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ScienceDirect

C. R. Physique 10 (2009) 379–390

COMPTES RENDUS PHYSIQUE

Metamaterials / Métamatériaux

# Metamaterials and invisibility

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Available online 9 May 2009

**Abstract**

Metamaterials have significantly extended the range of electromagnetic properties available to device designers. An interesting application of these new materials is to the problem of cloaking, where the goal is to render an object invisible to electromagnetic radiation within a certain frequency range. Here, I review the concepts behind recently-proposed invisibility cloaks, and the way in which metamaterials can allow these designs to be realized. ***To cite this article: B. Wood, C. R. Physique 10 (2009).***


**Résumé**

**Métamatériaux et invisibilité.** Les métamatériaux ont considérablement élargi le domaine des propriétés électromagnétiques accessibles pour la conception de système optiques. L'invisibilité est une application intéressante de ces nouveaux matériaux dont l'objet est de rendre un objet invisible pour un champ électromagnétique dans un certain domaine de fréquence. Dans cet article, je passe en revue les concepts sous jacents aux capes d'invisibilité récemment proposées et la manière dont les métamatériaux autorisent leur réalisation. ***Pour citer cet article : B. Wood, C. R. Physique 10 (2009).***


*Keywords:* Metamaterials; Invisibility; Cloaking

*Mots-clés :* Métamatériaux ; Invisibilité ; Capes d'invisibilité

## 1. Introduction

The desire to make objects or people invisible has a long history: to this end, many different devices have been imagined. However, it is only in relatively modern times that these plans have begun to move from fiction to reality. The recent development of metamaterials, and the new landscape of electromagnetic properties that has been made accessible to us as a result, have opened up new possibilities for invisibility, the most promising of which are described in this article.

To be invisible in the conventional sense is to be undetectable to electromagnetic radiation within a given frequency range; light must not be scattered or absorbed. An immediate and unfortunate consequence of this requirement is that an invisible person must also necessarily be blind (in the relevant frequency range)! In many applications, partial invisibility is sufficient. An example is stealth technology, where the aim is to hide an object from radar; since conventional radar detects light reflected by an object, it would be enough either to absorb all the light, or to redirect it away from the detector(s).

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1631-0705/$ – see front matter 
doi:10.1016/j.crhy.2009.01.002

Hiding an object from radar is an example of camouflage, which more generally involves modifying the appearance of an object so that it becomes indistinguishable from the background. It is a technique widely found in nature, and can be static or dynamic. Animals capable of dynamic camouflage must be able to change their appearance at will, according to their surroundings. The effect can be mimicked [1], using a camera, projector, and a retro-reflective material. The camera records an image of the background; this image is projected onto the object to be hidden, which is covered in the retro-reflective material, and consequently appears transparent to an observer. Of course, the projector is not hidden, and the method depends on the observer being in the right place. To circumvent these restrictions, an alternative approach is necessary, and this is where metamaterials can help; to date, three distinct ideas have emerged.

The first, due to Alú and Engheta, is to use a metamaterial coating to dramatically reduce the scattering from small objects [2,3]. The coating must be tailored according to the object to be hidden, and is designed so that the lowest-order (and hence, most significant) multipole terms in the scattered field are canceled. As the object size increases, higher-order multipoles become more important in the scattered field, and this method of cloaking becomes more difficult, although the possibility of concealing collections of particles in close proximity has been established [4], at least in theory.

The second technique employs the peculiar properties of negative-index media. These do not exist in nature, but have been made possible by the advent of metamaterials. In these materials, the real parts of the permittivity $\varepsilon$ and permeability $\mu$ are simultaneously negative. They can also be used to make superlenses [5]. Milton et al. showed that a polarizable object placed close to such a lens becomes effectively invisible [6–8]. As the losses in the superlens material are reduced, the field scattered by the object is damped more and more, until it disappears in the lossless limit. This phenomenon depends on the anomalous localized resonances associated with a superlens [9].

The third approach [10,11] is the most promising, and is the only one to have been demonstrated in an experiment [12]. It is therefore the focus of this review. It is based on the idea that a transformation of space can be mimicked by an appropriate transformation of $\varepsilon$ and $\mu$, at least as far as light is concerned. A transformation which takes a point in space and expands it to form a sphere can then be used to derive the properties of an invisibility cloak that hides any object within this sphere [10]; to fabricate the cloak according to this recipe requires metamaterials. Significantly, unlike the two other proposed methods, it can be used to hide objects which are much larger than the wavelength of light involved.

Schurig and coworkers constructed a prototype version of this cloak [12], designed to work at microwave frequencies. Since then, alternative implementations have been suggested: for visible light [13], static magnetic fields [14], and for sound waves [15,16]. Because the speed of sound is much slower than that of light, the acoustic cloaks do not have the same causality-imposed bandwidth limitation as the electromagnetic ones.

