Contents

Chapter 1
Topology

The notion of topology gives sense to the intuitive ideas of nearness and continuity. It appears that there are equivalent ways of defining a topology: In terms of open sets, or of closed sets or using as primitive notion the notion of neighbourhood of a point.

1.1 Preliminaries

1.1 De Morgan's laws: Let $A, B \subset X$. Then

$$C_X (A \cup B) = C_X A \cap C_X B$$

$$C_X (A \cap B) = C_X A \cup C_X B$$

where $C_X A = \{x \in X | x \notin A\}$ denotes the complement of A.

Proof: To proof that $C_X (A \cup B) = C_X A \cap C_X B$ is completed in 2 steps by proving both $C_X (A \cup B) \subseteq C_X A \cap C_X B$ and $C_X (A \cup B) \supseteq C_X A \cap C_X B$.

Part 1: Let $x \in C_X(A \cap B)$, then $x \notin A \cap B$. Because $A \cap B = \{y | y \in A \wedge y \in B\}$, it must be the case that $x \notin A$ or $x \notin B$. If $x \notin A$, then $x \in C_X A$, so $x \in C_X A \cup C_X B$. Similary, if $x \notin B$, then $x \in C_X B$, so $x \in C_X A \cup C_X B$. Hence, $\forall x : x \in C_X(A \cap B) \Rightarrow x \in C_X A \cup C_X B$, that is, $C_X (A \cup B) \subseteq C_X A \cap C_X B$.

Part 2: To prove the reverse direction, let $x \in C_X A \cup C_X B$, and for contradiction assume $x \notin C_X(A \cap B)$. Under that assumption, it must be the case that $x \in A \cap B$, so it follows that $x \in A$ and $x \in B$, and thus $x \notin C_X A$ and $x \notin C_X B$. However, that means $x \notin C_X A \cup C_X B$, in contradiction to the hypothesis that $x \in C_X A \cup C_X B$ therefore, the assumption $x \notin C_X(A \cap B)$ must not be the case, meaning that $x \in C_X(A \cap B)$. Thus, $\forall x : x \in C_X A \cup C_X B \Rightarrow x \in C_X(A \cap B)$, that is, $C_X (A \cup B) \supseteq C_X A \cap C_X B$.

□

1.2 Topological spaces

1.2 Definition: A **topological space** is a non-empty set X together with a family $\tau = (U_i \mid \forall i \in I, U_i \subset X)$ ("of subsets of X") satisfying the following axioms:

1. $\emptyset, X \in \{\tau\}$
2. The intersection of any finite number of sets in τ belongs to τ, i.e.

$$J \text{ finite }, J \subset I \Rightarrow \bigcap_{i \in J} U_i \in \tau$$

3. The union of any number of sets in τ belongs to τ, i.e.

$$J \subset I \Rightarrow \bigcup_{i \in J} U_i \in \tau$$

The elements of τ are called τ-open sets, or simply open sets in X. The pair (X, τ) is called a topological space.

1.3 Example: The family $\{\tau\} = \{\emptyset, X\}$, consisting of $\emptyset$ and X alone is itself a topology called the **indiscrete topology**. $\{\emptyset, X\}$ is then called an indiscrete topological space.

1.4 Example: Let $\tau = P(X)$ denote the family of all subsets of X (power set). Observe that $P(X)$ satisfies the axioms 1.-3. for a topology on X. This topology is called the **discrete topology**, the pair $\{X, P(X)\}$ is called a discrete topological space.

1.5 Example: Let $X = \mathbb{R}$ be the real line. A topology on $\mathbb{R}$ can be defined as follows: For any $x \in \mathbb{R}$, consider the open intervals (a, b) containing x, then τ is the family

$$\tau = \{U_i = (a_i, b_i) \mid a_i, b_i \in \mathbb{R}\,, a_i \leq b_i\}$$

If $a_i = b_i$, then $(a_i, b_i) = \emptyset$. This topology is referred as the **usual topology** on $\mathbb{R}$. Similarly, one can define the usual topology on $\mathbb{R}^n$.

