UNIVERSITY OF LONDON 291 0316 ZB

BSc EXAMINATION
for External Students

COMPUTING AND INFORMATION SYSTEMS AND CREATIVE COMPUTING

2910316 Mathematical Techniques of Operational Research

Eastern Zone

Duration: 2 hours 15 minutes

Date and Time: Friday 2 May 2008 : 2.30 – 4.45 pm

There are **FIVE** questions on this paper.

Answer **FOUR** questions only.

There are 100 marks available on this paper.

A hand held calculator may be used when answering questions on this paper but it must not be pre-programmed or able to display graphics, text or algebraic equations. The make and type of machine must be stated clearly on the front cover of the answer book.

GRAPH PAPER is required for this examination.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Q1. A company produces dry fertilizer to sell in 2 kilogram bags for use in small gardens. There are three types of bag, B_1, B_2, B_3 each containing a different combination of three basic components C_1, C_2, C_3. The given table below shows the amount, in kilograms, of each basic component required to produce a bag of fertilizer, together with the expected profit on the sale of each bag.

	B_1	B_2	B_3
C_1	0.6	0.6	0.8
C_2	0.8	0.4	1.2
C_3	0.6	1.0	0
Profit	\$1.2	\$0.9	\$1.5

Each week the company is guaranteed the following amounts of each basic component as follows:-

24000 kilograms of C_1 , 27000 kilograms of C_2, 21000 kilograms of C_3.

To find the production schedule, which will gain the maximum weekly profit, the company considers the following linear programming problem:-

Find $x_1, x_2, x_3 \;\varepsilon\; \mathbb{R}$ to maximize $z = 1.2x_1 + 0.9x_2 + 1.5x_3$ subject to:-

$$
\begin{aligned}
0.6x_1 + 0.6x_2 + 0.8x_3 &\leq 24000 \qquad (1)\\
0.8x_1 + 0.4x_2 + 1.2x_3 &\leq 27000 \qquad (2)\\
0.6x_1 + 1.0x_2 + \;\; 0x_3 &\leq 21000 \qquad (3)\\
x_1, x_2, x_3 &\geq 0 \qquad\qquad (4)
\end{aligned}
$$

(a) For the linear programming problem above [3]

  (i) define the decision variables x_1, x_2, x_3,

  (ii) explain how the objective function z is derived.

(b) Construct the initial tableau for the solution of this problem by the Simplex algorithm and state the augmented initial basic feasible solution $(x_1, x_2, x_3, x_4, x_5, x_6)^{\mathrm{T}}$. [3]

[Do not perform any iterations on this initial tableau.]

(question continues on next page)

(c) Applications of the Simplex algorithm lead to the following tableau in which x_4, x_5, x_6 are the slack variables.

Eqn	z	x_1	x_2	x_3	x_4	x_5	x_6	*RS*
0	1	0	0	0	$3/10$	$21/20$	$3/10$	41850
1	0	0	1	0	$9/2$	-3	$-1/2$	16500
2	0	0	0	1	$7/2$	$-3/2$	$-3/2$	12000
3	0	1	0	0	$-15/2$	5	$5/2$	7500

(i) Obtain the objective function z given by the above tableau, and use your expression for z to explain why the solution given by the above tableau is optimal. [3]

(ii) State the optimal solution and interpret it as a production schedule for the company. Also, give the expected profit to be gained from this schedule. [3]

(iii) Find if any of the available fertilizer remains unused at the end of a week's optimal production. [2]

(d) (i) Suppose the profit on the sale of bag B_2 is increased from \$0.9 to \$(0.9+w), where $w > 0$.
Show that the new objective function can be written as $z_0 = z + wx_2$ and find the new optimal profit. [4]

(ii) By changing the x_2 column of the given tableau in part (c) from $(0,1,0,0)^{\mathrm{T}}$ to $(-w,1,0,0)^{\mathrm{T}}$ and completing the required iteration, find the range of values of w for which the resulting solution remains optimal. [7]

Find the maximum possible profit.

Q2. Two players A and B, play a zero sum game in which A has strategies A_1, A_2, A_3, A_4, and B has strategies B_1, B_2, B_3, B_4, B_5. The payoff to A when A plays strategy A_i and B plays strategy B_j is given by the element (d_{ij}) in the following game matrix D_1.

	B_1	B_2	B_3	B_4	B_5
A_1	1	1	1	4	5
A_2	−5	−1	2	−5	1
A_3	2	1.5	1	−2	1
A_4	1	1	0	3	6

(a) For this game, find [3]

(i) the safest single strategy for each player,
(ii) the guaranteed minimum gain $\underline{v}$ to A,
(iii) the guaranteed maximum loss $\overline{v}$ to B.

