Discrete Mathematics

Topic 01 — Logic

Lecture 04 — Rules of Inference

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Outline

- Constructing arguments in propositional logic
- Normal forms

Outline

Notation

Single-line vs Double-line Arrows

For the purpose of this module the single line arrows (representing the IFTHEN and IFF connectives)

$$\rightarrow \qquad \text{and} \qquad \leftrightarrow$$

mean the same thing as the corresponding double-line arrow

$$\Rightarrow \qquad \text{and} \qquad \Leftrightarrow$$

I will use the double-lined arrows in places where I want to treat a complicated proposition as two smaller propositions. For example, I want to think of the proposition

$$(p \rightarrow q) \wedge \neg q \implies \neg p$$

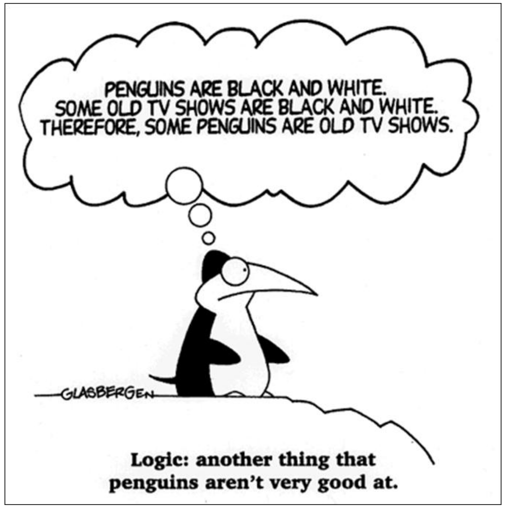
in terms of the two proposition $(p \rightarrow q) \wedge \neg q$ and $\neg p$.

Motivation

Remember the Socrates example when we started Logic.

"All men are mortal.
Socrates is a man.
Therefore, Socrates is mortal."

Compare with this argument...



Here we have two premises:

- All men are mortal
- Socrates is a man.

and the conclusion:

- Socrates is mortal.

Q: How do we get the conclusion from the premises?

A: We construct an argument, a sequence of propositions that follow from the rules of inference until we reach the conclusion.

Arguments

Definition 1 (Argument)

A argument in propositional logic is a sequence of propositions. All but the final proposition are called premises. The last statement is the conclusion. The argument is valid if the premises imply the conclusion.

- If the premises are $p_1, p_2, \ldots p_n$ and the conclusion is q then the argument is valid iff

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$$

  is a tautology.
- We could use truth tables to test if an argument is valid — construct the above expression, then build the truth table and check the output column.
- Alternatively, we could sequently apply inference rules to arrive at the conclusion.
- Inference rules are simple arguments that will be used to construct more complex argument forms.

Outline

Detachment (Modus Ponens)

Argument

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Corresponding Tautology

$$(p \rightarrow q) \wedge p \implies q$$

Example

Let

$p =$ "*It is snowing.*"

$q =$ "*I will study discrete maths.*"

Then the argument is

"*If it is snowing, then I will study discrete maths.*"
"*It is snowing.*"

Therefore "*I will study discrete maths.*"

Indirect Reasoning (Modus Tollens)

Argument

$$
\begin{array}{c}
p \rightarrow q \\
\neg q \\
\hline
\therefore \neg p
\end{array}
$$

Corresponding Tautology

$$(p \rightarrow q) \wedge \neg q \implies \neg p$$

Example

Let

$p =$ "*It is snowing.*"

$q =$ "*I will study discrete maths.*"

Then the argument is

"*If it is snowing, then I will study discrete maths.*"
"*I will not study discrete maths.*"

Therefore "*It is not snowing.*"

Chain Rule (Hypothetical Syllogism)

Argument

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology

$$(p \rightarrow q) \wedge (q \rightarrow r) \implies (p \rightarrow r)$$

Example

Let

$p =$ "*It is snowing.*"

$q =$ "*I will study discrete maths.*"

$r =$ "*I will get an A.*"

Then the argument is

"*If it is snowing, then I will study discrete maths.*"

"*If I will study discrete maths, then I will get an A.*"

Therefore "*If it is snowing, then I will get an A.*"

Chain Rule (Hypothetical Syllogism)

Argument

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Corresponding Tautology

$$(p \to q) \wedge (q \to r) \implies (p \to r)$$

Example

Let

$p =$ "*It is snowing.*"

$q =$ "*I will study discrete maths.*"

$r =$ "*I will get an A.*"

Then the argument is

"*If it is snowing, then I will study discrete maths.*"
"*If I will study discrete maths, then I will get an A.*"

Therefore "*If it is snowing, then I will get an A.*"

Disjunctive Simplification (Disjunctive Syllogism)

Argument

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline \therefore q \end{array} \qquad \begin{array}{c} p \vee q \\ \neg q \\ \hline \therefore p \end{array}$$