In this review, I discuss the technique – transformation optics – that allowed these cloaks (and other interesting devices) to be devised. After describing the "ideal" spherical and cylindrical cloaks, I examine the simplifications and approximations that are necessary for the practical realization of these designs. I then turn to the very promising application of cloaking to acoustic waves. Finally, I review the mathematical limitations on invisibility.

## 2. The metric for light

The key concept in developing the invisibility cloak is a simple one: it is the idea that light "sees" space differently. For light, the concept of distance is modified by the electromagnetic properties of the regions through which it travels. In geometrical optics, we are accustomed to the idea of the optical path; when traveling an infinitesimal distance $\mathrm{d}s$, the corresponding optical path length is $n\,\mathrm{d}s$, where $n$ is the local refractive index. This is the first hint that it may be possible to mimic a transformation of a region of space by using an equivalent transformation of electromagnetic properties.

To investigate this possibility, the first logical step is to determine how Maxwell's equations are affected by transforming to a new coordinate system [17,10,18]. In the frequency domain, the relevant equations in a source-free region are

$$\begin{aligned}\nabla \times \mathbf{E} &= i\omega\mathbf{B}\\ \nabla \times \mathbf{H} &= -i\omega\mathbf{D}\end{aligned} \tag{1}$$

These are supplemented by the constitutive equations

$$\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}$$
$$\mathbf{B} = \mu_0 \mu \mathbf{H} \tag{2}$$

where we allow the permittivity $\varepsilon$ and permeability $\mu$ to be anisotropic.

Consider now a change of coordinates which maps $(x, y, z)$ onto $(x', y', z')$. How do Maxwell's equations look in the new coordinates? The answer is that they look like (1); the derivatives are now taken with respect to the new coordinates, but the form is unchanged. However, the material parameters in the constitutive relations must be modified. The components of the permittivity become

$$\varepsilon'^{ij} = \frac{\Lambda^i_k \Lambda^j_l \varepsilon^{kl}}{|\det(\Lambda^i_k)|} \tag{3}$$

and the permeability transforms in the same way, for both are contravariant tensor densities of weight $+1$. Here $\Lambda$ is the Jacobian matrix for the transformation:

$$\Lambda^i_k = \frac{\partial x'^i}{\partial x^k} \tag{4}$$

Because of the invariance in form of Maxwell's equations, we can view the mapping as either a re-labeling of the same physical system, in which the three numbers needed to describe a given point in space have changed, or as a transformation of space, where $x'$, $y'$ and $z'$ are assumed to refer to the same coordinate system as $x$, $y$ and $z$. The latter interpretation proves to be very useful.

Pendry et al. pointed out [10] that if we wish to distort the fields in a certain region, we can perform a coordinate transformation that models the required distortion. We can then calculate the material parameters needed to achieve this effect with the help of (3). The result is a powerful technique – now known as transformation optics [10,12,18,19] – for designing electromagnetic devices.

Essentially the same idea of transforming space to manipulate light was proposed by Leonhardt, but in the context of geometrical optics in two dimensions [11]. The advantage of working in this simplified regime is that the transformations can then be elegantly expressed in terms of conformal maps.

Let us examine the kind of material parameters generated by this scheme. We will take as an example a simple transformation: the "squashing" of a volume of space in the $z$-direction [20], so that

$$z' = \begin{cases} z, & z < 0 \\ (a/b)z, & 0 < z < b \\ z - b + a, & z > b \end{cases} \tag{5}$$

while $x' = x$ and $y' = y$, as illustrated in Fig. 1. The Jacobian matrix is very simple, differing from the identity only in the $z'z$ component, which takes the value $a/b$ when $0 < z < b$ and 1 otherwise. The determinant therefore has the same value. Applying (3) gives diagonal material parameters, with the values

$$\varepsilon'^{11} = \varepsilon'^{22} = \mu'^{11} = \mu'^{22} = b/a$$
$$\varepsilon'^{33} = \mu'^{33} = a/b \tag{6}$$

The example illustrates that in order to produce the effect of compressing space, the components of the permittivity and permeability are reduced in the direction of compression (since $a/b < 1$), but increased in the perpendicular directions.

The material parameters prescribed by transformation optics are generally anisotropic. Both $\varepsilon$ and $\mu$ transform in the same way, and their components take non-negative values (unless the transformation is multi-valued, when space is mapped back onto itself [21]). However, these values may be less than one. These requirements, and the need to be able to tune the parameters very precisely, mean that devices based on this design paradigm would be impossible to realize without the help of metamaterials. Discussion of the role of metamaterials is postponed until Section 4; first, we explore the ideas behind invisibility.

## 3. An invisibility cloak

Transformation optics gives us a new way to control light, by mimicking the effect of a distortion of space. To use the technique to design an invisibility cloak, one must consider what kind of distortion could be used to hide an object.

