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Favre Averaging
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Documentation

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I certify that this submission is my original work and meets the Faculty's Expectations of Originality

Foreword

This documentation will present the Favre averaging concept. It will be brief for application purposes, for the detailed derivation please consult the AGARD report by Knight [3]. Instead, this documentation serves as reference for the expanded equation of the **Production** and **Dissipation**.

Introduction

Currently, the industry standard is using RANS models to solve fluid simulation. The main advantage is faster runtime, albeit lower solution accuracy.

The procedure proposed by Reynolds in which all quantities are expressed as the sum of mean and fluctuating part (*Reynolds Decomposition*) contains an unknown, the *Reynolds Stress Tensor*, $\tau_{ij} = -\overline{u_i'u_j'}$ [4]. In search of additional equations to close the model, Chou proposed a time evolution of the Reynolds stress by taking the moment of the Navier-Stokes equation [1].

$$N(u_i) = \rho\frac{\partial u_i}{\partial t} + \rho u_k\frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} + -\mu\frac{\partial^2 u_i}{\partial x_k \partial x_k} \tag{1}$$

$$Reynolds\ Stress\ Tensor\ Equation = \overline{u_i'N(u_j) + u_j'N(u_i)} \tag{2}$$

Despite the effort, the procedure generated more than 20 new unknowns. Because of the known high nonlinearity of the Navier-Stokes equation, higher moments will generate additional unknowns [5].

On the bright side, this new equation can be used to express a turbulent kinetic transport equation. Which in turn is used in many eddy viscosity turbulent model. Among the unknown terms in the turbulent kinetic transport equation [3], the *Pressure Diffusion & Turbulent Diffusion* are small for simple flows. Therefore, in the case of a turbulent channel flow, we are interested in the *Production & Dissipation* terms.

Turbulent Kinetic Energy

As the reader already know, the equations are presented using the Einstein Convention. The turbulent kinetic energy equation is

$$\frac{\partial \bar{\rho} k}{\partial t} + \frac{\partial \bar{\rho} k \tilde{u}_i}{\partial x_i} = A + B + C + D$$

Term	*Description*
A	Production
B	Diffusion
C	Pressure
D	Dissipation

Table 1: Terms in Turbulence Kinetic Energy Equation

Production - *A*

Similarly to the *Reynolds averaging*. The Favre averaging or density-weighted average has a **mean** and **fluctuating** part. They are known as the *Favre mean*, $\tilde{f}$, & *Favre fluctuation*, f'', respectively.

Some useful rules:

$$\tilde{f} = \frac{\overline{\rho f}}{\bar{\rho}} \tag{3}$$

$$f = \tilde{f} + f'' \tag{4}$$

$$\bar{\tilde{f}} = \tilde{f} \tag{5}$$

$$\overline{\rho f''} = 0 \tag{6}$$

Production:

$$A = \underbrace{-\overline{\rho u_i'' u_j''}}_{\text{part 1}} \underbrace{\frac{\partial \tilde{u}_i}{\partial x_j}}_{\text{part 2}} \tag{7}$$

Part 1:

$$-\underbrace{\overline{\rho u_i'' u_j''}}_{\text{part 1}} = -\overline{\rho u_i u_j} + \overline{2\rho u_i}\tilde{u}_j - \bar{\rho}\tilde{u}_i\tilde{u}_j = -\overline{\rho u_i u_j} + 2\bar{u}_i\overline{\rho u_j} - \bar{\rho}\frac{\overline{\rho u_i}}{\bar{\rho}}\frac{\overline{\rho u_j}}{\bar{\rho}} \tag{8}$$

Part 2:

$$\underbrace{\frac{\partial \tilde{u}_i}{\partial x_j}}_{\text{part 2}} = \frac{\partial\left(\frac{\overline{\rho u_i}}{\bar{\rho}}\right)}{\partial x_j} \tag{9}$$

