In many of the following problems we use that if $\mathbf{x}$ and $\mathbf{y}$ are both $l \times 1$ vectors then computing the scalar product $\mathbf{x}^T\mathbf{y}$ requires l (scalar) multiplications and $l-1$ (scalar) additions.

1. Multiplying a $n \times m$ matrix by a $m \times p$ matrix results in a $n \times p$ matrix, each of whose entries is a scalar product as described above. Hence the total number of multiplications in npm and the total number number of additions is $np(m-1)$. When $p = m = n$, these become n^3 and $n^2(n-1)$ respectively.

2. Algorithm 1 requires the evaluation of $m-1$ matrix products, each product being of two $n \times n$ matrices, followed by the evaluation of the product of a $n \times n$ matrix and a $n \times 1$ vector. Hence using the result of Question 1 gives a grand total of $(m-1)n^3 + n^2$ multiplications and $(m-1)n^2(n-1) + n(n-1)$ additions.
   Algorithm 2 requires the evaluation of m products, each of a $n \times n$ matrix by a $n \times 1$ vector. Again using the result of Question 1 gives a total of mn^2 multiplications and $mn(n-1)$ additions.
   Algorithm 2 requires fewer operations whenever $m > 1$.

3. *Gauss* Elimination: In the Elimination phase, the first pass of a row reduction algorithm (assuming $a_{11} \neq 0$) changes the augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} & \tilde{b}_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \tilde{a}_{n2} & \cdots & \tilde{a}_{nn} & \tilde{b}_n \end{pmatrix}$$

   where

$$\tilde{a}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \qquad \tilde{b}_i = b_i - \frac{a_{i1}}{a_{11}} b_1$$

   Each of the $n-1$ changed rows requires 1 division, n multiplications and n subtractions. The second and subsequent passes ignore the unchanged rows. Thus we can analyse the procedure recursively.
   Letting m_n^e be the number of multiplications/ divisions and a_n^e the number of additions/subtractions used in the elimination phase, we get the linear recurrences

$$\begin{aligned} m_n^e &= m_{n-1}^e + (1+n)(n-1), \qquad & m_1^e = 0 \\ a_n^e &= a_{n-1}^e + n(n-1), & a_1^e = 0 \end{aligned}$$

   which have solutions

$$m_n^e = \sum_{i=1}^{n} i^2 - n = \frac{(n-1)n(2n+5)}{6}, \qquad a_n^e = \frac{(n-1)n(n+1)}{3}$$

   respectively. Thus $m_n^e = \Theta(n^3)$ and $a_n^e = \Theta(n^3)$.
   In the Back Substitution phase, if the unknowns $x_n, x_{n-1}, \dots, x_{k+1}$ have been evaluated, we need to solve

$$\tilde{a}_{kk}x_k + (\tilde{a}_{k,k+1}x_{k+1} + \cdots + \tilde{a}_{k,n}x_n) = \tilde{b}_k$$

   to get x_k. This requires $n-k$ multiplications, $n-k-1$ additions, 1 subtraction and 1 division. Thus the total number of multiplications divisions m_n^b for this phase is given by

$$m_n^b = \sum_{k=1}^{n} n - k + 1 = \sum_{j=n}^{1} j = \frac{n(n+1)}{2} = \Theta(n^2)$$

Similarly a_n^b, the total number of additions/subtractions is

$$a_n^b = \sum_{k=1}^{n} n - k = \frac{(n-1)n}{2} = \Theta(n^2)$$

4. An efficient algorithm is as follows:

   Step 1 Use a row reduction algorithm to reduce the given matrix to triangular form. If the only operations used are adding a multiple of one row to another, then the determinant of the triangular matrix equals the determinant of the original matrix. If in addition,(i) row interchanges are used, we must keep track of the sign of the determinant or (ii) a row is multiplied by a constant(K), then the value of the determinant is also multiplied by K.

   Step 2 The determinant of a triangular matrix is given by the product of its diagonal entries.

   From Question 3, for step 1, $m_n = \Theta(n^3)$ and $a_n = \Theta(n^3)$, whereas for step 2, it is obvious that $m_n = n-1$ and $a_n = 0$. Thus the asymptotics for step 1 dominate the procedure.
   BEWARE! If you're doing a full analysis, notice that you are not dealing with an augmented matrix as in Question 3, but with a $n \times n$ matrix hence the recurrence relations for the elimination phase are

$$\begin{array}{rcll} m_n^e &=& m_{n-1}^e + (1+n-1)(n-1), & m_1^e = 0 \\ a_n^e &=& a_{n-1}^e + (n-1)(n-1), & a_1^e = 0 \end{array}$$

   which have solutions

$$m_n^e = \frac{(n-1)n(n+1)}{3} \qquad a_n^e = \frac{(n-1)n(2n-1)}{6}$$

   respectively.

5. *Cramer*'s Rule solves the system of equations

$$A\mathbf{x} = \mathbf{b}$$

   as follows. Define A_i as the matrix formed from A by replacing its i-th column by $\mathbf{b}$. Then

$$x_i = \frac{\det A_i}{\det A}, \qquad i = 1, 2, \ldots, n$$

   This requires evaluating $n+1$ determinants, and performing n divisions. Thus from Question 4 it requires $(n+1)\Theta(n^3) = \Theta(n^4)$ multiplications or additions. Compare this with *Gauss* Elimination which requires $\Theta(n^3)$ operations.