MA4016 - Engineering Mathematics 6

Solutions to Midterm

1. Solve the system of difference equations

$$\begin{aligned} x_n &= 2x_{n-1} + y_{n-1}, & x_0 &= 0, \\ y_n &= -6x_{n-1} - 5y_{n-1} + 5, & y_0 &= 6. \end{aligned}$$

Version 1: Substitution

$$\begin{aligned} x_{n+1} &= 2x_n + y_n \\ &= 2x_n - 6x_{n-1} - 5y_{n-1} + 5 \\ &= 2x_n - 6x_{n-1} - 5(x_n - 2x_{n-1}) + 5 \\ &= -3x_n + 4x_{n-1} + 5 \end{aligned}$$

This is a nonhomogeneous linear recurrence relation. Solve homogeneous first with characteristic equation

$$r^2 + 3r - 4 = 0 \quad \Rightarrow \quad r_1 = -4,\, r_2 = 1.$$

Thus

$$x_n^{hom} = c_1 + c_2(-4)^n.$$

Nonhomogeneous part is $5 = 5 \cdot 1^n$. Thus with $s = 1$, $m = 1$, $t = 0$ we have the ansatz

$$x_n^{part} = np_0 \quad \Rightarrow \quad p_0(n+1) = -3np_0 + 4(n-1)p_0 + 5 \quad \Rightarrow \quad p_0 = 1$$

and the general solution for x_n is

$$x_n = c_1 + c_2(-4)^n + n.$$

The general solution for y_n follows by

$$y_n = x_{n+1} - 2x_n = -c_1 - 6c_2(-4)^n + 1 - n.$$

With the initial conditions follows $c_1 = 1$ and $c_2 = -1$ and therefore

$$\begin{aligned} x_n &= 1 - (-4)^n + n \\ y_n &= -1 + 6(-4)^n + 1 - n. \end{aligned}$$

Version 2: with discrete Putzer algorithm (mind the nonhomogenity) ...

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2. (i) Suppose two algorithms solve the same problem. Algorithm A_1 solves the problem of size n using $(1 + \log n)(2^n + n^2)$ operations, while algorithm A_2 solves the same problem with $(n + 2^n)(n^2 + \log n)$ operations. Which algorithm of this two algorithms is the more efficient one in terms of operations used for large values of n?
(a) A_1 (b) A_2 (c) Either one
(d) It depends on n (e) Not computable from information given.

We try Θ-estimates first. If there is a difference in them, we can decide which one is more efficient.

$$\begin{aligned} (1 + \log n)(2^n + n^2) &= \Theta(\log n)\Theta(2^n) = \Theta(\log n\, 2^n) \\ (n + 2^n)(n^2 + \log n) &= \Theta(2^n)\Theta(n^2) = \Theta(n^2 2^n) \end{aligned}$$

With $\log n < n^2$ for $n > 1$ follows: **(a)** is the right answer.

(ii) The product of three matrices $A \in \mathbb{R}^{2n,n}$, $B \in \mathbb{R}^{n,1}$ and $C \in \mathbb{R}^{1,2n}$ for a positive integer n can be written as $(A \cdot B) \cdot C$. With the standard matrix multiplication algorithm, the number of scalar multiplications required is $\Theta(f(n))$ with $f(n)$ given by
(a) n^3 (b) n^2 (c) $n^{\log_2 7}$
(d) $n^2 \log n$ (e) Not computable from information given.

The matrix $D = A \cdot B$ has $2n \cdot 1 = 2n$ entries with n multiplications each. This gives $2n^2$ multiplications. The matrix $D \cdot C$ has $2n \cdot 2n = 4n^2$ entries with 1 multiplication each, thus $4n^2$ multiplication. Together we need $6n^2 = \Theta(n^2)$ multiplications. Answer **(b)** is right.

(iii) The complete solution of

$$a_n = 4a_{n-2}, \quad a_0 = 0,\, a_1 = 2$$

is given by
(a) 0 (b) $2n$ (c) $\frac{2}{3}(4^n - 1)$
(d) $2^{n-1}(1 - (-1)^n)$ (e) Not computable from information given.

The general solution is $a_n = c_1(-2)^n + c_2 2^n$ and initial conditions give $c_1 = -1/2$ and $c_2 = 1/2$. Thus

$$a_n = -\frac{1}{2}(-2)^n + \frac{1}{2}2^n = 2^{n-1}(1 - (-1)^n)$$

and therefore **(d)** is the right answer.

