

MA4016 - Engineering Mathematics 6 Solutions to Midterm

1. Solve the system of difference equations

$$x_n = 2x_{n-1} + y_{n-1},$$
 $x_0 = 0,$
 $y_n = -6x_{n-1} - 5y_{n-1} + 5,$ $y_0 = 6.$

Version 1: Substitution

$$x_{n+1} = 2x_n + y_n$$

$$= 2x_n - 6x_{n-1} - 5y_{n-1} + 5$$

$$= 2x_n - 6x_{n-1} - 5(x_n - 2x_{n-1}) + 5$$

$$= -3x_n + 4x_{n-1} + 5$$

This is a nonhomogeneous linear recurrence relation. Solve homogeneous first with characteristic equation

$$r^2 + 3r - 4 = 0 \implies r_1 = -4, r_2 = 1.$$

Thus

$$x_n^{hom} = c_1 + c_2(-4)^n.$$

Nonhomogeneous part is $5 = 5 \cdot 1^n$. Thus with s = 1, m = 1, t = 0 we have the ansatz

$$x_n^{part} = np_0 \implies p_0(n+1) = -3np_0 + 4(n-1)p_0 + 5 \implies p_0 = 1$$

and the general solution for x_n is

$$x_n = c_1 + c_2(-4)^n + n.$$

The general solution for y_n follows by

$$y_n = x_{n+1} - 2x_n = -c_1 - 6c_2(-4)^n + 1 - n.$$

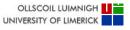
With the initial conditions follows $c_1 = 1$ and $c_2 = -1$ and therefore

$$x_n = 1 - (-4)^n + n$$

 $y_n = -1 + 6(-4)^n + 1 - n.$

Version 2: with discrete Putzer algorithm (mind the nonhomogenity) ...

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- 2. (i) Suppose two algorithms solve the same problem. Algorithm A₁ solves the problem of size n using (1 + log n)(2ⁿ + n²) operations, while algorithm A₂ solves the same problem with (n + 2ⁿ)(n² + log n) operations. Which algorithm of this two algorithms is the more efficient one in terms of operations used for large values of n?
 - (a) A_1 (b) A_2 (c) Either one
 - (d) It depends on n (e) Not computable from information given.

We try Θ -estimates first. If there is a difference in them, we can decide which one is more efficient.

$$(1 + \log n)(2^n + n^2) = \Theta(\log n)\Theta(2^n) = \Theta(\log n \, 2^n)$$
$$(n + 2^n)(n^2 + \log n) = \Theta(2^n)\Theta(n^2) = \Theta(n^2 2^n)$$

With $\log n < n^2$ for n > 1 follows: (a) is the right answer.

- (ii) The product of three matrices $A \in \mathbb{R}^{2n,n}$, $B \in \mathbb{R}^{n,1}$ and $C \in \mathbb{R}^{1,2n}$ for a positive integer n can be written as $(A \cdot B) \cdot C$. With the standard matrix multiplication algorithm, the number of scalar multiplications required is $\Theta(f(n))$ with f(n) given by
 - (a) n^3 (b) n^2 (c) $n^{\log_2 7}$
 - (d) $n^2 \log n$ (e) Not computable from information given.

The matrix $D=A\cdot B$ has $2n\cdot 1=2n$ entries with n multiplications each. This gives $2n^2$ multiplications. The matrix $D\cdot C$ has $2n\cdot 2n=4n^2$ entries with 1 multiplication each, thus $4n^2$ multiplication. Together we need $6n^2=\Theta(n^2)$ multiplications. Answer (b) is right.

(iii) The complete solution of

$$a_n = 4a_{n-2}, \quad a_0 = 0, a_1 = 2$$

is given by

(a) 0 (b)
$$2n$$
 (c) $\frac{2}{3}(4^n - 1)$

(d)
$$2^{n-1}(1-(-1)^n)$$
 (e) Not computable from information given.

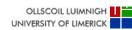
The general solution is $a_n=c_1(-2)^n+c_22^n$ and initial conditions give $c_1=-1/2$ and $c_2=1/2$. Thus

$$a_n = -\frac{1}{2}(-2)^n + \frac{1}{2}2^n = 2^{n-1}(1 - (-1)^n)$$

and therefore (d) is the right answer.

