# UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

**END OF SEMESTER ASSESSMENT PAPER**

MODULE CODE: MA4016

SEMESTER: Spring 2009

MODULE TITLE: Engineering Mathematics 6

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 80 %

EXTERNAL EXAMINER: Prof. J. Flavin

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions.**

1. An iterative procedure uses a test at the end of each iteration to determine whether it should finish (F) or which of 2 possible sub-processes (A or B) to do next. The transition probabilities are independent of the number of iterations so far and are as shown in the following table:

| | ↓ current | A | B | F |
|---|---|---|---|---|
| next | A | 0.2 | 0.3 | 0.0 |
| | B | 0.2 | 0.7 | 0.0 |
| | F | 0.6 | 0.0 | 1.0 |

Let $p_A(k)$ and $p_B(k)$ represent the probabilities that sub-processes A and B are performed on the $k$-th iteration respectively, and $p_F(k)$ the probability that the procedure finishes on or before the $k$-th iteration.

(a) Find a system of three recurrence equations which shows how the probabilities change from one iteration to the next. 2

(b) Solve this recurrence if the system performs sub-process B initially. 6

(c) Let $T$ be the random variable that measures the iteration on which the algorithm ends. Find an expression for Prob($T = k$), the probability that the procedure ends on the $k$-th iteration. 3

(d) What is the expected number of iterations till the procedure ends? 5

2. (a) Use the Master theorem to find the asymptotic solutions of the following *divide and conquer* recurrences:

(i)

$$T(n) = 4T\left(\frac{n}{2}\right) + n\log n$$

(ii)

$$T(n) = 4T\left(\frac{n}{3}\right) + n^2$$

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(b) Describe *Strassen*'s algorithm for multiplying two $n \times n$ matrices of real numbers. 4

Write down a *divide and conquer* recurrence relation for the number of scalar additions required by the algorithm, and find its asymptotic solution using the Master theorem. 4

3. State the Chinese Postman problem for a connected weighted graph. 4

   (a) For such a graph with 2 vertices of odd degree, how is the Chinese Postman problem solved. 4
   Hence find the length of the shortest circuit starting and ending at vertex A that visits each edge of the following graph: 4

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| A | - | 3 | - | 4 | 5 | - | 2 |
| B | 3 | - | 6 | 2 | 3 | - | - |
| C | - | 6 | - | 7 | - | 4 | 2 |
| D | 4 | 2 | 7 | - | - | - | 4 |
| E | 5 | 3 | - | - | - | 6 | 3 |
| F | - | - | 4 | - | 6 | - | 7 |
| G | 2 | - | 2 | 4 | 3 | 7 | - |

   What edges are traversed more than once in the circuit chosen?

   (b) For a connected weighted graph with 4 vertices of odd degree, describe how the Chinese Postman problem may be solved. 4

4. (a) What is a tree ? Define what is meant by a spanning tree of a connected graph. 2

   (b) Describe *Prim*'s algorithm for finding the minimal spanning tree of a weighted connected graph. 4

   (c) How can the algorithm be modified to take account of a specified subtree that must be included in the spanning tree ? 2

   (d) The table below gives the projected costs (in tens of millions of euro) of constructing a cable network between seven urban centres. Absence of an entry indicates that it is not feasible to construct such a link. Find the cheapest network that will link the seven centres. 5
   Assuming that it is necessary that A must be directly linked to F and G, find the cheapest network that will link the seven centres. 3

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| A | - | - | 7 | 4 | - | 8 | 6 |
| B | - | - | - | 7 | 9 | - | 3 |
| C | 7 | - | - | - | 5 | 7 | - |
| D | 4 | 7 | - | - | 4 | - | 8 |
| E | - | 9 | 5 | 4 | - | 5 | - |
| F | 8 | - | 7 | - | 5 | - | 9 |
| G | 6 | 3 | - | 8 | - | 9 | - |

5. (a) Prove or disprove that, for any statements $p$ and $q$,

$$[p \rightarrow q] \Leftrightarrow [\overline{p} \wedge q]$$

4

(b) Write the following argument is symbolic form, and hence prove or disprove its validity: 6
" If Seán studies, he will pass his exam. If Seán passes his exam, he will progress. Hence Seán won't progress unless he studies."

