


Discrete Mathematics

Number Theory

Topic 02 — Methods of Mathematical Proof

Mathematical Proofs

Lecture 02 — Proof by Contrapositive and by Cases

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Recurrence Relations

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Set Theory

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Outline

- Proof by Contrapositive
- Proof by Cases

Enumeration

Outline

1. Proof by Contrapositive 2
 - We prove a statement by first switching to the original statement to its contrapositive.
2. Proof by Cases 5
 - We prove a statement by breaking it up into smaller and easier cases, which we prove separately.
3. Proof by Contradiction 12
 - We prove a statement using the process:
 - assume reverse of statement . . .
 - derive conclusions from assumption . . .
 - show conclusions are contradictory . . .
 - hence assumption must be **False**, so original statement is **True**.
4. Proof by Construction 16
 - We prove the existence of something by giving the instructions needed to construct it.
5. Proof by Induction 20
 - Special proof technique used to prove a family of statements,

Proof by Contrapositive

Proof by Contrapositive

In a **proof by contrapositive** argument you prove the contrapositive of the claim rather than the claim itself.

Proof by Contrapositive (Formal Structure)

Given claim

$$P \implies Q$$

the contrapositive (and logically equivalent claim) is

$$\neg Q \implies \neg P$$

- 1 Assume $\neg Q$.
- 2 Demonstrate that $\neg P$ must follow from $\neg Q$.

Please, please, . . . , pretty please don't confuse this with proof by contradiction (covered later).

Example

Example 1

If x^2 is odd then x must be odd.

(by contrapositive)*.

The contrapositive is

If x is even, then x^2 is even.

We assume x is even. Hence we can write $x = 2k$ for some integer k . Now

$$x^2 = (2k)^2 = 4k^2 = 2 \underbrace{(2k^2)}_{\substack{\text{integer} \\ \text{even integer}}}$$

Hence the contrapositive is true, and so is the original statement. □

*The above proof is certainly doable by a direct proof. However, a direct proof requires a cumbersome proof by cases approach.

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Proof by Cases

Proof by Cases

In a **Proof by cases** argument you

- List of of the possible cases and analyse each separately.
- Need to ensure that the cases are exhaustive — cover all possibilities

Proof by Cases (Formal Structure)

Given claim

$$P \implies Q$$

- 1 Show that there exist a number of distinct cases C_1, C_2, \dots such that whenever P is true then at least one of the cases must be true.
- 2 Then, for each case, C , in C_1, C_2, \dots ,
 - 1 Assume case C .
 - 2 Demonstrate that Q must follow from C .

Example 2

Example 2

In a cave you find three boxes. One contains gold, the other two are empty. Each box has imprinted on it a clue as to its contents; the clues are:



A: The gold is not here



B: The gold is not here



C: The gold is in box B

Only one message is true; the other two are false. Which box has the gold?

- Notice that I changed the question to “Which box has the gold?”. I could have left it as “Prove that the gold is in box A.” since, for this problem the two versions are equivalent.

Example 2

II

In a proof by cases, there are three cases based on where the gold is located. In each case we check the truth value of the three messages[†]

Gold is in box A

Gold is in box B

Gold is in box C

[†]You might complain that in the direct proof we did earlier building a truth table is really a proof by cases. You would be correct.

Example 2

In a proof by cases, there are three cases based on where the gold is located. In each case we check the truth value of the three messages[†]

Gold is in box A

A: *“The gold is not here”*

B: *“The gold is not here”*

C: *“The gold is in box B”*

Gold is in box B

A: *“The gold is not here”*

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Gold is in box A

A: *"The gold is not here"* **F**

B: *"The gold is not here"* **T**

C: *"The gold is in box B"* **F**

Gold is in box B

A: *"The gold is not here"*

B: *"The gold is not here"*

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A: " <i>The gold is not here</i> "	F	} Exactly one message true? ✓
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Gold is in box B

A: "*The gold is not here*"
 B: "*The gold is not here*"
 C: "*The gold is in box B*"

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A: "*The gold is not here*"
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So in order that exactly one message is true, the gold must be in box A.

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Example 3

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Every group of 6 minions includes a group of 3 minions who all know each other or a group of 3 minions who are mutual strangers.



Call one of the minions Bob. There are five others. Either Bob knows three of them, or he does not know three of them.

CASE 1: Bob knows three of the five others ...

Say that Bob knows three of the five others. Of those five minions either there exists two minions who know each other or no two know each other.

CASE 1.1: Within the three minions, there exists two who know each other ...

Then those two and Bob form a mutually acquainted threesome.

CASE 1.2: No two of the three minions know each other ...

Then any three of the five minions are a mutually unacquainted threesome.

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Example 3

II

CASE 2: *Bob does not know three of the five others ...*

CASE 2.1: No two of the three minions know each other ...

Then those two and Bob form a mutually unacquainted threesome.

CASE 2.2: All pairs within the three minions know each other ...

Then any three of the five minions are a mutually acquainted threesome.

We have covered all possibilities, and in every instance come up either with a mutually acquainted threesome or a mutually unacquainted threesome.

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Examples

- a) Prove that for any integer n , the number $(n^3 - n)$ is even.
- b) Prove that every prime number greater than 3 is either one more or one less than a multiple of 6. Hint. Prove the contrapositive by cases.
- c) Let a, b, c, d be integers. If $a > c$ and $b > c$, then $\max(a, b) - c$ is always positive.