Rethinking Imaginary [1]

February 22, 2016

[1] Part of a year-long journey to learn — or rethink —- one thing a day.

Introduction

The complex number system is often taught by introducing $\sqrt{-1}$.

There is a lot of meaning lost in this notation. As a result, many are left wondering if imaginary numbers actually exist. The name "imaginary" doesn't help either.

Today, I hope to better explain the complex number system and set the stage for other natural extensions of numbers like the quarternions.

Visualizing Numbers

We become so accustomed to real numbers — integers, fractions, square roots — that we may forget what they look like.

They live on a line. The magnitude of a number defines how far we are on the line, and the sign indicates our direction from the origin.

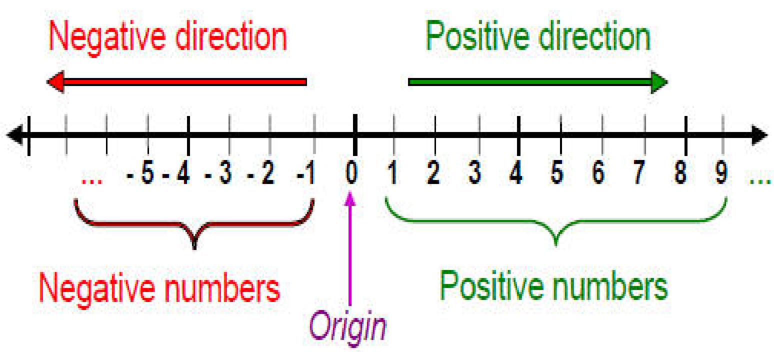

Figure 1: The real number line.

A 2 Dimensional Number System

To work in two dimensions, we need two variables.

One variable will describe how far we are from the origin, just as before. Let's call this the **radius**.

On the other hand, it is no longer possible to describe every direction by using positive and negative signs.

Instead, we will use a second variable to measure the **angle** of our direction from the origin.

Let's agree on the below convention for our system:

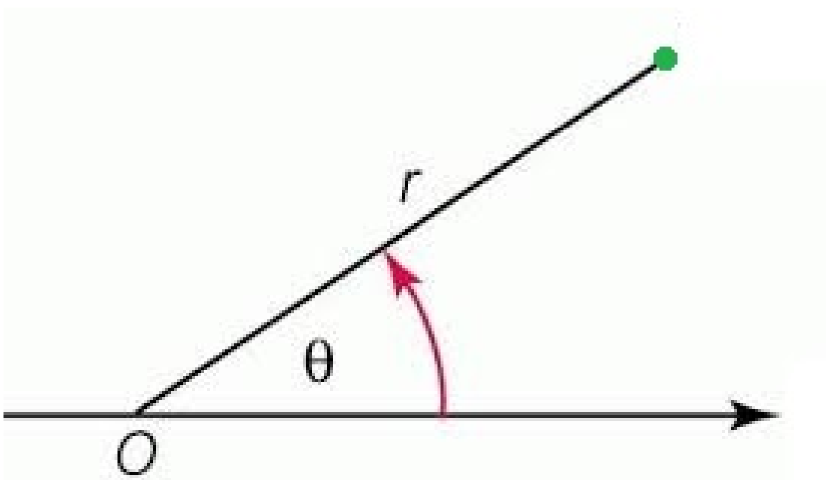

Figure 2: A 2D system.

In this system, we will let the horizontal axis be the real line.

When the angle is zero, we are on the positive side of the real line. When the angle is 180 degrees, we are on the negative side.

Adding

To add numbers in our system, connect lines together.

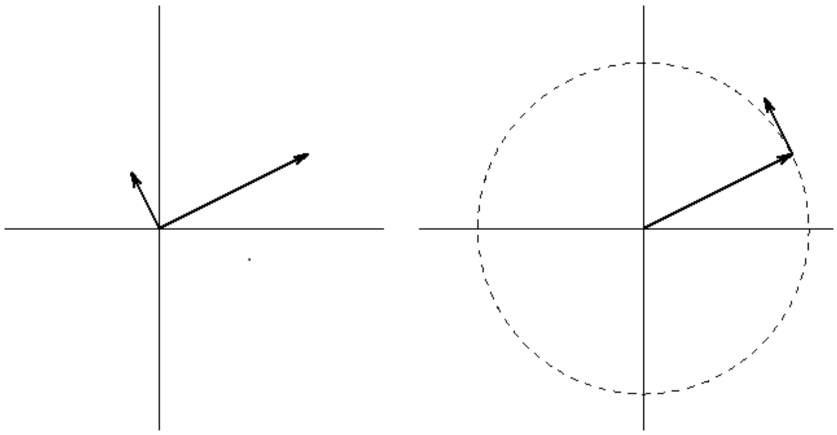

Figure 3: Add by connecting lines.

