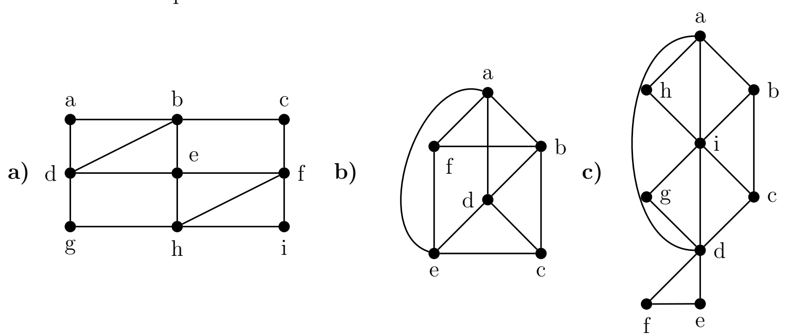
MA4016 - Engineering Mathematics 6

Solution Sheet 10: Euler, Hamilton and shortest paths

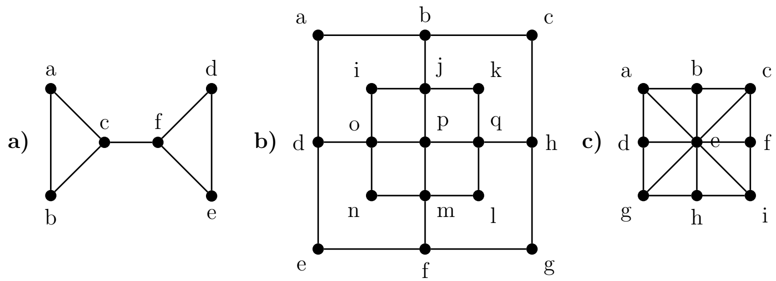
(April 16, 2010)

1. Determine whether the given graph has an Euler circuit and construct it when existent. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



a) The degrees of all vertices are even and therefore an Euler circuit exists: $(a, b, c, f, e, b, d, e, h, f, i, h, g, d, a)$.
b) Vertex f and vertex c have odd degree. Therefore no Euler circuit exists, but an Euler path from f to c exists: $(f, a, b, f, e, d, b, c, d, a, e, c)$.
c) Vertex b and vertex c have odd degree. Therefore no Euler circuit exists, but an Euler path from b to c exists: $(b, c, i, b, a, i, d, g, i, h, a, d, e, f, d, c)$

2. Determine whether the given graph has a Hamilton circuit and construct it when existent. If it does not, give an argument to show why no such circuit exists.
If no Hamilton circuit exists, determine whether the graph has a Hamilton path and construct such a path if one exists. If not, give an argument, why no Hamilton path exists.



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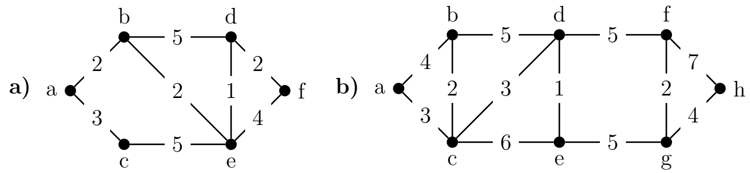
a) No Hamilton circuit exists because the two sub circuits $(a, b, c, a$ and $d, e, f, d)$ are connected by only the edge (c, f). But Hamilton Paths exist, e.g. (a, b, c, f, d, e).
b) No Hamilton circuits exist. Assuming a Hamilton circuit exists, all edges incident to vertices of degree 2 must be part of the circuit. This means the two circuits $a, b, c, h, g, f, e, d, a)$ and $(i, j, k, q, l, m, n, o, i)$ must be subsets of the Hamilton circuit. but a Hamilton circuit cannot contain a smaller circuit. Additionally it does not have a Hamilton path. Only two of the eight vertices of degree 2 can be endpoints of a Hamilton path, all others must be included. Thus one outer degree-2-vertex and one of the inner degree-2-vertex must be the endpoints. But if only one of the inner corners is the endpoint, p or one other corner cannot be included into a Hamilton circuit.
c) One of many Hamilton circuits is $(a, b, c, f, i, h, g, d, e, a)$.

