UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

MID TERM ASSESSMENT PAPER

MODULE CODE: MA4016 SEMESTER: Spring 2006

MODULE TITLE: Engineering Mathematics 6 DURATION OF EXAMINATION: 45 minutes

LECTURER: Dr. M. Burke PERCENTAGE OF TOTAL MARKS: 20 %

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Answer both questions. Each question is worth 10 marks. Each part of Question 2 carries equal marks. Use the Answer Sheet provided for Question 2.**

ANSWER SHEET

STUDENT'S NAME:
STUDENT'S ID NUMBER:

For each part of Question 2, place an "**X**" in the box of your choice.

Question	a	b	c	d	e	Do not write in this column
(i)	**X**					
(ii)		**X**				
(iii)			**X**			
(iv)		**X**				
(v)					**X**	

1. Solve the difference equation

$$\begin{aligned} x_{n+1} &= y_n, & x_0 = 1 \\ y_{n+1} &= -6x_n + 5y_n + n, & y_0 = 1 \end{aligned}$$

2. (i) Algorithm A1 solves a problem of size n using n^5 operations, while algorithm A2 solves the same problem with 2^n operations. Which algorithm is the more efficient of the two in terms of operations used for large n ?
   (a) A1 (b) A2 (c) Either one
   (d) It depends on n (e) Not computable from information given

   (ii) In an undirected *Eulerian* graph where the sum of the lengths of the edges is 58, the shortest cycle visiting each edge is
   (a) ≤ 58 (b) 58 (c) ≥ 58
   (d) 116 (e) Not computable from information given

   (iii) A 3-state *Markov* Chain with an absorbing state (F) accessible from the other two states has equilibrium probability $\lim_{n\to\infty} p_F(n) =$
   (a) 0 (b) $\frac{2}{3}$ (c) 1
   (d) $\frac{3}{2}$ (e) Not computable from information given

   (iv) For the *Markov* Chain shown in Fig 1, the probability of arriving at state B at time $n+1$ is given by $p_B(n+1) =$

   $(a)\ \frac{3}{4}p_A(n)+\frac{1}{3}p_C(n)$ $(b)\ \frac{3}{4}p_A(n)+\frac{1}{2}p_C(n)$ $(c)\ \frac{2}{3}p_A(n)+\frac{1}{2}p_C(n)$

   $(d)\ \frac{2}{3}p_A(n)+\frac{1}{3}p_C(n)$ $(e)\ p_B(n)$

   (v) The complete solution of $a_{n+2} = 4a_{n+1} - 3a_n,\ a_0 = 1$ is given by
   (a) 0 (b) 1 (c) $3(-1)^n - 2(-3)^n$
   (d) 3^n (e) Not computable from information given

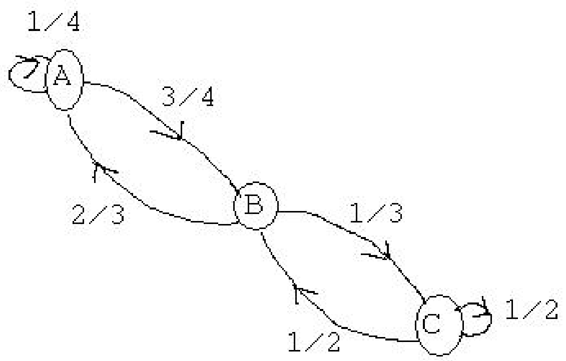

Figure 1: Markov Chain

Question 1 Solution

Either - convert to 2nd order difference equation:

$$x_{n+2} = y_{n+1} = -6x_n + 5y_n + n = -6x_n + 5x_{n+1} + n$$

$$\Rightarrow x_{n+2} - 5x_{n+1} + 6x_n = n, \qquad x_0 = 1, x_1 = y_0 = 1$$

Homogeneous Equation:

$$x^h_{n+2} - 5x^h_{n+1} + 6x^h_n = 0$$

has characteristic equation

$$\begin{aligned} K^2 - 5K + 6 &= 0 \\ \Rightarrow K &= 2 \text{ or } 3 \\ \Rightarrow x^h_n &= A2^n + B3^n \end{aligned}$$

Particular Solution: $x^p_n = Cn + D$. The (original) difference equation becomes

$$\begin{aligned} C(n+2) + D - 5(C(n+1) + D) + 6(Cn + D) &= n \\ \Rightarrow (2C)n + (-3C + 2D) &= 1n + 0 \\ \Rightarrow C &= \frac{1}{2} \\ D &= \frac{3}{4} \end{aligned}$$

The general solution is thus $x_n = A2^n + B3^n + \frac{1}{2}n + \frac{3}{4}$. The initial conditions (IC) give

$$\begin{aligned} 1 = x_0 &= A + B + \frac{3}{4} \\ 1 = x_1 &= 2A + 3B + \frac{5}{4} \\ \Rightarrow A &= 1 \\ B &= -\frac{3}{4} \end{aligned}$$

Hence

$$\begin{aligned} x_n &= 2^n - \frac{3^{n+1}}{4} + \frac{1}{2}n + \frac{3}{4} \\ y_n = x_{n+1} &= 2^{n+1} - \frac{3^{n+2}}{4} + \frac{1}{2}n + \frac{5}{4} \end{aligned}$$

Or - deal with as a system of equations:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} 0 \\ n \end{pmatrix}, \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

has solution

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sum_{j=0}^{n-1} \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}^{n-1-j} \begin{pmatrix} 0 \\ j \end{pmatrix}$$

The system matrix is diagonalisable:

$$\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3(2)^n - 2(3)^n & -2^n + 3^n \\ 6(2^n - 3^n) & -2^{n+1} + 3^{n+1} \end{pmatrix}$$

so

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= \begin{pmatrix} 2^{n+1} - 3^n \\ 2^{n+2} - 3^{n+1} \end{pmatrix} + \sum_{j=0}^{n-1} \begin{pmatrix} -2^{n-1-j} + 3^{n-1-j} \\ -2^{n-j} + 3^{n-j} \end{pmatrix} j \\ &= \begin{pmatrix} 2^{n+1} - 3^n \\ 2^{n+2} - 3^{n+1} \end{pmatrix} + \begin{pmatrix} -(2^n - n - 1) + \frac{3^n - 2n - 1}{4} \\ -2(2^n - n - 1) + 3\frac{3^n - 2n - 1}{4} \end{pmatrix} \\ &= \begin{pmatrix} 2^n - \frac{3^{n+1}}{4} + \frac{1}{2}n + \frac{3}{4} \\ 2^{n+1} - \frac{3^{n+2}}{4} + \frac{1}{2}n + \frac{5}{4} \end{pmatrix} \end{aligned}$$