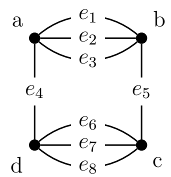
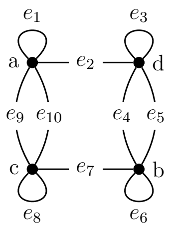
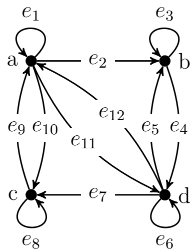
MA4016 - Engineering Mathematics 6

Solution Sheet 9: Graphs (April 09, 2010)

1. Represent the given graphs using an adjacency matrix and an incidence matrix (note the simplified graphical representations in **a)** and **b)**).

a) 

b) 

c) 

a) canonical ordering of vertices and edges

$$A = \begin{pmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

b) canonical ordering of vertices and edges

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

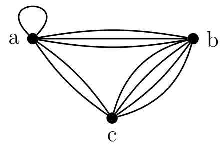
c) canonical ordering of vertices and edges

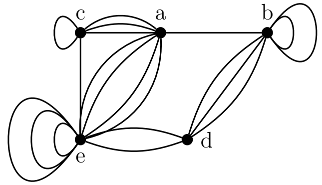
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

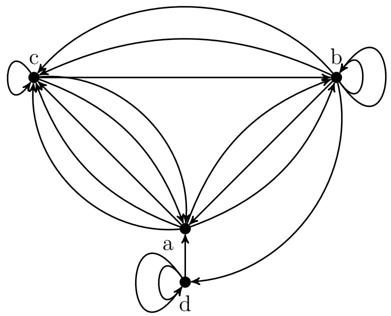
2. Draw the graphs represented by the given adjacency matrices.

a) $\begin{pmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{pmatrix}$ **b)** $\begin{pmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix}$ **c)** $\begin{pmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$

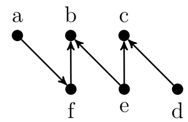
Assuming canonical ordering of the vertices we can draw in **a)** and **b)** undirected graphs, because the matrices are symmetric, and in **c)** a directed graph.

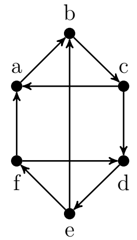
a) 

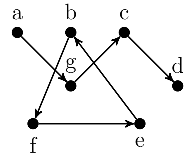
b) 

c) 

3. Determine whether each of the following graphs is strongly connected and if not, whether it is weakly connected.

a) 

b) 

c) 

a) not strongly connected, no directed path from f to a, but weakly connected.
b) strongly connected because directed Hamilton circuit a, b, c, d, e, f, a exists.
c) not connected, one weakly connected component with vertices $\{a, g, c, d\}$ and one strongly connected component $\{b, f, e\}$.

4. Find the number of paths of length n between
a) two adjacent vertices **b)** two non-adjacent vertices
in K_4 and $K_{3,3}$ if $n = 2,\ 3,\ 4,\ 5$.
We use the adjacency matrices of both graphs given by

$$A_{K_4} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_{K_{3,3}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We can compute the number of paths of length n between two vertices using the n-th power of the adjacency matrix. Thus we compute

$$A_{K_4}^2 = \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix} \quad \text{and} \quad A_{K_{3,3}}^2 = \begin{pmatrix} 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{pmatrix}$$

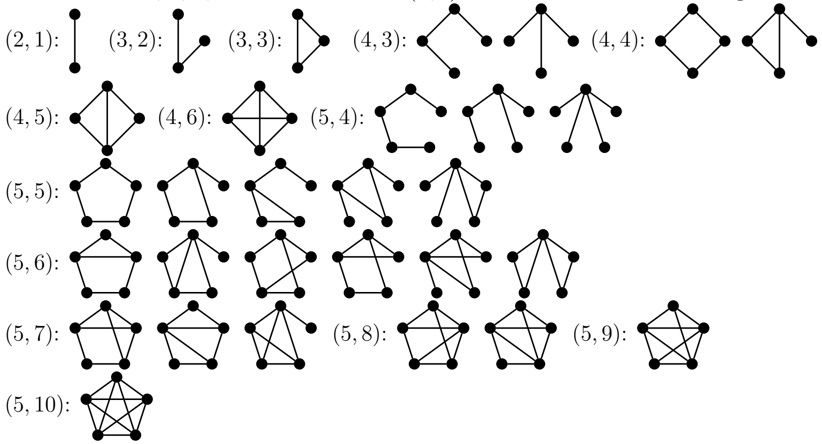
$$A_{K_4}^3 = \begin{pmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{pmatrix} \quad \text{and} \quad A_{K_{3,3}}^3 = \begin{pmatrix} 0 & 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 9 \\ 9 & 9 & 9 & 0 & 0 & 0 \\ 9 & 9 & 9 & 0 & 0 & 0 \\ 9 & 9 & 9 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{K_4}^4 = \begin{pmatrix} 21 & 20 & 20 & 20 \\ 20 & 21 & 20 & 20 \\ 20 & 20 & 21 & 20 \\ 20 & 20 & 20 & 21 \end{pmatrix} \quad \text{and} \quad A_{K_{3,3}}^4 = \begin{pmatrix} 27 & 27 & 27 & 0 & 0 & 0 \\ 27 & 27 & 27 & 0 & 0 & 0 \\ 27 & 27 & 27 & 0 & 0 & 0 \\ 0 & 0 & 0 & 27 & 27 & 27 \\ 0 & 0 & 0 & 27 & 27 & 27 \\ 0 & 0 & 0 & 27 & 27 & 27 \end{pmatrix}$$

