Contents

0.1 Required knowledge

1. Inner product $(\cdot, \cdot)$, norm $||\cdot||$.
2. Hilbert space H, Banach space.

Chapter 1
Abstract Fourier series

1.1 Complete Orthogonal System

1.1 Definition: Let H be a Hilbert space equiped with norm $||\cdot||_H = \sqrt{(\cdot,\cdot)_H}$ and $\{\Phi_\alpha\}_{\alpha\in I}$ be a system of it's elements where I is a set of indicies.

- We say that $\{\Phi_\alpha\}_{\alpha\in I}$ is an *orthogonal system* if $\forall\alpha \in I : \Phi_\alpha \neq 0$ and $\forall \alpha, \beta,\ \alpha \neq \beta : \left(\Phi_\alpha, \Phi_\beta\right)_H = 0$.
- We say that $\{\Phi_\alpha\}_{\alpha\in I}$ is an *orthonormal system* if it's an orthogonal system and $\forall\alpha : (\Phi_\alpha, \Phi_\alpha)_H = 1$.
- We say that $\{\Phi_\alpha\}_{\alpha\in I}$ is an *complete system* if $\forall\alpha \in I, \forall\Phi \in H : (\Phi, \Phi_\alpha)_H = 0 \Leftrightarrow \Phi = 0$.

1.2 Definition: A subset A of a normed topological space $(X, ||\cdot||_X)$ is called *dense* in X if $\forall\epsilon > 0,\ \forall x \in X,\ \forall y \in A :$ $||y - x|| \leq \epsilon$.

1.3 Definition: We say that $(X, ||\cdot||_X)$ is a *separable set* if $\exists A \subset X$, where A countable and dense in X.

1.4 Definition: We say that Hilbert or Banach space H is a *separable* if $\exists\{\Phi_n\}_{n=1}^\infty, \forall\epsilon > 0, \forall\Phi \in H, \exists\Phi_n : ||\Phi-\Phi_n||_H < \epsilon$, where $\exists\{\Phi_n\}_{n=1}^\infty$ is a counatble system in X.

1.2 Abstract Fourier series

1.5 Theorem (About the best approximation): Let H be a Hilbert space, $\{\Phi_k\}_{k=1}^\infty \subset H$ be an orthonormal system, $f \in H$, $\{c_k\}_{k=1}^\infty := \{(f, \Phi_k)_H\}_{k=1}^\infty$ and $\{a_k\}_{k=1}^\infty \subset \mathbb{R}$. $\forall n \in \mathbb{N}$ let $s_n := \sum_{k=1}^n c_k\Phi_k$ and $t_n := \sum_{k=1}^n a_k\Phi_k$. Then

1. $\forall n \in N : ||f - s_n||_H \leq ||f - t_n||_H$,
2. For given $n \in N : ||f - s_n||_H \leq ||f - t_n||_H \Leftrightarrow \forall k \in \{1, \dots, n\} : a_k = c_k$,
3. $\forall n \in N : ||f - s_n||_H \leq ||f - t_n||_H \Leftrightarrow \forall k \in \mathbb{N} : a_k = c_k$.

Proof:

$$\begin{aligned}
||f - t_n||_H^2 &= \left(f - \sum_{k=1}^n a_k\Phi_k, f - \sum_{k=1}^n a_k\Phi_k\right)_H \\
&= (f, f)_H - \left(\sum_{k=1}^n a_k\Phi_k, f\right)_H - \left(f, \sum_{k=1}^n a_k\Phi_k\right)_H + \sum_{k=1}^n\sum_{l=1}^n a_k\bar{a}_l\,(\Phi_k, \Phi_l)_H \\
&= ||f||_H^2 - \sum_{k=1}^n a_k\bar{c}_k - \sum_{k=1}^n \bar{a}_k c_k + \sum_{k=1}^n |a_k|^2 \\
&= ||f||_H^2 - \sum_{k=1}^n |c_k|^2 + \sum_{k=1}^n |c_k|^2 - \sum_{k=1}^n a_k\bar{c}_k - \sum_{k=1}^n \bar{a}_k c_k + \sum_{k=1}^n |a_k|^2 \\
&= ||f - s_n||_H^2 + \sum_{k=1}^n |c_k - a_k|^2
\end{aligned}$$