(iv) Suppose $f(n)$ is increasing and satisfies the recurrence relation

$$f(n) = 4f(n/2) + n^3 \log n$$

with $f(1) = 1$. Then $f(n)$ is estimated for large n by
(a) $\Theta(n^2)$ (b) $\Theta((n\log(n))^2)$ (c) $\Theta(n^3 \log n)$
(d) $\Theta(n^4)$ (e) Not computable from information given.

We have $a = 4$, $b = 2$, $\log_b a = 2$ and $g(n) = n^3 \log n > 0$ for large n. Master Theorem case 3 holds with $n^3 \log n = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$ and

$$4\left(\frac{n}{2}\right)^3 \log(n/2) = \frac{1}{2}n^3\log(n/2) < \frac{1}{2}n^3 \log n$$

Thus $ag(n/b) < cg(n)$ holds with $c = 1/2 < 1$. It follows

$$f(n) = \Theta(g(n)) = \Theta(n^3 \log n)$$

and therefore answer **(c)**.
Alternatively, iteration gives for $n = 2^k$

$$\begin{aligned}
f(n) = f(2^k) &= 4f(2^{k-1}) + 2^{3k} k \log 2\\
&= 4^2 f(2^{k-2}) + 2^{3k-1}(k-1)\log 2 + 2^{3k} k \log 2\\
&= 4^2 f(2^{k-2}) + \log 2\left[(2^{3k} + 2^{3k-1})k - 2^{3k-1}\right]\\
&= 4^3 f(2^{k-3}) + \log 2\left[(2^{3k} + 2^{3k-1} + 2^{3k-2})k - 1\cdot 2^{3k-1} - 2\cdot 2^{3k-2}\right]\\
&\vdots\\
&= 4^k f(1) + \log 2\left[k\sum_{j=0}^{k-1} 2^{3k-j} - \sum_{j=0}^{k-1} j2^{3k-j}\right]
\end{aligned}$$

We have

$$\sum_{j=0}^{k-1} 2^{3k-j} = 2^{3k}\sum_{j=0}^{k-1}\left(\frac{1}{2}\right)^j = 2^{3k}\frac{1-\left(\frac{1}{2}\right)^k}{1-1/2} = 2^{2k+1}(2^k - 1)$$

and

$$\begin{aligned}
\sum_{j=0}^{k-1} j2^{3k-j} &= 2^{3k}\sum_{j=0}^{k-1} j\left(\frac{1}{2}\right)^j = 2^{3k}\frac{\left(\frac{1}{2}\right)^k(-k/2 - k + 1/2) - 1/2}{(-1/2-1)^2}\\
&= \frac{1}{9}\left[2^{2k+1} - 2^{3k+1} - 3k2^{2k+1}\right]
\end{aligned}$$

Together follows with $n = 2^k$ and $k = \log n$

$$\begin{aligned}
f(n) = f(2^k) &= 4^k + \frac{\log 2}{9}2^{2k+1}\left[(9k+1)2^k - 6k - 1\right]\\
&= n^2 + \frac{2\log 2}{9}n^2\left[(9\log n + 1)n - 6\log n - 1\right]\\
&= \Theta(n^3 \log n).
\end{aligned}$$

For $2^{k-1} < n < 2^k$ we apply the usual steps and get finally $f(n) = \Theta(n^3 \log n)$.

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(v) The sum of the squares of the first n odd positive numbers is given by
(a) $\dfrac{n(n+1)(2n+1)}{6}$ (b) $\dfrac{n(4n^2-1)}{3}$ (c) $8n^2 - 15n + 8$
(d) $\dfrac{n(n+1)(n+2)}{6}$ (e) None of the given.

Let

$$a_n = \sum_{k=1}^{n}(2k-1)^2 = a_{n-1} + (2n-1)^2.$$

We have immediately $a_n^{hom} = c_1$ and as ansatz for the nonhomogeneous problem

$$a_n^{part} = n(p_0 + p_1 n + p_2 n^2).$$

It follows with the nonhomogeneous recurrence relation

$$a_n^{part} = \frac{n}{3}(4n^2 - 1)$$

and therefore $a_n = c_1 + n(4n^2-1)/3$. With the initial condition $a_1 = 1$ follows $c_1 = 0$ and the sum is given by **(b)**.