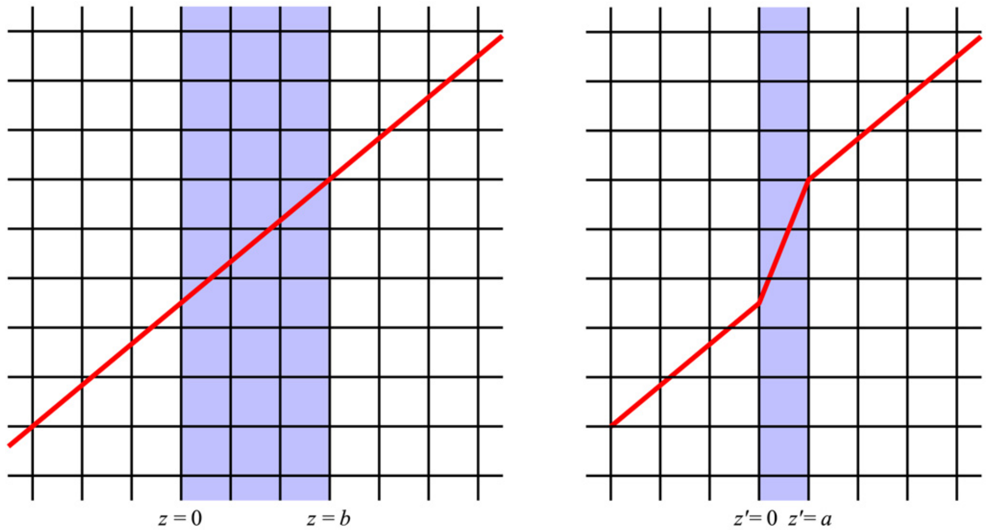


Fig. 1. A simple transformation. The shaded region is compressed in the $z$-direction by a factor $a/b$. The path of a light ray is bent by the compression.

The distortion must create a region of space isolated from the outside world (within which the hidden object can rest); it must also be of finite extent, so that an observer outside the cloak is unaware of the cloak's existence.

One suitable transformation [10] is to create a "hole" in space by expanding a single point to fill a sphere of radius $a$. To accommodate this sphere, the space around the point must be compressed: the space initially occupying a sphere of radius $b$ (where $b > a$) is squashed into the spherical shell between $a$ and $b$. More precisely, in spherical coordinates, we let

$$r' = a + \frac{b-a}{b} r \tag{7}$$

for $0 < r < b$ while leaving the $\theta$- and $\phi$-coordinates unchanged. To calculate the elements of the Jacobian matrix, first note that

$$x'^i = \frac{r'}{r} x^i = \left(\frac{a}{r} + \frac{b-a}{b}\right) x^i \tag{8}$$

Simple differentiation then gives

$$\begin{aligned} \Lambda^i_k &= \frac{r'}{r}\delta^i_k - \frac{a x^i x_k}{r^3} \\ \det\left(\Lambda^i_k\right) &= \frac{r'-a}{r}\left(\frac{r'}{r}\right)^2 \end{aligned} \tag{9}$$

Assuming that the region initially contains free space (so that $\varepsilon^{ij} = \mu^{ij} = \delta^{ij}$), the material parameters required to model this transformation are then given by (3) as

$$\varepsilon'^{ij} = \mu'^{ij} = \frac{b}{b-a}\left[\delta^{ij} + \frac{a(a-2r')}{r'^4} x'^i x'^j\right] \tag{10}$$

The key step in the transformation optics paradigm is to treat the primed coordinates as new values on the old Cartesian grid; the new material properties ensure that light behaves as though space itself has been transformed in the desired way. Naturally the parameters described in (10) only apply in the region $a < r' < b$. Outside the cloak, everything is as it was before; the difference is that the space inside the cloak has been electromagnetically isolated from the outside world. The form of the parameters is as we might have expected by analogy with the one-dimensional compression described in (5): space is effectively squashed in the radial direction, and so the radial components of the permittivity and permeability are less than one (and zero at the inner boundary), while the perpendicular components are greater

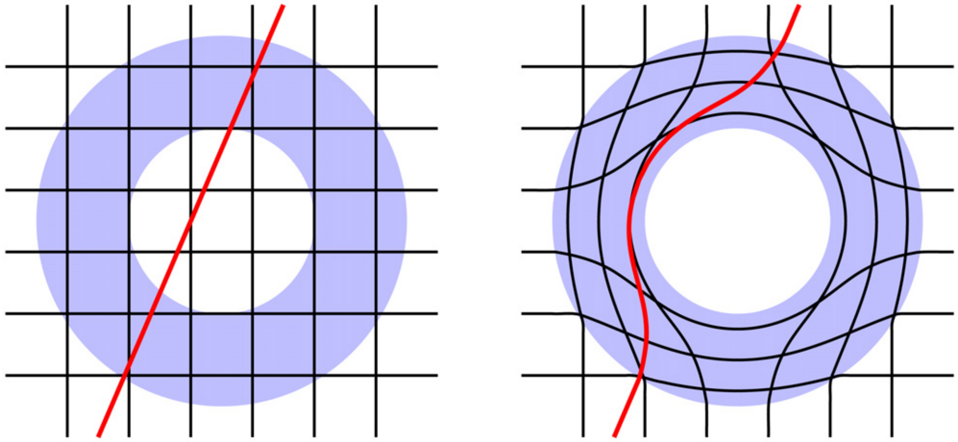

Fig. 2. A 2D representation of the transformation of space corresponding to a simple spherical cloak. On the left, a light ray passes through free space; on the right, space has been deformed by expanding a single point to fill the inner white sphere, which becomes the cloaked region. Within the outer radius of the cloak, space has been squashed to accommodate this region, and light is guided around it. Outside the outer radius, space is undisturbed, making the cloak and its contents undetectable.

