MA4016 Solutions: Problem Sheet 9

1. It is not in (e), it is in all the others.

2. 
   - $AB = \{xy, xyx\}$
   - $BA = \{xy, xxy\}$
   - $B^3 = \{\lambda, x, xx, xxx\}$
   - $B^+ = \{\lambda, x, xx, xxx, \ldots\} = \{x^n : n \in \mathbf{N}_0\}$
   - $A^* = \{\lambda, xy, xyxy, xyxyxy, \ldots\} = \{(xy)^n : n \in \mathbf{N}_0\}$

3. Given that $A^2 = A$ it is straightforward to show that $A^n = A,\ n \geq 1$, using induction. Hence $A^+ = A$.
   Next we need to show that $\lambda \in A$. When $A \neq \emptyset$,

   (a) if $|A| = 1$, then $A^2 = A$ implies
   $$xx = x \Rightarrow \|xx\| = 2\|x\| = \|x\| \Rightarrow \|x\| = 0 \Rightarrow x = \lambda$$
   (b) if $|A| > 1$: let $x$ be a minimum length string in $A$. Now $A^2 = A$ implies
   $$x \in A^2 \Rightarrow x = yz \quad y, z \in A \Rightarrow \|x\| = \|y\| + \|z\|$$
   If $\|x\| \neq 0$, the above implies that one of $y$ or $z$ has length less than $x$, which is a contradiction. Hence $\|x\| = 0 \Rightarrow x = \lambda$.

   In either case, $\lambda \in A$. And so $A^* = A$.

4. $G = (\{\mathrm{S}, 0, 1\}, \{0, 1\}, \mathrm{S}, \{\mathrm{S} \to 0\mathrm{S}1, \mathrm{S} \to 0, \mathrm{S} \to 1\})$.
   $0001111 = 0(0[0\{1\}1]1)1 \in A$. $00001111 \notin A$ since every element of $A$ must have an odd number of bits.

5. (a) Base Step: $\lambda \in A$. Recursive Step: $x \in A \Rightarrow 1x, x1, 0x0, 00x, x00 \in A$.
   $G = (\{\mathrm{S}, 0, 1\}, \{0, 1\}, \mathrm{S}, \{\mathrm{S} \to \mathrm{S}1, \mathrm{S} \to 1\mathrm{S}, \mathrm{S} \to 0\mathrm{S}0, \mathrm{S} \to 00\mathrm{S}, \mathrm{S} \to \mathrm{S}00, \mathrm{S} \to \lambda\})$.
   (b) Base Step: $0 \in A$. Recursive Step: $x \in A \Rightarrow 1x, x1 \in A$.
   $G = (\{\mathrm{S}, 0, 1\}, \{0, 1\}, \mathrm{S}, \{\mathrm{S} \to \mathrm{S}1, \mathrm{S} \to 1\mathrm{S}, \mathrm{S} \to 0\})$.
   (c) Base Step: $\lambda \in A$. Recursive Step: $x \in A \Rightarrow 0x, x1 \in A$.
   $G = (\{\mathrm{S}, 0, 1\}, \{0, 1\}, \mathrm{S}, \{\mathrm{S} \to \mathrm{S}1, \mathrm{S} \to 0\mathrm{S}, \mathrm{S} \to \lambda\})$.

6. 

| Grammar | Type |
|---|---|
| (a) | 2 |
| (b) | 3 |
| (c) | 0 |
| (d) | 2 |
| (e) | 2 |
| (f) | 0 |
| (g) | 3 |
| (h) | 0 |
| (i) | 2 |
| (j) | 2 |