

Logic

Discrete Mathematics

Number Theory

Topic 04 — Relations and Functions

Mathematical Proofs

Lecture 01 — Relations

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Recurrence
Relations

Set Theory

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Outline

- Defining a relation via Cartesian product
- Relation Terminology

Enumeration

Outline

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Cartesian product

Recall that the Cartesian product of two sets, A and B , is the set of all ordered pairs of all elements where the first element is from set A and the second element is from B .

Definition 1 (Cartesian product)

The **Cartesian product** of two sets A and B , denoted by $A \times B$ is

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

- The order within the pair matters, so $(a, b) \neq (b, a)$.
- But, since $A \times B$ is a set, the order between the pairs is not important.

$$\{(a, b), (c, d)\} = \{(c, d), (a, b)\}$$

- The set $A \times B$ has $|A||B|$ elements.

Example

Example 2

The Cartesian product of $A = \{0, 1, 2, 3\}$ and $B = \{0, 1, 4\}$ is

$\{(0, 0), (0, 1), (0, 4), (1, 0), (1, 1), (1, 4), (2, 1), (2, 1), (2, 4), (3, 0), (3, 1), (3, 4)\}$

Or in Python*...

cartesian_product .py

```
1 A = {0,1,2,3}
2 B = {0,1,4}
3
4 C = {(a,b) for a in A for b in B}
5
6 print (C)
```

```
1 {(0, 1), (0, 0), (3, 0), (3, 1), (1, 4), (2, 1), (2, 0), (2, 4), (0, 4), (1, 4)}
```

*From now on I will break with the upper/lower case programming naming convention that you are probably using in Processing/Java and follow conventions in Mathematics.

Relation

I

The formal definition of a relation is based on the Cartesian product between two sets, later we will see more initiative but less general definitions.

Definition 3 (Relation)

Given two sets A and B . **Any** subset of the Cartesian product between A and B is called a **relation**.

Example 4

The Cartesian product of the sets $A = \{0, 1, 2, 3\}$ and $B = \{0, 1, 4\}$ is

$\{(0, 0), (0, 1), (0, 4), (1, 0), (1, 1), (1, 4), (2, 1), (2, 1), (2, 4), (3, 0), (3, 1), (3, 4)\}$

So possible relations between A and B include

4096 possible relations!

- $R = \{(0, 0), (1, 1), (2, 4)\}$ (relation is based on $x \mapsto x^2, x < 3$)
- $R = \{(0, 0), (1, 1), (2, 0), (3, 1)\}$ (relation is based on $x \mapsto x \bmod 2$)
- $R = \{(0, 0), (1, 0), (2, 1), (3, 1)\}$ (relation is based on $x \mapsto \text{isPrime}(x)$)
- $R = \{(0, 0), (1, 4)\}$ (relation is ... unknown?)
- $R = \{\}$ (remember, empty set is a ...)

Relation

II

You should spend some time thinking about the consequences of the definition that we have just covered ...

- *Given two sets, A and B , how many distinct relations can we construct?*
 - Set A and B have $|A|$ and $|B|$ elements respectively.
 - The Cartesian product, $A \times B$, has $|A||B|$ elements.
 - Relation between A and B is **any** subset of $A \times B$.
 - Sets of size n have 2^n subsets.

Number of distinct relations between sets A and B is $2^{|A||B|}$

- *Relation vs. Cartesian product vs. power set of the Cartesian product*

Let R be a relation between sets A and B . Then

- R is a subset of the Cartesian product of A and B .

$$R \subseteq A \times B$$

- R is an element of the power set of the Cartesian product of A and B .

$$R \in \mathcal{P}(A \times B)$$

Example

Example 5

Let $A = \{2, 3, 5, 6\}$ and define a relation R from A to A by $(a, b) \in R$ if and only if a divides evenly into b .

The relation R is defined by

$$R = \{(a, b) \mid a \in A, b \in A, a \text{ divided evenly into } b\}$$

The set of pairs that qualify for membership of R is

$$R = \{(2, 2), (3, 3), (5, 5), (6, 6), (2, 6), (3, 6)\}$$

Definition 6 (Relation on a Set)

A relation from set A to A is called a **relation on** A .

Notation Warning — Divisibility

When explaining relations we will often use (as in the previous example) the idea of “divides”. Lets make sure we all agree on what this means ...

Definition 7 (Divides)

Let $a, b \in \mathbb{Z}$. We say that a **divides** b , denoted $a \mid b$, if and only if there exists an integer k such that $ak = b$.

