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BSc and Diploma in
Computing and Related subjects

Mathematics for computing
Volume 2

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This subject guides is for the use of the University of London External students registered for programmes in the field of Computing. The programmes currently available in these subject areas are:

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BSc (Honours) in Creative Computing

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Diploma in Creative Computing

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Contents

Preface

Introduction to Volume 2

This is Volume 2 of the subject guide for 2910102 [CIS102]. The guide often builds upon ideas and topics from Volume 1. In particular we use and develop further the notation you met in Volume 1. It is thus a good idea to have your copy of Volume 1 at hand for reference, when reading Volume 2.

A list of symbols is included as an Appendix. You are also reminded that there is a list of symbols in an Appendix to Volume 1.

The aims and objectives of the unit

Computer science depends upon the science of mathematics and without the mathematical ideas that underpin it, none of the marvels of modern computer technology would be possible. In order to study for a degree in Computing and Information Systems, you need to understand and feel easy with some essential mathematical ideas. The topics in this subject have been selected with that in mind and you will find that you use many of the ideas and skills introduced in this unit directly or indirectly in the other subjects in this degree programme. You will also gain experience of the way that mathematicians and computer scientists express their ideas, and you will learn how to use symbols to make statements precise.

Although this subject does not go very deeply into any one topic, we hope that you will gain sufficient confidence from studying it to consult mathematical and statistical textbooks to pursue areas that particularly interest you or about which you need more information. Currently, there are two optional half units available at Level 3 of this degree programme that develop aspects of this unit directly.

Using this subject guide effectively

The subject guide for 2910102 [CIS102] is in two volumes, and the material is presented in a convenient order of study. Taken together, the two volumes contain a complete account of the examinable topics in the unit. The chapters are not all of equal length and the time you should allow for studying them will depend very much on your previous mathematical experience.

It is very important to understand that you only learn mathematics by *doing* it. Ideas and methods that may seem complicated at first sight will become more familiar and natural with practice. You will then be able to apply the ideas you learn in this subject to your other units with more facility, and will find the examination questions easier to answer. The exercises at the ends of each chapter are therefore a crucially important part of the subject and we strongly recommend you to try all of them. Answers to the exercises in this volume are included as an Appendix. You can get extra practice, particularly on topics that you find difficult, by trying additional questions from an appropriate textbook.

Textbooks

Although this subject guide aims to give a complete account of the subject, we do encourage you to consult a textbook for alternative explanations, extra examples and exercises, and supplementary material. There are a number of books on Discrete Mathematics available. Many of them cover most (but not necessarily all) of the topics in the syllabus. They vary in style and the level of mathematical maturity and expertise they presuppose on the part of the reader. We recommend that you obtain a copy of **one** of the following two titles. References are given to them where appropriate in the subject guide.

Main textbooks

Epp, Susanna S, *Discrete Mathematics with Applications*, 3rd Edition, Brooks/Cole (2004). ISBN 0-534-49096-4;

Molluzzo, John C and Fred Buckley, *A First Course in Discrete Mathematics*, Wadsworth (1986) ISBN 0-534-05310-6; reprinted by Waveland Press Inc. (1997) ISBN 0-88133-940-7.

A list of other suitable textbooks known to the authors of this guide and in print at the time of writing is included in this volume as an Appendix.

Discrete Mathematics is a comparatively modern area of study and the notation in some topics has not been completely standardized. This means that you may find occasional differences between the symbols and terminology used in textbooks by different authors. The examination papers will follow the usage in this subject guide, however.

Assessment

Important: the information and advice given in the following section is based on the examination structure used at the time this guide was written. However, the university can alter the format, style or requirements of an examination paper without notice. Because of this, we strongly advise you to check the rubric/instructions on the paper when you actually sit the examination. You should also read the examiner's report from the previous year for advice on this.

This unit is examined by a three hour written paper. Currently, ten questions are set and full marks for the paper are awarded for complete answers to *all* of them. This means, therefore, that you should allow about 18 minutes for each question. In general, there is at least one question on the material in each chapter of the guide, but some questions may require knowledge or techniques from more than one chapter. Each question is marked out of 10 and so the total mark available on the paper is 100. The mark which appears on your transcript will be the percentage of 100 that you obtained on the paper.

You may answer the questions in any order. If you feel nervous, it may help you to start with a question on a topic you feel really confident about. For the most part, the questions are very similar to worked examples or exercises in the subject guide, so that any student who has understood the material and revised thoroughly for the examination should be able to answer most of them quite easily. Some parts of questions may contain a new twist that requires more careful thought. If you get stuck on part of a question, do not spend *too* long trying to figure it out, because this may mean that you do not have time to attempt another question that you could answer quite easily. Leave a space in your script and come back to the difficult bit when you have attempted as much as you can of the rest of the paper. Some questions may also ask you for a definition or the statement of a result proved in the subject guide. These need to be carefully learnt. You may express a definition in your own words, but make sure that your words cover all the points in the definition in this guide. A detailed marking scheme is given on the paper and that can be used as a guide to the length of answer expected. If there are just 1 or 2 marks awarded for a definition, for example, only one sentence is expected, not a half page discussion.

Although most students find that attempting past examination papers is reassuring, you are strongly advised not to spend *all* your revision time in this way, because every year the questions will be different! It is much better to revise thoroughly the ideas and skills taught in each chapter until you are quite confident that you really *understand* the material. When you have finished your revision, test yourself by answering the specimen examination questions to time. Suggested solutions are also included, so that you can check your answers and see the level of detail required.

In conclusion, we hope you will enjoy studying this unit and find the material interesting and challenging, as well as increasing your understanding and appreciation of your other units.

Chapter 1

Digraphs and relations

Essential reading

Epp Sections 10.1, 10.2, 10.3, 10.5 or M&B Section 5.1

Keywords

Digraphs, relations on a set, using digraphs to illustrate relations, equivalence relations, partitions, partial orders, relations between two sets, relations and cartesian products.

We start this chapter by widening our study of graphs to digraphs, that is graphs with directed edges. We then use digraphs to visualize and understand the concept of a relation. In particular we study two special kinds of relations known as equivalence relations and partial orders. These relations are important to mathematicians and computer scientists in that they help to classify the elements of large sets of objects and thus can help make such sets seem less abstract and easier to handle. Before you read the chapter it is a good idea to revise the terminology and main theorems on graph theory you met in Volume 1 of this subject guide.

1.1 Digraphs

Learning objectives for this section

- The definition of a digraph and directed paths and cycles
- The definition of a strongly connected digraph
- Using digraphs to illustrate relationships

Some problems require that we *direct* the edges of a graph, from one endpoint to the other, in order to express the relationship between the vertices. A graph in which every edge has a direction assigned to it is called a **digraph** (an abbreviation of **directed graph**). The directed edges are often called **arcs**.

Example 1.1 Using a digraph to model traffic flow
Suppose we want to model a plan for traffic flow in part of a city, where some streets allow traffic flow in only one direction. For this model, a digraph would be appropriate. We would represent each road junction by a vertex; a one-way street from junction u to junction v by an arc directed from u to v; and a street allowing

traffic flow in both directions between junctions x and y, by two arcs, one directed from x to y and the other from y to x.

Figure 1.1 is an example of how to draw a picture of a digraph. The direction of each arc is indicated by an arrow.

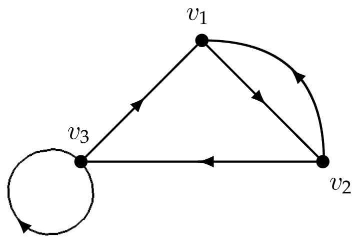


Figure 1.1: A picture of a digraph

Just as for undirected graphs, the most common way to represent a digraph within a computer is as a square array of numbers, each number representing the number of arcs joining a pair of vertices.

Definition 1.1 *Suppose that D is a digraph with n vertices, numbered $1, 2, \dots, n$. Then the* ***adjacency matrix*** *$\mathbb{A}(D)$ of D is a square $n \times n$ array, with rows and columns numbered $1, 2, \dots, n$, such that the entry in row i and column j is the number of arcs from vertex i to vertex j.*

Example 1.2 An adjacency matrix
The digraph in Figure 1.1 has the adjacency matrix

	v_1	v_2	v_3
v_1	0	1	0
v_2	1	0	1
v_3	1	0	1

When the labels of the rows and columns of the adjacency matrix are the vertices in their natural ascending order (as they are in this example), we omit the labels and give the adjacency matrix as:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Note that for digraphs the adjacency matrix is not necessarily symmetric.

Each arc in a digraph figures precisely once in the adjacency matrix of the digraph. Hence we have the following result.

Theorem 1.1 *The number of arcs in a digraph D is equal to the sum of the entries in its adjacency matrix $\mathbb{A}(D)$.*

We can also use an **adjacency list** to describe a digraph in much the same way as we did for undirected graphs. For each vertex v of the digraph, we list all the vertices w such that vw is an arc:

Example 1.3 An adjacency list
The digraph in Figure 1.1 has the adjacency list

$$\begin{array}{ll} v_1 : & v_2 \\ v_2 : & v_1, v_3 \\ v_3 : & v_1, v_3 \end{array}$$

In a digraph, we define the **outdegree** of u, denoted by **outdeg**(u), as the number of arcs directed *out of* (away from) the vertex u; and the **indegree** of u, denoted by **indeg**(u), as the number of arcs directed *into* (towards) the vertex u.

Example 1.4 In the digraph in Figure 1.1, indeg$(v_3) = 2$ and outdeg$(v_3) = 2$.

Because each arc in a digraph contributes 1 to the outdegree of its start vertex and 1 to the indegree of its end vertex, the sum of all the indegrees is equal to the sum of all the outdegrees of the vertices, and both these sums are equal to the sum of the entries in the adjacency matrix. We state this as a corollary to Theorem 1.1.

Corollary 1.2 *The number of arcs in a digraph D is equal to the sum of all the outdegrees of the vertices of D. The number of arcs in a digraph D is also equal to the sum of all the indegrees of the vertices of D.*

In a digraph we define **directed paths** and **directed cycles** as in graphs, but with the added condition that arcs must be used in their correct direction. We say a digraph D is **strongly connected** if for every pair of vertices x and y of D, there is a directed path from x to y and a directed path from y to x in D.

Example 1.5 A strongly connected graph
The digraph in Figure 1.1 contains a directed path $P = v_3v_1v_2$ and a directed cycle $C = v_2v_1v_2$. We leave it for you to check that there are directed paths from v_i to v_j for each of the nine pairs of vertices (v_i, v_j) where $i, j = 1, 2, 3$. The digraph of Figure 1.1 is thus strongly connected.

Digraphs can be used to illustrate a variety of "one-way" relationships, such as predator-prey models in ecology, the scheduling of tasks in a manufacturing process, family trees, flow diagrams for a computer program and many others.

Example 1.6 Using digraphs to model information
If you want to make a digraph to illustrate some information, it is important to explain what your vertices and arcs represent as well as drawing the digraph. As an example, suppose that we want to make a digraph to illustrate the following information.

Ralph is a dog. A dog is a mammal. A cat and a gnu are mammals too. Wanda is a fish. An admiral is a butterfly. Nemo is a fish. Fish, birds, insects and mammals are animals. James is a parrot. A parrot is a bird. Mungo is a cat. A butterfly is an insect. Garth is a gnu.

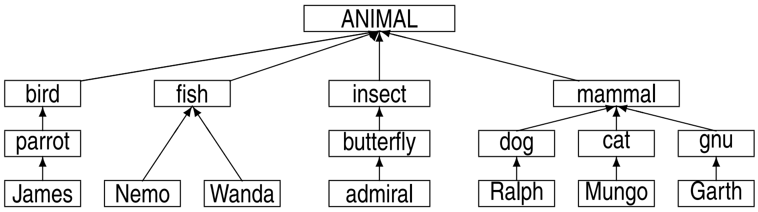


Figure 1.2: The digraph of Example 1.6

One possible solution would be to let the vertices of the digraph represent animals or classes of animals and to let an arc from vertex

a to vertex b mean that "a is a b". The resulting digraph would be as shown in Figure 1.2.

1.2 Relations

Learning objectives for this section

- The definition of a relation on a set X
- Using a digraph to illustrate a relation on a set X
- Reflexive, symmetric, transitive and anti-symmetric relations
- Using the relationship digraph to determine whether a relation is reflexive, symmetric, transitive or anti-symmetric
- Proving from the definition whether or not a given relation is reflexive, symmetric, transitive or anti-symmetric
- The definitions of an equivalence relation, a partial order and an order
- Listing the equivalence classes of an equivalence relation
- The definition of a partition of a set X
- The equivalence classes of an equivalence relation on X is a partition of X

Let S be a set. A **relation** $\mathcal{R}$ on S is a rule which compares any two elements $x, y \in S$ and tells us either that x is related to y or that x is not related to y. We write $x\mathcal{R}y$ to mean "x is related to y under the relation $\mathcal{R}$".

We are already familiar with many examples of relations defined in society, for example we could let S be the set of all people in London and say that two people x and y are related if x is a parent of y.

We have also seen several examples of relations in mathematics, for example "=", "$<$", and "$\leq$" are all relations on the set of integers. A different example is the following:

Example 1.7 Let $S = \{1, 2, 3\}$. We define a relation $\mathcal{R}$ on S by saying that x is related to y if $|x - y| \leq 1$, for all $x, y \in S$. Thus $1\mathcal{R}2$ but 1 is not related to 3.

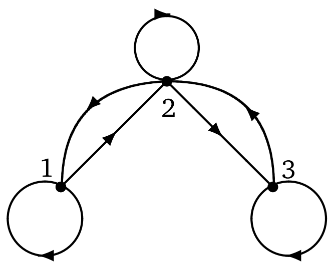


Figure 1.3: The relationship digraph of Example 1.7

1.2.1 Using digraphs to illustrate relations

Given a relation $\mathcal{R}$ on a set S, we can model $\mathcal{R}$ by defining the digraph D with $V(D) = S$ in which, for any two vertices x and y, there is an arc in D from x to y if and only if $x\mathcal{R}y$. We shall call D the **relationship digraph** corresponding to $\mathcal{R}$.

Example 1.8 Constructing a relationship digraph
In order to construct the relationship digraph of the relation from Example 1.7, we consider each of the nine ordered pairs (x, y) of elements from $S = \{1, 2, 3\}$; if $x\mathcal{R}y$ we draw an arc from x to y. Figure 1.3 shows the resulting relationship digraph.

Conversely, given a digraph D, we can define a relation $\mathcal{R}$ on the set $S = V(D)$ of vertices of D by saying $x\mathcal{R}y$ if and only if there is an arc in D from x to y.

Example 1.9 The relation defined by a digraph
The digraph given in Figure 1.1 defines the relation $\mathcal{R}$ on the set $S = \{v_1, v_2, v_3\}$ given by $v_1\mathcal{R}v_2$, $v_2\mathcal{R}v_3$, $v_2\mathcal{R}v_1$, $v_3\mathcal{R}v_3$, and $v_3\mathcal{R}v_1$.

1.2.2 Equivalence relations

We can see from Example 1.9 that the definition of a relation on a set may be rather arbitrary. Many relations which occur in practical situations however have a more “regular structure”.

Definition 1.2 *Let $\mathcal{R}$ be a relation defined on a set S. We say that $\mathcal{R}$ is:*

- ***reflexive***
  if for all $x \in S$, we have $x\mathcal{R}x$.
- ***symmetric***
  if for all $x, y \in S$ such that $x\mathcal{R}y$, we have $y\mathcal{R}x$.
- ***transitive***
  if for all $x, y, z \in S$ such that $x\mathcal{R}y$ and $y\mathcal{R}z$, we have $x\mathcal{R}z$.

In terms of the relationship digraph D of $\mathcal{R}$, it can be seen that:

- $\mathcal{R}$ is reflexive if *all* vertices of D are in a directed loop.
- $\mathcal{R}$ is symmetric if all arcs xy in D are in a directed cycle of length two.
- $\mathcal{R}$ is transitive if for all directed paths of length two $P = xyz$ of D, we have an arc xz; and for all directed cycles of length two $C = xyx$ of D, we have a loop xx and a loop yy.

Example 1.10 Using Figure 1.3 we deduce that the relation defined in Example 1.7 is **reflexive** because there is a directed loop at each vertex; **symmetric** because whenever there is an arc xy in the relation digraph, then there is also an arc yx back; **not transitive** as e.g. $1\mathcal{R}2$ and $2\mathcal{R}3$, but there is no arc from 1 to 3.

Example 1.11 Similarly, using Figure 1.1 we deduce that the relation defined in Example 1.9 is **not reflexive** because there is no directed loop at vertex v_1; **not symmetric** because there is an arc from vertex

v_3 to vertex v_1, but there is no arc back from v_1 to v_3; **not transitive** as e.g. $v_3 \mathcal{R} v_1$ and $v_1 \mathcal{R} v_2$, but there is no arc from v_3 to v_2.

Example 1.12 Consider the relation "$\leq$" defined on the set of integers $\mathbb{Z}$. For any integer a we have that $a \leq a$, so the "$\leq$"-relation is **reflexive** on $\mathbb{Z}$. We have e.g. $2 \leq 10$ in $\mathbb{Z}$ while 10 is not less than or equal to 2, so the "$\leq$"-relation is **not symmetric** on $\mathbb{Z}$. For any integers a, b and c where $a \leq b$ and $b \leq c$, we have that $a \leq c$ also, so the "$\leq$"-relation is **transitive** on $\mathbb{Z}$.

Example 1.13 Consider the relation "$<$" defined on $\mathbb{Z}$.
No integer a is strictly less than itself, so the "$<$"-relation is **not reflexive** on $\mathbb{Z}$. We have e.g. $2 < 10$ in $\mathbb{Z}$ while 10 is not less than 2, so the "$<$"-relation is **not symmetric** on $\mathbb{Z}$. For any integers a, b and c where $a < b$ and $b < c$, we have that $a < c$ also, so the "$<$"-relation is **transitive** on $\mathbb{Z}$.

Definition 1.3 *If a relation $\mathcal{R}$ defined on a set S is reflexive, symmetric and transitive, then we say that $\mathcal{R}$ is an* **equivalence relation** *on S.*

The relation "=" defined on any set S is an example of an equivalence relation on S. Figure 1.4 shows that the relation digraph for the "="-relation on $S = \{1, 2, 3, 4\}$ is reflexive, symmetric and transitive.

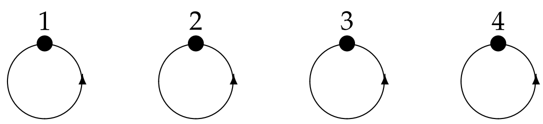


Figure 1.4: The relationship digraph of the "="-relation on $S = \{1, 2, 3, 4\}$

Indeed equivalence relations can be seen as generalisations of the "equality" relation. If two elements of a set are related by an equivalence relation, then they are in some sense equal.

Example 1.14 A relation on the set of 3-bit binary strings
Let S be the set of all 3-bit binary strings. Define a relation $\mathcal{R}$ on S by saying that two binary strings are related if they contain the same number of ones. Thus e.g. $100\mathcal{R}010$ but 100 is not related to 101.

Clearly any binary string has the same number of ones as itself, so $\mathcal{R}$ is **reflexive**. If binary strings a and b are such that $a\mathcal{R}b$, then a has the same number of ones as b; thus b has clearly got the same number of ones as a also, such that $b\mathcal{R}a$. $\mathcal{R}$ is thus **symmetric**. For any binary strings a, b and c where $a\mathcal{R}b$ and $b\mathcal{R}c$, we have that a has the same number of ones as b which in turn has the same number of ones as c; thus a and c must have the same number of ones and $a\mathcal{R}c$. This proves that $\mathcal{R}$ is **transitive**. The relation $\mathcal{R}$ is thus an **equivalence relation** because it is reflexive, symmetric and transitive.

Definition 1.4 *Let $\mathcal{R}$ be an equivalence relation defined on a set S and let $x \in S$. Then the* **equivalence class** *of x is the subset of S containing all elements of S which are related to x. We denote this by $[x]$. Thus*

$$[x] = \{y \in S \;:\; y\mathcal{R}x\}.$$

Example 1.15 The equivalence classes of the relation of Example 1.14
Consider Example 1.14 again. The equivalence class of a 3-bit binary string x is the set of all 3-bit binary strings which contain the

same number of ones as x. This gives us four distinct equivalence classes, namely the class [000] of strings with no ones

$$[000] = \{000\}\,,$$

the class of strings with precisely one one, denoted by [100], [010] or [001]:

$$[100] = \{100, 010, 001\}\,,$$

the class of strings with precisely two ones, denoted by [110], [011] or [101]:

$$[110] = \{110, 011, 101\}\,,$$

and finally the class [111] of strings with three ones

$$[111] = \{111\}\,.$$

The set of equivalence classes listed in Example 1.15 has the following two nice structural properties.

- Every element of S belongs to exactly one of the four distinct equivalence classes.
- Two elements of S are related if and only if they belong to the same equivalence class.

We shall see that these properties hold for all equivalence relations. We first need one more definition.

Definition 1.5 *Let S be a set and $T = \{S_1, S_2, \ldots, S_m\}$ be a set of non-empty subsets of S. Then T is said to be a* **partition** *of S if every element of S belongs to exactly one element of T.*

Example 1.16 A partition on the set of 3-bit binary strings
Let S be the set of 3-bit binary strings. Let T be the set whose elements are the four distinct equivalence classes of the equivalence relation of Example 1.14. Thus from Example 1.15 we get that

$$T = \{\{000\}, \{100, 010, 001\}, \{110, 011, 101\}, \{111\}\}\,.$$

Then T is a partition of S. Our promised result on equivalence relations is a generalisation of Example 1.16:

Theorem 1.3 *Let $\mathcal{R}$ be an equivalence relation on a set S. Then:*

- *The set of distinct equivalence classes of $\mathcal{R}$ on S is a partition of S.*
- *Two elements of S are related if and only if they belong to the same equivalence class.*

The proof of this result is beyond the scope of this course. Theorem 1.3 tells us that if we have an equivalence relation $\mathcal{R}$ defined on a set S, then we can think of the equivalence classes as the subsets of S containing all the elements which are "equivalent" to each other under this relation.

Note that it is useful to be able to recognize the relationship digraph of an equivalence relation. The relationship digraph corresponding to an equivalence relation falls into distinct components, one component for each equivalence class. Inside each component every vertex is incident to a directed loop and every pair of vertices are joined by a directed cycle of length two.

1.2.3 Partial orders

In the previous subsection we looked at equivalence relations as a generalisation of the relation "=". In this subsection we will consider relations which generalise the relation "$\leq$".

Definition 1.6 *We say that a relation $\mathcal{R}$ on a set S is* **anti-symmetric** *if for all $x, y \in S$ such that $x\mathcal{R}y$ and $y\mathcal{R}x$, we have $x = y$.*
Thus $\mathcal{R}$ is anti-symmetric if and only if the relationship digraph of $\mathcal{R}$ has no directed cycles of length two.

It follows that the relations described in Examples 1.7, 1.9 and 1.14 are not anti-symmetric: The relation $\mathcal{R}$ of Example 1.7 is not anti-symmetric as its relation digraph has directed cycles of length two, e.g. $1\mathcal{R}2$ and $2\mathcal{R}1$ while $1 \neq 2$. The relation $\mathcal{R}$ of Example 1.9 is not anti-symmetric as its relation digraph has a directed cycle of length two, namely $v_1\mathcal{R}v_2$ and $v_2\mathcal{R}v_1$ while $v_1 \neq v_2$. The relation $\mathcal{R}$ of Example 1.14 is not anti-symmetric as e.g. $001\mathcal{R}100$ and $100\mathcal{R}001$ while $001 \neq 100$.

Examples of relations which are anti-symmetric are "$\leq$" on any set of numbers, and "$\subseteq$" on the set of all subsets of a given set.

Example 1.17 An anti-symmetric relation on $\mathbb{Z}$
For any integer a and b, if $a \leq b$ and $b \leq a$, we see from the sandwich

$$a \leq b \leq a$$

that $a = b$, and the "$\leq$"-relation is thus anti-symmetric on the set of integers $\mathbb{Z}$.

Definition 1.7 *We say that a relation $\mathcal{R}$ on a set S is a* **partial order** *if it is reflexive, anti-symmetric and transitive.*

We say further that $\mathcal{R}$ is an **order** *if it is a partial order with the additional property that for any two elements $x, y \in S$, either $x\mathcal{R}y$ or $y\mathcal{R}x$.*

It follows that $\leq$ is an example of an order on any set of numbers. We show this for $\mathbb{Z}$ in the following example.

Example 1.18 The relation $\leq$ is an order on $\mathbb{Z}$
In Example 1.17 we showed that the relation "$\leq$" is anti-symmetric on $\mathbb{Z}$. In Example 1.12 we saw that "$\leq$" is reflexive and transitive on $\mathbb{Z}$. Hence the relation "$\leq$" is a partial order on $\mathbb{Z}$. Further, for every pair of integers a and b we have that either $a \leq b$ or $b \leq a$, so "$\leq$" is an order on $\mathbb{Z}$.

An example of a partial order which is not an order is the following. The subset-relation "$\subseteq$" is a partial order on $\mathcal{P}(U)$, the power set of a set U, but not an order.

Example 1.19 The partial order "$\subseteq$" on $\mathcal{P}(U)$ is not an order
Let $U = \{1, 2, 3\}$ and $S = \mathcal{P}(U)$ be the set of all subsets of U. Then "$\subseteq$' is a partial order on S.

To show this, you need to check that the "$\subseteq$"-relation is **reflexive**, i.e. you must check *for all* 8 subsets A of U that $A \subseteq A$; **transitive**, i.e. you must check *for all* subsets A, B and C of U satisfying $A \subseteq B$ and $B \subseteq C$, that $A \subseteq C$ also; **anti-symmetric**, i.e. you must verify that $A = B$ *for all* subsets A and B of U satisfying that $A \subseteq B$ and

$B \subseteq A$. We showed all these properties of the "$\subseteq$"-relation in Section 2.2 of Vol. 1.

The "$\subseteq$"-relation is not an order, however, since if we let $X = \{1, 2\}$ and $Y = \{1, 3\}$ then $X \nsubseteq Y$ and $Y \nsubseteq X$. Thus X and Y are two elements of S such that X is not related to Y, and Y is not related to X.

1.3 Relations and Cartesian products

Learning objectives for this section

- Relations from a set X to a set Y
- The cartesian product $X \times Y$ of two sets X and Y
- Relations on a set X as subsets of $X \times X$
- Defining a relation from a set X to a set Y as a subset of $X \times Y$
- The set X^n of all ordered n-tuples of the set X

In the previous section we considered relations between the elements of a single set S. We now extend this concept to define relations between the elements of two sets.

Definition 1.8 *Let X and Y be sets. Then a* **relation** *$\mathcal{R}$ from X to Y is a rule which compares any two elements $x \in X$ and $y \in Y$, and tells us either that x is related to y or that x is not related to y.*

Relations between two sets are fundamental to databases, as can be seen from the following example.

Example 1.20 A relation from a set of students to a set of courses
Let X be the set of all students registered at a college and Y be the set of all courses being taught at the college. Then the college database keeps a record of the relationship in which a student x is related to a course y if x is registered for y.

We can model a relation $\mathcal{R}$ between sets X and Y by a digraph D in much the same way as we did previously for relations on a single set: we put $V(D) = X \cup Y$ and draw an arc from a vertex $x \in X$ to a vertex $y \in Y$ if $x\mathcal{R}y$.

Example 1.21 The relationship digraph of a relation between two sets
Let $X = \{1, 2, 3\}$ and $Y = \{b, c\}$. Define $\mathcal{R}$ by $1\mathcal{R}b$, $1\mathcal{R}c$, $2\mathcal{R}c$, and $3\mathcal{R}c$. Then the relationship digraph for $\mathcal{R}$ is shown in Figure 1.5.

There is a strong similarity between Figure 1.5 and the figures drawn in Volume 1, Chapter 4 to illustrate *functions*. Indeed we can view a function $f : X \to Y$ as being a relation in which each $x \in X$ is related to a *unique* element of Y. We write $f(x)$ to represent the unique element of Y which is related to x.

We close this chapter by introducing one more mathematical structure which can be used to define a relation.

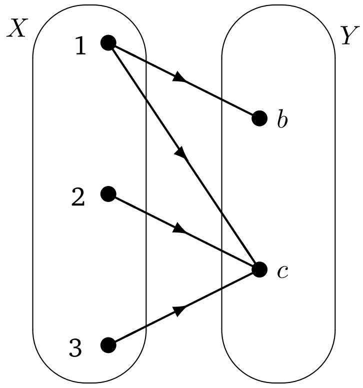


Figure 1.5: The relationship digraph of Example 1.21

Definition 1.9 *Let X and Y be sets. Then the* **cartesian product** $X \times Y$ *is the set whose elements are all ordered pairs of elements (x, y) where $x \in X$ and $y \in Y$.*

Example 1.22 A cartesian product
Let $X = \{1, 2, 3\}$ and $Y = \{b, c\}$. Then

$$X \times Y = \{(1, b), (1, c), (2, b), (2, c), (3, b), (3, c)\}.$$

We can also compute

$$Y \times X = \{(b, 1), (c, 1), (b, 2), (c, 2), (b, 3), (c, 3)\},$$

and we note that $X \times Y \neq Y \times X$.

Given a relationship $\mathcal{R}$ between X and Y, we can use the cartesian product of X and Y to help us define $\mathcal{R}$: we simply give the subset of $X \times Y$ containing all ordered pairs (x, y) for which $x\mathcal{R}y$. This is equivalent to listing the ordered pairs corresponding to the arcs in the relationship digraph of $\mathcal{R}$.

Example 1.23 Relations defined as subsets of a cartesian product

- The relation $\mathcal{R}$ in Example 1.21 is defined by the subset

  $$\{(1, b), (1, c), (2, c), (3, c)\}$$

  of $X \times Y$.
- In Example 1.7 we have $X = \{1, 2, 3\} = Y$, and $\mathcal{R}$ is defined by the subset

  $$\{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$$

  of $X \times X$.

We saw in Section 4.1.2 in Vol. 1 that we can generalise the idea of an ordered pair to an **ordered n-tuple**, $(x_1, x_2, \ldots, x_n)$. Similarly we can generalise the cartesian product of two sets to the cartesian product of n sets, for any $n \in \mathbb{Z}^+$.

When each of the sets in a cartesian product is the same, as in the second part of Example 1.23, we often use an abbreviated notation: we denote $X \times X$ by X^2 and, in general, we denote the set $X \times X \times \ldots \times X$, by X^n, where there are n X's altogether in the product.

Example 1.24 n-bit binary strings
An n-bit binary string is an example of an ordered n-tuple, even though we usually write it without the brackets and without commas between the entries. Let $B = \{0, 1\}$, i.e. B is the set of bits. Then the set of all n-bit binary strings is

$$\{a_1 a_2 \ldots a_n : a_1 \in B, a_2 \in B, \ldots, a_n \in B\},$$

and this set can be denoted by B^n.

Example 1.25 The set B^3
The set of all 3-bit binary strings is
$\{a_1 a_2 a_3 : a_1 \in B, a_2 \in B, a_3 \in B\}$, where $B = \{0, 1\}$, i.e.

$$B^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}.$$

1.4 Exercises on Chapter 1

1.4.1 True/False questions

For each of the following statements, decide whether it is true or false.

(a) A relation $\mathcal{R}$ on the set $X = \{1, 2, 3\}$ is reflexive if $1\mathcal{R}1$.

(b) If a relation $\mathcal{R}$ on a set X is symmetric, then $x\mathcal{R}y$ and $y\mathcal{R}x$ for all $x, y \in X$.

(c) If a relation $\mathcal{R}$ on a set X is transitive, then $x\mathcal{R}y$, $y\mathcal{R}z$ and $x\mathcal{R}z$ for all $x, y, z \in X$.

