2910102 Mathematics for Computing
Examiners' Report 2010–2011
Zone B

The overall performance on this paper was satisfactory, and the students are coping well with most of the subject. There were a few poor candidates, but many more that gained high marks.

Many candidates have problems with precise mathematical notation and exact definitions. Thus many did very poorly or left blank questions like 2(a) and 2(c) which require understanding of the notation for number sets, set operations and relations. Frequently it happens also that a candidate knows the answer to a question, but is unable to phrase it using proper mathematical terms, e.g. in 4(a) a substantial number of candidates could not define properly the floor function while it was clear from 4(b) that they know what it is. Candidates are advised to pay attention to the notation used in the subject guide and to strive to adapt this notation in their own work. When you have solved an exercise, always compare your notation as well as your answers to the model answers provided in the subject guide or by your lecturer. If your notation is substantially different, try to improve it.

Another frequent mistake made by a large number of candidates is not to show their working and calculations and not explaining the reasoning behind an answer. This generally results in the loss of marks, e.g. in a question like 1(a) the examiners want to see you perform the conversion and in question 8(a)(iv) they want to see you find the number of paths rather than just guessing the answer. Also questions requiring a short answer are best answered by initially giving a short answer to the question followed by your *reason* for this particular answer. Remember that examiners are usually more interested in how you arrived at an answer than in the answer itself!

What follows is some answers, hints, solutions and comments on the examination questions that may help you, when you are revising for your examination.

Question 1

(a) The answer is $(111110111)_2$. You must show all the successive divisions by 2 with remainders and explain how to get the answer.

(b) The answer is $(1000011)_2$. It is important to show all carries.

(c) The answer to (i) are the four integers $4, 5, 6$ and 7. Note that the positive integers less than 4 have at most 2 bits in binary notation. The answer to (ii) is 2^{n-1}.

(d) Let $y = 0.021021\ldots$. Then $1000y = 21.021021\ldots$.
Subtraction yields $999y = 1000y - y = 21$, and hence $y = 21/999 = 7/333$.

Question 2

(a) The correct answers here are
$A \cap \mathbb{Q} = \{2, \frac{1}{2}\}$, $A \cap \mathbb{R} = \{2, \frac{1}{2}, \pi\}$, $A \cap B = \{\frac{1}{2}\}$ and $A - B = \{2, \pi\}$.

Many candidates were unaware of the definitions of rational and real numbers, and some thought erroneously that the set $\mathbb{R}$ contains rational numbers only. Further, many confused the two symbols $\cap$ and $\cup$. Try to remember that $\cup$ stands for $\cup$nion.

(b) One good answer to this question is $\{100n : n \in \mathbb{N}\}$, but equivalent answers are possible. Many attempted to use Java or C++ notation to express a number with remainder 0 on division by 100. Some credit was given for such answers, but candidates should be warned not to mix mathematical and programming notation as they are not completely equivalent.

(c) (i) R is reflexive. To show this you must show that $(a, b)R(a, b)$ for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, i.e. that $a \geq a$ and $b \geq b$ for all $a, b \in \mathbb{Z}$, which is clearly true.

(ii) R is anti-symmetric. To show this you must show that $(a, b)R(x, y)$ and $(x, y)R(a, b)$ only when $(a, b) = (x, y)$. But $a \geq x$ and $x \geq a$ implies $x = a$ for all $a, x \in \mathbb{Z}$, and similarly $b = y$, so the required result follows.

(iii) R is transitive, for if $(a, b)R(x, y)$ and $(x, y)R(\alpha, \beta)$ then $a \geq x \geq \alpha$ and $b \geq y \geq \beta$, implying that $a \geq \alpha$ and $b \geq \beta$ such that $(a, b)R(\alpha, \beta)$.

(iv) As R is reflexive, anti-symmetric and transitive, it is a partial order.

(v) R is not an order as e.g. $(1, 2)$ and $(2, 1)$ are not related either way.

This part was very poorly answered in general, mainly because many candidates struggled with using the mathematical notation involved.

