Examiners' reports 2012
CO1102 Mathematics for computing
Zone A

General remarks

The overall performance on this paper was satisfactory, and students seem to be coping well with most of the subject.

However, as in last year's report, it must be pointed out that many candidates have problems with precise mathematical notation and exact definitions. Too many 'easy' marks on the paper were lost because of this, and a not insignificant number of failing candidates could have passed, had they been more proficient in reading and writing mathematics and had they mastered the terminology surrounding the topics taught. While some batches of scripts were better than others in this respect, it was disappointing that overall we could not identify any improvement over last year on this serious issue. For example, in a question like Question 1(c) a large number of candidates did not know that $\mathbb{R}$ denotes the set of real numbers while $\mathbb{Q}$ denotes the set of rational numbers; while in Question 4(b) the majority of candidates were not able to define properly the two logarithmic functions or explain O-notation. Moreover, in Question 4(b)(ii) quite a number of candidates were not able to work out $\log_2(16)$, giving the value of $\log_{10}(16)$ or $\ln(16)$ instead. In Question 9(a) a substantial number of candidates did not know the difference between a walk, a cycle and a path in a graph.

Further, many candidates are not sufficiently proficient in basic algebra; for example, Question 4(a) required the use of rules of indices and logarithms, and a large proportion of the candidates gave wrong or incomplete answers to this question. The problem with lack of confidence in manipulating expressions with indices also surfaced in Question 10(c). In this question we were pleased to see that a large number of candidates understand the structure of a proof by induction, but unfortunately poor algebra skills often prevent them from completing the induction step of their proofs. The questions on matrix algebra and algebra of functions were not answered well either. Many candidates did not appreciate the significance of the matrix product being non-commutative and thus failed to spot the false statement of Question 7(b)(iii). With respect to the algebra of functions in Question 4(c), a substantial number of candidates erroneously interpreted the composite function $f(f(x))$ as $f(x) \cdot f(x)$. Candidates are advised to ponder the difference between these two, in mathematics as well as in the context of function calls in computing. Candidates are also advised to pay attention to the notation used in the subject guide and to strive to adopt this notation in their own work. When you have solved an exercise, always compare your notation as well as your answers to the model answers provided in the subject guide or by your lecturer. If your notation is substantially different, try to improve it. If you are working in a study group, do spend some time discussing with each other how you present answers, and during examination revision, test one another in the definitions and notation so that you become confident in explaining the basics.

Another frequent mistake made by a large number of candidates is not to show their working and calculations and not to explain the reasoning behind an answer. This generally results in the loss of marks; for example, in a question like 1(a) the Examiners want to see you perform the conversion and in Question 1(b) they want you to demonstrate clearly that you are working in binary. Questions requiring a short answer are best answered by initially giving a short answer to the question followed by your **reason** for this particular answer. In a question like 9(b)(i) for example, the Examiners want you to explain briefly why a relation is an equivalence relation. Stating that the relation is reflexive, symmetric and transitive is only half the answer, you need to explain subsequently why this is the case. Remember that Examiners are usually more interested in how you arrived at an answer than in the answer itself!

What follows are some answers, hints, solutions and comments on the examination questions that may help you, when you are revising for your examination.

Comments on specific questions

Question 1

(a) The answer is $(11110111)_2$. You must show all the successive divisions by 2 with remainders and explain how to get the answer.

(b) The answer is $(101011)_2$. It is important to show all carries.

(c) The answer to (i) is the set of rational numbers in A, namely, $\{\frac{3}{2}, 2\}$. The answer to (ii) is the set of irrational numbers in A, namely, $\{\sqrt{2}\}$. The answer to (iii) is the union of all irrational numbers and rational numbers in $\mathbb{R}$, that is, all of $\mathbb{R}$.

(d) Let $y = 1.247247247\ldots$. Then $1000y = 1247.247247\ldots$.
Subtraction yields $999y = 1000y - y = 1246$, and hence $y = 1246/999$.