than one. The deformation described in (7) is illustrated in Fig. 2. The figure shows the path of a ray of light that would have crossed the cloaked region; instead, it is guided around it [18]. There are no reflections at the boundaries, which are perfectly matched. However, the cloaking is not limited to rays; the inner region is also invisible to waves [22] and to evanescent fields [10].

In order to function properly, the material in the cloaking shell must satisfy the prescription laid down in (10) at the frequency of interest. There is a limitation on the functional bandwidth which is easily demonstrated. Consider the path taken by a ray like the one shown in Fig. 2; with the cloak in place, the distance traveled by the ray clearly increases, but the time allowed to traverse this distance must remain unchanged. This requires that the speed of light be greater than $c$ in the cloak, which is allowed – but only if the system is also strongly dispersive around the operating frequency.

The spherical cloak is, of course, not the only possible cloak. The mapping described in (7) does not have to be linear [23]; going further, the point does not have to be mapped onto a sphere [24] – although the mathematical analysis then becomes more difficult. Alternatively, one could start with a line instead of a point. In this case, the simplest imaginable cloak is cylindrical in shape, derived from (7) but with $r$ now the radial cylindrical coordinate. The resulting cloak [12] has parameters

$$\begin{aligned}
\varepsilon_r &= \mu_r = \frac{r-a}{r} \\
\varepsilon_\theta &= \mu_\theta = \frac{r}{r-a} \\
\varepsilon_z &= \mu_z = \left(\frac{b}{b-a}\right)^2 \frac{r-a}{r}
\end{aligned} \tag{11}$$

where the primes have been dropped. Note that these are the "ordinary" components of the modified permittivity in the cylindrical coordinate system, rather than the contravariant components, and in this form the permittivity is diagonal. For these reasons, the double superscript indices have been replaced by a single subscript label. As for the spherical cloak, the radial components of the material parameters tend to zero at the inner boundary; unlike the spherical cloak, the azimuthal components tend to infinity here. The cylindrical mapping has also been extended to include cloaks of arbitrary cross-section [25].

The spherical and cylindrical cloaks have been investigated extensively, both theoretically and numerically. Ray-tracing calculations based on a Hamiltonian formalism [10,18] show that light rays are indeed guided around the cloaked region to emerge on the opposite side of the cloak on their initial trajectories, as indicated in Fig. 2. Full-wave simulations using finite-element methods [22,12,26] confirm that the cloak works for waves, not just rays, and demonstrate how phase fronts are guided around the cloaked region (Fig. 3). The radial symmetry of the cloak

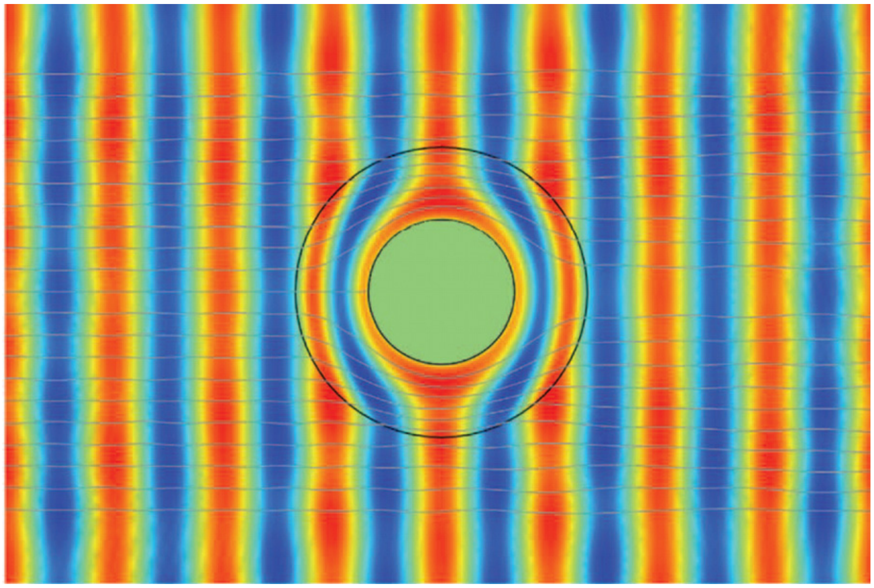

Fig. 3. Finite-element simulation [22] demonstrating the operation of a cylindrical cloak of the form described in (11). Inside the cloak is a metal cylinder, which would normally scatter incident radiation strongly.