Question 3

(a) The truth set of $\neg q \wedge p$ consists of all even integers not less than 10, i.e. $\{10, 12, 14, 16, \ldots\}$. Many candidates confused the two symbols $\wedge$ and $\vee$ here.

(b) A number of equivalent answers are possible here, e.g.

(i) $p \wedge q$;

(ii) $\neg r \wedge \neg p$;

(iii) $r \wedge \neg q$.

(c) (i) The truth table asked for is

p	q	r	$q \vee r$	$p \to (q \vee r)$	$\neg q$	$\neg r$	$\neg q \wedge \neg r$	$\neg p$	$(\neg q \wedge \neg r) \to \neg p$
0	0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	0	0	1	1
0	1	0	1	1	0	1	0	1	1
0	1	1	1	1	0	0	0	1	1
1	0	0	0	0	1	1	1	0	0
1	0	1	1	1	1	0	0	0	1
1	1	0	1	1	0	1	0	0	1
1	1	1	1	1	0	0	0	0	1

The table shows that the two statements are true under exactly the same conditions. They are thus logically equivalent. Many candidates computed the correct truth table, but failed to explain why it shows that the two given statements are logically equivalent.

(ii) The contrapositive of the given statement is
"If $n \neq 1$ then either $n \geq 2$ or n is not a positive integer."

Question 4

(a) A good answer here would be that $\lfloor x \rfloor = n \in \mathbb{Z}$ if $n \leq x < n+1$. Alternatively, using words, you could explain that $\lfloor x \rfloor$ is the largest integer less than or equal to x. This question was very badly answered by many candidates. It requires giving a relatively elaborate definition using exact language, a weak point for many.

(b) $f(3) = 0$ and $f(-3) = -1$. f is not one-to-one as e.g. $f(3) = 0 = f(2)$ while $2 \neq 3$. f is not onto either, as f takes on integer values only while the codomain is specified as $\mathbb{R}$.

(c) $f(x)$ is $O(x)$ if there is a positive constant k and a number N such that for all $x \geq N$ we have $|f(x)| \leq k|x|$. This was another question requiring a formal definition which very few candidates attempted.

(d) Using the definition from (c), $f(x)$ is $O(x)$ as $f(x) \leq \frac{x}{10}$ by the definition of the floor function, and so for $x \geq 0$ we have that $f(x) \leq \frac{x}{10}$ and also that $f(x) \geq 0$, such that $|f(x)| \leq \frac{1}{10}|x|$ for all $x \geq 0$.

Question 5

(a) $u_2 = u_1 + 4 = 1 + 4 = 5$, $u_3 = u_2 + 6 = 5 + 6 = 11$, $u_4 = u_3 + 8 = 11 + 8 = 19$, $u_5 = u_4 + 10 = 19 + 10 = 29$. Most candidates answered this question correctly, but some computed $u_n = u_{n-1} + 2(n-1)$ instead of $u_n = u_{n-1} + 2n$.

(b) Many candidates did not attempt this question or did poorly. Many attempts at the proof write down a good base and induction hypothesis, but then go wrong in the induction step because they mix up what they know and what they still need to prove. One good piece of advice, that will save many errors is, that you should never write down an =-symbol without specifically checking that you are absolutely sure you know that what is on the left hand side of it is equal to what is on the right hand side of it.

When proving an identity by induction the proof has 4 steps: the base, setting up the induction hypothesis, the induction step and a final remark stating why the identity holds by induction.

When proving the base case and the induction step, you must demonstrate that the left hand side (LHS) of the identity is equal to the right hand side (RHS) of the identity for the case in question. It is usually best to keep the two computations completely separate in order not to confuse what you know and what you have not proven yet.

Here we want to prove the identity $u_n = n^2 + n - 1$ for all $n \geq 1$:

Base case: When $n = 1$, $LHS = u_1 = 1$ and $RHS = 1^2 + 1 - 1 = 1$. The identity thus holds for $n = 1$.

