Composing Lattices and CRDTs

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Motivation

- Conflict-free Replicated Data Types (CRDTs)
- Convergence after concurrent updates. Favor AP under CAP
- Examples include counters, sets, mv-registers, maps, graphs
- State based CRDTs are rooted on join semi-lattices
- Current catalog design is heuristic and proofs are specific

- Here we revisit CRDTs looking at composition strategies
- Composition applies to lattices, operations and proofs

Type hierarchy

- set, provides identity comparison
- poset, provides partial order
- lattice, provides universal join a.k.a. LUB, merge
- lattice$_\perp$, adds a bottom element, least element

Partially ordered sets

$$\frac{A : \text{poset}}{A : \text{set}}$$

$a \sqsubseteq a$	Reflexive
$a \sqsubseteq b \wedge b \sqsubseteq c \Rightarrow a \sqsubseteq c$	Transitive
$a \sqsubseteq b \wedge b \sqsubseteq a \Rightarrow a = b$	Anti-symmetric

$$a \sqsubset b \equiv a \sqsubseteq b \wedge a \neq b \qquad a \parallel b \equiv a \not\sqsubseteq b \wedge b \not\sqsubseteq a$$

Join semi-lattices

$$\frac{A : \text{lattice}}{A : \text{poset}}$$

$a \sqcup a = a$	Idempotent
$a \sqcup b = b \sqcup a$	Commutative
$a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$	Associative
$a \sqsubseteq a \sqcup b \qquad b \sqsubseteq a \sqcup b$	Upper-bound
$a \sqsubseteq c \land b \sqsubseteq c \Rightarrow a \sqcup b \sqsubseteq c$	Minimal

$$a \sqsubseteq b \equiv a \sqcup b = b$$

Bounded join semi-lattices

$$\frac{A : \text{lattice}_\perp}{A : \text{lattice}}$$

$a \sqcup \perp = a$ Neutral

Singleton type

$$\frac{}{\mathbb{1} : \mathsf{lattice}_{\perp}}$$

$$\bullet \sqsubseteq \bullet \qquad \bullet \sqcup \bullet = \bullet \qquad \perp = \bullet$$

Boolean

$$\frac{}{\mathbb{B} : \mathsf{lattice}_{\perp}}$$

$$\mathsf{False} \sqsubseteq \mathsf{True} \qquad x \sqcup y = x \vee y \qquad \perp = \mathsf{False}$$

Natural numbers

$$\frac{}{\mathbb{N} : \text{lattice}_{\perp}}$$

$$n \sqsubseteq m = n \leq m \qquad n \sqcup m = \max(n, m) \qquad \perp = 0$$

Product

$$\frac{A : \text{lattice} \qquad B : \text{lattice}}{A \times B : \text{lattice}} \qquad \frac{A : \text{lattice}_\perp \qquad B : \text{lattice}_\perp}{A \times B : \text{lattice}_\perp}$$

$$(x_1, y_1) \sqsubseteq (x_2, y_2) = x_1 \sqsubseteq x_2 \wedge y_1 \sqsubseteq y_2$$

$$(x_1, y_1) \sqcup (x_2, y_2) = (x_1 \sqcup x_2, y_1 \sqcup y_2)$$

$$\perp = (\perp, \perp)$$

Product

$$\frac{A : \text{lattice} \qquad B : \text{lattice}}{A \times B : \text{lattice}} \qquad \frac{A : \text{lattice}_\perp \qquad B : \text{lattice}_\perp}{A \times B : \text{lattice}_\perp}$$

$$(x_1, y_1) \sqsubseteq (x_2, y_2) = x_1 \sqsubseteq x_2 \wedge y_1 \sqsubseteq y_2$$

$$(x_1, y_1) \sqcup (x_2, y_2) = (x_1 \sqcup x_2, y_1 \sqcup y_2)$$

$$\perp = (\perp, \perp)$$

E.g: Version Vectors, $\mathbb{N} \times \mathbb{N}$ $\quad [3, 5] \not\sqsubseteq [6, 0] \quad [3, 5] \sqsubseteq [6, 5]$

