Contents

Chapter 1
Topology

The notion of topology gives sense to the intuitive ideas of nearness and continuity. It appears that there are equivalent ways of defining a topology: In terms of open sets, or of closed sets or using as primitive notion the notion of neighbourhood of a point.

1.1 Preliminaries

1.1 De Morgan's laws: Let $A, B \subset X$. Then

$$C_X (A \cup B) = C_X A \cap C_X B$$

$$C_X (A \cap B) = C_X A \cup C_X B$$

where $C_X A = \{x \in X | x \notin A\}$ denotes the complement of A.

Proof: To proof that $C_X (A \cup B) = C_X A \cap C_X B$ is completed in 2 steps by proving both $C_X (A \cup B) \subseteq C_X A \cap C_X B$ and $C_X (A \cup B) \supseteq C_X A \cap C_X B$.

Part 1: Let $x \in C_X(A \cap B)$, then $x \notin A \cap B$. Because $A \cap B = \{y | y \in A \wedge y \in B\}$, it must be the case that $x \notin A$ or $x \notin B$. If $x \notin A$, then $x \in C_X A$, so $x \in C_X A \cup C_X B$. Similary, if $x \notin B$, then $x \in C_X B$, so $x \in C_X A \cup C_X B$. Hence, $\forall x : x \in C_X(A \cap B) \Rightarrow x \in C_X A \cup C_X B$, that is, $C_X (A \cup B) \subseteq C_X A \cap C_X B$.

Part 2: To prove the reverse direction, let $x \in C_X A \cup C_X B$, and for contradiction assume $x \notin C_X(A \cap B)$. Under that assumption, it must be the case that $x \in A \cap B$, so it follows that $x \in A$ and $x \in B$, and thus $x \notin C_X A$ and $x \notin C_X B$. However, that means $x \notin C_X A \cup C_X B$, in contradiction to the hypothesis that $x \in C_X A \cup C_X B$ therefore, the assumption $x \notin C_X(A \cap B)$ must not be the case, meaning that $x \in C_X(A \cap B)$. Thus, $\forall x : x \in C_X A \cup C_X B \Rightarrow x \in C_X(A \cap B)$, that is, $C_X (A \cup B) \supseteq C_X A \cap C_X B$.

□

1.2 Topological spaces

1.2 Definition: A **topological space** is a non-empty set X together with a family $\tau = (U_i \mid \forall i \in I, U_i \subset X)$ ("of subsets of X") satisfying the following axioms:

1. $\emptyset, X \in \{\tau\}$
2. The intersection of any finite number of sets in τ belongs to τ, i.e.

$$J \text{ finite }, J \subset I \Rightarrow \bigcap_{i \in J} U_i \in \tau$$

3. The union of any number of sets in τ belongs to τ, i.e.

$$J \subset I \Rightarrow \bigcup_{i \in J} U_i \in \tau$$

The elements of τ are called τ-open sets, or simply open sets in X. The pair (X, τ) is called a topological space.

1.3 Example: The family $\{\tau\} = \{\emptyset, X\}$, consisting of $\emptyset$ and X alone is itself a topology called the **indiscrete topology**. $\{\emptyset, X\}$ is then called an indiscrete topological space.

1.4 Example: Let $\tau = P(X)$ denote the family of all subsets of X (power set). Observe that $P(X)$ satisfies the axioms 1.-3. for a topology on X. This topology is called the **discrete topology**, the pair $\{X, P(X)\}$ is called a discrete topological space.

1.5 **Example:** Let $X = \mathbb{R}$ be the real line. A topology on $\mathbb{R}$ can be defined as follows: For any $x \in \mathbb{R}$, consider the open intervals (a, b) containing x, then τ is the family

$$\tau = \{U_i = (a_i, b_i) \mid a_i, b_i \in \mathbb{R}\,, a_i \leq b_i\}$$

If $a_i = b_i$, then $(a_i, b_i) = \emptyset$. This topology is referred as the **usual topology** on $\mathbb{R}$. Similarly, one can define the usual topology on $\mathbb{R}^n$.

