

Sum and Asymptotics

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Time Value of Money

Suppose I offer you two choices.

a.) 1 Million Baht now.

b.) 1,100,000 Baht 10 years from now.

Which one is a better choice?

An important factor in deciding between the two is the interest rate(r). What we need to compare is if we get 1 Million Baht now and put it safe in the bank how much is would it be in the next 10 years.

After 1 year, your 1 Million Baht will become

$$1,000,000 \xrightarrow{1 \text{ year}} (1+r) \times 1,000,000$$

Then, 1 year later(on your 2nd year).

$$(1+r)1,000,000 \xrightarrow{1 \text{ more year}} (1+r)^2 \times 1,000,000$$

This means after 10 years your 1,000,000 Baht will become

$$1,000,000 \xrightarrow{10 \text{ years}} (1+r)^{10} \times 1,000,000$$

The current interest rate is at 2%. This means $r = 0.02$. So after 10 years the value of 1,000,000 Baht is

$$1,000,000 \times (1.02)^{10} = 1,218,994$$

So, we can conclude that the 1,000,000 is a better deal because the value of both deals 10 years from now of the first deal is better than the second one.

In the previous analysis, we compared the future value of the money together. We can look at this from another angle by comparing the [present value](#)(PV) of the deal instead.

The present value of the first deal is easy. The present value of 1,000,000 Baht now is, by definition, 1,000,000.

For the second deal, we need to find how much 1,100,000 Baht next 10 year worth now. What we need to find is how much money do we need today to put in the bank so that we will get 1,100,000 Baht in the next 10 year.

$$PV_{1.1M,10year} \times (1+r)^{10} = 1,100,000$$

So, the present value of 1.1M Baht 10 year from now is

$$PV_{1.1M,10year} = \frac{1,100,000}{(1+r)^{10}} = 902,383$$

Or, in general,

$$PV_{m,t} = \frac{m}{(1+r)^t}.$$

So, the first deal is much better.

Monthly Interest Rate

Before we go to the next topic let us think about how to calculate monthly interest rate. We can just equate the money after 1 yearly interest rate and the money after 12 monthly interest payment.

$$M_0(1+r_y) = M_0(1+r_m)^{12}$$

Thus,

$$r_m = (1+r_y)^{\frac{1}{12}} - 1$$

For example, at the current 0.02 interest rate. The monthly interest rate is 0.00165.

Geometric Sum

Let us consider a bit more complicated deal. Supposed you are buying a car of 1,000,000 Baht and the dealer give you two choices

a.) Pay cash now and you will get 40,000 Baht discount.

b.) Or you can pay 20,000 per month for 50 months.

Which deal should you take given that you have enough cash now?

The real question we need to decide is whether the present value of paying 20k for 50 months is. The present value of this deal is just the sum of the present value of each payment.

Let us label each payment from 0 to 49 where 0th payment indicate the payment now at month 0 and the nth payment is the payment n-month in the future.

The present value of the nth payment is given by

$$\frac{20,000}{(1+r_m)^n}.$$

So, the sum of the total payment is given by

$$\frac{20,000}{(1+r_m)^0} + \frac{20,000}{(1+r_m)^1} + \frac{20,000}{(1+r_m)^2} + \dots + \frac{20,000}{(1+r_m)^{49}}$$

Factoring out the 20,000

$$20,000 \left(\frac{1}{(1+r_m)^0} + \dots + \frac{1}{(1+r_m)^{49}} \right)$$

So we will need to evaluate the sum that looks like

$$S = 1 + x + x^2 + x^3 + \dots + x^{49} = \sum_{i=0}^{49} x^i$$

The sum above is called geometric sum. The \sum symbol reads sigma is the notation for the sum. This is one of the few sums that has a closed form formula. Finding the closed form formula is quite simple using the method of perturbation.

$$S = 1 + x + x^2 + \dots + x^m$$

multiplying by x on both sides

$$\begin{aligned} Sx &= x + x^2 + x^3 + \dots + x^m + x^{m+1} \\ Sx &= S - 1 + x^{m+1} \end{aligned}$$

