Discrete Math

Counting

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Two Rules of Counting

Count (1) **everything** (2) **exactly once**. There is really no fixed rule to do it. It's all essentially common sense. It is not hard but this is really something that can not be taught. You will need to practice it a little bit to be good at it.

Boring Definition

Def: **Set** is an *unordered* collection of item. The order doesn't matter. Duplicates are not allowed.

For example, the set $\{3, 2, 1\}$ and the set $\{1, 2, 3\}$ are the same set.

Def: **Cardinality** of a set is the number of items in the set. We use $|S|$ to denote the cardinality of set S.

For example, the cardinality of the set $S = \{1, 2, 3\}$ is $|S| = 3$.

Def: **Sequence** is an *ordered* collection of item. The order matters.

For example, the sequence

$$(1, 2, 3) \neq (3, 2, 1).$$

Furthermore, dupicate elements are allowed ex: $(1, 1, 2)$ is also a sequence.

Def: A **permutation** of a set S is a sequence that contains every element in S exactly once.

For example, let $S = \{a, b, c\}$

$$(a, b, c) \text{ is a permutation of } S$$
$$(b, a, c) \text{ is also a permutation of } S$$

.

Product Rule

Consider counting how many ways you can have a meal in a day. You have the following choices for breafast

$$B = \{\text{Jok, Kao mun kai, Boiled Rice, Pancake}\}$$

Then the following choice for Lunch

$$L = \{\text{Noodles, Som tum, Omlette Rice}\}$$

Then the following choices for dinner

$$D = \{\text{Boat Noodles, Chicken Basil}\}$$

Then the number of way you can have these meal in a day is

$$\#\ \text{Meal} = |B| \times |L| \times |D|$$

This is the common sense number one called the product rule.

Here is some excercise you can do:

Example: How many password of 8 character no numbers are?
Solution: 26^8

Example: How many license plate of the type NNCCNNN are?
Solution: $10 \times 10 \times 44 \times 44 \times 10^4$

Example: Counting IPv4 Addresses.
Solution: 256^4

Example: Counting Bacteria. Bacteria-algae DNA contains about 10^5 links. Each link has 4 possibilities.
Solution: 4^{10^5} a lot.

Example: How many ways are there to place knight bishop and pawn on an 8×8 chessboard such that no two share the same row nor column.
Solution: $8 \times 8 \times 7 \times 7 \times 6 \times 6$

Example: How many ways are there to arrange 10 different numbers?
Solution: Since there are 10 different ways to pick the first number, then 9 different ways to

pick the second number and so on. The number of ways to arrange 10 numbers is

$$10 \times 9 \times 8 \times \ldots \times 2 \times 1 = 10!$$

Example: How many binary string of 32 bits are there?
Solution: 2^{32}

Counting Another

Example: How many ways are there to pick numbers from a list of 32 distinct number.
Solution: Let us not count this directly. Instead, we can form a one to one map between the binary string of 32 bits number and the numbers we picked. All we need to do is to have the i-th digit represent whether we pick such number(1) or not(0). Since we can count the number of binary string of 32 bits and the numbers we picked, the cardinality of both sets are equal. Thus, there are 2^{32} ways to pick numbers from 32 distinct number.

This is a very powerful technique which needs some imagination. You will see some more difficult example later.

Sum Rule

Common sense: If you count multiple things, count each one add them up.

Example: How many alphanumeric password are there that is 6-8 character long?
Solution: All we need to do is count how many 6 character password are there then 7 then 8. After that we just need to add them up. So the answer is $62^6 + 62^7 + 62^8$.

Subtraction Rule

Another common sense: If you overcount, subtract off.

Example: Let us count the number of 8 alphanumeric password with at least 1 number.
Solution: If we count all the password then we will overcount the one with no number in. But, that can be counted easily. So, the number of password with at least 1 character is $62^8 - 52^8$.

