1 Solving Linear Recurrences

It is straightforward to verify that the recurrence

$$\mathbf{x}_{n+1} = A\mathbf{x}_n + \mathbf{f}_n, \qquad \mathbf{x}_0 \text{ given} \tag{1}$$

where $\mathbf{x}_n$ is a m-vector and A a $m \times m$ constant matrix, has solution

$$\mathbf{x}_n = A^n\mathbf{x}_0 + \sum_{j=0}^{n-1} A^{n-1-j}\mathbf{f}_j \tag{2}$$

As it stands, this formula gives limited information about the nature of A^n. To get a better handle on this, we need to consider the eigenstructure of the matrix.

2 Eigenvalues and Eigenvectors

Let A be a $m \times m$ matrix whose elements are members of the field K, ($K = \mathbf{R}$ or $\mathbf{C}$), $\lambda \in K$ and $\mathbf{e} \neq \mathbf{0}$ a m-vector such that

$$A\mathbf{e} = \lambda\mathbf{e} \tag{3}$$

then λ is an *eigenvalue* of A, and $\mathbf{e}$ a corresponding eigenvector.
From Eq(3)

$$(\lambda I - A)\mathbf{e} = \mathbf{0}$$

and so λ must satisfy the CHARACTERISTIC equation:

$$det(\lambda I - A) = 0 \tag{4}$$

It can be shown that

$$det(\lambda I - A) = \lambda^m + a_{m-1}\lambda^{m-1} + \ldots + a_1\lambda + a_0 = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_m) \tag{5}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of A. This is called the characteristic polynomial of A.
Note: In theory, Eq(4) can be used to find λ 's – than Eq(3) used to find the corresponding $\mathbf{e}$'s.
Example:

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

$$\begin{aligned} det(\lambda I - A) &= det\begin{pmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{pmatrix} \\ &= \lambda^2 + 3\lambda + 2 \\ &= 0 \;\Rightarrow\; \lambda = -1 \text{ or } -2 \end{aligned}$$

For $\lambda_1 = -1$

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\mathbf{e}_1 = (-1)\mathbf{e}_1 \;\Rightarrow\; \mathbf{e}_1 = \alpha_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \alpha_1 \in K$$

For $\lambda_2 = -2$

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\mathbf{e}_2 = (-2)\mathbf{e}_2 \;\Rightarrow\; \mathbf{e}_2 = \alpha_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \alpha_2 \in K$$

3 Similarity

Two $m \times m$ matrices R and S are said to be *similar* if there exist an *invertible* matrix P such that

$$R = P^{-1}SP$$

Similar matrices have the same *spectrum* (set of eigenvalues):

$$\begin{aligned}
det(\lambda I - R) &= det(\lambda I - P^{-1}SP) \\
&= det(\lambda P^{-1}P - P^{-1}SP) \\
&= det\left(P^{-1}(\lambda I - S)P\right) \\
&= detP^{-1}det(\lambda I - S)detP \\
&= det(\lambda I - S)
\end{aligned}$$

4 Diagonalisability

Recall that a diagonal matrix is a $m \times m$ matrix, all of whose off-diagonal entries are zero. We denote a diagonal matrix by $diag\{d_1, d_2, \ldots, d_m\}$ where $d_1, d_2, \ldots, d_m$ are the diagonal entries.

The $m \times m$ matrix A is said to be *diagonalisable* if it is similar to a diagonal matrix. It can be shown that the diagonal entries of the diagonal matrix are the eigenvalues of A.

Not every square matrix is diagonalisable. A necessary and sufficient condition for A to be diagonalisable is that its eigenvectors form a *linearly independent* set. Let A have spectrum $\lambda_1, \lambda_2, \ldots, \lambda_m$ with associated eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ respectively. Then, for $i = 1, 2, \ldots, m$ we have $A\mathbf{e}_i = \lambda_i \mathbf{e}_i$. Consider

$$\begin{aligned}
A\,E = A\,[\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m] &= [A\mathbf{e}_1, A\mathbf{e}_2, \ldots, A\mathbf{e}_m] \\
&= [\lambda_1\mathbf{e}_1, \lambda_2\mathbf{e}_2, \ldots, \lambda_m\mathbf{e}_m] \\
&= [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m]\, diag\{\lambda_1, \lambda_2, \ldots, \lambda_m\} \\
&= E\,\Lambda
\end{aligned}$$

where $E = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m]$ and $\Lambda = diag\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$.
Thus, if A is diagonalisable

$$A = E\,\Lambda\,E^{-1} \tag{6}$$

Λ is unique up to ordering of the eigenvalues.