(d) A relation $\mathcal{R}$ on $X = \{1, 2, 3\}$ is transitive if $1\mathcal{R}2$, $2\mathcal{R}3$ and $1\mathcal{R}3$.

(e) If a relation $\mathcal{R}$ on X is symmetric then it cannot be anti-symmetric.

(f) If a relation $\mathcal{R}$ on the set $X = \{1, 2, 3\}$ is symmetric and $1\mathcal{R}2$, then $2\mathcal{R}1$.

(g) If a relation $\mathcal{R}$ on the set $X = \{1, 2, 3\}$ is symmetric and transitive and $1\mathcal{R}2$, then $1\mathcal{R}1$.

(h) A relation $\mathcal{R}$ on a set X is a partial order if it is not an equivalence relation.

(i) If $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e, f, g\}$ then $X \times Y$ has more than 20 elements.

(j) For all sets X and Y we have that $X \times Y \neq Y \times X$.

1.4.2 Longer exercises

For each of the relations given in Questions 1, 2 and 3:

(a) Draw the relationship digraph.

(b) Determine whether the relation is either reflexive, symmetric, transitive or anti-symmetric. For the cases when one of these properties does not hold, justify your answer by giving an example to show that it does not hold.

(c) Determine which of the relations is an equivalence relation. For the case when it is an equivalence relation, calculate the distinct equivalence classes and verify that they give a partition of S.

(d) Determine which of the relations is a partial order or an order.

Question 1
Let $S = \{0, 1, 2, 3\}$. Define a relation $\mathcal{R}_1$ between the elements of S by "x is related to y if the product xy is even".

Question 2
Let $S = \{0, 1, 2, 3\}$. Define a relation $\mathcal{R}_2$ between the elements of S by "x is related to y if $x - y \in \{0, 3, -3\}$".

Question 3
Let $S = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$. Define a relation $\mathcal{R}_3$ between the elements of S by "X is related to Y if $X \subseteq Y$".

Question 4
Let S be a set and $\mathcal{R}$ be a relation on S. Explain what it means to say that $\mathcal{R}$ is

(a) reflexive;

(b) symmetric;

(c) transitive;

(d) anti-symmetric;

(e) an equivalence relation;

(f) a partial order;

(g) an order.

Question 5
Let $X = \{0, 1, 2\}$ and $Y = \{3, 4\}$. Define a relation $\mathcal{R}$ from X to Y by "x is related to y if $x = y - 3$", for $x \in X$ and $y \in Y$.

(a) Draw the relationship digraph corresponding to $\mathcal{R}$.

(b) Determine the set $X \times Y$.
Determine the subset of $X \times Y$ corresponding to $\mathcal{R}$.

(c) Is $\mathcal{R}$ a *function* from X to Y? Justify your answer.

Question 6
Let $X = \{0, 1, 2, 3, \ldots 9\}$. Define a relation $\mathcal{R}$ from X by "x is related to y if x and y gives the same remainder on division by 3".

(a) Show that $\mathcal{R}$ is an equivalence relation.

(b) List the equivalence classes of $\mathcal{R}$.

(c) Is the set of equivalence classes of $\mathcal{R}$ a partition of X?

Chapter 2

Sequences, series and induction

Essential reading

Epp Sections 4.1, 4.2, 4.3, 4.4, 8.1, 8.2 or M&B Section 3.1

Keywords

Sequences, proof by induction, series, the Sigma notation, finding sums of finite series.

In this chapter we study number sequences and their sums, especially the ones defined by initial terms and recurrence relation. We further introduce Sigma notation, which is a convenient shorthand notation for writing long sums of numbers.

The method of proof by induction is particularly important for computer scientists. This method of proof can be used to prove properties of series and sequences, which is why we introduce it in this chapter. It is sometimes possible to guess the sum of the first n terms of a finite sequence by computing it for a few specific values of n, but once we have guessed the value, we require a proof to verify it in general for all n. We show you how a proof by induction can be used to do this.

2.1 Sequences

Learning objectives for this section

- Sequences defined by recurrence relation and initial term(s)
- How to compute a given term in a sequence defined by recurrence relation and initial term(s)
- Recognizing an arithmetic progression and finding a recurrence relation for it
- Recognizing a geometric progression and finding a recurrence relation for it
- Defining and computing the terms of the Fibonacci sequence

A sequence is simply a list of numbers as, for example,

(a) $2, 5, 8, 11, 14, \ldots$

(b) $5, 0.5, 0.05, 0.005, 0.0005, \ldots$

(c) $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots.$

Formally, a **sequence** is a function from the set $\mathbb{Z}^+$ into $\mathbb{R}$. The first term in the sequence is called the **initial** term and is the image of 1, the second term is the image of 2, the third is the image of 3, and so on. We usually denote the terms of the sequence by a letter with a subscript, thus

$$u_1, u_2, u_3, \ldots.$$

If this denotes the sequence (a) given above, the initial term is $u_1 = 2$; then $u_2 = 5$, $u_3 = 8$, and so on.

Sequences are important because they arise naturally in a wide variety of practical situations, whenever a process is repeated and the result recorded. When the process is random as, for example, when the air temperature is recorded at a weather station or a die is rolled, there is no way of predicting for certain what the next term of the sequence will be, no matter how many earlier terms we have knowledge of. There are processes, however, that give rise to sequences where the terms fall into a pattern as, for example, when the value of a sum of money invested at a fixed rate of compound interest is calculated at regular intervals or the sequence of values of a variable in a loop in a computer program.

Example 2.1 Compound interest
Suppose that on 1 January 2001 we put £100 into a savings account at a fixed interest of 5% per annum and that we have decided not to make any withdrawals from this account. The balance of the account on 1 January 2001, 2002, 2003, 2004, etc. can conveniently be given as a sequence

$$b_1, b_2, b_3, \ldots,$$

where the initial term $b_1 = 100$, $b_2 = 105$, $b_3 = 110.25$, $b_4 = 115.76(25)$, etc.

Example 2.2 Investigating loops
Suppose we want to investigate the following **while**-loop, e.g. for debugging it. The loop takes as input a positive integer n.

$t := 1;$
$x := n;$
while $x > 0$ **do**
  begin $t := t * x;$ $x := x - 1$ **end**;
output t.

The two variables t and x are changed during the iteration of the loop. If we want to investigate what the loop does, we could fix a value for n and compute the two sequences of values for t and x. Let e.g.

$$t_0, t_1, t_2, \ldots,$$

be the sequence where t_0 is the initial value of t, t_1 is the value of t after one iteration of the loop, t_2 is the value of t after two iterations of the loop, etc. We could compute a similar sequence

$$x_0, x_1, x_2, \ldots,$$

for the successive values of x. Using $n = 5$, for example, we get

$$t_0 = 1, t_1 = 5, t_2 = 20, t_3 = 60, t_4 = 120, t_5 = 120$$

and

$$x_0 = 5, x_1 = 4, x_2 = 3, x_3 = 2, x_4 = 1, x_5 = 0.$$

For finite, correct loops like this, the sequences would be finite, but for practical debugging, they would become infinite, if you have accidentally made an infinite loop.

2.1.1 Recurrence relations

On this course we are mainly concerned with the type of sequence, where we can continue the sequence when we know the pattern and the first few terms. In this subsection, our objective is to find a way of expressing the relationship between the terms of this kind of sequence.

To be able to continue the intended sequence, you must be given sufficient data to be sure of the relationship between its terms, as for example can be seen by considering the sequence $1, 2, 4, \ldots$. There are at least two logical ways of continuing this sequence, for example:

- $1, 2, 4, 7, 11, 16, \ldots$,
  here you get the second term by adding 1 to the first term, then you add 2 to get the third term, add 3 to get the fourth, 4 to get the fifth and so on.
- $1, 2, 4, 8, 16, 32, \ldots$,
  here you get a term by multiplying the previous term by 2.

Reconsider the sequence $2, 5, 8, 11, 14, \ldots$, from the beginning of the section. We have enough terms of this sequence to convince us that each term is found by adding 3 to the preceding term. So the terms are calculated successively by the rules:

$$\begin{aligned} u_1 &= 2, \\ u_2 &= u_1 + 3 = 5, \\ u_3 &= u_2 + 3 = 8, \ldots . \end{aligned}$$

We can express this relationship between the terms *in general* by $u_{n+1} = u_n + 3$, for all $n \in \mathbb{Z}^+$. This is called the **recurrence relation** for this sequence.

Definition 2.1 *A sequence for which the recurrence relation is of the form $u_{n+1} = u_n + d$, where d is a constant, is known as an **arithmetic progression (A.P.)**.*

Note that a recurrence relation on its own does not define a sequence. In order to specify a sequence you need to give the initial term(s) as well as a recurrence relation. For example, in order to define the A.P. above, it is necessary to specify the initial term $u_1 = 2$ as well as the recurrence relation $u_{n+1} = u_n + 3$.

Example 2.3 The significance of the initial term
The sequence given by the recurrence relation $u_{n+1} = u_n + 3$ and initial term $u_1 = 2$ is the sequence

$$2, 5, 8, 11, 14, 17, \ldots,$$

while the sequence given by the recurrence relation $u_{n+1} = u_n + 3$ and initial term $u_1 = 1$ is the sequence

$$1, 4, 7, 10, 13, 16, \ldots,$$

so we note that even though the two sequences have the same recurrence relation, they are not equal.

Reconsider the sequence $5, 0.5, 0.05, 0.005, 0.0005, \ldots$ from the beginning of the section. In this sequence we obtain each term by *multiplying* the preceding term by 0.1. This time, the terms are calculated successively by the rules:

$$\begin{aligned} u_1 &= 5, \\ u_2 &= (0.1)\, u_1 = 0.5, \\ u_3 &= (0.1)\, u_2 = 0.05, \ldots . \end{aligned}$$

The recurrence relation for this sequence is $u_{n+1} = (0.1)\, u_n$, for all $n \in \mathbb{Z}^+$.

Definition 2.2 *A sequence for which the recurrence relation is of the form $u_{n+1} = ru_n$, where r is a constant, is called a* ***geometric progression (G.P.)***.

We remind you once more that the recurrence relation on its own does not define a sequence. In the G.P. above, it is necessary to specify the initial term $u_1 = 5$ as well as the recurrence relation $u_{n+1} = (0.1)u_n$ in order for the sequence to be well-defined.

2.1.2 The Fibonacci sequence

The last of the three sequences given at the beginning of the section,

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots,$$

is known as the **Fibonacci sequence**. The terms are called **Fibonacci numbers** and we shall denote them by $F_0, F_1, F_2, \ldots$ (Note that it is customary to start this sequence at term 0 instead of term 1). The sequence has so many interesting properties that it has fascinated mathematicians for centuries. Recently a number of applications have been found for computer science.

Learning activity

It is vital for any student to be able to gather rudimentary information about a new subject quickly and efficiently. Practise this by finding information about the Fibonacci numbers; there is lots of stuff written about them. Use search engines to find information on the world wide web and also try finding historical mathematics books in the library. We are convinced that you will find something to interest you about Fibonacci numbers as they are connected to almost anything from rabbits to Pythagoras Theorem.

Careful consideration of the Fibonacci sequence tells us that each term is the sum of the previous two. So, starting from the initial terms $F_0 = 0, F_1 = 1$, the terms are calculated successively by the rules:

$$\begin{aligned} F_2 &= F_0 + F_1\, (= 0 + 1 = 1)\,, \\ F_3 &= F_1 + F_2\, (= 1 + 1 = 2)\,, \\ F_4 &= F_2 + F_3\, (= 1 + 2 = 3)\,, \\ F_5 &= F_3 + F_4\, (= 2 + 3 = 5)\,, \ldots . \end{aligned}$$

The recurrence relation for the Fibonacci sequence is thus $F_{n+2} = F_{n+1} + F_n$, where $n \geq 0$. Notice carefully, however, that for this sequence we need knowledge of *two* initial terms, F_0 *and* F_1, in order to be able to use the recurrence relation to calculate the terms.

2.2 Induction

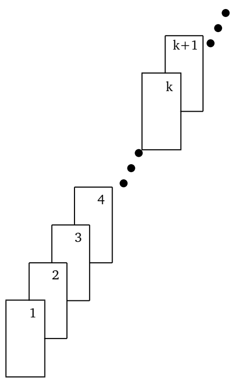
Learning objectives for this section

- Understanding the Principle of Induction
- Proving a result for all positive integers by induction
- Proving by induction a formula for the n'th term of a sequence, in particular when the sequence is given by initial term(s) and recurrence relation

A technique that is often useful in proving results "for all positive integers n" is called **Proof by Induction**. It is based on a fundamental property of the integers known as the "Principle of Induction", which we shall introduce below.

2.2.1 The Principle of Induction

The basic idea contained in the Principle of Induction is that of a "domino effect". Imagine an infinite number of dominoes with the numbers $1, 2, 3, \ldots$ lined up in a long row such that if the first k dominoes (with numbers $1, 2, 3, \ldots, k$) are knocked over, then the next domino (with number $k + 1$) will be knocked over too.



Now suppose that somebody knocks over domino 1, this will then knock over domino 2. As dominoes 1 and 2 have fallen, these will in turn knock over number 3. Next the fact that numbers 1, 2 and 3 have fallen will mean that number 4 falls. Subsequently numbers 1, 2, 3 and 4 will bring down number 5, and so on. The key idea is that all dominoes will fall:

The Principle of Induction *says that* ***if*** *(i) somebody knocks over domino 1 and (ii) the dominoes are arranged properly* ***then*** *all dominoes fall.*

In terms of subsets of positive integers and using mathematical notation the principle is expressed in the following way, where the set S corresponds to the set of dominoes knocked over in the domino effect:

The Principle of Induction
Suppose that S is a subset of $\mathbb{Z}^+$ and that we have the following information about S:
(i) $1 \in S$;
(ii) whenever the integers $1, 2, \ldots, k \in S$, then $k + 1 \in S$ also.
Then we may conclude that $S = \mathbb{Z}^+$.

The Principle of Induction can at first seem very abstract, so let us try to understand why the two properties (i) and (ii) of the set S should indeed by "domino effect" give a set S which is all of the positive integers $\mathbb{Z}^+$.

So assume properties (i) and (ii) hold. We note first that $1 \in S$ by property (i), so

$$S = \{1, \ldots\}.$$

Now, since $1 \in S$, then $2 \in S$ by property (ii), so we thus have that

$$S = \{1, 2 \ldots\}.$$

We have now established that $1, 2 \in S$, so therefore by using property (ii) again we get $3 \in S$ and thus

$$S = \{1, 2, 3 \ldots\}.$$

Similarly, since $1, 2, 3 \in S$, then $4 \in S$ by (ii) and

$$S = \{1, 2, 3, 4 \ldots\},$$

and so on. Thus the two conditions together show that $\mathbb{Z}^+ \subseteq S$. But we know that $S \subseteq \mathbb{Z}^+$ and hence $S = \mathbb{Z}^+$ by Rule 2.5 of Volume 1 of this subject guide.

2.2.2 Proof by induction

Now suppose that we wish to prove that a certain statement is true "for all $n \in \mathbb{Z}^+$". Imagine each of the dominoes in the "domino effect"-picture corresponding to one instance of the statement you are going to prove, e.g. domino 4 corresponds to the statement for $n = 4$. Knocking over a domino corresponds to proving the result for the number on that domino.

In order to make sure all the dominoes fall (i.e. that the statement holds for all n), we must make sure that (i) we can knock over domino 1, and (ii) that the dominoes are lined up properly, so that knocking over $1, 2, 3, \ldots, k$ ensures that number $k + 1$ falls.

Part (i) is proven in the **base case** of a proof by induction.

Part (ii) is proven in two steps by first *making the assumption* that somebody has knocked over dominoes $1, 2, 3, \ldots, k$ for some arbitrary domino k (this is known as the **induction hypothesis**) and then using this assumption to show that domino $k + 1$ falls also (this is known as the **induction step**).

More formally, let S be the subset of $\mathbb{Z}^+$ for which the statement holds. We can prove that $S = \mathbb{Z}^+$, by establishing the following *three* steps which together make up the proof method known as proof by induction.

The 3-step method of Proof by Induction

Base case *Verify that the result is true when $n = 1$ so that $1 \in S$.*

Induction hypothesis *For some arbitrarily fixed integer $k \geq 1$, assume that the result is true for all the integers $1, 2, \ldots, k$.*

Induction step *Use the hypothesis that the result is true when $n = 1, 2, \ldots, k$ to prove that the result also holds when $n = k + 1$.*

We now use the 3-step method of Proof by Induction to prove a result about a sequence which we studied in the previous section. You must learn how to give a proof by induction by following the 3-step method. Pay attention to how such proofs are presented in the examples that follow and try to adapt a similar layout for your own proofs.

Example 2.4 Consider the sequence

$$2, 5, 8, 11, 14, \ldots .$$

We saw above that the recurrence relation for this sequence is

$$u_{n+1} = u_n + 3,$$

starting from the initial term $u_1 = 2$. We can calculate successively:

$$\begin{aligned} u_2 &= u_1 + 3 = u_1 + 3 \times 1 \\ u_3 &= u_2 + 3 = u_1 + 3 + 3 = u_1 + 3 \times 2 \\ u_4 &= u_3 + 3 = u_1 + 3 + 3 + 3 = u_1 + 3 \times 3, \end{aligned}$$

and it would be reasonable to guess that a formula that would give us the value of u_n directly in terms of n might be

$$u_n = u_1 + 3(n-1) = 2 + 3(n-1) = 3n - 1,$$

for all $n \in \mathbb{Z}^+$. We can use the method of Proof by Induction to prove that this guess is correct. We state the result as a formal proposition first.

Proposition *The sequence given by the recurrence relation*

$$u_{n+1} = u_n + 3$$

and the initial term $u_1 = 2$ is also given by

$$u_n = 3n - 1 \textit{ for all } n \in \mathbb{Z}^+$$

Proof.
Base case The formula is correct when $n = 1$, since $RHS = 3\,(1) - 1 = 2$ and $LHS = u_1 = 2$ also.

Induction hypothesis Suppose that $u_n = 3n - 1$ is true for $n = 1, 2, 3, \ldots, k$. Thus in particular we know that $u_k = 3k - 1$.

Induction step We prove that $u_n = 3n - 1$ is also true when $n = k + 1$. To do this, we must calculate the value of u_{k+1} from u_k (using the recurrence relation and the induction hypothesis) and check that the result agrees with the formula, i.e. we check that we get $u_{k+1} = 3(k + 1) - 1$.

Putting $n = k$ in the recurrence relation, gives

$$u_{k+1} = u_k + 3. \tag{2.1}$$

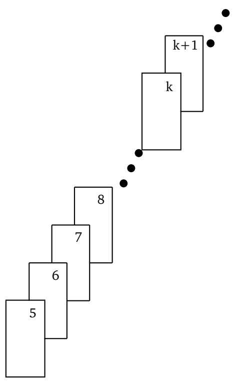
Using the induction hypothesis to substitute for u_k in Equation (2.1) gives

$$u_{k+1} = (3k - 1) + 3 = 3k + 2 = 3\,(k + 1) - 1.$$

Thus the formula holds when $n = k + 1$. Hence it holds for all $n \geq 1$ by induction. ∎

Induction from a base other than 1

We can use induction to prove results "for all $n \geq n_0$", for any integer n_0; the base case is then $n = n_0$ and the rest of the proof is as above. This is because having another base than $n = 1$, e.g. $n = 5$ for your induction, just means that the dominoes in your "domino effect"-picture have consecutive integers starting from $n = 5$ onwards rather than from $n = 1$ onwards.



In order to make sure all the dominoes fall (i.e. that the statement holds for all n), we must make sure that (i) we can knock over the first domino (i.e. domino n_0, in this case $n_0 = 5$), and (ii) that the dominoes are lined up properly, so that knocking over $5, 6, 7, \ldots, k$ ensures that number $k + 1$ falls.

Notice that your base case is always the *least* value of n for which the statement is true. In the following two examples, this least value of n is $n = 0$.

Example 2.5 We prove the following proposition by induction.

Proposition $3^n + 1$ *is even for all* $n \geq 0$.

Proof.
Base case When $n = 0$, $3^n + 1 = 3^0 + 1 = 1 + 1 = 2$, which is even, so the result holds for $n = 0$.

Induction hypothesis Suppose that $3^n + 1$ is even for $n = 1, 2, 3, \ldots, k$. Thus in particular we know that $3^k + 1$ is even.

Induction step We must prove that $3^n + 1$ is also even when $n = k + 1$. Now,

$$3^{k+1} + 1 = 3(3^k + 1) - 2.$$

But $3^k + 1$ is even by the induction hypothesis, so $3(3^k + 1)$ is also even and so is $3(3^k + 1) - 2$. Thus $3^n + 1$ is even for $n = k + 1$. Hence it holds for all $n \geq 0$, by induction. ∎

Example 2.6 A sequence is determined by the recurrence relation

$$u_n = 4u_{n-1} - 3u_{n-2}$$

and the initial terms $u_0 = 0$, $u_1 = 2$. We shall prove for all $n \geq 0$ that

$$u_n = 3^n - 1.$$

Notice that we could not calculate u_2 and subsequent terms of this sequence unless we had been given the values of *two* initial terms. Thus for the base case, we must verify the formula is correct for *both* u_0 and u_1. Also, note that the recurrence relation connects u_n with two previous terms, not just with u_{n-1}.

Proof.
Base cases When $n = 0$, the formula $u_n = 3^n - 1$ gives $u_0 = 3^0 - 1 = 0$; and when $n = 1$, it gives $u_1 = 3^1 - 1 = 2$. Hence the formula agrees with the initial terms given for $n = 0$ and $n = 1$.

Induction hypothesis Suppose for some $k \geq 0$ that the formula $u_n = 3^n - 1$ holds for $n = 0, 1, 2, \ldots, k$.

Induction step We prove the formula also holds for $n = k + 1$. From the recurrence relation, we have

$$u_{k+1} = 4u_k - 3u_{k-1}. \tag{2.2}$$

By the induction hypothesis, the result is true when $n = k$ and $n = k - 1$. Hence $u_k = 3^k - 1$ and $u_{k-1} = 3^{k-1} - 1$. Substituting these into Equation (2.2) gives

$$\begin{aligned} u_{k+1} &= 4\left(3^k - 1\right) - 3\left(3^{k-1} - 1\right) \\ &= 4\left(3^k\right) - 4 - 3^k + 3 \\ &= 4\left(3^k\right) - 3^k - 1 \\ &= (4 - 1)\,3^k - 1 = 3^{k+1} - 1. \end{aligned}$$

Thus the formula also holds when $n = k + 1$ and hence holds for all $n \in \mathbb{N}$ by induction. ∎

We shall find further applications of proof by induction in the next section and in the next chapter which is about a special kind of graph called a tree.

2.3 Series and the Sigma notation

Learning objectives for this section

- Expressing a finite series in Sigma notation
- How to write out all terms of a finite series given in Sigma notation

- Arithmetic with finite series given in Sigma notation
- The sum of n terms of an A.P and a G.P.
- The "standard sums" of Theorem 2.1

A finite **series** is what we get when we add together a finite number of terms of a sequence. A handy notation for writing series uses the capital Greek letter Sigma $\sum$ as follows:

$$u_1 + u_2 + u_3 + \ldots + u_n = \sum_{r=1}^{n} u_r$$

We read the right hand side as "the sum of u_r from $r = 1$ to $r = n$". The integers 1 and n are known respectively as the **lower** and **upper limits of summation**; the variable r is called the **index of summation**.

2.3.1 Expressing sums in Sigma notation

We need to be able to convert to Sigma notation a finite series given as a 'long' sum and vice versa. Consider the general finite series

$$u_m + u_{m+1} + u_{m+2} + \ldots + u_n = \sum_{r=m}^{n} u_r. \tag{2.3}$$

Converting Sigma notation to 'long' notation

In order to get the LHS of Equation (2.3) if the RHS is given, use the following two-step algorithm:

for $r := m$ **to** $n - 1$ **do** **write**$(u_r, +)$;
write(u_n).

Converting 'long' sums to Sigma notation

Giving a rule for finding the RHS of Equation (2.3) from a given LHS is harder, as in general the same series may be written in Sigma notation in more than one way.

Example 2.7 Sigma notation is not uniquely determined
Let the sum S be the sum of all odd integers between 1 and 10, that is

$$S = 1 + 3 + 5 + 7 + 9,$$

then

$$S = \sum_{r=1}^{5}(2r - 1)$$

as can be seen by using the algorithm from the previous subsection. For $r = 1$ we have $2r - 1 = 1$, for $r = 2$ we have $2r - 1 = 3$, for $r = 3$ we have $2r - 1 = 5$, for $r = 4$ we have $2r - 1 = 7$ and for $r = 5$ we have $2r - 1 = 9$, Thus

$$\sum_{r=1}^{5}(2r - 1) = 1 + 3 + 5 + 7 + 9 = S.$$

However, we could equally well have chosen to write this as

$$S = \sum_{r=0}^{4}(2r + 1)$$

which you can verify yourself in a similar way.

Here are a few steps to try when attempting to convert 'long' sums to Sigma notation. We use

$$T = 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16$$

as an example.

1. Begin by identifying the variable index and the limits of the sum. Look for a variable r running from some lower limit ℓ to some upper limit u in steps of 1 as the series progresses. For our example T, a little thought gives that each term is $2r$ where r runs from $\ell = 1$ to $u = 8$.
2. Next find a function f, such that the r'th term of the series is $f(r)$. For our example T, we already identified that the r'th term is $2r$.
3. The Sigma notation for the sum is then $\sum_{r=\ell}^{u} f(r)$.

   In our example
   $$T = \sum_{r=1}^{8} 2r.$$

We give some further sums written in both 'long' notation and Sigma notation in the following example. Try to practise converting sums between the two notations yourself. You can make up your own sums or find some in the textbooks.

Example 2.8 Consider the following sums.

(a) $1 + 2 + 3 + \ldots + n$
Here, we can put $u_r = r$; then $r = 1$ gives the first term in the sum and $r = n$ gives the last term. So we can write

$$1 + 2 + 3 + \ldots + n = \sum_{r=1}^{n} r.$$

(b) $1^2 + 2^2 + 3^2 + \ldots + n^2$
Here, we can put $u_r = r^2$; then $r = 1$ gives the first term in the sum and $r = n$ gives the last term. So we can write

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum_{r=1}^{n} r^2.$$

(c) $1 + 2 + 4 + 8 + \ldots + 2^n$
We notice that $1 + 2 + 4 + 8 + \ldots 2^n = 2^0 + 2^1 + 2^2 + \ldots + 2^n$, so here, we can put $u_r = 2^r$; then $r = 0$ gives the first term in the sum and $r = n$ gives the last term. So we can write

$$1 + 2 + 4 + 8 + \ldots + 2^n = \sum_{r=0}^{n} 2^r.$$

(d) $\sum_{r=1}^{n}(3r-1)$.
We write out the sum using the 2-step algorithm. For $r = 1$ we compute $3r - 1 = 2$, for $r = 2$ we compute $3r - 1 = 5$, for $r = 3$ we compute $3r - 1 = 8$, and so on, up to finally for $r = n$ we get $3r - 1 = 3n - 1$. So we can write

$$\sum_{r=1}^{n}(3r-1) = 2 + 5 + 8 + \ldots + (3n-1).$$

2.3.2 Some standard series and their sums

Induction is a useful method for verifying a formula for the sum of a finite number of terms of a sequence because it is very easy to obtain a recurrence relation between the sum of the first $n + 1$ terms and the sum of the first n terms of a sequence. To see this let

$$u_1, u_2, u_3, \ldots$$

be a sequence and for any positive integer n let S_n be the sum of the first n terms, that is

$$S_n = u_1 + u_2 + \ldots + u_n,$$

and similarly

$$S_{n+1} = u_1 + u_2 + \ldots + u_n + u_{n+1}.$$

We can thus express S_{n+1} in terms of S_n:

$$\begin{aligned} S_{n+1} &= u_1 + u_2 + \ldots + u_n + u_{n+1} \\ &= (u_1 + u_2 + \ldots + u_n) + u_{n+1} \\ &= S_n + u_{n+1}. \end{aligned}$$

This idea is used in the induction step of the proofs of parts *(b)* and *(c)* of the following theorem, which gives the sum of some commonly occurring finite series.

Theorem 2.1 *Let n be a positive integer. Then*

(a) $\sum_{r=1}^{n} 1 = n$.

(b) $\sum_{r=1}^{n} r = n(n+1)/2$.

(c) $\sum_{r=1}^{n} r^2 = n(n+1)(2n+1)/6$.

(d) $\sum_{r=0}^{n} x^r = \frac{x^{n+1}-1}{x-1}$, *for any $x \in \mathbb{R}$ with $x \neq 1$.*

Proof of Theorem 2.1 (a).
In this sum we have $u_r = 1$, for $r = 1, 2, \ldots, n$. So we are adding $1 + 1 + \ldots + 1$, giving n altogether. ∎

Proof of Theorem 2.1 (b).
Let S_n denote the sum of the first n integers, so that $S_n = 1 + 2 + \ldots + n$. We prove by induction that

$$S_n = n(n+1)/2, \text{ for all } n \geq 1.$$

Base case The LHS is $S_1 = u_1 = 1$ and the RHS gives $1\,(1+1)\,/2 = 1$, so the formula holds when $n = 1$.

Induction hypothesis Suppose that $S_n = n\,(n+1)\,/2$, for $n = 1, 2, \ldots, k$; then in particular we know that $S_k = k\,(k+1)\,/2$.

Induction step We must prove that $S_n = n\,(n+1)\,/2$ is also true when $n = k+1$; that is, we find S_{k+1} from S_k and check that the result agrees with the formula.

But $S_k = 1+2+3+\ldots+k$ and $S_{k+1} = 1+2+3+\ldots+k+(k+1)$, so

$$S_{k+1} = S_k + (k+1)\,. \tag{2.4}$$

Using the induction hypothesis to substitute for S_k in Equation (2.4) gives

$$\begin{aligned} S_{k+1} &= S_k + (k+1) \\ &= k\,(k+1)\,/2 + (k+1) \\ &= (k+1)\,(k/2) + (k+1) \\ &= (k+1)\,(k/2+1) \\ &= (k+1)\,(k/2+2/2) \\ &= (k+1)\,(k+2)\,/2. \end{aligned}$$

On the other hand, putting $n = k+1$ in the formula gives $S_{k+1} = (k+1)\,(k+2)\,/2$.
Thus the formula also holds for $n = k+1$ and hence it holds for all $n \geq 1$, by induction.∎

Proof of Theorem 2.1 (c).
Let T_n denote the sum of the squares of the first n integers, so that $T_n = 1^2 + 2^2 + \ldots + n^2$. We shall prove by induction that T_n is given by the formula

$$T_n = n\,(n+1)\,(2n+1)\,/6, \text{ for all } n \geq 1.$$

Base case When $n = 1$, $T_1 = 1^2$. The formula gives $T_1 = 1\,(1+1)\,(2+1)\,/6 = 1$. Hence the formula holds when $n = 1$.

Induction hypothesis Suppose $T_n = n\,(n+1)\,(2n+1)\,/6$ is true for $n = 1, 2, \ldots, k$; then, in particular, we have $T_k = k\,(k+1)\,(2k+1)\,/6$.

Induction step We prove that the formula also holds for $n = k+1$; that is, we calculate T_{k+1} from T_k and check that the result agrees with the formula.

From the recurrence relation we have

$$T_{k+1} = T_k + (k+1)^2\,.$$

Using the induction hypothesis to substitute for T_k in this equation gives

$$\begin{aligned} T_{k+1} &= k\,(k+1)\,(2k+1)\,/6 + (k+1)^2 \\ &= (k+1)\,[k\,(2k+1)\,/6 + (k+1)] \\ &= (k+1)\,[\left(2k^2+k\right)/6 + (k+1)] \\ &= (k+1)\,[\left(2k^2+k\right)/6 + (6k+6)\,/6] \\ &= (k+1)\,\left[2k^2+7k+6\right]/6 \\ &= (k+1)\,(k+2)\,(2k+3)\,/6. \end{aligned}$$

On the other hand, putting $n = k + 1$ in the formula gives $T_{k+1} = (k + 1)(k + 2)(2k + 3)/6$.