parameters allows the wave equation to be separated and solved inside the cloaking shell. This permits a scattering analysis of the cloaks [27,28]. General solutions to the wave equation in the inner region, cloaking shell, and outside world are matched at the boundaries; the different angular orders decouple to give an expression for the scattering coefficients for each order. These relate the amplitude of the scattered and incident waves. For both spherical and cylindrical cloaks, the scattered field and the field inside the cloaked region have thus been shown to be zero. The analysis for the cylindrical cloak is slightly complicated by the divergence of the azimuthal material parameters at the inner boundary. To deal with this, it is convenient [29,28] to consider a slightly perturbed cloak, where the inner radius has been increased by a small amount $\delta$. The boundary conditions can then once again be used to match the fields in the three regions, and to relate the incident and scattered wave amplitudes; the ideal cloak is described by the limit $\delta \to 0$. Such deviations from the ideal are obviously also important when considering practical applications.

## 4. Cloaking in practice

Even with the help of metamaterials, achieving the design specifications laid down in (10) and (11) is a daunting task. However, further simplifications are possible, if we are prepared to accept less-than-perfect performance [12,13]. An immediate simplification is achieved in the case of the cylindrical cloak if we choose to work with a single polarization, so that either the electric or magnetic field is aligned with the $z$-axis. Only three (instead of six) parameters are then significant: either $\varepsilon_z$, $\mu_r$ and $\mu_\theta$, in the case of transverse electric (TE) polarization, or $\varepsilon_r$, $\varepsilon_\theta$ and $\mu_z$ for transverse magnetic (TM).

In the geometric limit, where these parameters are assumed to vary slowly on the scale of the wavelength, the behavior of light inside the cloak is governed only by the products $\varepsilon_z\mu_r$ and $\varepsilon_z\mu_\theta$ (for TE waves) or $\varepsilon_r\mu_z$ and $\varepsilon_\theta\mu_z$ (for TM).

This can be seen by considering Maxwell's equations in cylindrical coordinates; for TE polarization, they reduce to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\mu_\theta}\frac{\partial E_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\mu_r}\frac{\partial E_z}{\partial \theta}\right)+\frac{\omega^2}{c^2}\varepsilon_z E_z=0 \tag{12}$$

For an axially-symmetric cloak, $\mu_r$ is independent of $\theta$, while in the geometric limit the $r$-dependence of $\mu_\theta^{-1}$ is neglected; these parameters can therefore be taken outside the partial derivatives, leaving only the products $\varepsilon_z\mu_r$ and $\varepsilon_z\mu_\theta$.

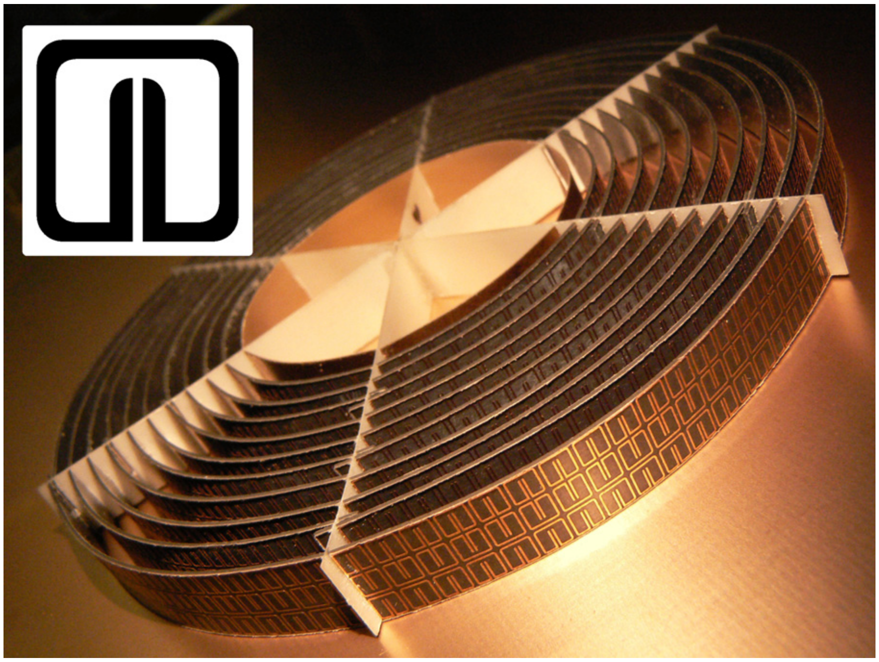

Fig. 4. The first experimentally-demonstrated cloak [12]. The insert shows a schematic of the split-ring resonator element that is used to construct the metamaterial cloaking shell.

