

Logic

Discrete Mathematics

Number Theory

Topic 01 — Logic

Mathematical
Proofs

Lecture 04 — Rules of Inference

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Recurrence
Relations

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Set Theory

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Outline

- Constructing arguments in propositional logic
- Normal forms

Enumeration

1. Building Arguments

2

- Our final topic on logic deals with constructing and validating arguments. We start by giving examples of valid and non-valid arguments and define various concepts that we will need to breakdown an argument.

2. Inference Rules for Propositional Logic

6

- Breaking down arguments take effort. To simplify things we will collect some standard arguments which we will use, like lego bricks, when working with complicated arguments.

3. Using the Rules of Inference to Build Valid Arguments

15

- In our final topic in logic, we will use the properties of logical operators to construct a valid argument.
- This is a relatively advanced topic and could be ignored until you are comfortable with the earlier topics in logic.

Notation

Single-line vs Double-line Arrows

For the purpose of this module the single line arrows (representing the IFTHEN and IFF connectives)

\rightarrow and \leftrightarrow

mean the same thing as the corresponding double-line arrow

\Rightarrow and \Leftrightarrow

I will use the double-lined arrows in places where I want to treat a complicated proposition as two smaller propositions. For example, I want to think of the proposition

$$(p \rightarrow q) \wedge \neg q \Rightarrow \neg p$$

in terms of the two proposition $(p \rightarrow q) \wedge \neg q$ and $\neg p$.

Motivation

Remember the Socrates example when we started Logic.

“All men are mortal.

Socrates is a man.

Therefore, Socrates is mortal.”

Here we have two premises:

- All men are mortal
- Socrates is a man.

and the conclusion:

- Socrates is mortal.

Q: How do we get the conclusion from the premises?

A: We construct an argument, a sequence of propositions that follow from the rules of inference until we reach the conclusion.

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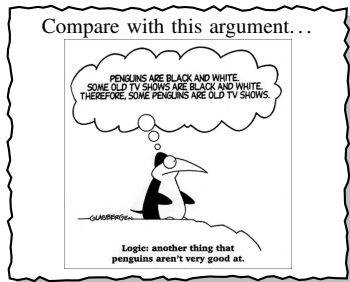
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and the conclusion:

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Arguments

Definition 1 (Argument)

A argument in propositional logic is a sequence of propositions. All but the final proposition are called **premises**. The last statement is the **conclusion**. The argument is **valid** if the premises imply the conclusion.

- If the premises are p_1, p_2, \dots, p_n and the conclusion is q then the argument is valid iff

$$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$$

is a tautology.

- We could use truth tables to test if an argument is valid — construct the above expression, then build the truth table and check the output column.
- Alternatively, we could sequentially apply **inference rules** to arrive at the conclusion.
- Inference rules are simple arguments that will be used to construct more complex argument forms.

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Detachment (Modus Ponens)

Argument

$$\begin{array}{c} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Corresponding Tautology

$$(p \rightarrow q) \wedge p \implies q$$

Example

Let

$p = \text{"It is snowing."}$

$q = \text{"I will study discrete maths."}$

Then the argument is

"If it is snowing, then I will study discrete maths."

"It is snowing."

Therefore *"I will study discrete maths."*

Indirect Reasoning (Modus Tollens)

Argument

$$\begin{array}{c} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology

$$(p \rightarrow q) \wedge \neg q \implies \neg p$$

Example

Let

$p = \text{"It is snowing."}$

$q = \text{"I will study discrete maths."}$

Then the argument is

"If it is snowing, then I will study discrete maths."

"I will not study discrete maths."

Therefore *"It is not snowing."*

Chain Rule (Hypothetical Syllogism)

Argument

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology

$$(p \rightarrow q) \wedge (q \rightarrow r) \implies (p \rightarrow r)$$

Example

Let

$p = \text{"It is snowing."}$

$q = \text{"I will study discrete maths."}$

$r = \text{"I will get an A."}$

Then the argument is

"If it is snowing, then I will study discrete maths."

"If I will study discrete maths, then I will get an A."

Therefore *"If it is snowing, then I will get an A."*

Chain Rule (Hypothetical Syllogism)

Argument

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology

$$(p \rightarrow q) \wedge (q \rightarrow r) \implies (p \rightarrow r)$$

Example

Let

$p = \text{"It is snowing."}$

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$r = \text{"I will get an A."}$

Then the argument is

"If it is snowing, then I will study discrete maths."

"If I will study discrete maths, then I will get an A."

Therefore *"If it is snowing, then I will get an A."*

Disjunctive Simplification (Disjunctive Syllogism)

Argument

$$\begin{array}{cc}
 p \vee q & p \vee q \\
 \neg p & \neg q \\
 \hline
 \therefore q & \therefore p
 \end{array}$$

Corresponding Tautology

$$\begin{aligned}
 (p \vee q) \wedge (\neg p) &\implies q \\
 (p \vee q) \wedge (\neg q) &\implies p
 \end{aligned}$$

Example

Let

$p = \text{"I will study discrete maths."}$

$q = \text{"I will study programming."}$

Then the argument is

"I will study discrete maths or I will study programming."