- Be careful in writing about the relation “divides.” The vertical line symbol use for this relation, if written carelessly, can look like division. While $a \mid b$ is either **True** or **False**, a/b is a number[†].
- Even worse. We, mathematicians, use the same symbol “|” for “such that” in set builder notation and for “divides”.
 - Usually this is not a problem as the intended meaning for “|” will be clear from the context.
 - Use alternative symbols: “|” is replaced by “:” in set builder notation.

[†]Also the direction is different. “ $a \mid b$ ” means “ a divides (evenly) into b ”, while “ a/b ” means “the value of a divided by b ”.

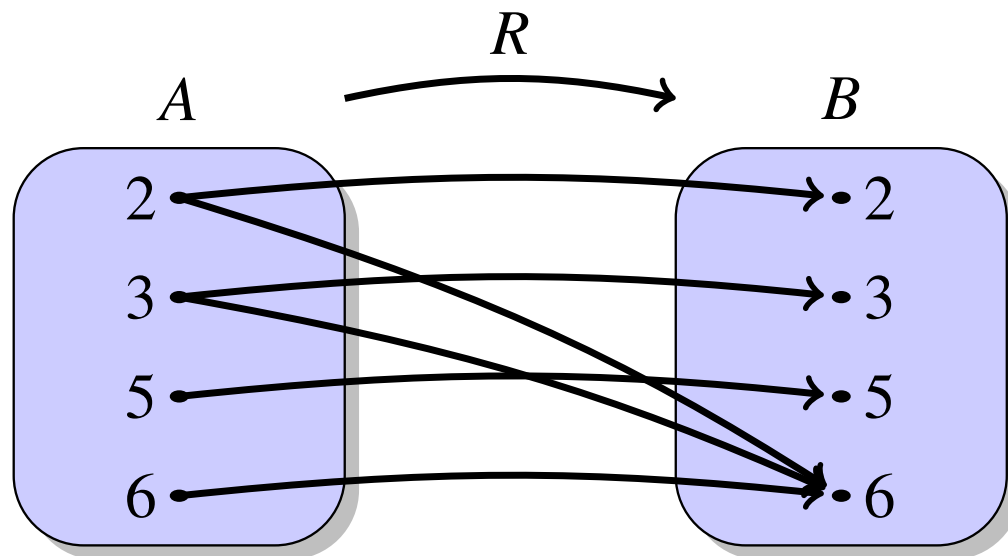
Graphical Representation of Relations I — Venn Diagrams

Representing relations graphically can help in identifying its properties ...

Consider the relation R from A into A , where $A = \{2, 3, 5, 6\}$ and $(a, b) \in R$ if and only if a divides evenly into b .

$$R = \{(2, 2), (3, 3), (5, 5), (6, 6), (2, 6), (3, 6)\}$$

- Draw set A .
- This relation is from A to A , so we make a copy of set A and called it B .
- Indicate each of the ordered pairs in R using an arrow.



Things we are interested in seeing ...