1.6 Definition: Let (X, τ) be a topological space. A subset A of X is **closed** if its complement $C_X A$ is an open set.

1.7 Theorem: The family $\overline{\tau} = (A_i \mid \forall i \in I)$ of closed subsets of X satisfying the following conditions:

1. $\emptyset$ and X are closed sets, i.e. $\emptyset, X \in \overline{\tau}$
2. The union of any finite number of sets in $\overline{\tau}$ belongs to $\overline{\tau}$, i.e.

$$J \text{ finite}\,, J \subset I \Rightarrow \bigcup_{i \in J} U_i \in \overline{\tau}$$

3. The intersection of any number of sets in $\overline{\tau}$ belongs to $\overline{\tau}$, i.e.

$$J \subset I \Rightarrow \bigcap_{i \in J} U_i \in \overline{\tau}$$

We denote this topological place $(X, \overline{\tau})$.

Proof: Follows from De'Morgan laws and definition of the topological space of the open sets.

□

1.3 Neighbourhood spaces

1.8 Definition: Let $x \in X$ be a point in a topological space X. Any subset U of X containing an open set A such that $x \in A$ is called a **neighbourhood** of x denoted by $U = U(x)$. In particular, any open set U is a neighbourhood of each of its points. The class of all neighbourhoods of $x \in X$, denoted by $\mathfrak{B}(x)$, is called the **fundamental neighbourhood system** of x.

1.4 Types of points

1.9 Definition: Let (X, τ) be a topological space and $A \subset X$. A point $x \in A$ is said to be an **isolated point** of A if there exists an open set containing x which contains no other points of A.

1.10 Definition: Let (X, τ) be a topological space and $A \subset X$. A point $x \in A$ is said to be an **accumulation point** of A if every open set containing x contains at least one other point from A. (Basically the opposite to the isolated point of A).

1.11 Definition: Let $(X, \{\tau\})$ be a topological space and $A \subset X$. A point $x \in X$ is said to be an **adherent point** of A if every neighbourhood of x contains at least one other point of A.

1.12 Definition: Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be a **limit point** of A if every neighbourhood of x contains infinitely many points of A.

1.13 Definition: Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be a **condensation point** of A if every neighbourhood of x contains uncountable1 many points of A.

1.14 Definition: Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be a **interior point** of A if there exists an open set U containing x that is completely a subset of A.

1.15 Definition: Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be a **boundary point** of A if every neighbourhood of x contains at least one point of A and at least one point of $C_X A$.

1.16 Remark: Isolated and accumulation points of A are always in A, while adherent, limit and condensation points don't have to be in A. If a point is isolated it can't be accumulated and vice versa. Every interior point of A is also a condensation point of A, every condensation point of A is also a limit point of A, and every limit point of A is also an adherent point of A. Isolated and accumulation points are both adherent but can't be limit or condensation points.

1.17 Theorem: Union of set of interior points and boundary points is a set of adherent points.

1.18 Example: The set $\mathbb{N}$ in usual topology on $\mathbb{R}$ has no accumulation point, i.e. all points of $\mathbb{N}$ are isolated.

1.19 Example: The set $A = (0, 1] \cup \{2\} \subset \mathbb{R}$ which has the usual topology on $\mathbb{R}$ has limit point every point of the interval $[0, 1]$. Notice that 2 is an isolated nonlimit point and 0 is a limit point but does not belongs to A.

1.20 Definition: The set of all limit points of A, denoted by A' is called the **derived set** of A.

1.21 Example: Discrete topology where adherent points are not limit points.

1.5 Closure

1.22 Definition: Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be an **adherent point** of A if every neighbourhood of x contains at least one other point of A.

1.23 Definition: Let (X, τ) be a topological space and $A \subset X$. The set of all adherent points of A, denoted by $\overline{A}$, is called a **closure** of A.

1.24 Theorem: Let (X, τ) be a topological space and $A \subset X$. The closure of a set A is the intersection of all closed supersets of A and $\overline{A}$ is the smallest closed superset of A.