Question 2

(a) The truth set of p consists of all the factors of 36, namely, $\{\pm1, \pm2, \pm3, \pm4, \pm6, \pm9, \pm12, \pm18, \pm36\}$.

(b) (i) $p \to q \vee r$ with truth set $\{\pm1, \pm2, \pm3, \pm4, \pm9\}$, as this statement is true unless p is true while $q \vee r$ is false.

(ii) $q \wedge r \to p$ with truth set $\mathbb{Z}$, because this statement is true unless $q \wedge r$ is true while p is false, which never happens.

(c) The truth table asked for is

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$	$p \to q$	$q \to p$	$p \leftrightarrow q$
0	0	0	1	1	1	1	1	1	1
0	1	0	1	0	0	0	1	0	0
1	0	0	0	1	0	0	0	1	0
1	1	1	0	0	0	1	1	1	1

The table shows that the two statements are true under exactly the same conditions. They are thus logically equivalent. Many candidates computed the correct truth table, but failed to explain why it shows that the two given statements are logically equivalent.

(d) By part c) the required network is the one corresponding to $(p \wedge q) \vee (\neg p \wedge \neg q)$. Make sure you draw the gates clearly and label your network.

Question 3

(a) In part (i), remember to give the key to the shading on your Venn diagram. In part (iv) the reason why Y is not a subset of X is that in the membership table the column for X does not always have a 1 when there is a 1 in the column for Y.

(b) The set is $\{-1/4, -1/2, -1, -2, -4, -8\}$. Many candidates got the signs wrong in this set, working out $(-2)^n$ instead of -2^n.

(c) Several answers are possible here, for example,
$\{\frac{2n-1}{2n} | n \in \mathbb{Z}, 1 \leq n \leq 50\}$ or $\{1 - \frac{1}{2n} | n \in \mathbb{Z}, 1 \leq n \leq 50\}$.

(d) $\mathcal{P}(S) = \{\emptyset, \{0\}, \{2\}, \{4\}, \{0,2\}, \{0,4\}, \{2,4\}, S\}$.

Question 4

(a) $4 \cdot 2^x - 2^{x+1} = 4 \cdot 2^x - 2 \cdot 2^x = 2 \cdot 2^x = 2^{x+1}$,

$$\frac{\ln(2) + \ln(2^2) + \ln(2^3) + \ln(2^4) + \ln(2^5)}{\ln(4)} = \frac{(1+2+3+4+5)\ln(2)}{2\ln(2)} = \frac{15}{2}.$$

(b) (i) $\log_2(x) = y$ when $x = 2^y$ and $\log_4(x) = z$ when $x = 4^z$.

(ii) $\log_2(16) = 4$ because $16 = 2^4$. Similarly $\log_2(\frac{1}{16}) = -4$

(iii) The best answer is to give a formal definition here: $f(x)$ is $O(\log_2(x))$ if there is a positive constant k and a number N such that for all $x \geq N$ we have $|f(x)| \leq k|\log_2(x)|$. Part credit was also given for good attempts at explaining O-notation less formally.

(iv) As $\log_4(x) = \dfrac{\log_2(x)}{\log_2(4)} = \frac{1}{2}\log_2(x)$ for all $x > 0$,
we have that $\log_4(x)$ is $O(\log_2(x))$.

(c) (i) The table for $g(x)$ is

x	1	2	3	4	5	6
$g(x)$	2	3	1	6	4	5

(ii) g is the inverse of f because $g(f(x)) = f(g(x)) = x$ for all $x \in \{1,2,3,4,5,6\}$. A good answer to this question verifies this by computing a table showing that $g \circ f$ is indeed the identity function on $\{1,2,3,4,5,6\}$.

Question 5

(a) Recall $\lfloor x \rfloor = n \in \mathbb{Z}$ if $n \leq x < n+1$. Thus $f(-6) = -3$ and $f(6) = 3$. f is not one-to-one as for example, $f(5) = 3 = f(6)$ while $5 \neq 6$. f is onto, and to get full marks here you would need to show that this is so by finding a preimage for any $n \in \mathbb{Z}$. For example, take any $n \in \mathbb{Z}$, then $2n-1 \in \mathbb{R}$ and $f(2n-1) = n$.