Lexicographic product

$$\frac{A : \text{lattice} \qquad B : \text{lattice}_\perp}{A \boxtimes B : \text{lattice}} \qquad \frac{A : \text{lattice}_\perp \qquad B : \text{lattice}_\perp}{A \boxtimes B : \text{lattice}_\perp}$$

$$(x_1, y_1) \sqsubseteq (x_2, y_2) = x_1 \sqsubseteq x_2 \lor (x_1 = x_2 \land y_1 \sqsubseteq y_2)$$

$$(x_1, y_1) \sqcup (x_2, y_2) = \begin{cases} (x_1, y_1) & \textbf{if } x_2 \sqsubset x_1 \\ (x_2, y_2) & \textbf{if } x_1 \sqsubset x_2 \\ (x_1, y_1 \sqcup y_2) & \textbf{if } x_1 = x_2 \\ (x_1 \sqcup x_2, \perp) & \textbf{otherwise} \end{cases}$$

$$\perp = (\perp, \perp)$$

Lexicographic product

$$\frac{A : \text{lattice} \qquad B : \text{lattice}_\perp}{A \boxtimes B : \text{lattice}} \qquad \frac{A : \text{lattice}_\perp \qquad B : \text{lattice}_\perp}{A \boxtimes B : \text{lattice}_\perp}$$

$$(x_1, y_1) \sqsubseteq (x_2, y_2) = x_1 \sqsubseteq x_2 \vee (x_1 = x_2 \wedge y_1 \sqsubseteq y_2)$$

$$(x_1, y_1) \sqcup (x_2, y_2) = \begin{cases} (x_1, y_1) & \textbf{if } x_2 \sqsubset x_1 \\ (x_2, y_2) & \textbf{if } x_1 \sqsubset x_2 \\ (x_1, y_1 \sqcup y_2) & \textbf{if } x_1 = x_2 \\ (x_1 \sqcup x_2, \perp) & \textbf{otherwise} \end{cases}$$

$$\perp = (\perp, \perp)$$

E.g: Dictionary order, $\{a, b, c\} \boxtimes \{a, b, c\}$ $\quad ac \sqsubseteq bc \quad ac \sqsubseteq ba$

Linear sum

$$\frac{A : \text{lattice} \qquad B : \text{lattice}}{A \oplus B : \text{lattice}} \qquad \frac{A : \text{lattice}_\perp \qquad B : \text{lattice}}{A \oplus B : \text{lattice}_\perp}$$

$$\begin{aligned}
\text{Left } x &\sqsubseteq \text{Left } y &&= x \sqsubseteq y \\
\text{Right } x &\sqsubseteq \text{Right } y &&= x \sqsubseteq y \\
\text{Left } x &\sqsubseteq \text{Right } y &&= \text{True} \\
\text{Right } x &\sqsubseteq \text{Left } y &&= \text{False}
\end{aligned}
\qquad
\begin{aligned}
\text{Left } x &\sqcup \text{Left } y &&= \text{Left } (x \sqcup y) \\
\text{Right } x &\sqcup \text{Right } y &&= \text{Right } (x \sqcup y) \\
\text{Left } x &\sqcup \text{Right } y &&= \text{Right } y \\
\text{Right } x &\sqcup \text{Left } y &&= \text{Right } x
\end{aligned}$$

$$\perp = \text{Left } \perp$$

Linear sum

$$\frac{A : \text{lattice} \qquad B : \text{lattice}}{A \oplus B : \text{lattice}} \qquad \frac{A : \text{lattice}_\perp \qquad B : \text{lattice}}{A \oplus B : \text{lattice}_\perp}$$

$$\begin{aligned}
\text{Left } x &\sqsubseteq \text{Left } y &&= x \sqsubseteq y \\
\text{Right } x &\sqsubseteq \text{Right } y &&= x \sqsubseteq y \\
\text{Left } x &\sqsubseteq \text{Right } y &&= \text{True} \\
\text{Right } x &\sqsubseteq \text{Left } y &&= \text{False}
\end{aligned}
\qquad
\begin{aligned}
\text{Left } x &\sqcup \text{Left } y &&= \text{Left } (x \sqcup y) \\
\text{Right } x &\sqcup \text{Right } y &&= \text{Right } (x \sqcup y) \\
\text{Left } x &\sqcup \text{Right } y &&= \text{Right } y \\
\text{Right } x &\sqcup \text{Left } y &&= \text{Right } x
\end{aligned}$$

$$\perp = \text{Left } \perp$$

E.g: $\{a, b, c\} \oplus \mathbb{N}$ $\qquad a \sqsubseteq c \qquad 5 \sqsubseteq 6 \qquad b \sqsubseteq 1$

Function

$$\frac{A : \text{set} \qquad B : \text{lattice}}{A \to B : \text{lattice}} \qquad \frac{A : \text{set} \qquad B : \text{lattice}_\perp}{A \to B : \text{lattice}_\perp}$$

$$f \sqsubseteq g = \forall x \in A \cdot f(x) \sqsubseteq g(x) \qquad (f \sqcup g)(x) = f(x) \sqcup g(x)$$

$$\perp(x) = \perp$$

E.g: $\{a, b, c\} \to \mathbb{N}$

$$\{a \mapsto 0, b \mapsto 2, c \mapsto 0\} \sqsubseteq \{a \mapsto 3, b \mapsto 2, c \mapsto 0\}$$