Proof: Let $(U|i \in I)$ be the family of all closed supersets of A. If $x \in \overline{A}$, then x is adherent point and belongs to a closed superset of A, i.e. $\exists i_0 : x \in U_{i_0}$. Hence $x \in \bigcap_i U_i$ and $\overline{A} \subset \bigcap_i U_i$. Conversely, $y \in \bigcap_i U_i$, implies $y \in U_i$ for every i. Thus y is an adherent point, i.e $y \in \overline{A}$ and if we take all such y we have $\bigcap_i U_i \subset \overline{A}$. Accordingly $\overline{A} = \bigcap_i U_i$, while not forgetting axiom: intersection of closed sets is a closed set, hence $\overline{A}$ is closed. If U is a closed superset of A, it is in the family $(U|i \in I)$ and because $A = \bigcap_i U_i$ we have $\overline{A} \subset U$.

□

1.25 Theorem: Let (X, τ) be a topological space and $A \subset X$. Every point of a closed set A is its adherent point, i.e $A = \overline{A}$.

Proof: Because $\overline{A}$ is the intersection of all closed supersets of A we have $A \subset \overline{A}$. But if A is closed and $\overline{A}$ is the smallest closed superset of A then $\overline{A} \subset A$. Therefore $A = \overline{A}$.

□

1.26 Example: The empty set $\emptyset$ is closed, since there is no point in which is not an accumulation point. And as every accumulation point is also adherent we have $\overline{\emptyset} = \emptyset$.

1.6 Interior

1.27 Definition: Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be a **interior point** of A if there exists an open set U containing x that is completely a subset of A.

1.28 Definition: Let (X, τ) be a topological space and $A \subset X$. The set of all interior points of A, denoted by $\mathring{A}$, is called an **interior** of A.

1.29 Theorem: Let (X, τ) be a topological space and $A \subset X$. The interior of a set A is the union of all open subsets of A and $\mathring{A}$ is the largest open subset of A.

Proof: Let $(U|i \in I)$ be the family of all open subsets of A. If $x \in \mathring{A}$, then x is interior point and belongs to an open subset of A, i.e. $\exists i_0 : x \in U_{i_0}$. Hence $x \in \bigcup_i U_i$ and $\mathring{A} \subset \bigcup_i U_i$. Conversely, $y \in \bigcup_i U_i$, implies $y \in U_i$ for some i. Thus y is an interior point, i.e $y \in \mathring{A}$ and if we take all such y we have $\bigcup_i U_i \subset \mathring{A}$. Accordingly $\mathring{A} = \bigcup_i U_i$, while not forgetting axiom: uion of open sets is an open set, hence $\mathring{A}$ is open. If U is an open subset of A, it is in the family $(U|i \in I)$ and because $\mathring{A} = \bigcup_i U_i$ we have $U \subset \mathring{A}$.

□

1.30 **Theorem:** Let (X, τ) be a topological space and $A \subset X$. Every point of an open set A is its interior point, i.e $A = \mathring{A}$.

Proof: Because $\mathring{A}$ is the union of all open subsets of A we have $\mathring{A} \subset A$. But if A is open and $\mathring{A}$ is the largest open subset of A then $A \subset \mathring{A}$. Therefore $A = \mathring{A}$.

□

1.7 Boundary

1.31 **Definition:** Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be a **boundary point** of A if every neighbourhood of x contains at least one point of A and at least one point of $C_X A$.

1.32 **Definition:** Let (X, τ) be a topological space and $A \subset X$. The set of all boundary points of A, denoted by ∂A, is called a **boundary** of A.

1.33 **Theorem:** Let (X, τ) be a topological space and $A \subset X$. The set A is said to be closed subset of $X \Leftrightarrow A$ contains all of its boundary points.

Proof: First assume A is closed. Assume, for a contradiction, that there is $x \in \partial A$ such that $x \notin A$. Then $x \in C_X A$ which is open so there exists an open set U containing x that is entirely in $C_X A$. Then this set contains no points in A contradicting the definition of the boundary point x.

Now assume that all x from ∂A are also in A. Assume, for a contradiction, that A is open. If A is open, then for every point of A exists an open set U eentirely in A. But then there is no point satisfying the definition of boundary point and we arrive to contradiction.

