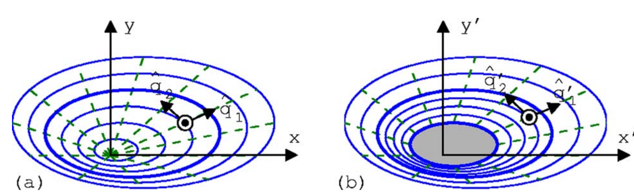

FIG. 1. (Color online) Coordinate systems for the eccentric elliptic annular cloak. (a) The (q_1, q_2, q_3) system and the associated unit vectors. The solid (blue) and dashed (green) curves represent constant q_1 and q_2 contours, respectively. (b) The (q_1', q_2', q_3') system after transformation and the unit vectors. The region shown shaded in gray is completely excluded from the system.

a)Electronic mail: kwon22@psu.edu.
b)Electronic mail: dhw@psu.edu.

To create a cloak, we define a spatial transformation that maps the elliptic cylindrical volume $q_1 \leqslant s$ $(s>1)$ in the (q_1,q_2,q_3) system into an eccentric elliptic annulus $1 \leqslant q_1' \leqslant s$ in the (q_1',q_2',q_3') system via

$$(q_1',q_2',q_3') = \left(1 + \frac{s-1}{s}q_1, q_2, q_3\right), \quad 0 \leqslant q_1 \leqslant s, \tag{4}$$

$$(q_1',q_2',q_3') = (q_1,q_2,q_3), \quad q_1 > s. \tag{5}$$

The transformed coordinate system is illustrated in Fig. 1(b), where the elliptic region $q_1'<1$ (indicated by the gray shaded area) is excluded from the system completely. Let the position vector in the original system be written as $\mathbf{r}=\hat{x}x+\hat{y}y+\hat{z}z$. The unit vectors in the original and transformed coordinate systems defined by $\hat{q}_i=(\partial\mathbf{r}/\partial q_i')/|\partial\mathbf{r}/\partial q_i'|$ and $\hat{q}_i'=(\partial\mathbf{r}'/\partial q_i')/|\partial\mathbf{r}'/\partial q_i'|$ $(i=1,2,3)$, respectively, are given by

$$\hat{q}_1 = \hat{q}_1' = \frac{\hat{x}(x_c + a\cos q_2') + \hat{y}(y_c + b\sin q_2')}{\sqrt{(x_c + a\cos q_2')^2 + (y_c + b\sin q_2')^2}}, \tag{6}$$

$$\hat{q}_2 = \hat{q}_2' = \frac{-\hat{x}a\sin q_2' + \hat{y}b\cos q_2'}{\sqrt{(a\sin q_2')^2 + (b\cos q_2')^2}}, \tag{7}$$

$$\hat{q}_3 = \hat{q}_3' = \hat{z}, \tag{8}$$

where $\mathbf{r}'=\hat{x}x'+\hat{y}y'+\hat{z}z'$ is the position vector in the (q_1',q_2',q_3') system. The unit vectors involved in the transformation in the two coordinate systems are also shown in Fig. 1. Following the notations in Ref. 16, the scale factors $Q_i=|\partial\mathbf{r}/\partial q_i'|/|\partial\mathbf{r}'/\partial q_i'|$ $(i=1,2,3)$ of the transformation [Eq. (4)] are found to be

$$Q_1 = \frac{s}{s-1},$$

$$Q_2 = \frac{q_1}{q_1'},$$

$$Q_3 = 1. \tag{9}$$

Finally, the elements of the permittivity tensor $\bar{\bar{\varepsilon}}'$ in the (q_1',q_2',q_3') system[16] can be cast in the following forms:

$$\varepsilon_{11}' = \frac{1}{\hat{q}_3 \cdot \hat{q}_1 \times \hat{q}_2}\frac{q_1'-1}{q_1'}, \tag{10}$$

$$\varepsilon_{12}' = \varepsilon_{21}' = -\frac{\hat{q}_1 \cdot \hat{q}_2}{\hat{q}_3 \cdot \hat{q}_1 \times \hat{q}_2}, \tag{11}$$

$$\varepsilon_{22}' = \frac{1}{\hat{q}_3 \cdot \hat{q}_1 \times \hat{q}_2}\frac{q_1'}{q_1'-1}, \tag{12}$$

$$\varepsilon_{33}' = \hat{q}_3 \cdot \hat{q}_1 \times \hat{q}_2\left(\frac{s}{s-1}\right)^2\frac{q_1'-1}{q_1'}, \tag{13}$$

and $\varepsilon_{13}'=\varepsilon_{23}'=\varepsilon_{31}'=\varepsilon_{32}'=0$. Furthermore, the permeability tensor $\bar{\bar{\mu}}'$ is equal to $\bar{\bar{\varepsilon}}'$. Specific expressions for the quantities $\hat{q}_1\cdot\hat{q}_2$ and $\hat{q}_3\cdot\hat{q}_1\times\hat{q}_2$ in terms of transformed coordinates may be obtained from Eqs. (6)–(8). However, we choose to leave the expressions in the above format to explicitly indicate the effect of the coordinate system on the cloak material parameters.