Set

$$\frac{A : \text{set}}{\mathcal{P}(A) : \text{lattice}_{\perp}}$$

$$\mathcal{P}(A) \cong A \to \mathbb{B}$$

$$a \sqsubseteq b = a \subseteq b \qquad a \sqcup b = a \cup b \qquad \perp = \emptyset$$

Set

$$\frac{A : \text{set}}{\mathcal{P}(A) : \text{lattice}_\perp}$$

$$\mathcal{P}(A) \cong A \to \mathbb{B}$$

$$a \sqsubseteq b = a \subseteq b \qquad a \sqcup b = a \cup b \qquad \perp = \emptyset$$

E.g: $\mathcal{P}(\{a, b, c\})$ $\qquad \{b\} \sqsubseteq \{a, b\}$ $\qquad \{a, b\} \parallel \{b, c\}$

Multiset

$$\frac{A : \text{set}}{\mathcal{M}(A) : \text{lattice}_\perp}$$

$$\mathcal{M}(A) \cong A \to \mathbb{N}$$

$$a \sqsubseteq b = a \subseteq b \qquad a \sqcup b = a \cup b \qquad \perp = \emptyset$$

Map

$$\frac{K : \text{set} \qquad V : \text{lattice}}{K \hookrightarrow V : \text{lattice}_{\bot}}$$

$$K \hookrightarrow V \cong K \rightarrow \mathbb{1} \oplus V$$

$$f \sqsubseteq g = \ldots \qquad f \sqcup g = \ldots \qquad \bot = \ldots$$

Map

$$\frac{K : \text{set} \qquad V : \text{lattice}}{K \hookrightarrow V : \text{lattice}_\bot}$$

$$K \hookrightarrow V \cong K \to \mathbb{1} \oplus V$$

$$f \sqsubseteq g = \dots \qquad f \sqcup g = \dots \qquad \bot = \dots$$

E.g: $\{a, b, c\} \hookrightarrow \mathbb{N}$

$$\{b \mapsto 2\} \sqsubseteq \{a \mapsto 3, b \mapsto 2\} \qquad \{b \mapsto 2\} \parallel \{a \mapsto 3, b \mapsto 1\}$$

Set of maximal elements (antichain)

$$\frac{A : \text{poset}}{\mathcal{A}(A) : \text{lattice}_{\perp}}$$

$$\mathcal{A}(A) = \{\text{maximal}(a) \mid a \in \mathcal{P}(A)\} \qquad \cong \mathcal{O}(A)$$

$$\text{maximal}(a) = \{x \in a \mid \nexists y \in a \cdot x \sqsubset y\}$$

$$a \sqsubseteq b = \forall x \in a \cdot \exists y \in b \cdot x \sqsubseteq y \qquad = \downarrow a \subseteq \downarrow b$$

$$a \sqcup b = \text{maximal}(a \cup b)$$

$$\perp = \emptyset$$

Catalog

CRDT	Lattice	Ops	Comments
P Counter	$I \hookrightarrow \mathbb{N}$	inc	*version vector*
PN Counter	$(I \hookrightarrow \mathbb{N}) \times (I \hookrightarrow \mathbb{N})$	inc dec	*in riak-dt*
	$I \hookrightarrow \mathbb{N} \boxtimes \mathbb{N}$		*in cassandra*
G Set	$\mathcal{P}(E)$	add	*grow only*
2P Set	$\mathcal{P}(E) \times \mathcal{P}(E)$	add rmv	*tombstones*
	$E \hookrightarrow \mathbb{B}$		
LWW elem Set	$E \hookrightarrow \mathbb{N} \boxtimes \mathbb{B}$	add rmv	*by H.G. Roh*
OR Set	$E \hookrightarrow I \hookrightarrow \mathbb{N} \boxtimes \mathbb{B}$	add rmv	*add/rmv wins*
Opt OR Set	$I \hookrightarrow \mathbb{N} \hookrightarrow E \oplus \mathbb{1}$	add rmv	*compactable*

Catalog

CRDT	Lattice	Ops	Comments
LWW Reg	$\mathbb{N} \boxtimes V$	assign	*unique timestamps*
	$\mathbb{N} \boxtimes \mathcal{P}(V)$		*allowing collisions*
MV Reg	$\mathcal{A}((I \hookrightarrow \mathbb{N}) \boxtimes V)$	assign	*BloomL Dom Set*
MV KVS	$K \hookrightarrow \mathcal{A}((I \hookrightarrow \mathbb{N}) \boxtimes V)$	put	*Dynamo like*
Sequences	t.b.d.		
Graphs	t.b.d.		
Trees	t.b.d.		