As an analogy, imagine walking in some direction for a period of time. When you decide to change directions, your final position can be found by connecting your paths together.

Since the lines are not going in the same direction, the final distance from the origin is not just the sum of the length of the two lines. Instead, the result obeys laws of triangles.

Scaling

Let's take a hint from 1D. When we want to compress or extend a number on the real line, we multiply it by a real number.

Therefore, to scale any number in its given direction, we will multiply by a real number.

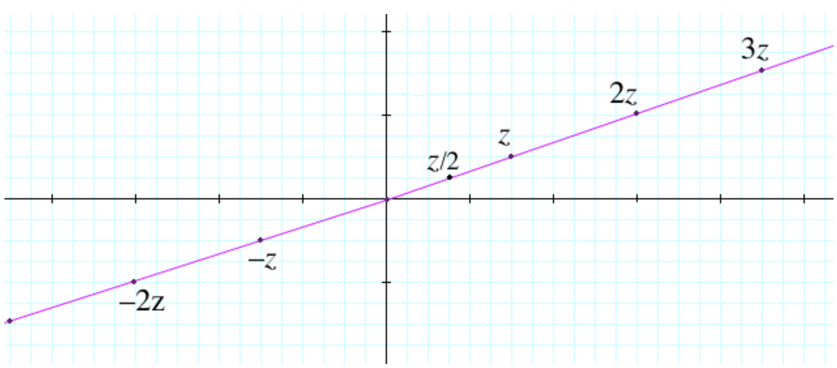

Figure 4: Multiplying any 2D point by a real number.

Rotation

Let's invent an element that rotates.

With rotation at our disposal, any position is just a real number rotated to a desired degree.

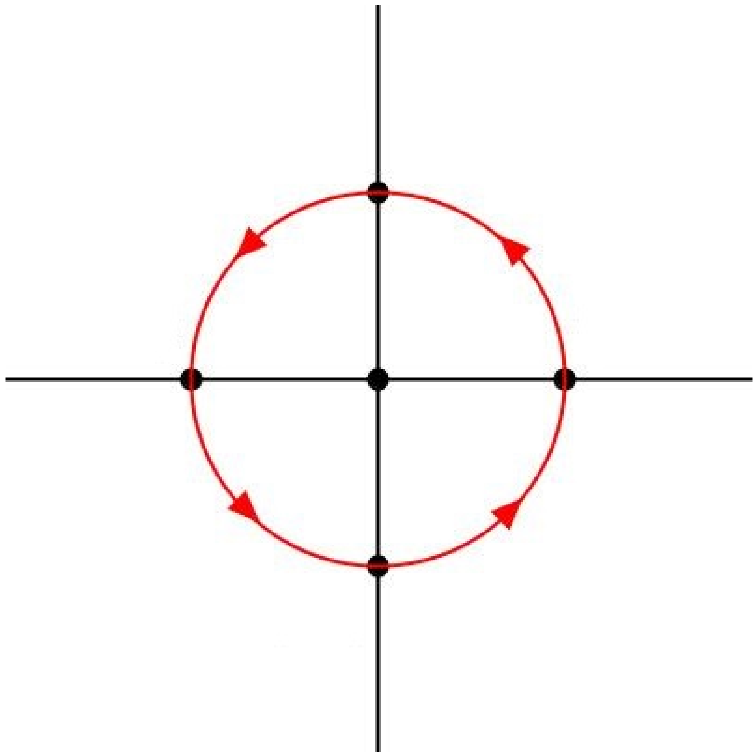

Figure 5: Spinning around a real number in 2D.

This allows us to decompose any point. For example:

- Point 1° Rotated = (Point on Real Line) x (1° Rotation)

It is important that rotation acts the same every time, no matter which way we are currently facing.

So we expect that repeating the rotation again would only add one extra degree:

- Point 2° Rotated = (Point on Real Line) x (1° Rotation)^2

In other words, multiplying rotations should have an **additive** effect to angles.

Back to the Imaginary

Now we can try understand the meaning behind $\sqrt{-1}$.

If we multiply by one degree 180 times, we make a full 180 turn and end up on the negative side of the real line:

- Negative Point = (Positive Point) x (1° Rotation)^{180}

A 1° rotation is not the square root of negatives. It is the 180^{th} root!