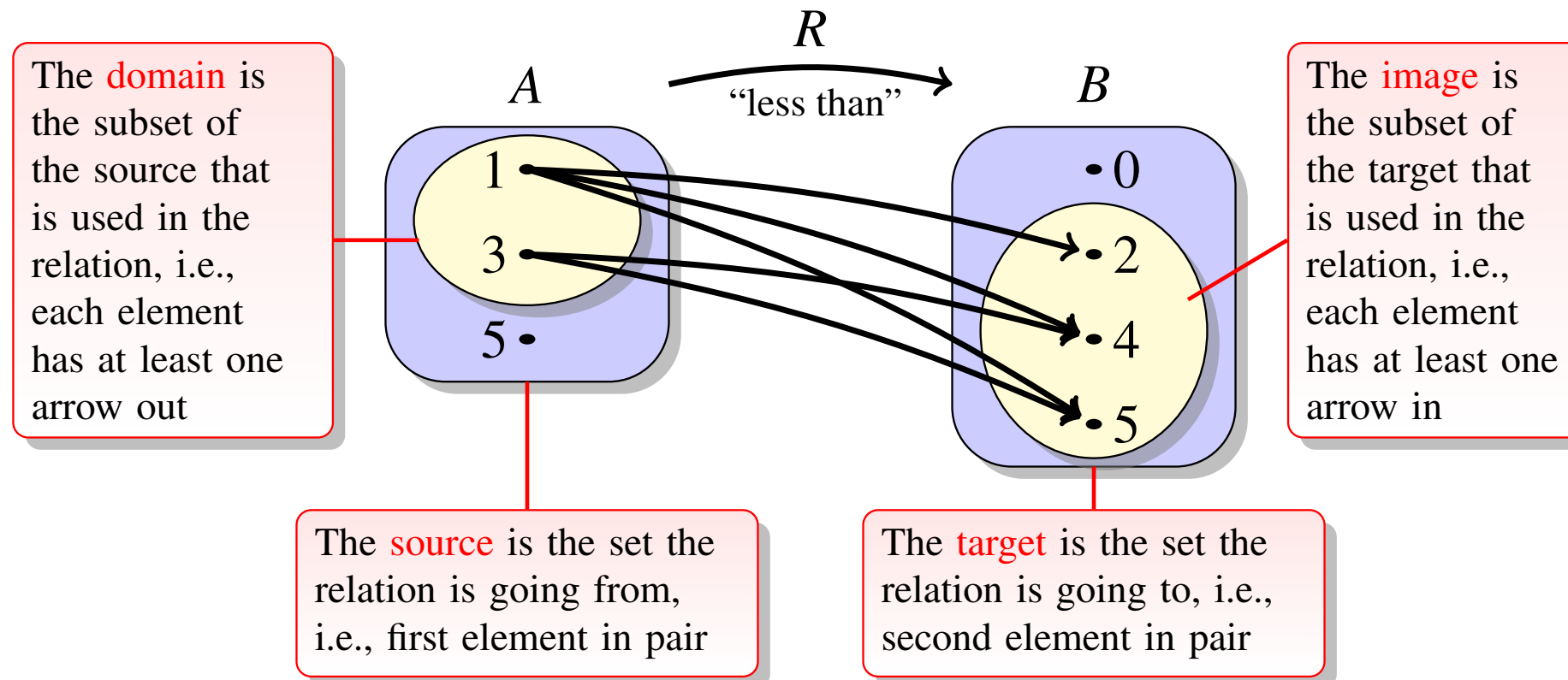
- Is there an arrow from every element in the first set?
- Is there an arrow to every element in the second set?
- Are there multiple arrows from some elements?
- Are there multiple arrows into some elements?

Relation Terminology

Consider the relation “is less than” from set $A = \{1, 3, 5\}$ to set $B = \{0, 2, 4, 5\}$. We have

$$R = \{(a, b) \mid a \in A, b \in B, a < b\} = \{(1, 2), (1, 4), (1, 5), (3, 4), (3, 5)\}$$

In python $R = \{ (a,b) \text{ for } a \text{ in } A \text{ for } b \text{ in } B \text{ if } a < b \}$



Relation Terminology

II

Given relation R from set S to set T we have:

- The **source**, S , is the set that the relation is going from.
- The **target**, T , is the set that the relation is going to.
- The **domain** of R , denoted by $\text{Dom}(R)$, is the subset of the source for which there is at least one arrow leaving each element.

$$\text{Dom}(R) = \{s \mid s \in S, \underbrace{\exists t \in T((s, t) \in R)}_{\text{exists at least one arrow leaving each element}}\} \subseteq S$$

- The **image** of R , denoted by $\text{Im}(R)$, is the subset of the target for which there is at least one arrow entering each element.

$$\text{Im}(R) = \{t \mid t \in T, \underbrace{\exists s \in S((s, t) \in R)}_{\text{exists at least one arrow entering each element}}\} \subseteq T$$

Relation Terminology — Into vs. Onto

III

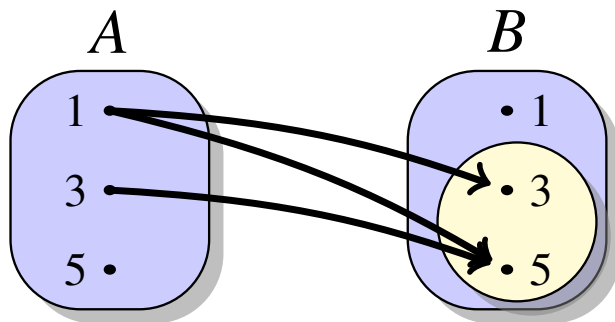
From our definitions, we have that the image of a relation is a subset of its target, i.e.,

$$\text{Im}(R) \subseteq T$$

This gives us two possibilities ...

