MA4016 - Engineering Mathematics 6

Problem Sheet 6: Discrete Mathematics (March 12, 2010)

1. Are all elements of the sequence f_n, $n = 1, 2, \dots$ with

$$f_n = n^2 - n + 41$$

primes?

n	f_n	prime?	n	f_n	prime?	n	f_n	prime?	n	f_n	prime?
1	41	$\surd$	11	151	$\surd$	21	461	$\surd$	31	971	$\surd$
2	43	$\surd$	12	173	$\surd$	22	503	$\surd$	32	1033	$\surd$
3	47	$\surd$	13	197	$\surd$	23	547	$\surd$	33	1097	$\surd$
4	53	$\surd$	14	223	$\surd$	24	593	$\surd$	34	1163	$\surd$
5	61	$\surd$	15	251	$\surd$	25	641	$\surd$	35	1231	$\surd$
6	71	$\surd$	16	281	$\surd$	26	691	$\surd$	36	1301	$\surd$
7	83	$\surd$	17	313	$\surd$	27	743	$\surd$	37	1373	$\surd$
8	97	$\surd$	18	347	$\surd$	28	797	$\surd$	38	1447	$\surd$
9	113	$\surd$	19	383	$\surd$	29	853	$\surd$	39	1523	$\surd$
10	131	$\surd$	20	421	$\surd$	30	911	$\surd$	40	1601	$\surd$

but $f_{41} = 41^2 - 41 + 41 = 41 \cdot 41 = 1681$ is composite. Nevertheless, the number of primes in the sequence f_n is above average. There are e.g. 581 primes in $\{f_1, \dots, f_{1000}\}$ compared to 168 primes in $\{1, \dots, 1000\}$.

2. If the product of two integers is $2^7 3^8 5^2 7^{11}$ and their greatest common divisor is $2^3 3^4 5$, what is their least common multiple?

$$\begin{aligned}
a \cdot b = 2^7 3^8 5^2 7^{11} \quad &\Rightarrow \quad a_1 + b_1 = 7 && a_2 + b_2 = 8\\
& \quad a_3 + b_3 = 2 && a_4 + b_4 = 11\\
\gcd(a,b) = 2^3 3^4 5 \quad &\Rightarrow \quad \min\{a_1,b_1\} = 3, && \max\{a_1,b_1\} = 7-3 = 4\\
& \quad \min\{a_2,b_2\} = 4, && \max\{a_2,b_2\} = 8-4 = 4\\
& \quad \min\{a_3,b_3\} = 1, && \max\{a_3,b_3\} = 2-1 = 1\\
& \quad \min\{a_4,b_4\} = 0, && \max\{a_4,b_4\} = 11-0 = 11
\end{aligned}$$

and therefore $\operatorname{lcm}(a,b) = 2^4 3^4 5 \cdot 7^{11}$.

University of Limerick
Department of Mathematics and Statistics
Dr. S. Franz
OLLSCOIL LUIMNIGH
UNIVERSITY OF LIMERICK

3. Show that whenever $n \geq 3$, $f_n > \alpha^{n-2}$, where f_n is the n-th Fibonacci number and $\alpha = (1+\sqrt{5})/2$.
A proof with strong induction can be found in Rosen, chapter 4.3, example 6. A direct proof uses the explicit formula for the Fibonacci numbers.

$$\begin{aligned}
f_n &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)\\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n-2}\left(\frac{1-\sqrt{5}}{2}\right)^2\right]\\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\frac{1}{2\sqrt{5}}\left[(3+\sqrt{5}) - \underbrace{(3-\sqrt{5})}_{>0}\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n-2}\right]\\
&\geq \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\frac{1}{2\sqrt{5}}\left[(3+\sqrt{5}) - (3-\sqrt{5})\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)\right]\\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\frac{4}{1+\sqrt{5}} > \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}.
\end{aligned}$$

4. How many divisions are required to find $\gcd(34, 55)$ using the Euclidean algorithm? What is the bound from Lamé's theorem?
8 divisions are needed and Lamé's theorem gives upper bound of $5 \cdot 2 = 10$ divisions. 34 and 55 are two consecutive Fibonacci numbers.

5. Apply the extended Euclidean algorithm to find the greatest common divisor and s, t in

a)

$$\gcd(1529, 14038) = 1529s + 14038t, \quad s, t \text{ integers},$$

step	x	y	s_0	s_1	t_0	t_1	r	q	s	t
1	14038	1529	1	0	0	1	277	9	1	-9
2	1529	277	0	1	1	-9	144	5	-5	46
3	277	144	1	-5	-9	46	133	1	6	-55
4	144	133	-5	6	46	-55	11	1	-11	101
5	133	11	6	-11	-55	101	1	12	138	-1267
6	11	1	-11	138	101	-1267	0	11	-1529	14038
7	**1**	$\underline{0}$	**138**	11	**-1267**	14038				

$$\gcd(1529, 14038) = 1 = 138 \cdot 14038 - 1267 \cdot 1529.$$

b)

$$\gcd(1529, 14039) = 1529s + 14039t, \quad s, t \text{ integers},$$

step	x	y	s_0	s_1	t_0	t_1	r	q	s	t
1	14039	1529	1	0	0	1	278	9	1	-9
2	1529	278	0	1	1	-9	139	5	-5	46
3	278	139	1	-5	-9	46	0	2	11	-101
4	**139**	$\underline{0}$	**-5**	11	**46**	-101				

$$\gcd(1529, 14039) = 139 = -5 \cdot 14039 + 46 \cdot 1529.$$