If we adjust the individual components while maintaining the value of these products, light rays will follow the same paths as in the ideal case, but with some reflections at the cloak boundary, which is no longer perfectly matched. For example, we are free to set $\mu_\theta = 1$, which implies that

$$\begin{aligned} \varepsilon_z &= \left(\frac{b}{b-a}\right)^2 \\ \mu_r &= \left(\frac{r-a}{r}\right)^2 \end{aligned} \tag{13}$$

in order to preserve the products. These parameters are much easier to realize, since only $\mu_r$ is a function of $r$. Despite the simplification, making a cloak remains a challenging task. The cloak calls for a material with a strong diamagnetic response in one direction, with a constant dielectric response in a perpendicular direction. Fortunately, metamaterials are able to provide the desired behavior.

The split-ring resonator (SRR) [30,31] is the generic metamaterial element, and for good reason. It consists of one or two concentric rings, not necessarily rounded, that have been cut in one or more places; an example is shown in the inset to Fig. 4. The usefulness of the SRR stems from the wide range of responses that can be achieved with designs based on this simple structure. These have been amply documented elsewhere [32–34], and are a consequence of the resonant nature of the basic element: when the magnetic flux through the ring changes, it induces a current, which results in a build-up of charge around the gap. The combination of inductance and capacitance can be modeled as an *LC* circuit. Close to the natural oscillation frequency, the amplitude of the response is large, and the phase changes rapidly with frequency. In a metamaterial based on SRRs, this translates to a wide range of values of the effective permeability for a magnetic field applied perpendicular to the rings [31], going far beyond those available in “natural” materials and even becoming negative for some frequencies.

The resonant frequency depends on the precise geometry of the rings; when working with light at a fixed frequency, the effective permeability can thus be tuned by making very small adjustments to the basic geometrical parameters. In this way, the variable anisotropic permeability set out in (13) can be obtained. The SRRs also show a polarization response when an electric field is applied in the plane of the rings; thus, we can simultaneously tune $\mu_r$ and keep $\varepsilon_z$ constant.

An additional challenge for anyone wishing to construct a cloak using the designs described above is that they are all based on spherical or cylindrical coordinates, whereas many of the effective medium theories that are used to characterize metamaterials are appropriate for systems which are periodic in Cartesian coordinates [35]. A way around this obstacle is to note that spherical and cylindrical systems are still described by orthogonal coordinates, and thus resemble Cartesian coordinates locally; using the standard material parameters should therefore be permissible as long as the coordinate system changes slowly on a scale set by the size of the metamaterial unit cell.

These ideas combined to enable the construction and testing of a prototype 2D cylindrical cloak for microwaves [12], shown in Fig. 4. Simulations showed [22,29] that the simplified cloak, while not invisible, should reduce scattering significantly, even when losses and the stepwise change in material parameters are taken into account, and the experimental results confirmed this. In a planar geometry, a waveguide was used to direct light at a metal cylinder, and the resulting electric field strength was probed over a wide area. The light scattered by the cylinder was reduced dramatically when it was encased in the cloak; the field map revealed the phase distortion within the cloak to be similar to the ideal case shown in Fig. 3.

The approach which proved successful for microwaves must be adapted slightly when working in other regimes. SRRs are excellent microwave-frequency metamaterials, giving access to a wide range of values of $\mu$. However, their usefulness does not extend to visible light, where tuning the magnetic response is much more challenging. The appropriate simplification is then one which requires no magnetic response at all. By working with TE-polarized light, it is possible to design a cylindrical cloak with reduced parameters that meets this criterion [13].

For TE polarization, the important quantities in the geometrical limit are the products $\varepsilon_\theta \mu_z$ and $\varepsilon_r \mu_z$, which are determined by (11). One can set $\mu_z = 1$, which leaves $\varepsilon_\theta$ constant and transfers all the $r$-dependence into $\varepsilon_r$. A simple design which incorporates all these features, based on embedding radially-oriented metal wires in a dielectric has been proposed [13].

While the reduced-parameter cloaks give approximately the correct ray behavior, they suffer from reflections because the outer boundary is no longer perfectly matched to the surroundings. This situation can be improved by considering alternative transformations [36,24]. These all perform the same basic task – of mapping a sphere or cylinder of radius $b$ onto a spherical or cylindrical shell, contained between $r = a$ and $r = b$ – but are not restricted to the linear form described in (7). For example, in a cylindrical cloak, the outer boundary can be made reflectionless by ensuring that $\varepsilon_z/\mu_\theta = 1$ (for TE modes) and $\mu_z/\varepsilon_\theta = 1$ (for TM modes) when $r = b$. Of course, this is true for the ideal cloak; the aim is to choose a transformation that preserves one of these relationships even after the parameters have been simplified. A version of the non-magnetic optical cloak has been designed by applying this principle, and finite-element simulations show a remarkable improvement over the linear version [36]. Instead of concentrating on impedance matching, we could instead worry about the bandwidth. The argument in Section 3 proved that the ideal cloaks are necessarily dispersive, but reduced-parameter cloaks need not be so; at the expense of permitting light that travels through the cloak to suffer a phase lag, the cloaking effect might be extended to a greater range of frequencies [37].