□

1.6 Definition: Let H be a Hilbert space, $\{\Phi_k\}_{k=1}^{\infty} \subset H$ be an orthonormal system, $f \in H$ and $\{c_k\}_{k=1}^{\infty} := \{(f, \Phi_k)_H\}_{k=1}^{\infty}$. Then

$$\sum_{k=1}^{\infty} c_k \Phi_k$$

is called an *abstract Fourier series* with respect to the othonotmal system $\{\Phi_k\}_{k=1}^{\infty}$. Number c_k is called *k-th Fourier coeficient*. Abstract Fourier series constructed by this fenition will be denoted as

$$f \sim \sum_{k=1}^{\infty} c_k \Phi_k.$$

1.7 Theorem (Bessel inequality and Parseval equality): Let H be a Hilbert space, $\{\Phi_k\}_{k=1}^{\infty} \subset H$ be an orthonormal system, $f \in H$, $f \sim \sum_{k=1}^{\infty} c_k \Phi_k$ and $s_n := \sum_{k=1}^{n} c_k \Phi_k$ for all $n \in \mathbb{N}$. Then

$$\sum_{k=1}^{\infty} |c_k|^2 < \infty \text{ and } \sum_{k=1}^{\infty} |c_k|^2 \leq ||f||_H^2 \qquad \text{(Bessel inequality)},$$

and

$$\sum_{k=1}^{\infty} |c_k|^2 = ||f||_H^2 \Leftrightarrow \lim_{n \to \infty} ||f - s_n||_H = 0 \qquad \text{(Parseval equality)}.$$

Proof: Let $n \in \mathbb{N}$. In the proof of theorem about the best approximation let $a_k = c_k$, then

$$0 \leq ||f - s_n||_H^2 = ||f||_H^2 - \sum_{k=1}^{\infty} |c_k|^2.$$

Therefore the series $\sum_{k=1}^{\infty} |c_k|^2$ is bounded by $||f||_H^2$ and by Bolzano-Weierstrass theorem is convergent[1]. From the equation $||f - s_n||_H^2 = ||f||_H^2 - \sum_{k=1}^{\infty} |c_k|^2$ we have Parseval equality.

□

1.8 Riesz–Fischer theorem (reversed Parseval equality): Let H be a Hilbert space, $\{\Phi_k\}_{k=1}^{\infty} \subset H$ be an orthonormal system and $\{c_k\}_{k=1}^{\infty} \in \ell^2$. Then

1. $\sum_{k=1}^{\infty} c_k \Phi_k$ converges in H (in other words $\exists f \in H$ such that $f = \lim_{n \to \infty} \sum_{k=1}^{n} c_k \Phi_k$),
2. $c_k = (f, \Phi_k)$,
3. $\sum_{k=1}^{\infty} |c_k|^2 = ||f||_H^2$.

Proof: Basically, we want to show that for $\boldsymbol{c} \in \ell^2$, $\exists f \in H$ such, that $c_k = (f, \Phi_k)$ and $||f - s_n||_H^2 \to 0$ where $s_n = \sum_{k=1}^{n} c_k \Phi_k$. However s_n is Cauchy sequence[2], becasue

$$||s_{n+p} - s_n||_H^2 = \left\| \sum_{k=n+1}^{n+p} c_k \Phi_k \right\|_H^2 = \sum_{k=n+1}^{n+p} |c_k|^2 < \epsilon$$

and H is also a Banach space, hence s_n converges to $f \in H$. Now we will show that $c_k = (f, \Phi_k)$. But

$$|c_k - (f, \Phi_k)| = |(s_n, \Phi_k) - (f, \Phi_k)| = |(s_n - f, \Phi_k)| \leq ||s_n - f||_H ||\Phi_k||_H = ||s_n - f||_H \to 0,$$

where we used Cauchy-Schwarz inequality and orthonormality of $\{\Phi_k\}_{k=1}^{\infty}$. Now we have all the requirements for the Parseval equality, hence third statement holds.

□

[1] This is not right, it's more like nonegative bounded sequence is convergent while Bolzano-Weierstrass theorem says that every bounded sequence has a convergent subsequence.

[2] s_n is Cauchy sequence $\Leftrightarrow \forall \epsilon > 0,\ \exists n_0,\ \forall m, n > n_0 : |s_m - s_n| < \epsilon$.

1.9 Characteristic of complete orthonormal system: Let H be a Hilbert space and $\{\Phi_k\}_{k=1}^{\infty} \subset H$ an orthonormal system. Then following statements are equivalent

1. $\{\Phi_k\}_{k=1}^{\infty}$ is complete system,
2. $\forall f \in H: \ ||f - s_n||_H \longrightarrow 0$ where $s_n := \sum_{k=1}^{\infty} c_k \Phi_k$,
3. $\forall f \in H: \ \sum_{k=1}^{\infty} |c_k|^2 = ||f||_H^2$,
4. span $\{\Phi_1, \ldots, \Phi_n\}$ for $n \in \mathbb{N}$ is dense in H,

where $c_k := (f, \Phi_k)$.