$$A_{K_4}^5 = \begin{pmatrix} 60 & 61 & 61 & 61 \\ 61 & 60 & 61 & 61 \\ 61 & 61 & 60 & 61 \\ 61 & 61 & 61 & 60 \end{pmatrix} \quad \text{and} \quad A_{K_{3,3}}^5 = \begin{pmatrix} 0 & 0 & 0 & 81 & 81 & 81 \\ 0 & 0 & 0 & 81 & 81 & 81 \\ 0 & 0 & 0 & 81 & 81 & 81 \\ 81 & 81 & 81 & 0 & 0 & 0 \\ 81 & 81 & 81 & 0 & 0 & 0 \\ 81 & 81 & 81 & 0 & 0 & 0 \end{pmatrix}$$

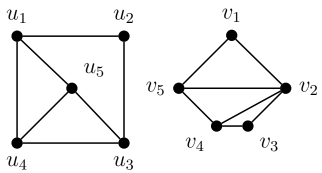
Thus for two adjacent vertices in K_4 (this is equivalent to two different vertices) we have 2 paths of length 2, 7 paths of length 3, 20 paths of length 4 and 81 paths of length 5. For two non-adjacent vertices (starting and ending at the same point) we have 3 paths of length 2, 6 paths of length 3, 6 paths of length 4 and 60 paths of length 5.
In $K_{3,3}$ we have for an even length n no paths for adjacent vertices and for non-adjacent vertices 3^{n-1} paths. For an odd length n this is vice versa.

5. How many nonisomorphic connected simple graphs are there with n vertices when $n = 2,\ 3,\ 4,\ 5$?
The answer is 1, 2, 6, 21. These are with (n, e) where e is the number of edges



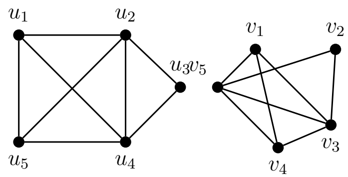
6. Determine whether the given pairs of graphs are isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

a)



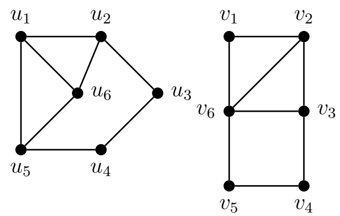
not isomorphic because deg(v_2)=4 and no vertex in the left graph has degree 4

b)



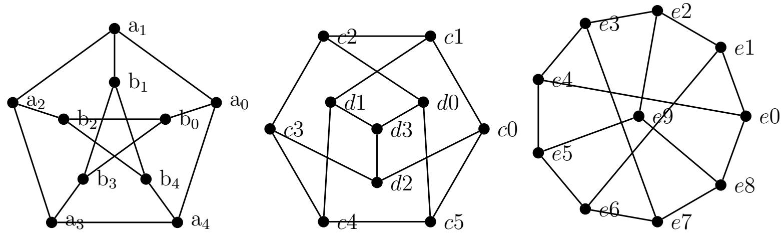
isomorphic, $f(u_1) = v_1$, $f(u_2) = v_5$, $f(u_3) = v_2$, $f(u_4) = v_3$, $f(u_5) = v_4$

c)



not isomorphic, deg(v_6)=4 and no vertex in the left graph has degree 4

d) and **e)**



These are three isomorphic representation of the Petersen graph. We will show this with adjacency matrices (using "." instead of "0" for better readability). The idea is to look for simple paths of maximal length (Hamilton-paths) and use them as bijective mapping. Comparing the adjacency matrices then shows which rows and columns have to be swapped.