$$\text{Im}(R) \subset T$$

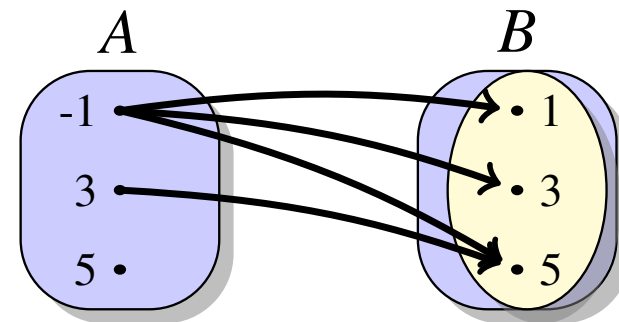
Example, consider relation “is less than” from set $A = \{1, 3, 5\}$ ~~to~~ **into** set $B = \{1, 3, 5\}$ is



A relation, R , in which the image is a proper subset of the target is said to be an **into** relation.

$$\text{Im}(R) = T$$

Example, consider relation “is less than” from set $A = \{-1, 3, 5\}$ ~~to~~ **onto** set $B = \{1, 3, 5\}$ is



A relation, R , in which the image is equal to the target is said to be an **onto** relation.

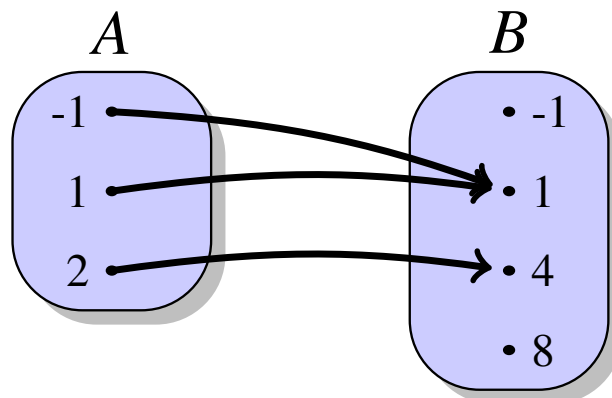
Relation Terminology — Injective (one-to-one)

III

Definition 8 (Injective)

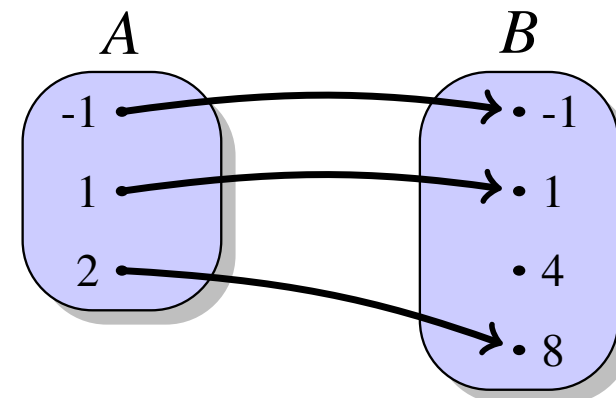
A relation is said to be **injective** (or **one-to-one**) if there is at most one arrow into every element in the target set.

Consider the relation “is square root of” from set $A = \{-1, 1, 2\}$ to set $B = \{-1, 1, 4, 8\}$.



Not injective, since there exists at least one element in the target, (1), which has more than one incoming arrows.

Consider the relation “is cube root of” from set $A = \{-1, 1, 2\}$ to set $B = \{-1, 1, 4, 8\}$.



Is injective, since there is at most one arrow into each element in the target.

Review Exercises 1 (Relation Definition)

Question 1:

Consider the sets $A = \{0, 1, \dots, 6\}$ and $B = \{0, 1, \dots, 12\}$. Draw each of the following relations, and specify the domain and image of R from A to B and whether it is into or onto, and injective or not.

- a) $(a, b) \in R$ iff $a \mid b$
- b) $(a, b) \in R$ iff $a > b$
- c) $(a, b) \in R$ iff number of primes less than a is equal to number of primes less than b
- d) $(a, b) \in R$ iff number of factors of a is equal to number of factors of b .
- e) $(a, b) \in R$ iff number of letters in writing a in English is equal number of letters in writing b in English.

Question 2:

Let R be the relation from \mathbb{N} to \mathbb{N} where $(a, b) \in R$ iff $b = a + 2$. Is R onto?

Question 3:

Let R be the relation from \mathbb{Z} to \mathbb{Z} where $(a, b) \in R$ iff $b = a + 2$. Is R onto?

Question 4:

Let R be the relation from \mathbb{N} to \mathbb{N} where $(a, b) \in R$ iff $b = a^2$. Is R one-to-one?

Question 5:

Let R be the relation from \mathbb{Z} to \mathbb{Z} where $(a, b) \in R$ iff $b = a^2$. Is R one-to-one?