Thus the formula also holds for $n = k + 1$ and hence it holds for all $n \geq 1$, by induction.■

It is very tempting after the use of the induction hypothesis to use brute force calculations in order to verify that the two sides of the formula are equal. This is usually not a good strategy though, because such calculations often become very heavy. We use a much better strategy in order to get line two of the computation for T_{k+1} above: we *looked for a common factor* between the RHS and the expression we got after using the hypothesis. In our case only very little thought was required in order to spot that $(k + 1)$ is a factor of both the RHS and $T_{k+1} = k(k + 1)(2k + 1)/6 + (k + 1)^2$. Hence we were able to perform the labour-saving step of *taking this factor outside a bracket*.

Proof of Theorem 2.1 (d).
Let

$$S = 1 + x + x^2 + \ldots + x^n,$$

where $x \neq 1$. Multiplying through by x, gives

$$xS = x + x^2 + x^3 + \ldots + x^{n+1}.$$

Subtracting the two equations, we have

$$xS - S = x^{n+1} - 1.$$

Thus

$$(x - 1)S = x^{n+1} - 1,$$

and when $x \neq 1$ we can divide both sides by $x - 1$ to obtain the required result.■

Note Part (d) of Theorem 2.1 can also be proved by induction. We leave this as an exercise.

Note also that part (d) can be used to find the sum of the first n terms of any geometric progression. Parts (a) and (b) together can be used to find the sum of arithmetic progressions, as we shall see an example of in Example 2.9 a bit further on.

2.3.3 Rules of arithmetic for sums

The Sigma notation is not just a convenient shorthand for writing sums. Most importantly, it gives us a way of working out the sum of a complicated expression by turning it into simpler sums. We can do this by applying combinations of the following three simple rules.

RULE 1: Expressing a sum as a difference of known sums

For example when computing the sum

$$11 + 12 + \cdots + 20$$

we can use that

$$11 + 12 + \cdots + 20 = (1 + 2 + 3 + \cdots + 20) - (1 + 2 + 3 + \cdots + 10).$$

In Sigma notation this is

$$\sum_{r=11}^{20} r = \sum_{r=1}^{20} r - \sum_{r=1}^{10} r,$$

which can be computed easily by using the standard sum of Theorem 2.1(b) above:

$$\begin{aligned}\sum_{r=11}^{20} r &= \sum_{r=1}^{20} r - \sum_{r=1}^{10} r \\ &= 20 \times 21/2 - 10 \times 11/2 \\ &= 155.\end{aligned}$$

RULE 2: Taking out a common factor

For example

$$5\,(1) + 5\,(3) + 5\left(3^2\right) + \ldots + 5\left(3^{n-1}\right) = 5\left(1 + 3 + 3^2 + \ldots + 3^{n-1}\right).$$

Thus

$$\sum_{r=0}^{n-1} 5\left(3^r\right) = 5\sum_{r=0}^{n-1} 3^r.$$

The common factor 5 can be taken outside the sigma sign because it can be taken outside the bracket in the long version of the sum.

RULE 3: Splitting a sum into two (or more) components

For example

$$\begin{aligned}\left(1+1^2\right) + \left(2+2^2\right) + \quad \ldots \quad &+ \left(n+n^2\right) \\ = \quad &(1+2+3+\ldots+n) + \left(1^2+2^2+3^2+\ldots+n^2\right)\end{aligned}$$

Thus in Sigma notation

$$\sum_{r=1}^{n}\left(r+r^2\right) = \sum_{r=1}^{n} r + \sum_{r=1}^{n} r^2$$

where both sums on the right hand side can be computed using the standard sums of Theorem 2.1.

We formalise these rules in the following theorem.

Theorem 2.2

(a) $\sum_{r=m}^{n} u_r = \sum_{r=1}^{n} u_r - \sum_{r=1}^{m-1} u_r.$

(b) $\sum_{r=m}^{n} cu_r = c\sum_{r=m}^{n} u_r$ *, where c is a constant.*

(c) $\sum_{r=m}^{n} (u_r + w_r) = \sum_{r=m}^{n} u_r + \sum_{r=m}^{n} w_r.$

Proof of Theorem 2.2 (a).
follows immediately by writing out the sums. ∎

Proof of Theorem 2.2 (b).

$$\begin{aligned}\sum_{r=m}^{n} cu_r &= cu_m + cu_{m+1} + cu_{m+2} + \ldots + cu_n \\ &= c(u_m + u_{m+1} + u_{m+2} + \ldots + u_n) \\ &= c\sum_{r=m}^{n} u_r. \blacksquare\end{aligned}$$

Proof of Theorem 2.2 (c).

$$\begin{aligned}\sum_{r=m}^{n} (u_r + w_r) &= (u_m + w_m) + (u_{m+1} + w_{m+1}) + \ldots + (u_n + w_n) \\ &= (u_m + u_{m+1} + ... + u_n) + (w_m + w_{m+1} + \ldots + w_n) \\ &= \sum_{r=m}^{n} u_r + \sum_{r=m}^{n} w_r. \blacksquare\end{aligned}$$

By using Theorem 2.2 to take out factors and split up the sums, we may reduce a complicated sum to simpler sums for which we already know a formula.

Example 2.9 Manipulating sums in Sigma notation
We now find the formula for the sum in Example 2.8(d).

$$\begin{aligned}\sum_{r=1}^{n}(3r-1) &= \sum_{r=1}^{n}3r + \sum_{r=1}^{n}(-1), \text{ by Theorem 2.2(c)}, \\ &= 3\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1, \text{ by Theorem 2.2(b)},\end{aligned}$$

Hence, by Theorem 2.1 (a) and (b),

$$\begin{aligned}\sum_{r=1}^{n}(3r-1) &= 3n\,(n+1)\,/2 - n \\ &= n\,[3\,(n+1)\,/2 - 1] \\ &= n\,[(3n+3)\,/2 - 2/2] \\ &= n\,(3n+1)\,/2.\end{aligned}$$

2.4 Exercises on Chapter 2

2.4.1 True/False questions

In each of the following questions, decide whether the given statements are true or false.

(a) $\sum_{i=1}^{n} r = \dfrac{n(n+1)}{2}$.

(b) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$.

(c) $\sum_{i=1}^{s} 1 = s$.

(d) $\sum_{i=3}^{s} 1 = s - 3$.

(e) $\sum_{r=1}^{n}(2r-3) = \frac{(2n-3)(2n-2)}{2}$.

(f) $\sum_{i=-2}^{3} 2^{-i} = \frac{1}{4} + \frac{1}{2} + 1 + 2 + 4 + 8$.

(g) $\sum_{r=1}^{n} 2 \times 2^r = \sum_{r=1}^{n} 4^r$.

(h) If $u_n = u_{n-1} + 1$ for all $n = 1, 2, 3, \ldots$, then $u_n = n$ for all $n = 1, 2, 3, \ldots$.

(i) Suppose that $u_0 = 0$ and $u_n = u_{n-1} + n$ for all $n = 1, 2, 3, \ldots$, then $u_8 = 36$.

(j) Suppose that $u_0 = 0$ and $u_n = u_{n-1} + (n-1)$ for all $n = 1, 2, 3, \ldots$, then $u_8 = 35$.

2.4.2 Longer exercises

Question 1
For each of the following sequences, (i) calculate the next term of the sequence and (ii) find a recurrence relation that gives u_{n+1} in terms of u_n.

(a) $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$;

(b) $2, 7, 12, 17, 22, \ldots$

Question 2
Determine the value of u_n, for $n = 1, 2, 3, 4$, for the sequences determined by each of the following recurrence relations.

(a) $u_{n+1} = 5u_n + 2, u_1 = 0$;

(b) $u_{n+2} = u_{n+1} - u_n, u_1 = 0$ and $u_2 = 1$.

Question 3
A sequence is determined by the recurrence relation $u_{n+1} = 3u_n + 2$ and initial term $u_1 = 2$. Prove by induction that $u_n = 3^n - 1$, for all $n \in \mathbb{Z}^+$.

Question 4
Let n be a positive integer and x be a real number with $x \neq 1$. State, without proving, the formulae for

(a) $\sum_{r=1}^{n} 1$;

(b) $\sum_{r=1}^{n} r$;

(c) $\sum_{r=0}^{n} x^r$.

Question 5
Use the formulae you stated in Question 4 to evaluate

(a) $\sum_{i=11}^{30} i$;

(b) $\sum_{i=1}^{20}(2^i + 1)$;

(c) $\sum_{i=1}^{10}(4i + 3)$.

Question 6
Let $s_n = 1 + 3 + 5 + \ldots + (2n - 1)$ for $n \in \mathbb{Z}^+$.

(a) Express s_n using $\sum$ notation.

(b) Calculate s_1, s_2 and s_3.

(c) Find a recurrence relation which expresses s_{n+1} in terms of s_n.

(d) Use induction to prove that $s_n = n^2$ for all $n \geq 1$.

Question 7
Prove Part (d) of Theorem 2.1 by induction.

Chapter 3

Trees

Essential reading

Epp Sections 11.5, 11.6 or M&B Section 8.3

Keywords

Properties of trees, recursive construction of all trees, spanning trees, rooted trees, binary trees, binary search trees.

In this chapter we study a type of graph known as a tree. A number of practical situations can be modelled by trees. We will use "tree diagrams" to analyse counting problems in the next chapter. They are used in a similar way in Statistics, for calculating the probabilities of compound events and in Decision Theory. In Computer Science trees are used extensively for data structures and in the design of sorting and searching procedures. In particular a special kind of rooted tree called a binary search tree is important for storing data in such a way that retrieval of records is very efficient. We study binary search trees in the last section of this chapter.

3.1 Properties of trees

Learning objectives for this section

- The definition of a tree
- A recursive algorithm for constructing all trees
- A tree on n vertices has $n-1$ edges
- In any tree there is a unique path between any pair of vertices
- Spanning trees for a graph G
- How to find all non-isomorphic spanning trees for a small graph

We may have a good intuitive idea of what kind of structure should be called a "tree", but in order to use trees in Graph Theory and for designing algorithms, we must have a proper definition.

Definition 3.1 *A* ***tree*** *is a connected graph that contains no cycles.*

Since a tree contains no cycles, it has no loops (because a loop is a cycle of length 1) and no multiple edges (because two edges with the same endpoints form a cycle of length 2). Thus all trees are simple graphs.

Example 3.1 Some small trees
In Figure 3.1, we illustrate all non-isomorphic trees that have five or fewer vertices.

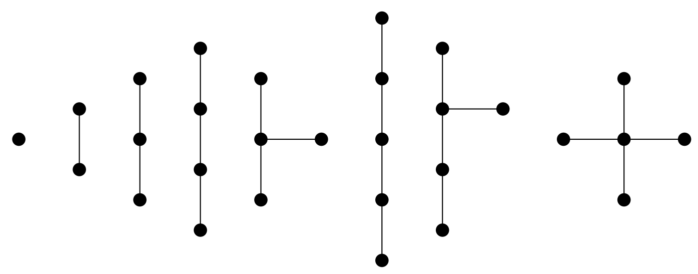


Figure 3.1: All trees on at most five vertices

A tree that contains only vertices of degree one or two is called a **path graph**. Thus, in Figure 3.1, the first four graphs and the sixth one are all examples of path graphs. The **length** of a path graph is the number of *edges* in it.

A few points to consider

By considering the trees in Figure 3.1, conjecture answers to the following questions before reading on.

(a) Do all trees with at least two vertices contain a vertex of degree one? If so, what is the least number of vertices of degree one which must exist in all trees with at least two vertices?

(b) What kind of graph do we obtain if we delete a vertex of degree one from a tree?

(c) What is the connection between the number of edges and the number of vertices of a tree?

(d) How many distinct paths are there joining two given vertices in a tree?

To be sure that our answers to these questions are correct, we need to prove that they hold for *all* trees. It is not sufficient just to check that they hold for several examples. We will give these proofs below.

Lemma 3.1 *Let T be a tree with at least two vertices. Then T has at least two vertices of degree one.*

Proof. Let $P = v_1 v_2 \dots v_m$ be a path of maximum length in T. Since T is connected and has at least two vertices, P has length at least one. Thus $v_1 \neq v_m$.

Since T has no cycles, the only vertex of the path P which is adjacent to v_1 in T is v_2.[1]

[1] Suppose there exists another vertex v_i in P such that $v_1 v_i$ is an edge of T. Then there is a path $v_i \dots v_2$ in P because P is connected, and thus $v_1 v_i \dots v_2 v_1$ is a cycle in T. But T has no cycles, so we know such a vertex v_i cannot exist.

Since P is a path of maximum length in T, no vertex of $V(T) - V(P)$ is adjacent to v_1.[2]

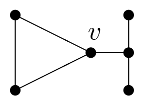
Thus the only vertex of T which is adjacent to v_1 is v_2. Hence $deg_T(v_1) = 1$. Applying a similar argument to v_m we deduce that $deg_T(v_m) = 1$. Thus T has at least two vertices of degree one. ∎

[2] Suppose there exists a vertex $w \in V(T) - V(P)$ such that wv_1 is an edge of T, then $wv_1v_2 \dots v_m$ is a path in T, and has longer length than P. Such a path cannot exist because P was chosen to be a path of longest possible length in T. Hence we know there can be no vertex $w \in V(T) - V(P)$ such that wv_1 is an edge of T.

We next consider the question of what happens when we delete a vertex of degree one from a tree. We first need to define precisely what we mean by *deleting a vertex from a graph*.

Definition 3.2 *Let G be a graph and v be a vertex of G. Then $G - v$ is the graph we obtain by deleting v and all edges incident to v from G.*

Example 3.2 Deleting a vertex from a graph
Let G be the following graph and let v be the vertex indicated.



The graph $G - v$ is then

Note that deleting a random vertex from a connected graph may sometimes disconnect the graph as the example above shows. However, when G is a tree and the vertex v has degree 1, then $G - v$ is still connected. We prove this next.

Lemma 3.2 *Let T be a tree and v be a vertex of T of degree one. Then $T - v$ is a tree.*

Proof. To show that $T - v$ is a tree we need to show that (a) $T - v$ has no cycles and is (b) connected.

(a) If $T - v$ had a cycle C, then C would also be a cycle in T. This is impossible since T has no cycles. Hence $T - v$ has no cycles.

(b) To see that $T - v$ is connected, we choose two vertices x and y of $T - v$. Since T is connected, there is a path P from x to y in T. Since $deg_T(v) = 1$ and P does not repeat vertices, P cannot pass through v. Thus P is also a path from x to y in $T - v$. Since x and y can be any vertices of $T - v$, it follows that $T - v$ is connected.

Thus $T - v$ is a tree. ∎

Lemmas 3.1 and 3.2 give us a useful recursive procedure for investigating trees. We first use this to give an algorithm for constructing all trees.

3.1.1 Recursive construction of all trees

Algorithm for constructing trees

Initial Step *Let T_1 be the tree with one vertex and put $S_1 = \{T_1\}$.*

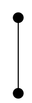
Recursive Step *Suppose we have constructed a set S_i containing all trees with i vertices, for some integer $i \geq 1$. For each tree $T_i \in S_i$ and each vertex $u \in V(T_i)$ we construct a tree T on $i + 1$ vertices by adding a new vertex v to T_i and joining v to u by a single edge. Let S_{i+1} be the set of all trees constructed in this way.*

We can be sure that this algorithm will give us all trees, since Lemmas 3.1 and 3.2 tell us that if T is a tree on $i + 1$ vertices, then T has a vertex of degree one and so can be constructed from a tree on i vertices by the recursive step in the algorithm. Note however that we have glossed over one difficulty. When we present the set S_i containing all trees on i vertices we would like it to contain only one copy of each distinct (i.e. non-isomorphic) tree. To do this we require a 'sub-routine' for checking when two trees are isomorphic. This can be done easily by inspection when the trees are small. It is not so easy for large trees.

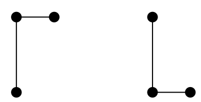
Example 3.3 Constructing all trees
We now use the tree construction algorithm to verify that the trees in Figure 3.1 are indeed all trees on 5 vertices or less:

1. The set S_1 contains the unique tree with one vertex:

2. The set S_2 contains all trees which can be formed by adding one vertex to a tree in S_1. Thus S_2 also consists of one tree only:

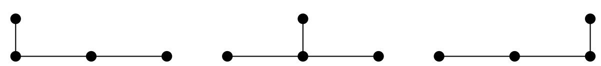


3. The set S_3 contains all trees which can be formed by adding one vertex to a tree in S_2. Thus the candidates for S_3 are the two trees:

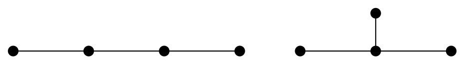


   However, these two graphs are isomorphic (both are a path graph of length 2). Hence S_3 also consists of just one tree:

4. The set S_4 contains all trees which can be formed by adding one vertex to a tree in S_3. Thus the candidates for S_4 are the three trees:



However, the first and the last of these graphs are isomorphic (both are a path graph of length 3). Hence S_4 consists of just two trees:



5. The three non-isomorphic trees of S_5 are found in a similar manner by adding 1 vertex to a tree in S_4 in every possible way and then weeding out isomorphic 'copies'.

6. Continuing in this manner, we can construct all trees.

3.1.2 The number of edges in a tree

We saw in the previous subsection that finding all trees on n vertices becomes harder and harder as n gets bigger because the process of adding a vertex in every possible way and then weeding out isomorphic trees becomes more and more tedious. We next use Lemmas 3.1 and 3.2 to establish the relationship between the number of edges and the number of vertices of a tree. The proof is by induction on the number of vertices in the tree and is a very nice example of how induction can be used to prove a general result for all possible trees without us having to draw them all.

Theorem 3.3 *Let T be a tree with n vertices. Then T has $n-1$ edges.*

Proof. We shall use induction on the number of vertices n.

Base Case. If T is a tree with one vertex then T has no edges[3]. Thus the theorem is true when $n = 1$.

[3] Because T has no cycles and hence no loops.

Induction Hypothesis. Suppose that $k \geq 1$ is an integer, and that all trees with n vertices have $n-1$ edges when $n = 1, 2, \dots, k$.

Induction Step. Let T be any tree on $k+1$ vertices[4].

[4] Notice that we are making no assumption here about how T looks, it could be **any** tree on $k+1$ vertices.

Since $k+1 \geq 2$, it follows from Lemma 3.1 that T has a vertex of degree one. Using Lemma 3.2, we can delete this vertex and obtain a tree T_1 with k vertices.

By the induction hypothesis T_1 has $k-1$ edges. Since T has exactly one more edge than T_1, we may conclude that T has k edges. Thus the theorem holds for any tree T on $k+1$ vertices. By the Principle of Induction, the theorem thus holds for all trees. ∎

We close this section by answering our final question in the points we encouraged you to think about on page 32.

Lemma 3.4 *Let T denote a tree with at least two vertices. Then there is exactly one path connecting any pair of vertices of T.*

Intuitively this holds because T has no cycles. Formally:

Proof. T has at least two vertices, so let v and w be two vertices of T. Since T is connected, there is a path P from v to w in T. Let

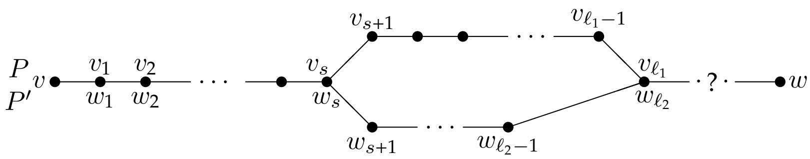
$$P = vv_1 \dots v_m w.$$

Suppose there is another path P' from v to w in T. Let

$$P' = vw_1 \dots w_n w.$$

P and P' share the first vertex and perhaps others, but they are different, so at some point they will differ. Let $v_s = w_s$ be the last vertex before P and P' differ for the *first* time.

P and P' share the last vertex, so they will meet again after splitting. Suppose this happens for the *first* time in $v_{\ell_1} = w_{\ell_2}$ such that ℓ_1, ℓ_2 are the smallest indices for which this occurs.



Note that in the place of the '?' the two paths are not necessarily the same.

Then we have a cycle $v_s v_{s+1} \dots v_{\ell_1}[= w_{\ell_2}]w_{\ell_2 - 1} \dots w_s$. But this is impossible as T is a tree and thus has no cycles. Hence there cannot be a second path P' from v to w in T. ∎

3.1.3 Spanning trees

We say that a graph H is a **subgraph** of a graph G if its vertices are a subset of the vertex set of G, its edges are a subset of the edge set of G, and each edge of H has the same end-vertices in G and H.

Definition 3.3 *If H is a subgraph of G such that $V(H) = V(G)$, then H is called a **spanning subgraph** of G. If H is a spanning subgraph which is also a tree, then H is said to be a **spanning tree** of G.*

Example 3.4 In Figure 3.2, the graphs T_1 and T_2 are both spanning trees of the graph G.

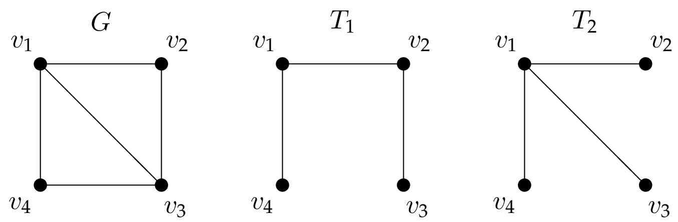


Figure 3.2: A graph G and two of its spanning trees

3.2 Rooted trees

Learning objectives for this section

- The definition of a rooted tree
- The terminology associated with rooted trees: Node, internal/external node, leaf, ancestor, descendant, parents, child, etc.
- The height of a rooted tree
- The definition of a binary tree, and in particular a binary search tree

- A balanced binary tree has 2^i nodes on all levels i apart from the highest level

---

In a number of applications, trees are used to model procedures that can be divided up into a sequence of stages, such as counting problems or searching or sorting algorithms. In this type of application, one vertex is singled out to represent the *start* of the process. A tree in which one vertex has been singled out in this way is called a **rooted tree** and the chosen vertex is called the **root** of the tree.

The topic of rooted trees has its own special terminology, which is used particularly in Computer Science. First, you will find that the vertices of a rooted tree are often referred to as **nodes**. The other terms are mainly derived from the similarity between a rooted tree and a *family tree*, depicting the descendants of a single ancestor.

We arrange the vertices of a rooted tree T in *levels*. We first put the root on level 0. For any positive integer i, the set of vertices in level i is the set of all vertices of the tree which are joined to the root by a path of length i. (The levels in the tree then correspond to *generations* in a family tree.) We know, from Lemma 3.4, that there is a unique path in the tree from the root r to any other vertex. Thus each vertex belongs to exactly one level.

The **height** of T is the length of a longest path in T starting at the root. Thus, if T has height h, then its vertices lie on levels $0, 1, 2, \ldots, h$.

Let x and y be vertices of T. If the unique path from the root r to x in T passes through y, then y is called an **ancestor** of x and x is called a **descendant** of y. If the vertices y and x are adjacent on the path from r to x, then y is called the **parent** of x and x is called the **child** of y.

Example 3.5 The terminology of rooted trees

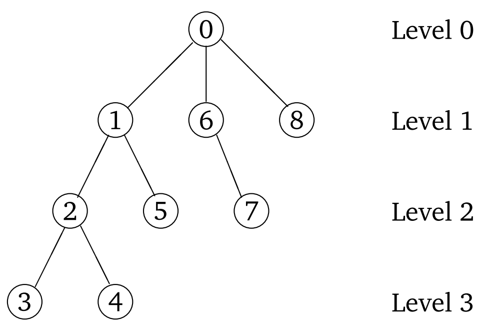


Figure 3.3: A rooted tree

In the tree shown in Figure 3.3,

- the vertex 0 is the root;
- the vertices $1, 6, 8$ are the children of 0;
- the vertex 6 is the parent of 7;
- the vertex 1 is an ancestor of $2, 3, 4$ and 5;
- vertex 1 is *not* an ancestor of any of the vertices $0, 6, 7, 8$;

- the vertex 4 is a descendant of $0, 1$ and 2.

Theorem 3.5 *In a rooted tree, every vertex other than the root has exactly one parent.*

Proof. Let T be a tree rooted at r and let x be any other vertex of T. Then there is a unique path in T starting at r and ending at x, by Lemma 3.4. The parent of x must be a vertex of this path (because a parent is an ancestor) and it must be adjacent to x. Thus because the path is unique, x has a unique parent. ∎

It is not necessary for every vertex in the tree to have a child. The vertices with no children are called **end vertices** or **external nodes** of the tree. The other vertices are called **internal nodes**.

Example 3.6 The terminology of rooted trees (cont.)
The vertices $3, 4, 5, 7, 8$ are the external nodes of the rooted tree shown in Figure 3.3 and $0, 1, 2, 6$ are the internal nodes.

We can use the parent-child relationship to define the idea of *levels* in a rooted tree more precisely. Starting from the root at level 0, we obtain the **level** of each vertex x by adding 1 to the level of its parent. The **height** of the tree is the greatest of the levels, so for example the height of the rooted tree shown in Figure 3.3 is 3.

3.2.1 Binary trees

A **binary tree** is a rooted tree in which each internal node has exactly two children, one of which we call the **left child** and the other we call the **right child**. Binary trees arise as models of procedures in which two possibilities occur at each stage.

Example 3.7 A model using binary trees

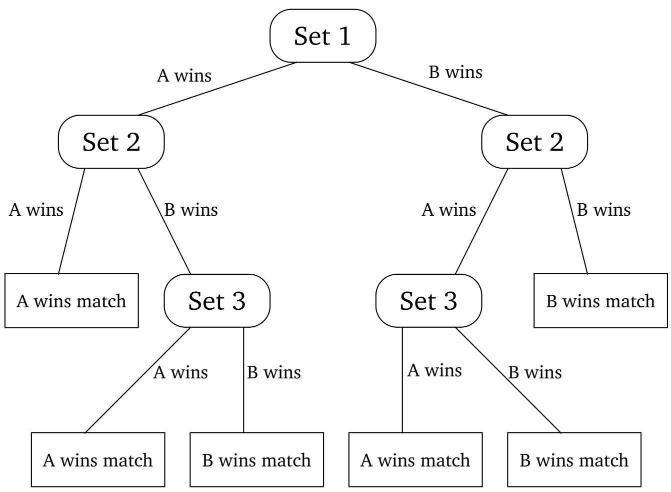


Figure 3.4: A tennis match

In a tennis match, two players A and B play up to three sets and the winner is the first player to win two sets. A binary tree to model the possible outcomes is shown in Figure 3.4.

The internal nodes in the binary tree of Figure 3.4, depicted in oval boxes, each represent the start of a set and the edges joining this node to its left and right child represent respectively the two possible outcomes for that set, a win for A or a win for B. The external vertices, depicted in rectangular boxes, represent the final outcome of the match as a result of the wins recorded on the unique path that leads from the root to that box.

The binary tree illustrated in Figure 3.4 has the property that its external nodes are *all on the highest level or the highest two levels*. Such a binary tree is called **balanced**.

If T is a balanced binary tree of height h, then we can compute the exact number of vertices on all levels apart from the highest level, that is on all levels i, for $0 \leq i \leq h-1$.

Theorem 3.6 *Let T be a balanced binary tree of height h. Then the number of vertices of T on level i is equal to 2^i, for all integers i, $0 \leq i \leq h-1$. Furthermore*

$$2^h + 1 \leq |V(T)| \leq 2^{h+1} - 1.$$

Proof. Since each vertex at level i has exactly two children at level $i+1$ for all levels $i = 0 \leq i \leq h-2$, it follows that the number of vertices at level $i+1$ is exactly twice the number of vertices at level i.

The only vertex at level 0 is the root, so the number of vertices at level 0 is $1 = 2^0$, hence we have exactly 2^i vertices at level i for all i where $0 \leq i \leq h-1$.

The number of vertices at level h is not fixed, but we know it must be at least[5] two and at most[6] 2^h. The number of vertices $|V(T)|$ in the tree is the sum of the number of vertices at all levels, thus

$$1+2+2^2+\ldots+2^{h-1}+2 \leq |V(T)| \leq 1+2+2^2+\ldots+2^{h-1}+2^h. \quad (3.1)$$

[5] If only one node at level $h-1$ has children.

[6] If all nodes at level $h-1$ have children.

From Theorem 2.1(d) we know that $1 + 2 + 2^2 + \ldots + 2^{h-1} = 2^h - 1$. Thus equation (3.1) becomes

$$(2^h - 1) + 2 \leq |V(T)| \leq (2^h - 1) + 2^h,$$

which in turn gives the required result that

$$2^h + 1 \leq |V(T)| \leq 2^{h+1} - 1. \blacksquare$$

3.3 Binary search trees

Learning objectives for this section

- How to store and find records in a binary search tree
- How to compute the height of a binary search tree with n records
- The maximum number of comparisons needed in order to find a record in a binary search tree with n records

We conclude the chapter by giving a description of one way in which binary trees can be used to construct an efficient storage and retrieval solution for a list of records which is to be kept on a computer.

Example 3.8 Storing a list of records
Suppose that a mail order company has a list of past customers stored on its computer in alphabetical order. When it receives an order, it wants to be able to check quickly whether this is from an existing customer, or whether this is a new customer that needs to be added to the database.

Let us assume we have N past customer records numbered $1, 2, 3, \ldots, N$, where record 1 holds the information of the customer first in alphabetical order, record 2 holds the information for the customer second in alphabetical order, etc.

One traditional way of storing the customer records would be in a line of N records as in Figure 3.5, and a simple check whether a new order is from a past customer would be to ask the computer to begin at the beginning of the list and compare the customer's name and address with each record on the existing database in turn until a match is found.

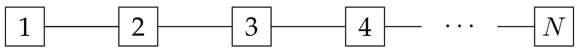


Figure 3.5: A simple linear list

This search method is not very efficient though, for if the order is from a new customer, then a computer using this simple method would have to check the name against every single record to verify that the name is not already on the database. Even in the case of an existing record, depending on the size of the database, the computer might have to make several thousand comparisons before the matching record is located. We shall see next how a binary tree can be used to design a much better storage solution for the company.

3.3.1 Subtrees of a binary tree

Binary trees have the important 'recursive' structural property that *if you take any internal node and all its descendants*, then these vertices and the edges joining them also form a binary tree, because the property that each internal node has exactly two children is preserved.

Suppose x is an internal node. Then the binary subtree that is formed by taking the *left child of x and all its descendants* is called the **left subtree** of x; similarly, the binary subtree formed by taking the *right child of x and all its descendants* is called the **right subtree** of x.

Example 3.9 Figure 3.6 shows the left subtree of the root "Set 1" of the binary tree in Figure 3.4.

We shall now show how this recursive property of binary trees can be used to effectively store an ordered list of records in a binary tree. The resulting binary tree is going to be balanced and retrieval

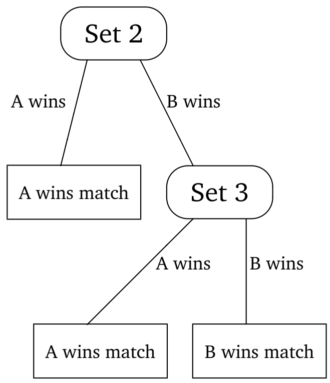


Figure 3.6: The left subtree of the root in the 'tennis match'-tree of Figure 3.4

of records will generally be much more efficient than the simple method we used in Example 3.8.

3.3.2 Storing data in a binary search tree

We first explain how data can be stored efficiently at the internal nodes of a binary tree. An ordered list of records is stored by a recursive procedure in which we first store the 'middle' record at the root and then recursively store the list of all smaller records in the left subtree and the list of all larger records in the right subtree. The resulting binary tree structure is known as a **binary search tree** or simply a **BST**.