The advantage of working with cylindrical, rather than spherical, cloaks is that by choosing to work with a particular polarization, one can eliminate three of the six material parameters. Thus cylindrical cloaks have been favored when practical schemes are designed, despite the requirement of infinite-valued parameters at the inner boundary in the ideal case. The same reduction in complexity is achieved automatically when one works at zero frequency. Electric and magnetic fields decouple, and one can consider constructing an electrostatic or magnetostatic cloak, and dealing with either $\varepsilon$ or $\mu$ – but not both. However, such cloaks would require new metamaterials, in particular because of the absence of natural materials with $\varepsilon < 1$ or $\mu < 1$ at zero frequency. Progress has been made in the magnetostatic case [14,38], where arrays of superconducting plates have been shown to provide tunable diamagnetism in one direction; combining these with thin layers of high-permeability material could lead to a DC magnetic cloak [14].

## 5. Acoustic cloaking

Cloaking is not restricted to electromagnetism. It extends to any set of equations that remain invariant in form under a change of coordinates. This includes linear surface liquid waves [41], and electrical conductivity, for which the establishment of cloaked solutions predates that for electromagnetism [39,40] (see Section 6); it also includes acoustics.

With acoustic cloaking in mind, Milton et al. considered the general elastodynamic wave equation, and showed that changing coordinates had some undesirable effects: in order for the transformation to lead to a new solution, the mass density was required to be anisotropic, and new terms appear in the equation which couple the stress with the velocity [42]. An anisotropic mass density is not an impossible hurdle – one could envisage a material containing cavities in which masses are connected to the walls by springs, with different constants according to the direction [43]. However, the additional terms in the equation are problematic, and suggest that a more general class of equations need to be considered, bringing an extra degree of complexity.

On the other hand, Leonhardt's electromagnetic cloaking scheme was based on the 2D Helmholtz equation, which is isomorphic to the geometric limit of the 2D acoustic wave equation; an equivalent geometric-limit acoustic cloak should therefore be possible. In fact, Cummer and Schurig went further [16]. They considered the linearized (small-amplitude) equations for motion for an inviscid fluid with zero shear modulus; at constant frequency, these are

$$i\omega\rho\mathbf{v} = \nabla p \tag{14}$$

$$i\omega p = \lambda\nabla\cdot\mathbf{v} \tag{15}$$

where $p$ is the pressure, $\mathbf{v}$ is the fluid velocity, $\rho$ is the density and $\lambda$ is the bulk modulus. They compared these equations – for $z$-invariant systems, expressed in cylindrical coordinates – with the reduced form of Maxwell's equations appropriate for TE polarization (and $z$-invariance) in the same basis, and found them to be equivalent. The required variable exchange is

$$\left[p, v_r, v_\phi, \rho_r, \rho_\phi, \lambda^{-1}\right] \leftrightarrow [E_z, -H_\phi, H_r, \mu_\phi, \mu_r, \varepsilon_z] \tag{16}$$

An acoustic version of the cloak described in (11) can then be obtained immediately. The resulting parameters are [16]

$$\rho_r = \frac{r}{r-a}$$

$$\rho_\phi = \frac{r-a}{r}$$

$$\lambda^{-1} = \left(\frac{a}{b-a}\right)^2 \frac{r-a}{r} \tag{17}$$

Note that the mass density is still required to be anisotropic.

The effectiveness of the cloak has been confirmed in simulations, using a finite-difference code [16] and a uniform-layer approximation [44]. The latter calculation may give a better idea of the performance of any eventual metamaterial-based realization.

Despite initial appearances to the contrary, acoustic cloaking is not restricted to two dimensions. It was pointed out by Chen and Chan [15] that the 3D acoustic equation is isomorphic to the conductivity equation, for which cloaking had already been established [39,40,23]. The acoustic equation itself is obtained by combining (14) and (15):

$$\nabla\cdot\left(\frac{1}{\rho}\nabla p\right) = -\frac{\omega^2}{\lambda}p \tag{18}$$

By analogy with the conductivity equation, Chen and Chan were able to write down the form-preserving transformations for the density and bulk modulus:

$$\left[\rho'^{-1}\right]^{ij} = \frac{\Lambda^i_k \Lambda^j_l [\rho^{-1}]^{kl}}{|\det(\Lambda^i_k)|} \tag{19}$$

$$\lambda' = \frac{\lambda}{|\det(\Lambda^i_k)|} \tag{20}$$

The bulk modulus transforms as a scalar density, while the inverse of the mass density transforms exactly like $\varepsilon$ and $\mu$. The standard point transformation then gives the formula for a spherical cloak as

$$\rho_r = \left(\frac{b-a}{b}\right)\frac{r^2}{(r-a)^2}$$
$$\rho_\theta = \rho_\phi = \frac{b-a}{b}$$
$$\lambda = \left(\frac{b-a}{b}\right)^3 \frac{r^2}{(r-a)^2} \tag{21}$$

while the line transformation reproduces (17) precisely. The same prescription for a spherical cloak has also been obtained using an approach based on scattering [45]. In both the cylindrical and spherical acoustic cloaks, the radial component of the mass density tends to infinity as $r \to a$. This singular behavior is also found in the cylindrical electromagnetic cloak, but not in the spherical version.