Induction hypothesis: Suppose the identity holds for some $k \geq 1$, i.e.

$$u_k = k^2 + k - 1.$$

Induction step: We must prove that the identity also holds for $n = k + 1$:

$$RHS = (k+1)^2 + (k+1) - 1 = k^2 + 3k + 1.$$

$$\begin{aligned} LHS &= u_{k+1} = u_k + 2(k+1) \text{ by the recurrence relation} \\ &= k^2 + k - 1 + 2(k+1) \text{ by the induction hypothesis} \\ &= k^2 + 3k + 1 = RHS. \end{aligned}$$

Hence the result is also true for $n = k + 1$ and thus for all $n \geq 1$ by induction.

(c) Quite a number of candidates answered this question correctly, realising that what was required was to substitute into the sum the expression for u_n proven in (b):
$$\sum_{n=1}^{100}(u_n - (n-1)^2) = \sum_{n=1}^{100}(n^2 + n - 1 - (n-1)^2) = \sum_{n=1}^{100}(3n - 2)$$
$$= 3\sum_{n=1}^{100} n - 2\sum_{n=1}^{100} 1 = 3 \cdot 100 \cdot 101/2 - 200 = 14950.$$

Question 6

(a) The sum of the degrees of the vertices of G is twice the number of edges of G.

(b) $5 + 1 + 1 + 1 + 1 = 9$ is odd, and by (a) the sum of the degree sequence of a graph should be even. Hence there cannot exist a graph with this degree sequence.

(c) Quite a number of candidates thought this graph exists because the sum of the degree sequence given is even. However, note carefully that the result in (a) states that IF the graph G exists, THEN the sum of the degree sequence is even. The converse statement is not true in general. E.g. a simple graph with the degree sequence given in the exercise would have 5 vertices, and as one of them has degree 4 it means the graph is connected and there thus cannot be a vertex of degree 0.

(d) Part (d) is a generalisation of part (c) and this graph does not exist either: a simple graph with the degree sequence given in the exercise would have n vertices, and as one of them has degree $n - 1$ it means the graph is connected and there thus cannot be a vertex of degree 0.

(e) Some candidates provided excellent answers here, while many other candidates realised that this graph cannot exist, but then failed to give a good argument. The largest simple graph on 40 vertices is the complete graph K_{40} which has 780 edges. Hence it is not possible to construct a simple graph on 40 vertices with 800 edges.

Question 7

(a) (i) In this question the task was to construct a binary search tree to store an ordered list of 12 integers. The algorithm in the subject guide shows how to do this by putting the 12 integers in records numbered 1 to 12 and then storing these in the tree. The root is the record labelled $\lfloor (12+1)/2 \rfloor = 6$. In this case the 6th record is 12, so 12 is stored at the root of the tree. Following the algorithm in the subject guide we get at level 1 of the tree the records 6 and 18, at level 2 we have records 2, 8, 14 and 22; at level 3 we have 4, 10, 16, 20 and 24 and at level 4 we have terminators only. Many candidates confused the record labels 1-12 with the 12 numbers in the list, but generally this question was well answered. A number of candidates used an alternative algorithm using the ceiling function rather than the floor function to construct the binary search tree. Some credit was given for this even though none of these candidates justified their alternative algorithm. It is always important to show your working, but if you are using an alternative method to the one taught on the course, it is of paramount importance to explain and justify the method such that it is clear that you understand that what you are doing is indeed a viable alternative method. E.g. with the alternative method used in this exercise the examiners will be looking for evidence that you have not merely confused the floor function with the ceiling function and happened by accident to create a binary search tree in the process.

(ii) The height of the tree is 4, so a maximum of 4 comparisons are needed in order to find an existing record.

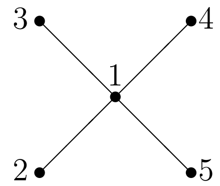
(b) (i) The root is the record labelled $\lfloor (600+1)/2 \rfloor = 300$.

(ii) For the records at level 1: in the left subtree the records stored are numbers 1-299 and in the right subtree the records stored are numbers 301-600, hence the root of the left subtree is $\lfloor (299+1)/2 \rfloor = 150$ and the root of the right subtree is $\lfloor (301+600)/2 \rfloor = 450$.

