Chapter 1

Set theory

1.1 Set Theory

A set, in mathematics, is a collection of distinct entities, called its elements, considered as a whole. The early study of sets led to a family of paradoxes and apparent contradictions. It therefore became necessary to abandon "nave" conceptions of sets, and a precise definition that avoids the paradoxes turns out to be a tricky matter. However, some unproblematic examples from nave set theory will make the concept clearer. These examples will be used throughout this article:

- A = the set of the numbers 1, 2 and 3.
- B = the set of primary light coloursred, green and blue.
- C = the empty set (the set with no elements).
- D = the set of all books in the British Library.
- E = the set of all positive integers, 1, 2, 3, 4, and so on.

Note that the last of these sets is infinite.

A set is the collection of its elements considered as a single, abstract entity. Note that this is different from the elements themselves, and may have different properties. For example, the elements of D are flammable (they are books), but D itself is not flammable, since abstract objects cannot be burnt.

1.1.1 Notation - Listing Method

By convention, a set can be written by **listing** its elements, separated by commas, between braces.

Using the sets defined above:

- A = $\{1, 2, 3\}$
- B = $\{red, green, blue\}$
- C = $\{\}$

This is impractical for large sets (D), and impossible for infinite ones (E).

1.1.2 Notation - Building Method

Thus a set can also be described by naming a particular property that is shared by all its elements and only by them. A common notation uses a bar (—) to separate a variable name (e.g. "x") from a property of the variable that elements of the set must have. For example:

- D = $\{x|$ x is a book and x is in the British Library $\}$
- E = $\{x|$ x is a positive integer$\}$

A simple translation of this notation is that "*D is the set of all x, where x is a book and x is in the British Library*" or "*E is the set of all x, where x is a positive integer*".

1.2 Definition of a set

A set is defined completely by its elements. Formally, sets X and Y are the same set if they have the same elements; that is, if every element of X is also an element of Y, and vice versa. For example, suppose we define:

$$F = \{x|(\text{x is an integer) and } 0 < x < 4)\}$$

The equivalence of empty sets has a ***metaphysical*** consequence for some theories of the metaphysics of properties that define the property of being x as simply the set of all x, then if the two properties are uninstantiated or coextensive they are equivalent - under this theory, because there are no unicorns and there are no pixies, the property of being a unicorn and being a pixie are the same - but if there were a unicorns and pixies, we could tell them apart. (See Universals for more on this.)

1.2.1 Elements and subsets

The $\in$ sign indicates set membership. If x is an element (or "member") of a set X, we write $\in X$; e.g. $3 \in A$. (We may also say "X contains x" and "A contains 3")

A very important notion is that of a subset. X is a subset of Y, written $X \subseteq Y$ (sometimes simply as$X \subset Y$), if every element of X is also an element of Y. From before $C \subseteq A \subseteq E$.

1.2.2 Sets containing Sets

Sets can of course be elements of other sets; for example we can form the set $G = \{A, B, C, D, E\}$, whose five elements are the sets we considered earlier. Then, for instance, $A \in G$. (Note that this is very different from saying $A \subseteq G$)

1.2.3 Set operations

Suppose X and Y are sets. Various operations allow us to build new sets from them.

Union The union of X and Y, written $X \cup Y$, contains all the elements in X and all those in Y. Thus $A \cup B = \{1, 2, 3, red, green, blue\}$. As A is a subset of E, the set $A \cup E$ is just E.

Intersection The intersection of X and Y, written XY, contains all the elements that are common to both X and Y. Thus 1,2,3,red,green,blue 2,4,6,8,10 = 2.

Set difference The difference X minus Y, written XY or $X\|Y$, contains all those elements in X that are not also in Y. For example, EA contains all integers greater than 3. AB is just A; red, green and blue were not elements of A, so no difference is made by excluding them.

1.2.4 Important Operations in Set Theory

- Union ($\cup$) - also known as the OR operator. A union signifies a bringing together. The union of the sets A and B consists of the elements that are in either A or B.
- Intersection ($\cap$) - also known as the AND operator. An intersection is where two things meet. The intersection of the sets A and B consists of the elements that in both A and B.
- Complement (^{c}) - The complement of the set A consists of all of the elements in the universal set that are not elements of A.

2.a Describe the following set by the listing method

$$\{2r + 1 : r \in Z^{+} and r \leq 5\}$$

2.b Let A,B be subsets of the universal set U.

3.a Let n be an element of the set $\{10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$, and p and q be the propositions: p : n is even, $q : n > 15$. Draw up truth tables for the following statements and find the values of n for which they are true: (i) $p \vee \neg q$ (ii) $\neg p \wedge q$

1.2.5 Universal Set and the Empty Set

The first is the ***universal set***, typically denoted U. This set is all of the elements that we may choose from. This set may be different from one setting to the next.

For example one universal set may be the set of real numbers whereas for another problem the universal set may be the whole numbers $\{0, 1, 2, \ldots\}$.

The other set that requires consideration is called the ***empty set***. The empty set is the unique set is the set with no elements. We write this as $\{\}$ and denote this set by $\emptyset$.

1.3 Number Sets

The font that the symbols are written in (i.e. $\mathbb{N}$, $\mathbb{R}$) is known as ***blackboard font***.