Acoustic cloaks, therefore, are certainly possible in theory, although they will require new acoustic metamaterials to provide the necessary tunable mass density and bulk modulus. They have an advantage over electromagnetic cloaks, which are constrained by causality to have a narrow operating bandwidth; acoustic cloaks do not suffer from this, because acoustic wave packets are permitted to travel faster in the cloak than in the surrounding medium.

## 6. Limitations on invisibility

An important issue is the extent to which the electromagnetic properties of the materials in a given region are determined by the fields on some surface bounding the region. If such information were sufficient to determine these properties completely, then something which is not free space could never be made to look like free space, and perfect invisibility would be impossible in principle – even at a single frequency. A large amount of research [46–49,39,50,51] has been devoted to this mathematical problem.

Several of these studies have emerged from a consideration of electric impedance tomography (EIT). The aim of this technique is to probe the spatial variations in conductivity $\sigma$ within some region $\Omega$ by applying a known static voltage $u$ to the surface bounding the region ($\partial\Omega$) and recording the resulting current $\sigma\nabla u$. The equation satisfied by these quantities inside $\Omega$ is

$$\nabla \cdot (\sigma\nabla u) = 0 \tag{22}$$

The current–voltage relationship provides a Dirichlet-to-Neumann map $\Lambda_\sigma$ on $\partial\Omega$:

$$\Lambda_\sigma : u|_{\partial\Omega} \to (\sigma\nabla u)\cdot\nu|_{\partial\Omega} \tag{23}$$

Here $\nu$ is the unit outward normal to $\partial\Omega$. In order for EIT to work, it must be possible to determine $\sigma$ from a knowledge of $\Lambda_\sigma$. If this can be done, then cloaking (in this situation) is impossible, even in theory. The question of whether or not the mapping can be used to determine the form of the conductivity is known as the Calderón problem [46]. It turns out [50] that the Dirichlet-to-Neumann map does determine $\sigma$, but only under certain conditions: namely, that $\sigma$ must be known to be scalar-valued, positive and finite (and "smooth" to some extent, depending on the dimension of the problem). However, if these conditions are broken, cloaking is possible; in fact, a cloaking scheme equivalent to that described in Section 2 for electromagnetic waves had already been established in the context of EIT [39,40]. The singularity and anisotropy of the material parameters required for the ideal cloak mean that the Dirichlet-to-Neumann map does not uniquely determine the conductivity.

The construction proposed by Leonhardt for a 2D electromagnetic cloak [11] works in the geometric optics limit, where the propagation of light is governed by the Helmholtz equation,

$$\left(\nabla^2 + k^2 n^2\right)\psi = 0 \tag{24}$$

for which a similar uniqueness theorem has been established [48]. Although it relies only on a scalar refractive index, Leonhardt's construction does not set upper and lower bounds on this quantity and is therefore not subject to the uniqueness constraint. The treatment of the uniqueness problem has been extended to include the acoustic wave equation, with the same results [52]; uniqueness can be proved, but only for well-behaved compressibilities and densities.

The EIT problem is analogous to Maxwell's equations at zero frequency. For finite frequencies, a rigorous treatment of cloaking [23] once again shows that invisibility is theoretically possible with singular, anisotropic material parameters, at least for passive objects at a given frequency. Once sources are included in the cloaked region, the problem becomes more difficult, and finite-energy solutions do not exist. Some authors claim that this means that the cloaks of Section 2 cannot hide active objects [23], while others claim that the infinite polarization at the inner boundary prevents radiation from leaving the inner cloaked region [53,54]. It is acknowledged, however, that lining the inner surface of the cloak with a perfectly reflecting material provides a remedy to this difficulty. (For the cylindrical cloak, which requires the azimuthal components of the material parameters to be infinite at the inner boundary, a reflective lining also has a beneficial effect on the scattering coefficients [23].)

## 7. Conclusions

Invisibility is a topic that arouses a great deal of interest, both within the research community and beyond. Metamaterials have allowed us to contemplate a wide range of novel invisibility devices. In particular, those designs based on the transformation optics paradigm show outstanding promise, with potential applications in electromagnetic and acoustic cloaking. Many different forms of cloak are possible, and each may be adapted, simplified and approximated according to the priorities of the final application: maximizing bandwidth, minimizing back-scattering, keeping the cloaking shell thin, etc. It is to be expected that the needs of cloaking will provide a driving force for the pursuit of new metamaterials for some time to come.

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