1.6 **Definition:** Let (X, τ) be a topological space. A subset A of X is **closed** if its complement $C_X A$ is an open set.

1.7 **Theorem:** The family $\overline{\tau} = (A_i \mid \forall i \in I)$ of closed subsets of X satisfying the following conditions:

1. $\emptyset$ and X are closed sets, i.e. $\emptyset, X \in \overline{\tau}$
2. The union of any finite number of sets in $\overline{\tau}$ belongs to $\overline{\tau}$, i.e.

$$J \text{ finite}\,, J \subset I \Rightarrow \bigcup_{i \in J} U_i \in \overline{\tau}$$

3. The intersection of any number of sets in $\overline{\tau}$ belongs to $\overline{\tau}$, i.e.

$$J \subset I \Rightarrow \bigcap_{i \in J} U_i \in \overline{\tau}$$

We denote this topological place $(X, \overline{\tau})$.

Proof: Follows from De'Morgan laws and definition of the topological space of the open sets.

□

1.3 Neighbourhood spaces

1.8 **Definition:** Let $x \in X$ be a point in a topological space X. Any subset U of X containing an open set A such that $x \in A$ is called a **neighbourhood** of x denoted by $U = U(x)$. In particular, any open set U is a neighbourhood of each of its points. The class of all neighbourhoods of $x \in X$, denoted by $\mathfrak{B}(x)$, is called the **fundamental neighbourhood system** of x.

1.4 Types of points

1.9 **Definition:** Let (X, τ) be a topological space and $A \subset X$. A point $x \in A$ is said to be an **isolated point** of A if there exists an open set containing x which contains no other points of A.

1.10 **Definition:** Let (X, τ) be a topological space and $A \subset X$. A point $x \in A$ is said to be an **accumulation point** of A if every open set containing x contains at least one other point from A. (Basically the opposite to the isolated point of A).

1.11 **Definition:** Let $(X, \{\tau\})$ be a topological space and $A \subset X$. A point $x \in X$ is said to be an **adherent point** of A if every neighbourhood of x contains at least one other point of A.

1.12 **Definition:** Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be a **limit point** of A if every neighbourhood of x contains infinitely many points of A.

1.13 **Definition:** Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be a **condensation point** of A if every neighbourhood of x contains uncountable1 many points of A.

1.14 **Definition:** Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be a **interior point** of A if there exists an open set U containing x that is completely a subset of A.

1.15 **Definition:** Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be a **boundary point** of A if every neighbourhood of x contains at least one point of A and at least one point of $C_X A$.

1.16 **Remark:** Isolated and accumulation points of A are always in A, while adherent, limit and condensation points don't have to be in A. If a point is isolated it can't be accumulated and vice versa. Every interior point of A is also a condensation point of A, every condensation point of A is also a limit point of A, and every limit point of A is also an adherent point of A. Isolated and accumulation points are both adherent but can't be limit or condensation points.

1.17 **Theorem:** Union of set of interior points and boundary points is a set of adherent points.

1.18 **Example:** The set $\mathbb{N}$ in usual topology on $\mathbb{R}$ has no accumulation point, i.e. all points of $\mathbb{N}$ are isolated.

1.19 **Example:** The set $A = (0, 1] \cup \{2\} \subset \mathbb{R}$ which has the usual topology on $\mathbb{R}$ has limit point every point of the interval $[0, 1]$. Notice that 2 is an isolated nonlimit point and 0 is a limit point but does not belongs to A.

1.20 **Definition:** The set of accumulation points of A, denoted by A' is called the **derived set** of A.

1.5 Closure

1.21 **Definition:** Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be an **adherent point** of A if every neighbourhood of x contains at least one other point of A.

1.22 **Definition:** Let (X, τ) be a topological space and $A \subset X$. The set of all adherent points of A, denoted by $\overline{A}$, is called a **closure** of A.