(b) (i) State both sets of inequalities that are used to see if one strategy dominates another strategy in any payoff matrix for a zero sum game between two players. [4]

(ii) Explain how the matrix D_1 in part (a) may be reduced to the

$$\text{matrix } D_2 = \begin{bmatrix} 1 & 1 & 1 & 4 \\ -5 & -1 & 2 & -5 \\ 2 & 1.5 & 1 & -2 \end{bmatrix}$$

(c) Using the reduced matrix D_2 of part (b) suppose that A plays a mixed strategy $\mathbf{p} \in \mathbb{R}^3$ with a minimum expected gain of v_2 against each of B's strategies, while B plays a mixed strategy $\mathbf{q} \in \mathbb{R}^4$ with a maximum expected loss of v_1 against each of A's strategies. Describe how the problem, of finding the optimal values of $\mathbf{p}$ and $\mathbf{q}$, can be formulated as a pair of dual linear programming problems in new variables $\mathbf{y}$ and $\mathbf{x}$. [7]

(question continues on next page)

(d) Using the payoff matrix D_2, construct the initial tableau to solve the reduced problem by the Simplex method. [4]
[Do not perform any iterations on this initial tableau.]

(e) The final tableau, for the solution of this linear programming formulation of B's problem in the reduced game, is given below, where x_5, x_6 and x_7 are the slack variables. [7]

Eqn	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	*RS*
0	1	0	$3/110$	0	0	$28/55$	$3/55$	$21/55$	$52/55$
1	0	0	$-1/110$	0	1	$9/55$	$-1/55$	$-7/55$	$1/55$
2	0	0	$13/22$	1	0	$4/11$	$2/11$	$3/11$	$9/11$
3	0	1	$49/110$	0	0	$-1/55$	$-6/55$	$13/55$	$6/55$

(i) Use the above tableau to find $\mathbf{p}$ and $\mathbf{q}$.
State the value of the game.

(ii) Suppose the reduced game is played 1612 times using the mixed strategies given by $\mathbf{p}$ and $\mathbf{q}$.
How many times is player A expected to choose A_2?
How many times is player B expected to choose B_1?
What is the maximum total number of points that B could possibly win by choosing B_1 during the 1612 games?

Q3.(a) Consider the following linear programming problem **P**.

Find $x_1, x_2, x_3 \ \varepsilon\ \mathbb{R}$ to maximize $z = 20x_1 - 17x_2 - 9x_3$
subject to :-

$$\begin{aligned} 6x_1 - 5x_2 - 3x_3 &\le 91 \qquad (1) \\ -4x_1 + 4x_2 - 1x_3 &\ge 60 \qquad (2) \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

(i) By introducing, ***and identifying***, slack, surplus and artificial variables, prepare the constraints (1) and (2) for the solution of **P** by the Simplex algorithm. [3]

(ii) Using the equation $z_0 = z - M\bar{x}_6$, modify the objective function for the solution of **P** by the Big M method. [3]

(iii) Complete the solution of **P** by the Big M method. [7]
State the values of x_1, x_2, x_3 and z in the optimal solution to **P**.
(All entering and leaving variables must be identified.)

(b)(i) Use a graphical method to find both the minimum and maximum value of $z = 2x - 3y$ [10]
subject to:-

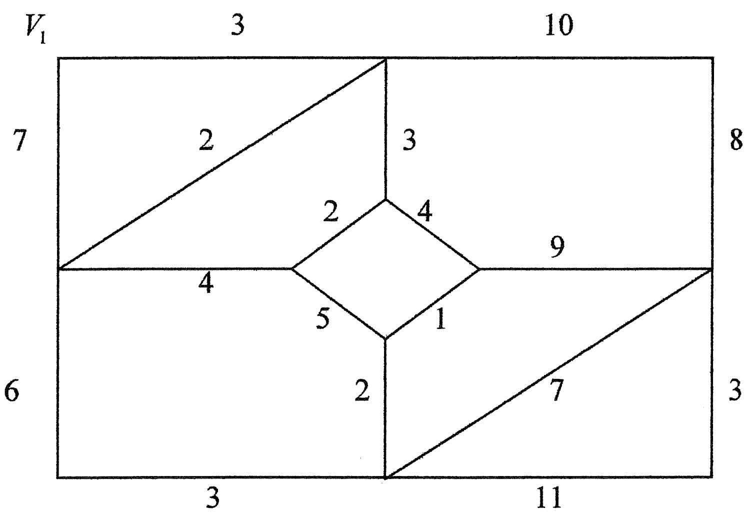
$$\begin{aligned} 2y - x &\le 15 \\ 17 \le x + 2y &\le 29 \\ 5x - 2y &\le 49 \\ x, y &\ge 0 \end{aligned}$$

Graph paper must be used with a scale of 10 mm to 1 unit for both axes.
A value line $z = 6$ should be drawn, and the feasible region clearly identified by lettering the vertices of the region A,B,C and D.
(Note: The correct graphical work will be contained in the first quadrant by marking each axis from -2 to +16.)