Corresponding Tautology

$$(p \vee q) \wedge (\neg p) \implies q$$

$$(p \vee q) \wedge (\neg q) \implies p$$

Example

Let

$$p = \text{“}\textit{I will study discrete maths.}\text{”}$$

$$q = \text{“}\textit{I will study programming.}\text{”}$$

Then the argument is

“*I will study discrete maths or I will study programming.*”
“*I will not study discrete maths.*”

Therefore “*I will study programming.*”

Disjunctive Addition

Argument

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology

$$p \implies (p \vee q)$$

Example

Let

$p =$ "*I will study discrete maths.*"

$q =$ "*I will get high.*"

Then the argument is

$$\frac{\text{“\textit{I will study discrete maths.}”}}{\text{Therefore “\textit{I will study discrete maths or I will get high.}”}}$$

Conjunctive Simplification

Argument

$$\frac{p \wedge q}{\therefore p} \qquad \frac{p \wedge q}{\therefore q}$$

Corresponding Tautology

$$(p \wedge q) \implies p$$

$$(p \wedge q) \implies q$$

Example

Let

$p =$ "*I will study discrete maths.*"

$q =$ "*I will get high.*"

Then the argument is

$$\frac{\text{"\textit{I will study discrete maths and I will get high.}"}}{\text{Therefore "\textit{I will study discrete maths.}"}}$$

Resolution

Argument

$$\begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

Corresponding Tautology

$$(\neg p \vee r) \wedge (p \vee q) \implies (q \vee r)$$

Example

Let

$p =$ "*I will study discrete maths.*"

$p =$ "*I will study programming.*"

$p =$ "*I will study databases.*"

Then the argument is

"*I will not study discrete maths or I will study programming.*"
"*I will study discrete maths or I will study databases.*"

Therefore "*I will study programming or I will study databases.*"

Outline

Example 2 I

A valid argument is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.

Example 2

From the single proposition

$$p \wedge (p \to q)$$

show that q is a conclusion.

Method 1

Construct argument using inference rules …

	Step	Reason
1)	$p \wedge (p \to q)$	Premise
2)	p	Conjunctive Simplification from (1)
3)	$p \to q$	Conjunctive Simplification from (1)
$\therefore$	q	Detachment (Modus Ponens) from (2) and (3)

Example 2 II

Method 2

Construct an expression of the form

$$(\text{premise } 1) \wedge (\text{premise } 2) \wedge \cdots \wedge (\text{premise } n) \implies (\text{conclusion})$$

and verify that the expression is a tautology (using a truth table).

p	q	$(p \to q)$	$p \wedge (p \to q)$	$p \wedge (p \to q) \Rightarrow q$
F	**F**	**T**	**F**	**T**
F	**T**	**T**	**F**	**T**
T	**F**	**F**	**F**	**T**
T	**T**	**T**	**T**	**T**

Example 3 I

Example 3

With these hypotheses:

a) *"It is not sunny this afternoon and it is colder than yesterday."*

b) *"We will go swimming only if it is sunny."*

c) *"If we do not go swimming, then we will take a canoe trip."*

d) *"If we take a canoe trip, then we will be home by sunset."*

Using the inference rules, construct a valid argument for the conclusion:

- *"We will be home by sunset."*

General procedure . . .

STEP 1 Choose propositional variables.

STEP 2 Translation into propositional logic.

STEP 3 Construct the valid argument (OR verify related tautology using truth table.)

Example 3 II

STEP 1 *Choose propositional variables.*

- s = *"It is Sunny this afternoon."*
- c = *"It is Colder than yesterday."*
- w = *"We will go sWimming"*
- t = *"We will take a canoe Trip."*
- h = *"We will be Home by sunset."*

STEP 2 *Translation into propositional logic.*

Premises ...

a) *"It is not sunny this afternoon and it is colder than yesterday."* $\neg s \land c$
b) *"We will go swimming only if it is sunny."* $w \rightarrow s$
c) *"If we do not go swimming, then we will take a canoe trip."* $\neg w \rightarrow t$
d) *"If we take a canoe trip, then we will be home by sunset."* $t \rightarrow h$

and conclusion

- *"We will be home by sunset."* h

Example 3 III

STEP 3 *Construct the valid argument* *(Note the truth table here would have 32 rows)*

We have

Premises (a)	Premises (b)	Premises (c)	Premises (d)	Conclusion
$\neg s \wedge c$	$w \rightarrow s$	$\neg w \rightarrow t$	$t \rightarrow h$	h

And our argument is ...

	Step	Reason
1)	$\neg s \wedge c$	Premise (a)
2)	$\neg s$	Conjunctive Simplification from (1)
3)	$w \rightarrow s$	Premise (b)
4)	$\neg w$	Indirect Reasoning (Modus Tollens) from (2) and (3)
5)	$\neg w \rightarrow t$	Premise (c)
6)	t	Detachment (Modus Ponens) from (4) and (5)
7)	$t \rightarrow h$	Premise (d)
$\therefore$	h	Detachment (Modus Ponens) from (6) and (7)