1.23 **Theorem:** Let (X, τ) be a topological space and $A \subset X$. The closure of a set A is the intersection of all closed supersets of A and $\overline{A}$ is the smallest closed superset of A.

Proof: Let $(U|i \in I)$ be the family of all closed supersets of A. If $x \in \overline{A}$, then x is adherent point and belongs to a closed superset of A, i.e. $\exists i_0 : x \in U_{i_0}$. Hence $x \in \bigcap_i U_i$ and $\overline{A} \subset \bigcap_i U_i$. Conversely, $y \in \bigcap_i U_i$, implies $y \in U_i$ for every i. Thus y is an adherent point, i.e $y \in \overline{A}$ and if we take all such y we have $\bigcap_i U_i \subset \overline{A}$. Accordingly $\overline{A} = \bigcap_i U_i$, while not forgetting axiom: intersection of closed sets is a closed set, hence $\overline{A}$ is closed. If U is a closed superset of A, it is in the family $(U|i \in I)$ and because $A = \bigcap_i U_i$ we have $\overline{A} \subset U$.

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1.24 **Theorem:** Let (X, τ) be a topological space and $A \subset X$. Every point of a closed set A is its adherent point, i.e $A = \overline{A}$.

Proof: Because $\overline{A}$ is the intersection of all closed supersets of A we have $A \subset \overline{A}$. But if A is closed and $\overline{A}$ is the smallest closed superset of A then $\overline{A} \subset A$. Therefore $A = \overline{A}$.

□

1.6 Interior

1.25 **Definition:** Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be a **interior point** of A if there exists an open set U containing x that is completely a subset of A.

1.26 **Definition:** Let (X, τ) be a topological space and $A \subset X$. The set of all interior points of A, denoted by $\mathring{A}$, is called an **interior** of A.

1.27 **Theorem:** Let (X, τ) be a topological space and $A \subset X$. The interior of a set A is the union of all open subsets of A and $\mathring{A}$ is the largest open subset of A.

Proof: Let $(U|i \in I)$ be the family of all open subsets of A. If $x \in \mathring{A}$, then x is interior point and belongs to an open subset of A, i.e. $\exists i_0 : x \in U_{i_0}$. Hence $x \in \bigcup_i U_i$ and $\mathring{A} \subset \bigcup_i U_i$. Conversely, $y \in \bigcup_i U_i$, implies $y \in U_i$ for some i. Thus y is an interior point, i.e $y \in \mathring{A}$ and if we take all such y we have $\bigcup_i U_i \subset \mathring{A}$. Accordingly $\mathring{A} = \bigcup_i U_i$, while not forgetting axiom: uion of open sets is an open set, hence $\mathring{A}$ is open. If U is an open subset of A, it is in the family $(U|i \in I)$ and because $\mathring{A} = \bigcup_i U_i$ we have $U \subset \mathring{A}$.

□

1.28 **Theorem:** Let (X, τ) be a topological space and $A \subset X$. Every point of an open set A is its interior point, i.e $A = \mathring{A}$.

Proof: Because $\mathring{A}$ is the union of all open subsets of A we have $\mathring{A} \subset A$. But if A is open and $\mathring{A}$ is the largest open subset of A then $A \subset \mathring{A}$. Therefore $A = \mathring{A}$.

□

1.7 Boundary

1.29 Definition: Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be a **boundary point** of A if every neighbourhood of x contains at least one point of A and at least one point of $C_X A$.

1.30 Definition: Let (X, τ) be a topological space and $A \subset X$. The set of all boundary points of A, denoted by ∂A, is called a **boundary** of A.

1.31 Theorem: Let (X, τ) be a topological space and $A \subset X$. The set A is said to be closed subset of $X \Leftrightarrow A$ contains all of its boundary points.

Proof: First assume A is closed. Assume, for a contradiction, that there is $x \in \partial A$ such that $x \notin A$. Then $x \in C_X A$ which is open so there exists an open set U containing x that is entirely in $C_X A$. Then this set contains no points in A contradicting the definition of the boundary point x.