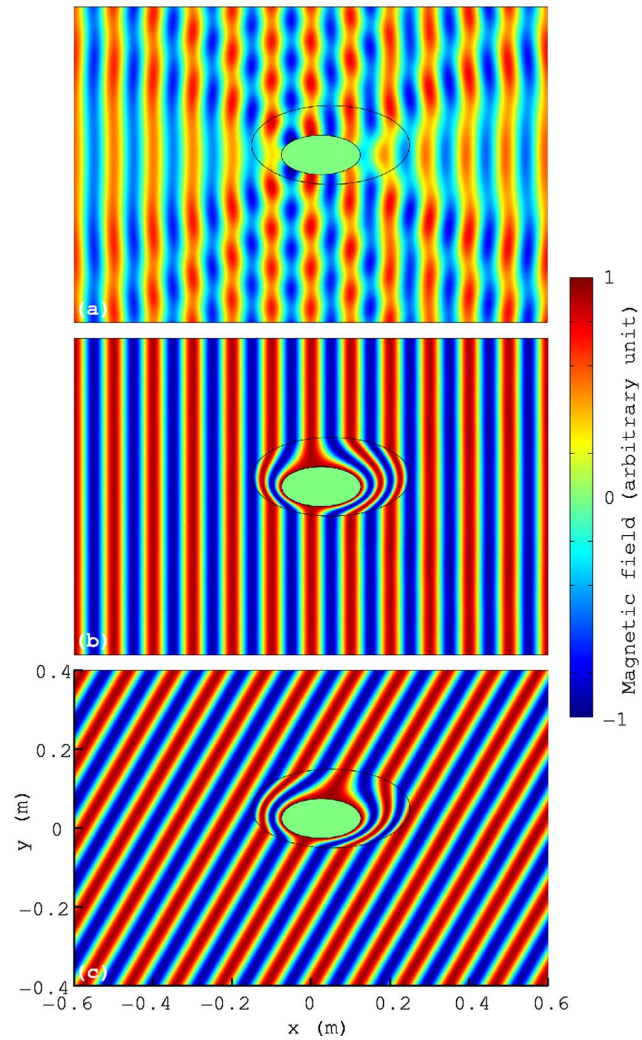

FIG. 2. (Color online) Snapshots of the total magnetic field for two-dimensional TM mode scattering from an elliptic PEC cylinder. (a) PEC cylinder exposed directly to the incoming plane wave propagating in the $+\hat{x}$ direction. (b) The same cylinder surrounded by the eccentric elliptic annular cloak. (c) The cloaked cylinder subject to the illumination by a plane wave making an angle of $-30°$ from the $+\hat{x}$ direction.

Full-wave finite-element simulations using COMSOL Multiphysics were performed to verify the behavior of the cloak. For this purpose, it is convenient to express $\bar{\bar{\varepsilon}}'$ in Cartesian coordinates by the tensor $\bar{\bar{\varepsilon}}$, which may be written as $\bar{\bar{\varepsilon}}=\Sigma_{i=1}^3\Sigma_{j=1}^3\varepsilon_{ij}'\hat{q}_i'(\hat{q}_j')^T|\hat{q}_3'\cdot\hat{q}_1'\times\hat{q}_2'|^{-1}$. The unit vectors that appear in this expression represent column vectors in the (x,y,z) system.

Figure 2 shows a cut in the x-y plane of the time-harmonic simulation results for TM-mode scattering by a perfect electric conducting (PEC) elliptic cylinder. Only a portion of the computation domain close to the scatterer is depicted. Perfectly matched layers are placed on all four sides of the computation domain to remove spurious reflections that would otherwise be caused by boundary truncation. The semiaxes of the cylinder are given by $a=0.1$ m and $b=0.05$ m, respectively, and it is centered at $(x_c,y_c)=(0.025,0.025\text{ m})$. The frequency of the time harmonic incident wave is assumed to be 3 GHz so that the scatterer is 2.0 and 1.0 wavelengths wide in the $\hat{x}$ and $\hat{y}$ directions, respectively. The value of parameter s is set to 2 so that the outer elliptic boundary is 4.0 and 2.0 wavelengths thick, and centered at $(0.05,0.05\text{ m})$. The scatterer is illuminated from the $-\hat{x}$ direction by a TM-mode plane wave having a $\hat{z}$-polarized magnetic field. The cloak's two elliptic boundaries (inner and outer) are shown as black contours in the figure. The PEC cylinder is directly exposed to the incoming wave in Fig. 2(a), where strong scattering is observed, especially in the forward and backward directions. However, when the PEC cylinder is covered by a cloak with the material properties given in Eqs. (10)–(13), it is seen that the incident field is completely guided around the cloaked object without creating any scattering. In Fig. 2(c), the cloaked cylinder is illuminated by a plane wave having the incident wave vector in the direction of $(\hat{x}\sqrt{3}-\hat{y})/2$, making an angle of $-30°$ with respect to the $+\hat{x}$ direction. In this configuration, the propagation direction is not parallel to either of the major axes of the ellipses. It is also neither parallel nor normal to the line connecting the centers of the cloak's inner and outer elliptic boundaries. The incident field is again bent smoothly around the scatterer by the cloak to emerge on the other side without any distortion in amplitude or phase.