Apply quotient rule on the numerator:

$$\frac{\partial \tilde{u}_i}{\partial x_j} = \frac{\bar{\rho}\,\frac{\partial \overline{\rho u_i}}{\partial x_j} - \overline{\rho u_i}\,\frac{\partial \bar{\rho}}{\partial x_i}}{\bar{\rho}^{\,2}} \tag{10}$$

Apply the product rule on the first term:

$$\frac{\partial \tilde{u}_i}{\partial x_j} = \frac{\bar{\rho}\left[\,\overline{\frac{\partial \rho}{\partial x_j}u_i} + \overline{\rho\frac{\partial u_i}{\partial x_j}}\,\right] - \overline{\rho u_i}\,\frac{\partial \bar{\rho}}{\partial x_j}}{\bar{\rho}^{\,2}} \tag{11}$$

Finally, the production is:

$$\mathbf{A} = \left(-\overline{\rho \mathbf{u_i u_j}} + \mathbf{2\bar{u}_i}\overline{\rho \mathbf{u_j}} - \bar{\rho}\frac{\overline{\rho \mathbf{u_i}}}{\bar{\rho}}\frac{\overline{\rho \mathbf{u_j}}}{\bar{\rho}}\right) \times \frac{\bar{\rho}\left[\,\overline{\frac{\partial \rho}{\partial \mathbf{x_j}}\mathbf{u_i}} + \overline{\rho\frac{\partial \mathbf{u_i}}{\partial \mathbf{x_j}}}\,\right] - \overline{\rho \mathbf{u_i}}\,\frac{\partial \bar{\rho}}{\partial \mathbf{x_j}}}{\bar{\rho}^{\,\mathbf{2}}} \tag{12}$$

Dissipation - *D*

First, the laminar stress tensor must be defined and not to be confused with the mean molecular viscous stress, $\bar{\tau}_{ij}$.

$$\tau_{ij} = \lambda \frac{\partial u_k}{x_k}\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{13}$$

Dissipation:

$$D = \overline{\tau_{ij}\frac{\partial u_i''}{\partial x_j}} = \overline{\left(\underbrace{\lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}}_{\text{part 1}} + \mu(\underbrace{\frac{\partial u_i}{\partial x_j}}_{\text{part 2}} + \underbrace{\frac{\partial u_j}{\partial x_i}}_{\text{part 3}})\right)\frac{\partial u_i''}{\partial x_j}} \tag{14}$$

Note: the δ_{ij} is the Kronecker delta, and λ is the Stokes Law, $\lambda = -\frac{2}{3}\mu$.

Before starting part 1, we can use the Favre definition to simplify the unknown derivative - last term in (14).

$$\frac{\partial u_i''}{\partial x_j} = \left(\frac{\partial u_i}{\partial x_j} - \underbrace{\frac{\partial \tilde{u}_i}{\partial x_j}}_{(4)}\right) \tag{15}$$

Part 1:

$$\overline{\left(\lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\frac{\partial u_i''}{\partial x_j}} = \overline{\left(-\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial \tilde{u}_i}{\partial x_j}\right)} \tag{16}$$

Further simplification:

$$\overline{\left(\lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\frac{\partial u_i''}{\partial x_j}} = -\overline{\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij}\frac{\partial u_i}{\partial x_j}} + \overline{\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij}}\frac{\partial \tilde{u}_i}{\partial x_j} \tag{17}$$

Part 2:

$$\overline{\mu\frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial \tilde{u}_i}{\partial x_j}\right)} = \mu\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}} - \mu\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial \tilde{u}_i}{\partial x_j}} \tag{18}$$

Further simplification:

$$\overline{\mu\frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial \tilde{u}_i}{\partial x_j}\right)} = \mu\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}} - \mu\frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \tilde{u}_i}{\partial x_j} \tag{19}$$

