CIS 102 2002

Question 1 (a) Write the hexadecimal numeral $(1A9.5)_{16}$ in expanded form and hence find its decimal equivalent. [3]

(b) Working in base 2, compute the following binary multiplication, *showing all your working.*

$$(1011)_2 \times (111)_2.$$

[3]

(c) (i) Use the method of repeated division to convert the decimal number 459 to binary, *showing all your working.* [3]

(ii) The positive integer n is such that $2^7 \leq n < 2^8$. How many bits are required to express n in binary notation? [1]

Question 2 (a) (i) Let $S = \{w, x, y, z\}$. Describe briefly how the subsets of S can each be represented by a unique 4-bit binary string. [2]

(ii) Make a list of all 4-bit binary strings which have 1 as their first bit. Use this list to find all the subsets of S containing the element w. [3]

(iii) What is the total number of subsets of S? [1]

(b) (i) Draw a Venn diagram to show three subsets A, B, C of a universal set $\mathcal{U}$ intersecting in the most general way. Shade the regions contained in the subset X defined by the membership table below.

A	B	C	X
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

[2]

(ii) Describe the subset X in terms of the sets A, B, C, using any appropriate set operations. [2]

Question 3 (a) Let $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and let p, q be the following propositions concerning the integer n.

$$p\text{: } n \text{ is even;} \qquad q\text{: } n < 5.$$

Find the values of n for which each of the following compound statements is true.

$$\text{(i) } \neg p; \quad \text{(ii) } p \wedge q; \quad \text{(iii) } \neg p \vee q; \quad \text{(iv) } p \ominus q.$$

[4]

(b) (i) Let p, q be propositions. Construct the truth table for $p \rightarrow q$. [2]

(ii) Use truth tables to prove that $\neg q \rightarrow \neg p = p \rightarrow q$. [2]

(iii) Write the contrapositive of the statement:

If $n = 23$, then n is prime.

[2]

Question 4 (a) The function $f : \mathbf{Z} \to \mathbf{Z}$ is defined by the rule

$$f(m) = |m + 2|,$$

where $|m|$ denotes the absolute value of m.

(i) Calculate $f(-3)$. [1]

(ii) Calculate the set of ancestors (pre-images) of 10. [2]

(iii) Decide whether the function f has the *one-to-one* property, justifying your answer. [1]

(iv) Decide whether the function f has the *onto* property, justifying your answer. [1]

(b) Give the conditions to be satisfied by a function $f : X \to Y$ for it to have an inverse function $f^{-1} : Y \to X$. [1]

(c) The following functions f are both invertible. In each case, give a formula for the inverse function f^{-1}.

(i) $f : \mathbf{R}^+ \cup \{0\} \to \mathbf{R}^- \cup \{0\}$ defined by $f(x) = x^3$. [2]

(ii) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 4x + 3$. [2]

Question 5 (a) (i) Given a real number x, say how $\lfloor x \rfloor$, the *floor* of x, is defined. [1]

(ii) Find the value of $\lfloor 2.97 \rfloor$, $\lfloor -2.97 \rfloor$. [2]

(iii) Find an example of a real number x such that $\lfloor 2x \rfloor \neq 2\lfloor x \rfloor$, justifying your answer. [2]

(b) (i) Copy and complete the following table of values for the function $f(x) = \log_2 x$.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
$f(x)$							

[3]

(ii) Deduce the value of $\lfloor \log_2 12 \rfloor$ and $\lfloor \log_2 \frac{1}{3} \rfloor$. [2]

Question 6

A company operates an express coach service between seven cities, $c_1, c_2, \ldots, c_7$. The number of other cities to which each city is directly linked by a coach is given in the following table.