Similarly, a rotation 90 degrees from the origin could be squared to reach 180 degrees:

- Negative Point = (Positive Point) x (90° Rotation)^2

Therefore, the square root of negatives in our system is a 90° rotation.

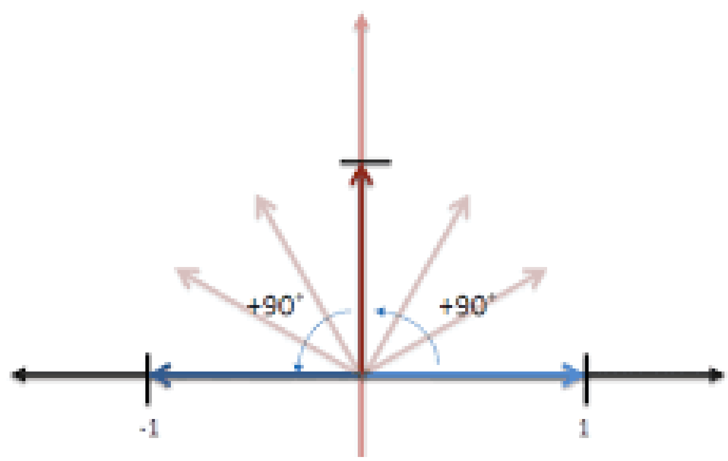

Figure 6: Rotating from positive to negative.

A Missing Detail

When we rotated over and over, we used a unit for rotation, but didn't define it precisely. Now we will solidify this idea.

Recap: Scaling

It is important to remember that when we multiply two real numbers, their values scale in a normal way. We didn't change this.

Multiplying Any Two Points

Since every point can be broken down into a radius and its rotation, the multiplication of two points can also be broken down:

- A = 10 x (1° Rotation)^3
- B = 5 x (1° Rotation)^{10}

Then:

- A x B = 10 x (1° Rotation)^3 x 5 x (1° Rotation)^{10} = 50 x (1° Rotation)^{13}

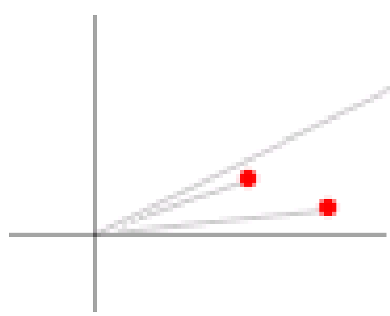

Figure 7: Multiplication of two lines yields a product of radii and sum of angles. Not fully to scale.

In summary, angles add and radii multiply.

Repeated Multiplication

If we multiply a number by itself over and over, we get an image like this, with increasingly growing radius at a steady rate of rotation.

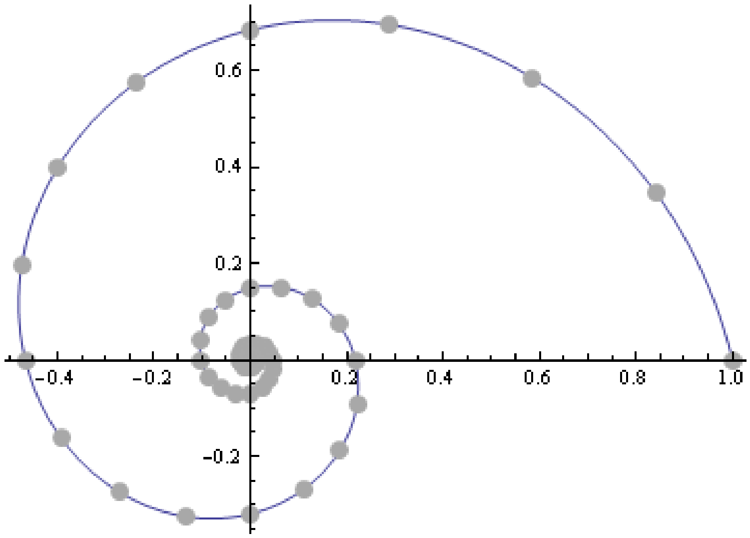

Figure 8: Exponentiation adds angles and multiplies radii.

Therefore, a number purely for rotation would need a radius of 1, since 1x1 = 1.

The Square Root of Negative One

As a result of the previous section, the exact square root of -1 would be a line with a radius of 1 that was rotated by ninety degrees:

- 1 x (90° Rotation) x 1 x (90° Rotation) = 1 x (180° Rotation) = -1

Textbooks commonly refer to this number as i.

Converting to a Coordinate System

The 2D system introduced with a radius and angle is formally known as the **polar coordinates** and has applications beyond the complex numbers since it simplifies working with circles and spheres.