Algorithm for storing data in a binary search tree
Suppose we have to store N records and that the first record to be stored is #1 and the last record is #N:

1. **The root** r is the record # $\lfloor (1 + N)/2 \rfloor$.
2. We then store all the records that come *before* the root in the *left* subtree of r and all records that come *after* the root in the *right* subtree of r in the following way:
   If the first record in the subtree is #a and the last record is #b, then the **root of the subtree** is
   $$\# \lfloor (a + b)/2 \rfloor .$$
3. For each of the two subtrees we then divide the remaining records into two halves again, those that come before the root of the subtree and those that come after the root of the subtree. These are then stored in left and right subtrees to the roots of the subtrees by the process in step 2 again.
4. We continue dividing and adding new subtrees until each subtree consists of only one record. This completes the internal vertices of the binary tree.
5. The external nodes are empty boxes representing positions in the list between each existing pair of records (and before and after the present first and last record) in which a new record could be entered in correct alphabetical order.

Example 3.10 Storing data in a binary search tree

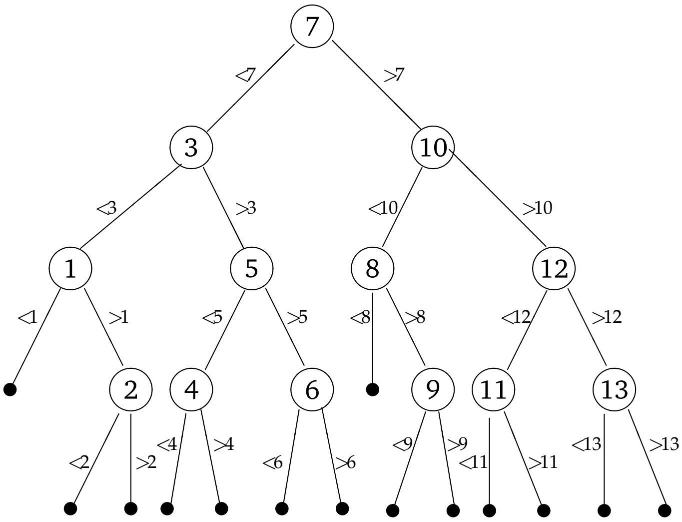


Figure 3.7: A binary search tree storing 13 records numbered $1, 2, \ldots, 13$

Figure 3.7 shows how we would store 13 records in a binary tree constructed as described above.

We imagine that the records have been put in alphabetical order and then numbered from 1 to 13. The root is at # $\lfloor (1+13)/2 \rfloor = 7$.

Records $\{1, 2, 3, 4, 5, 6\}$ are stored in the left subtree and $\{8, 9, 10, 11, 12, 13\}$ in the right subtree of #7. The roots of the left and right subtrees are records # $\lfloor (1+6)/2 \rfloor = 3$ and # $\lfloor (8+13)/2 \rfloor = 10$ respectively.

The left subtree of #3 contains the records $\{1, 2\}$ and its right subtree contains the records $\{4, 5, 6\}$. These subtrees are therefore rooted at #1 and #5 respectively. Similarly, the left subtree of #10 contains the records $\{8, 9\}$ and its right subtree contains the records $\{11, 12, 13\}$. These subtrees are therefore rooted at #8 and #12 respectively.

The left subtree of #1 is empty and is therefore an external node representing a position in which a data item that precedes #1 in alphabetical order could be placed. The right subtree of #1 contains just the record #2. Its subtrees consist of two external nodes representing the positions in which a new data item could be placed between #1 and #2 or between #2 and #3. The remaining subtrees are constructed in a similar way.

3.3.3 The height of a binary search tree

When storing the records at the internal nodes of a binary search tree, at any given node we always divide into two parts, of as equal size as possible, the list of records to be stored in the subtree rooted at this node.

Example 3.11 Consider the node 12 in the BST of Figure 3.7. The subtree rooted at 12 is to contain an odd number of records, namely the sublist $\{11, 12, 13\}$. 12 becomes the root and the two others are divided equally among the two subtrees of 12 which therefore have the same height.

Consider the node 8 in the BST of Figure 3.7. The subtree rooted at 8 is to contain an even number of records, namely the sublist $\{8, 9\}$. 8 becomes the root and the other node is to be divided equally among the two subtrees of 8, which is only possible by putting it in one of the subtrees and none in the other. The right subtree of 8 will thus get one more internal node than the left and the right subtree has height 1 while the left subtree has height 0.

Consider also the node 3 in the BST of Figure 3.7. The subtree rooted at 3 is to contain an even number of records, namely the sublist $\{1, 2, 3, 4, 5, 6\}$. 3 becomes the root and the other five are to be divided equally among the two subtrees of 3, i.e. the right subtree will thus get one more internal node than the left subtree. We note that the right and left subtrees of 3 have the same height though.

We note that when the list to be stored in a subtree has an odd number of records, this will make the right and left subtree of the node have the same number of internal nodes and thus the same height. When the list to be stored in a subtree has an even number of records, this will make the right subtree have one more internal node than the left subtree of the node and the right subtree may therefore have one more level than the left subtree. We may thus conclude that in a binary search tree all the external nodes occur either on just the highest level or on the highest two levels. In other words:

Lemma 3.7 *A binary search tree is balanced.*

This property enables us to calculate from the size of the data set to be stored at the internal nodes of the BST, the number of levels that the binary search tree will have.

Suppose the database contains N records, which we must store in the internal nodes of a binary tree of height h. The BST has thus got levels $0, 1, 2, \ldots, h$.

We start by estimating the number of internal nodes in a binary search tree of height h. All internal nodes occur at levels $0, 1, 2, \ldots, h-1$. Hence the *maximum* number of internal nodes in the tree is

$$1 + 2^1 + 2^2 + \ldots + 2^{h-1} = \tfrac{2^h - 1}{2-1} = 2^h - 1.$$

This maximum occurs when *all* the vertices at level $h-1$ are internal nodes.

Some of the vertices on level $h-1$ may be external nodes, as we had in Figure 3.7 for example. However, because a binary search tree is always balanced, we know that no external node will occur at any level before $h-1$. Hence all nodes on levels $0, 1, \ldots, h-2$ are internal, so there are always a minimum of

$$1 + 2^1 + 2^2 + \ldots + 2^{h-2} = 2^{h-1} - 1$$

internal nodes.

We must store N records in the BST and all of these must be at internal nodes. Thus using the lower limit on the number of internal nodes in a tree of height h and the upper limit found in the previous two paragraphs, we must have that

$$2^{h-1} - 1 < N \leq 2^h - 1$$

in order to be able to store the N records in the tree. This inequality enables us to find the height of tree needed. Adding 1 to each part of the inequality, gives

$$2^{h-1} < N + 1 \leq 2^h.$$

Recall that if $y = 2^x$, then $x = \log_2 y$. Hence the height h of the tree is the positive integer h such that

$$h - 1 < \log_2 (N + 1) \leq h.$$

Using the ceiling function, we can express this as

Theorem 3.8 *The height h of a binary search tree with N records stored at internal nodes is*

$$h = \lceil \log_2 (N + 1) \rceil .$$

3.3.4 Finding a record in a binary search tree

The algorithm for retrieving a record from a BST is recursive.

Algorithm for retrieving a record from a BST

We start the search by comparing our target name with the root of the BST. There are three possible conclusions: (1) the computer tells us that the target matches the root, in which case the search is concluded; or (2) that the target comes alphabetically before the root, in which case we search for the record in the left subtree; or (3) that it comes alphabetically after the root, in which case we search for the record in the right subtree.

If we do not get a match with the root, then we know which subtree should contain the target and we search that subtree only. Thus after just one comparison, we reduce the total number of records to be searched by 50%. We now repeat this procedure in the indicated subtree, comparing our target with its root and then, if we don't get a match, moving to its left or right subtree as indicated.

Example 3.10 (cont.) Retrieving a record from a binary search tree
Suppose our target matches record 9 in Figure 3.7. The search is conducted as follows.

The computer compares the target with the root 7. Since $9 > 7$, it goes to the right subtree[7] of 7.

Now the computer compares the target with 10. Since $9 < 10$, it moves to the left subtree[8] of 10 and compares 9 with 8. Since $9 > 8$, it moves to the right subtree of 8 and achieves a match with 9. Thus the record is identified in 4 comparisons.

Now suppose we have a target that is not on our existing database and comes alphabetically between the records entered at 3 and 4. The computer compares it with 7 and moves to the left subtree of 7; it compares it with 3 and moves to the right subtree of 3; it compares it with 5 and moves to the left subtree of 5; it compares it with 4 and moves to the left child of 4. But this is an empty box. This tells us that the target is not on the current list and should be inserted immediately before the record 4.

[7] Note that the left subtree of 7 can be completely disregarded after this comparison, so with just one comparison, we have halved the number of records to be searched.

[8] The number of records to be searched has been halved once more.

Notice that only 4 comparisons[9] were needed to establish that a record was not in the list. This is considerably better than the 13 comparisons which were needed in the simple linear search method of checking all records which we considered in Example 3.8.

[9] With records 7, 3, 5 and 4.

The maximum number of comparisons needed to find a record in a BST

The computer makes its first comparison with the root at level 0 and then makes one comparison at each level until either the target is matched or it is verified that it is not in the data set. Hence at worst, the computer makes a comparison on each of the levels $0, 1, \ldots, h-1$, giving h comparisons altogether. Thus combining this with Theorem 3.8 we have the following Theorem.

Theorem 3.9 *Suppose we store an ordered list of N records labelled $1, 2, 3, \ldots, N$ at the internal nodes of a binary search tree of height h. The maximum number of comparisons needed to retrieve a record from the binary search tree is*

$$h = \lceil \log_2 (N+1) \rceil .$$

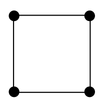
Example 3.10 (cont.) Consider again the binary search tree of Figure 3.7. We have $N = 13$ here. We need a binary tree of height 4 to store this data because $2^3 < 13 + 1 \leq 2^4$. The internal nodes of the binary search tree are on levels $0, 1, 2, 3$ and the height of the tree is $h = 4$. As we saw, a worst case requires four comparisons to match the target.

3.4 Exercises on Chapter 3

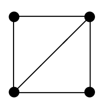
3.4.1 True/False questions

In each of the following questions, decide whether the given statements are true or false.

1. A tree is a connected graph without loops.
2. A connected graph without loops is a tree.
3. A connected graph without cycles is a tree.
4. If a graph G is a tree, then G is simple.
5. A path graph of length n is a tree with n vertices.
6. A tree with n vertices has $n-1$ edges.
7. The following graph has two non-isomorphic spanning trees.



8. The following graph has two non-isomorphic spanning trees.



9. Suppose that a binary search tree is designed to store an ordered list of 32 records at its internal nodes.
   (i) The height of the binary search tree is 5.
   (ii) The root of the binary search tree is 17.
   (iii) The records at level 1 of the binary search tree are 8 and 24.
   (iv) Record 10 is an ancestor of record 20 in the tree.
10. A binary search tree of height h is balanced; this means that it has 2^i records at all levels.

3.4.2 Longer exercises

Question 1
Let G be a graph.

(a) What two properties must G satisfy in order to be a *tree*?

(b) Suppose that every pair of vertices of G are joined by a unique path in G. Must G be a tree? Justify your answer.

Question 2

(a) Draw the tree T with $V(T) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E(T) = \{v_1v_2, v_2v_3, v_2v_4, v_4v_5, v_4v_6\}$.

(b) Construct all the non-isomorphic trees on seven vertices which can be obtained by attaching a new vertex of degree one to T.

(c) Explain briefly why the trees you obtain in (b) are not isomorphic to each other.

Question 3
Let G be the connected graph with $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_2\}$.

(a) Find all spanning trees of G.

(b) How many non-isomorphic spanning trees does G have?

Question 4
Let T be a rooted tree with root r.

(a) Explain how the nodes of T are partitioned into *levels*.

(b) What does it mean to say that T has *height* h?

(c) What does it mean to say that a node of v is an *external node*?

Question 5
A *ternary tree* is a rooted tree in which each internal node has exactly three children.

(a) Draw a ternary tree of height 2 in which each external node lies on level 2.

(b) Let T be a ternary tree of height h in which all external nodes lie on level h. Determine the number of nodes on level i for all integers i, $0 \leq i \leq h$.

Question 6
A restaurant offers a set meal consisting of a choice of one of two starters S_1, S_2; followed by a choice of one of three main courses M_1, M_2, M_3; followed by a choice of one of two desserts D_1, D_2.

(a) Construct a rooted tree to model the outcomes of ordering a meal. (Make the first three levels represent the three different courses.)

(b) How many different meals can be chosen?

Question 7

(a) Design a binary search tree for an ordered list of 11 records.

(b) What is the maximum number of comparisons that the computer would have to make to match any existing record?

(c) Which existing records would require the maximum number of comparisons?

Question 8
A mail order company has 5,000,000 records on its database. Calculate the maximum number of comparisons that would need to be made to match a target with any record in the database.

Chapter 4

Counting methods and probability

Essential reading

Epp Sections 6.1, 6.2, 6.3, 6.4 or M&B Sections 3.2, 3.3, 3.4, 3.5, 4.1

Keywords

Counting in which order is important, the Multiplication Principle, the Addition Principle, permutations and the factorial notation, counting in which order is not important, the Principle of Inclusion-Exclusion, counting using sets, probability.

Many of the problems that arise in discrete mathematics, probability and their applications involve counting the number of ways in which some operation can be performed or counting the number of items satisfying some given specifications. We shall consider two types of counting problems depending on whether or not the items we are counting are affected by order.

4.1 The basic counting methods

Learning objectives for this section

- Illustrating a counting problem by a rooted tree
- Using the Multiplication Principle
- How to distinguish between combinations and permutations
- The number of permutations of length r chosen from a set of n objects
- The number of combinations of r objects chosen from a set of n distinct objects
- Factorial notation
- Deciding whether an (ordered) sequence or an (unordered) set is the correct type of solution to a given counting problem
- Counting problems involving permutations and combinations

4.1.1 Counting in which order is important

Consider the following two related examples.

Example 4.1 From the menu for a school meal, you may select one of three starters, one of five main dishes and one of two desserts. How many different meals consisting of a starter, a main course and a dessert is it possible to order?

Example 4.2 You toss a coin six times in succession; when it comes down showing "heads", you record a 1 and when it comes down showing "tails", you record a 0. How many different sequences of 0's and 1's is it possible to obtain?

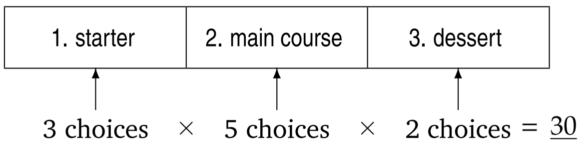
You can easily solve these small problems by enumerating the possibilities (i.e. making a list and checking that you haven't left any out). But we would really like a general method for solving problems of this type, larger as well as smaller ones.

The problem described in Example 4.1 can be divided into stages.

Stage 1 Select one of the three starters;

Stage 2 Select one of the five main dishes.
Each of the 3 choices at Stage 1 is followed by one of 5 choices at Stage 2, giving $3 \times 5 = 15$ different combinations of starter and main dish.

Stage 3 Select one of the two desserts.
Each of these 15 combinations of starter and main dish can be followed by any one of 2 choices for dessert at Stage 3, giving a total of $3 \times 5 \times 2 = 30$ different meals.



This approach gives the following useful counting principle.

The Multiplication Principle.
Suppose that a process can be completed in k stages. If there are m_1 choices at Stage 1 and each of these can be followed by m_2 choices at Stage 2, ..., and finally each of the m_{k-1} choices at Stage $(k-1)$ can be followed by m_k choices at Stage k, then the number of different outcomes of the process is given by the product $m_1 m_2 \ldots m_k$.

We can illustrate the above process by means of a *rooted tree*, as described in Chapter 3.

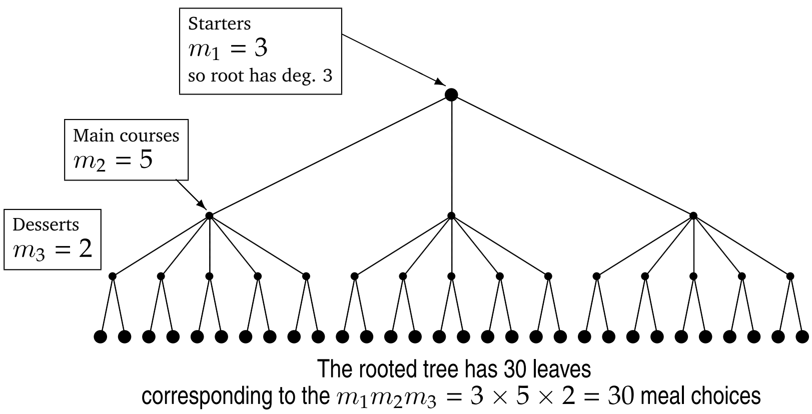


Figure 4.1: There are 30 possible school meals

The root of the tree corresponds to the beginning of the process, the edges leaving the root correspond to the m_1 choices of Stage 1, the vertices at the first level to the m_1 different outcomes from Stage 1, and so on. The resulting tree will have $m_1 m_2 \ldots m_k$ external vertices, all belonging to level k, corresponding to the different outcomes from the whole process. Figure 4.1 shows the rooted tree depicting the 30 possible school meals of Example 4.1.

Now let us turn to the solution of Example 4.2. We need to calculate the number of possible sequences of six symbols, each of which is either a 1 or a 0. We can regard this as a 6-*stage* problem in which we have just two choices at *stage 1* and each of these is followed by two choices at *stage 2*, and so on. Thus we may apply the *Multiplication Principle* with $m_1 = m_2 = \ldots = m_6 = 2$.
Hence the number of ways of completing the 6 stages is $(2 \times 2 \times 2 \times 2 \times 2 \times 2) = 2^6$, which gives the total number of different sequences as 64.

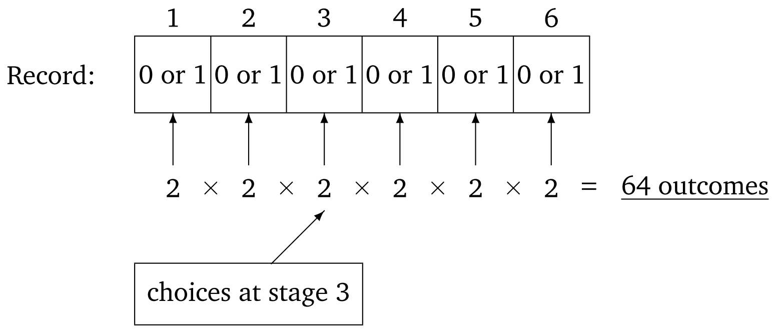


Figure 4.2: There are 64 possible outcomes from tossing a coin 6 times.

Example 4.3 We now draw the tree diagram to illustrate Example 4.2.

(a) The full tree has 7 levels (height 6).

(b) It is a binary tree.

(c) The first three levels of the tree are shown in Figure 4.3.

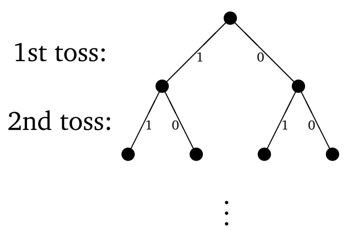


Figure 4.3: The start of the binary tree illustrating Example 4.2.

(d) Due to space restrictions we leave it to you to draw the remaining four levels of the tree. There will be 64 external vertices at level 7.

The number of binary strings

A finite sequence of zeros and ones is known as a **binary string**. Each digit (0 or 1) in the string is called a **bit** and a binary string of n

bits is called an n**-bit binary string**. Thus 01 is a 2-bit binary string and 10110 is a 5-bit binary string. The complete list of solutions to Example 4.2 gives all the 6-bit binary strings. Generalizing the method of Example 4.2 by which we showed that there are exactly 2^6 6-bit binary strings, we can prove the following result.

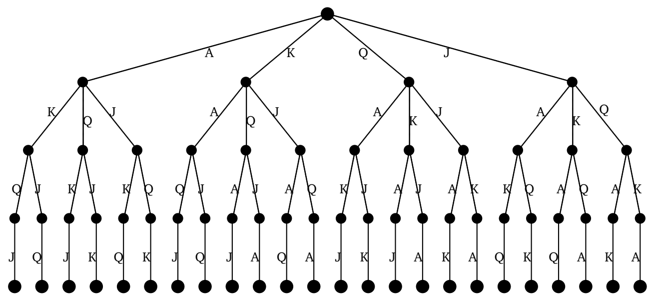
Theorem 4.1 *There are 2^n different n-bit binary strings.*

Proof. We can construct any n-bit binary string in n stages, where for $i = 1, 2, \ldots, n$, stage i consists of choosing the ith digit in the sequence. At each stage, we have exactly 2 choices, regardless of which choice we made at the previous stage. Thus we may apply the Multiplication Principle with $m_1 = m_2 = \ldots = m_n = 2$, giving 2^n n-bit binary strings altogether.∎

Permutations and the factorial notation

Example 4.4 Four playing cards, the Ace, King, Queen and Jack of hearts, are shuffled and then dealt, face up, in a row. In how many different orders can these cards be laid down?

This is a 4-stage problem; there are 4 possibilities for the card that is turned up first; when this has been decided, it leaves just 3 possibilities for the card that is turned up next; this in turn leaves 2 possibilities for the card that is turned up third and just 1 possibility for the final card. Thus the total number of different arrangements is $4 \times 3 \times 2 \times 1 = 24$, by the Multiplication Principle. Here is the corresponding tree diagram:



Any arrangement of a set of n distinct items that puts these items in an order is called a **permutation** of the set. Thus we have shown above that there are 24 different permutations of a set containing 4 elements.

Example 4.5 Suppose that we repeat Example 4.4 using all the 13 cards in one suit. We now have a 13-stage problem and the total number of permutations of the cards is $13 \times 12 \times \ldots \times 3 \times 2 \times 1$. We denote this number by the symbol 13!, read "13 **factorial**".

In general, for any $n \geq 1$, the notation $n!$, read "n **factorial**", stands for the product $n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$. For algebraic reasons, it is convenient to define

$$0! = 1.$$

We shall explain later why this is so.

The algorithm we considered in Example 2.2 computes $n!$ by simply iterating a while-loop which multiplies together all the positive integers between n and 1.

Notice that for $n \geq 1$, factorials are defined recursively, using the recurrence relation

$$n! = n \times (n-1)!$$

By using this recurrence relation it is possible to make a recursive function to calculate $n!$.

Generalizing Examples 4.4 and 4.5, we have the following result.

Theorem 4.2 *Suppose that we have a set of n distinct items. The number of ordered lists without repetition that we can make from these is $n!$.*

This Theorem follows from the Multiplication Principle as Figure 4.4 indicates. An alternative statement of Theorem 4.2 is:

Theorem 4.2' *There are $n!$ permutations of a set with n distinct items.*

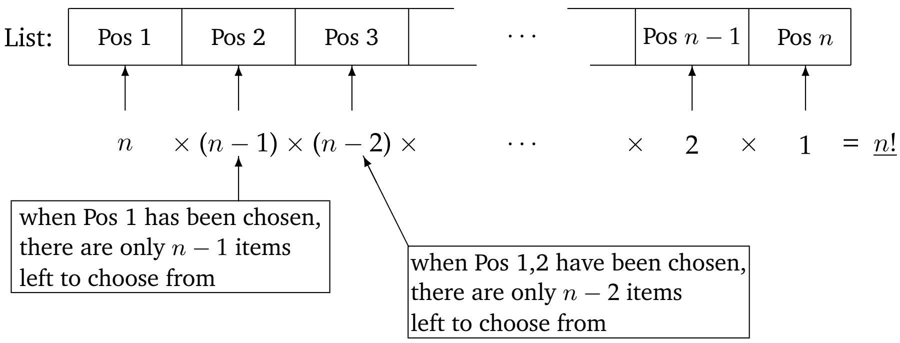


Figure 4.4: There are $n!$ ways of making an ordered list of n distinct objects

Example 4.6 The number of ordered lists of r elements chosen from n
Suppose that in Example 4.5, we lay down just four of the 13 cards in a row. Using the Multiplication Principle we see that the number of different arrangements possible is

$$13 \times 12 \times 11 \times 10 = \frac{13!}{9!}.$$

The following result generalizes Example 4.6 above.

Theorem 4.3 *Suppose that n and r are positive integers, with $r \leq n$. Then the number of ordered lists of r distinct items chosen from a set of n distinct elements is*

$$n\,(n-1)\,(n-2)\ldots(n-(r-1)) = \frac{n!}{(n-r)!}.$$

Proof. Choosing r objects from n is an r-stage process where we have n choices at stage 1, $n-1$ choices at stage 2, $n-2$ choices at stage 3, and so on up to stage r where we have $n-(r-1)$ choices. Hence the required number of lists is

$$(n-1)\,(n-2)\ldots(n-(r-1))\,,$$

which is the left hand side. To see that the right hand side is also equal to this is a straightforward calculation:

$$\begin{aligned}\frac{n!}{(n-r)!} &= \frac{n(n-1)(n-2)\ldots(n-r+1)\quad(n-r)(n-r-1)\ldots 2\cdot 1}{(n-r)(n-r-1)\ldots 2\cdot 1}\\ &= n(n-1)(n-2)\ldots(n-r+1)\\ &= n(n-1)(n-2)\ldots(n-(r-1))\,.\blacksquare\end{aligned}$$

4.1.2 Counting in which order is not important

We next consider the problem of counting the number of ways of choosing a subset from a given set. Recall that sets are unordered, so that for example

$$\{a, n, t\} = \{t, a, n\}.$$

This means that when we are selecting subsets, *the order in which the items are selected is not important.*

Example 4.7 We consider two fundamentally different problems:

(a) A President, Secretary and Treasurer for the student Computing Society are elected from among 45 students. Given that no student can fill more than one of these posts, how many different selections are possible?

(b) A set of three students is to be chosen from a class of 45 students to represent the class on the Staff-Student Consultative Committee. How many different selections are possible?

In (a), the order of selection is important, because it is not unimportant which of the elected persons gets the post of President, who gets the post of Secretary and who gets the post of Treasurer. Hence by the Multiplication Principle, the number of different selections is $45 \times 44 \times 43$.

In (b), there are 45 ways of choosing the first student, then 44 ways of choosing the second and finally 43 ways of choosing the third. This gives us $45 \times 44 \times 43$ ways of choosing an *ordered* list of three students, just as for (a) above. But this time we are not interested in the order in which the students are chosen; all that interests us is the set of students selected. Suppose that A, B, C, is an ordered list of three students. Then we would get the *same* set $\{A, B, C\}$ of students from any of the following six ordered lists:

$$A, B, C;\quad A, C, B;\quad B, A, C;\quad B, C, A;\quad C, A, B;\quad C, B, A.$$

You will recognize that any of the six possible permutations of a set of three items will give rise to the same unordered set. Thus among the $45 \times 44 \times 43$ different ordered lists of three students, each possible set of three students occurs exactly $6 = 3!$ times. Thus the number of different sets of three students is

$$\frac{45 \times 44 \times 43}{3!} = \frac{45!}{42!\,3!}.$$

In general, we call a subset of r items chosen from a set of n distinct items a **combination of r items chosen from n items**. The following result generalizes Example 4.7.

Theorem 4.4 *The number of different combinations of r items chosen from n items is $\frac{n!}{(n-r)!\,r!}$.*

Proof. There are $\frac{n!}{(n-r)!}$ different ordered lists that we could make of r items chosen from n items. But given any subset of r items, we can form $r!$ different ordered lists of its elements. Thus the elements of each subset of r items occur on $r!$ different lists. Hence the number of different subsets that can be chosen is $\frac{n!}{(n-r)!\,r!}$. ∎

We denote the number of combinations of r items chosen from n items by the **binomial coefficient** $\binom{n}{r}$, read "n **choose** r". Thus

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} \tag{4.1}$$

Other books may use different notations for $\binom{n}{r}$. Among these are $C(n,r)$, C_r^n and ${}^nC_r$, but as the $\binom{n}{r}$ is the most universally recognized, we shall not be using other symbols on this course. You are also encouraged not to use these other notations for binomial coefficients.

Example 4.8 Bill knows 7 children.

(a) He has 4 cinema tickets and thus wants to take 3 of the children to the cinema. In how many possible ways can he select 3 children to take to the cinema?

Answer: $\binom{7}{3}$.
This selection is unordered as we are selecting a set of 3 children to accompany Bill. The order in which the children are chosen is not important as the end result is the same: they all end up with one cinema ticket each.

(b) The next day Bill has three tasks to do: (1) he has to do the dishes; (2) he has to eat lunch at a nice restaurant and (3) he has to solve his maths assignment. In how many possible ways can he select 3 children to help him with his tasks?

Answer: $7 \times 6 \times 5$.
This selection is ordered because the three tasks are very different, some are much more pleasant than others (e.g. solving the maths assignment is obviously everybody's favourite, while nobody wants to do dishes!). This is thus a 3-stage problem where stage i, for $i = 1,2,3$, is choosing somebody to help with task (i). The answer thus follows from Theorem 4.3.

We can immediately deduce the following identity for binomial coefficients from equation (4.1).

Lemma 4.5 *Let n and r be integers with $0 \le r \le n$. Then*

$$\binom{n}{n-r} = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}.$$

Proof. Computing $\binom{n}{n-r}$ and $\binom{n}{r}$ using the equation (4.1) both yield the fraction on the right hand side. ∎

Although Lemma 4.5 follows immediately from equation (4.1), we would expect that there should also be an underlying logical reason why Lemma 4.5 holds. This logical reason is illustrated by the following example.

Example 4.9 It is required to make cloth covers for the notice board for each of the 17 different sports clubs in the Athletic Union. There is only sufficient green cloth for 13 boards and the remainder are to be covered in blue.

There are $\binom{17}{13}$ ways of choosing 13 boards to be covered in green, (leaving 4 boards to be covered blue); or, looked at the other way round, there are $\binom{17}{4}$ ways of choosing 4 boards to be covered in blue (leaving 13 boards to be covered in green). This is just two different ways of counting all ways of getting 13 green boards and 4 blue, so it is clear that both expressions should give the same answer. Thus,

$$\binom{17}{13} = \binom{17}{4} = \frac{17 \times 16 \times 15 \times 14}{4 \times 3 \times 2 \times 1} = 2380.$$

We can now give a reason why we should put $0! = 1$. There is only one way of selecting a subset of size 0 from a set of n items (we choose none of them); similarly there is only one way of choosing a subset of size n from a set of n items (we choose all of the items). Thus we would expect $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$. You will see this holds in equation (4.1) if we define $0!$ to be equal to 1. With this definition, which must have seemed rather mysterious when it was first given, the expression $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$ holds for all values of r where $0 \leq r \leq n$.

4.1.3 Summary of strategies for counting problems

When you are faced with a counting problem, you need to analyse it very carefully before applying one of the formulae we have derived in this section. If you rely solely on memorising the formulae you may use the wrong one! We have encountered three main types of counting problems:

Summary of counting strategies

Sequences with repetition allowed

In this type, we are required to find the number of possible sequences where each term is drawn from a given set and repetitions are allowed.

Examples of this type: Binary strings; codes, such as telephone numbers, car numbers, codewords, etc.; integers which can be formed from a given set of digits.

Solution to this type: If the number of choices for each term of the sequence is a constant number n, say, then the solution is n^r, where r is the length of the sequence.

Sequences with no repetitions allowed

In this type, we are required to find the number of possible sequences where each term is drawn from a given set and repetitions are not allowed.

Examples of this type: Permutations of a set; lists; arrangements of objects in a row; integers with distinct digits; teams in which each person is chosen to perform a distinct task, etc.

Solution to this type: The number of different arrangements of r distinct objects chosen from a set of n distinct objects is $\frac{n!}{(n-r)!}$

Choosing subsets of a given set

In this type, we are interested only in choosing the subset and the order in which the elements are selected is not important.

Examples of this type: Teams, groups and subsets where the members are not chosen to fulfill different functions; hands of cards, etc.