(iv) Suppose f(n) is increasing and satisfies the recurrence relation

$$f(n) = 4f(n/2) + n^3 \log n$$



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with f(1) = 1. Then f(n) is estimated for large n by (b) $\Theta((n \log(n))^2)$ (c) $\Theta(n^3 \log n)$ (d) $\Theta(n^4)$ (e) Not computable from information given.

We have a = 4, b = 2, $\log_b a = 2$ and $g(n) = n^3 \log n > 0$ for large n. Master Theorem case 3 holds with $n^3 \log n = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$ and

$$4\left(\frac{n}{2}\right)^3\log(n/2) = \frac{1}{2}n^3\log(n/2) < \frac{1}{2}n^3\log n$$

Thus aq(n/b) < cq(n) holds with c = 1/2 < 1. It follows

$$f(n) = \Theta(g(n)) = \Theta(n^3 \log n)$$

and therefore answer (c).

Alternatively, iteration gives for $n=2^k$

$$\begin{split} f(n) &= f(2^k) = 4f(2^{k-1}) + 2^{3k}k\log 2 \\ &= 4^2f(2^{k-2}) + 2^{3k-1}(k-1)\log 2 + 2^{3k}k\log 2 \\ &= 4^2f(2^{k-2}) + \log 2\left[(2^{3k} + 2^{3k-1})k - 2^{3k-1}\right] \\ &= 4^3f(2^{k-3}) + \log 2\left[(2^{3k} + 2^{3k-1} + 2^{3k-2})k - 1 \cdot 2^{3k-1} - 2 \cdot 2^{3k-2}\right] \\ &\vdots \\ &= 4^kf(1) + \log 2\left[k\sum_{j=0}^{k-1} 2^{3k-j} - \sum_{j=0}^{k-1} j2^{3k-j}\right] \end{split}$$

We have

$$\sum_{j=0}^{k-1} 2^{3k-j} = 2^{3k} \sum_{j=0}^{k-1} \left(\frac{1}{2}\right)^j = 2^{3k} \frac{1 - \left(\frac{1}{2}\right)^k}{1 - 1/2} = 2^{2k+1} (2^k - 1)$$

and

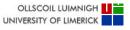
$$\sum_{j=0}^{k-1} j 2^{3k-j} = 2^{3k} \sum_{j=0}^{k-1} j \left(\frac{1}{2}\right)^j = 2^{3k} \frac{\left(\frac{1}{2}\right)^k \left(-k/2 - k + 1/2\right) - 1/2}{(-1/2 - 1)^2}$$
$$= \frac{1}{9} \left[2^{2k+1} - 2^{3k+1} - 3k2^{2k+1}\right]$$

Together follows with $n = 2^k$ and $k = \log n$

$$f(n) = f(2^k) = 4^k + \frac{\log 2}{9} 2^{2k+1} \left[(9k+1)2^k - 6k - 1 \right]$$
$$= n^2 + \frac{2\log 2}{9} n^2 \left[(9\log n + 1)n - 6\log n - 1 \right]$$
$$= \Theta(n^3 \log n).$$

For $2^{k-1} < n < 2^k$ we apply the usual steps and get finally $f(n) = \Theta(n^3 \log n)$.

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(v) The sum of the squares of the first n odd positive numbers is given by

(a)
$$\frac{n(n+1)(2n+1)}{6}$$
 (b) $\frac{n(4n^2-1)}{3}$ (c) $8n^2-15n+8$ (d) $\frac{n(n+1)(n+2)}{6}$ (e) None of the given.

Let

$$a_n = \sum_{k=1}^{n} (2k-1)^2 = a_{n-1} + (2n-1)^2.$$

We have immediately $a_n^{hom} = c_1$ and as ansatz for the nonhomogeneous problem

$$a_n^{part} = n(p_0 + p_1 n + p_2 n^2).$$

It follows with the nonhomogeneous recurrence relation

$$a_n^{part} = \frac{n}{3}(4n^2 - 1)$$

and therefore $a_n = c_1 + n(4n^2 - 1)/3$. With the initial condition $a_1 = 1$ follows $c_1 = 0$ and the sum is given by (b).