1. Using T for true and F for False, we get

$f(n) =$	$O(n)$	$O(n^2)$	$O(n^5)$	$O(n \log_{10} n)$
n^2	F	T	T	F
$n \log_2 n$	F	T	T	T
$n^{2.5}$	F	F	T	F
1.1^n	F	F	F	F

$f(n) =$	$\Omega(n)$	$\Omega(n^2)$	$\Omega(n^5)$	$\Omega(n \log_{10} n)$
n^2	T	T	F	T
$n \log_2 n$	T	F	F	T
$n^{2.5}$	T	T	F	T
1.1^n	T	T	T	T

$f(n) =$	$\Theta(n)$	$\Theta(n^2)$	$\Theta(n^5)$	$\Theta(n \log_{10} n)$
n^2	F	T	F	F
$n \log_2 n$	F	F	F	T
$n^{2.5}$	F	F	F	F
1.1^n	F	F	F	F

2. Recall that in the master theorem, the recurrence is of the form

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where $a \geq 1$ and $b > 1$. In addition, in case 3, the forcing function must satisfy the Regularity Condition: there exists $c < 1$ such that

$$af(n/b) \leq cf(n)$$

for all n large enough.

(1) $a = 2$ and $b = 2$

$n^{\log_b a}$		$f(n)$	Case	$T(n) =$
n^1	$<$	$n \log n$	3?	N/A (see below)

Regularity condition

$$\begin{aligned} 2n/2 \log(n/2) &\leq cn \log n \\ \Leftrightarrow \log n - \log 2 &\leq c \log n \end{aligned}$$

This is not true for $c < 1$ whenever $n > 2^{\frac{1}{1-c}}$

(2) $a = 8$ and $b = 4$

$n^{\log_b a}$		$f(n)$	Case	$T(n) =$
$n^{1.5}$	$>$	n	1	$\Theta(n^{1.5})$

(3) $a = 2$ and $b = 4$

$n^{\log_b a}$		$f(n)$	Case	$T(n) =$
$n^{0.5}$	$<$	$n^{0.6}$	3?	$\Theta(n^{0.6})$ (see below)

Regularity condition

$$\begin{aligned} 2(n/4)^{0.6} &\le cn^{0.6} \\ \Leftrightarrow 0.8706n^{0.6} &\le cn^{0.6} \end{aligned}$$

This is true for all n when c is chosen so that $0.8706 < c < 1$.

(4) $a = 1/2$ and $b = 2$. Theorem N/A since $a < 1$.

(5) $a = 3$ and $b = 3$

$n^{\log_b a}$		$f(n)$	Case	$T(n) =$
n^1	$\approx$	$n/2$	2	$\Theta(n \log n)$

(6) $a = 4$ and $b = 2$

$n^{\log_b a}$		$f(n)$	Case	$T(n) =$
n^2	$>$	$\log n$	1	$\Theta(n^2)$

3.

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

Here $a = 2$ and $b = 2$, Hence

$$n^{\log_2 2} = n = O(n) = f(n)$$

which is case 2 of the Master theorem, and so we conclude that $T(n) = \Theta(n \log n)$

4. (a) For instance

$$\begin{aligned} z_{11} &= p_1 + p_6 - p_5 + p_7 \\ &= (x_{11} + x_{22})(y_{11} + y_{22}) + x_{22}(-y_{11} + y_{21}) - (x_{11} + x_{12})y_{22} + (x_{12} - x_{22})(y_{21} + y_{22}) \\ &= x_{11}(y_{11} + y_{22} - y_{22}) + x_{12}(-y_{22} + y_{21} + y_{22}) + x_{22}(y_{11} + y_{22} - y_{11} + y_{21} - y_{21} - y_{22}) \\ &= x_{11}y_{11} + x_{12}y_{21} \end{aligned}$$

which is correct. Similarly for the rest.

(b) To multiply

$$\begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 3 & -7 \\ -1 & 2 \end{pmatrix}$$

compute

$$\begin{aligned} p_1 &= (2+5)(3+2) = 35 \\ p_2 &= (1+5)3 = 18 \\ p_3 &= 2(7-2) = 10 \\ p_4 &= (-2+1)(3+7) = -10 \\ p_5 &= (2+4)2 = 12 \\ p_6 &= 5(-3-1) = -20 \\ p_7 &= (4-5)(-1+2) = -1 \end{aligned}$$

then

$$\begin{aligned} z_{11} &= 35 - 20 - 12 - 1 = 2 \\ z_{12} &= 10 + 12 = 22 \\ z_{21} &= 18 - 20 = -2 \\ z_{22} &= 35 - 18 + 10 - 10 = 17 \end{aligned}$$

which gives the answer

$$\begin{pmatrix} 2 & 22 \\ -2 & 17 \end{pmatrix}$$

To multiply

$$\begin{pmatrix} 1 & 0 & 2 \\ 4 & 5 & 4 \\ 2 & -2 & 3 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 0 & 1 & -2 \\ 1 & 1 & -1 \end{pmatrix}$$

embed zeroes to convert to the product of two $2^r \times 2^r$ matrices, getting

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 4 & 5 & 4 & 0 \\ 2 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then compute (all the 2×2 matrix multiplications are done using the *Strassen* algorithm)