Example: How many binary string are there that start with 1 *or* end with 00.
Solution: We could count the nubmer of binary strings that start with 1. That is just 2^9. Then we can count the number of binary strings that end with 00: 2^8. But, if we just add the two we would double count those then we would double count those that start with 1 *and* end with 00. So all we need is just count those that start with 1 *and* end with 00 (2^7) then subtract it off from the first sum. So the answer is

$$2^9 + 2^8 - 2^7$$

Division Rule

Common sense: If you count everything twice(or anything with multiple factor) all you need to do is divide.

Example: We can count the number of people in the room by counting the nubmer of ears then divide by 2.

Example: How many ways are there to place 2 rooks on the chessboard such that they do not share the same row nor column?
Solution: If the two rooks are different then it is easy. It would be just $8 \cdot 8 \cdot 7 \cdot 7$. However, this counts each of what we want twice. So all we need to do is divide the answer by two. So the answer is

$$\frac{8 \cdot 8 \cdot 7 \cdot 7}{2}$$

Example: How many ways are there to place 2 rooks and 2 bishop on the chessboard such that none of them share the same row nor column?
Solution: If all the chess pieces are different that is easy to count $(8 \cdot 7 \cdot 6 \cdot 5)^2$. We then need to divide by 2 for double counting the rook. Then divide it by another two for double counting the

bishop. This yields the final answer of

$$\frac{(8 \cdot 7 \cdot 6 \cdot 5)^2}{2 \cdot 2}$$

Example: Let us consider sitting 10 people in a round table. How many ways are there for them to sit. If all adjacent ppl are the same then we consider them to be the same way.
Solution: We can translate this in to the problem of counting the permutation of 10 people. Then we assign the first person to seat no 1 the second to seat no 2 ans so on. There are 10! permutaions. However, such mapping count each way of sitting by 10 times. For example, $(10, 9, 8, \ldots, 1)$ and $(1, 10, 9, 8, \ldots, 2)$ results in the same seating but they both got counted in 10!. So, we need to divide the 10! by 10. So the answer is

$$10!/10 = 9!$$

Example: Now what if there are two people out of the 10 demand that they must sit next to each other. How many way of sitting are there?
Solution: For this we need to be a bit creative. We can just lump the two picky people(A, B) together. Then we have 9 object to arrange in a circle 9!/9. Then, we can unpack the two people. There are two ways to do that: A sits before B or B sits before A. So, the answer is

$$9!/9 \times 2$$

Choosing

One of the most common theme that occur in counting problem is *choosing*. But, essentially this is just a division rule in disguise.

Example: How many ways are there to *choose* 5 yoyo from 22 flavor?
Solution: So first we can arrange the 22 yoyo (22!). But we overcount the permutation of the 5 yoyo we pick by a factor of 5!. We also overcount the permutation of the $22 - 5$ flavors we don't pick by $(22 - 5)!$. So, the number of way to choose 5 flavors from 22 flavors is just

$$\frac{22!}{5!(22-5)!} \equiv \binom{22}{5}$$

The problem of choosing comes up so often that we invent a new notation for them

$$\frac{n!}{k!(n-k)!} \equiv \binom{n}{k}$$

which reads n choose k which is the number of choosing k things from n things ignoring the order. Normally, when we face the problem that involve choosing items we just use n choose k, we normally don't do the division rule. But, it is good that you know it is just a division rule in disguise.

Example: How many ways are there to give 3 yoyo of the same flavor to 8 students?
Solution: All we need is to choose 3 students from 8 students to give the yoyo choose 3 student from the 8 student $\binom{8}{3}$.

