MA4016 Problem Sheet 2: Recurrence Relations

1. Find the general solution of the following

   (a) $a_{n+1} = \frac{1}{3} a_n$

   (b) $a_{n+2} + 4a_{n+1} + 4a_n = 0$

   (c) $a_{n+2} = 3a_{n+1} - 2a_n$

   (d) $a_{n+2} + a_n = 0$

   (e) $a_{n+3} + 3a_{n+2} + 3a_{n+1} + a_n = 0$

   (f) $a_{n+1} = (n+1)a_n$

   (g)

$$\begin{aligned} a_{n+1} &= a_n - b_n \\ b_{n+1} &= 2a_n + 4b_n \end{aligned}$$

   Hint: This is a system of two 1st order recurrences. By eliminating one of the variables in terms of the other, it can be transformed into a 2nd order recurrence in terms of the remaining variable, say a_n. Solve this recurrence, for a_n and then obtain an expression for b_n using the original equations.

2. Find the complete solution of the following

   (a) $a_{n+1} = 2a_n + 2(n+1), \qquad a_0 = 0$

   (b) $a_{n+2} - 3a_{n+1} + 2a_n = (-1)^n, \qquad a_0 = 1,\ a_1 = 1$

   (c) $a_{n+2} = a_{n+1} + a_n, \qquad a_0 = 0,\ a_1 = 1$

   (d) $a_{n+2} - 2a_{n+1} + a_n = n^2 + 1, \qquad a_0 = 0,\ a_1 = 2$

   (e) $a_{n+2} = 2a_{n+1} + 10a_n, \qquad a_0 = 1,\ a_1 = 1$

   (f) $a_{n+1} = (1 + a_n)/a_n, \qquad a_0 = 1$

   (g)

$$\begin{aligned} a_{n+1} &= a_n - b_n & a_0 &= 2 \\ b_{n+1} &= 2a_n + 4b_n + n & b_0 &= 0 \end{aligned}$$

3. Solve the following systems of linear recurrences

   (a) $\mathbf{x}_{n+1} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x}_n \qquad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

   (b) $\mathbf{x}_{n+1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix} \mathbf{x}_n \qquad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

   (c) $\mathbf{x}_{n+1} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x}_n + \begin{pmatrix} 0 \\ n \end{pmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

   by (i) diagonalisation, if appropriate, and (ii) using the Discrete Putzer Algorithm.