(b) (i) The root is $\lfloor \frac{200+1}{2} \rfloor = 100$.

(ii) The left subtree contains records 1 to 99, so the root is 50. The right subtree contains records 101 to 200, so the root is 150. This records 50 and 150 are at level 1 of the tree.

(iii) The root of the tree T_n is $\lfloor \frac{n+1}{2} \rfloor = r$ is given. But the floor function is not one-to-one (compare part (a)), and it is thus not possible to determine n. A small calculation shows n could be either $2r$ or $2r-1$.

Question 6

(a) (i) The total number of committees possible is $\binom{15}{6} = 5005$. If a committee has all 5 women in it, it contains the women plus one man. Hence there are 10 such committees and so the required probability is $P(\text{5 women, 1 man}) = 10/5005 = 2/1001$.

(ii) $P(\text{at least one woman}) = 1 - P(\text{no women}) = 1 - P(\text{6 men}) = 1 - \frac{\binom{10}{6}}{5005} = 137/143$.

(b) (i) $\lfloor \frac{600}{5} \rfloor = 120$.

(ii) 150 numbers are divisible by 4, 120 numbers are divisible by 5, 100 numbers are divisible by 6, 30 numbers are divisible by both 4 and 5, 50 numbers are divisible by both 4 and 6, 20 numbers are divisible by both 5 and 6, and finally 10 numbers are divisible by all three. Hence, by the Principle of Inclusion-Exclusion $150 + 120 + 100 - 30 - 50 - 20 + 10 = 280$ numbers are divisible by 4, 5 or 6.

Question 7

(a) The solution is $(x, y, z) = (1, -1, 2)$.
Note that you are required to solve the system using Gaussian elimination, hence you must clearly demonstrate that you have done so, and any other solution method gets little or no credit. It is also a good idea to write down all the elementary row operations you do on the augmented matrix.

(b) Many candidates provided wrong answers to this question, especially part (iii). It is important to note that matrix addition is commutative while matrix multiplication is not, so care has to be taken when multiplying out brackets.

(i) True, as due to the Distributive Law and commutativity of matrix addition we have $(A+B)C = AC + BC = BC + AC$. Note that it is necessary to work with three general matrices when proving this rule; it is not sufficient to provide an example of three matrices satisfying it.

(ii) True, B is the 2×2 identity matrix.

(iii) False, $(A+B)^2 = A^2 + AB + BA + B^2$ and $AB + BA \neq 2AB$ except in very rare cases! In order to gain full marks in this part, a counterexample must then be given here. For example, let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then

$$(A+B)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

while

$$A^2 + 2AB + B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

(c) If the number of columns of A differs from the number of rows of B, then the product AB cannot be computed.

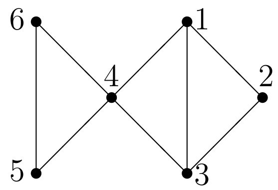
Question 8

(a) (i) Non-isomorphic graphs with degree sequence $4, 3, 3, 3, 2, 1$ exist. A good answer to this question gives two such graphs and explains why they are not isomorphic.

(ii) G has 6 vertices, one vertex for each number in the degree sequence.

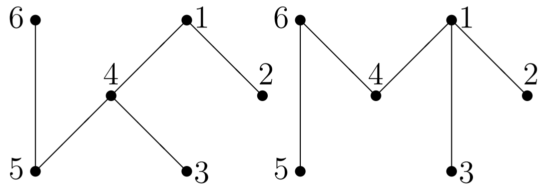
(iii) The sum of the degrees of the vertices of G is 16 and this is twice the number of edges of G, so G has 8 edges.

(b) (i) The graph G is



(ii) H is a spanning tree of G if H is a tree (that is, connected and acyclic) and contains all the vertices of G.