Example: Now what if the 3 yoyo are all of different flavor?
Solution: Well we can first choose the 3 student from 8 students to give yoyo $\binom{8}{3}$. Then for the 3 student we can arrange them so we can assign yoyo. That's another 3!. So the answer is

$$\binom{8}{3} \times 3!$$

Example: Now how about if I have 2 cola yoyo and 1 grape yoyo?
Solution: We can choose 2 studnet to give the 2 cola $\binom{8}{2}$. Then we choose 1 student from the 7 student. So

$$\binom{8}{2} \times 7$$

Example: How many squares are there if we have 5 vertical line and 7 horizontal lines.
Solution: Each different square is form by 2 vertical lines and 2 horizontal lines so the nubmer of square is then

$$\binom{5}{2}\binom{7}{2}$$

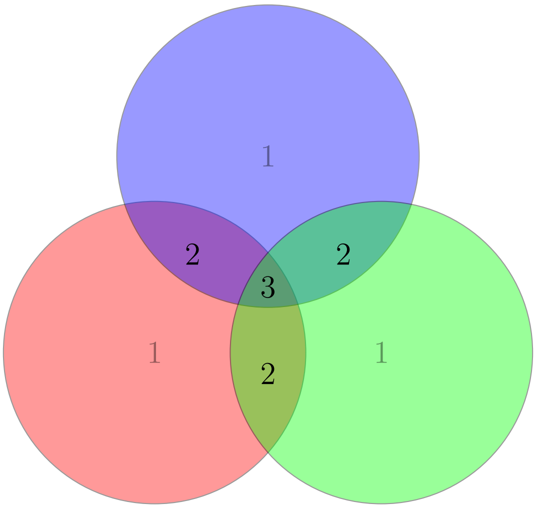
Example: How many 8 character(uppercase) password are there such that no to character are used twice?

Solution: $\binom{26}{8}$

Inclusion-Exclusion 3 Ways

If we want to find the size of $|A \cup B \cup C|$,

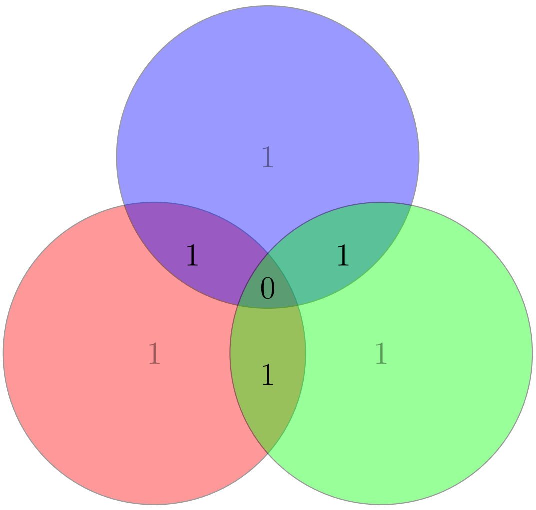
- The first attempt would be to just add up $|A|+|B|+|C|$ but this means we over count the intersection like mad.



- The second attempt we would be subtracting off the intersection that is

  $$|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$$

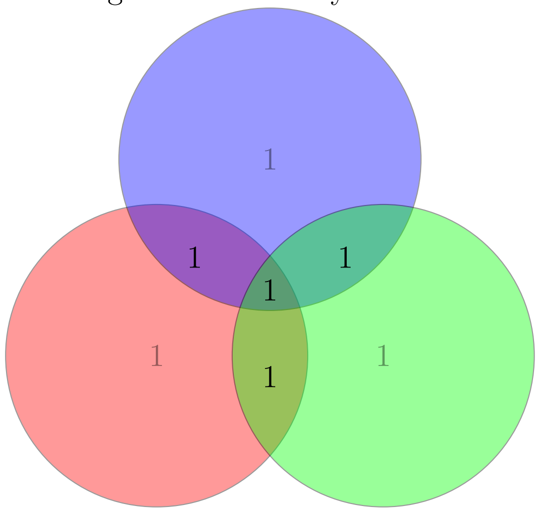
  But, this creates another trouble since we subtract off the center(the insection of the three) three times which means we are left with



- But this is very easy to fix since all we need to do is to *add* the intersection of the three back. Specifically,

  $$\begin{aligned} |A \cup B \cup C| = &|A| + |B| + |C| \\ &- |A\cap B| - |A\cap C| - |B\cap C| \\ &+ |A\cap B\cap C| \end{aligned}$$

  which gives us exactly what we want



Example: How many permutation of the set $\{0, 1, 2, \ldots, 9\}$ has 42 or 04, or 60.