Now assume that all x from ∂A are also in A. Assume, for a contradiction, that A is open. If A is open, then for every point of A exists an open set U eentirely in A. But then there is no point satisfying the definition of boundary point and we arrive to contradiction.

□

1.32 Theorem: Let (X, τ) be a topological space and $A \subset X$. The set A is said to be open subset of $X \Leftrightarrow A$ does not contain any of its boundary points.

Proof: If A is open, then for every point of A exists an open set U eentirely in A. But then there is no point satisfying the definition of boundary point, hence A does not contain any of its boundary points.

Now assume that A does not contain any of its boundary points. Assume, for a contradiction, that A is closed. But we already proved that closed set contains all of its boundary points, therefore we arrive to contradiction.

□

1.33 Theorem: Let $(X, \{\tau\})$ be a topological space and $A \subset X$. The boundary of a set A is given by $\partial A = \overline{A} \backslash \mathring{A}$. Furthermore $\overline{A} = \mathring{A} \cup \partial A$ and Furthermore $A = \partial A \Leftrightarrow \mathring{A} = \emptyset$.

1.8 Continuous maps

1.34 Definition: Let (X_i, τ_i), $i = 1, 2$, be topological spaces. A map $f : X_1 \to X_2$ is **continuous** at a point $x_0 \in X_1$ if for any neighborhood $V(f(x_0))$ of $f(x_0)$ there exists a neighborhood $U(x_0)$ of x_0 such that $f(U) \subset V$, i.e.

$$f \text{ is continuous at } x_0 \in X_1 \Leftrightarrow \forall V(f(x_0))\, \exists U(x_0) \,:\, f(U(x_0)) \subset V(x_0).$$

1.35 Definition: Let (X_i, τ_i), $i = 1, 2$, be topological spaces. A map $f : X_1 \to X_2$ is called an **open map** if, for any open set $U \subset X_1$, $f(U)$ is an open set in X_2.

1.36 Definition: Let $(X_i, \{\tau\}_i)$, $i = 1, 2$, be topological spaces and $f : X_1 \to X_2$ a map. Let $U \in X_2$ and denote $f^{-1}(U) = \{x \in X_1 | f(x) \in U\}$.

1.37 Definition: Two topological spaces $(X_i, \{\tau\}_i)$, $i = 1, 2$, are called **homeomorphic** if there exists a bijective map $f : X_1 \to X_2$ such that f and f^{-1} are continuous. The map f is called a **homeomorphism**.

1.9 Properties of topological spaces

1.38 **Definition:** Let (X, τ) be a topological space. (X, τ) is a **Hausdorff space** if it satisfies the following additional axiom: For every pair of distinct points $x_1, x_2 \in X$ there are disjoint neighborhoods $U_1(x_1), U_2(x_2)$, i.e:

$$\forall x_1, x_2 \in X, x_1 \neq x_2,\ \exists U_1(x_1), U_2(x_2)\ :\ U_1(x_1) \cap U_2(x_2) = \emptyset.$$

1.39 **Definition:** Let (X, τ) be a topological space. A **base** for the topology τ of a topological space (X, τ) is a family $\mathfrak{B}$ of open subsets of X such that every open set is equal to a union of sets from $\mathfrak{B}$.

1.40 **Definition:** A topological space (X, τ) is called **second countable**, if there exist a countable base $\mathfrak{B}$ for the topology τ.

1.41 **Definition:** Let (X, τ) be a topological space. A family $\mathfrak{U} = (A_i | i \in I)$ of subsets of X is called a **cover** of X, if

$$\forall i \in I :\ A_i \neq \emptyset$$

$$X = \bigcup_{i \in I} A_i$$

If, for $J \subset I$, $X = \bigcup_{j \in J} A_j$, then $(A_j | j \in J)$ is called a **subcover**. In particular it is a **finite subcover** if the index set J is finite.

1.42 **Definition:** A topological space (X, τ) is **compact**, if it is a Hausdorff space and if every open cover of X has a finite subcover.