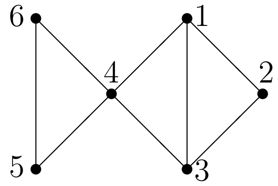
(iii) For example:



These two spanning trees are not isomorphic as in one of them the two vertices of degree 2 are adjacent while in the other they are not. Yet both have the degree sequence $3, 2, 2, 1, 1, 1$.

Question 9

(a) The graph G is



(i) For example, 1234 is a path of length 3.

(ii) For example, 1231 is a 3-cycle.

(iii) For example, 1232 is a walk of length 3 which is neither a path nor a cycle.

(b) (i) R is reflexive, symmetric and transitive, hence it is an equivalence relation. A good answer here also explains why the relation has the three properties, that is, it is reflexive because every walk has the same length as itself, it is symmetric because if walk W_1 has the same length as walk W_2 then W_2 also has the same length as W_1; and it is transitive because if W_1 has the same length as walk W_2 and W_2 has the same length as W_3 then W_1 has the same length as walk W_3 also.

(ii) The walk 123 is of length 2, hence the equivalence class containing it consists of all walks of length 2 in the graph.

(iii) Compute

$$A^2 = \begin{pmatrix} 3 & 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 1 & 1 & 1 \\ 1 & 2 & 1 & 4 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}.$$

Each walk of length 2 is recorded in A^2 exactly once, hence the cardinality of the equivalence class containing the walk 123 is the sum of the entries in A^2, that is, 46.

Question 10

(a) $s_0 = 2^0 = 1$, $s_1 = 2^0 + 2^2 = 1 + 4 = 5$, $s_2 = 2^0 + 2^2 + 2^4 = 1 + 4 + 16 = 21$, $s_3 = 2^0 + 2^2 + 2^4 + 2^6 = 1 + 4 + 16 + 64 = 85$ and $s_4 = 2^0 + 2^2 + 2^4 + 2^6 + 2^8 = 1 + 4 + 16 + 64 + 256 = 341$.

(b) $s_{n+1} = s_n + 2^{2n+2}$ for $n > 0$ and $s_0 = 1$.

(c) Many candidates did not attempt this question or did poorly. Many attempts at the proof start by writing down a good base and induction hypothesis, but then go wrong in the induction step because they mix up what they know and what they still need to prove. One good piece of advice, that will save many errors is that you should never write down an =-symbol without specifically checking that you are absolutely sure you know that what is on the left-hand side of it is equal to what is on the right-hand side of it. Further, getting through the induction step successfully requires the correct use of rules of indices.

When proving an identity by induction the proof has four steps: the base, setting up the induction hypothesis, the induction step and a final remark stating why the identity holds by induction.

When proving the base case and the induction step, you must demonstrate that the left-hand side (LHS) of the identity is equal to the right-hand side (RHS) of the identity for the case in question. It is usually best to keep the two computations completely separate in order not to confuse what you know and what you have not proven yet.

Here we want to prove the identity $3s_n = 2^{2n+2} - 1$ for all $n \geq 0$:

Base case: When $n = 0$, $LHS = 3s_0 = 3 \cdot 1 = 3$ and $RHS = 2^{0+2} - 1 = 3$. The identity thus holds for $n = 0$.

Induction hypothesis: Suppose the identity holds for some $k \geq 0$, that is,

$$3s_k = 2^{2k+2} - 1.$$

Induction step: We must prove that the identity also holds for $n = k + 1$, namely, that

$$3s_{k+1} = 2^{2(k+1)+2} - 1.$$

$RHS = 2^{2(k+1)+2} - 1 = 2^{2k+4} - 1.$

$$\begin{aligned} LHS &= 3s_{k+1} = 3(s_k + 2^{2k+2}) \text{ by the recurrence relation we found in (b)} \\ &= 3s_k + 3 \cdot 2^{2k+2} \\ &= 2^{2k+2} - 1 + 3 \cdot 2^{2k+2} \text{ by the induction hypothesis} \\ &= 4 \cdot 2^{2k+2} - 1 = 2^{2k+4} - 1 = RHS. \end{aligned}$$

Hence the result is also true for $n = k + 1$ and thus for all $n \geq 0$ by induction.