Grouping S

$$S(x-1) = x^{m+1} - 1$$

$$S = \frac{x^{m+1} - 1}{x - 1}$$

Which means

$$1 + x + x^2 + \dots + x^m = \frac{x^{m+1} - 1}{x - 1} \quad (1)$$

In our case,

$$x = \frac{1}{1+r_m} = 0.99835$$

Thus,

$$\frac{1}{(1+r_m)^0} + \dots + \frac{1}{(1+r_m)^{49}} = 48.311$$

So, the present value of the 50 payment is 960,621. But, the present value of the cash payment is only 960,000. So, paying cash now is a slightly better deal.

So, by the time you get your own car, show me that you learn something from this course and do the calculation first.

Infinite Geometric Sum

We can consider Equation 1 in the case where we sum it up to infinity. If $|x| < 1$, then the sum becomes

$$\sum_{i=1}^{\infty} x^i = \frac{1}{1-x}$$

Saving Money

Suppose after you graduate you save 10K a month and put it in the bank. How much do you have after 5 years? This is left for the reader as an exercise. All you need to do is to write out what happen to the 10k you put in at i-th month at the end of 5 years.

Variation on Geometric sum

Consider

$$\sum_{i=1}^n ix^i = 1x + 2x^2 + 3x^3 + \dots$$

You can actually find this sum quite easily by just differentiating both side of Equation 1 with respect to x and manipulating the expression a little bit. Specifically,

$$\begin{aligned} 1 + x^2 + x^3 + \dots + x^m &= \frac{x^{m+1} - 1}{x - 1} \\ \frac{d}{dx} (1 + x + x^2 + x^3 + \dots + x^m) &= \frac{d}{dx} \left(\frac{x^{m+1} - 1}{x - 1} \right) \\ 0 + 1 + 2x + 3x^2 + \dots + mx^{m-1} &= \frac{(x - 1)(m + 1)x^m + x^{m+1} - 1}{(x - 1)^2} \\ 0 + 1 + 2x + 3x^2 + \dots + mx^{m-1} &= x^m \left(\frac{m + 1}{x - 1} \right) - \frac{x^{m+1} - 1}{(x - 1)^2} \\ 0 + x + 2x^2 + 3x^3 + \dots + mx^m &= x^{m+1} \left(\frac{m + 1}{x - 1} \right) - x \frac{x^{m+1} - 1}{(x - 1)^2} \end{aligned}$$

If you ever need something like

$$\sum_{i=1}^n i^2 x^i = 1x + 2^2 x^2 + 3^2 x^3 + \dots$$

All we need to do is just keep differentiating it.

Sum Properties

Here are some obvious property of sum. All of them are obvious if you just write it out.

First, let us consider multiplying a constant to a sum

$$\sum_{i=0}^n 5i = 5 \times 1 + 5 \times 2 + 5 \times 3 + 5 \times 4 + \dots$$

We can see that if we write it out we can just factor out the 5 from each term. We can then conclude that we can just pull the constant out from the sum.

$$\sum_{i=0}^n cf(i) = c \times \sum_{i=0}^n f(i)$$

Let us consider another situation where we add the terms in the sum:

$$\begin{aligned} \sum_{i=0}^n (i^2 + i) &= 1^2 + 1 + 2^2 + 2 + 3^2 + 3 + \dots \\ &= 1^2 + 2^2 + 3^2 + \dots + 1 + 2 + 3 + \dots \\ &= \sum_{i=0}^n i^2 + \sum_{i=0}^n i \end{aligned}$$

So, we can also distribute the sum inside the expression.

$$\sum_{i=0}^n f(i) + g(i) = \sum_{i=0}^n f(i) + \sum_{i=0}^n g(i)$$

One thing that students get wrong a lot even though it looks trivial is the sum of constants:

$$\sum_{i=1}^n 3 = \underbrace{3 + 3 + 3 + \dots + 3}_{\text{There are } n \text{ of these}} = 3n.$$

So, the answer is $3n$ not 3.