(iii) At levels 0 - 8 there are $\sum_{r=0}^{8} 2^r = 511$ records stored, so at level 9 we have just 89 records stored. Many candidates here failed to read the question properly and thought the answer to the question was 512, the maximum number of records which can be stored at level 9, rather than the actual number of records stored there.

Question 8

(a) (i) The graph G is



(ii) Computing

$$A^2 = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

we see that there are no paths of length 2 between vertex 1 and any other vertex while between any other pair of distinct vertices there is precisely one path of length 2.

(iii) The entries in A^3 show the number of *walks* of length 3 between any pair of vertices.

$$A^3 = \begin{pmatrix} 0 & 4 & 4 & 4 & 4 \\ 4 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(iv) As a path is a walk, we can see from A^3 above that the only candidates for paths of length 3 start or end at vertex 1. However, the graph is a tree and in a tree there is a unique path between any pair of vertices. The paths from 1 to any other vertex have length 1 and we saw in part (a)(ii) that the paths between a pair of vertices other than 1 all have length 2. Hence there are no paths of length 3 in G. Many candidates guessed the right answer here, but failed to justify it.

(b) (i) A function $f : V(H) \to V(G)$ is an isomorphism when f is a one-to-one correspondence that preserves adjacency, i.e. whenever xy is an edge of H then $f(x)f(y)$ is an edge of G. Most candidates were not able to provide this definition, leaving the question blank.

(ii) Define f by the following table:

x	a	b	c	d	e
$f(x)$	2	3	1	4	5

Next check what happens to the edges of H under this one-to-one correspondence.

edge xy of H	ca	cb	cd	ce
$f(x)f(y)$	12	13	14	15

We see that the edges of H map to the edges of G, and so f preserves adjacency and is an isomorphism. It was good to see that many candidates were able to provide an isomorphism in this question. However, most forgot to show that the one-to-one correspondence provided actually preserves adjacency, and this was required for full marks here.

Question 9

A substantial number of candidates did very well on this question. Though hardly any candidate was able to describe the sample space in (a) they correctly computed its size in (b) and went on to compute the probabilities in (c).

(a) The sample space consists of all possible set lunches. Let S be the set containing the 3 starters, M be the set containing the 4 mains and let D be the set containing the 5 desserts. Then a large lunch is an element of the set $S \times M \times D$, a medium lunch is an element of the set $(S \times M) \cup (M \times D)$ and a small lunch is an element of M. So a set lunch is an element of $(S \times M \times D) \cup (S \times M) \cup (M \times D) \cup M$.

(b) The cardinality of $(S \times M \times D) \cup (S \times M) \cup (M \times D) \cup M$ is $(3 \times 4 \times 5) + (3 \times 4) + (4 \times 5) + 4 = 96$.

(c) Using the figure computed in (b) we have

(i) $(3 \times 4 \times 5)/96 = 5/8$;

(ii) $|M| + |S \times M| = 4 + 12 = 16$, so $P(\text{no dessert}) = 16/96 = 1/6$;

(iii) There are 2 starters, 4 mains and 3 desserts without nuts, giving 4 small, $8 + 12 = 20$ medium and 24 large lunches without nuts.
Hence $P(\text{no nuts}) = 48/96 = 1/2$.

Question 10

(a) The solution is $(x, y, z) = (\frac{1}{2}, 1, 0)$.
Note that you are required to solve the system using Gaussian elimination, hence you must clearly demonstrate that you have done so, and any other solution method gets little or no credit. It is also a good idea to write down all the elementary row operations you do on the augmented matrix.

(b) Most candidates were able to compute the three matrices

$$\text{(i) } AB = \begin{pmatrix} 14 \\ 18 \\ 14 \end{pmatrix}; \qquad \text{(ii) } X = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}; \qquad \text{(iii) } Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In (ii), that $AX = B$ follows from part (a). Appreciating that there is a connection between matrix equations and systems of linear equations is important, and the examiners were pleased to see that quite a number of candidates spotted the connection here. The matrix Y needed in (iii) is the 3×3 identity matrix, and most candidates answered this question correctly.