□

1.34 **Theorem:** Let (X, τ) be a topological space and $A \subset X$. The set A is said to be open subset of $X \Leftrightarrow A$ does not contain any of its boundary points.

Proof: If A is open, then for every point of A exists an open set U eentirely in A. But then there is no point satisfying the definition of boundary point, hence A does not contain any of its boundary points.

Now assume that A does not contain any of its boundary points. Assume, for a contradiction, that A is closed. But we already proved that closed set contains all of its boundary points, therefore we arrive to contradiction.

□

1.35 **Theorem:** Let $(X, \{\tau\})$ be a topological space and $A \subset X$. The boundary of a set A is given by $\partial A = \overline{A} \backslash \mathring{A}$. Furthermore $\overline{A} = \mathring{A} \cup \partial A$ and $A = \partial A \Leftrightarrow \mathring{A} = \emptyset$.

1.36 **Example:** Consider the usual topology on $\mathbb{R}$ and the set $\mathbb{Q}$ of rational numbers. Every real number $x \in \mathbb{R}$ is an adherent point of $\mathbb{Q}$, WHY?. Moreover, $\mathring{\mathbb{Q}} = (C_{\mathbb{R}}\mathbb{Q})^{\circ} = \emptyset$, since every open subset of $\mathbb{R}$ contains both rational and irrational points, hence there are no interior points of $\mathbb{Q}$ and $C_{\mathbb{R}}\mathbb{Q}$, furhtemore $\partial\mathbb{Q} = \overline{\mathbb{Q}} \backslash \mathring{\mathbb{Q}} = \mathbb{R}$.

1.8 Continuous maps

1.37 **Definition:** Let (X_i, τ_i), $i = 1, 2$, be topological spaces. A map $f : X_1 \longrightarrow X_2$ is **continuous** at a point $x_0 \in X_1$ if for any neighborhood $V(f(x_0))$ of $f(x_0)$ there exists a neighborhood $U(x_0)$ of x_0 such that $f(U) \subset V$, i.e.

$$f \text{ is continuous at } x_0 \in X_1 \Leftrightarrow \forall V(f(x_0))\, \exists U(x_0) \;:\; f(U(x_0)) \subset V(x_0).$$

1.38 **Definition:** Let (X_i, τ_i), $i = 1, 2$, be topological spaces. A map $f : X_1 \longrightarrow X_2$ is called an **open map** if, for any open set $U \subset X_1$, $f(U)$ is an open set in X_2.

1.39 Definition: Let $(X_i, \{\tau\}_i)$, $i = 1,2$, be topological spaces and $f : X_1 \to X_2$ a map. Let $U \in X_2$ and denote $f^{-1}(U) = \{x \in X_1 | f(x) \in U\}$.

1.40 Definition: Two topological spaces $(X_i, \{\tau\}_i)$, $i = 1,2$, are called **homeomorphic** if there exists a bijective map $f : X_1 \to X_2$ such that f and f^{-1} are continuous. The map f is called a **homeomorphism**.

1.9 Properties of topological spaces

1.41 Definition: Let (X, τ) be a topological space. (X, τ) is a **Hausdorff space** if it satisfies the following additional axiom: For every pair of distinct points $x_1, x_2 \in X$ there are disjoint neighborhoods $U_1(x_1), U_2(x_2)$, i.e:

$$\forall x_1, x_2 \in X, x_1 \neq x_2,\ \exists U_1(x_1), U_2(x_2)\ :\ U_1(x_1) \cap U_2(x_2) = \emptyset.$$

1.42 Definition: Let (X, τ) be a topological space. A **base** for the topology τ of a topological space (X, τ) is a family $\mathfrak{B}$ of open subsets of X such that every open set is equal to a union of sets from $\mathfrak{B}$.

1.43 Definition: A topological space (X, τ) is called **second countable**, if there exist a countable base $\mathfrak{B}$ for the topology τ.

