Question 1

(a) Working in base 2 perform the following calculation, showing all your working. [3 Marks]

$$110101_2 + 10111_2 - 100001_2$$

(b) Express the following hexadecimal number as a decimal number: $(A32.C)_{16}$. [3 Marks]

(c) Convert the following decimal number into base 2, showing all your working: $(253)_{10}$. [2 Marks]

(d) Express the recurring decimal $0.4242424\ldots$ as a rational number in its simplest form. [2 Marks]

Question 2

(a) Let $S = \{w, x, y, z\}$. Describe briefly how the subsets of S can each be represented by a unique 4-bit binary string. [2 Marks]

(b) Make a list of all 4-bit binary strings which have 1 as their first bit. Use this list to find all the subsets of S containing the element w. [3 Marks]

(c) What is the total number of subsets of S? [1 Mark]

(d) Draw a Venn Diagram to show the three subsets A,B and C of a universal set $\mathcal{U}$ intersecting in the most general way. Shade the regions contained in the subset X defined by the membership table below. [2 Marks]

A	B	C	X
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1

(e) Describe the subset X in terms of the sets A,B,C using the appropriate set operations. [2 Marks]

Question 3

(a) Let $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and let p, q be the following propositions concerning the integer n.

p : n is even

q : $n < 5$.

Find the values of n for which each of the following compound statements is true.

(i) $\neg p$

(ii) $p \wedge q$

(iii) $\neg p \vee q$

(iv) $p \otimes q$.

(b) Let p and q be propositions.

(i) Construct the truth table for $p \rightarrow q$. [2 Marks]

(ii) Use truth tables to prove that $\neg q \rightarrow \neg p = p \rightarrow q$. [2 Marks]

(c) Let p, q be the following propositions:

p : this apple is red,

q : this apple is ripe.

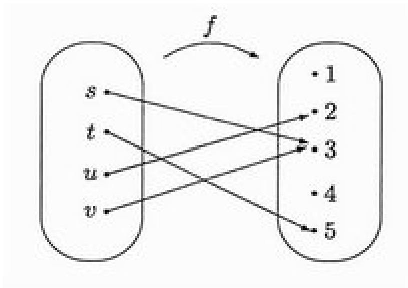
Express the following statements in words as simply as you can. [2 Marks]

(i) $p \rightarrow q$

(ii) $p \wedge \neg q$.

Question 4

(a) A function f is represented by the arrow diagram shown below.



(i) Give the domain, co-domain and range of f. [3 Marks]

(ii) Say why f does not have the one-to-one property and why it does not have the onto property, giving a specific counter-example in each case. [2 Marks]

(b) (i) State the conditions to be satisfied by a function $f : X \to Y$ for it to have an inverse function $f^{-1} : Y \to X$. [3 Marks]

(ii) Define f^{-1} when $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c, d\}$, and f is given by the table below. [2 Marks]

x	1	2	3	4
f(x)	b	c	d	a

Question 5

(a) Let $\mathcal{S}$ be a set and let $\mathcal{R}$ be a relation on $\mathcal{S}$ Explain what it means to say that $\mathcal{R}$ is

(i) reflexive

(ii) symmetrix

(iii) anti-symmetric

(iv) transitive [4 Marks]

(b) Let $\mathcal{S}$ be the set $\{2, 3, 4, 5, 6, 7\}$ and a relation $\mathcal{R}$ is defined between the elements of $\mathcal{S}$ by

x is related to y if $|x - y| \in \{0, 3\}$.

(i) Draw the relationship digraph. [3 Marks]

(ii) Determine whether or not $\mathcal{R}$ is reflexive, symmetric or transitive. In cases where one of these properties does not hold give an example to show that it does not hold.n [3 Marks]

Question 6

(a) Let G be a graph and let v be a vertex of G. Say what is meant by the degree of v. [1 Mark]

(b) A graph is called k-regular if each of its vertices has degree k. Construct an example of:

(i) a 2-regular graph with 5 vertices, [2 Marks]

(ii) a 3-regular graph with 6 vertices. [2 Marks]

(c) State, without proving, a result connecting the degrees of the vertices of a graph G with the number of its edges. [1 Mark]

(d) Use the result in part (c) to find the number of edges of a 3-regular graph with 10 vertices [2 Marks]

(e) Explain why it is not possible to construct a 3-regular graph with 9 vertices. [2 Marks]

Question 7

(a) A sequence is given by the recurrence relation

$$u_{n+1} = u_n + n \text{ and } u_1 = 0$$

(i) Calculate u_3, u_4, and u_5. [2 Marks]

(ii) Use induction to prove the following [5 Marks]

$$u_n = \frac{n(n-1)}{2} \text{ for all n} = 1$$

(b) Write the following in σ notation

$$1 + 4 + 7 + 10 + \ldots + (3n - 2)$$

Evaluate this when n = 100. [3 Marks]

Question 8

(a) What properties must a graph satisfy in order for it to be a tree? [2 Marks]

(b) Design a balanced binary search tree for an ordered list of 15 records. Label the records $1, 2, \ldots, 15$ in your tree. [4 Marks]

(c) A binary search tree is designed to store an ordered list of 50000 records, numbered $1, 2, 3 \ldots 50000$ at its internal nodes.

(i) Draw levels 0, 1 and 2 of this tree, showing which number record is stored at the root and at each of the nodes at level 1 and 2, making it clear which records are at each level.

(ii) What is the maximum number of comparisons that would have to be made in order to locate an existing record from the list of 50000?

[4 Marks]

Question 9

(a) The code to open a combination lock is an ordered sequence of four digits chosen from the set $\{1, 2, 3, 4, 5, 6, 7\}$. How many different codes are possible

(i) if repetition is allowed?

(ii) if repetition is not allowed? [2 Marks]

(b) Twelve balls numbered $\{1, 2, 3, \ldots, 12\}$ are placed in a container and three balls are drawn at random without replacement. How many different selections of three balls are possible, if the order of selection is not important? [2 Marks]

(c) In the experiment described in part (b), let A be the event that the number on each ball drawn is at most 5. Let B be the event that the number on each ball drawn is odd. Calculate the probability of each of the events A , B, $A \cup B$ and $A \cap B$. [6 Marks]

Question 10

Consider the following matrices **A** and **B** which are given as

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

(a) Show that **AB** and **BA** are not equal. [3 Marks]

(b) Find matrices C, D and E such that

(i) $\mathbf{B} + \mathbf{C} = \mathbf{A}$

(ii) $\mathbf{BD} = \mathbf{B}$

(iii) **B** - **E** $=$ **A**. [3 Marks]

(c) (i) Write down the augmented matrix for the following system of equations.

(ii) Use Gaussian elimination to solve the system.

$$2x + y - z = 2$$

$$x - y + z = 4$$

$$x + 2y + 2z = 10$$

[4 Marks]