1.

(a) Base step: $n = 0$. Statement is 5 divides 0, which is correct.
Inductive step: Assume 5 divides $k^5 - k$. Then

$$\begin{aligned}(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\ &= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)\end{aligned}$$

is also divisible by 5. Hence the statement is true.

(b) Base step: $n = 0$. Statement is 3 divides $2 + 1$, which is correct.
Inductive step: Assume 3 divides $2^{2k+1} + 1$. Then

$$\begin{aligned}2^{2(k+1)+1} + 1 &= 4[2^{2k+1}] + 1 \\ &= 4[2^{2k+1} + 1] - 4 + 1 \\ &= 4[2^{2k+1} + 1] - 3\end{aligned}$$

is also divisible by 3. Hence the statement is true.

(c) Base step: $n = 0$. The left hand side (LHS) of the inequality is $(1-a)^0 = 1$. The RHS is 1.
Inductive step: Assume $(1-a)^k \geq 1 - ka$. When $n = k+1$ the LHS is

$$\begin{aligned}(1-a)^{k+1} &= (1-a)^k(1-a) \\ &\geq (1-ka)(1-a) \text{ by assumptions} \\ &= 1 - (k+1)a + ka^2 \\ &\geq 1 - (k+1)a\end{aligned}$$

which is the RHS . Hence the statement is true.

2.

(a) Base step: $n = 1$. The LHS of the equality is $1 \times 2 = 2$. While the RHS is $1 \times 2 \times 3/3 = 2$.
Inductive step: Assume $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$.
When $n = k+1$ the LHS is

$$\begin{aligned}1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + k(k+1) + (k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= (k+1)(k+2)(\frac{k}{3} + 1) \\ &= \frac{(k+1)(k+2)(k+3)}{3}\end{aligned}$$

which is the RHS. Hence the statement is true.

(b) Base step: $n = 1$. The LHS of the equality is $1/(2!) = 1/2$. While the RHS is $1 - 1/(2!) = 1/2$.
Inductive step: Assume $\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$. When $n = k+1$ the LHS is

$$\begin{aligned}\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} &= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} \text{ (by assumption)} \\ &= 1 - \frac{k+2}{(k+2)!} + \frac{k+1}{(k+2)!} \\ &= 1 - \frac{1}{(k+2)!}\end{aligned}$$

which is the RHS. Hence the statement is true.

(c) Base step: $n = 3$. The left hand side (LHS) of the inequality is $2^3 = 8$. The RHS is $2 \times 3 + 1 = 7$.
Inductive step: Assume $2^k \geq 2k + 1$. When $n = k + 1$ the LHS is

$$\begin{aligned} 2^{k+1} &= 2^k 2 \\ &\geq (2k+1)2 \text{ (by assumption)} \\ &= 4k+2 \\ &\geq 2k+3 \text{ (when } k \geq 3) \\ &= 2(k+1)+1 \end{aligned}$$

which is the RHS . Hence the statement is true.

3. Let S_n represent the sum of the first n terms of each series.

(a) We evaluate the first few partial sums.

n	S_n
1	$\frac{1}{2}$
2	$\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$
3	$\frac{2}{3} + \frac{1}{12} = \frac{3}{4}$
4	$\frac{3}{4} + \frac{1}{20} = \frac{4}{5}$
$\vdots$	$\vdots$
n	$\frac{n}{n+1}$

The formal proof of this by induction is akin to Question 2 parts (a) or (b) and is left as an exercise.

(b) Again evaluating the first few partial sums.

n	S_n
1	$\frac{2}{3}$
2	$\frac{2}{3} + \frac{4}{3} = 2 = \frac{8}{4}$
3	$2 + \frac{12}{5} = \frac{22}{5}$
4	$\frac{22}{5} + \frac{64}{15} = \frac{52}{6}$
5	$\frac{26}{3} + \frac{160}{21} = \frac{114}{7}$
$\vdots$	$\vdots$
n	$\frac{u_n}{n+2}$

where u_n satisfies

$$u_{n+1} = 2u_n + 2n + 2, \qquad u_1 = 2$$

The solution of this forced first order linear recurrence is $u_n = 2^{n+2} - 2(n+2)$, and hence $S_n = \frac{2^{n+2}}{n+2} - 2$. Formally this may be proven as follows.

Base step: $n = 1$. The first term of the sum is $2/3$. While the formula gives $8/3 - 2 = 2/3$.
Inductive step: Assume

$$4\left(\frac{1}{2\times 3}\right) + 8\left(\frac{2}{3\times 4}\right) + \cdots + 2^{k+1}\left(\frac{k}{(k+1)(k+2)}\right) = \frac{2^{k+2}}{k+2} - 2$$

When $n = k+1$ the LHS is

$$4\left(\frac{1}{2\times 3}\right)+8\left(\frac{2}{3\times 4}\right)+\cdots+2^{k+1}\left(\frac{k}{(k+1)(k+2)}\right)+2^{k+2}\left(\frac{k+1}{(k+2)(k+3)}\right)$$

$$\begin{aligned}
&= \frac{2^{k+2}}{k+2} - 2 + 2^{k+2}\left(\frac{k+1}{(k+2)(k+3)}\right)\\
&= \frac{2^{k+2}}{k+2}\left(1 + \frac{k+1}{k+3}\right) - 2\\
&= \frac{2^{k+2}}{k+2}\,\frac{2k+4}{k+3} - 2\\
&= \frac{2^{k+3}}{k+3} - 2
\end{aligned}$$

which is the RHS. Hence the expression for S_n is true.

4. Let R_n represent the number of regions into which the plane is divided by n lines satisfying the given conditions. The formula is obviously true for $n = 0$. Assume it's true for $n = k$. Consider a collection of k lines satisfying the given conditions. Add an extra line which also satisfies the given conditions. Starting at one end of this line, it

   - divides the existing region into two (i.e. adds one additional region), and
   - each time it cuts an existing line and enters a new existing region, it divides that region in two (i.e. adds one additional region). This happens k times in all.

   Therefore

$$\begin{aligned}
R_{k+1} &= R_k + 1 + k\\
&= \frac{k^2+k+2}{2} + 1 + k\\
&= \frac{k^2+3k+4}{2}\\
&= \frac{(k+1)^2+(k+1)+2}{2}
\end{aligned}$$

   which completes the induction.

5. The flaw in the argument appears in the base step. When $n = 1$ the LHS of the expression is not defined and is not the first term of the summation as indicated in the "proof". There is no value of n for which the base step works. The inductive step is flawless.