(ii) Find the minimum and maximum value of $z = 3x + 2y$ subject to the same constraints in part (b)(i) above. [2]

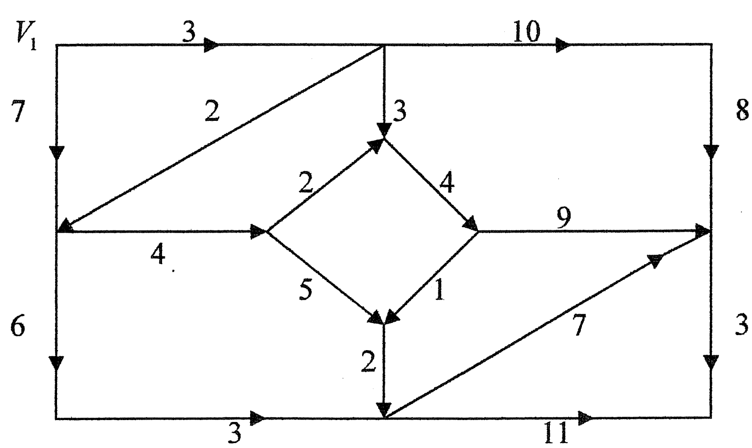
Q4.(a)(i) Describe the three steps of Prim's algorithm for finding a minimum weight spanning tree of a connected network N. [5]

(ii) Starting from the vertex V_1 and using Prim's algorithm, obtain the minimum weight spanning tree for the network given below. [6]
Letter each vertex $V_2, V_3, \ldots, V_{12}$ **in the order found** by the algorithm.
State the weight of the tree.

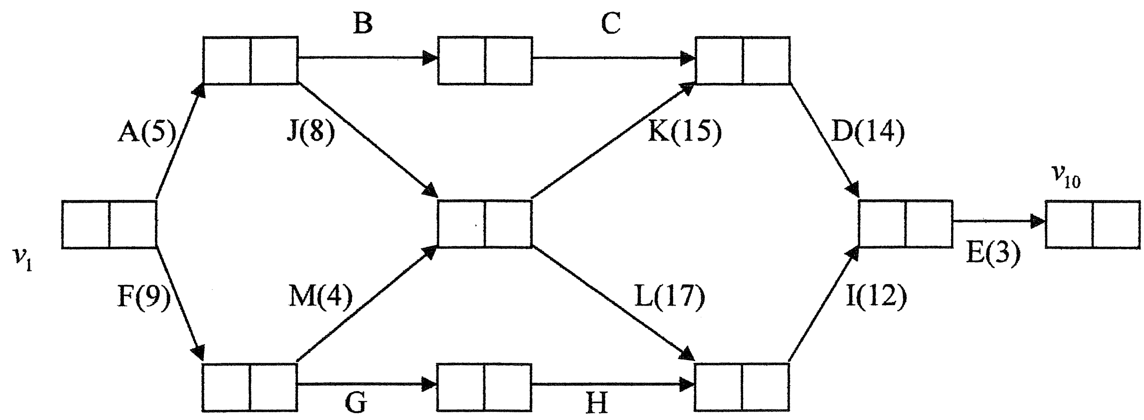


(b)(i) An acyclic directed network N has n vertices with an acyclic ordering $V_1, V_2, \ldots, V_n$. State an algorithm for finding the **shortest distance** from V_1 to every other vertex in the network N. [8]

(ii) In the figure below, starting from the vertex V_1, give the network an acyclic labelling $V_1, V_2, \ldots, V_{12}$ and draw the tree of shortest directed paths from the vertex V_1 to all the other vertices, showing all distances. [6]



Q5.(a) The C.P.A.-network below is the initial attempt to create a sequence of activities required to refit a shop space in 45 hours.
At this stage of planning the activities B,C,G and H are unknown but will not be allowed to affect the nine main activities A,D,E,F,I,J,K,L and M.



Each arc of the network is lettered to denote an activity and show the number of hours required to complete that activity using only one worker. Activities A and F both start together when the refit begins.

(i) Define the early event time ET(v) and late event time LT(v) for an event v in a C.P.A.-network. [2]

(ii) Draw the C.P.A.-network for the refit and give your network an acyclic labelling $v_1, v_2, ..., v_{10}$ where v_1 represents the starting event of the work. Also insert the ET(v) and LT(v) for each event without knowing B,C,G,H. Show that the shortest completion time is the required 45 hours and indicate any possible critical paths. [7]

(iii) Find the maximum number of hours x, y such that B+C $\leq x$ and G+H $\leq y$, without affecting the shortest possible time of 45 hours. [2]

(iv) Suppose that each activity B,C,G and H can be given 6 hours to complete. [7]

Show that the complete refit would need three workers W_1, W_2, W_3; with W_3 working only on activities B,C,G and H; in order to complete the refit in the shortest possible time of 45 hours.

State which activities and how many hours would be done by each worker. Also state which day of the refit, worker W_3 would start and finish each of the activities B,C,G and H.

(question continues on next page)

(b) Draw the C.P.A.-network for the following project with the activities B,A and C starting from the same event at the beginning of the project. [7]

The network should be drawn with only ***seven*** events and using ***two and only two*** dummy activities to show that the shortest completion time for the project is 20 hours.

Activities	*Duration(hours)*	*PrecedingActivities*
B	9	–
A	7	–
C	12	–
D	3	A
E	6	B,D
F	5	C,D
G	3	E,F

Give the network an acyclic labelling $v_1, v_2, v_3, v_4, v_5, v_6, v_7$.
Also enter every ET(v) and LT(v).

END OF PAPER