Inflation

- Inflation: $x \sqsubseteq f(x)$
- Inflations advance state with robustness to join
- New state subsumes old state
- Immune to replays of the past
- Not same as monotone functions: $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$

Inflation - strict and non strict

$$\frac{\forall x \in a \cdot x \sqsubseteq f(x)}{f : A \xrightarrow{\sqsubseteq} A}$$

$$\frac{\forall x \in a \cdot x \sqsubset f(x)}{f : A \xrightarrow{\sqsubset} A}$$

$$\frac{f : A \xrightarrow{\sqsubset} A}{f : A \xrightarrow{\sqsubseteq} A}$$

Primitive inflations

$$\mathrm{id}(x) = x \qquad \frac{}{\mathrm{id} : A \xrightarrow{\sqsubseteq} A}$$

$$\underline{\mathrm{True}}(x) = \mathrm{True} \qquad \frac{}{\underline{\mathrm{True}} : \mathbb{B} \xrightarrow{\sqsubseteq} \mathbb{B}}$$

$$\mathrm{succ}(x) = x + 1 \qquad \frac{}{\mathrm{succ} : \mathbb{N} \xrightarrow{\sqsubseteq} \mathbb{N}}$$

$$\mathrm{insert}_x(a) = a \cup \{x\} \qquad \frac{}{\mathrm{insert}_x : \mathcal{P}(A) \xrightarrow{\sqsubseteq} \mathcal{P}(A)}$$

Sequential composition

$$(f \bullet g)(x) = f(g(x))$$

$$\frac{f : A \xrightarrow{\sqsubseteq} A \quad g : A \xrightarrow{\sqsubseteq} A}{f \bullet g : A \xrightarrow{\sqsubseteq} A}$$

$$\frac{f : A \xrightarrow{\sqsubseteq} A \quad g : A \xrightarrow{\sqsubset} A}{f \bullet g : A \xrightarrow{\sqsubset} A} \qquad \frac{f : A \xrightarrow{\sqsubset} A \quad g : A \xrightarrow{\sqsubseteq} A}{f \bullet g : A \xrightarrow{\sqsubset} A}$$

Products

$$(f \times g)(x, y) = (f(x), g(y))$$

$$\frac{f : A \xrightarrow{\sqsubseteq} A \quad g : B \xrightarrow{\sqsubseteq} B}{f \times g : A \times B \xrightarrow{\sqsubseteq} A \times B}$$

$$\frac{f : A \xrightarrow{\sqsubseteq} A \quad g : B \xrightarrow{\sqsubset} B}{f \times g : A \times B \xrightarrow{\sqsubset} A \times B} \qquad \frac{f : A \xrightarrow{\sqsubset} A \quad g : B \xrightarrow{\sqsubseteq} B}{f \times g : A \times B \xrightarrow{\sqsubset} A \times B}$$

Lexicographic products

$$(f \boxtimes g)(x, y) = (f(x), g(y))$$

$$\frac{f : A \xrightarrow{\sqsubseteq} A \quad g : B \xrightarrow{\sqsubseteq} B}{f \boxtimes g : A \boxtimes B \xrightarrow{\sqsubseteq} A \boxtimes B}$$

$$\frac{f : A \xrightarrow{\sqsubseteq} A \quad g : B \xrightarrow{\sqsubset} B}{f \boxtimes g : A \boxtimes B \xrightarrow{\sqsubset} A \boxtimes B} \qquad \frac{f : A \xrightarrow{\sqsubset} A \quad g : B \longrightarrow B}{f \boxtimes g : A \boxtimes B \xrightarrow{\sqsubset} A \boxtimes B}$$

Linear sum

$$\begin{aligned}(f \oplus g)(\mathsf{Left}\ x) &= \mathsf{Left}\ f(x)\\ (f \oplus g)(\mathsf{Right}\ x) &= \mathsf{Right}\ g(x)\end{aligned}$$

$$\frac{f : A \xrightarrow{\sqsubseteq} A \quad g : B \xrightarrow{\sqsubseteq} B}{f \oplus g : A \oplus B \xrightarrow{\sqsubseteq} A \oplus B}$$

$$\frac{f : A \xrightarrow{\sqsubset} A \quad g : B \xrightarrow{\sqsubset} B}{f \oplus g : A \oplus B \xrightarrow{\sqsubset} A \oplus B}$$