Part 3 is similar to Part 2:

$$\overline{\mu\frac{\partial u_j}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial \tilde{u}_i}{\partial x_j}\right)} = \mu\overline{\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}} - \mu\frac{\partial \bar{u}_j}{\partial x_i}\frac{\partial \tilde{u}_i}{\partial x_j} \tag{20}$$

Finally, the dissipation is:

$$\mathbf{D} = -\overline{\frac{\mathbf{2}}{\mathbf{3}}\mu\frac{\partial \mathbf{u_k}}{\partial \mathbf{x_k}}\delta_{\mathbf{ij}}\frac{\partial \mathbf{u_i}}{\partial \mathbf{x_j}}} + \overline{\frac{\mathbf{2}}{\mathbf{3}}\mu\frac{\partial \mathbf{u_k}}{\partial \mathbf{x_k}}\delta_{\mathbf{ij}}}\frac{\partial \tilde{\mathbf{u}}_\mathbf{i}}{\partial \mathbf{x_j}} + \mu\overline{\frac{\partial \mathbf{u_i}}{\partial \mathbf{x_j}}\frac{\partial \mathbf{u_i}}{\partial \mathbf{x_j}}} - \mu\frac{\partial \bar{\mathbf{u}}_\mathbf{i}}{\partial \mathbf{x_j}}\frac{\partial \tilde{\mathbf{u}}_\mathbf{i}}{\partial \mathbf{x_j}} + \mu\overline{\frac{\partial \mathbf{u_j}}{\partial \mathbf{x_i}}\frac{\partial \mathbf{u_i}}{\partial \mathbf{x_j}}} - \mu\frac{\partial \bar{\mathbf{u}}_\mathbf{j}}{\partial \mathbf{x_i}}\frac{\partial \tilde{\mathbf{u}}_\mathbf{i}}{\partial \mathbf{x_j}}$$

Nondimensionalize the *P* & *D*

To validate the work, we have to nondimensionalize the data as per K.M.M. [2].

Starting with equation [7],

$$A = -\overline{\rho u_i'' u_j''}\frac{\partial \tilde{u}_i}{\partial x_j}\left[\frac{kg}{m \cdot s^3}\right] \tag{21}$$

Similarly for Dissipation with equation [14],

$$D = \overline{\tau_{ij}\frac{\partial u_i''}{\partial x_j}}\left[\frac{kg}{m \cdot s^3}\right] \tag{22}$$

From K.M.M. documentation, they used the kinematic viscosity over the frictional velocity, μ/u_τ. The additional term to take in consideration here is the compressibility or the variation in density over time. The nondimentionalizing factor (NF) has been found to be the following,

$$NF = \frac{\nu}{\rho u_\tau^4} = \frac{\mu}{\rho^2 u_\tau^4}\left[\frac{m \cdot s^3}{Kg}\right] \tag{23}$$

Turbulence Model

- Zero-Equation Models (Algebraic) ⟶Mixing Length
- One-Equation Models ⟶Spalart-Allmaras turbulence model
- Two-Equation Models ⟶$k - \epsilon,\ k - \omega$
- Stress-Transport Models ⟶Transition SST

References

[1] P. Y. Chou. On velocity correlations and the solutions of the equations of turbulent fluctuation. page 16, 1945.

[2] John Kim, Parviz Moin, and Robert Moser. The turbulence statistics in fully developed channel flow at low reynolds number. *Journal of Fluid Mechanics*, 177, 05 1987.

[3] Doyle D. Knight. Numerical simulation of compressible turbulent flows using the reynolds-averaged navier-stokes equations. *AGARD FDP Special Course, R-819*, page 52, 06 1997.

[4] Osborne Reynolds. Iv. on the dynamical theory of incompressible viscous fluids and the determination of the criterion. *Royal Society*, page 42, 01 1895.

[5] D.C. Wilcox. *Turbulence Modeling for CFD*. Number v. 1 in Turbulence Modeling for CFD. DCW Industries, 2006.