Solution: $9! + 9! + 9! - 8! - 8! - 8! + 7!$

Sticks and Stones

Example: Consider the problem of splitting 20 identical yoyo to 10 student. Such that everyone gets at least one yoyo.

Solution: $\binom{19}{9}$

Example: How about now, some kids do not behave so well so those bad kids don't get the yoyo. How many ways are there to split 20 identical yoyo to 10 students?

Solution: $\binom{29}{9}$

Example: There are 20 books on the shelf. How many ways are there to select 6 books so that no two adjacent book are selected?

Solution: $\binom{20-6+1}{6}$

Random Problem

Example: How many 12 bit string are there that doesn't have 01?
Solution: 13.

Example: How many binary questions you can ask to distinguish 5 items?
Solution: $2^{5-1} - 1$.

Example: Suppose the pizza shop have 9 toppings, how many ways are there to order 3 pizza?
Solution: GLHF.

Example: The three deck of cards in 1409 are mixed up. Assuming that the same card from the different deck is distinguishable how many ways are there to arrage these cards?
Solution: GLHF

Example: Let us consider a square grid. You start at (0,0) then the treasure is at (30,40). On each step you can either go up 1 or go right 1. How many ways are there to get the reasure?
Solution: GLHF

Binomial Theorem

Let us consider the following polynomial

$$(a+b)^5$$

We want to know the coefficient in front of the term a^2b^3. We can consider the multiplication

$$(a+b)_1(a+b)_2(a+b)_3(a+b)_4(a+b)_5$$

To get a^2b^3 all you need to do is count how many way are there to pick two brakets to pick and a from so the coefficient is $\binom{5}{2}$

The same principle can be applied for a general case of

$$(a+b)^n$$

and you want to find the coefficent in front of $a^k b^{n-k}$. This is just $\binom{n}{k}$. With this we can write

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

This is called binomial theorem. But, do not bother memorizing it though you can get it from the first principle in 1 minute.

As an exercise try to figure out the coefficient in front of $a^2b^3c^5$ in

$$(a+b+c)^{10}$$

Combinatoric Proof

As you may find in many of the example above that you find a different way to get the answer. Counting the same thing using different methods has to give us the same number if we count it correctly. This is actually one of my favorite kind of proof.

For example, let us count the number of ways to pick k from n people in the room. Assuming that one of the person in the room is Alice.

One way to count this is as simple as the number of ways to choose k people from n people

$$\binom{n}{k}.$$

Another way to count this is to say that we can either pick or not pick Alice. If we pick alice then we just need to choose $k-1$ people from $n-1$ people. But if we don't pick alice then we need to choose k people from $n-1$ people. If we add the two then we have the number of ways to choose k people from n people. This means the number of ways to choose is

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

The two ways are counting the same thing so they must be equal. So, by just this reasoning we have obtain Pascal's identity.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

The reason it is called the pascal identity since this is exactly how you get each number on the Pascal's triangle. (Wikipedia it if you don't know what it looks like)

Let us look at a more colorful one

Theorem:

$$\sum_{r=0}^{n} \binom{n}{r} \binom{2n}{n-r} = \binom{3n}{n}$$

Proof: If you try to do this with algebra you will probably give up after three terms. Induction doesn't help much either. Let us do a combinatoric proof them.

Let us try to count the number of ways to pick n yoyo from a jar of $3n$ distinguishable yoyo(let us say you mark them all with numbers) which comprises of n cola yoyo, n grape yoyo, n pineapple yoyo.

