University of Limerick
Department of Mathematics and Statistics
Dr. S. Franz

OLLSCOIL LUIMNIGH
UNIVERSITY OF LIMERICK

MA4016 - Engineering Mathematics 6

Problem Sheet 3: Recursion (February 19, 2010)

1. Determine whether each of these proposed definitions is a valid recursive definition of a function f from the set of non-negative integers to the set of integers. If f is well defined, find a formula for $f(n)$ when n is an non-negative integer and prove that your formula is valid.

a) $f(0) = 1,\ f(n) = -f(n-1)$ for $n \geq 1$

b) $f(0) = 1,\ f(1) = 0,\ f(2) = 2,\ f(n) = 2f(n-3)$ for $n \geq 3$

c) $f(0) = 0,\ f(1) = 1,\ f(n) = 2f(n+1)$ for $n \geq 2$

d) $f(0) = 0,\ f(1) = 1,\ f(n) = 2f(n-1)$ for $n \geq 1$

e) $f(0) = 2,\ f(n) = \begin{cases} f(n-1) & \text{for } n \geq 1 \text{ and } n \text{ is odd} \\ 2f(n-2) & \text{for } n \geq 2 \text{ and } n \text{ is even} \end{cases}$

2. a) A robot can take steps of 1 meter, 2 meters, or 3 meters. Write a recursive algorithm to calculate the number of ways the robot can walk n meters.

b) Give a proof using mathematical induction that your algorithm for part **a)** is correct.

c) What is the complexity of this algorithm in terms of additions?

In the following exercises f_n is the nth Fibonacci number ($f_0 = 0,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$).

3. Show that

a)
$$f_{n+2}^2 - f_{n+1}^2 = f_n f_{n+3} \quad \text{for all } n \geq 1$$

b)
$$f_n^2 = f_{n-1} f_{n+1} + (-1)^{n+1} \quad \text{for all } n \geq 2$$

c)
$$f_n^2 = f_{n-2} f_{n+2} + (-1)^n \quad \text{for all } n \geq 3$$

d)
$$\sum_{k=1}^{n} f_k^2 = f_n f_{n+1} \quad \text{for all } n \geq 1$$

4. Show that the number of ways to tile a $2 \times n$ board with 1×2 rectangular pieces is f_{n+1}.