City	c_1	c_2	c_3	c_4	c_5	c_6	c_7
Number of connections	3	2	3	5	4	4	1

(a) Describe how such a communications network can be modelled by a *graph*, saying what the vertices represent and a rule for determining when two vertices are adjacent. [2]

(b) Calculate how many pairs of cities have a direct coach link between them, giving a brief explanation of your method. [3]

(c) What is meant by saying that a graph is *simple*? Say why a graph model of this communications network would be simple. [3]

(d) Is it possible to construct a graph with degree sequence 4, 4, 4, 3, 3, 2, 1? Either construct an example of such a graph, or say why it is not possible to do so. [2]

Question 10 A deck of cards has 52 playing cards. The deck has 4 suits (Spades, Hearts, Diamonds and Clubs), with 13 different cards in each suit. We can represent each card by its suit and number (from 1 to 13).

In an experiment two cards are chosen at random and removed from the deck of cards. Let A be the event that both cards are Spades, and let B be the event that one card is an Ace (with number 1) and the other card is a picture card (with number 11, 12 or 13).

(a) Describe a sample space to model the out comes of this experiment, giving the total number of possibilities. You may assume that the order in which the cards are choosen is not important. [4]

(b) Calculate the probabilities of A, B, $A \cap B$ and $A \cup B$. [4]

(c) Are A and B independent events? Justify your answer. [2]

Question 11 (a) Define matrices A, $\mathbf{x}$ and $\mathbf{b}$ to express the following system of equations as a matrix equation $A\mathbf{x} = \mathbf{b}$. [2]

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 3 \\ 2x & & & + & z & = & 4 \\ x & - & y & & & = & 3 \end{array}$$

Use Gaussian elimination to solve this system of equations. [5]

(b) Let

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}.$$

Calculate $2A - C$, AB and C^2. [3]

Question 12 Let G be a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and adjacency matrix $\mathbf{A}(G)$ given by

$$\mathbf{A}(G) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

(a) Find the degree of v_1. [1]

(b) Say how the number of edges in G is related to the sum of the enteries in its adjacency matrix $\mathbf{A}(G)$.

Hence find the number of edges of the graph G. [3]

(c) Calculate the *last row* of the matrix $\mathbf{A}(G)^2$. Hence, or otherwise, find the number of walks of length 2 starting at v_5 and ending at each of the vertices v_1, v_2, v_3, v_4. [4]

(d) Find the number of walks of length 3 starting at v_5 and ending at [illegible]. [2]

Question 7 (a) Let S be a set and let $\mathcal{R}$ be a relation on S. Explain what it means to say that $\mathcal{R}$ is:

(i) reflexive;

(ii) symmetric;

(iii) antisymmetric;

(iv) transitive.

[4]

(b) Let $S = \{1, 2, 3, 4, 6, 12\}$. Define a relation $\mathcal{R}$ between the elements of S by

"x is related to y whenever x divides y".

(i) Draw the relationship digraph. [2]

(ii) Determine whether or not $\mathcal{R}$ is reflexive, symmetric, antisymmetric or transitive. For the case(s) when one of these properties does not hold justify your answer by giving an example to show that the property does not hold. [2]

(iii) State, giving reasons, whether or not $\mathcal{R}$ is an equivalence relation and whether or not it is a partial order. [2]

Question 8 (a) A sequence is defined by the recurrence relation

$$x_{n+2} = 3x_{n+1} - 2x_n$$

and initial terms and $x_1 = 1$ and $x_2 = 3$.

(i) Calculate x_3 and x_4, showing your working. [2]

(ii) Prove by induction that $x_n = 2^n - 1$ for all $n \geq 1$. [6]

(b) Evaluate $\sum_{n=1}^{50} (5n - 2)$. [2]

Question 9 (a) What properties must a graph satisfy in order for it to be a *tree*? [2]

(b) (i) Design a balanced binary search tree for an ordered list of 11 records. Label the records $1, 2, \ldots, 11$ in your tree. [4]

(ii) What is the height of the tree that you have constructed? [1]

(iii) What is the maximum number of comparisons that would have to be made to match any existing record? This number should include the final comparison that determines the match. [1]

(c) Calculate the least height of a binary search tree that has 1000 records stored at its internal nodes. [2]