125 However, it is quite easy to see that the sum
 126 doesn't expand for the product.

$$\sum_{i=0}^n f(i)g(i) \neq \left(\sum_{i=0}^n f(i)\right) \times \left(\sum_{i=0}^n g(i)\right)$$

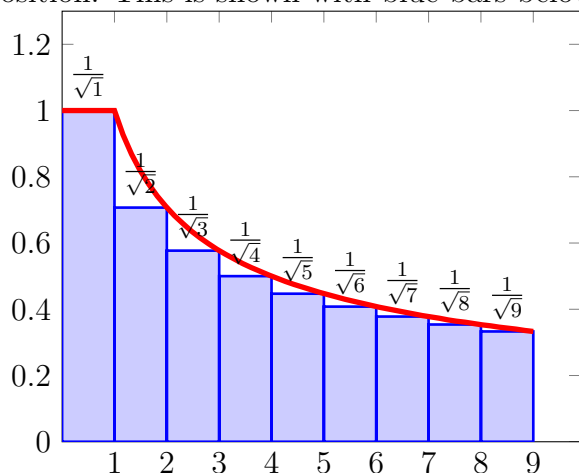
127 The counter example for this is left for the reader
 128 as an exercise.

129 Integral Bound

130 The geometric sum we consider above is one of a
 131 very few sum that has a closed form. The sum,
 132 in general, has no closed form formula. For ex-
 133 ample, let us consider

$$\sum_{x=1}^9 \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{9}}$$

134 This sum has no close form but first let us
 135 consider what exactly does the sum represent.
 136 We can draw number of bars of width 1 and each
 137 with height equal to $\frac{1}{\sqrt{x}}$ where x is the right side
 138 x position. This is shown with blue bars below.



139 From the figure, we can see that the sum we
 140 want to calculate is just the area of the blue bars.
 141 This is because the area of i -th bar is exactly
 142 $1 \times \frac{1}{\sqrt{x}}$.

143 Even though there is no closed form formula
 144 for this sum we can see from the figure that the
 145 area of the blue bars is *less than* the area under
 146 the red line. The red line (ignoring the first parts)
 147 is given simply by
 148

$$f(x) = \frac{1}{\sqrt{x}}$$

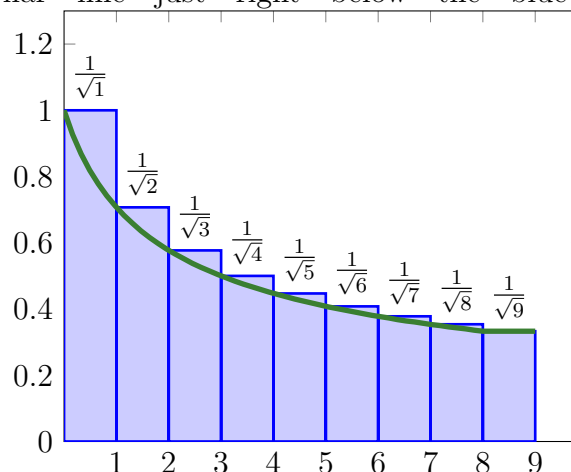
149 You can try a few points to see that this is indeed
 150 correct.

151 So we know that

$$\begin{aligned} \sum_{x=1}^9 \frac{1}{\sqrt{x}} &\leq \int_{x=1}^{x=9} \frac{1}{\sqrt{x}} dx + 1 \times \frac{1}{\sqrt{1}} \\ &\leq 2x^{\frac{1}{2}} \Big|_{x=1}^{x=9} + 1 \\ &\leq 2(3 - 1) + 1 \\ &\leq 5 \end{aligned}$$

152 where the last term comes from the area under
 153 the first flat red region.

154 We can do a similar trick for lower
 155 bound of the sum by considering a sim-
 156 ilar line just right below the blue bars.