Solution to this type: The number of ways of selecting a subset of size r from a set of size n is $\frac{n!}{(n-r)!\,r!} = \binom{n}{r}$.

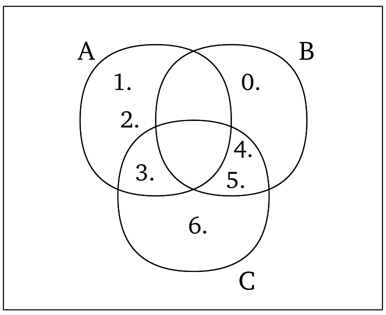
4.2 Counting using sets

Learning objectives for this section

- Definition of what it means for sets $A_1, A_2, \ldots, A_n$ to be pairwise disjoint
- Definition of what it means for sets $A_1, A_2, \ldots, A_n$ to partition a set S
- Statement and use of the Addition Principle
- Statement and use of the Principle of Inclusion-Exclusion of Theorem 4.6 to count the number of elements in the union of two arbitrary sets A and B
- Using Venn diagrams (or the Principle of Inclusion-Exclusion extended to three sets) to find the size of a union of three finite sets
- Using the Addition Principle and Theorem 4.6 in conjunction with the basic counting methods for permutations and combinations we learnt in the previous section

In Volume 1, Chapter 2, we defined the **size** or **cardinality** of a finite set S to be the number of elements in S and denoted this by $|S|$. In this section, we shall find a way of counting the number of elements in the union of two or more sets.

Example 4.10 Finding the cardinality of a union of sets
Let $A = \{1, 2, 3\}$, $B = \{0, 4, 5\}$ and $C = \{3, 4, 5, 6\}$.



Then:

(a) $A \cup B = \{0, 1, 2, 3, 4, 5\}$, so that $|A \cup B| = 6 = |A| + |B|$.

(b) $A \cup C = \{1, 2, 3, 4, 5, 6\}$, so that $|A \cup C| = 6$.
But in this case, $|A| + |C| = 3 + 4 = 7$.

(c) $B \cup C = \{0, 3, 4, 5, 6\}$, so that $|B \cup C| = 5$.
But in this case, $|B| + |C| = 3 + 4 = 7$.

We see that in Example 4.10(a), we have the simple formula $|A \cup B| = |A| + |B|$ *only* because the sets A and B are *disjoint*. This is an example of our next counting principle. Before stating this, we need two definitions.

Definition 4.1 *A collection of subsets $A_1, A_2, \ldots, A_n$ is called* ***pairwise disjoint*** *if no pair of the subsets has any element in common, that is $A_i \cap A_j = \emptyset$, if $i \neq j$ where $i, j = 1, 2, \ldots, n$.*

Example 4.11 Pairwise disjoint sets

(a) In an ordinary pack of 52 playing cards, let C denote the set of cards in the suit clubs, D denote the set in the suit diamonds, H denote the set in the suit hearts and S denote the set in the suit spades. Then the sets C, D, H, S are *pairwise disjoint*.

(b) Let A_k be the set of natural numbers of which the last digit is a k, $k = 0, 1, \ldots, 9$, so that $A_0 = \{0, 10, 20, \ldots\}$, $A_1 = \{1, 11, 21, \ldots\}$, and so on. Then the sets $A_0, A_1, \ldots, A_9$ are *pairwise disjoint*.

You may have noticed that the union of the four sets of cards in Example 4.11(a), gives the complete pack. Thus the sets C, D, H, S are a **partition** of the complete set of playing cards (see Definition 1.5).

The sets $A_0, A_1, \ldots, A_9$ of Example 4.11(b) are a partition of the set of natural numbers $\mathbb{N}$, because they are pairwise disjoint and every natural number ends in one of the digits $0, 1, \ldots, 9$ and thus belongs to one of the sets.

The following principle uses a partition of a set to count the number of elements in the set.

The Addition Principle

Suppose that $X_1, X_2, \ldots, X_n$ is a partition of a finite set X. If X_1 contains m_1 elements, X_2 contains m_2 elements, and so on, then the size of X is $m_1 + m_2 + \ldots + m_n$, that is

$$|X| = m_1 + m_2 + \ldots + m_n.$$

The following extension of Example 4.7(b) illustrates how the Addition Principle can be used to solve problems.

Example 4.12 Choosing committees

(a) A set of three students is to be chosen from a class of 45 students to represent the class on the Staff-Student Consultative Committee. How many different selections are possible?

(b) Suppose the class contains 25 male students and 20 female students. How many of the selections in (a) contain at least one student of each sex?

We have seen in Example 4.7 that the solution to (a) is $\binom{45}{3}$.

To solve (b) we first let S be the set whose elements are all the possible selections of three students from the class. (So, if X is the set of all 45 students, then S is the set of all subsets of X of size 3.) Next, for $0 \leq i \leq 3$, we let S_i be the subset of S which contains all selections of 3 students with exactly i male students. Then

S_0, S_1, S_2, S_3 is a partition of S since each selection of 3 students has either 0, 1, 2, or 3 male students in it. Hence, by the Addition Principle,

$$|S| = |S_0| + |S_1| + |S_2| + |S_3|.$$

The number we want to find is the number of selections with at least one student of each sex. With the notation we have just introduced, this number is

$$|S_1| + |S_2|.$$

We know that $|S| = \binom{45}{3}$. Since S_0 is the set of selections which contain exactly three female students, we have $|S_0| = \binom{20}{3}$. Similarly, since S_3 is the set of selections which contain exactly three male students, $|S_3| = \binom{25}{3}$. Thus the number of selections with at least one student of each sex is given by

$$|S_1| + |S_2| = |S| - |S_0| - |S_3| = \binom{45}{3} - \binom{20}{3} - \binom{25}{3}.$$

We next consider more general examples where we do not have disjoint subsets. In Example 4.10(b), the sets A and C overlap, with $A \cap C = \{3\}$. So when we add the sizes of A and C, the element 3 is counted twice, once in $|A|$ and once in $|C|$. Thus $|A \cup C| = |A| + |C| - 1$.

Similarly, in Example 4.10(c), the sets B and C overlap, with $|B \cap C| = \{4, 5\}$. Thus when we add the sizes of B and C, both the elements 4 and 5 are counted twice, and hence $|B \cup C| = |B| + |C| - 2$.

These examples suggest the following counting theorem.

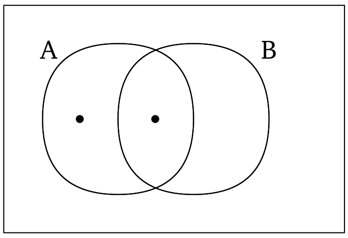
Theorem 4.6 *Let A and B be finite sets. Then*

$$|A \cup B| = |A| + |B| - |A \cap B|\,. \tag{4.2}$$

Theorem 4.6 is called the **Principle of Inclusion-Exclusion** in many text books because the counting method works by first including all regions of each set and then excluding the regions that were counted several times because of overlaps. The **Principle of Inclusion-Exclusion** is easy to understand pictorially as we shall illustrate next. The principle also extends to unions of more than two sets in a natural way.

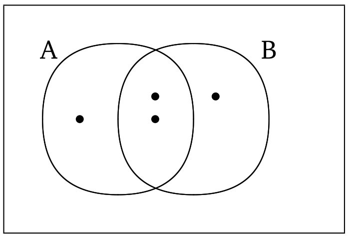
Developing the Principle of Inclusion-Exclusion

1. To count the elements of $A \cup B$, we first compute the cardinality of A, $|A|$. On the Venn diagram we have thus counted all the regions covered by A. We put a dot in these regions:



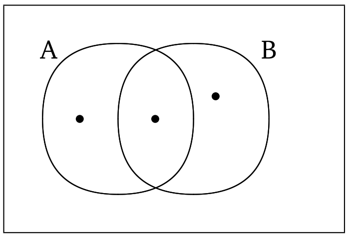
$$|A \cup B| = |A| + \cdots$$

2. We next count the cardinality of B. On the Venn diagram we have thus counted all the regions covered by B also. We put a dot in these regions too. Hence our "Venn diagram" for $|A| + |B|$ looks like:



$$|A \cup B| = |A| + |B| - \cdots$$

3. Looking at the "Venn diagram" for $|A| + |B|$, we see that we have indeed calculated every region of $A \cup B$, but unfortunately there are two dots in the region $A \cap B$, indicating that we have counted this region twice. We can delete one of the two dots again if we subtract $|A \cap B|$. Hence the "Venn diagram" for $|A| + |B| - |A \cap B|$ looks like



$$|A \cup B| = |A| + |B| - |A \cap B|$$

4. We see on the "Venn diagram" for $|A| + |B| - |A \cap B|$ that we have counted every region of $A \cup B$ precisely once. Hence

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Here is a more formal proof for Theorem 4.6:

Proof. We need to check that each element of $A \cup B$ is counted once on the right hand side of (4.2). If $x \in A \cup B$, then x belongs either (i) to just one of A, B or (ii) to both the subsets A, B. In case (i), $x \notin A \cap B$ and hence x is counted once in $|A| + |B|$ and 0 times in $|A \cap B|$, so it is counted once altogether. In case (ii), $x \in A \cap B$ and hence x is counted twice in $|A| + |B|$ and once in $|A \cap B|$. Hence x is counted $2 - 1 = 1$ time altogether. Thus each element of $A \cup B$ has been counted once on the right hand side of (4.2). Hence the right hand side of equation (4.2) counts exactly the number of elements in $|A \cup B|$.■

Example 4.13 Using the Principle of Inclusion-Exclusion
How many 6-bit binary strings either begin or end with a 0?

Let A be the set of all 6-bit binary strings which begin with a 0 and B be the set of all 6-bit binary strings which end with a 0. We need to calculate $|A \cup B|$.
Using the Multiplication Principle, we have $|A| = 2^5$ since we have one choice for the first bit and two choices for each of the remaining five bits. Similarly $|B| = 2^5$.
Since $A \cap B$ is the set of 6-bit binary strings which begin and end with a 0, we can also use the Multiplication Principle to deduce that $|A \cap B| = 2^4$.
Thus the number of 6-bit binary strings which either begin or end with a 0 is given by

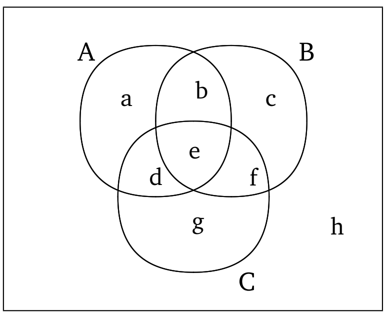
$$|A \cup B| = |A| + |B| - |A \cap B| = 2^5 + 2^5 - 2^4 = 48.$$

Example 4.14 Using Venn diagrams to find the size of a union of three sets
A school teaches evening classes in English, Maths and History to 80 adults. At the end of the year 38 pass the English exam, 36 pass the Maths exam, and 35 pass the History exam. 17 pass both the History and the Maths exams, 13 pass in both English and History and 15 pass in Mathematics and English. 2 students pass all three subjects. We want to know how many fail all subjects.

Let A be the set of all those who pass English, B the set of all those who pass Maths and C the set of all those who pass History. We thus want to find the number of elements in $(A \cup B \cup C)'$.

We can do this by using a Venn diagram:

We start by drawing a Venn diagram which divides the universal set of students in the school into 8 regions, which we label $a, b, c, \ldots, h$:

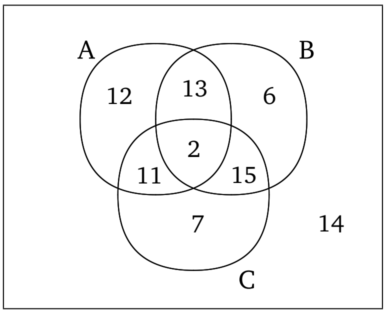


Next we use the information given to substitute the sizes of each of the 8 regions instead of the region label:

- 2 students pass all three subjects, so region e has 2 elements;
- 17 pass both the History and the Maths exams, so region f has $17 - 2 = 15$ elements;
- 13 pass in both English and History, so region d has $13 - 2 = 11$ elements;

- 15 pass in Maths and English, so region b has 15 − 2 = 13 elements;
- 38 pass in English, so region a has 38 − 13 − 11 − 2 = 12 elements;
- 36 pass the Maths exam, so region c has 36 − 13 − 15 − 2 = 6 elements;
- 35 pass the History exam, so region g has 35 − 11 − 15 − 2 = 7 elements.
- This means that $|A \cup B \cup C|$ = 12 + 13 + 6 + 11 + 2 + 15 + 7 = 66, and since the school has 80 students, we thus have that region h has 80 − 66 = 14 elements.

The final Venn diagram looks like this:



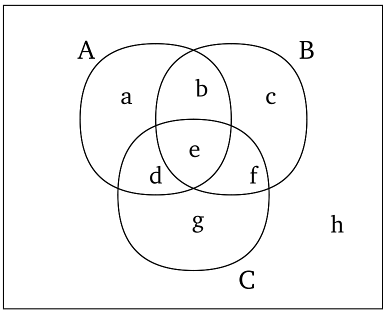
We wanted to now how many failed all three subjects. These are the 14 students in $(A \cup B \cup C)'$.

You can also answer other interesting questions by using this Venn diagram:

- How many passed Maths only? Answer: 6;
- How many passed just one subject? Answer: 12 + 6 + 7 = 25;
- How many passed precisely two subjects? 11 + 13 + 15 = 39.

Example 4.15 Finding the size of a region by Venn diagram
A Science department teaches Abstract Algebra, Boolean Algebra and Computing to 36 students who all pass at least one subject. None of them pass the Computing exam though. 14 of them pass in two subjects, and 29 pass in Boolean Algebra. We want to find the number of students who pass the Abstract Algebra exam only, and we also want to know the number of students failing the Abstract Algebra exam.

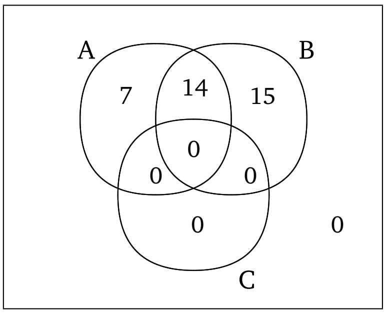
Let A be the set of students passing Abstract Algebra, B be the set of students passing Boolean Algebra and C be the set of students passing Computing. We draw a Venn diagram with 8 regions as in Example 4.14



and fill in the information given:

- none of the students fail all subjects, so $h = 0$;
- they all fail Computing, so $|C| = d + e + f + g = 0$; hence $d = e = f = g = 0$;
- 14 pass two subjects, so $b + d + f = 14$, and thus $b = 14$;
- $|B| = 29$, so $b + c + e + f = 29$, whence $c = 15$;
- all students pass at least one subject, so $|A \cup B \cup C| = 36$,
- which finally yields that $a = 7$.

The final Venn diagram is:



We wanted to find the number of students passing Abstract Algebra only. We see on the Venn diagram that there are 7 such students.

We also wanted to know how many failed Abstract Algebra, these are all the elements of A'. From the Venn diagram we see that $|A'| = 15$.

Example 4.16 The Principle of Inclusion-Exclusion for three sets
Let A, B and C be finite sets. We often want to find the number of elements in $A \cup B \cup C$, $A \cap B \cap C$ or some other region in the Venn diagram for A, B and C. The method used in the previous two examples is very useful, but sometimes the information given makes the equations for determining the number of elements in regions $a, b, c, \ldots, h$ hard to solve. It is thus useful to have a Principle of Inclusion-Exclusion for three sets which is basically a universal identity giving you an extra equation to help you determine the number of elements in regions $a, b, c, \ldots, h$. It is easy enough to develop this from Theorem 4.6:

$$\begin{aligned} |A \cup B \cup C| &= |(A \cup B) \cup C| \\ &= |A \cup B| + |C| - |(A \cup B) \cap C| \\ &\quad \text{by Theorem 4.6} \end{aligned}$$

$$\begin{aligned}
&= |A| + |B| - |A \cap B| + |C| - |(A \cup B) \cap C| \\
&\quad \text{by Theorem 4.6 again} \\
&= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)| \\
&\quad \text{by Rule 2.21(i) in Volume 1} \\
&= |A| + |B| - |A \cap B| + |C| - (|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|) \\
&\quad \text{by Theorem 4.6 one more time} \\
&= |A| + |B| - |A \cap B| + |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.
\end{aligned}$$

We thus have:

The Principle of Inclusion-Exclusion for three sets
Let A, B and C be finite sets. Then

$$|A \cup B \cup C| = |A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|+|A \cap B \cap C| .$$

You will not be required to state the Principle of Inclusion-Exclusion for three sets in the examination, but it is a very useful result to know. Recall the exercise from Example 4.14:

A school teaches evening classes in English, Maths and History to 80 adults. At the end of the year 38 pass the English exam, 36 pass the Maths exam, and 35 pass the History exam. 17 pass both the History and the Maths exams, 13 pass in both English and History and 15 pass in Mathematics and English. 2 students pass all three subjects. We want to know how many fail all subjects.

Let A be the set of all those who pass English, B the set of all those who pass Maths and C the set of all those who pass History. We thus want to find $80 - |A \cup B \cup C|$.

By using the Principle of Inclusion-Exclusion for three sets we can now solve this in just one calculation:

$$\begin{aligned}
|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 38 + 36 + 35 - 15 - 13 - 17 + 2 \\
&= 66,
\end{aligned}$$

so $80 - 66 = 14$ students fail in all three subjects.

4.3 Probability

Learning objectives for this section

- Describing the sample space of a given experiment
- Computing the probability $P(x)$ of an outcome x of a simple experiment
- Events in a sample space S
- Computing $P(E) = |E|/|S|$ for any event E of a sample space S
- What it means for two events to be disjoint
- Computing $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any events A and B of a sample space S

- Determining whether two events are independent

Suppose an experiment is performed which can have one of a finite number of possible outcomes. We shall call the *set* of all possible outcomes the **sample space** of the experiment. We say that the sample space is **equiprobable** if each of the outcomes is equally likely. In this course we will only be concerned with equiprobable sample spaces.

Example 4.17 The sample space of an experiment
We can consider the process described in Example 4.2 as an experiment: A coin is tossed six times in succession; when it comes down showing "heads", we record a 1 and when it comes down showing "tails", we record a 0. Each outcome of this experiment is a 6-bit binary string. The sample space of the experiment is the set of all 6-bit binary strings. The sample space is equiprobable if the coin is "fair", that is to say a head and a tail are equally likely on each throw.

Let S be an equiprobable sample space and let $x \in S$, that is x is one possible outcome of the experiment. Then the **probability** that x occurs is given by $1/|S|$. We will denote this number by $P(x)$.

Example 4.18 The size of a sample space
We saw in Example 4.2 that the number of 6-bit binary strings is $2^6 = 64$. Thus the size of the sample space in Example 4.17 is 64. Hence the probability that the outcome of the experiment is any particular 6-bit string is $2^{-6} = 1/64$. In particular, the probability that the outcome of the experiment is the string $x = 111111$, i.e. that we throw six consecutive heads, is $1/64$. We write

$$P(111111) = 1/64.$$

We next extend our definition of probability to cover more complicated statements than just the occurrence of a single outcome of the experiment. Let A be a statement that one of several possible outcomes occurs. Then the **probability** that A is true is given by the ratio of the number of outcomes in A with the total number of outcomes in S. We shall denote this ratio by $P(A)$. Formally we can think of A as being a subset of S, and put

$$P(A) = |A|/|S|.$$

We shall refer to statements involving subsets of S as **events**.

Example 4.19 Some events and their probabilities
We return to Examples 4.17 and 4.18 where a coin is tossed six times in succession; when it comes down showing "heads", we record a 1 and when it comes down showing "tails", we record a 0. The sample space is the set of all 6-bit binary strings. The sample space has size 2^6. Each outcome of this experiment is a 6-bit binary string and it has probability $1/2^6$ of occurring.

Let A be the event that the first bit in the outcome of the experiment is a 1. Let B be the event that exactly one of the bits in the outcome of the experiment is a 1.

The number of outcomes in A is the number of 6-bit binary strings which begin with a 1. Using the Multiplication Principle this number is 2^5. Thus $P(A) = |A|/|S| = 2^5/2^6 = 1/2$.

The number of outcomes in B is the number of 6-bit binary strings which contain exactly one bit 1. We can determine this number by either of the following two methods. Note that the first of these methods will only be suitable for small examples like this, for larger examples it is necessary to employ the more theoretical second method.

- *Method 1:* List all such strings and count how many there are. The strings are

  $$100000,\ \ 010000,\ \ 001000,\ \ 000100,\ \ 000010,\ \ 000001$$

  giving a total of six strings with exactly one 1.
- *Method 2:* Calculate the number of ways of choosing the bit which is equal to 1 from the six available bits. This gives $\binom{6}{1} = 6$ strings with exactly one 1.

Thus $P(B) = |B|/|S| = 6/2^6 = 3/32$.

Let A and B be events in a given sample space S. We denote the event that either A, or B, or both, occur by $A \cup B$. We denote the event that both A and B occur by $A \cap B$. It can be seen that these definitions are consistent with our alternative view of events as subsets of the sample space. It can also be seen that the above definition for the probability of an event has the following three properties.

Axiom 1 For any event A, $P(A) \geq 0$.

Axiom 2 $P(S) = 1$.

Axiom 3 For *disjoint* events A and B, $P(A \cup B) = P(A) + P(B)$.

These properties are known as the **Axioms of Probability**.

The following analogue of Theorem 4.6 extends Axiom 3 to the case when A and B are not necessarily disjoint.

Theorem 4.7 *Let A and B be events in an equiprobable sample space S. Then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof. Using Theorem 4.6 we have

$$\begin{aligned}
P(A \cup B) &= \frac{|A \cup B|}{|S|} \\
&= \frac{|A| + |B| - |A \cap B|}{|S|} \\
&= \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|} \\
&= P(A) + P(B) - P(A \cap B). \blacksquare
\end{aligned}$$

Example 4.20 Computing the probability of a union of events
Let A and B be the events described in Example 4.19.

We have seen that $P(A) = 2^5/2^6 = 1/2$ and $P(B) = 6/2^6 = 3/32$. Since $A \cap B$ is the event that the first bit is a 1 and that there is exactly one 1 in the six-bit string, the number of outcomes in $A \cap B$ is one[1]. Thus

$$P(A \cap B) = \frac{|A \cap B|}{|S|} = 1/2^6 = 1/64.$$

[1] The only such outcome is the six bit string 100000.

The event $A \cup B$ is that either the first bit is a 1 or there is exactly one 1 in the string, or both. Using Theorem 4.7 we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 1/2 + 3/32 - 1/64 = 37/64.$$

4.3.1 Independent events

We say that two events A and B of a sample space S are **independent** if they satisfy the equation $P(A \cap B) = P(A)P(B)$. Intuitively this means that the likelihood that one of the events occurs is not affected by the occurrence, or non-occurrence, of the other event.

Example 4.21 Checking whether two events are independent
Let A and B be the events described in Examples 4.19 and 4.20.

We have seen that $P(A) = 1/2$, $P(B) = 3/32$ and $P(A \cap B) = 1/64$. Thus

$$P(A)P(B) = (1/2) \times (3/32) = 3/64 \neq P(A \cap B).$$

Hence A and B are not independent. (Intuitively the likelihood that "exactly one digit is a 1" will change if we know whether or not "the first digit is a 1".)

Let C be the event that the second digit of a 6-bit binary string is a 1. Then using the Multiplication Principle we can show that the number of outcomes in C is 2^5 and the number of outcomes in $A \cap C$ is 2^4.

Thus $P(A) = P(C) = 1/2$, $P(A \cap C) = 1/4$ and

$$P(A)P(C) = (1/2) \times (1/2) = 1/4 = P(A \cap C).$$

Hence A and C are independent. (Intuitively the likelihood that "the second digit is a 1" will not change if we know whether or not "the first digit is a 1".)

4.4 Exercises on Chapter 4

4.4.1 Multiple choice questions

In each of the following questions, decide which answer is correct.

1. A fair coin is tossed 12 times.
   (a) How many possible outcomes are there?
      (i) 12^2

   (ii) 2^{12}
   (iii) $12!$
   (iv) $\binom{12}{2}$
   (v) Other.

(b) How many outcomes have precisely 5 heads?
   (i) 2^5
   (ii) 2^7
   (iii) $12 \times 11 \times 10 \times 9 \times 8$
   (iv) $\binom{12}{7}$
   (v) Other.

(c) How many outcomes have heads in toss 5, 6 and 7?
   (i) 2^3
   (ii) 2^9
   (iii) $12 \times 11 \times 10$
   (iv) $\binom{12}{3}$
   (v) Other.

(d) How many outcomes have more heads than tails?
   (i) Half of them
   (ii) 2^6
   (iii) $\binom{12}{0} + \binom{12}{1} + \binom{12}{2} + \binom{12}{3} + \binom{12}{4} + \binom{12}{5}$

2. Consider all rearrangements of the word MATHS.

(a) How many rearrangements are possible?
   (i) 5^5
   (ii) 2^5
   (iii) $5!$
   (iv) $\binom{5}{2}$
   (v) Other

(b) How many rearrangements contain the substring AMS?
   (i) 5×4
   (ii) 3
   (iii) $3!$
   (iv) $3!2!$
   (v) $\binom{5}{2}$

(c) How many rearrangements do not contain the substring HAT?
   (i) 5×4
   (ii) $3!$
   (iii) $5! - 3!$
   (iv) $\binom{5}{2}$
   (v) Other

(d) How many rearrangements contain the substrings MA and SH?
   (i) 4×4
   (ii) 4×2
   (iii) $3!$
   (iv) $2!2!$
   (v) $\binom{5}{3}$

3. A petshop has 10 dogs, 12 cats and 20 rodents for sale. Bill wants to buy 3 pets. How many different choices has he got if
   (a) he has no restrictions on what kind of pets he wants?
      (i) $42 \times 41 \times 40$
      (ii) 42^3
      (iii) $\binom{42}{3}$
      (iv) $42!/3!$
      (v) $42!/39!$
   (b) he wants a dog, a cat and a rodent?
      (i) $10 \times 12 \times 20$
      (ii) $(10 \times 12 \times 20)/3!$
      (iii) $(10 \times 12 \times 20)/3$
   (c) he wants a cat and two rodents?
      (i) $12 \times 20 \times 19$
      (ii) $(12 \times 20 \times 19)/3!$
      (iii) $\binom{12}{1}\binom{20}{2}$
      (iv) Other
   (d) he wants 3 dogs?
      (i) $10 \times 9 \times 8$
      (ii) $\binom{10}{3}$
      (iii) $(10 \times 9 \times 8)/3$
      (iv) $\binom{42}{3} - \binom{32}{3}$
      (v) Other
   (e) he wants no cats?
      (i) $\binom{42}{3} - \binom{12}{3}$
      (ii) $30 \times 29 \times 28$
      (iii) $\binom{30}{3}$
      (iv) Other
   (f) he does not want both a cat and a dog?
      (i) $\binom{32}{3} + \binom{30}{3} - \binom{20}{3}$
      (ii) $\binom{32}{3} + \binom{30}{3}$
      (iii) $\binom{20}{3}$
      (iv) Other

4. Consider the set of integers between 101 and 999 (both inclusive).
   (a) How many are there?
      (i) 897
      (ii) 898
      (iii) 899
      (iv) 900
      (v) 901
      (vi) Other
   (b) How many are odd?
      (i) 448
      (ii) 449
      (iii) 450
      (iv) 451
      (v) 550

(vi) Other

(c) How many are even?

(i) 448
(ii) 449
(iii) 450
(iv) 451
(v) 550
(vi) Other

(d) How many are divisible by 10?

(i) 88
(ii) 89
(iii) 90
(iv) 99
(v) 100
(vi) Other

(e) How many are larger than 799?

(i) 198
(ii) 199
(iii) 200
(iv) 201
(v) Other

(f) How many are both even and larger than 799?

(i) 99
(ii) 100
(iii) 101
(iv) Other

4.4.2 Longer exercises

Note that in the following questions you may leave your answers in the form of powers of integers or products of integers; it is not necessary to evaluate them.

Question 1
The College Student Union is electing a new committee, consisting of a President, a Secretary, and a Treasurer. There are 5 nominations for the office of President, 2 nominations for Secretary and 3 for Treasurer. No person is nominated for more than one office. How many different committees would it be possible to elect? Explain how you obtain your answer.

Question 2
The secret code for a combination lock is formed by choosing an ordered sequence of 3 digits from the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(a) Describe the sample space.

(b) How many different secret codes can be formed if repetitions of digits are allowed?

(c) How many different secret codes can be formed if no repetitions are allowed?

(d) What is the probability that a burglar will guess the secret code on his first attempt in case (b)?

Question 3
A football club has 25 members. It needs to select a team of 11 players for a match.

(a) Describe the sample space.

In how many ways can the team be chosen if

(b) all members are equally eligible for selection?
(c) a given member has already been selected as captain and must be included in the team?

Question 4 Suppose you throw a die three times, noting the number obtained on each throw, so that you end up with a sequence of three numbers in the order in which they are thrown.

(a) Describe the sample space of this experiment.
(b) How many possible ordered sequences can result?
(c) What is the probability of throwing three 6's?
(d) What is the probability that none of the throws is a 6?
(e) What is the probability that at least one of the throws is a 6?

Let A be the event that the first throw is a 1, and B be the event that the sum of the throws is at most 4.

(f) Calculate the probabilities $P(A)$, $P(B)$, $P(A \cap B)$ and $P(A \cup B)$.
(g) Are A and B independent events? Justify your answer.

Question 5

(a) Find the number of integers between 1 and 1000 (both inclusive) which are divisible by 3 or 5.[2]
(b) Suppose that 30% of all households have both a cat and a dog, 60% have a dog and 40% have a cat.
   (i) What is the probability that a household has a cat?
   (ii) What is the probability that a household has neither dogs nor cats?

Question 6

Find the number of integers between 1 and 1000 (both inclusive) which are *not* divisible by 3, 5 or 11.[3]

4.4.3 Further exercises and worked examples

Further exercises and worked examples can be found in Epp Sections 6.1, 6.2, 6.3 and 6.4. We strongly recommend that you do some more exercises on combinations, permutations and probability than are given here in order to become better at recognizing which formula/solution method is relevant for which types of problem. The summary of subsection 4.1.3 may help. Also remember the Principle of Inclusion-Exclusion.

[2] **Help:** Let A be the set of integers between 1 and 1000 (both inclusive) which are divisible by 3. Let B be the set of integers between 1 and 1000 (both inclusive) which are divisible by 5. Then $A \cap B$ is the set of integers between 1 and 1000 (both inclusive) which are divisible by 15. Now use the Principle of Inclusion-Exclusion.

[3] **Help:** Read margin note 2 for Question 5(a) above. Further, let C be the set of integers between 1 and 1000 (both inclusive) which are divisible by 11. Then $A \cap C$ is the set of integers between 1 and 1000 (both inclusive) which are divisible by 33. What are $B \cap C$ and $A \cap B \cap C$? Now use the Principle of Inclusion-Exclusion for three sets which we developed in Example 4.16. You could also use the Venn diagram-method of Example 4.14.

Chapter 5

Systems of linear equations and matrices

Essential reading

M&B Sections 6.1, 6.2. Only part of this chapter is covered in Epp (Section 11.3)

Keywords

Systems of linear equations; Gaussian elimination. Matrix algebra.

It often happens that information has to be stored in a list or a table. Matrices are a convenient way of representing such data, as you have already experienced with the incidence matrix of graphs and digraphs. We thus study the basics of matrix algebra in this chapter. We shall also see that a system of n linear equations in n unknowns can be conveniently expressed in the form of a matrix equation. The equations in this form can be solved in an algorithmic manner. The method is suitable for programming into a computer, which distinguishes it from the methods for solving equations you learnt in school. The algorithm for solving systems of linear equations is known as Gaussian Elimination.