We note that it can be shown that eigenvectors corresponding to distinct eigenvalues are linearly independent; hence, if A has m distinct eigenvalues, it is diagonalisable.
Example (Cont'd): $A = E\,\Lambda\,E^{-1}$

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

We also note that the following corollary of Eq(6) which may be proven using induction

$$A^n = E\,\Lambda^n\,E^{-1} \qquad n = 0, 1, \ldots \tag{7}$$

where it is also straightforward to show that

$$\Lambda^n = diag\{\lambda_1^n, \lambda_2^n, \ldots, \lambda_m^n\} \tag{8}$$

Example(Cont'd): $A^n = E\,\Lambda^n\,E^{-1}$

$$\begin{aligned}
\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}^n &= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & (-2)^n \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2(-1)^n - (-2)^n & (-1)^n - (-2)^n \\ -2(-1)^n + 2(-2)^n & -(-1)^n + 2(-2)^n \end{pmatrix}
\end{aligned}$$

Thus the solution of

$$\mathbf{x}_{n+1} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\mathbf{x}_n, \qquad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is given by

$$\begin{aligned}
\mathbf{x}_n &= \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 2(-1)^n - (-2)^n \\ -2(-1)^n + 2(-2)^n \end{pmatrix}
\end{aligned}$$

5 Jordan Canonical Form

Although every square matrix is not diagonalisable, it is possible to show that every matrix A is similar to a Jordan Form matrix J, i.e.

$$A = P^{-1}\,J\,P$$

where J is a block diagonal matrix

$$J = diag\{J_1, J_2, \ldots, J_s\} = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J_s \end{pmatrix}$$

with each block J_i being of size $m_i \times m_i$ with $\sum m_i = m$ and of form

$$J_i = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

where λ belongs to the spectrum of A. (The same eigenvalue may appear in more than one block of J). The Jordan Form of A is unique up to the ordering of the blocks. If A is diagonalisable, then its Jordan Form coincides with its diagonalised form.

It is straightforward to establish that

$$A^n = P^{-1}\,J^n\,P \tag{9}$$

where $J^n = diag\{J_1^n, J_2^n, \ldots, J_s^n\}$ and

$$J_i^n = \begin{pmatrix} \lambda^n & c_n(1)\lambda^{n-1} & c_n(2)\lambda^{n-2} & \cdots & c_n(m_i-1)\lambda^{n-m_i-1} \\ 0 & \lambda^n & c_n(1)\lambda^{n-1} & \cdots & c_n(m_i-2)\lambda^{n-m_i-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^n \end{pmatrix} \tag{10}$$

where $c_n(j) = \binom{n}{j}$. Example:

$$\hat{A} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$$

has eigenvalue $\lambda = -1$ (multiplicity 2) and associated eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence it is not diagonalisable since there are not two linear independent eigenvectors. However ($\hat{A} = P^{-1}\, J\, P$)

$$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and ($\hat{A}^n = P^{-1}\, J^n\, P$)

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}^n &= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} (-1)^n & n(-1)^{n-1} \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} (-1)^n + n(-1)^{n-1} & n(-1)^{n-1} \\ -n(-1)^{n-1} & (-1)^n - n(-1)^{n-1} \end{pmatrix} \end{aligned}$$

6 The Discrete Putzer Algorithm

The Cayley-Hamilton Theorem states that every square matrix obeys its own characteristic equation, i.e.

$$A^m + a_{m-1}A^{m-1} + \ldots + a_1 A + a_0 I = \mathbf{0} = (A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_m I)$$

Example: $\hat{A}$ in the previous example has characteristic polynomial $\lambda^2 + 2\lambda + 1$. Hence

$$\hat{A}^2 + 2\hat{A} + I = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -2 & -4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

This relationship is the basis of a number of methods for calculating A^n, none of which require knowledge of the Jordan Form of A. One of these is the Discrete Putzer Algorithm. Define the sequence of matrices M_i by

$$M_0 = I \qquad M_{i+1} = (A - \lambda_{i+1} I) M_i, \quad i = 0, 1, \ldots, m-1 \tag{11}$$

and the scalar sequences $\{s_i(n)\}$ by

$$s_1(n) = \lambda_1^n \tag{12}$$

$$s_i(n) = \sum_{j=0}^{n-1} \lambda_i^{n-1-j} s_{i-1}(j), \quad i = 2, 3, \ldots, m \tag{13}$$

Then it may be shown that

$$A^n = \sum_{i=1}^{m} s_i(n) M_{i-1} \tag{14}$$

Example: Again for $\hat{A}$, $\lambda_1 = -1, \lambda_2 = -1$ and so

$$\begin{aligned} M_0 &= I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ M_1 &= (\hat{A} - \lambda_1 I) I = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \\ s_1(n) &= \lambda_1^n = (-1)^n \\ s_2(n) &= \sum_{j=0}^{n-1} \lambda_2^{n-1-j} s_1(j) = \sum_{j=0}^{n-1} (-1)^{n-1-j}(-1)^j = n(-1)^{n-1} \end{aligned}$$

Hence

$$\begin{aligned} \hat{A}^n &= \sum_{i=1}^{2} s_i(n) M_{i-1} = s_1(n) M_0 + s_2(n) M_1 \\ &= (-1)^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + n(-1)^{n-1} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} (-1)^n + n(-1)^{n-1} & n(-1)^{n-1} \\ -n(-1)^{n-1} & (-1)^n - n(-1)^{n-1} \end{pmatrix} \end{aligned}$$