$$\begin{aligned}
p_1 &= \left[\begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}\right]\left[\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 16 & 4 \\ 16 & 9 \end{pmatrix} \\
p_2 &= \left[\begin{pmatrix} 2 & -2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 25 & 3 \\ 0 & 0 \end{pmatrix} \\
p_3 &= \begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix}\left[\begin{pmatrix} 2 & 0 \\ -2 & 0 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix} \\
p_4 &= \left[-\begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 2 & -2 \\ 0 & 0 \end{pmatrix}\right]\left[\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ -2 & 0 \end{pmatrix}\right] = \begin{pmatrix} 11 & -1 \\ -18 & -9 \end{pmatrix} \\
p_5 &= \left[\begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}\right]\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ -8 & 0 \end{pmatrix} \\
p_6 &= \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}\left[-\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} -12 & 0 \\ 0 & 0 \end{pmatrix} \\
p_7 &= \left[\begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & -1 \\ 0 & 4 \end{pmatrix}
\end{aligned}$$

Then

$$\begin{aligned}
z_{11} &= \begin{pmatrix} 16 & 4 \\ 16 & 9 \end{pmatrix} + \begin{pmatrix} -12 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} -3 & 0 \\ -8 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 24 & 13 \end{pmatrix} \\
z_{12} &= \begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ -8 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -6 & 0 \end{pmatrix} \\
z_{21} &= \begin{pmatrix} 25 & 3 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -12 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 13 & 3 \\ 0 & 0 \end{pmatrix} \\
z_{22} &= \begin{pmatrix} 16 & 4 \\ 16 & 9 \end{pmatrix} - \begin{pmatrix} 25 & 3 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 11 & -1 \\ -18 & -9 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

Hence the solution of the augmented problem is

$$\begin{pmatrix} 7 & 13 & 0 & 0 \\ 24 & 13 & -6 & 0 \\ 13 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Strip away the embedded zeroes to get

$$\begin{pmatrix} 7 & 13 & 0 \\ 24 & 13 & -6 \\ 13 & 3 & 5 \end{pmatrix}$$

(c) Applying the Master theorem, $18n^2 = O(n^{\log_2 7}) = O(n^{2.807})$, so Case 1 holds and we conclude that $a_n = \Theta(n^{2.807})$. (Compare this with the standard algorithm where $a_n = \Theta(n^3)$).

5. The first few *Fibonacci* numbers are

$k =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$F_k =$	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987

(i) $k = 16$

$V_{377} < V_{610}$; discard values in positions 611 to 987; $k = 15$

$V_{233} > V_{377}$; discard values in positions 1 to 233; renumber remaining as follows

old	234	235	⋯	322	⋯	377	⋯	466	⋯	610
new	1	2	⋯	89	⋯	144	⋯	233	⋯	377

$k = 14$

$V_{144} < V_{233}$; discard values in positions 234 to 377; $k = 13$

$V_{89} > V_{144}$; discard values in positions 1 to 89; renumber remaining as follows

original	323	324	⋯	343	⋯	356	⋯	377	⋯	411	⋯	466
old	90	91	⋯	110	⋯	123	⋯	144	⋯	178	⋯	233
new	1	2	⋯	21	⋯	34	⋯	55	⋯	89	⋯	144

$k = 12$

$V_{55} < V_{89}$; discard values in positions 90 to 144; $k = 11$

$V_{34} < V_{55}$; discard values in positions 56 to 89; $k = 10$

$V_{21} > V_{34}$; discard values in positions 1 to 21; renumber remaining as follows

original	344	345	⋯	356	⋯	364	⋯	377
old	22	23	⋯	34	⋯	42	⋯	55
new	1	2	⋯	13	⋯	21	⋯	34

$k = 9$

$V_{13} = V_{21}$; discard values in positions 1 to 13 and positions 22 to 34 ; renumber remaining as follows

original	357	358	359	⋯	361	⋯	364
old	14	15	16	⋯	18	⋯	21
new	1	2	3	⋯	5	⋯	8

$k = 6$

$V_3 = V_5$; discard values in positions 1 to 3 and positions 6 to 8 ; renumber remaining as follows

original	360	361
old	4	5
new	1	2

$k = 3$

$V_{360} = 0 < V_{361}$. This is the minimum.

(ii) If the minimum appears as the first entry in the list, then the algorithm will do n look ups if $n \leq 3$ or otherwise $m - 2$ look ups where $n = F_m$ (Prove this by induction).

(iii) If $F_{m-1} < n < F_m$, append $F_m - n$ DUD values (each smaller or larger than the value in position n ?) to the end of the list of values, and apply the algorithm.

(iv) The algorithm is indeed of the "divide and conquer" variety. At each generic iteration of the algorithm, the number of values under consideration goes from F_m to F_{m-1} but from Problem Sheet 2 Question 2 (c),

$$F_m = \frac{1}{\sqrt{5}}\left(\phi^m - \overline{\phi}^m\right)$$

where $\phi = (1+\sqrt{5})/2 \approx 1.618$, and $\overline{\phi} = -1/\phi$. Therefore for large m

$$F_{m-1} \approx \frac{F_m}{\phi}$$

Hence if $T(n)$ represents the number of look ups to search a table of $n = F_m$ entries, we have the approximate relationship

$$T(n) = T\left(\frac{n}{\phi}\right) + 1$$

Applying the Master theorem, $1 = \Theta(n^{\log_\phi 1})$ hence Case 2 applies and so we conclude that $T(n) = \Theta(\log n)$. Compare this with the answer given in part(ii).