Section 7 Sequences, Series and Proof by Induction

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The 3-step method of Proof by Induction

Base case *Verify that the result is true when* $n = 1$ *so that* $1 \in S$.

Induction hypothesis *For some arbitrarily fixed integer* $k \geq 1$, *assume that the result is true for all the integers* $1, 2, \ldots, k$.

Induction step *Use the hypothesis that the result is true when* $n = 1, 2, \ldots, k$ *to prove that the result also holds when* $n = k + 1$.

Section 7 Sequences, Series and Proof by Induction

- Series

Theorem 2.1 *Let n be a positive integer. Then*

(a) $\sum_{r=1}^{n} 1 = n.$

(b) $\sum_{r=1}^{n} r = n(n+1)/2.$

(c) $\sum_{r=1}^{n} r^2 = n(n+1)(2n+1)/6.$

(d) $\sum_{r=0}^{n} x^r = \frac{x^{n+1}-1}{x-1}$, *for any* $x \in \mathbb{R}$ *with* $x \neq 1.$

2001

Question 9

(a) Calculate the terms u_3 and u_4 of the sequence defined for $n \geq 2$ by the recurrence relation

$$u_{n+1} = u_n + 3u_{n-1},$$

when $u_1 = 1$ and $u_2 = 4$. [2]

2001

(b) For each of the following sequences, find a recurrence relation that gives u_{n+1} in terms of u_n.

(i) 12, 1.2, 0.12, 0.012, 0.0012, …;

(ii) 3, 7, 11, 15, 19, …. [2]

2001

(c) Use the formula $\sum_{r=1}^{n} r = n(n+1)/2$ to evaluate the following sums.

(i) $1 + 2 + 3 + \cdots + 100$; [2]

2001

(ii) $21 + 22 + 23 + \cdots + 100;$ [2]

2001

(iii) $5 + 10 + 15 + 20 + 25 + \cdots + 100.$ [2]

2002

Question 8 (a) A sequence is defined by the recurrence relation

$$x_{n+2} = 3x_{n+1} - 2x_n$$

and initial terms and $x_1 = 1$ and $x_2 = 3$.

(i) Calculate x_3 and x_4, showing your working. [2]

2002

(ii) Prove by induction that $x_n = 2^n - 1$ for all $n \geq 1$. [6]

2002

(b) Evaluate $\sum_{n=1}^{50}(5n-2)$. [2]

2003

Question 7 (a) A sequence is given by the recurrence relation

$$u_{n+1} = u_n + n \quad and u_1 = 0.$$

(i) Calculate u_3, u_4, and u_5. [2]

2003

(ii) Use induction to prove that

$$u_n = \frac{n(n-1)}{2} \; for all n \geq 1.$$

[5]

2003

(b) Write the following in $\sum$ notation

$$1 + 4 + 7 + 10 + ... + (3n - 2).$$

Evaluate this when $n = 100$. [3]

2004

Question 6 (a) Consider the sequence given by $1, 4, 7, 10, 13, \ldots$

State a recurrence relation which expresses the nth term, u_n, in terms of the $(n-1)th$ term, u_{n-1}. [2]

2004

(b) Another sequence is defined by the recurrence relation $u_n = u_{n-1} + 2n - 1$ and $u_1 = 1$.

(i) Calculate u_2, u_3, u_4 and u_5.

(ii) Prove by induction that $u_n = n^2$ for all $n \geq 1$.

2004

(iii) Find the sum of the first 50 terms of this sequence.

You may assume the formula for $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, [8]

2005

Question 5 Let the sequence u_n be defined by the recurrence relation

$$u_{n+1} = u_n + 2n, \quad for n = 1, 2, 3, ...and let u_1 = 1.$$

(a) Calculate u_2, u_3, u_4 and u_5, showing all your working. [2]

2005

(b) Prove by mathematical induction that the nth term, where $n \geq 0$, is given by

$$u_n = n^2 - n + 1.$$

[5]

2005

(c) Showing all your working, find the sum of the first 100 terms of this sequence.

[3]

2006

Question 3

(a) (i) Write down the first three and last three terms of the series given by

$$\sum_{k=1}^{33}(3k-1).$$

(ii) Write the terms of this sequence as a recurrence relation that gives u_{n+1} in terms of u_n and give the value of the initial term. [3]

2006

(b) Write the following series in $\sum$ notation:

(i) $2 + 5 + 8 + ...200$

(ii) $101 + 104 + 107 + ... + 299$.

Use the formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ to evaluate the first of these two sums. [4]

2006

(c) It can be proved by induction that the seies $3 + 7 + 11 + ...$ has the sum to r terms given by S_r, where

$$S_r = 2r^2 + r.$$

Use this result to evaluate the following sums:

(i) $3 + 7 + 11 + ... + 399$

(ii) $403 + 407 + 411 ... + 999.$ [3]