1.44 Definition: Let (X, τ) be a topological space. A family $\mathfrak{U} = (A_i | i \in I)$ of subsets of X is called a **cover** of X, if

$$\forall i \in I : A_i \neq \emptyset$$

$$X = \bigcup_{i \in I} A_i$$

If, for $J \subset I$, $X = \bigcup_{j \in J} A_j$, then $(A_j | j \in J)$ is called a **subcover**. In particular it is a **finite subcover** if the index set J is finite.

1.45 Definition: A topological space (X, τ) is **compact**, if every open cover of X has a finite subcover.

1.46 Definition: Let (X, τ) be a topological space. A set $S \subset X$ is **compact**, if every open cover of S has a finite subcover.

1.47 Note: Some authors require that a compact topological space be Hausdorff as well, and use the term quasi-compact to refer to a non-Hausdorff compact space. The modern convention seems to be to use compact in the sense given here.

1.48 A compact subset of a Hausdorff space is closed: Suppose X is a Hausdorff space. If K is a compact subset of X, then K is a compact set in X.

Proof: Let X be a Hausdorff space, and $S \subset X$ a compact subset. To show that S is closed, its enough to show that the complement $U = C_X S = X \backslash S$ is open. For U to be open is sufficient to demonstrate that for each $x \in U$, there exists an open set V with $x \in V$ and $V \subset U$.

Fix $x \in U$. For each $y \in S$, using the definition of Hausdorff space, we can choose disjoint open sets A and B with $x \in A(y)$ and $y \in B(y)$.

Since every $y \in S$ is an element of $B(y)$, the family $\{B(y) \mid y \in S\}$ is an open cover of S. Since S is compact, this open cover has a finite subcover. Therefore we can choose $y_1, , y_n \in S$ such that $S \subset B(y_1) \cup \cdots \cup B(y_n)$.

Notice that $A(y_1) \cap \cdots \cap A(y_n)$, being a finite intersection of open sets, is open, and contains x. Call this neighborhood of x by the name V. All we need to do is show that $V \subset U$.

For any point $z \in S$, we have $z \in B(y_1) \cup \cdots \cup B(y_n)$, and therefore $z \in B(y_k)$ for some k. Since $A(y_k)$ and $B(y_k)$ are disjoint, $z \notin A(y_k)$, and therefore $z \notin A(y_1) \cap \cdots \cap A(y_n) = V$. Thus S is disjoint from V, and V is contained in U.

□

1.49 Remark: This theorem is giving us closed intervals as we know them in $\mathbb{R}$ to use in all Hausdorff spaces. But this time closed means compact.

1.50 Definition: Lt $\{X, \tau\}$ be a topological space. A family $A = \{A_i\}_{i \in I}$ of subsets of a set X is said to have the **finite intersection property**, if every finite subfamily $\{A_1, A_2, \ldots . A_n\}$ of A satisifes $\bigcap_{i=1}^{n} A_i \neq \emptyset$.

1.51 Example: Consider $\mathbb{R}$ as a topological space with usual topology, then the family of closed intervals $A = \left\{ \left[n + \frac{1}{n}, n + 1 - \frac{1}{n}\right] \,\middle|\, n \in \mathbb{N} \right\}$ does not have the finite intersection property, howerer for $n \longrightarrow \infty$ the intervals will touch, hence intersect.

1.52 Theorem: A topological space $\{X, \tau\}$ is compact if and only if for every family $A = \{A_i\}_{i \in I}$ of closed subsets of X, where A has the finite intersection property, $\bigcap_{i=1}^{n} A_i = \emptyset$.

Proof: Let $\{X, \tau\}$ be a topological space and assume that the intersection of any family of closed subsets having the finite intersection property is non-empty. Let $\{F_i\}_{i \in I}$ be some open cover of X and assume that it has no finite subcover. Hence for each finite $J \subseteq I$ there is some $x \in X$ such that $x \notin \bigcup_{i \in J} F_i$. Equivalently, for each finite $J \subseteq I$ there is some $x \in \bigcap_{i \in J} C_X F_i$. Hence the family $\{C_X F_i\}_{i \in I}$ is a family of closed subsets with the finite intersection property. It follows that there is some $x \in \bigcap_{i \in I} C_X F_i$, i.e., $x \notin \bigcup_{i \in I} F_i$, contradicting that $\{F_i\}_{i \in I}$ is a cover of X. We conclude that $\{F_i\}_{i \in I}$ must have a finite subcover, so X is compact.

