1.

(a) The following are (not necessarily minimal) representations of the connectives shown using Nand only.

(i)
$$\neg p \Leftrightarrow \neg(p \wedge p) \Leftrightarrow p \uparrow p$$

(ii)
$$p \vee q \Leftrightarrow \neg(\overline{p \vee q}) \Leftrightarrow \neg(\overline{p} \wedge \overline{q}) \Leftrightarrow \overline{p} \uparrow \overline{q} \Leftrightarrow (p \uparrow p) \uparrow (q \uparrow q)$$

(iii)
$$p \wedge q \Leftrightarrow \neg(\overline{p \wedge q}) \Leftrightarrow \neg(p \uparrow q) \Leftrightarrow (p \uparrow q) \uparrow (p \uparrow q)$$

(iv)
$$p \rightarrow q \Leftrightarrow \overline{p} \vee q \Leftrightarrow (\overline{p} \uparrow \overline{p}) \uparrow (q \uparrow q) \Leftrightarrow ((p \uparrow p) \uparrow (p \uparrow p)) \uparrow (q \uparrow q)$$

(v)
$$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow ((p \rightarrow q) \uparrow (q \rightarrow p)) \uparrow ((p \rightarrow q) \uparrow (q \rightarrow p)) \Leftrightarrow \cdots$$

using part(iv), this leads to a representation involving 23 Nand operators.

(b) The following are (not necessarily minimal) representations of the connectives shown using Nor only.

(i)
$$\neg p \Leftrightarrow \neg(p \vee p) \Leftrightarrow p \downarrow p$$

(ii)
$$p \vee q \Leftrightarrow \neg(\overline{p \vee q}) \Leftrightarrow \neg(p \downarrow q) \Leftrightarrow (p \downarrow q) \downarrow (p \downarrow q)$$

(iii)
$$p \wedge q \Leftrightarrow \neg(\overline{p \wedge q}) \Leftrightarrow \neg(\overline{p} \vee \overline{q}) \Leftrightarrow \overline{p} \downarrow \overline{q} \Leftrightarrow (p \downarrow p) \downarrow (q \downarrow q)$$

(iv)
$$p \rightarrow q \Leftrightarrow \overline{p} \vee q \Leftrightarrow (\overline{p} \downarrow q) \downarrow (\overline{p} \downarrow q) \Leftrightarrow ((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q)$$

(v)
$$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow ((p \rightarrow q) \downarrow (p \rightarrow q)) \downarrow ((q \rightarrow p) \downarrow (q \rightarrow p)) \Leftrightarrow \cdots$$

using part(iv), this leads to a representation involving 23 Nor operators.

2. (a) $(p \wedge q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ are logically equivalent if $((p \wedge q) \rightarrow r) \leftrightarrow (p \rightarrow (q \rightarrow r))$. Construct the truth table and verify that they are logically equivalent.

(b) (i) Where they differ, the left hand segment has the structure $(p \wedge q) \rightarrow r$ while the right hand segment is of the form $p \rightarrow (q \rightarrow r)$.

(ii) Both segments compute the same quantity. In terms of the total number of comparisons made, the segment on the right is more efficient.

3.

(a) The atoms or primitives are:
p: Clare win the "All Ireland" this year.
q: Des [McInerney] will be happy.
r: [Des's] being home in Clarecastle.
The paragraph reads

$$[p \to q] \wedge [r \to q] \Rightarrow [p \to (r \to q)]$$

The argument is valid since (see Question 2(a))

$$[p \to q] \wedge [r \to q] \Leftrightarrow [(p \wedge r) \to q] \Leftrightarrow [p \to (r \to q)]$$

(b) The atoms are:
p: Lisa [is/will] be in College [on/this] Friday.
q:[Lisa] will attend her 2 o'clock lecture.
r: [Lisa] will catch a later bus [home].
The paragraph reads

$$[p \to (r \to q)] \wedge [p \wedge \overline{r}] \Rightarrow [\overline{q}]$$

The argument is false. The truth assignment

$$p : \text{True} \qquad q : \text{True} \qquad r : \text{False}$$

creates a counterexample.

(c) The atoms are (modulo grammar):
p: The Punt is strong.
q: Exports fall.
r: Unemployment will rise.
s: Interest rates drop.
The paragraph reads:

$$[p \to q] \wedge [q \to r] \wedge [\overline{p} \to s] \Rightarrow [s \to \overline{r}]$$

The argument is invalid. The hypotheses can be rewritten as (using the contrapositive form and interchanging the order of the first two sentences)

$$[\overline{r} \to \overline{q}] \wedge [\overline{q} \to \overline{p}] \wedge [\overline{p} \to s]$$

and further, using the Law of the Syllogism (twice), this implies

$$[\overline{r} \to s]$$

which is the converse of what is to be proven.

4. (a) true; (b) false; (c) true; (d) true; (e) true;
(f) false; (g) false; (h) false; (i) false; (j) true ($x = 0$ or -1).

5.

(a) $\forall\, x\, \exists\, y \;\; y < x$

(b) $\exists!\, x \;\; x = x^2$. Define $\exists!$ by

$$[\exists!\, x \;\; p(x)] \Leftrightarrow [\exists\, x \;\; p(x)] \wedge [\forall\, x\, \forall\, y\, [(p(x) \wedge p(y)) \to (x = y)]]$$

(c) $\forall\, x\, \exists!\, y \;\; xy = 1$