(c) The connective *NOR* is defined by $(p \downarrow q) \Leftrightarrow \overline{p \vee q}$ for any statements $p$ and $q$. Express the programming construction " If $p$ then $q$ else $r$" in terms of the statements $p,\ q,\ r$ and *NOR*. 6

6. The language $A \subset \{0,\ 1\}^*$ is defined by:
In each string of $A$ there are twice as many 1's as 0's.

(a) Find a phrase-structure grammar $G = (\Sigma, \mathrm{T}, \mathrm{S}, \mathrm{P})$ that generates $A$, and classify the grammar using *Chomsky*'s classification. 10

(b) Construct a finite state machine which recognises all occurrences of 101101 in strings from the language $A$. 6

7. (a) Describe the main features of a *Turing* Machine. 4

(b) Construct a "$\times 3$" Machine, i.e. a *Turing* Machine which takes as tape input a string of symbols representing an integer $n$ and produces as tape output the string of symbols representing $3 \times n$. 8

(c) Illustrate the operation of the "$\times 3$" Machine on the input string representing the number *two*. 4

MA4016 **Exam Notes**

## The Master theorem

Let $a \geq 1$ and $b > 1$ be constants. Let $f(n)$ be an asymptotically positive function, and let $T(n)$ be defined on the nonnegative integers by the recurrence relation:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where $n/b$ stands for either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$.

There are 3 cases

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and $n$ large enough, then $T(n) = \Theta(f(n))$

## *Strassen* Algorithm

The basis of the *Strassen* algorithm (SA) for matrix multiplication is as follows:
To multiply the (square block) $2 \times 2$ matrices

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}, \text{ say}$$

compute

$$\begin{aligned}
p_1 &= (x_{11} + x_{22})(y_{11} + y_{22}) \\
p_2 &= (x_{21} + x_{22})y_{11} \\
p_3 &= x_{11}(y_{12} - y_{22}) \\
p_4 &= (-x_{11} + x_{21})(y_{11} + y_{12}) \\
p_5 &= (x_{11} + x_{12})y_{22} \\
p_6 &= x_{22}(-y_{11} + y_{21}) \\
p_7 &= (x_{12} - x_{22})(y_{21} + y_{22})
\end{aligned}$$

then

$$\begin{aligned} z_{11} &= p_1 + p_6 - p_5 + p_7 \\ z_{12} &= p_3 + p_5 \\ z_{21} &= p_2 + p_6 \\ z_{22} &= p_1 - p_2 + p_3 + p_4 \end{aligned}$$

## Classification of Grammars

Let $G = (\Sigma, \mathrm{T}, \mathrm{S}, \mathrm{P})$ be a phrase structure grammar, where each production is of the form

$$w_1 \longmapsto w_2 \quad \text{or} \quad \mathrm{S} \longmapsto \lambda$$

The nonterminal symbols are $\mathrm{N} = \Sigma \setminus \mathrm{T}$.

A grammar is classified according to the restrictions on its $w_1 \longmapsto w_2$ productions as follows

| Type | Name | Restriction |
|---|---|---|
| 3 | (Regular) | $w_1 \in \mathrm{N}$ and $w_2 \in \mathrm{T}$ or $w_2 \in \mathrm{TN}$ |
| 2 | (Context Free) | $w_1 \in \mathrm{N}$ |
| 1 | (Context Sensitive) | $\|\|w_1\|\| \leq \|\|w_2\|\|$ |
| 0 | | No restrictions |

## Some Geometric Series Identities

For $|x| < 1$,

$$\sum_{k=0}^{\infty} x^k = (1-x)^{-1}$$

$$\sum_{k=0}^{\infty} kx^{k-1} = (1-x)^{-2}$$