- $\mathbb{N}$ Natural Numbers $(0, 1, 2, 3)$ (Not used in the CIS102 Syllabus)
- $\mathbb{Z}$ Integers $(-3, -2, -1, 0, 1, 2, 3, \ldots)$
  - $\mathbb{Z}^{+}$ Positive Integers
  - $\mathbb{Z}^{-}$ Negative Integers
- $\mathbb{Q}$ Rational Numbers
- $\mathbb{R}$ Real Numbers

(a)

(b)

(c)

1.3.1 Complement and universal set

The universal set (if it exists), usually denoted U, is a set of which everything conceivable is a member. In pure set theory, normally sets are the only objects considered (unlike here, where we have also considered numbers, colours and books, for example); in this case U would be the set of all sets. (Non-set objects, where they are allowed, are called 'urelemente' and are discussed below.)

In the presence of a universal set we can define X, the complement of X, to be UX. X contains everything in the universe apart from the elements of X. We could alternatively have defined it as

$$X = \{x|\tilde{(}x \in X)\}$$

and U as the complement of the empty set.

1.3.2 Cardinality

The cardinality $\|X\|$ of X is, roughly speaking, its size. The empty set has size 0, while B = $\{red, green, blue\}$ has size 3.

This method of expressing cardinality works for finite sets but is not helpful for infinite ones. A more useful notion is that of two sets having the same size: if a direct one-to-one correspondence can be found between the elements of X and those of Y, they have the same size.

Consider the sets, both infinite, of positive integers $\{1, 2, 3, 4, \ldots\}$ and of even positive integers $\{2, 4, 6, 8, \ldots\}$. One is a subset of the other. Nevertheless they have the same cardinality, as is shown by the correspondence mapping n in the former set to 2n in the latter. We can express this by writing $\|X\| = \|Y\|$, or by saying that X and Y are "equinumerous".

1.3.3 Power set

The power set of X, $P(X)$, is the set whose elements are all the subsets of X. Thus

$$P(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

. The power set of the empty set $P(\{\}) = \{\{\}\}$.

Note that in both cases the cardinality of the power set is strictly greater than that of base set: No one-to-one correspondence exists between the set and its power set.

Rules of Inclusion, Listing and Cardinality

For each of the following sets, a set is specified by the rules of inclusion method and listing method respectively. Also stated is the cardinality of that data set.

Worked example 1

- $\{x : x \text{ is an odd integer } 5 \leq x \leq 17\}$
- $x = \{5, 7, 9, 11, 13, 15, 17\}$
- The cardinality of set x is 7.

Worked example 2

- $\{y : y \text{ is an even integer } 6 \leq y < 18\}$
- $y = \{6, 8, 10, 12, 14, 16\}$
- The cardinality of set y is 6.

Worked example 3

A perfect square is a number that has a integer value as a square root. 4 and 9 are perfect squares ($\sqrt{4} = 2$, $\sqrt{9} = 3$).

- $\{z : z \text{ is an perfect square } 1 < z < 100\}$
- $z = \{4, 9, 16, 25, 36, 49, 64, 81\}$
- The cardinality of set z is 8.

Exercises

For each of the following sets, write out the set using the listing method. Also write down the cardinality of each set.

- $\{s : s \text{ is an negative integer } -10 \leq s \leq 0\}$
- $\{t : t \text{ is an even number } 1 \leq t \leq 20\}$
- $\{u : u \text{ is a prime number } 1 \leq u \leq 20\}$
- $\{v : v \text{ is a multiple of 3 } 1 \leq v \leq 20\}$

Power Sets

Worked Example

Consider the set Z:

$$Z = \{a, b, c\}$$

Q1 How many sets are in the power set of Z?

Q2 Write out the power set of Z.

Q3 How many elements are in each element set?

Solutions to Worked Example

Q1 There are 3 elements in Z. So there is $2^3 = 8$ element sets contained in the power set.

Q2 Write out the power set of Z.

$$\mathcal{P}(Z) = \{\{0\}, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}$$

Q3 * One element set is the null set - i.e. containing no elements
- Three element sets have only elements
- Three element sets have two elements
- One element set contains all three elements
- 1+3+3+1=8

Exercise

For the set $Y = \{u, v, w, x\}$, answer the questions from the previous exercise

Complement of a Set

Consider the universal set U such that

$$U = \{2, 4, 6, 8, 10, 12, 15\}$$

For each of the sets A,B,C and D, specify the complement sets.

Set	Complement
$A = \{4, 6, 12, 15\}$	$A' = \{2, 8, 10\}$
$B = \{4, 8, 10, 15\}$	
$C = \{2, 6, 12, 15\}$	
$D = \{8, 10, 15\}$	

Set Operations

Consider the universal set U such that

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

and the sets

$$A = \{2, 5, 7, 9\}$$

$$B = \{2, 4, 6, 8, 9\}$$

(a) $A - B$

(b) $A \otimes B$

(c) $A \cap B$

(d) $A \cup B$

(e) $A' \cap B'$

(f) $A' \cup B'$

Venn Diagrams

Draw a Venn Diagram to represent the universal set $\mathcal{U} = \{0, 1, 2, 3, 4, 5, 6\}$ with subsets:
$A = \{2, 4, 5\}$
$B = \{1, 4, 5, 6\}$

Find each of the following

(a) $A \cup B$

(b) $A \cap B$

(c) $A - B$

(d) $B - A$

(e) $A \otimes B$