Maps

$$\mathrm{map}(f)(m) = \{(k, f(v)) \mid (k, v) \in m\}$$

$$\frac{f : V \xrightarrow{\sqsubseteq} V}{\mathrm{map}(f) : (K \hookrightarrow V) \xrightarrow{\sqsubseteq} (K \hookrightarrow V)}$$

$$\mathrm{apply}_k(f)(m) = \begin{cases} m\{k \mapsto f(v)\} & \textbf{if } (k, v) \in m \\ m\{k \mapsto f(\bot)\} & \textbf{otherwise} \end{cases}$$

$$\frac{f : V \xrightarrow{\sqsubseteq} V}{\mathrm{apply}_k(f) : (K \hookrightarrow V) \xrightarrow{\sqsubseteq} (K \hookrightarrow V)}$$

$$\frac{f : V \xrightarrow{\sqsubset} V}{\mathrm{apply}_k(f) : (K \hookrightarrow V) \xrightarrow{\sqsubset} (K \hookrightarrow V)}$$

P Counter

$$\text{PCounter}(I) = I \hookrightarrow \mathbb{N}$$

$$\begin{aligned} \text{inc}_i(a) &= \text{apply}_i(\text{succ})(a) \\ \text{value}(a) &= \sum \{v \mid (c, v) \in a\} \end{aligned}$$

OR Set - add wins

$$\mathrm{ORSet}^{+}(E, I) = E \hookrightarrow I \hookrightarrow \mathbb{N} \boxtimes \mathbb{B}$$

$$\begin{aligned}
\mathrm{add}_{e,i}(a) &= \mathrm{apply}_e(\mathrm{apply}_i(\mathrm{succ} \boxtimes \underline{\mathrm{False}}))(a) \\
\mathrm{rmv}_e(a) &= \mathrm{apply}_e(\mathrm{map}(\mathrm{id} \boxtimes \underline{\mathrm{True}}))(a)
\end{aligned}$$

$$\mathrm{member}_e(a) = \exists (e, m) \in a \cdot \exists i, n \cdot (n, \mathrm{False}) \in m(i)$$

OR Set - remove wins

$$\text{ORSet}^{-}(E, I) = E \hookrightarrow I \hookrightarrow \mathbb{N} \boxtimes \mathbb{B}$$

$$\begin{aligned}
\text{rmv}_{e,i}(a) &= \text{apply}_e(\text{apply}_i(\text{succ} \boxtimes \underline{\text{False}}))(a) \\
\text{add}_e(a) &= \text{apply}_e(\text{map}(\text{id} \boxtimes \underline{\text{True}}))(a)
\end{aligned}$$

$$\text{member}_e(a) = \exists (e, m) \in a \cdot \nexists i, n \cdot (n, \text{False}) \in m(i)$$

MV Register

$$\mathrm{MVReg}(V, I) = \mathcal{A}((I \hookrightarrow \mathbb{N}) \boxtimes V) \qquad V : \text{poset}$$

$$\begin{aligned}
\mathrm{assign}_{v,i}(a) &= \{\mathrm{apply}_i(\mathrm{succ})(\bigsqcup\{c \mid (c, v') \in a\}) \boxtimes v\} \\
\mathrm{values}(a) &= \{v \mid (c, v) \in a\}
\end{aligned}$$

MV Register with reconcile

$$\mathsf{MVRegisterR}(V, I) = \mathcal{A}((I \hookrightarrow \mathbb{N}) \boxtimes V) \qquad V : \text{lattice}$$

$$\begin{aligned}
\mathsf{assign}_{v,i}(a) &= \{\mathsf{apply}_i(\mathsf{succ})(\bigsqcup\{c \mid (c, v') \in a\}) \boxtimes v\} \\
\mathsf{reconcile}_i(a) &= \{\bigsqcup\{c \mid (c, v) \in a\} \boxtimes \bigsqcup\{v \mid (c, v) \in a\}\} \\
\mathsf{values}(a) &= \{v \mid (c, v) \in a\}
\end{aligned}$$

Discussion

- Now $\sqcup, \sqsubseteq$ are derived, inflation class also
- But no semantics captured. Though can help checking it
- Convergence related to known updates, not stopping updates: $\mathrm{upds}(a) \subseteq \mathrm{upds}(b) \Rightarrow a \sqsubseteq b$. But, does not imply correctness
- Correction often requires global invariants, e.g. id uniqueness

- Are these lattice compositions universal?
- Embedding the id in the state, looking relevant for protocols
- Scalability: Id management, GC.