5.1 Systems of linear equations

Learning objectives for this section

- Recognizing linear and non-linear equations
- Writing down the augmented matrix of a system of linear equations
- Using Gaussian elimination to reduce a matrix to row echelon form
- Using Gaussian elimination and back substitution to find all solutions to a system of linear equations

Definition 5.1 *A* ***linear equation*** *in n unknowns $x_1, x_2, \ldots, x_n$ is an equation of the form*

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are given real numbers.

Example 5.1 The equation

$$x_1 + 2x_2 - x_3 = \tfrac{1}{2}$$

is a linear equation.
The equations $x_1 x_2 = 3$ or $x_1^2 + x_2 = 4$ or $\log(x_1) + x_2 = 7$ are not linear.

Definition 5.2 *A* ***system of linear equations*** *in n unknowns $x_1, x_2, \ldots, x_n$ is a collection of one or more linear equations involving the same unknowns. A* ***solution*** *for this system is an assignment of real numbers $x_1 = k_1$, $x_2 = k_2, \ldots, x_n = k_n$ which satisfies each of the equations in the system. To* ***solve*** *the system we want to find all the solutions to the system.*

Example 5.2 A system of linear equations and its solution
Consider the following system of two equations in two unknowns.

$$3x_1 + 6x_2 = 60 \qquad (5.1)$$
$$2x_1 - x_2 = 10. \qquad (5.2)$$

Then $\frac{1}{3}\times$ Equation (5.1) is

$$x_1 + 2x_2 = 20. \qquad (5.3)$$

Now Equation (5.2) $-$ $2\times$ Equation (5.3) gives

$$-5x_2 = -30.$$

Therefore

$$x_2 = 6.$$

Substituting back into Equation (5.3) gives

$$x_1 = 20 - 2x_2 = 20 - 12 = 8.$$

So the only solution to this system is $x_1 = 8$ and $x_2 = 6$.

We can solve a system of m equations with n unknowns using a method similar to the one of Example 5.2 by first reducing to $m-1$ equations with $n-1$ unknowns, then $m-2$ equations with $n-2$ unknowns, and so on. To describe accurately the general procedure we use **matrices**.

Definition 5.3 *An $m \times n$* ***matrix*** *is a rectangular array of real numbers with m rows and n columns.*

Example 5.3 The matrix

$$A = \begin{pmatrix} 0 & 1 & -2 \\ -\frac{1}{2} & 3 & 5 \end{pmatrix}$$

is a 2×3 matrix.

We shall usually use upper case letters such as A, B, X to represent matrices. The only exception being for matrices with just one row or one column which we represent by bold face letters, such as **b**. Note that other books may use a different notation. Some authors use overlined or underlined letters, such as $\overline{b}$ or $\underline{b}$, or even underlined bold face letters like $\underline{\mathbf{b}}$ for matrices with one row/column.

We write $A = (a_{i,j})$ to mean that the entry in the ith row and the jth column of A is labelled $a_{i,j}$. Sometimes we also use the notation $A(i,j)$ for the entry in the ith row and the jth column of A.

Example 5.3 (cont.) In this matrix $a_{1,2} = 1$ and $a_{2,1} = -\frac{1}{2}$.

Definition 5.4 *We can write a system of m equations in n unknowns as:*

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \ldots & + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \ldots & + & a_{2,n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \ldots & + & a_{m,n}x_n & = & b_m \end{array}$$

*where $a_{i,j}$ and b_i are given real numbers for $1 \leq i \leq m$ and $1 \leq j \leq n$. The **matrix of coefficients** for this system is the $m \times n$ matrix A in which the entry in the ith row and jth column is the coefficient of the unknown x_j in the ith equation of the system. Thus, using the above representation for the system, we have $A = (a_{i,j})$, that is*

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & & \ddots & \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{pmatrix}$$

*The **augmented matrix** for this system is the $m \times (n+1)$ matrix $(A : \mathbf{b})$ which we get by adding the extra column*

$$\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

to the right of A, that is the augmented matrix for the system is

$$(A : \mathbf{b}) = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} & : & b_1 \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} & : & b_2 \\ \vdots & & \ddots & & : & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} & : & b_m \end{pmatrix}$$

Example 5.4 The augmented matrix of a system of linear equations
For the system of equations of Example 5.2 we have the matrix of coefficients

$$A = \begin{pmatrix} 3 & 6 \\ 2 & -1 \end{pmatrix}$$

and the augmented matrix

$$(A : \mathbf{b}) = \begin{pmatrix} 3 & 6 & : & 60 \\ 2 & -1 & : & 10 \end{pmatrix}$$

We shall solve the system of equations of Example 5.2 by using the following three operations on the rows of the augmented matrix $(A : \mathbf{b})$ rather than working with the equations (5.1) and (5.2).

- Interchange two rows.
- Multiply a row by a non-zero real number.
- Add multiples of one row to another row.

To describe which operation we are using we denote the first and second rows of $(A : \mathbf{b})$ by R_1 and R_2 respectively. We first multiply R_1 by $1/3$, so $R_1 := \frac{1}{3}R_1$:

$$\begin{pmatrix} 1 & 2 & : & 20 \\ 2 & -1 & : & 10 \end{pmatrix}.$$

We next subtract twice R_1 from R_2, so $R_2 := R_2 - 2R_1$:

$$\begin{pmatrix} 1 & 2 & : & 20 \\ 0 & -5 & : & -30 \end{pmatrix}$$

From the final augmented matrix we can read off the solution as follows:

$$R_2 \Rightarrow -5x_2 = -30 \Rightarrow x_2 = 6.$$

Now substituting this into the equation corresponding to R_1 we get:

$$R_1 \Rightarrow x_1 + 2x_2 = 20 \Rightarrow x_1 = 20 - 2x_2 = 20 - (2 \times 6) = 8.$$

You can check that these values for x_1, x_2 satisfy both the equations of Example 5.2 by substituting into the equations:

$$\begin{aligned} 3 \cdot 8 + 6 \cdot 6 &= 60 \\ 2 \cdot 8 - 6 &= 10 \end{aligned}$$

Example 5.5 We now solve the system

$$\begin{array}{rcrcrcl} & & x_2 & + & 2x_3 & = & 3 \\ 2x_1 & + & 3x_2 & & & = & 1 \\ x_1 & + & x_2 & - & x_3 & = & 1. \end{array}$$

The augmented matrix is

$$(A : \mathbf{b}) = \begin{pmatrix} 0 & 1 & 2 & : & 3 \\ 2 & 3 & 0 & : & 1 \\ 1 & 1 & -1 & : & 1 \end{pmatrix}$$

Like the previous example we number the rows of the augmented matrix R_1, R_2 and R_3. We first interchange the first row (R_1) with the third row (R_3) to bring a one into the upper left hand corner of the augmented matrix, that is $R_1 \leftrightarrow R_3$:

$$\begin{pmatrix} 1 & 1 & -1 & : & 1 \\ 2 & 3 & 0 & : & 1 \\ 0 & 1 & 2 & : & 3 \end{pmatrix}$$

We next subtract twice the first row from the second row so that all entries in the first column, below the first entry, become equal to zero, that is $R_2 := R_2 - 2R_1$:

$$\begin{pmatrix} 1 & 1 & -1 & : & 1 \\ 0 & 1 & 2 & : & -1 \\ 0 & 1 & 2 & : & 3 \end{pmatrix}$$

Finally, we subtract the second row from the third row so that the entry in the second column, below the second entry, becomes equal to zero. $R_3 := R_3 - R_2$:

$$\begin{pmatrix} 1 & 1 & -1 & : & 1 \\ 0 & 1 & 2 & : & -1 \\ 0 & 0 & 0 & : & 4 \end{pmatrix}.$$

The system of equations corresponding to the final augmented matrix is

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & 1 \\ 0x_1 & + & x_2 & + & 2x_3 & = & -1 \\ 0x_1 & + & 0x_2 & + & 0x_3 & = & 4. \end{array}$$

Now $R_3 \Rightarrow 0x_1 + 0x_2 + 0x_3 = 4$. Since we cannot have $0x_1 + 0x_2 + 0x_3 = 4$, there are no solutions to this system of equations.

Example 5.6 We solve a system with the same matrix of coefficients as in Example 5.5, but a different column **b**.

$$\begin{array}{rcrcrcr} & & x_2 & + & 2x_3 & = & 3 \\ 2x_1 & + & 3x_2 & & & = & 1 \\ x_1 & + & x_2 & - & x_3 & = & -1 \end{array}$$

The augmented matrix is

$$(A : \mathbf{b}) = \begin{pmatrix} 0 & 1 & 2 & : & 3 \\ 2 & 3 & 0 & : & 1 \\ 1 & 1 & -1 & : & -1 \end{pmatrix}.$$

Proceeding as in Example 5.5:

$R_1 \leftrightarrow R_3$

$$\begin{pmatrix} 1 & 1 & -1 & : & -1 \\ 2 & 3 & 0 & : & 1 \\ 0 & 1 & 2 & : & 3 \end{pmatrix}$$

$R_2 : = R_2 - 2R_1$

$$\begin{pmatrix} 1 & 1 & -1 & : & -1 \\ 0 & 1 & 2 & : & 3 \\ 0 & 1 & 2 & : & 3 \end{pmatrix}$$

$R_3 := R_3 - R_2$

$$\begin{pmatrix} 1 & 1 & -1 & : & -1 \\ 0 & 1 & 2 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{pmatrix}$$

The system of equations corresponding to the final augmented matrix is

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & -1 \\ & & x_2 & + & 2x_3 & = & 3 \\ & & & & 0 & = & 0. \end{array}$$

Note that the third equation holds trivially for all values of x_1, x_2, x_3, so we have found that the original system is equivalent to a system of two linear equations in three unknowns. When there are less equations than unknowns, the system has either infinitely many solutions or no solution at all. This system has infinitely many solutions.

We can write down the general solution by working our way *backwards* through the system of equations corresponding to the final augmented matrix:

$$R_2 \Rightarrow x_2 + 2x_3 = 3.$$

So

$$x_2 = 3 - 2x_3.$$

We let $x_3 = r$ where r is any real number. Then $x_2 = 3 - 2r$. Now

$$R_1 \Rightarrow x_1 + x_2 - x_3 = -1.$$

So

$$x_1 = -1 - x_2 + x_3 = -1 - (3 - 2r) + r = -4 + 3r.$$

Thus the general solution is $x_1 = -4 + 3r$, $x_2 = 3 - 2r$ and $x_3 = r$, where $r \in \mathbb{R}$. We have infinitely many solutions as we get a different solution for x_1, x_2, x_3 for each real number r.

5.1.1 Gaussian elimination

The algorithm we used to solve the systems of linear equations given in Examples 5.4, 5.5 and 5.6 is called **Gaussian elimination**. It gives a general method for solving all such systems.

The Gaussian elimination algorithm
We start with the augmented matrix $(A : \mathbf{b})$ for the system. We then reduce $(A : \mathbf{b})$ to a 'simpler' augmented matrix $(A^* : \mathbf{b}^*)$ by using the following steps:

Step 1 Choose the leftmost column of $(A : \mathbf{b})$ which contains a non-zero entry. Call it the **pivot column**.

Step 2 Interchange the first row with another row, if necessary, so that the top entry in the pivot column is non-zero. Call this entry the **pivot**.

Step 3 Multiply the first row by a suitable real number so that the pivot becomes equal to 1.

Step 4 Subtract multiples of the first row from the other rows so that all entries in the pivot column, below the pivot entry, become equal to zero.

Step 5 Now disregard the first row and return to step 1 with the resulting smaller matrix.

Step 6 Stop either when there are no more rows or all columns contain only zeros.

The output of this algorithm will be an augmented matrix with a very simple type of structure. To describe this structure precisely we need one further definition.

Definition 5.5 *Let M be an $m \times n$ matrix. Then an entry of M is said to be a **leading entry** if it is the first non-zero entry in some row.*

Example 5.7 The following matrix M has three leading entries, which are given in bold face.

$$M = \begin{pmatrix} \mathbf{1} & 0 & 2 & 3 \\ 0 & 0 & \mathbf{3} & 5 \\ 0 & \mathbf{4} & 2 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Row 4 has no leading entry as it consists entirely of entries zero.

The output from Gaussian elimination is an augmented matrix with the four properties listed below:

P1 All rows which consist entirely of zeros occur at the bottom.

P2 All leading entries are equal to 1.

P3 If a column contains a leading entry then all entries in that column below the leading entry are zero.

P4 In any two consecutive non-zero rows, the leading entry in the upper row occurs to the left of the leading entry in the lower row.

Definition 5.6 *A matrix which satisfies properties* **P1** to **P4** is said to be in **row echelon form**.

Given an augmented matrix in row echelon form $(A^* : \mathbf{b}^*)$, we can easily write down the solution to the corresponding system of equations. There are three possible cases:

Case 1: *Some leading entry occurs in the final column* $\mathbf{b}^*$.
In this case we deduce that there is no solution. For example:

$$\begin{pmatrix} 1 & 2 & 1 & : & 4 \\ 0 & 0 & 0 & : & 1 \\ 0 & 0 & 0 & : & 0 \end{pmatrix}.$$

$R_2 \Rightarrow 0x_1 + 0x_2 + 0x_3 = 1 \Rightarrow$ No solution.

Case 2: *The final column does not contain a leading entry but all other columns do contain a leading entry.*
In this case the system has a unique solution. We obtain this solution by the process of **back substitution**: we use the final non-zero row to determine the unknown, x_n, corresponding to the last column of A^*, then use the second to last row and substitute for x_n to determine x_{n-1} and so on. For example:

$$\begin{pmatrix} 1 & 2 & 1 & : & -1 \\ 0 & 1 & -1 & : & 2 \\ 0 & 0 & 1 & : & 5 \end{pmatrix}$$

$R_3 \Rightarrow x_3 = 5.$
$R_2 \Rightarrow x_2 = 2 + x_3 = 2 + 5 = 7.$
$R_1 \Rightarrow x_1 = -1 - 2x_2 - x_3 = -1 - 14 - 5 = -20.$
So the unique solution is $x_1 = -20, x_2 = 7, x_3 = 5$.

Case 3: *The final column and some other columns do not contain leading entries.*
In this case the system has infinitely many solutions. We set the unknowns corresponding to the columns of A^* which do not contain leading entries equal to arbitrary real numbers, then use back substitution to solve for the remaining unknowns. For example:

$$\begin{pmatrix} 1 & 2 & 3 & -1 & : & 1 \\ 0 & 0 & 0 & 1 & : & 3 \\ 0 & 0 & 0 & 0 & : & 0 \end{pmatrix}$$

$R_2 \Rightarrow x_4 = 3.$
We can choose x_3 and x_2 freely because columns 3 and 2 of the augmented matrix in row echelon form contain no leading entries, so set $x_3 = r$ where $r \in \mathbb{R}$ and $x_2 = s$ where $s \in \mathbb{R}$.

$R_1 \Rightarrow x_1 = 1 - 2x_2 - 3x_3 + x_4 = 1 - 2s - 3r + 3 = 4 - 2s - 3r.$
So the general solution is $x_1 = 4 - 2s - 3r, x_2 = s, x_3 = r, x_4 = 3$, for $r, s \in \mathbb{R}$.

5.2 Matrix algebra

Learning objectives for this section

- Writing a system of linear equations as a matrix equation
- The rules for when two matrices can be added and subtracted
- The rules for when two matrices can be multiplied
- Adding, subtracting and multiplying matrices of suitable sizes
- The identity matrices I_n and their properties
- Interpreting powers of the adjacency matrix of a graph/digraph

Definition 5.7 *Let A and B be two $m \times n$ matrices. Then $A = B$ if and only if each entry of A is equal to the corresponding entry of B.*

More formally if $A = (a_{i,j})$ and $B = (b_{i,j})$, then $A = B$ if $a_{i,j} = b_{i,j}$ for all $1 \le i \le m$ and $1 \le j \le n$.

5.2.1 Addition and subtraction of matrices

Definition 5.8 *Let A and B be two $m \times n$ matrices. Then $A + B$ is the $m \times n$ matrix obtained by adding each entry of A to the corresponding entry of B.*

More formally if $A = (a_{i,j})$ and $B = (b_{i,j})$, then $A + B$ is the $m \times n$ matrix given by $A + B = (c_{i,j})$ where $c_{i,j} = a_{i,j} + b_{i,j}$ for all $1 \le i \le m$ and $1 \le j \le n$.

Note that in order to add together two matrices they must have the same size.

Example 5.8 Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 & 4 \\ 2 & 1 & 3 \end{pmatrix}$. Then

$$A + B = \begin{pmatrix} \mathbf{1} & 2 & -1 \\ 3 & \boxed{4} & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{0} & -1 & 4 \\ 2 & \boxed{1} & 3 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 1 & 3 \\ 5 & \boxed{5} & 3 \end{pmatrix}.$$

Subtraction of matrices is defined in a similar manner and can also be performed only when the two matrices are of the same size.

Example 5.9 Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 & 4 \\ 2 & 1 & 3 \end{pmatrix}$. Then

$$A - B = \begin{pmatrix} \mathbf{1} & 2 & -1 \\ 3 & \boxed{4} & 0 \end{pmatrix} - \begin{pmatrix} \mathbf{0} & -1 & 4 \\ 2 & \boxed{1} & 3 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 3 & -5 \\ 1 & \boxed{3} & -3 \end{pmatrix}.$$

5.2.2 Multiplication of a matrix by a constant

Definition 5.9 *Let A be an $m \times n$ matrix and $r \in \mathbb{R}$. Then rA is the $m \times n$ matrix obtained by multiplying each entry in A by r. Equivalently, if $A = (a_{i,j})$ then $rA = (c_{i,j})$ where $c_{i,j} = ra_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.*

Example 5.10 Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \end{pmatrix}$ and $r = -2$. Then

$$-2A = \begin{pmatrix} -2 & -4 & 2 \\ -6 & -8 & 0 \end{pmatrix}.$$

5.2.3 Multiplication of two matrices

We next define a rule for multiplying two matrices together. To motivate this definition, we consider the system of equations

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 - x_2 &= 2. \end{aligned}$$

We can write this system of equations as a matrix equation involving two 2×1 matrices:

$$\begin{pmatrix} x_1 + 2x_2 \\ 3x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We would like to rewrite this matrix equation in the form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

To do this we need to define matrix multiplication in such a way that

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{b} = A\mathbf{x} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 - x_2 \end{pmatrix}.$$

Definition 5.10 *Let $A = (a_{i,j})$ be an $m \times n$ matrix and $B = (b_{i,j})$ be an $n \times p$ matrix. Then the matrix product $AB = (x_{i,j})$ is the $m \times p$ matrix defined by*

$$x_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,n}b_{n,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq p$.

Note that in order to construct the product AB we need the number of columns of A to be equal to the number of rows of B.

Example 5.11 A matrix multiplication

Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 1 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}$.

The product AB is defined as A has 2 columns and B has 2 rows. The resulting matrix will be a 3×2-matrix as A has 3 rows and B

has 2 columns:

$$AB = \begin{pmatrix} (1 \times 5) + (2 \times 2) & (1 \times 1) + (2 \times 3) \\ (4 \times 5) + (1 \times 2) & (4 \times 1) + (1 \times 3) \\ (0 \times 5) + (2 \times 2) & (0 \times 1) + (2 \times 3) \end{pmatrix} = \begin{pmatrix} 9 & 7 \\ 22 & 7 \\ 4 & 6 \end{pmatrix}.$$

The product BA is not defined in this example as B has 2 columns while A has 3 rows.

To obtain the entry in the ith row and jth column of AB we multiply together the corresponding terms in the ith row of A and the jth column of B, and then add together the resulting products.

5.2.4 Powers of square matrices

A matrix is said to be **square** if it has the same number of rows as it has columns. Square matrices are special in that they are the only matrices which can be multiplied by themselves.

Example 5.12 A square matrix multiplied by itself

Consider the 2×2-matrix

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}.$$

Then A can be multiplied by itself:

$$AA = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -5 & 9 \end{pmatrix}.$$

Note that the size of AA is the same as the size of A. In fact, if A and B are two square matrices of the same size, the size of AB is the same as the size of A (and the size of B). We can thus define powers of square matrices:

Definition 5.11 *Let n and k be two positive integers and suppose that A is a square $n \times n$-matrix. We define*

$$A^1 := A,$$

and for $k \geq 2$ we define

$$A^k := AA^{k-1}.$$

Example 5.13 The powers of a square matrix

Consider the 2×2-matrix

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$$

from Example 5.12 above. We have

$$A^2 = AA = \begin{pmatrix} 4 & 0 \\ -5 & 9 \end{pmatrix}.$$

We can thus compute

$$A^3 = AA^2 = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ -5 & 9 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ -19 & 27 \end{pmatrix}.$$

Further,

$$A^4 = AA^3 = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 8 & 0 \\ -19 & 27 \end{pmatrix} = \begin{pmatrix} 16 & 0 \\ -65 & 81 \end{pmatrix}.$$

In this way we can compute all positive powers of the matrix A.

Walks in graphs and powers of adjacency matrices

Definition 5.12 *A **walk** in a graph G is an alternating sequence of vertices and edges of the form*

$$v_1 e_1 v_2 e_2 v_3 \ldots e_{k-1} v_k,$$

where e_i is an edge joining v_i to v_{i+1}. The vertices and edges in the walk are not necessarily distinct.
Note that when the graph is simple, we may list the walk without the edges,

$$v_1 v_2 v_3 \ldots v_k,$$

as there is a maximum of one edge between any pair of vertices in a simple graph.

If you compare this definition to the definition of a path in Section 5.2.1 of Volume 1, you will see that a **path** in a graph is a walk in which all vertices (and all edges) are distinct.

Definition 5.13 *A **directed walk** in a digraph D is an alternating sequence of vertices and arcs of the form*

$$v_1 e_1 v_2 e_2 v_3 \ldots e_{k-1} v_k,$$

where e_i is an arc from v_i to v_{i+1}. The vertices and arcs in the directed walk are not necessarily distinct.

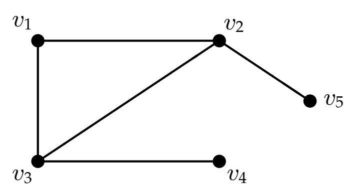
A **directed path** in a digraph is a directed walk in which all vertices (and all arcs) are distinct.

Definition 5.14 *The **length of a walk** in a graph is the number of edges in it. The **length of a directed walk** in a digraph is the number of arcs in it.*

Suppose you want to find the number of walks of length 1 between two vertices v_i and v_j in a simple (di)graph G with vertices $v_1, v_2, \ldots, v_n$. Such a walk is of the form $v_i e v_j$ where e is an edge (arc) from v_i to v_j. The number of such walks is thus the number of choices of e. If $\mathbb{A} = (a_{i,j})$ denotes the adjacency matrix of G with respect to the natural ordering of vertices, then the entry $a_{i,j}$ is the number of edges (arcs) from v_i to v_j, that is $a_{i,j}$ is the number of walks of length 1 from v_i to v_j.

Example 5.14 The number of walks of length 2 in a graph

Consider the following graph G.



The graph G has adjacency matrix

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & \mathbf{0} & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the natural ordering of vertices: v_1, v_2, v_3, v_4, v_5.

The entry $\mathbb{A}(4,3) = 1$, so there is precisely one walk of length 1 from v_4 to v_3 as can be easily seen from the picture of the graph. The entry $\mathbb{A}(2,4) = 0$, so there is no walk of length 1 from v_2 to v_4 which can also be seen from the picture.

The adjacency matrix of a graph is a square matrix, so we can compute powers of it. Let us compute $\mathbb{A}^2$ for our graph G:

$$\mathbb{A}^2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & \mathbf{0} & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Looking at the picture of G, there is precisely 1 walk of length 2 between v_1 and v_2, namely the path $v_1v_3v_2$; we note that entry $\mathbb{A}^2(1,2) = 1$.

There is no walk of length 2 between v_4 and v_5; we note that entry $\mathbb{A}^2(4,5) = 0$.

There are three walks of length 2 between v_2 and v_2, namely $v_2v_1v_2$, $v_2v_3v_2$ and $v_2v_5v_2$; we note that entry $\mathbb{A}^2(2,2) = 3$.

We leave it for you to check that $\mathbb{A}^2(i,j)$ is the number of walks of length 2 between v_i and v_j for all other values of i and j also.

To understand why this is so, note that any walk of length 2 from v_i to v_j is made up of a walk of length 1 from v_i to some intermediate vertex v_m followed by a walk of length 1 from v_m to v_j. By the Multiplication Principle the number of such walks via v_m is the number of walks of length 1 from v_i to v_m, namely $\mathbb{A}(i,m)$, times the number of walks of length 1 from v_m to v_j, namely $\mathbb{A}(m,j)$. Using the Addition principle, we get all possible walks of length 2 from v_i to v_j by just adding the result for every possible intermediate vertex v_m. Hence the number of walks of length 2 from v_i to v_j is

$$\mathbb{A}(i,1)\mathbb{A}(1,j) + \mathbb{A}(i,2)\mathbb{A}(2,j) + \ldots + \mathbb{A}(i,5)\mathbb{A}(5,j),$$

and comparing this sum with Definition 5.10, this is exactly the entry in the ith row and jth column of $\mathbb{A}^2$, that is $\mathbb{A}^2(i,j)$.

It is possible to generalise the result of Example 5.14 to walks of any length by using induction. You can find this proof in Section 11.3 in Epp. You will not be asked to prove the result in the examination, but you must know how to compute the number of walks between any pair of vertices in a simple graph:

Theorem 5.1 *Let G be a simple graph with vertices $v_1, v_2, \ldots, v_n$, and let $\mathbb{A}$ be the adjacency matrix of G with respect to the natural ordering of the vertices, then for all positive integers k, the number of **walks** of length k in G from v_i to v_j is $\mathbb{A}^k(i,j)$.*

If a graph is simple, it has no loops, hence a walk of length 2 between two *distinct* vertices will be a path of length 2. Thus we get the following result:

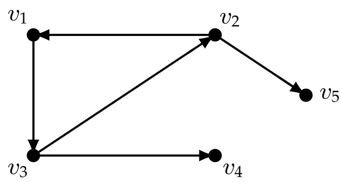
Corollary 5.2 *Let G be a simple graph with vertices $v_1, v_2, \ldots, v_n$, and let $\mathbb{A}$ be the adjacency matrix of G with respect to the natural ordering of the vertices, then the number of **paths** of length 2 in G from v_i to v_j where $i \neq j$ is $\mathbb{A}^2(i,j)$.*

The result for digraphs is similar:

Theorem 5.3 *Let D be a digraph with vertices $v_1, v_2, \ldots, v_n$, and let $\mathbb{A}$ be the adjacency matrix of G with respect to the natural ordering of the vertices, then for all positive integers k, the number of **directed walks** of length k in D from v_i to v_j is $\mathbb{A}^k(i,j)$.*

Example 5.15 The number of walks in a digraph

Consider the following digraph D.



The digraph D has adjacency matrix

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the natural ordering of vertices: v_1, v_2, v_3, v_4, v_5.

Let us compute $\mathbb{A}^2$ for our digraph D:

$$\mathbb{A}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence there is just one directed walk of length 2 from v_1 to v_4 for example, while there is no directed walk of length 2 from v_3 to v_2.

Let us also compute $\mathbb{A}^3$ for our graph D:

$$\begin{aligned} \mathbb{A}^3 = \mathbb{A}\mathbb{A}^2 \quad &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence there is just one directed walk of length 3 from v_2 to v_4 for example, while there is no directed walk of length 3 from v_2 to v_3.

5.2.5 Rules of arithmetic for matrices

Most of the rules of arithmetic you are used to from your experience with numbers hold also for matrices:

Rule 5.4 *If A,B and C are matrices of the same size then*

$$(A + B) + C = A + (B + C).$$

Rule 5.5 *If A,B and C are matrices such that the products AB and BC both exist (that is, the number of columns of A is the same as the number of rows of B and the number of columns of B is the same as the number of rows of C), then*

$$(AB)C = A(BC).$$

Rule 5.6 *Suppose that A and B are matrices of the same size,*

(a) If C is a matrix such that CA and CB exist (that is, the number of columns of C is the same as the number of rows of A and B), then

$$C(A + B) = CA + CB.$$

(b) If D is a matrix such that AD and BD exist (that is, the number of rows of D is the same as the number of columns of A and B), then

$$(A + B)D = AD + BD.$$

Rule 5.7 *If A and B are matrices of the same size then*

$$A + B = B + A.$$

Proofs of these rules are straightforward from the definitions of the various arithmetic operations on matrices. You will not be required to prove these rules in the examination, but knowing them can ease computations with matrices, and you are therefore advised to learn and use them.

It is important to realise that Rule 5.7 does *not* have a multiplicative equivalent. Unlike multiplication of numbers, matrix multiplication need not always be *commutative*, i.e. the two products AB and BA may yield different matrices. You have already seen in Example 5.11 that the product BA is not necessarily defined even though AB can be computed. Worse still, even when AB and BA are both defined, they are not necessarily equal.

Example 5.16 Matrix multiplication is not commutative
If A and B are $n \times n$ matrices, then we can calculate both the product AB and the product BA. However, the two products AB and BA more often than not yield different matrices. To see an example of this let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

so clearly $AB \neq BA$ in this case.

5.2.6 Identity matrices

Definition 5.15 *The $n \times n$ **identity matrix** I_n is the $n \times n$ matrix with 1's along its diagonal and 0's elsewhere. Thus $I_n = (\delta_{i,j})$ where*

$$\delta_{i,j} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$$

So, for example,

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Lemma 5.8 *Let A be an $m \times n$ matrix. Then*

$$AI_n = A = I_m A.$$

Proof. We show that $AI_n = A$. To do this we have to show that AI_n has the same size as A and that the corresponding entries in AI_n and A are equal. Since A is $m \times n$ and I_n is $n \times n$ we have AI_n is $m \times n$. Hence AI_n has the same size as A. Let $A = (a_{i,j})$, $I_n = (\delta_{i,j})$, $AI_n = (x_{i,j})$. Then

$$x_{i,j} = \sum_{k=1}^{n} a_{i,k}\delta_{k,j} = a_{i,j}\delta_{j,j} = a_{i,j}$$

since $\delta_{k,j} = 0$ if $k \neq j$ and $\delta_{j,j} = 1$. Thus AI_n and A have the same entries. Hence $AI_n = A$.■

Example 5.17 Properties of identity matrices We illustrate Lemma 5.8 by the 2×3 matrix

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Recall the two identity matrices

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then

$$\begin{aligned} I_2 A &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1\cdot 3 + 0\cdot 1 & 1\cdot 2 + 0\cdot 0 & 1\cdot 0 + 0\cdot 1 \\ 0\cdot 3 + 1\cdot 1 & 0\cdot 2 + 1\cdot 0 & 0\cdot 0 + 1\cdot 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} = A. \end{aligned}$$

Also

$$
\begin{aligned}
AI_3 &= \begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3\cdot 1 + 2\cdot 0 + 0\cdot 0 & 3\cdot 0 + 2\cdot 1 + 0\cdot 0 & 3\cdot 0 + 2\cdot 0 + 0\cdot 1 \\ 1\cdot 1 + 0\cdot 0 + 1\cdot 0 & 1\cdot 0 + 0\cdot 1 + 1\cdot 0 & 1\cdot 0 + 0\cdot 0 + 1\cdot 1 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} = A.
\end{aligned}
$$

5.3 Exercises on Chapter 5

5.3.1 True/False questions

In each of the following questions, decide whether the given statements are true or false.

1. The equation $x_1 + 2x_2 - \pi x_3 = 0$ is linear.

Let

$$
A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{pmatrix}; \qquad B = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 5 \end{pmatrix}.
$$

2. $a_{1,3} = 0$.
3. A has 2 leading entries.
4. A is in row echelon form.
5. B is in row echelon form.
6. $BC = CB$.
7. $A + BC$ can be computed.
8. AC can be computed.
9. CA can be computed.
10. $A^2 = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 9 \end{pmatrix}$.