157 Since the area under the green curve is less
 158 than the area under the blue bars. The lower
 159 bound of the sum can be found by just finding
 160 the area under the green curve. The green curve
 161 is given by
 162

$$g(x) = \frac{1}{\sqrt{x+1}}.$$

163 You can get this by just plugging in a few points.
 164 Using the above function we can bound the sum
 165 from below by the area under the green curve.
 166 Look carefully on the limits and compare to the
 167 graph above. Note also that last term of the right
 168 hand side comes from the area of the last piece.

$$\begin{aligned}
\sum_{x=1}^9 \frac{1}{\sqrt{x}} &\geq \int_{x=0}^{x=8} \frac{1}{\sqrt{x+1}} dx + 1 \times \frac{1}{\sqrt{9}} \\
&\geq \int_{x=1}^{x=9} \frac{1}{\sqrt{u}} du + \frac{1}{3} \\
&\geq 2(3-1) + \frac{1}{3} \\
&\geq 4\frac{1}{3}
\end{aligned}$$

Combining the result from the lower bound and the upper bound we can conclude that

$$5 \geq \sum_{x=1}^9 \frac{1}{\sqrt{x}} \geq 4\frac{1}{3}$$

You can work this out for general upper limit n where you will find that

$$2(\sqrt{n} - 1) + 1 \geq \sum_{x=1}^n \frac{1}{\sqrt{x}} \geq 2(\sqrt{n} - 1) + \frac{1}{\sqrt{n}}$$

The exact formula for the bounds is not given here since I want you to just draw the picture rather than just memorizing it. There are fine details which you will be more likely to screw up on the limit and the additional piece than to get it correctly. The situation also works a bit differently for increasing function as well.

It is a good exercise for the reader to try to bound

$$\sum_{i=1}^n \sqrt{i}$$

This sum comes up when you try to find the first n prime number using naive method.

Products

A similar thing that you will find that you would need to compute is the product. We have a special symbol (\prod) for that.

$$\prod_{i=1}^{i=n} f(i) = f(1) \times f(2) \times f(3) \times \dots \times f(n)$$

For example,

$$\prod_{i=1}^{i=n} i = 1 \times 2 \times 3 \times \dots \times n = n!$$

The product above is also called the factorial of n denoted by $n!$.

The product is quite difficult to work with since they typically grow really fast. We can turn it into a sum using the property of logarithm. For example, taking \ln of the above expression gives us

$$\begin{aligned}
\ln\left(\prod_{i=1}^{i=n} i\right) &= \ln(1 \times 2 \times 3 \times \dots \times n) \\
&= \ln 1 + \ln 2 + \ln 3 + \dots + \ln n \\
&= \sum_{i=1}^{i=n} \ln i
\end{aligned}$$

which we can then find a bound using the integral bound with the fact that $\int \ln x \, dx = x \ln x - x + c$ to show that

$$n \ln n - n + 1 \leq \ln n! \leq n \ln n - n + 1 + \ln n$$

which means that as n gets large $n!$ is bounded by

$$\ln n! \underset{\text{as } n \text{ gets large}}{=} n \ln n - n$$

Or exponentiate both sides give

$$n! \underset{\text{as } n \text{ gets large}}{=} n^n e^{-n}$$

Asymptotics Behavior

The concept of knowing how the function behaves as n gets large is quite useful.

Def: Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ we say that f is asymptotically equal to g ,

$$f \sim g,$$

if and only if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$$

For example, from the factorial, we can say that

$$n! \sim n^n e^{-n}$$

Let us do a more typical example to illustrate the asymptotic behavior. Consider

$$f(n) = n^2 + n + 2, g(n) = n^2$$

the limit of the ratio of the two as the argument get large is

$$\lim_{n \rightarrow \infty} \frac{n^2 + n + 2}{n^2} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2} = 1$$

We can say that $f \sim g$ or f is asymptotically equal to g .

However, these two function

$$f(n) = n^2 + n + 2, g(n) = 3n^2$$

are *not* asymptotically equal since

$$\lim_{n \rightarrow \infty} \frac{n^2 + n + 2}{3n^2} = \lim_{n \rightarrow \infty} \frac{n^2}{3n^2} = \frac{1}{3} \neq 1$$

Asymptotic Comparison

There is a whole zoo of asymptotic comparison. Don't worry about memorizing these though. The table will be given in the exam.

