UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

MID TERM ASSESSMENT PAPER

MODULE CODE: MA4016

SEMESTER: Spring 2008

MODULE TITLE: Engineering Mathematics 6

DURATION OF EXAMINATION: 45 minutes

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 20 %

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Answer both questions. Each question is worth 10 marks. Each part of Question 2 carries equal marks. Use the Answer Sheet provided for Question 2.**

ANSWER SHEET

STUDENT'S NAME:
STUDENT'S ID NUMBER:

For each part of Question 2, place an "**X**" in the box of your choice.

Question	a	b	c	d	e	Do not write in this column
(i)		**X**				
(ii)		**X**				
(iii)					**X**	
(iv)	**X**					
(v)				**X**		

1. Solve the difference equation

$$\begin{aligned} x_{n+1} &= y_n, & x_0 = 0 \\ y_{n+1} &= -12x_n + 7y_n + 2^n, & y_0 = 1 \end{aligned}$$

2. (i) Algorithm A1 solves a problem of size n using $O(n^3)$ operations, while algorithm A2 solves the same problem with $O(n^2 \log n)$ operations. Which algorithm is the more efficient of the two in terms of operations used for large n ?
(a) A1 (b) A2 (c) Either one
(d) It depends on n (e) Not computable from information given

(ii) The standard polynomial-time algorithm to compute the determinant of a $n \times n$ matrix requires a number of scalar multiplications which is $\Theta(f(n))$ where $f(n) =$
(a) n (b) n^3 (c) n^4
(d) $n!$ (e) Not computable from information given

(iii) A 3-state *Markov* Chain with an absorbing state (F) accessible only from one of the other two states has equilibrium probability $\lim_{n\to\infty} p_F(n) =$
(a) 0 (b) $\frac{2}{3}$ (c) 1
(d) $\frac{3}{2}$ (e) Not computable from information given

(iv) For the *Markov* Chain shown in Fig 1, the probability of arriving at state C at time $n+1$ is given by $p_C(n+1) =$

$$(a)\ \frac{1}{3}p_B(n)+\frac{1}{2}p_C(n) \quad (b)\ -\frac{1}{6}p_B(n)+\frac{1}{2}p_C(n) \quad (c)\ \frac{1}{3}p_B(n)+p_C(n)$$

$$(d)\ 1-(p_A(n)+p_B(n)) \quad (e)\ p_C(n)$$

(v) The complete solution of $a_{n+2} = 4a_{n+1} - 3a_n,\ a_0 = 1, a_1 = 3$ is given by
(a) 0 (b) 1 (c) $3(-1)^n - 2(-3)^n$
(d) 3^n (e) Not computable from information given

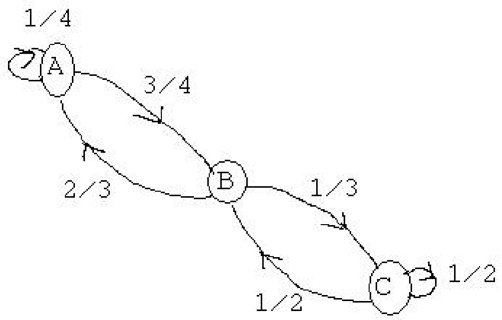

Figure 1: Markov Chain

Question 1 Solution

Either - convert to 2nd order difference equation:

$$x_{n+2} = y_{n+1} = -12x_n + 7y_n + 2^n = -12x_n + 7x_{n+1} + 2^n$$

$$\Rightarrow x_{n+2} - 7x_{n+1} + 12x_n = 2^n, \qquad x_0 = 0, x_1 = y_0 = 1$$

Homogeneous Equation:

$$x^h_{n+2} - 7x^h_{n+1} + 12x^h_n = 0$$

has characteristic equation

$$\begin{aligned} C^2 - 7C + 12 &= 0 \\ \Rightarrow C &= 3 \text{ or } 4 \\ \Rightarrow x^h_n &= A3^n + B4^n \end{aligned}$$

Particular Solution: $x^p_n = C2^n$. The (original) difference equation becomes

$$\begin{aligned} C2^{n+2} - 7C2^{n+1} + 12C2^n &= 2^n \\ \Rightarrow 4C - 14C + 12C &= 1 \\ \Rightarrow C &= \frac{1}{2} \end{aligned}$$

The general solution is thus $x_n = A3^n + B4^n + \frac{1}{2}2^n$. The initial conditions (IC) give

$$\begin{aligned} 0 = x_0 &= A + B + \frac{1}{2} \\ 1 = x_1 &= 3A + 4B + 1 \\ \Rightarrow A &= -2 \\ B &= \frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} x_n &= -2(3^n) + \frac{3}{2}(4^n) + \frac{1}{2}(2^n) \\ y_n = x_{n+1} &= -6(3^n) + 6(4^n) + 2^n \end{aligned}$$

Or - deal with as a system of equations:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -12 & 7 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} 0 \\ 2^n \end{pmatrix}, \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

has solution

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -12 & 7 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=0}^{n-1} \begin{pmatrix} 0 & 1 \\ -12 & 7 \end{pmatrix}^{n-1-j} \begin{pmatrix} 0 \\ 2^j \end{pmatrix}$$

The system matrix is diagonalisable:

$$\begin{pmatrix} 0 & 1 \\ -12 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ -12 & 7 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & 4^n \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 4(3^n) - 3(4^n) & -3^n + 4^n \\ 12(3^n - 4^n) & -3^{n+1} + 4^{n+1} \end{pmatrix}$$

so

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= \begin{pmatrix} -3^n + 4^n \\ -3^{n+1} + 4^{n+1} \end{pmatrix} + \sum_{j=0}^{n-1} \begin{pmatrix} -3^{n-1-j} + 4^{n-1-j} \\ -3^{n-j} + 4^{n-j} \end{pmatrix} 2^j \\ &= \begin{pmatrix} -3^n + 4^n \\ -3^{n+1} + 4^{n+1} \end{pmatrix} + \begin{pmatrix} -3^n + \frac{1}{2}(4^n) + \frac{1}{2}(2^n) \\ -3^{n+1} + 2(4^n) + 2^n \end{pmatrix} \\ &= \begin{pmatrix} -2(3^n) + \frac{3}{2}(4^n) + \frac{1}{2}(2^n) \\ -6(3^n) + 6(4^n) + 2^n \end{pmatrix} \end{aligned}$$