5.3.2 Longer exercises

Question 1
Use Gaussian elimination to find all solutions to the following systems of equations.

(a)

$$
\begin{aligned}
2x_1 + x_2 &= 2 \\
x_1 - x_2 &= 1 \\
x_1 + 2x_2 &= 1
\end{aligned}
$$

(b)

$$\begin{aligned} 2x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 + 2x_2 &= 0 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 - x_2 + x_3 &= 1 \\ x_1 - 2x_2 \phantom{{}+x_3} &= -1 \end{aligned}$$

Question 2
Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Express the matrix equation $A\mathbf{x} = \mathbf{b}$ as a system of linear equations.

Question 3
Consider the following system of equations:

$$\begin{array}{rcrcrcl} & & x_2 & + & x_3 & = & 2 \\ x_1 & + & x_2 & + & 2x_3 & = & 5 \\ x_1 & - & 2x_2 & + & x_3 & = & 5 \end{array}$$

(a) Write the system as a matrix equation $A\mathbf{x} = \mathbf{b}$.

(b) Write down the augmented matrix of the system.

Question 4
Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Calculate $B + 2C$ and AB.
Find a matrix Y such that $A + Y = A$.

Question 5
Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

Find matrices C, D and E such that

(a) $B + C = A$:

(b) $BD = B$;

(c) $B - E = 2A$.

Appendix A

Additional references

There are a number of books available on Discrete Mathematics. This list contains a selection of those known to the authors. Other editions (earlier or later) can be used instead of the editions listed below.

The books in this list are additional to the main textbooks for the course (by Epp and by Molluzzo and Buckley), for which details have been given in the Preface.

Additional references

Albertson, M.O. and Hutchinson, J.P., *Discrete Mathematics with Algorithms*, Wiley, 1998, [ISBN 0-471-61278-2]. Good background book, showing links with computer science.

Barnet, S., *Discrete Mathematics: Numbers and Beyond*, Addison-Wesley, 1988, [ISBN 0-201-34292-8]. Does not cover all the syllabus, but useful for some sections, particularly number bases, counting methods, applications of graphs.

Eccles, Peter J., *An Introduction to Mathematical Reasoning: Numbers, sets and functions*, Cambridge U.P., 1997, [ISBN 0-521-59718-8]. A good introduction to the basics of the subject.

Garnier, R. and Taylor, J., *Discrete Mathematics for new Technology*, The Institute of Physics, 1997, [ISBN 0-750-30135-X]. Good coverage and simply explained.

Goodaire, Edgar C. and Parmenter M.M., *Discrete Mathematics with Graph Theory*, Prentice Hall, 1998, [ISBN 0-13-602798-8].

Grimaldi, R.P., *Discrete and Combinatorial Mathematics* 4^{th} Ed. (though earlier editions are just as useful), Addison-Wesley, 1999, [ISBN 0-201-30424-4]. A comprehensive coverage of the whole subject area, but more demanding than Epp.

Johnsonbaugh, Richard, *Discrete Mathematics*, Prentice Hall, 2000, [ISBN 0-13-089008-1].

Mattson Jr., Harold F., *Discrete Mathematics with Applications*, Wiley, 1993, [ISBN 0-471-59966-2(pbk);ISBN 0-471-60672-3(hbk)]. A more demanding treatment, but good for links with computer science.

Mizrahi and Sullivan, M., *Finite Mathematics, an applied approach* 8^{th} Ed., Wiley, 1998, [ISBN 0-471-32202-4]. Covers only part of the syllabus with an emphasis on business applications. Good for topics covered in Vol 2 of this subject guide. Earlier editions are just as useful.

Ross, Kenneth A. and Wright, Charles R.B., *Discrete Mathematics* 5^{th} Ed., Prentice Hall, 2002, [ISBN 0-13-178448-X].

Simpson, A., *Discrete Mathematics by Example*, McGraw-Hill Education, 2001, [ISBN 0-07-709840-4].

Appendix B

Solutions to Exercises

B.1 Exercises on Chapter 1

B.1.1 Solutions to True/False questions

(a) False.
In order to conclude that $\mathcal{R}$ is reflexive, we must check that $x\mathcal{R}x$ holds for all $x \in X$, not just $x = 1$.

(b) False.
The definition states that in order for $\mathcal{R}$ to be symmetric on X, we have that *if* $x\mathcal{R}y$ for $x, y \in X$ *then* $y\mathcal{R}x$ also. This does not imply that all x and y in X are related under $\mathcal{R}$.

(c) False.
The definition states that in order for $\mathcal{R}$ to be transitive on X, we have that *if* $x\mathcal{R}y$ and $y\mathcal{R}z$ for $x, y, z \in X$ *then* $x\mathcal{R}z$ also. This does not imply that all x and y in X are related under $\mathcal{R}$.

(d) False.
The definition states that in order for $\mathcal{R}$ to be transitive on X we must have *for all* $x, y, z \in X$ that if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$ also, so it is not sufficient to check that $1\mathcal{R}2$, $2\mathcal{R}3$ and $1\mathcal{R}3$.

(e) False.
because e.g. the relation "=" on the set $X = \{1, 2, 3\}$ is both symmetric and anti-symmetric.

(f) True.
If $1\mathcal{R}2$ then $2\mathcal{R}1$ also because $\mathcal{R}$ is symmetric.

(g) True.
If $1\mathcal{R}2$ then $2\mathcal{R}1$ also, because $\mathcal{R}$ is symmetric. Now, using that $1\mathcal{R}2$ and $2\mathcal{R}1$, it follows that $1\mathcal{R}1$ because $\mathcal{R}$ is transitive.

(h) False.
There are relations which are neither partial orders nor equivalence relations.

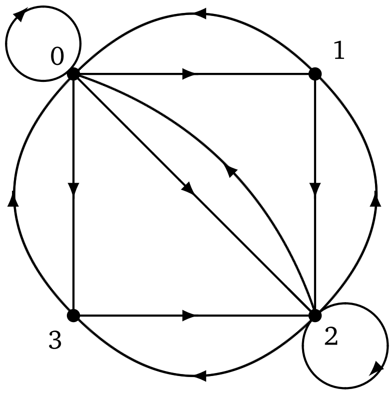
(i) True.
As $|X| = 4$ and $|Y| = 7$ we have $|X \times Y| = 4 \cdot 7 = 28$.

(j) False.
When $X = Y$ we have that $X \times Y = Y \times X$.

B.1.2 Solutions to Longer exercises

Question 1

(a) The relationship digraph is shown below.



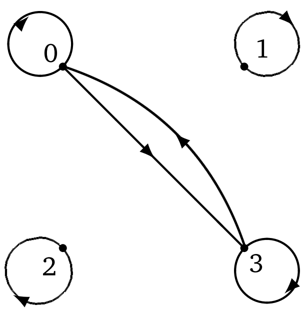
(b) The relation:
is not reflexive e.g. 1 is not related to 1;
is symmetric;
is not transitive e.g. 1 is related to 2, and 2 is related to 3, but 1 is not related to 3;
is not anti-symmetric e.g. 1 is related to 2, and 2 is related to 1, but $1 \neq 2$.

(c) $\mathcal{R}_1$ is not an equivalence relation because it is neither reflexive nor transitive.

(d) $\mathcal{R}_1$ is not a partial order or an order.

Question 2

(a) The relationship digraph is shown below.



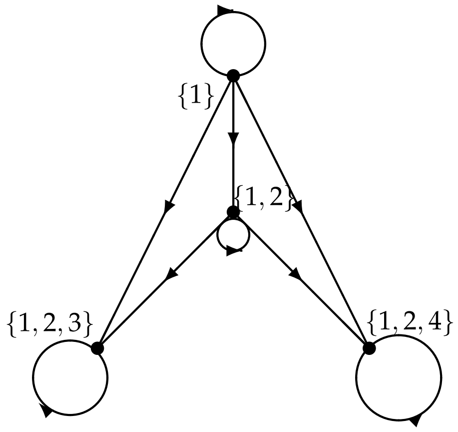
(b) The relation:
is reflexive, symmetric and transitive;
is not anti-symmetric e.g. 0 is related to 3, and 3 is related to 0, but $0 \neq 3$.

(c) $\mathcal{R}_2$ is an equivalence relation. The three equivalence classes are $\{0, 3\}$, $\{1\}$, and $\{2\}$. These form a partition of $\{0, 1, 2, 3\}$.

(d) $\mathcal{R}_2$ is not a partial order or an order as it is not anti-symmetric.

Question 3

(a) The relationship digraph is shown below.



(b) The relation:
is reflexive, transitive and anti-symmetric;
is not symmetric e.g. $\{1\}$ is related to $\{1,2\}$, but $\{1,2\}$ is not related to $\{1\}$.

(c) $\mathcal{R}_3$ is not an equivalence relation as it is not symmetric.

(d) $\mathcal{R}_3$ is a partial order but not an order (since $\{1,2,3\}$ and $\{1,2,4\}$ are not related to each other).

Question 4

(a) $x\mathcal{R}x$ for all $x \in S$.

(b) $x\mathcal{R}y$ implies $y\mathcal{R}x$ for all $x, y \in S$.

(c) $x\mathcal{R}y$ and $y\mathcal{R}z$ implies $x\mathcal{R}z$ for all $x, y, z \in S$.

(d) $x\mathcal{R}y$ and $y\mathcal{R}x$ implies $x = y$ for all $x, y \in S$.

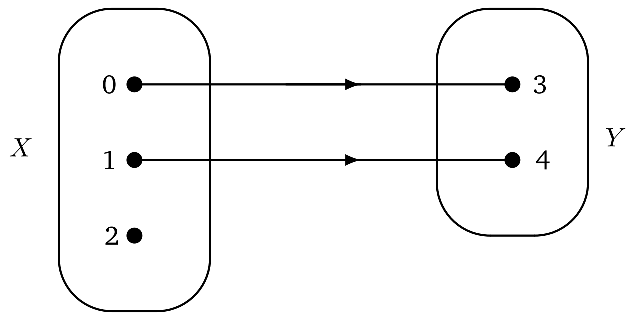
(e) $\mathcal{R}$ is reflexive, symmetric and transitive.

(f) $\mathcal{R}$ is reflexive, anti-symmetric and transitive.

(g) $\mathcal{R}$ is a partial order and, for all $x, y \in S$, either $x\mathcal{R}y$ or $y\mathcal{R}x$.

Question 5

(a) The relationship digraph is shown below.



(b) $X \times Y = \{(0,3), (0,4), (1,3), (1,4), (2,3), (2,4)\}$.
The subset of $X \times Y$ corresponding to $\mathcal{R}$ is $\{(0,3), (1,4)\}$.

(c) $\mathcal{R}$ is not a function from X to Y as $2 \in X$ and 2 has no corresponding image in Y.

Question 6

(a) $\mathcal{R}$ is clearly both reflexive and symmetric, for every number has the same (unique) remainder on division by 3 as itself and if a has the same remainder on division by 3 as does b, then b has the same remainder on division by 3 as a also. Similarly $\mathcal{R}$ is transitive, for if $a\mathcal{R}b$ and $b\mathcal{R}c$, then a, b and c all have the same remainder on division by 3, so $a\mathcal{R}c$ also.

(b) There are three equivalence classes of $\mathcal{R}$ on $X = \{0, 1, 2, 3, \ldots 9\}$, they are

$$\begin{aligned} [0] &= \{0, 3, 6, 9\} \\ [1] &= \{1, 4, 7\} \\ [2] &= \{2, 5, 8\} \end{aligned}$$

(c) The set of equivalence classes of $\mathcal{R}$ is a partition of X, for the three classes are clearly disjoint, and

$$X = [0] \cup [1] \cup [2].$$

B.2 Exercises on Chapter 2

B.2.1 Solutions to True/False questions

(a) False, $\displaystyle\sum_{i=1}^{n} r = nr$.

(b) True, prove this by induction (cf. Theorem 2.1).

(c) True, this sum is the sum of s 1s.

(d) False, $\displaystyle\sum_{i=3}^{s} 1 = s - 2$.

(e) False, $\displaystyle\sum_{r=1}^{n}(2r - 3) = 2\sum_{r=1}^{n} r - \sum_{r=1}^{n} 3 = n(n+1) - 3n = n^2 - 2n$.

(f) False, $\displaystyle\sum_{i=-2}^{3} 2^{-i} = 4 + 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}$.

(g) False, as $2 \times 2^r \neq 4^r$.

(h) False, as no initial term is given, the recurrence relation does not allow you to compute the terms of the sequence.

(i) True, $u_8 = u_7 + 8 = u_6 + 15 = u_5 + 21 = u_4 + 26 = u_3 + 30 = u_2 + 33 = u_1 + 35 = u_0 + 36 = 36 = 36$.

(j) False, $u_8 = u_7 + 7 = u_6 + 13 = u_5 + 18 = u_4 + 22 = u_3 + 25 = u_2 + 27 = u_1 + 28 = u_0 + 28 = 28$.

B.2.2 Solutions to Longer exercises

Question 1

(a) (i) $\frac{1}{8}$ (ii)$u_{n+1} = \frac{1}{2}u_n$

(b) (i) 27 (ii)$u_{n+1} = u_n + 5$

Question 2

(a) $u_1 = 0$, $u_2 = 2$, $u_3 = 12$, $u_4 = 62$.

(b) $u_1 = 0$, $u_2 = 1$, $u_3 = 1$, $u_4 = 0$.

Question 3
Proof: (By induction)
Base Case: Set $n = 1$: $u_1 = 2 = 3^1 - 1$, so the formula is true when $n = 1$.
Induction Hypothesis: Assume that $u_n = 3^n - 1$ is true for all integers $n = 1, 2, \ldots, k$. (Note that in particular we thus assume $u_k = 3^k - 1$.)
Induction Step: We must now prove that $u_{k+1} = 3^{k+1} - 1$.

$$\begin{aligned} u_{k+1} &= 3u_k + 2 \text{ (by the recurrence relation)} \\ &= 3\left[3^k - 1\right] + 2 \text{ (by the induction hypothesis)} \\ &= \left[3 \times 3^k - 3\right] + 2 \\ &= 3^{k+1} - 1 \end{aligned}$$

Hence the formula also holds when $n = k + 1$, and so, by the principle of induction, $u_n = 3^n - 1$ for all $n \geq 1$.

Question 4

(a) $\sum\limits_{r=1}^{n} 1 = n$

(b) $\sum\limits_{r=1}^{n} r = \frac{n(n+1)}{2}$

(c) $\sum\limits_{r=0}^{n} x^r = \frac{x^{n+1}-1}{x-1}$, when $x \neq 1$,
$\sum\limits_{r=0}^{n} x^r = \sum\limits_{r=0}^{n} 1 = (n + 1) \times 1 = n + 1$, when $x = 1$.

Question 5

(a) $\sum\limits_{i=11}^{30} i = \sum\limits_{i=1}^{30} i - \sum\limits_{i=1}^{10} i = \frac{30\times31}{2} - \frac{10\times11}{2} = 410$

(b) $\sum\limits_{i=1}^{20}(2^i + 1) = \sum\limits_{i=1}^{20} 2^i + \sum\limits_{i=1}^{20} 1 = \frac{2^{21}-1}{2-1} + 20 = 2^{21} + 19$

(c) $\sum\limits_{i=1}^{10}(4i + 3) = \sum\limits_{i=1}^{10}(4i) + \sum\limits_{i=1}^{10} 3 = 4\sum\limits_{i=1}^{10} i + 3\sum\limits_{i=1}^{10} 1 = 4 \times \frac{10\times11}{2} + 3 \times 10$
$= 4 \times 55 + 30 = 250.$

Question 6

(a) $s_n = \sum\limits_{i=1}^{n}(2i - 1)$.

(b) $s_1 = 1$, $s_2 = 4$, $s_3 = 9$,

(c) $s_{n+1} = s_n + (2n + 1)$

(d) **Proof:** (By induction)
Base Case: Set $n = 1$: $s_1 = 1 = 1^2$, so the formula is true when $n = 1$.
Induction Hypothesis: Assume that $s_n = n^2$ is true for all integers $n = 1, 2, \ldots, k$. In particular we assume $s_k = k^2$
Induction Step: (We prove that $s_{k+1} = (k+1)^2$.

$$\begin{aligned} s_{k+1} &= s_k + (2k+1) \text{ (by the recurrence relation)} \\ &= k^2 + 2k + 1 \text{ (by the induction hypothesis)} \\ &= (k+1)^2 \end{aligned}$$

Hence the formula also holds when $n = k + 1$, and so, by the Principle of Induction, $s_n = n^2$ for all $n \geq 1$.

Question 7

Let $S_n = \sum_{r=0}^{n} x^r$ where $x \in \mathbb{R}$ and $x \neq 1$. We prove by induction that

$$S_n = \frac{x^{n+1} - 1}{x - 1} \text{ for all } n \geq 1. \qquad \text{(B.1)}$$

Base case: For $n = 1$,

$$\begin{aligned} \text{LHS of (B.1)} &= S_1 = \sum_{r=0}^{1} x^r = x^0 + x^1 = 1 + x, \\ \text{RHS of (B.1)} &= \tfrac{x^{1+1}-1}{x-1} = \tfrac{x^2-1}{x-1} = \tfrac{(x-1)(x+1)}{x-1} = x + 1. \end{aligned}$$

Hence the result holds for $n = 1$.

Induction hypothesis: Suppose that $S_n = \frac{x^{n+1}-1}{x-1}$, for $n = 1, 2, \ldots, k$; then in particular we know that $S_k = \frac{x^{k+1}-1}{x-1}$.

Induction Step:

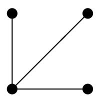
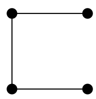
$$\begin{aligned} S_{k+1} &= \sum_{r=0}^{k+1} x^r \\ &= \sum_{r=0}^{k} x^r + x^{k+1} \\ &= S_k + x^{k+1} \\ &= \frac{x^{k+1} - 1}{x - 1} + x^{k+1} \text{ (by the induction hypothesis)} \\ &= \frac{x^{k+1} - 1}{x - 1} + \frac{(x-1)x^{k+1}}{x - 1} \\ &= \frac{x^{k+1} - 1}{x - 1} + \frac{(x^{k+2} - x^{k+1})}{x - 1} \\ &= \frac{(x^{k+1} - 1) + (x^{k+2} - x^{k+1})}{x - 1} \\ &= \frac{x^{k+2} - 1}{x - 1}. \end{aligned}$$

Thus the formula also holds for $n = k + 1$ and hence it holds for all $n \geq 1$, by induction.

B.3 Exercises on Chapter 3

B.3.1 Solutions to True/False questions

1. True, a tree is a connected graph with no cycles, hence it has no loops (1-cycles).
2. False, in order to be a tree it is not enough to have no loops, it must have no *cycles*.
3. True, this is the definition of a tree.
4. True, a tree has no cycles, hence it has no loops and no multiple edges, and so it is simple.
5. False, a path graph of length n is a tree with $n + 1$ vertices (it has n edges).
6. True, cf. Theorem 3.3.
7. False, any spanning tree of the graph is a path graph of length 3, so they are all isomorphic.
8. True, two non-isomorphic spanning trees of this graph are e.g.


and


They are non-isomorphic because they have different degree sequences.

9. (i) False, the height is $\lceil \log_2(32 + 1) \rceil = 6$ as $2^5 < 33 \leq 2^6$.
   (ii) False, the root is $\lfloor \frac{1+32}{2} \rfloor = 16$.
   (iii) True, the root of the left subtree of node 16 is $\lfloor \frac{1+15}{2} \rfloor = 8$, and the root of the right subtree of node 16 is $\lfloor \frac{17+32}{2} \rfloor = 24$.
   (iv) False, 10 is in the left subtree of the root (record 16), while 20 is in the right subtree.
10. False, only levels $0, 1, 2, \ldots h - 1$ need be full, at level h you can have between 2 and 2^h nodes.

B.3.2 Solutions to Longer exercises

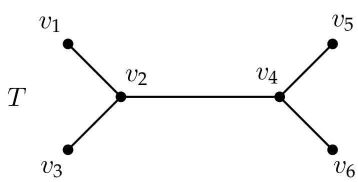
Question 1

(a) G must be *connected* and have *no cycles*.

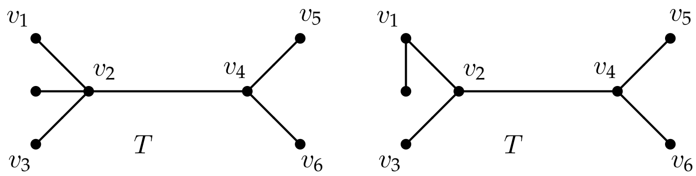
(b) G must be a tree: it is connected since every pair of vertices is joined by a path; it has no cycles since, if it did, then two vertices on a cycle would be joined by two different paths.

Question 2

(a) The tree is shown below.



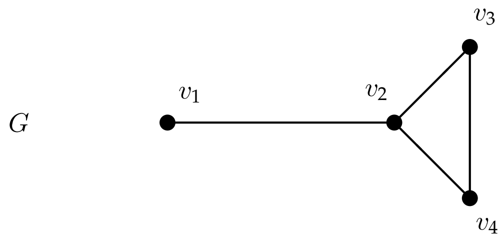
(b) The two non-isomorphic trees are shown below.



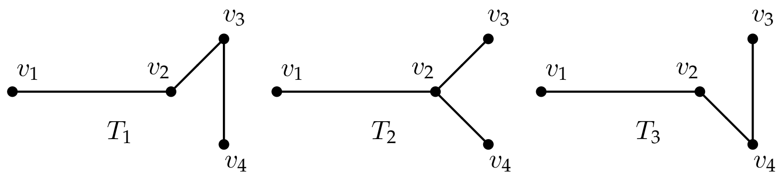
(c) To show T_1 and T_2 are not isomorphic to each other you can give any structural difference between them e.g. T_1 and T_2 have different degree sequences, or T_1 has a path of length four and T_2 does not.

Question 3

(a) The graph G is shown below.



(b) The three spanning trees of G are shown below.



(c) G has exactly *two* non-isomorphic spanning trees since T_1 and T_3 are isomorphic to each other.

Question 4

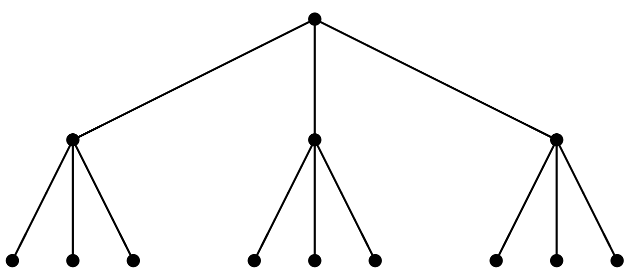
(a) The nodes on level i are the nodes which are joined to r by a path of length i.

(b) The height of T is the length of a longest path starting at node r.

(c) An external node of T is a node with no children.

Question 5

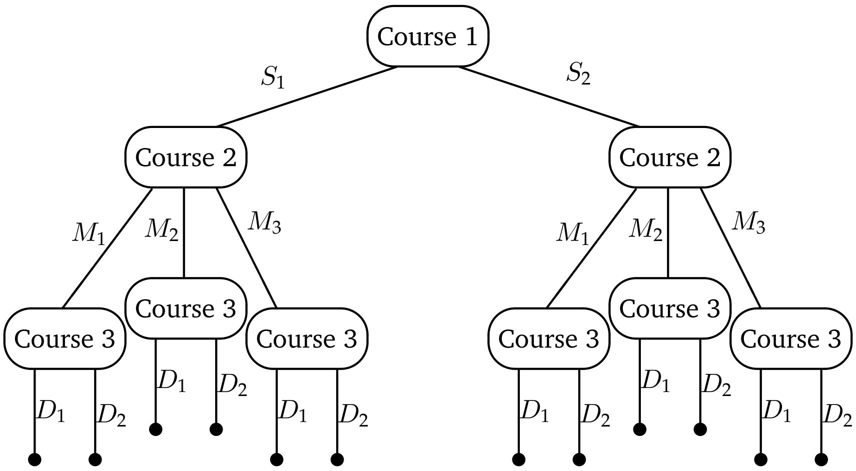
(a) The ternary tree is shown below.



(b) The number of nodes on level i is 3^i for all i, $0 \leq i \leq h$.

Question 6

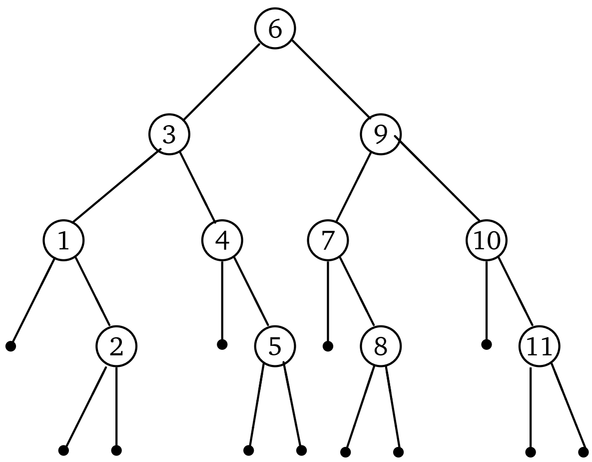
(a) The rooted tree is shown below.



(b) The number of different meals is equal to the number of external nodes in this tree. Thus there are twelve different meals.

Question 7

(a) The binary search tree is shown below.



(b) The maximum number of comparisons is 4.

(c) Records 2, 5, 8, 11 require four comparisons.

Question 8 The maximum number of comparisons is given by

$$\lceil \log_2(5,000,001) \rceil = 23.$$

B.4 Exercises on Chapter 4

B.4.1 Solutions to Multiple Choice questions

Question 1

(a) Answer: 2^{12}.
This is a 12 stage process with 2 choices at each stage, so the total number of possible outcomes is 2^{12}.

(b) Answer: $\binom{12}{7}$.
These outcomes are represented by strings of length 12 where we have to choose a set of 7 entries to call tails, the remaining 5 entries are all heads. If you thought the answer was 'Other' because you thought the answer was $\binom{12}{5}$, remember that $\binom{12}{5} = \binom{12}{7}$.

(c) Answer: 2^9.
This is a 12 stage process with 2 choices at stages 1,2,3,4,8,9,10,11 and 12 and just 1 choice at stages 5,6 and 7, so the total number of possible outcomes is 2^9.

(d) Answer: $\binom{12}{0} + \binom{12}{1} + \binom{12}{2} + \binom{12}{3} + \binom{12}{4} + \binom{12}{5}$.
A string with more heads than tails has either 0,1,2,3,4 or 5 tails. Note that
$\binom{12}{0} + \binom{12}{1} + \binom{12}{2} + \binom{12}{3} + \binom{12}{4} + \binom{12}{5} = \left(2^{12} - \binom{12}{6}\right)/2.$

Question 2

(a) Answer: 5!.
There are 5! permutations of the 5 letters.

(b) Answer: 3!.
There are 3! permutations of the 3 items AMS, T, H.

(c) Answer: $5! - 3!$.
There are 5! permutations of the 5 letters and 3! permutations of HAT, S and M, hence $5! - 3!$ permutations do not contain HAT.

(d) Answer: 3!.
There are 3! permutations of the 3 items MA, SH, T.

Question 3

(a) Answer: $\binom{42}{3}$.
There are 42 animals to choose from and he wants a set of 3, order does not matter.

(b) Answer: $10 \times 12 \times 20$.
There are $10 \times 12 \times 20$ ways of choosing a dog, a cat and a rodent. You can view this as a three stage process where stage 1 is choosing a dog, stage two is choosing a cat and stage 3 is choosing a rodent.

(c) Answer: $\binom{12}{1}\binom{20}{2}$.
There are $\binom{12}{1}\binom{20}{2}$ ways of choosing a cat and two rodents. You can view this as a two stage process where the first stage is to choose a cat and the second stage is to choose two rodents.

(d) Answer: $\binom{10}{3}$.
There are $\binom{10}{3}$ ways of choosing 3 dogs among the 10 available dogs, as order does not matter.

(e) Answer: $\binom{30}{3}$.
There are $\binom{30}{3}$ ways of choosing a selection containing only dogs and rodents, that is no cats.

(f) Answer: $\binom{32}{3} + \binom{30}{3} - \binom{20}{3}$.
There are $\binom{32}{3}$ ways of making a selection without a dog, and there are $\binom{30}{3}$ ways of making a selection without a cat. The $\binom{20}{3}$ selections which have neither cats nor dogs are

included in both these figures. Hence by the Principle of Inclusion-Exclusion there are $\binom{32}{3} + \binom{30}{3} - \binom{20}{3}$ ways of choosing a selection containing not both a cat and a dog.

Question 4

(a) Answer: 899.
There are $999 - 101 + 1 = 899$ numbers.

(b) Answer: 450.
There are $899 - \lfloor 899/2 \rfloor = 450$ odd numbers.

(c) Answer: 449.
There are $\lfloor 899/2 \rfloor = 449$ even numbers (Cf. Examples 4.11 and 4.12 in Volume 1 of this subject guide).

(d) Answer: 89.
There are $\lfloor 899/10 \rfloor = 89$ numbers divisible by 10 (Cf. Examples 4.11 and 4.12 in Volume 1 of this subject guide).

(e) Answer: 200.
There are $999 - 800 + 1 = 200$ numbers larger than 799.

(f) Answer: 100.
There are $\lfloor 200/2 \rfloor = 100$ even numbers larger than 799.

B.4.2 Solutions to Longer exercises

Question 1

There are 5 choices for president, 2 for secretary, and 3 for treasurer, so, by the multiplication principle, there are $5 \times 2 \times 3$ different committees.

Question 2

(a) The sample space is the set of all ordered sequences of 3 digits chosen from S. The sample space has thus got 9^3 elements of type abc where $a, b, c \in S$.

(b) 9^3

(c) $9 \times 8 \times 7$

(d) $1/9^3$

Question 3

(a) The sample space is the set of all possible teams $\{m_1, m_2, \ldots, m_{11}\}$ where each m_i is a club member.

(b) $\binom{25}{11} = \frac{25\times24\times\ldots\times12}{14\times13\times\ldots\times1}$, this is the whole sample space.

(c) $\binom{24}{10} = \frac{24\times23\times\ldots\times11}{14\times13\times\ldots\times1}$

Question 4

(a) The sample space is the set of all ordered sequences of 3 numbers chosen from $D = \{1, 2, 3, 4, 5, 6\}$. The sample space has thus got 6^3 elements of type (a, b, c) where $a, b, c \in D$.

(b) 6^3.

(c) $\frac{1}{6^3}$, as the only 'favourable' outcome here is $(6, 6, 6)$

(d) $\frac{5^3}{6^3}$

(e) $1 - \frac{5^3}{6^3}$

(f) (i) $P(A) = \frac{6^2}{6^3} = \frac{1}{6}$,
as the 'favourable' outcomes here are $(1, b, c)$ where $b, c \in D$.

(ii) $P(B) = \frac{4}{6^3}$,
for as the sum is ≤ 4, the only favourable outcomes here are $(2,1,1), (1,2,1), (1,1,2)$ and $(1,1,1)$.

(iii) $P(A \cap B) = \frac{3}{6^3}$
as only three outcomes satisfy the criteria for both event A and event B, namely $(1,2,1), (1,1,2)$ and $(1,1,1)$.

