MA4016 - Engineering Mathematics 6

Problem Sheet 5: Master theorem (March 05, 2010)

1. For each of the following recurrence relations, give an estimate for $T(n)$ if the Master theorem applies, or state why the theorem does not apply.

a) $T(n) = 2T(n/2) + n\log n$
$g(n) = n\log n$, $a = 2$, $b = 2$, $\log_b a = 1$
Case II) $n\log n = \Theta(n\log^k n)$ for $k = 1$. Therefore $f(n) = \Theta(n\log^2 n)$.

b) $T(n) = 8T(n/4) + n$ $g(n) = n$, $a = 8$, $b = 4$, $\log_b a = 3/2$
Case I) $n = \mathcal{O}(n^{3/2-\varepsilon})$ for $\varepsilon = 1/2$. Therefore $f(n) = \Theta(n^{3/2})$.

c) $T(n) = 2T(n/4) + n^{0.6}$
$g(n) = n^{0.6}$, $a = 2$, $b = 4$, $\log_b a = 1/2$
Case III) $n^{0.6} = \Omega(n^{1/2+\varepsilon})$ for $\varepsilon = 0.1$.

$$c > \frac{2\left(\frac{n}{4}\right)^{0.6}}{n^{0.6}} = \frac{2}{4^{0.6}} \approx 0.871$$

Thus with $c = 0.9$ follows $f(n) = \Theta(n^{0.6})$.

d) $T(n) = \frac{1}{2}T(n/2) + n^2$
$a = 1/2 < 1$

e) $T(n) = 3T(n/3) + n/2$
$g(n) = n/2$, $a = 3$, $b = 3$, $\log_b a = 1$
Case II) $n/2 = \Theta(n\log^k n)$ with $k = 0$. Therefore $f(n) = \Theta(n\log n)$.

f) $T(n) = 4T(n/2) + \log n$
$g(n) = \log n$, $a = 4$, $b = 2$, $\log_b a = 2$
Case I) $\log n = \mathcal{O}(n^{2-\varepsilon})$ with $\varepsilon = 1$. Therefore $f(n) = \Theta(n^2)$.

2. Find $f(n)$ when $n = 4^k$, where f satisfies the recurrence relation

$$f(n) = 5f(n/4) + 6n,$$

with $f(1) = 1$ and estimate f if f is an increasing function.
$f(n) = f(4^k) == 5f(4^{k-1}) + 6\cdot 4^k$. With $f_k = f(4^k)$, $f(1) = f_0 = 1$ follows

$$f_k = 5f_{k-1} + 6\cdot 4^k.$$

$f_k^h = \alpha 5^k$, $f_k^p = p_0 4^k = -24\cdot 4^k \Rightarrow f_k = \alpha 5^k - 24\cdot 4^k$.
with initial condition follows

$$f_k = 5^{k+2} - 6\cdot 4^{k+1} = f(4^k) = f(n) = 25\cdot 5^{\log_4 n} - 24n = 25n^{\log_4 5} - 24n$$

Applying Master Theorem ($a = 5$, $b = 4$, $g(n) = 6n$, case I) gives $f(n) = \Theta(n^{\log_4 5})$

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3. Suppose that the function f satisfies the recurrence relation

$$f(n) = 2f(\sqrt{n}) + \log_2 n$$

whenever n is a perfect square greater than 1 and $f(2) = 1$.

a) Find $f(16)$.
$f(16) = 2f(4) + 4 = 4f(2) + 4 + 4 = 12$

b) Find a big-$\mathcal{O}$ estimate for $f(n)$.
Hint: Make the substitution $m = \log_2 n$.
$f(2^m) = 2f(2^{m-1}) + m$. Let $\tilde{g}(m) = f(2^m)$. Then

$$\tilde{g}(m) = 2\tilde{g}(m/2) + m$$

and Master Theorem ($a = 2$, $b = 2$, $g(m) = m$, case II) gives

$$\tilde{g}(m) = \Theta(m\log m)$$

with $m = \log_2 n$ follows

$$f(n) = f(2^m) = \tilde{g}(m) = \Theta(m\log m) = \Theta(\log n\cdot\log\log n).$$

4. An efficient algorithm for evaluating polynomials is called Horner's method. It uses the representation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = (((((a_n)x + a_{n-1})x + a_{n-2})x + \ldots)x + a_1)x + a_0.$$

Write an iterative pseudo code algorithm to compute the value of a polynomial given by a_0, a_1, ..., a_n at x. How many multiplications and additions are used to evaluate a polynomial of degree n at a position x?

```
procedure horner(x,a_0,a_1,...,a_n: real)
 y:=a_n
 for i:=n-1 downto 0
     y:=y*x+a_i
 end
 return(y)
end
```

Multiplications: n, Additions: n. Standard algorithm (represented by left side) uses $2n-1$ multiplications and n additions.

5. What sequences of pseudorandom numbers is generated using the linear congruential generator

$$x_{n+1} = (4x_n + 1) \mod 7, \qquad \text{with seeds } x_0 = 1, 2 \text{ and } 3?$$

Explain this behaviour.
For $x_0 = 1$ we obtain sequence $\overline{1, 5, 0}$, for $x_0 = 2$ the sequence $\overline{2}$ and for $x_0 = 3$ the sequence $\overline{3, 6, 4}$, with $\overline{\{a\}}$ indicating the subsequence $\{a\}$ that repeats itself. We have three different sequences with period less then $m = 7$. The reason is, that $a - 1 = 3$ is not divisible by $m = 7$ as needed for maximal period.