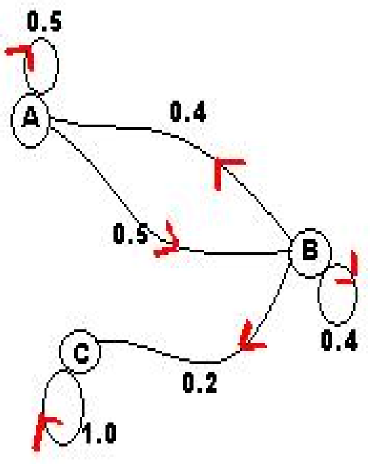
$\mathrm{Prob}\{T = k\}$ represents the probability that the procedure ends at time k exactly.

1.

(a)

Figure 1: The 3-state Markov Chain of Question 1



(b)

$$\begin{aligned}
p_A(k+1) &= 0.5p_A(k) + 0.4p_B(k), \qquad p_A(0) = 1 &(1)\\
p_B(k+1) &= 0.5p_A(k) + 0.4p_B(k), \qquad p_B(0) = 0 &(2)\\
p_C(k+1) &= 0.2p_B(k) + p_C(k), \qquad p_C(0) = 0
\end{aligned}$$

In addition we have

$$p_A(k) + p_B(k) + p_C(k) \equiv 1 \tag{3}$$

and

$$\mathrm{Prob}\{T = k\} = 0.2p_B(k-1), \qquad \mathrm{Prob}\{T = 0\} = 0 \tag{4}$$

(c) From Equations (1) and (2), we have $p_A(k) = p_B(k), k \geq 1$. Also $p_A(1) = p_B(1) = 0.5$. Hence we may rewrite Equations (1) and (2) as

$$\begin{aligned}
p_A(k+1) &= 0.9p_A(k), \qquad p_A(1) = 0.5\\
p_B(k+1) &= 0.9p_B(k), \qquad p_B(1) = 0.5
\end{aligned}$$

which have solutions $p_A(k) = p_B(k) = 0.5(0.9)^{k-1}, \;\; k \geq 1$. Thus we have

$$p_A(k) = \begin{cases} 1, & \text{if } k = 0;\\ 0.5(0.9)^{k-1}, & \text{if } k \geq 1.\end{cases}$$

$$p_B(k) = \begin{cases} 0, & \text{if } k = 0;\\ 0.5(0.9)^{k-1}, & \text{if } k \geq 1.\end{cases}$$

Using Equation (3), we get

$$p_C(k) = \begin{cases} 0, & \text{if } k = 0;\\ 1 - (0.9)^{k-1}, & \text{if } k \geq 1.\end{cases}$$

and from Equation (4)

$$\mathrm{Prob}\{T = k\} = \begin{cases} 0, & \text{if } k = 0, 1;\\ 0.5(0.9)^{k-2}, & \text{if } k \geq 2.\end{cases}$$

(d) The expected number of iterations before the procedure ends is

$$\begin{aligned}
\sum_{k=0}^{\infty} k\text{Prob}\{T=k\} &= \sum_{k=2}^{\infty} k\ 0.1(0.9)^{k-2} \\
&= 0.1\left[\sum_{k=2}^{\infty}(k-1)(0.9)^{k-2} + \sum_{k=2}^{\infty}(0.9)^{k-2}\right] \\
&= 0.1[(1-0.9)^{-2} + (1-0.9)^{-1}] = 11
\end{aligned}$$

where we have used $\sum_{n=0}^{\infty} x^n = (1-x)^{-1}$ and $\sum_{n=0}^{\infty} nx^{n-1} = (1-x)^{-2}$ whenever $|x| < 1$.

2.

$$\begin{aligned}
p_A(k+1) &= 0.4p_A(k) + 0.8p_B(k), \qquad p_A(0) = 1 \qquad (5)\\
p_B(k+1) &= 0.4p_A(k), \qquad p_B(0) = 0 \qquad (6)\\
p_F(k+1) &= 0.2p_A(k) + 0.2p_B(k) + p_F(k), \qquad p_F(0) = 0
\end{aligned}$$

In addition we have

$$p_A(k) + p_B(k) + p_F(k) \equiv 1 \qquad (7)$$

and

$$\text{Prob}\{T=k\} = 0.2p_A(k-1) + 0.2p_B(k-1) \qquad (8)$$

We can rewrite equations (5) and (6) as a system of equations whose solution is

$$\begin{pmatrix} p_A(k) \\ p_B(k) \end{pmatrix} = \begin{pmatrix} 0.4 & 0.8 \\ 0.4 & 0.0 \end{pmatrix}^k \begin{pmatrix} 1.0 \\ 0.0 \end{pmatrix} = Q^k \mathbf{V}_0^{Tr}$$