$$A_1 = \begin{array}{c} a_0\\ a_1\\ a_2\\ a_3\\ a_4\\ b_0\\ b_1\\ b_2\\ b_3\\ b_4 \end{array} \left[\begin{array}{cccccccccc}
. & 1 & . & . & 1 & 1 & . & . & . & .\\
1 & . & 1 & . & . & . & 1 & . & . & .\\
. & 1 & . & 1 & . & . & . & 1 & . & .\\
. & . & 1 & . & 1 & . & . & . & 1 & .\\
1 & . & . & 1 & . & . & . & . & . & 1\\
1 & . & . & . & . & . & . & 1 & 1 & .\\
. & 1 & . & . & . & . & . & . & 1 & 1\\
. & . & 1 & . & . & 1 & . & . & . & 1\\
. & . & . & 1 & . & 1 & 1 & . & . & .\\
. & . & . & . & 1 & . & 1 & 1 & . & .
\end{array}\right]$$

$$A_2 = \begin{array}{c} c_0\\ c_1\\ c_2\\ c_3\\ c_4\\ c_5\\ d_0\\ d_1\\ d_2\\ d_3 \end{array} \left[\begin{array}{cccccccccc}
. & 1 & . & . & . & 1 & . & . & 1 & .\\
1 & . & 1 & . & . & . & . & 1 & . & .\\
. & 1 & . & 1 & . & . & 1 & . & . & .\\
. & . & 1 & . & 1 & . & . & . & 1 & .\\
. & . & . & 1 & . & 1 & . & 1 & . & .\\
1 & . & . & . & 1 & . & 1 & . & . & .\\
. & . & 1 & . & . & 1 & . & . & . & 1\\
. & 1 & . & . & 1 & . & . & . & . & 1\\
1 & . & . & 1 & . & . & . & . & . & 1\\
. & . & . & . & . & . & 1 & 1 & 1 & .
\end{array}\right] = \begin{array}{c} c_0\\ d_2\\ c_3\\ c_2\\ c_1\\ c_5\\ d_3\\ c_4\\ d_0\\ d_1 \end{array} \left[\begin{array}{cccccccccc}
. & 1 & . & . & 1 & 1 & . & . & . & .\\
1 & . & 1 & . & . & . & 1 & . & . & .\\
. & 1 & . & 1 & . & . & . & 1 & . & .\\
. & . & 1 & . & 1 & . & . & . & 1 & .\\
1 & . & . & 1 & . & . & . & . & . & 1\\
1 & . & . & . & . & . & . & 1 & 1 & .\\
. & 1 & . & . & . & . & . & . & 1 & 1\\
. & . & 1 & . & . & 1 & . & . & . & 1\\
. & . & . & 1 & . & 1 & 1 & . & . & .\\
. & . & . & . & 1 & . & 1 & 1 & . & .
\end{array}\right]$$

see also http://isu.indstate.edu/ge/Graphs/PETE/pete.gif

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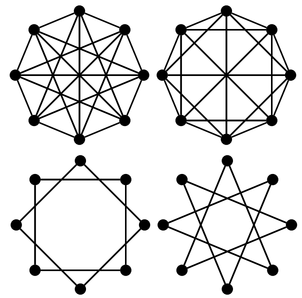
$$A_3 = \begin{array}{c} e_0\\ e_1\\ e_2\\ e_3\\ e_4\\ e_5\\ e_6\\ e_7\\ e_8\\ e_9 \end{array} \left[\begin{array}{cccccccccc}
. & 1 & . & . & 1 & . & . & . & 1 & .\\
1 & . & 1 & . & . & . & 1 & . & . & .\\
. & 1 & . & 1 & . & . & . & . & . & 1\\
. & . & 1 & . & 1 & . & . & 1 & . & .\\
1 & . & . & 1 & . & 1 & . & . & . & .\\
. & . & . & . & 1 & . & 1 & . & . & 1\\
. & 1 & . & . & . & 1 & . & 1 & . & .\\
. & . & . & 1 & . & . & 1 & . & 1 & .\\
1 & . & . & . & . & . & . & 1 & . & 1\\
. & . & 1 & . & . & 1 & . & . & 1 & .
\end{array}\right] = \begin{array}{c} e_0\\ e_1\\ e_2\\ e_3\\ e_4\\ e_8\\ e_6\\ e_9\\ e_7\\ e_5 \end{array} \left[\begin{array}{cccccccccc}
. & 1 & . & . & 1 & 1 & . & . & . & .\\
1 & . & 1 & . & . & . & 1 & . & . & .\\
. & 1 & . & 1 & . & . & . & 1 & . & .\\
. & . & 1 & . & 1 & . & . & . & 1 & .\\
1 & . & . & 1 & . & . & . & . & . & 1\\
1 & . & . & . & . & . & . & 1 & 1 & .\\
. & 1 & . & . & . & . & . & . & 1 & 1\\
. & . & 1 & . & . & 1 & . & . & . & 1\\
. & . & . & 1 & . & 1 & 1 & . & . & .\\
. & . & . & . & 1 & . & 1 & 1 & . & .
\end{array}\right]$$

f)



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Let us take a look at the complements of these graphs—the missing edges to a complete graph K_8.

The complement of the first graph consists of two circuits of length 4 while the complement of the second graph is a circuit of length 8. Thus the complements are not isomorphic and so are the graphs.