Definiton	Definition	Intuition
Asym. Equal	$f \sim g$ iff $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$	$f \underbrace{\equiv}_{x \rightarrow \infty} g$
Big Oh	$f \in O(g)$ iff $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} < \infty$	$f \underbrace{\leq}_{x \rightarrow \infty} g$
Little Oh	$f \in o(g)$ iff $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$	$f \underbrace{<}_{x \rightarrow \infty} g$
Little Omega	$f \in \omega(g)$ iff $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} \rightarrow \infty$	$f \underbrace{>}_{x \rightarrow \infty} g$
Big Omega	$f \in \Omega(g)$ iff $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} > 0$	$f \underbrace{\geq}_{x \rightarrow \infty} g$
Theta	$f \in \Theta(g)$ iff $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = c, c \neq 0$	$f \underbrace{=}_{x \rightarrow \infty} g$

Let us look at some example.

Example:

$$f(x) = x^3 + 2x^2 + 1, g(x) = x^4$$

Before doing any calculaton, we can guess that $f(x)$ must be asymptotically strictly less than $g(x)$ since g grows like x^4 and $f(x)$ grows like x^3 eventually g will beat f . So we expect that $f \in o(g)$

The limit we need to calculate is

$$\lim_{x \rightarrow \infty} \frac{x^3 + 2x^2 + 1}{x^4} = \lim_{x \rightarrow \infty} \frac{x^3}{x^4} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

Now let us decide which comparison operator is applicable for f

	Reason
$f \not\sim g$	$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0 \neq 1$
$f \in O(g)$	$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0 < \infty$
$f \in o(g)$	$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0 = 0$
$f \notin \omega(g)$	$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0 \not\rightarrow \infty$
$f \notin \Omega(g)$	$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0 \not\geq 0$
$f \notin \Theta(g)$	$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0 \neq c, c \neq 0$

Here are some exercise you should do

a.) $f(x) = x^3 + 2x^2 + 3, g(x) = x^3$

b.) $f(x) = 3x^2 + 1, g(x) = x^2$

c.) $f(x) = 3, g(x) = 1$

d.) $f(x) = x, g(x) = \log x$

Here are some that you will typically find in computer science. We can order them asymptotically

$$e^x > x^2 > x \log x > x > \log x$$

So when you find something like this

$$f(x) = x^3 + \log x + x \log x$$

All the need to care is the term that grows the fastest. So all we care is the x^3 term. Normally you can just read the Big Oh off the expression. In this case, it is $f \in O(x^3)$

Double Sum/Double Product

Sometimes the sum that you are facing is double sum. You can think about it as nested loop. Consider the following expression

$$\sum_{j=1}^4 \sum_{i=1}^3 i \times j$$

The symbol looks quite complicated but all we need to do is just write it out

$$\begin{aligned} \sum_{j=1}^4 \sum_{i=1}^3 i \times j &= \sum_{j=1}^4 (1j + 2j + 3j) \\ &= (1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1) + \\ &\quad (1 \cdot 2 + 2 \cdot 2 + 3 \cdot 2) + \\ &\quad (1 \cdot 3 + 2 \cdot 3 + 3 \cdot 3) + \\ &\quad (1 \cdot 4 + 2 \cdot 4 + 3 \cdot 4) \end{aligned}$$

There are lots of simplification we can do with double sum. For example, we can use the property we found before to distribute the sum.

$$\begin{aligned} \sum_{j=1}^4 \sum_{i=1}^3 i \times j &= \sum_{j=1}^4 (1j + 2j + 3j) \\ &= \sum_{j=1}^4 1j + \sum_{j=1}^4 2j + \sum_{j=1}^4 3j \\ &= 1 \sum_{j=1}^4 j + 2 \sum_{j=1}^4 j + 3 \sum_{j=1}^4 j \\ &= \left(\sum_{j=1}^4 j \right) \times (1 + 2 + 3) \\ &= \left(\sum_{j=1}^4 j \right) \times \left(\sum_{i=1}^3 i \right) \end{aligned}$$

So, we learn one very important property that the product of the sum is a sum of product. The above is a very good way for you to test if the property you guess works or not. Just write it out. But, there is a much simpler way to approach the answer.