The proof in the other direction is analogous. Conversely, assume that X is compact and let $\{F_i\}_{i \in I}$ be some family of closed subsets with the finite intersection property. Assume that $\bigcap_{i \in I} F_i$ is empty. Then $\{C_X F_i\}_{i \in I}$ is an open cover of X. By compactness, it has a finite subcover, so $\bigcup_{i \in J} C_X F_i = X$ for some finite $J \subseteq I$. But then $\bigcup_{i \in J} F_i = \emptyset$, contradicting that $\{F_i\}_{i \in I}$ has the finite intersection property. We conclude that $\bigcap_{i \in I} F_i \emptyset$, i.e .

$\square$

1.53 Remark: One can understnad this theorem as every family A has to have also the "infinite" intersection property.

1.54 Remark: The above theorem is essentially the definition of a compact space rewritten using de Morgan's laws. The usual definition of a compact space is based on open sets and unions. The above characterization, on the other hand, is written using closed sets and intersections.

1.55 Example: Consider $\mathbb{R}$ as a topological space with usual topology, then the family of open intervals $A = \left\{ \left(n - \frac{1}{n}, n + 1 + \frac{1}{n}\right) \,\middle|\, n \in \mathbb{N} \right\}$ has the finite intersection property, howerer for $n \longrightarrow \infty$ the intervals will stop intersecting, hence $\mathbb{R}$ is not compact.

1.56 A closed subset in a compact space is compact: Suppose $\{X, \tau\}$ is a topological space. Let $C \subset K \subset X$ where C is closed relative to K and K is compact. Then C is compact.

Proof: Suppose C is not compact. Then there is a cover of C which has no finite subcover say, call this cover S. Define $M = X \backslash C$. Then $M \cup S$ is a cover of K. But then clearly, this cover has no finite subcover hence K is not compact.

$\square$

1.57 Definition: Let $\{\tau, X\}$ be a topological space. Space X is called **locally compact** if every point x of X has a compact neighbourhood, i.e., there exists an open set $U \subset X$ and a compact set $K \subset X$, such that $x \in U \subseteq K$.

1.58 Definition: Let $\{\tau, X\}$ be a topological space. A **refinement** of a cover of a space X is a new cover of the same space such that every set in the new cover is a subset of some set in the old cover. I.e., the cover $V = \{V_\beta : \beta \in B\}$ is a refinement of the cover $U = \{U_\alpha : \alpha \in A\}$ if and only if, for any V_β in V, there exists some U_α in U such that $V_\beta \subseteq U_\alpha$.

1.59 Definition: Let $\{\tau, X\}$ be a topological space. An open cover of a space X is **locally finite** if every point of the space has a neighborhood that intersects only finitely many sets in the cover. I.e., $U = \{U_\alpha : \alpha \in A\}$ is locally finite if and only if, for any $x \in X$, there exists some neighbourhood $V(x)$ of x such that the set

$$\{\alpha \in A : U_\alpha \cap V(x) \neq \emptyset\}$$

is finite.

1.60 Definition: A topological space $\{\tau, X\}$ is said to be **paracompact** if every open cover has a locally finite open refinement.

1.61 **Definition:** Let $\{\tau, X\}$ be a topological space. X is said to be **disconnected** if it is the union of at least two disjoint non-empty open sets. Otherwise, X is said to be **connected**.

1.62 **Theorem:** A topological space $\{\tau, X\}$ is connected if the only subsets of X which are both open and closed are $\emptyset$ and X.