This matrix has characteristic polynomial $\lambda^2 - 0.4\lambda - 0.32 \Rightarrow \lambda = 0.8,\ -0.4$ with associated eigenvalues $(2,1)^T$ and $(1,-1)$ respectively. It is diagonalisable and

$$\begin{aligned}
\begin{pmatrix} 0.4 & 0.8 \\ 0.4 & 0.0 \end{pmatrix}^k &= \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} (0.8)^k & 0 \\ 0 & (-0.4)^k \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix} \\
&= \begin{pmatrix} \frac{2}{3}(0.8)^k + \frac{1}{3}(-0.4)^k & \frac{2}{3}(0.8)^k - \frac{2}{3}(-0.4)^k \\ \frac{1}{3}(0.8)^k - \frac{1}{3}(-0.4)^k & \frac{1}{3}(0.8)^k + \frac{2}{3}(-0.4)^k \end{pmatrix}
\end{aligned}$$

Therefore

$$\begin{pmatrix} p_A(k) \\ p_B(k) \end{pmatrix} = \begin{pmatrix} \frac{2}{3}(0.8)^k + \frac{1}{3}(-0.4)^k \\ \frac{1}{3}(0.8)^k - \frac{1}{3}(-0.4)^k \end{pmatrix}$$

and from equation (7)

$$p_F(k) = 1 - p_A(k) - p_B(k) = 1 - (0.8)^k$$

Also from equation (8),

$$\text{Prob}\{T=k\} = \begin{cases} 0, & \text{if } k = 0; \\ 0.2(0.8)^{k-1}, & \text{otherwise.} \end{cases}$$

The expected number of iterations before the procedure ends is therefore

$$\begin{aligned}
\sum_{k=0}^{\infty} k\text{Prob}\{T=k\} &= \sum_{k=0}^{\infty} k\ 0.2(0.8)^{k-1} \\
&= 0.2(1-0.8)^{-2} = 5
\end{aligned}$$

Alternatively

$$\begin{aligned}
p_F(k) &= \mathbf{r}(I-Q)^{-1}(I-Q^k)\mathbf{V}_0^{Tr} \\
&= \begin{pmatrix} 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} 25/7 & 20/7 \\ 10/7 & 15/7 \end{pmatrix} \begin{pmatrix} 1-\frac{2}{3}(0.8)^k - \frac{1}{3}(-0.4)^k & -\frac{2}{3}(0.8)^k + \frac{2}{3}(-0.4)^k \\ -\frac{1}{3}(0.8)^k + \frac{1}{3}(-0.4)^k & 1-\frac{1}{3}(0.8)^k - \frac{2}{3}(-0.4)^k \end{pmatrix} \begin{pmatrix} 1.0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1-\frac{2}{3}(0.8)^k - \frac{1}{3}(-0.4)^k \\ -\frac{1}{3}(0.8)^k + \frac{1}{3}(-0.4)^k \end{pmatrix} \\
&= 1 - (0.8)^k
\end{aligned}$$

and

$$
\begin{aligned}
E(T) &= \left[\mathbf{r}(I-Q)^{-1}\right]\left[(I-Q)^{-1}\mathbf{V}_0^{Tr}\right] \\
&= \begin{pmatrix} 1 & 1 \end{pmatrix}\left[\begin{pmatrix} 25/7 & 20/7 \\ 10/7 & 15/7 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \\
&= 5
\end{aligned}
$$

3. (a)

$$
\begin{aligned}
p_A(k+1) &= 0.6p_A(k) + 0.675p_B(k), \qquad p_A(0)=1 \qquad (9)\\
p_B(k+1) &= 0.4p_A(k), \qquad p_B(0)=0 \qquad (10)\\
p_F(k+1) &= 0.325p_B(k) + p_F(k), \qquad p_F(0)=0
\end{aligned}
$$

In addition we have

$$p_A(k) + p_B(k) + p_F(k) \equiv 1 \qquad (11)$$

(b) We can rewrite equations (9) and (10) as a system of equations whose solution is

$$\begin{pmatrix} p_A(k) \\ p_B(k) \end{pmatrix} = \begin{pmatrix} 0.6 & 0.675 \\ 0.4 & 0 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = Q^k\,\mathbf{V}_0^{Tr}$$

