Day 1: The Cantor Set[1]

February 17, 2016

[1] Part of a year-long journey to learn one thing a day.

Introduction

It only takes a straight line and some imagination. Today's topic will illustrate that simplicity can be an effective tool for discovery.

Making the Set

The Cantor Set is formed by repeatedly splitting a line and removing pieces from it. A process like this is called a **finite subdivision rule**. The goal of finite subdivision is to systematically divide a shape into smaller, more interesting pieces. One of my favourite examples of this is the Sierpinski triangle.

Figure 1: Pretty things happen when you remove stuff systematically.

By applying subdivision rules, you can generate other amazing fractals, and cool shapes stuff like mountains.

The steps for creating the Cantor Set are simple:

1. Take a line from 0 to 1, and split it into thirds[2].
2. Remove the middle of the line, between $1/3$ to $2/3$.
3. Do this again to the remaining pieces. And again, and again.

[2] We don't need to choose thirds, but that is a common choice. Also, we will focus on the line from 0 to 1, but any line works.

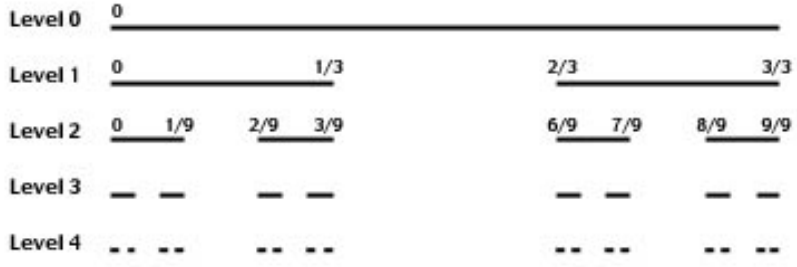

Figure 2: A few iterations. Each time you could zoom in and start all over with a smaller line.

That's it.

What's Left?

We cut a line into lines, and those lines into lines. So what?

After many iterations, do we have anything left?

- We started by cutting out a length of 1/3.
- Then we cut the two remaining segments by 1/3 of their length. That's two times 1/9.
- Then we cut two remaining segments from those two segments. That's four times 1/27.

See the pattern? Stuff Cut Out $= \frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \frac{2^3}{3^4} + ...$

You might remember that this is a geometric series. Summed together, the length of all the lines we cut out is one, which is the length of our original line!

But there isn't nothing left after infinite subdivisions. Remember we keep end points. Take 1/3, for example. Once we cut the line there, it became an end. The middle third of any future line wil never be that end. The same idea applies to other cutting points at any iteration.

What is less easy to see is that we also have infinitely many other points. Let's try to show that.

How Many Points Are Left?

What is remarkable about the Cantor Set is that there are as many numbers left behind as there were to begin with.

Does that sound impossible?

You might have heard before that the numbers between 0 and 1 are not "countable," but what does that really mean?

The plan is to show that any number left over can be matched with a number we had to begin with.

The numbers won't be the same, since we removed some numbers. However, if we can always provide a unique number from the Cantor Set for any unique number we originally had, we must have the same count of numbers.

Review

Before we go deeper, it is important to review how to write a number in a different base. It is normal to not have learned this. We will be working in Binary and Ternary to take advantage of how numbers are represented, so it is important to realize that there are infinitely many different ways to write numbers. Here is an introduction to this idea.

Number Crunching Time

We are now going to use base 3 notation (ternary) to efficiently summarize what gets deleted in an iteration of building the Cantor Set.

Let's start in the first iteration. The end point $1/3$ written in ternary is 0.1_3 because it is one times the first negative exponent of three. Similarly, $2/3$ is twice that: 0.2_3.

By removing the middle third in the first iteration, we are removing the numbers between $1/3$ and $2/3$. In ternary, we are removing numbers that are bigger than 0.1_3 and less than 0.2_3. In other words, we remove any number that has 1 in its first decimal place in base 3 (except $1/3$, which we will talk about later).

Here are some examples of numbers we remove: $0.11_3, 0.12_3, 0.121_3, 0.122_3$

We now have two lines remaining. The lines are a third of the length of the original line. This means any cutting that happens will now occur one decimal down in ternary. That is, cutting a third into thirds creates ninths.

Let's just focus on the first line from 0 to 0.1_3. Anything that is true about that line should be true for the other line from 0.2_3 to 1, since they are just translations of each other.

The line from 0 to 0.1_3 is cut at a third of a third, or 0.01_3 in ternary. It is also cut at the point twice that: 0.02_3. Again, by the same way of thinking, the terms with 1, now in the second decimal place, will be removed. We remove anything between 0.01_3 and 0.02_3.

For the other line from 0.2_3 to 1, the same logic applies at the second decimal place. That is, we remove anything between $0.2 + 0.01_3$ and $0.2 + 0.02_3$.

This can continue forever. Try it out yourself!

One Last Detail

We need to talk about end points like $1/3$ and $1/9$. In ternary, these points are 0.1_3 and 0.01_3. This can also be written without any ones as $0.222..._3$ and $0.0222..._3$, just like how the number 1 can be represented as $0.999...$ repeating forever.

Similarly, any other end point with a one is just a combination of numbers we have already looked at. For example, the first end point of the second line is $(2/3 + 1/9)$, which is $0.2_3 + 0.0222..._3 = 0.2222..._3$.

Going Back Up a Level

We now have a convenient way of describing any number in our set: it must consist entirely of 0s and 2s in ternary.

With only these numbers, we can actually match any number we originally had in the following way:

- Take any number between 0 and 1.
- Its binary representation consists of 0s and 1s.
- Take any code of 0s and 1s and switch the 1s to 2s.
- Since we have all numbers with 0s and 2s, we have just as many numbers, just coded differently.

This shows that there are as many points in the original line as there are in its sliced up set. Since this does not agree with common sense, the numbers are **uncountable**.

Conclusion

The Cantor set has as many points as the interval it was made from, yet it contains no interval with length!

Thanks for joining me for Day 1.