Proof: The proof is done by contradiction. Let $A \subset X$ be both open and closed, but $A \neq X$ and $A \neq \emptyset$. Then as A is open, there exists a set B such that $B \notin \tau$ (B is closed) and $A \cup B = X$. Hence $B = X \backslash A$. However as A is also closed, there exists a set C such that $C \in \tau$ (C is opne) and $A \cup C = X$. Therefore $C = X \backslash A$. One can see that $B = C$ and we arrive at contradiction $B \notin \tau$ while $C \in \tau$.

□

1.63 **Definition:** Let $\{\tau, X\}$ be a topological space. The union A of all connected subsets of X containing a point $x \in X$ is a connected subset of X. Then, A is called the **maximal connected** subset containing x (or **componet** of X containing x), denoted by $C(x)$.

1.64 **Definition:** Let $\{\tau, X\}$ be a topological space. A **path** from a point $x \in X$ to a point $y \in X$ in a topological space X is a continuous function f from the unit interval $[0, 1]$ to X with $f(0) = x$ and $f(1) = y$.

1.65 **Definition:** Let $\{\tau, X\}$ be a topological space. A **path-component** of X is an equivalence class of X under the equivalence relation which makes x equivalent to y if there is a path from x to y.

1.66 **Definition:** Let $\{\tau, X\}$ be a topological space. The space X is said to be **path-connected** (or pathwise connected or 0-connected) if there is exactly one path-component, i.e. if there is a path joining any two points in X.

1.67 **Definition:** Let $\{\tau, X\}$ be a topological space. A space X is said to be **arc-connected** or arcwise connected if any two distinct points can be joined by an arc, which by definition is a path $f : [0, 1] \longrightarrow X$ that is also a topological embedding. Explicitly, a path $f : [0, 1] \longrightarrow X$ is called an arc if the surjective map $f : [0, 1] \longrightarrow \operatorname{Im} f$ is a homeomorphism, where its image $\operatorname{Im} f := f([0, 1])$ is endowed with the subspace topology induced on it by X.

1.68 **Example:** Let X be a finite set with topology $\{\tau, X\}$ that is path-connected. However such space is not arc-connected because there can't be a homeomorphism from an uncountable interval $[0, 1]$ to a finite set X.

1.69 **Definition:** Let $\{\tau, X\}$ be a topological space. A space is called **totally disconnected** if its only non-empty connected subsets consists of single points, i.e. $C(x) = \{x\}$.

1.70 **Theorem:** Let $\{\tau_i, X_i\}$, $i = 1, 2$ be topological space and let $f : X_1 \longrightarrow X_2$ be a continuous map. If $A \subseteq X_1$ and A is connected, then $f(A)$ is connected.

Proof:

□

1.71 **Theorem:** Let $\{\tau_i, X_i\}$, $i = 1, 2$ be topological space and let $f : X_1 \longrightarrow X_2$ be a continuous map. If $A \subseteq X_1$ and A is compact, then $f(A)$ is compact.

Proof:

□

1.72 **Theorem:** Let $\{\tau_i, X_i\}$, $i = 1, 2, 3$ be topological space and let $f : X_1 \longrightarrow X_2$ and $g : X_2 \longrightarrow X_3$ be continuous maps. Then the map $g \circ f : X_1 \longrightarrow X_3$ is continuous.

Proof:

□

1.73 **Theorem:** If a topological space $\{\tau, X\}$ is path-connected, then it is also connected.

Proof: We do this proof by contradiction. Suppose X is not connected. Then, there exist nonempty disjoint open subsets $U, V \subseteq X$ such that $X = U \cup V$. Pick a point $x \in U$ and a point $y \in V$.

By assumption, there exists a continuous function $f : [0, 1] \longrightarrow X$ such that $f(0) = x$ and $f(1) = y$. Consider the subsets $f^{-1}(U)$ and $f^{-1}(V)$. These are disjoint in $[0, 1]$ and their union is $[0, 1]$. By the continuity of f, they are both open in

$[0, 1]$. Finally, since $0 \in f^{-1}(U)$ and $1 \in f^{-1}(V)$, they are both nonempty. We have thus expressed $[0, 1]$ as a union of two disjoint nonempty open subsets, a contradiction to the fact that $[0, 1]$ is connected. This completes the proof.

$\square$

simply connected