This matrix has characteristic polynomial $\lambda^2 - 0.6\lambda - 0.27 \Rightarrow \lambda = 0.9,\ -0.3$
We compute the k-th power of this matrix using the Discrete *Putzer* Algorithm:

$$
\begin{aligned}
M_0 = I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
M_1 = (Q-\lambda_1 I)M_0 = Q - 0.9I &= \begin{pmatrix} -0.3 & 0.675 \\ 0.4 & -0.9 \end{pmatrix} \\
s_1(k) = \lambda_1^k &= (0.9)^k \\
s_2(k) = \sum_{j=0}^{k-1} \lambda_2^{k-1-j} s_1(j) &= \sum_{j=0}^{k-1} (-0.3)^{k-1-j}(0.9)^j = \frac{1}{4}(-0.3)^{k-1}(1-(-3)^k)
\end{aligned}
$$

Hence

$$
\begin{aligned}
Q^k = \sum_{i=1}^{m} s_i(k)M_{i-1} &= s_1(k)M_0 + s_2(k)M_1 \\
&= (0.9)^k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4}(-0.3)^{k-1}(1-(-3)^k)\begin{pmatrix} -0.3 & 0.675 \\ 0.4 & -0.9 \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{4}(0.9)^k + \frac{1}{4}(-0.3)^k & \frac{9}{16}(0.9)^k - \frac{9}{16}(-0.3)^k \\ \frac{1}{3}(0.9)^k - \frac{1}{3}(-0.3)^k & \frac{1}{4}(0.9)^k + \frac{3}{4}(-0.3)^k \end{pmatrix}
\end{aligned}
$$

Therefore

$$\begin{pmatrix} p_A(k) \\ p_B(k) \end{pmatrix} = \begin{pmatrix} \frac{3}{4}(0.9)^k + \frac{1}{4}(-0.3)^k \\ \frac{1}{3}(0.9)^k - \frac{1}{3}(-0.3)^k \end{pmatrix}$$

and from equation (11)

$$p_F(k) = 1 - p_A(k) - p_B(k) = 1 - \frac{13}{12}(0.9)^k + \frac{1}{12}(-0.3)^k$$

Alternatively

$$
\begin{aligned}
p_F(k) &= \mathbf{r}(I-Q)^{-1}(I-Q^k)\mathbf{V}_0^{Tr} \\
&= \begin{pmatrix} 0.0 & 0.325 \end{pmatrix}\begin{pmatrix} \frac{100}{13} & \frac{135}{26} \\ \frac{40}{13} & \frac{40}{13} \end{pmatrix}\begin{pmatrix} 1-\frac{3}{4}(0.9)^k - \frac{1}{4}(-0.3)^k & -\frac{9}{16}(0.9)^k + \frac{9}{16}(-0.3)^k \\ -\frac{1}{3}(0.9)^k + \frac{1}{3}(-0.3)^k & 1-\frac{1}{4}(0.9)^k - \frac{3}{4}(-0.3)^k \end{pmatrix}\left(\begin{matrix} 1 \\ 0 \end{matrix}\right. \\
&= \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1-\frac{3}{4}(0.9)^k - \frac{1}{4}(-0.3)^k \\ -\frac{1}{3}(0.9)^k + \frac{1}{3}(-0.3)^k \end{pmatrix} \\
&= 1 - \frac{13}{12}(0.9)^k + \frac{1}{12}(-0.3)^k
\end{aligned}
$$

(c) We have that

$$
\begin{aligned}
\text{Prob}\{T=k\} &= 0.325 p_B(k-1) \\
&= \begin{cases} 0, & \text{if } k=0; \\ 0.325(\frac{1}{3}(0.9)^{k-1} - \frac{1}{3}(-0.3)^{k-1}), & \text{otherwise.} \end{cases}
\end{aligned}
$$

The expected number of iterations before the procedure ends is therefore

$$
\begin{aligned}
\sum_{k=0}^{\infty} k\text{Prob}\{T=k\} &= \sum_{k=0}^{\infty} k\ 0.325(\frac{1}{3}(0.9)^{k-1} - \frac{1}{3}(-0.3)^{k-1}) \\
&= 0.325 \times \frac{1}{3}\left((1-0.9)^{-2} - (1+0.3)^{-2}\right) = 140/13
\end{aligned}
$$

Again, alternatively

$$
\begin{aligned}
E(T) &= \left[\mathbf{r}(I-Q)^{-1}\right]\left[(I-Q)^{-1}\mathbf{V}_0^{Tr}\right] \\
&= \begin{pmatrix} 1 & 1 \end{pmatrix} \left[\begin{pmatrix} 100/13 & 135/26 \\ 40/13 & 40/13 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\
&= \frac{140}{13}
\end{aligned}
$$