(iv) $P(A \cup B) = \frac{1}{6} + \frac{4}{6^3} - \frac{3}{6^3} = \frac{1}{6} + \frac{1}{6^3}$

(g) A and B are not independent since

$$P(A \cap B) = \frac{3}{6^3} \neq P(A)P(B) = \frac{1}{6} \times \frac{4}{6^3}.$$

Question 5

(a) We want $|A \cup B|$ where A is the set of integers between 1 and 1000 (both inclusive) which are divisible by 3 and B is the set of integers between 1 and 1000 (both inclusive) which are divisible by 5. Then $A \cap B$ is the set of integers between 1 and 1000 (both inclusive) which are divisible by 15. Now $|A| = \lfloor 1000/3 \rfloor = 333$, $|B| = \lfloor 1000/5 \rfloor = 200$, and $|A \cap B| = \lfloor 1000/15 \rfloor = 66$. So by the Principle of Inclusion-Exclusion (Theorem 4.6) we have

$$|A \cup B| = |A| + |B| - |A \cap B| = 333 + 200 - 66 = 467.$$

(b) (i) Let C be the event that a household has cat(s). We are given the information that 40% of all households are in this event. Thus $P(C) = 0.4$

(ii) Let D be the event that a household has dog(s). We want $1 - P(C \cup D)$. We are given in the exercise that $|C \cap D| = 30\%$ of all households, $|D| = 60\%$ of all households and $|C| = 40\%$ of all households. So by the extension of the Principle of Inclusion-Exclusion (Theorem 4.7) we have

$$\begin{aligned} 1 - P(C \cup D) &= 1 - (P(C) + P(D) - P(C \cap D)) \\ &= 1 - (0.4 + 0.6 - 0.3) \\ &= 0.3. \end{aligned}$$

Question 6

We want $1000 - |A \cup B \cup C|$ where A is the set of integers between 1 and 1000 (both inclusive) which are divisible by 3, B is the set of integers between 1 and 1000 (both inclusive) which are divisible by 5 and C is the set of integers between 1 and 1000 (both inclusive) which are divisible by 11. Then $A \cap B$ is the set of integers between 1 and 1000 (both inclusive) which are divisible by 15, $A \cap C$ is the set of integers between 1 and 1000 (both inclusive) which are divisible by 33, $B \cap C$ is the set of integers between 1 and 1000 (both inclusive) which are divisible by 55, and $A \cap B \cap C$ is the set of integers between 1 and 1000 (both inclusive) which are divisible by 165.

Now $|A| = \lfloor 1000/3 \rfloor = 333$, $|B| = \lfloor 1000/5 \rfloor = 200$, $|C| = \lfloor 1000/11 \rfloor = 90$. Further $|A \cap B| = \lfloor 1000/15 \rfloor = 66$, $|A \cap C| = \lfloor 1000/33 \rfloor = 30$, $|B \cap C| = \lfloor 1000/55 \rfloor = 18$, and

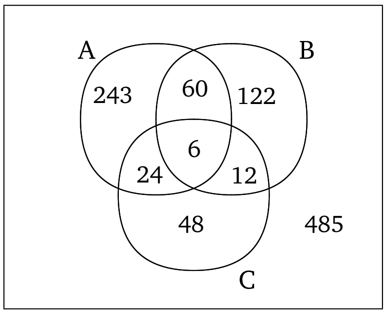
$|A \cap B \cap C| = \lfloor 1000/165 \rfloor = 6$. So by the Principle of Inclusion-Exclusion for three sets developed in Example 4.16 we have

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 333 + 200 + 90 - 66 - 30 - 18 + 6 \\ &= 515. \end{aligned}$$

Hence the required result is

$$1000 - |A \cup B \cup C| = 1000 - 515 = 485.$$

If you solved the exercise by using a Venn diagram as in Example 4.14, your diagram should look like this:



B.5 Exercises on Chapter 5

B.5.1 Solutions to True/False questions

1. True.
   The equation is linear because it is in the form $a_1x_1 + a_2x_2 + a_3x_3 = k$, where a_1, a_2, a_3, k are constants.
2. False.
   $a_{1,3} = 2$.
3. True.
   $a_{1,1} = 1$ and $a_{3,2} = 1$ are leading entries, row 2 has no leading entry.
4. False.
   A is not in row echelon form. Property **P1** does not hold as row 2 is all-zero and row 3 below row 2 has non-zero entries.
5. True.
   B is in row echelon form. Properties **P1** - **P4** all hold for this matrix.
6. False.
   BC is a 3×3 matrix while CB is a 2×2 matrix.
7. True.
   A is a 3×3 matrix and so is BC, hence they may be added.
8. False.
   A is a $3 \times \mathbf{3}$ matrix and C is a $\mathbf{2} \times 3$, hence they are not of compatible size for AC to be computed.
9. True.
   C is a $2 \times \mathbf{3}$ matrix and A is a $\mathbf{3} \times 3$ matrix hence they are of compatible size. CA is a 2×3 matrix.

10. False.

$$A^2 = AA = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 8 \\ 0 & 0 & 0 \\ 0 & 3 & 9 \end{pmatrix}$$

B.5.2 Solutions to Longer exercises

Question 1

(a) $\begin{pmatrix} 2 & 1 & :2 \\ 1 & -1 & :1 \\ 1 & 2 & :1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -1 & :1 \\ 2 & 1 & :2 \\ 1 & 2 & :1 \end{pmatrix} \xrightarrow[R_3:=R_3-R_1]{R_2:=R_2-2R_1}$
$\begin{pmatrix} 1 & -1 & :1 \\ 0 & 3 & :0 \\ 0 & 3 & :0 \end{pmatrix} \xrightarrow{R_2:=\frac{1}{3}R_2} \begin{pmatrix} 1 & -1 & :1 \\ 0 & 1 & :0 \\ 0 & 3 & :0 \end{pmatrix} \xrightarrow{R_3:=R_3-3R_2}$
$\begin{pmatrix} 1 & -1 & :1 \\ 0 & 1 & :0 \\ 0 & 0 & :0 \end{pmatrix}$

Solution:
$R_2 \;\Rightarrow\; x_2 = 0$
$R_1 \;\Rightarrow\; x_1 - x_2 = 1$. So $x_1 = 1 + x_2 = 1 + 0 = 1$.
Hence $x_1 = 1$ and $x_2 = 0$.

(b) $\begin{pmatrix} 2 & 1 & :2 \\ 1 & -1 & :1 \\ 1 & 2 & :1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -1 & :1 \\ 2 & 1 & :2 \\ 1 & 2 & :1 \end{pmatrix} \xrightarrow[R_3:=R_3-R_1]{R_2:=R_2-2R_1}$
$\begin{pmatrix} 1 & -1 & :1 \\ 0 & 3 & :0 \\ 0 & 3 & :0 \end{pmatrix} \xrightarrow{R_2:=\frac{1}{3}R_2} \begin{pmatrix} 1 & -1 & :1 \\ 0 & 1 & :0 \\ 0 & 3 & :0 \end{pmatrix} \xrightarrow{R_3:=R_3-3R_2}$
$\begin{pmatrix} 1 & -1 & :1 \\ 0 & 1 & :0 \\ 0 & 0 & :0 \end{pmatrix}$

Solution:
$R_3 \;\Rightarrow\; 0x_1 + 0x_2 = -1$. Hence there is no solution to this system of equations.

(c) $\begin{pmatrix} 1 & 1 & 1 & :2 \\ 2 & -1 & 1 & :1 \\ 1 & -2 & 0 & :-1 \end{pmatrix} \xrightarrow[R_3:=R_3-R_1]{R_2:=R_2-2R_1}$
$\begin{pmatrix} 1 & 1 & 1 & :2 \\ 0 & -3 & -1 & :-3 \\ 0 & -3 & -1 & :-3 \end{pmatrix} \xrightarrow[R_3:=R_3-R_2]{R_2:=-\frac{1}{3}R_2} \begin{pmatrix} 1 & 1 & 1 & :2 \\ 0 & 1 & \frac{1}{3} & :1 \\ 0 & 0 & 0 & :0 \end{pmatrix}$

Solution:
$R_2 \;\Rightarrow\; x_2 + \frac{1}{3}x_3 = 1$. Let $x_3 = r$. Then $x_2 = 1 - \frac{1}{3}r$.
$R_1 \;\Rightarrow\; x_1 + x_2 + x_3 = 2$. So

$$x_1 = 2 - x_2 - x_3 = 2 - (1 - \frac{1}{3}r) + r = 1 + \frac{4}{3}r.$$

Hence $x_1 = 1 + \frac{4}{3}r$, $x_2 = 1 - \frac{1}{3}r$, and $x_3 = r$ for any $r \in \mathbb{R}$.

Question 2

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ -x_1 + x_2 &= 2 \\ 3x_1 + 2x_2 &= 3 \end{aligned}$$

Question 3

(a) The system written as a matrix equation $A\mathbf{x} = \mathbf{b}$:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 5 \end{pmatrix}.$$

(b) The augmented matrix of the system is

$$\begin{pmatrix} 0 & 1 & 1 & : & 2 \\ 1 & 1 & 2 & : & 5 \\ 1 & -2 & 1 & : & 5 \end{pmatrix}$$

Question 4

$$B + 2C = \begin{pmatrix} -1 & 2 \\ 5 & 6 \end{pmatrix} \quad AB = \begin{pmatrix} 7 & 10 \\ 2 & 2 \\ 9 & 14 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Question 5

(a) $B + C = A$, so

$$C = A - B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

(b) $BD = B$, so

$$D = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) $B - E = 2A$, so

$$E = B - 2A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 4 & 0 \\ 4 & 2 & 4 \\ 0 & 4 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -3 & 1 \\ -3 & -1 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

Appendix C

Specimen examination questions

Important: *the information given in this Appendix is based on the examination structure used at the time this guide was written. Note that the university can alter the format, style or requirements of an examination paper* without notice. *Because of this, we strongly advise you to check the rubric/instructions on the paper when you actually sit the examination. You should also read the examiner's report from the previous year for advice on this.*

This unit is currently examined by a three hour written paper with ten questions. There is no choice of questions on the examination paper, full marks will be awarded for correct solutions to all ten questions. You will be expected to be able to answer questions on all the material covered in both *volumes of the subject guide.*

The following specimen questions are on the topics covered in Volume 2 of the subject guide. See Appendix C of Volume 1 for specimen examination questions on the topics covered in Volume 1.

Question 1

Let $X = \{a, b, c\}$ and let $\mathcal{R}$ be the relation defined on the set $X \times X$ by the following subset:

$$\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, c)\}.$$

(a) Draw a digraph to illustrate the relation $\mathcal{R}$. [2]

(b) Decide whether $\mathcal{R}$ is (i) reflexive; (ii) symmetric; (iii) transitive; (iv) anti-symmetric. In each case, say why $\mathcal{R}$ has that property or give a counter-example to show that the property does not hold. [6]

(c) Say whether $\mathcal{R}$ is (i) an equivalence relation; (ii) a partial order, giving a reason for your answer in each case. [2]

Question 2

(a) A sequence is defined by the recurrence relation $u_n = 3u_{n-1} - 1$ and the initial term $u_1 = 2$.

(i) Use the recurrence relation to calculate the terms u_2 and u_3, showing your working. [2]

(ii) Prove by induction that

$$u_n = \frac{3^n + 1}{2} \text{ for all } n \geq 1.$$

[5]

(b) State without proof, the formula for $\sum_{j=1}^{n} j$. [1]

Use this to evaluate $\sum_{j=1}^{40}(3j - 5)$. [2]

Question 3

(a) Let T be a rooted tree. What property must T possess for it to be called a **binary** tree? [1]

(b) (i) Design a binary search tree for an ordered list of 10 records $1, 2, \ldots, 10$. [4]

(ii) What is the height of the tree you have constructed? [1]

(iii) What is the maximum number of comparisons that would have to be made to locate an existing record? List all the existing records that require the maximum number of comparisons. [2]

(c) It is required to store $7,000$ records at the internal nodes of a binary search tree T. Find the height of T. [2]

Question 4

A deck of cards contains 52 playing cards. The deck has 4 suits (Hearts, Clubs, Diamonds and Spades), with 13 different cards in each suit. We can represent each card by its suit and number (from 1 to 13).

(a) 4 cards are chosen at random and removed from the deck of cards. Assuming that the order in which the cards are chosen does not matter, how many ways of choosing these four cards are possible? [2]

(b) How many of these combinations have all the cards from the suit Hearts? Hence determine the probability that all 4 cards come from the suit Hearts. [3]

(c) What is the probability that the 4 cards that are removed from the pack all have the same number? [5]

Question 5

(a) Define matrices A, $\mathbf{x}$ and $\mathbf{b}$ to express the following system of equations as a matrix equation $A\mathbf{x} = \mathbf{b}$. [2]

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 1 \\ 2x_1 & - & x_2 & = & 5 \end{array}$$

Use Gaussian elimination to solve this system of equations. [4]

(b) Let

$$A = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 1 & -3 \end{pmatrix}, \; C = \begin{pmatrix} -1 & -1 \\ 3 & 2 \\ 1 & 2 \end{pmatrix}.$$

Calculate AB and $B + 2C$. [2]

(c) Find two 2×2 matrices X and Y such that $XY \neq YX$. [2]

Question 6

The matrix A given below is the adjacency matrix of a graph G w.r.t. the natural ordering of its vertex set $V(G) = \{v_1, v_2, \ldots, v_5\}$.

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

(a) Use A to construct a set of adjacency lists for G. [2]

(b) Find $\deg(v_1)$. [1]

(c) Find the number of edges in G. [2]

(d) Calculate the *second row* of the matrix A^2. Hence, or otherwise, find the number of walks of length 2 starting at v_2 and terminating at each of v_1, v_3, v_4 and v_5. [3]

(e) Calculate the number of walks of length 3 starting at v_2 and terminating at v_1. [2]

Question 7

Let S be the set $\{m, a, t, h, s\}$.

(a) (i) Describe briefly how each subset of S can be represented by a unique 5-bit binary string.

(ii) Write down the string corresponding to the subset $\{h, a, t\}$.

(iii) Write down the subset corresponding to the string 01101.

(iv) What is the total number of subsets of S? [4]

(b) R is a relation defined on S in precisely the following cases:

$$mRm, \quad mRa, \quad mRt, \quad tRm, \quad tRt, \quad hRt, \quad sRs.$$

(i) Draw the relationship digraph for R on S.

(ii) The relation R is not reflexive. Which minimal set of pairs should be added to R to make it reflexive?

(iii) The relation R is not symmetric. Which minimal set of pairs should be added to R to make it symmetric?

(iv) The relation R is not transitive. Which minimal set of pairs should be added to R to make it transitive?

(v) Is the relation R anti-symmetric? Justify your answer. [6]

Question 8

A language school offers three courses, one in German, one in French and one in Spanish. Students at the school may be enrolled for any number of courses. The number of students in the school attending the German, French and Spanish classes are given in the following table.

German	334
French	440
Spanish	278
German & Spanish	132
German & French	178
French & Spanish	164
German, French & Spanish	20

(a) Use a Venn diagram or the principle of inclusion-exclusion to determine the number of students attending either German, French or Spanish. [4]

(b) The total number of students enrolled in the school is 660. Suppose that we pick a student at random from the school role. Expressing each answer as a fraction without simplification, determine the probability that the selected student is enrolled on

(i) at least one of the courses;

(ii) none of the courses;

(iii) the French course, but not on the two other courses;

(iv) exactly two of the courses. [6]

Question 9

(a) In a tournament with five players A, B, C, D, E, every player plays every other player precisely once, with the following results:
A beats C and E; B beats A and E; C beats B;
D beats A, B and C; E beats C and D.

Draw a digraph to model this information. Take care to describe what the vertices of your digraph represent and when there is an arc directed from vertex a to vertex b. [3]

(b) Check that the sum of the outdegrees equals the sum of the indegrees in the graph you constructed in (a) above. Explain why this is so for any digraph. [3]

(c) In another tournament there are 6 competitors, where each competitor plays every other precisely once. Each game has a winner, there are no draws. Alice and Bob are two spectators, who both keep scores. Alice decides to count the wins of each competitor, while Bob counts their losses. Here are their results:

	Alice (# wins)	Bob (# losses)
Player 1	2	3
Player 2	2	3
Player 3	4	1
Player 4	3	2
Player 5	3	1
Player 6	1	4

(i) Say why they cannot both have got the scores right.

(ii) Explain why Bob has got them wrong.

(iii) Has Alice got them right? [4]

Question 10

Consider the three matrices

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 0 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -3 \\ 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Find the following matrices:

(i) BC;

(ii) $BC + A$;

(iii) A^2;

(iv) $(AB)C + A^2$. [5]

(b) Let D be a 2×4-matrix and let E be a 4×3-matrix. Let X and Y denote matrices in the set of matrices $\mathcal{M} = \{A, B, C, D, E\}$. The relation $\mathcal{R}$ on $\mathcal{M}$ is defined by:

$$X\mathcal{R}Y \text{ if } YX \text{ is a valid product of matrices.}$$

(i) Draw the digraph of the relation $\mathcal{R}$ on $\mathcal{M}$.

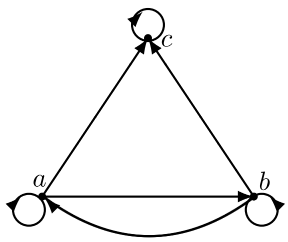
(ii) Write down the adjacency matrix of this digraph. [5]

Appendix D

Solutions to Specimen examination questions

Question 1

(a) The relationship digraph is shown below.



(b) (i) $\mathcal{R}$ is reflexive because $x\mathcal{R}x$, for all $x \in X$.

(ii) $\mathcal{R}$ is not symmetric as e.g. $a\mathcal{R}c$ but c is not related to a.

(iii) $\mathcal{R}$ is transitive because for all $x, y, z \in X$, whenever both $x\mathcal{R}y$ and $y\mathcal{R}z$, then also $x\mathcal{R}z$.

(iv) $\mathcal{R}$ is not anti-symmetric because we have both $a\mathcal{R}b$ and $b\mathcal{R}a$, where $a \neq b$.

(c) (i) $\mathcal{R}$ is not an equivalence relation as it is not symmetric.

(ii) $\mathcal{R}$ is not a partial order, because it is not anti-symmetric.

Question 2

(a) (i)
$$\begin{array}{rcl} u_2 &=& 3u_1 - 1 = 3(2) - 1 = 5, \\ u_3 &=& 3u_2 - 1 = 3(5) - 1 = 14. \end{array}$$

(ii) **Proof:** (By induction)

Base Case: Set $n = 1$: $u_1 = \frac{3^1+1}{2} = \frac{4}{2} = 2$, so the formula is true when $n = 2$.

Inductive Hypothesis: Assume that $u_k = \frac{3^k+1}{2}$ is true for all integers $1, 2, \ldots, k$.

Induction Step: (seek to prove that $u_{k+1} = \frac{3^{k+1}+1}{2}$)

$$\begin{aligned} u_{k+1} &= 3u_k - 1 \quad \text{(from recurrence relation)} \\ &= 3\left[\frac{3^k+1}{2}\right] - 1 \quad \text{(substituting hypothesis)} \\ &= \left[\frac{3\times 3^k+3}{2}\right] - 1 \\ &= \left[\frac{3^{k+1}+3}{2}\right] - 1 \\ &= \frac{3^{k+1}+3-2}{2} \\ &= \frac{3^{k+1}+1}{2}. \end{aligned}$$

Thus the formula also holds when $n = k + 1$, and hence $u_n = \frac{3^n+1}{2}$ for all $n \geq 1$ by induction.

(b)

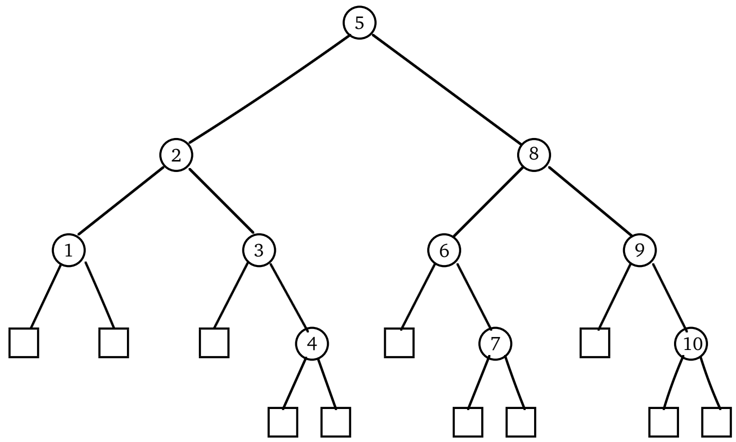
$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}.$$

$$\begin{aligned}
\sum_{j=1}^{40}(3j-5) &= 3\sum_{j=1}^{40} j - 40(5) \\
&= 3\left[\frac{40(40+1)}{2}\right] - 200 \\
&= 3(820) - 200 = 2460 - 200 = 2260
\end{aligned}$$

Question 3

(a) Each internal node must have exactly two children.

(b) (i) The binary tree is shown below.



(ii) Tree has height 4.

(iii) Maximum number of comparisons is 4. Existing records which require four comparisons are $4, 7, 10$.

(c) Since $2^{12} < 7000 < 2^{13}$, the height of the tree is 13. (Or, use the formula that the height of tree required to store N records is $\lceil \log_2(N+1) \rceil$. Then $\log_2 7001 = \frac{\log_{10} 7001}{\log_{10} 2} \approx 12.77$. Hence the height of the tree is 13.)

Question 4

(a)

$$\begin{aligned}
\binom{52}{4} &= \frac{52!}{4! \cdot (52-4)!} \\
&= \frac{52!}{4! \cdot 48!} \\
&= \frac{52 \cdot 51 \cdot 50 \cdot 49}{4 \cdot 3 \cdot 2} \\
&= 13 \cdot 17 \cdot 25 \cdot 49 \ (= 270725).
\end{aligned}$$

(b) Number of combinations that are all Hearts is

$$\begin{aligned}\binom{13}{4} &= \frac{13!}{4!\cdot(13-4)!}\\ &= \frac{13!}{4!\cdot 9!}\\ &= \frac{13\cdot 12\cdot 11\cdot 10}{4\cdot 3\cdot 2}\\ &= 13\cdot 1\cdot 11\cdot 5\ (=715).\end{aligned}$$

Probability that all three cards are hearts is thus $\frac{715}{270725}$ $(=\frac{11}{4165})$.

(c) Number of selections where all the cards are of a given number (e.g. 1:1:1:1) is $\binom{4}{4}=1$.

There are 13 different numbers, so we need to consider the total number of combinations where the cards have the same number, i.e. combinations 1:1:1:1, 2:2:2:2, 3:3:3:3, ... ,13:13:13:13.

By the addition principle, the total number of choices is

$$\underbrace{\binom{4}{4}+\binom{4}{4}+\cdots+\binom{4}{4}}_{13} = 13\cdot\binom{4}{4},$$

which is $13\cdot 1=13$.

Hence the probability of picking three cards of the same number is $\frac{13}{270725}$ $=(\frac{1}{20825})$.

Question 5

(a) $A=\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}$, $\mathbf{x}=\begin{pmatrix}x_1\\ x_2\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}1\\ 5\end{pmatrix}$.

$$\begin{pmatrix}1 & 1 & :1\\ 2 & -1 & :5\end{pmatrix}\to\begin{pmatrix}1 & 1 & :1\\ 0 & -3 & :3\end{pmatrix}\to\begin{pmatrix}1 & 1 & :1\\ 0 & 1 & :-1\end{pmatrix}\to\begin{pmatrix}1 & 0 & :2\\ 0 & 1 & :-1\end{pmatrix}$$

Solution: $x_1=2$ and $x_2=-1$.

(b) $AB=\begin{pmatrix}4 & -4\end{pmatrix}$, $B+2C=\begin{pmatrix}-1 & 0\\ 5 & 4\\ 3 & 1\end{pmatrix}$.

(c) e.g. $X=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $Y=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$.

Question 6

(a)

v_1: v_2, v_3, v_5
v_2: v_1, v_4, v_5
v_3: v_1, v_4
v_4: v_2, v_3, v_5
v_5: v_1, v_2, v_4

(b) $\deg(v_1)=3$.

(c) Number of edges is 7 (half the sum of the entries in A).

(d)

$$A^2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 3 & 2 & 1 & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Hence, number of walks of length 2 from v_2 to v_1, v_3, v_4, v_5 is $1, 2, 1, 2$ respectively.

(e) Number of walks of length 3 from v_2 to v_1 is given by the entry in cell $(2, 1)$ of the matrix A^3, which is

$$\begin{pmatrix} 1 & 3 & 2 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = 7.$$

Question 7

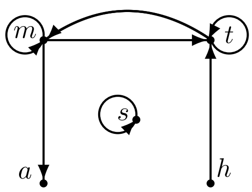
(a) (i) Let the first bit in the binary string correspond to the element m, the second bit correspond to the element a, etc. If m is in the subset, we let the first bit be a 1 and if m is not in the subset, we let the first bit be a 0. Similarly, if a is in the subset, we let the second bit be a 1 and if a is not in the subset, we let the second bit be a 0, etc.

(ii) The string corresponding to the subset $\{h, a, t\}$ is 01110.

(iii) The subset corresponding to the string 01101 is $\{a, t, s\}$.

(iv) The total number of subsets of S is $2^5 = 32$.

(b) (i) The relationship digraph for R on S is shown below.



(ii) The relation R will be reflexive if you add the two pairs aRa and hRh.

(iii) The relation R will be symmetric if you add the two pairs aRm and tRh.

(iv) The relation R will transitive if you add the three pairs hRm, tRa and hRa.

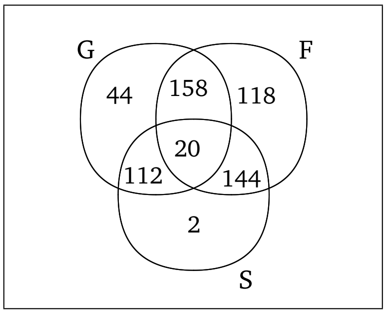
(v) R is not anti-symmetric as mRt and tRm but $m \neq t$.

Question 8

(a) Let G, F, S denote the sets of students attending German, French and Spanish respectively. Then by the Principle of Inclusion-Exclusion for three sets which we developed in Example 4.16 we get

$$\begin{aligned} |G \cup F \cup S| &= |G| + |F| + |S| - |G \cap S| - |G \cap F| - |F \cap S| + |G \cap F \cap S| \\ &= 334 + 440 + 278 - 132 - 178 - 164 + 20 \\ &= 598 \end{aligned}$$

The Venn diagram developed like we did in Example 4.14 looks like this:



(b) (i) By (a) the number of students enrolled on at least one of the courses is 598. Hence the probability that a student is enrolled on at least one course is 598/660 = 299/330.

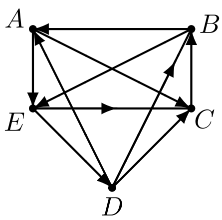
(ii) By (a) the number of students enrolled on at least one of the courses is 598, hence the number of students not enrolled on any of the courses is 660 − 598 = 62. Thus the probability that a student is enrolled on no course is 62/660 = 31/330.

(iii) By the Venn diagram we constructed in (a), 118 students study French only. Hence the probability that a student studies the French course, but not the two other courses is 118/660 = 59/330.

(iv) By the Venn diagram from (a), 158 + 112 + 144 = 414 students are enrolled on exactly two of the courses. Hence the probability that a student is enrolled on exactly two courses is 414/660 = 69/110.

Question 9

(a) The digraph modelling the tournament is given below. The vertices represent the players, and there is an arc directed from vertex a to vertex b if player a beats player b.



(b) The sum of the outdegrees equals the sum of the indegrees (both are 10) in the graph constructed in (a). The sum of the outdegrees equals the sum of the indegrees for any digraph (both are equal to the number of arcs in the digraph). This is so because an arc contributes precisely 1 to the outdegree of the vertex it leaves and 1 to the indegree of the vertex it goes to.

(c) (i) Each player plays 5 games, hence the number of wins and losses for each player must sum to 5. But the sum for player 5 is just 4, hence at least one of the two numbers for player 5 must be wrong.

(ii) Every player plays every other player precisely once in this tournament. Hence the number of games played is $\binom{6}{2} = 15$, Bob has counted just 14 losses, so he must have got at least one of his figures wrong as every game has a winner and a loser.

(iii) It is not possible to determine whether Alice has got all the scores right. Her numbers sum to 15, but she could have made more than one error so that her sum is right while individual scores might still be wrong.

Question 10

(a) (i) $BC = \begin{pmatrix} 0 & -6 \\ -3 & 12 \end{pmatrix}$;

(ii) $BC + A = \begin{pmatrix} 0 & -6 \\ -3 & 12 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -6 \\ -4 & 15 \end{pmatrix}$;

(iii) $A^2 = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -5 & 9 \end{pmatrix}$;

(iv)

$$\begin{aligned}(AB)C + A^2 = A(BC + A) &= \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & -6 \\ -4 & 15 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -12 \\ -14 & 51 \end{pmatrix}.\end{aligned}$$

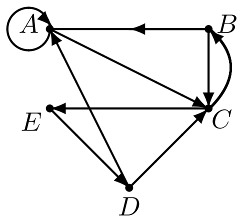
(b) (i) The valid products are

$$AA, CA, AB, CB, BC, EC, AD, CD, DE,$$

hence the relation consists of the 9 pairs

$$A\mathcal{R}A, A\mathcal{R}C, B\mathcal{R}A, B\mathcal{R}C, C\mathcal{R}B, C\mathcal{R}E, D\mathcal{R}A, D\mathcal{R}C, E\mathcal{R}D.$$

The digraph of the relation $\mathcal{R}$ is:



(ii) The adjacency matrix of the digraph is

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

w.r.t. the natural alphabetical ordering of the vertices.

Appendix E

List of Symbols

Sets

$\{a, b, c\}$	set with elements a, b, c
$a \in X$	a is an element of X
$a \notin X$	a is not an element of X
$\emptyset$	the empty set
$\mathbb{N}$	the set of natural numbers $\{0, 1, 2, 3, \ldots\}$
$\mathbb{Z}$	the set of integers
$\mathbb{Z}^+$	the set of positive integers $\{1, 2, 3, \ldots\}$
$\{x \in \mathbb{Z} : x \text{ has property } p\}$	the set of integers with property p
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathcal{P}(X)$	the power set of X, i.e. the set of all subsets of X
A'	the complement of set A
$A \cup B$	the union of sets A and B
$A \cap B$	the intersection of sets A and B
$A - B$	the set difference of A minus B
$A \times B$	the cartesian product of A and B
A^2	the cartesian product $A \times A$
A^n	the cartesian product $A \times A \times \ldots \times A$
$\lvert A \rvert$	the cardinality of the set A

Relations

$x\mathcal{R}y$	x is related to y under relation $\mathcal{R}$
$x = y$	x is equal to y
$x \neq y$	x is not equal to y
$x > y$	x is strictly greater than y
$x \geq y$	x is greater than or equal to y
$x < y$	x is strictly less than y
$x \leq y$	x is less than or equal to y
$A \subseteq B$	A is a subset of B
$A \subset B$	A is a proper subset of B

Functions

$\lvert x \rvert$	the absolute value of the real number x
$\lfloor x \rfloor$	the floor of the real number x
$\lceil x \rceil$	the ceiling of the real number x
$\log_2(x)$	the logarithm of the real number x to the base 2
$n!$	$0! = 1$,
	$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$ for positive integers n
$\binom{n}{r}$	the binomial coefficient $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Graph Theory

$V(G)$	the set of vertices of (di)graph G
$E(G)$	the set of edges of graph G
$\deg_G(v)$	the degree of vertex v in graph G
$\deg(v)$	same as $\deg_G(v)$,
	used when it is obvious which graph G is meant
$v_1 v_2 \ldots v_n$	(directed) path through distinct vertices $v_1, v_2, \ldots, v_n$
k-cycle	cycle of length k
$\mathbb{A}(D)$	the adjacency matrix of (di)graph D
$\mathrm{indeg}_D(v)$	the indegree of vertex v in graph D
$\mathrm{outdeg}_D(v)$	the outdegree of vertex v in graph D

Probability

$P(x)$	the probability of outcome x
$P(E)$	the probability of event E

Matrices

$A = (a_{i,j})$	matrix A with entry $a_{i,j}$ in row i and column j
$A(i,j)$	the entry in row i and column j of matrix A
$(A : \mathbf{b})$	the augmented matrix of the system of equations
	corresponding to the matrix equation $A\mathbf{x} = \mathbf{b}$

Notes

Notes

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