"I will not study discrete maths."

Therefore *"I will study programming."*

Disjunctive Addition

Argument

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology

$$p \implies (p \vee q)$$

Example

Let

$p = \text{"I will study discrete maths."}$

$q = \text{"I will get high."}$

Then the argument is

$\text{"I will study discrete maths."}$

Therefore $\text{"I will study discrete maths or I will get high."}$

Conjunctive Simplification

Argument

$$\frac{p \wedge q}{\therefore p} \quad \frac{p \wedge q}{\therefore q}$$

Corresponding Tautology

$$(p \wedge q) \implies p$$

$$(p \wedge q) \implies q$$

Example

Let

$p = \text{"I will study discrete maths."}$

$q = \text{"I will get high."}$

Then the argument is

$\text{"I will study discrete maths and I will get high."}$

Therefore $\text{"I will study discrete maths."}$

Resolution

Argument

$$\frac{\neg p \vee r \quad p \vee q}{\therefore q \vee r}$$

Corresponding Tautology

$$(\neg p \vee r) \wedge (p \vee q) \implies (q \vee r)$$

Example

Let

$p = \text{"I will study discrete maths."}$

$p = \text{"I will study programming."}$

$p = \text{"I will study databases."}$

Then the argument is

"I will not study discrete maths or I will study programming."

"I will study discrete maths or I will study databases."

Therefore *"I will study programming or I will study databases."*

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Example 2

A valid argument is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.

Example 2

From the single proposition

$$p \wedge (p \rightarrow q)$$

show that q is a conclusion.

Method 1

Construct argument using inference rules ...

	Step	Reason
1)	$p \wedge (p \rightarrow q)$	Premise
2)	p	Conjunctive Simplification from (1)
3)	$p \rightarrow q$	Conjunctive Simplification from (1)
\therefore	q	Detachment (Modus Ponens) from (2) and (3)

Example 2

Method 2

Construct an expression of the form

$$(\text{premise 1}) \wedge (\text{premise 2}) \wedge \cdots \wedge (\text{premise } n) \implies (\text{conclusion})$$

and verify that the expression is a tautology (using a truth table).

p	q	$(p \rightarrow q)$	$p \wedge (p \rightarrow q)$	$p \wedge (p \rightarrow q) \Rightarrow q$
F	F	T	F	T
F	T	T	F	T
T	F	F	F	T
T	T	T	T	T

Example 3

Example 3

With these hypotheses:

- a) *“It is not sunny this afternoon and it is colder than yesterday.”*
- b) *“We will go swimming only if it is sunny.”*
- c) *“If we do not go swimming, then we will take a canoe trip.”*
- d) *“If we take a canoe trip, then we will be home by sunset.”*

Using the inference rules, construct a valid argument for the conclusion:

- *“We will be home by sunset.”*

General procedure ...

STEP 1 Choose propositional variables.

STEP 2 Translation into propositional logic.

STEP 3 Construct the valid argument (OR verify related tautology using truth table.)

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- *“We will be home by sunset.”*

General procedure ...

- STEP 1 Choose propositional variables.
- STEP 2 Translation into propositional logic.
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Example 3

STEP 1 Choose propositional variables.

- $s = \text{"It is Sunny this afternoon."}$
- $c = \text{"It is Colder than yesterday."}$
- $w = \text{"We will go sWimming"}$
- $t = \text{"We will take a canoe Trip."}$
- $h = \text{"We will be Home by sunset."}$

STEP 2 Translation into propositional logic.

Premises ...

- | | | |
|-----|---|------------------------|
| (a) | "It is not sunny this afternoon and it is colder than yesterday." | $\neg s \wedge c$ |
| (b) | "We will go swimming only if it is sunny." | $w \rightarrow s$ |
| (c) | "If we do not go swimming, then we will take a canoe trip." | $\neg w \rightarrow t$ |
| (d) | "If we take a canoe trip, then we will be home by sunset." | $t \rightarrow h$ |

and conclusion

- | | | |
|---|------------------------------|-----|
| • | "We will be home by sunset." | h |
|---|------------------------------|-----|

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|---|---------------------------------------|-----|
| • | $\text{"We will be home by sunset."}$ | h |
|---|---------------------------------------|-----|

Example 3

STEP 3 Construct the valid argument

(Note the truth table here would have 32 rows)

We have

Premises				Conclusion
(a)	(b)	(c)	(d)	
$\neg s \wedge c$	$w \rightarrow s$	$\neg w \rightarrow t$	$t \rightarrow h$	h

And our argument is ...

Step	Reason
1) $\neg s \wedge c$	Premise (a)
2) $\neg s$	Conjunctive Simplification from (1)
3) $w \rightarrow s$	Premise (b)
4) $\neg w$	Indirect Reasoning (Modus Tollens) from (2) and (3)
5) $\neg w \rightarrow t$	Premise (c)
6) t	Detachment (Modus Ponens) from (4) and (5)
7) $t \rightarrow h$	Premise (d)
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