Proof: First let's prove 1. $\Rightarrow$ 2.: We already disscused that s_n is Cauchy sequence in Banach space, hence $||s_n - z||_H \longrightarrow 0$.. However

$$(z, \Phi_k) = \lim_{n\to\infty} (s_n, \Phi_k) = c_k = (f, \Phi_k) \quad \forall k \in \mathbb{N}$$

therefore $(z - f, \Phi_k) = 0$ and because of system $\{\Phi_k\}_{k=1}^{\infty}$ is complete we have $f = z$, what means $||s_n - f||_H \longrightarrow 0$. 2. $\Leftrightarrow$ 3. we have from Parselval equality. Now we will show that 3. $\Rightarrow$ 1.: Consider $f \in H$ such, that $c_k = (f, \Phi_k) = 0, \forall k \in \mathbb{N}$. Hence from Parseval equality we have $||f||_H = 0$ and $\{\Phi_k\}_{k=1}^{\infty}$ is complete system. Statement 2. $\Rightarrow$ 4. is true, because s_n is a countable linear combination of elemetns from $\{\Phi_k\}_{k=1}^{\infty}$ and if $\forall f \in H: \ ||f - s_n||_H \longrightarrow 0$ then $\{\Phi_k\}_{k=1}^{\infty}$ has to be dense in H. Finally 4. $\Rightarrow$ 1.: Let $z \in H$ be such that $(z, \Phi_k), \forall k \in \mathbb{N}$ and we want to show that $z = 0$. From our assumption there exists $\{t_n\}_{n=1}^{\infty}$, $t_n \in \text{span}\{\Phi_1, \ldots, \Phi_n\}$ for given $n \in \mathbb{N}$ such that $t_n \longrightarrow z$ in H. Therefore

$$||z||_H^2 = (z, z) = \left(\lim_{n\to\infty} t_n, z\right) = \lim_{n\to\infty} (t_n, z) = 0.$$

□

1.10 Corollary: Every Hilbert space in which exists a complete ortonormal system is separable.

Proof: From charakteristic of complete orthonormal systems, statement 1. $\Rightarrow$ 4.. As a countable, dense subset of H we can take linear combinations of $\{\Phi_k\}_{k=1}^{\infty}$ with rational coeficients.

□

1.11 Theroem (On the existence of a complete system in a separable Hilbert space): In every separable Hilbert space there exists a complete ortonormal system.

Proof: We will distinguish two cases. First, let H be Hilbert's space of finite dimension. Then just take any orthonormal basis. Now let H have an infinite dimension. Due to the assumed separability, there is a large dense subset in it. Let's arrange its elements in the sequence $\{x_j\}_{j=1}^{\infty}$ and remove all trivial and lineary dependent elements. Complete ortonormal system will be now constructed by using Gram-Schmidt ortogonalization process. Let $\Phi_1 := x_j / ||x_j||_H$. Next we will take

$$y_2 = x_2 - (x_2, \Phi_1)\, \Phi_1 \quad \text{and} \quad \Phi_2 = \frac{y_2}{||y_2||_H}.$$

Rest of elemetns will be constructed by indution.

$$y_{n+1} = x_{n+1} - \sum_{j=1}^{n} \left(x_{n+1}, \Phi_j\right) \Phi_j \quad \text{and} \quad \Phi_{n+1} = \frac{y_{n+1}}{||y_{n+1}||_H}.$$

We can see that $(x_1, x_2, \ldots, x_n) \subset \text{span}\,(\Phi_1, \Phi_2, \ldots, \Phi_n)$, hecne span $(\Phi_1, \Phi_2, \ldots, \Phi_n)$ is dense in H and orthonormal. Therefore by statement 4. $\Rightarrow$ 1. of theorem "characteristic of complete orthonormal system" $\{\Phi_k\}_{k=1}^{\infty}$ is also complete.

□

1.12 Solved problem: Consider a Hilert space ℓ^2 defined as

$$\ell^2 := \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{R} \,\middle|\, \sum_{k=1}^{\infty} x_k < \infty \right\},$$

with an inner product $\left(\{x_k\}_{k=1}^{\infty}, \{y_k\}_{k=1}^{\infty}\right) = \sum_{k=1}^{\infty} x_k y_k$. For every $n \in \mathbb{N}$ we define $\Phi_n \in \ell^2$ by enumeration as $(\Phi_n)_k = \delta_{nk}$. Then system $\{\Phi_n\}_{n=1}^{\infty}$ is a complete ortonormal system in ℓ^2. Therefore ℓ^2 is also separable.