Consider the same sum above

$$\sum_{j=1}^4 \left(\sum_{i=1}^3 i \times j \right)$$

inside the sum over i , j is effectively as constant. So we can just pull it out. So, we have

$$\sum_{j=1}^4 \left(\sum_{i=1}^3 i \times j \right) = \sum_{j=1}^4 \left(j \times \sum_{i=1}^3 i \right)$$

Now if we look at the outer sum over j , for each sum, the sum over i never change. So, it is a constant with respect to the sum over j . We can then just pull it out. So we have

$$\sum_{j=1}^4 \left(\sum_{i=1}^3 i \times j \right) = \sum_{j=1}^4 \left(j \times \sum_{i=1}^3 i \right) \\ = \left(\sum_{j=1}^4 j \right) \times \left(\sum_{i=1}^3 i \right)$$

268 Do not bother memorizing this. All you need
 269 to do is just write it out and you will get the
 270 property in less than a minute.

271 Let us look at the product counter part.

$$\prod_{i=1}^{i=2} \left(\prod_{j=1}^{j=3} 2^i 3^j \right)$$

272 The symbol again looks ridiculously complicated.
 273 Do not panic all we need to do is to just write it
 274 out to see what it means.

$$\prod_{i=1}^{i=2} \left(\prod_{j=1}^{j=3} 2^i 3^j \right) = \prod_{i=1}^{i=2} (2^i 3^1 \times 2^i 3^2 \times 2^i 3^3) \\ = (2^1 3^1 \times 2^1 3^2 \times 2^1 3^3) \times \\ (2^2 3^1 \times 2^2 3^2 \times 2^2 3^3) \times \\ (2^3 3^1 \times 2^3 3^2 \times 2^3 3^3)$$

275 Let us see if we can simplify it like the sum

$$\prod_{i=1}^{i=2} \left(\prod_{j=1}^{j=3} 2^i 3^j \right) = \prod_{i=1}^{i=2} (2^i 3^1 \times 2^i 3^2 \times 2^i 3^3) \\ = \prod_{i=1}^{i=2} [(2^i \times 2^i \times 2^i) \times (3^1 \times 3^2 \times 3^3)] \\ = \prod_{i=1}^{i=2} [2^{3i} \times (3^1 \times 3^2 \times 3^3)] \\ = \left[2^{3 \times 1} \left(\prod_{j=1}^{j=3} 3^j \right) \right] \times \left[2^{3 \times 2} \left(\prod_{j=1}^{j=3} 3^j \right) \right] \\ = 2^{3 \times 1} \times 2^{3 \times 2} \times \left(\prod_{j=1}^{j=3} 3^j \right)^2 \\ = \left(\prod_{i=1}^{i=2} 2^i \right)^3 \times \left(\prod_{j=1}^{j=3} 3^j \right)^2$$

276 So we really can simplify it but the result is a
 277 bit different from the sum. Because of the power
 278 or 2 and 3 that is hanging around. You may have
 279 expected that

$$\prod_{i=1}^{i=2} \left(\prod_{j=1}^{j=3} 2^i 3^j \right) = \left(\prod_{i=1}^{i=2} 2^i \right) \times \left(\prod_{j=1}^{j=3} 3^j \right).$$

280 But, it is wrong. The correct expression is

$$\prod_{i=1}^{i=2} \left(\prod_{j=1}^{j=3} 2^i 3^j \right) = \left(\prod_{i=1}^{i=2} 2^i \right)^3 \times \left(\prod_{j=1}^{j=3} 3^j \right)^2$$

281 However, please do not memorize the formula.
 282 You will definitely get it wrong if you try to mem-
 283 orize. There are just way too many details. For
 284 example, if you do the exercise above with i and
 285 j start from 0 not 1. The power on the sum will
 286 be changed by 1. When you see a double sum
 287 or a product or a mixed of the two all you need
 288 to do is to write it out carefully and group the
 289 terms as much as possible.