First way to compute it is that we do not even care what the colors are so the number of ways is

$$\binom{3n}{n}$$

Now, if we are about the color then we know that we can choose $r = 0, 1, \ldots, n$ cola yoyo first. There are $\binom{n}{r}$ ways to choose r cola yoyo. Then all we need to do is to get $n - r$ yoyo from $2n$ non-cola yoyo. So, the number of ways is

$$\sum_{r=0}^{n} \binom{n}{r} \binom{2n}{n-r}$$

Since we are counting the same thing the two quantity must be equal

$$\sum_{r=0}^{n} \binom{n}{r} \binom{2n}{n-r} = \binom{3n}{n}$$

□

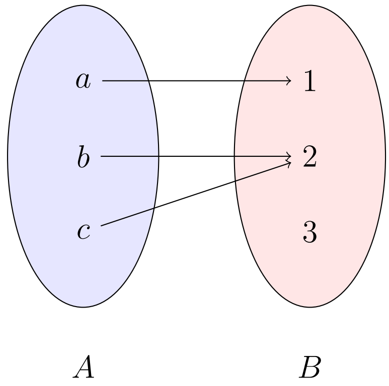
Relations

Def: A function $f : A \to B$ is called *injective* if for all element in B there is *at most* one element in A maps to it.

Def: A function $f : A \to B$ is called *surjective* if for all element in B there is *at least* one element in A maps to it.

Def: A function $f : A \to B$ is called *bijective/one-to-one* if for all element in B there is *exactly* one element in A maps to it. In other words, the function is bijective if and only if the function is both surjective and injective.

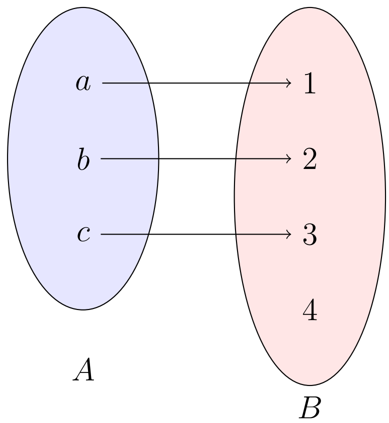
Example: Let us consider a function f which maps $A = \{a, b, c\}$ to $B = \{1, 2, 3\}$.



This function is not injective since 2 has both b and c maps to it.

Also, this function is not surjective since 3 has nothing maps to it.

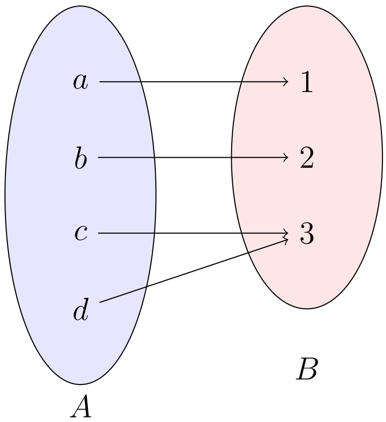
Example: Let us consider a function g which maps $A = \{a, b, c\}$ to $B = \{1, 2, 3, 4\}$.



The function g is injective since every element in B has at either 1 or zero element in A maps to it.

The function g is not surjective since there is nothing maps to 4.

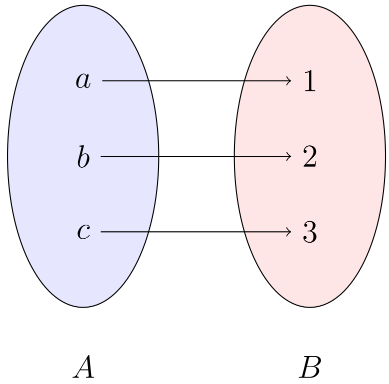
Example: Let us consider a function h which maps $A = \{a, b, c, d\}$ to $B = \{1, 2, 3\}$.



The function h is not injective since both c and d map to 3.

The function h is surjective since there is every element in B has something maps to it.

Example: Let us consider a function q which maps $A = \{a, b, c\}$ to $B = \{1, 2, 3\}$.



The function q is injective all element in B has at most one element in A maps to it.

The function q is surjective since there is every element in B has something maps to it.

Therefore, q is bijective.