1.13 Definition: Let (P_1,ρ_1), (P_2,ρ_2) be metric spaces and $I : P_1 \longrightarrow P_2$ a map between them. We say that I is *isometry* if $\forall x,y \in P_1 : \rho_2\,(I(x),I(y)) = \rho_1(x,y)$. Furthemore, we say that spaces (P_1,ρ_1), (P_2,ρ_2) are *isometric*, if there exists an isometry mapping P_1 onto P_2.

Isometries of the plane are linear transformations preserving length such as rotation, translation or reflection.

1.14 Corollary: An isometry is always one-to-one by the following argument

$$\forall x,y \in P_1, x \neq y \Rightarrow \rho_1(x,y) \neq 0 \Rightarrow \rho_2(I(x),I(y)) \neq 0 \Rightarrow I(x) \neq I(y).$$

Isometry I is always "one-to-one" and if (P_1,ρ_1), (P_2,ρ_2) are isometric, it is also "onto". Then exists $I^{-1} : P_2 \longrightarrow P_1$ mapping P_2 onto P_1.

1.15 Theroem (On the relation of separable Hilbert spaces and ℓ^2): Every infinitely dimensional separable Hilbert space is isometric to ℓ^2.

Proof: We know that in the separable Hilbert space H there exists a complete orthonormal system. Let I be a map that assigns to every $f \in H$ a sequence of its Fourier coefficients $\{c_k\}_{k=1}^{\infty}$ with respect to the given system. Such I is defined on the whole H, and is mapping whole H onto ℓ^2 (if $f \in H$ is finite we have that $\sum_{k=1}^{\infty}|c_k|^2 < \infty$). Therefore, from Parseval's equality

$$||f||_H^2 = \sum_{k=1}^{\infty}|c_k|^2 = ||\{c_k\}_{k=1}^{\infty}||_{\ell^2} = ||I(f)||_{\ell^2}^2.$$

Finally, because of linearity of the inner product we have

$$\forall f,g \in H : ||I(f) - I(g)||_{\ell^2}^2 = ||I(f-g)||_{\ell^2}^2 = ||f-g||_H^2.$$

$\square$

1.3 Orthogonal projections of Hilbert space

1.16 Definition: We say that M is a *closed subspace* of H if

1. $M \subset H$
2. $\forall \alpha,\beta \in \mathbb{C},\ \forall u,v \in M : \alpha u + \beta v \in M$
3. $\{u_n\}_{n=1}^{\infty} \subset M$ and $||u_n - u||_H \longrightarrow 0 \quad \Rightarrow u \in M$

1.17 Solved problem: Let $\Omega \subset \mathbb{R}^N$ be a non-empty open set such, that $\lambda_N(\Omega) < \infty$. Define set

$$M = \left\{ f \in L^2(\Omega) \middle| \int_\Omega f\,\mathrm{d}x \right\},$$

then M is a subspace $L^2(\Omega)$. Furthemore if $\{f_n\}_{n=1}^{\infty} \subset M$ and $f_n \longrightarrow f$ in $L^2(\Omega)$ we have from Holder inequality

$$\left|\int_\Omega f\mathrm{d}x\right| \leq \left|\int_\Omega (f-f_n)\mathrm{d}x\right| + \left|\int_\Omega f_n\mathrm{d}x\right| = \left|\int_\Omega (f-f_n)\mathrm{d}x\right| \leq \int_\Omega |f-f_n|\mathrm{d}x$$
$$\leq ||f-f_n||_{L^2(\Omega)}|1||_{L^2(\Omega)} \longrightarrow 0.$$

Hence $f \in M$ and M is closed subspace of $L^2(\Omega)$.

1.18 Theorem (About orthogonal projection): Let H be a Hilber space and M it's closed subspace.
Then $\forall f \in M,\ \exists! f_M \in M : ||f - f_M||_H = \inf_{z\in M}||f-z||_H$. We define map $P : H \longrightarrow M$ such, that $P(f) = f_M$. Then following holds:

1. $P(H) = M$
2. $P \circ P = P$
3. for every $z \in M$ holds $z \in P(f) \Leftrightarrow (f-z,y)_H = 0, \forall y \in M$
4. $\forall f \in H : ||f||_H^2 = ||P(f)||_H^2 + ||f - P(f)||_H^2$

Proof:

$\square$