The inspection of Eqs. (10)–(13) reveals that the elements of $\bar{\bar{\varepsilon}}'$ and $\bar{\bar{\mu}}'$ are equal to those of the concentric circular annular cloaks[14] slightly modified by the generalized coordinate system definitions. Moreover, these modifying factors are general to the nonorthogonal coordinate systems associated with 2D eccentric cloak geometries where $\hat{q}_3\cdot\hat{q}_1=\hat{q}_3\cdot\hat{q}_2=0$, yet $\hat{q}_1\cdot\hat{q}_2\neq 0$. If we choose $x_c=y_c=0$ and $a=b$ in the present formulation, the coordinate systems become orthogonal and the expressions in Eqs. (10)–(13) simply reduce to those of the well-known concentric circular annular cloak parameters.

Since the eccentric elliptic annular geometry is devoid of any rotational symmetry property, it is interesting to inspect the cloak parameter values at its nonreflecting outer boundary. Due to the presence of the nonunity factor $\hat{q}_3\cdot\hat{q}_1\times\hat{q}_2$, the cloak parameters do not satisfy the condition $\varepsilon'_{22}=\varepsilon'_{33}=1/\varepsilon'_{11}$ at the outer boundary $(q'_1=s)$ as the parameters of the concentric circular annular cloaks do.[1] However, the elliptic outer interface is still nonreflecting from any angle of incidence. To understand this better, it is more convenient to express the material parameters in terms of unit vectors that are normal and tangential to the outer boundary at the point of observation. For this purpose, let an orthogonal set of unit vectors be defined by $\hat{q}''_1=\hat{q}'_2\times\hat{q}'_3$, $\hat{q}''_2=\hat{q}'_2$, and $\hat{q}''_3=\hat{q}'_3$. On the outer boundary of the cloak shown in Fig. 2, the elements of the material tensor $\bar{\bar{\varepsilon}}'$ expressed in terms of the vectors $\hat{q}''_i$ $(i=1,2,3)$ are plotted in Fig. 3 with respect to q'_2. It is observed that $\varepsilon''_{33}=1/\varepsilon''_{11}$ holds with both parameters fixed at constant values. However, ε''_{22} varies as a function of position and ε''_{12} is generally nonzero, where the set of conditions $\varepsilon''_{22}=1/\varepsilon''_{11}$, $\varepsilon''_{12}=0$ are satisfied only at four distinct points along the boundary when $\hat{q}'_1$, $\hat{q}'_2$, $\hat{q}'_3$ are orthogonal. It is noted that the nondiagonal nature of the material tensors implies that the principle axes of the unit cells in a metamaterial realization of the cloak need to be rotated away from the unit vector directions.[13] This generalization demonstrates that the relationship $\varepsilon''_{22}=\varepsilon''_{33}=1/\varepsilon''_{11}$ together with zero off-diagonal terms for $\bar{\bar{\varepsilon}}''$ (and $\bar{\bar{\mu}}''=\bar{\bar{\varepsilon}}''$) (Ref. 1) is not a unique condition for nonreflecting interfaces.

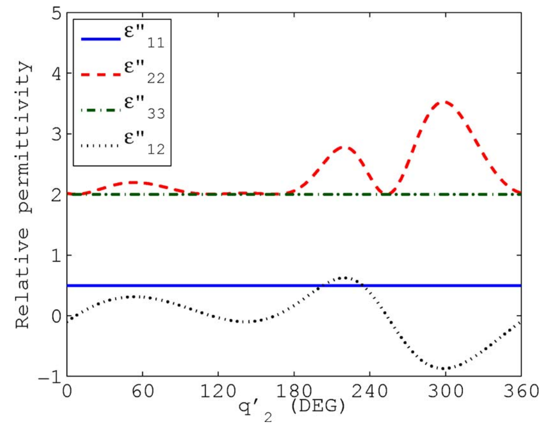

FIG. 3. (Color online) Cloak material parameters on the outer boundary of the cloak shown in Fig. 2 in terms of an orthogonal set of unit vectors $\hat{q}''_1$, $\hat{q}''_2$, and $\hat{q}''_3$.

The properties of two-dimensional electromagnetic cloaks, which are devoid of rotational symmetry, have been investigated by considering an eccentric elliptic annular cloak. The validity of the design was confirmed by full-wave simulations of plane wave scattering by a PEC cylinder with and without the cloak. The design example presented in this paper represents an important generalization of conventional concentric annular cloak designs toward the design of arbitrarily shaped and unsymmetrical electromagnetic cloaks.

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