MA4016 - Engineering Mathematics 6

Solution Sheet 4: Recurrence Relations (February 26, 2010)

1. Solve the homogeneous recurrence relations for the given initial conditions.

a) $a_n = 2a_{n-1} + 8a_{n-2},\ n \geq 2,\ a_0 = 4,\ a_1 = 10$
general solution: $a_n = c_1(-2)^n + c_2 4^n$
final solution: $a_n = (-2)^n + 3 \cdot 4^n$

b) The Lucas sequence: $L_n = L_{n-1} + L_{n-2},\ n \geq 2,\ L_0 = 2,\ L_1 = 1$
same recurrence relation as Fibonacci sequence. Therefore general solution

$$a_n = c_1 \left(\frac{1-\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1+\sqrt{5}}{2}\right)^n$$

and with initial conditions

$$a_n = \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^n.$$

c) $a_n = a_{n-1} + a_{n-2} - a_{n-3},\ n \geq 3,\ a_0 = 0,\ a_1 = 3,\ a_2 = 2$
characteristic polynomial $r^3 - r^2 - r + 1 = (r-1)(r^2-1)$ has roots $r_1 = 1,\ r_2 = 1,\ r_3 = -1$.
general solution: $a_n = c_1 + c_2 n + c_3(-1)^n$
final solution: $a_n = 1 + n - (-1)^n$

d) $a_n = 5a_{n-2} - 4a_{n-4},\ n \geq 4,\ a_0 = 3,\ a_1 = 2,\ a_2 = 6,\ a_3 = 8$
general solution: $a_n = c_1(-2)^n + c_2(-1)^n + c_3 + c_4 2^n$
final solution: $a_n = (-1)^n + 1 + 2^n$

2. What is the general form of the solution of a homogeneous linear recurrence relation if its characteristic equation has the roots

a) $1, 1, 1, 1, -2, -2, -2, 3, 3, -4$?
$a_n = (c_{0,1} + c_{1,1}n + c_{2,1}n^2 + c_{3,1}n^3) + (c_{0,2} + c_{1,2}n + c_{2,2}n^2)(-2)^n + (c_{0,3} + c_{1,3}n)3^n + c_{0,4}(-4)^n$

b) $-1, -1, -1, 2, 2, 5, 5, 7$?
$a_n = (c_{0,1} + c_{1,1}n + c_{2,1}n^2)(-1)^n + (c_{0,2} + c_{1,2}n)2^n + (c_{0,3} + c_{1,3}n)5^n + c_{0,4}7^n$

3. What is the ansatz function for the general solution of the nonhomogeneous linear recurrence relation

$$a_n = 8a_{n-2} - 16a_{n-4} + F(n) \text{ if}$$

a) $F(n) = n^3$?	**b)** $F(n) = (-2)^n$?	**c)** $F(n) = n2^n$?
d) $F(n) = n^2 4^n$?	**e)** $F(n) = (n^2-2)(-2)^n$?	**f)** $F(n) = n^4 2^n$?
g) $F(n) = 2$?	**h)** $F(n) = n + 2^n$?	**i)** $F(n) = 2^n + (-2)^2$?

a) $F(n) = n^3, \Rightarrow s = 1,\ m = 0,\ t = 3, \Rightarrow a_n^p = (p_0 + p_1 n + p_2 n^2 + p_3 n^3)1^n$
b) $F(n) = (-2)^n, \Rightarrow s = -2,\ m = 2,\ t = 0, \Rightarrow a_n^p = n^2 p_0(-3)^n$
c) $F(n) = n2^n, \Rightarrow s = 2,\ m = 2,\ t = 1, \Rightarrow a_n^p = n^2(p_0 + p_1 n)2^n$
d) $F(n) = n^2 4^n, \Rightarrow s = 4,\ m = 0,\ t = 2, \Rightarrow a_n^p = (p_0 + p_1 n + p_2 n^2)4^n$
e) $F(n) = (n^2-2)(-2)^n, \Rightarrow s = -2,\ m = 2,\ t = 2, \Rightarrow a_n^p = n^2(p_0 + p_1 n + p_2 n^2)(-2)^n$
f) $F(n) = n^4 2^n, \Rightarrow s = 2,\ m = 2,\ t = 4, \Rightarrow a_n^p = n^2(p_0 + p_1 n + p_2 n^2 + p_3 n^3 + p_4 n^4)2^n$
g) $F(n) = 2, \Rightarrow s = 1,\ m = 0,\ t = 0, \Rightarrow a_n^p = p_0$
h) $F(n) = n + 2^n, \Rightarrow \left.\begin{array}{l} F_1(n) = n,\ s = 1,\ m = 0,\ t = 1 \\ F_2(n) = 2^n,\ s = 2,\ m = 2,\ t = 0 \end{array}\right\}, \Rightarrow a_n^p = (p_0 + p_1 n) + n^2 p_2 2^n$
i) $F(n) = 2^n + (-2)^n, \Rightarrow \left.\begin{array}{l} F_1(n) = 2^n,\ s = -2,\ m = 2,\ t = 0 \\ F_2(n) = (-2)^n,\ s = 2,\ m = 2,\ t = 0 \end{array}\right\}, \Rightarrow a_n^p = n^2(p_0 2^n + p_1(-2)^n)$

4. Use a nonhomogeneous recurrence relation to find a formula for $\sum_{k=1}^n k^2 = a_n$.
$a_n = a_{n-1} + n^2,\ a_n^h = c_0,\ a_n^p = n(p_0 + p_1 n + p_2 n^2) = n(n+1)(2n+1)/6$
final solution: $a_n = n(n+1)(2n+1)/6$

5. Solve the simultaneous recurrence relations

$$\begin{aligned} a_n &= 3a_{n-1} + 2b_{n-1}, \quad a_0 = 1 \\ b_n &= a_{n-1} + 2b_{n-1}, \quad\quad b_0 = 2 \end{aligned}$$

a) by elimination.

$$\begin{aligned} a_{n+1} &= 3a_n + 2b_n \\ &= 2a_n + 2(a_{n-1} + 2b_{n-1}) \\ &= 3a_n + 2a_{n-1} + 2(a_n - 3a_{n-1}) \\ &= 5a_n - 4a_{n-1} \end{aligned}$$

has solution $a_n = c_1 + c_2 4^n$ and therefore $b_n = (a_{n+1} - 3a_n)/2 = -c_1 + c_2 4^n/2$.
final solution: $a_n = 2 \cdot 4^n - 1$, $b_n = 4^n + 1$.

b) with the discrete Putzer algorithm.
$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$.
Sequence of matrices:

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

Sequence of scalars:

$$s_1(n) = 1, \quad s_2(n) = \sum_{j=0}^{n-1} \lambda_2^{n-1-j} s_1(j) = \sum_{k=0}^{n-1} 4^k = \frac{4^n - 1}{3}$$

$$A^n = \sum_{i=1}^{2} s_1(n) M_{i-1} = \frac{1}{3} \begin{pmatrix} 2 \cdot 4^n + 1 & 2 \cdot 4^n - 2 \\ 4^n - 1 & 4^n + 2 \end{pmatrix}$$

and the solution follows by

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = A^n \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \cdot 4^n + 1 & 2 \cdot 4^n - 2 \\ 4^n - 1 & 4^n + 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 4^n - 1 \\ 4^n + 1 \end{pmatrix}$$