3. Find the shortest distances and shortest paths between a and all other vertices in the following graphs.



We use Dijkstra's algorithm

a)

a	b	c	d	e	f	S
$[0, a]$	∞	∞	∞	∞	∞	$\emptyset$
	$[2, a]$	$[3, a]$	∞	∞	∞	$\{a\}$
		$[3, a]$	$[7, b]$	$[4, b]$	∞	$\{a, b\}$
			$[7, b]$	$[4, b]$	∞	$\{a, b, c\}$
			$[5, e]$		$[8, e]$	$\{a, b, c, e\}$
					$[7, d]$	$\{a, b, c, d, e\}$
						$\{a, b, c, d, e, f\}$

b)

a	b	c	d	e	f	g	h	S
$[0, a]$	∞	∞	∞	∞	∞	∞	∞	$\emptyset$
	$[4, a]$	$[3, a]$	∞	∞	∞	∞	∞	$\{a\}$
	$[4, a]$		$[6, c]$	$[9, c]$	∞	∞	∞	$\{a, c\}$
			$[6, c]$	$[9, c]$	∞	∞	∞	$\{a, b, c\}$
				$[7, d]$	$[11, d]$	∞	∞	$\{a, b, c, d\}$
					$[11, d]$	$[12, e]$	∞	$\{a, b, c, d, e\}$
						$[12, e]$	$[18, f]$	$\{a, b, c, d, e, f\}$
							$[16, g]$	$\{a, b, c, d, e, f, g\}$
								$\{a, b, c, d, e, f, g, h\}$

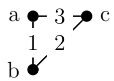
4. Find the shortest distances from vertex A to each of the other vertices for the following graph given by its adjacency matrix.

	A	B	C	D	E	F	G	H	I	J	K
A	-	3	-	-	5	-	-	4	-	-	-
B	3	-	2	-	5	7	-	-	-	-	-
C	-	2	-	3	-	2	6	-	-	-	-
D	-	-	3	-	-	-	7	-	-	-	2
E	5	5	-	-	-	4	-	7	-	-	-
F	-	7	2	-	4	-	4	5	4	3	-
G	-	-	6	7	-	4	-	-	-	4	6
H	4	-	-	-	7	5	-	-	2	-	-
I	-	-	-	-	-	4	-	2	-	6	-
J	-	-	-	-	-	3	4	-	6	-	5
K	-	-	-	2	-	-	6	-	-	5	-

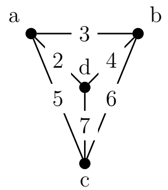
A	B	C	D	E	F	G	H	I	J	K	S
$[0,A]$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	$\emptyset$
	$[3,A]$	∞	∞	$[5,A]$	∞	∞	$[4,A]$	∞	∞	∞	$\{A\}$
		$[5,B]$	∞	$[5,A]$	$[10,B]$	∞	$[4,A]$	∞	∞	∞	$\{A,B\}$
		$[5,B]$	∞	$[5,A]$	$[9,H]$	∞		$[6,H]$	∞	∞	$\{A,B,H\}$
			$[8,C]$	$[5,A]$	$[7,C]$	$[11,C]$		$[6,H]$	∞	∞	$\{A-C,H\}$
			$[8,C]$		$[7,C]$	$[11,C]$		$[6,H]$	∞	∞	$\{A-C,E,H\}$
			$[8,C]$		$[7,C]$	$[11,C]$			$[12,I]$	∞	$\{A-C,E,H,I\}$
			$[8,C]$			$[11,C/F]$			$[10,F]$	∞	$\{A-C,E,F,H,I\}$
						$[11,C/F]$			$[10,F]$	$[10,D]$	$\{A-F,H,I\}$
						$[11,C/F]$				$[10,D]$	$\{A-F,H-J\}$
						$[11,C/F]$					$\{A-F,H-K\}$
											$\{A-K\}$

5. Is the shortest path between two vertices in a weighted graph unique if the weights of the edges are distinct?

No, consider shortest paths from a to c in



6. Solve the travelling salesman problem for this graph by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight (=length).



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WLOG start the Hamilton circuits at a. Then listing all possibilities gives (a,b,c,d,a), (a,b,d,c,a), (a,c,b,d,a), (a,c,d,b,a), (a,d,b,c,a), (a,d,c,b,a). We can ignore the order of the vertices and compute the total weight for the remaining three Hamilton circuits.

Hamilton circuit	total weight
(a,b,c,d,a)	3+6+7+2=18
(a,b,d,c,a)	3+4+7+5=19
(a,c,b,d,a)	5+6+4+2=17

Thus the solution to the TSP is (a,c,b,d,a).