The polar coordinate system can be converted into the typical way of working with complex numbers by splitting the radius into its horizontal and vertical coordinates.

Since we found out that i is unit length on the vertical axis, this means that any 2D point has a component that is real and a component that is made up of i.

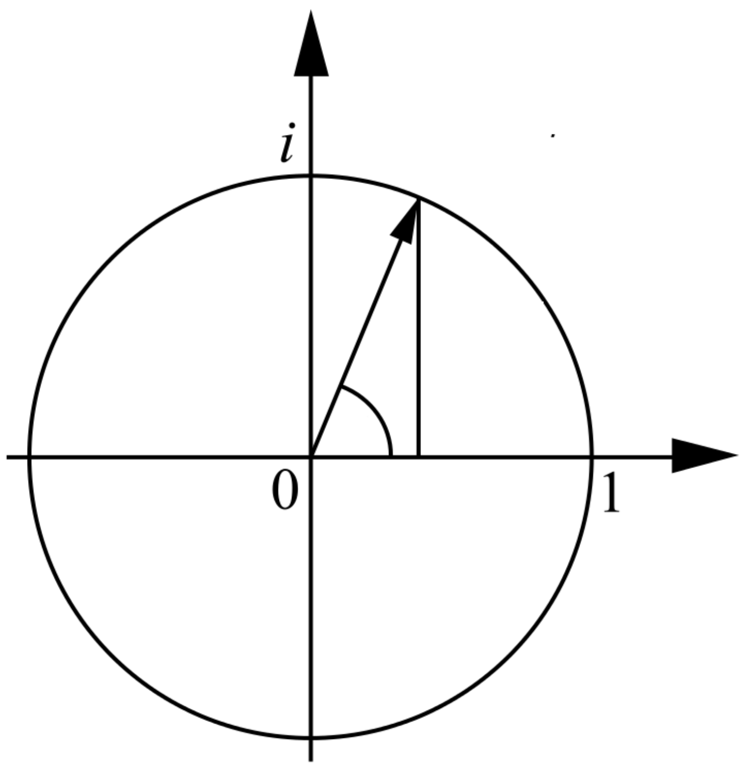

Figure 9: Thinking about our system in coordinates. We label the unit 90 degree rotation as i.

Conclusion

We now have a different view of what it means to be negative, in terms of **angles** rather than sign.

To find a square root of a negative, we need a new definition of multiplication which includes **rotation**.

With a new definition of multiplication, exponentiation is also redefined. Not only can exponentiation repeat multiplication, it can also repeat rotations.

All together, this makes the complex number system an excellent tool for describing natural phenomena in 2D.

As a bonus, complex numbers are deeply connected to 2D fractals and can be used to make incredible art.

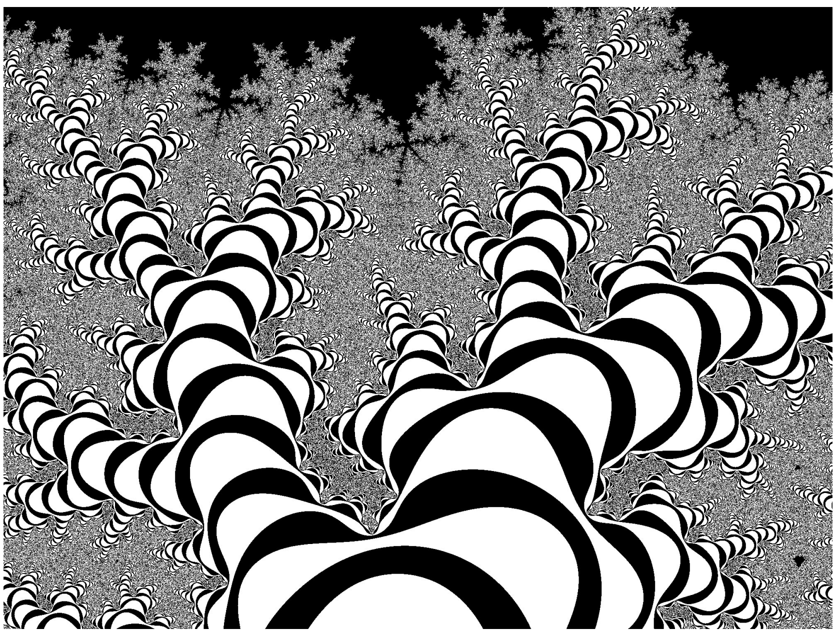

Figure 10: Zooming in on a fractal called the Mandelbrot set. Credit to online artist entheogen02.

More to come!