

## MA4016 - Engineering Mathematics 6

## Problem Sheet 5: Master theorem (March 05, 2010)

- 1. For each of the following recurrence relations, give an estimate for T(n) if the Master theorem applies, or state why the theorem does not apply.
  - a)  $T(n) = 2T(n/2) + n \log n$   $g(n) = n \log n, a = 2, b = 2, \log_b a = 1$ Case II)  $n \log n = \Theta(n \log^k n)$  for k = 1. Therefore  $f(n) = \Theta(n \log^2 n)$ .
  - b)  $T(n) = 8T(n/4) + n \ g(n) = n, \ a = 8, \ b = 4, \log_b a = 3/2$ Case I)  $n = \mathcal{O}(n^{3/2-\varepsilon})$  for  $\varepsilon = 1/2$ . Therefore  $f(n) = \Theta(n^{3/2})$ .
  - $\begin{array}{ll} \mathbf{c}) & T(n) = 2T(n/4) + n^{0.6} \\ & g(n) = n^{0.6}, \ a = 2, \ b = 4, \ \log_b a = 1/2 \\ & \text{Case III)} \ n^{0.6} = \Omega(n^{1/2 + \varepsilon}) \ \text{for} \ \varepsilon = 0.1. \end{array}$

$$c > \frac{2\left(\frac{n}{4}\right)^{0.6}}{n^{0.6}} = \frac{2}{4^{0.6}} \approx 0.871$$

Thus with c = 0.9 follows  $f(n) = \Theta(n^{0.6})$ .

- d)  $T(n) = \frac{1}{2}T(n/2) + n^2$ a = 1/2 < 1
- e) T(n) = 3T(n/3) + n/2  $g(n) = n/2, a = 3, b = 3, \log_b a = 1$ Case II)  $n/2 = \Theta(n \log^k n)$  with k = 0. Therefore  $f(n) = \Theta(n \log n)$ .
- $$\begin{split} \mathbf{f}) \quad T(n) &= 4T(n/2) + \log n \\ g(n) &= \log n, \ a = 4, \ b = 2, \ \log_b a = 2 \\ \text{Case I) } \log n &= \mathcal{O}(n^{2-\varepsilon}) \text{ with } \varepsilon = 1. \text{ Therefore } f(n) = \Theta(n^2). \end{split}$$
- **2.** Find f(n) when  $n = 4^k$ , where f satisfies the recurrence relation

$$f(n) = 5f(n/4) + 6n,$$

with f(1)=1 and estimate f if f is an increasing function.  $f(n)=f(4^k)==5f(4^{k-1})+6\cdot 4^k$ . With  $f_k=f(4^k)$ ,  $f(1)=f_0=1$  follows

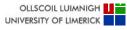
$$f_k = 5f_{k-1} + 6 \cdot 4^k.$$

 $f_k^h = \alpha 5^k, f_k^p = p_0 4^k = -24 \cdot 4^k \Rightarrow f_k = \alpha 5^k - 24 \cdot 4^k.$  with initial condition follows

$$f_k = 5^{k+2} - 6 \cdot 4^{k+1} = f(4^k) = f(n) = 25 \cdot 5^{\log_4 n} - 24n = 25n^{\log_4 5} - 24n$$

Applying Master Theorem (a = 5, b = 4, g(n) = 6n, case I) gives  $f(n) = \Theta(n^{\log_4 5})$ 

University of Limerick Department of Mathematics and Statistics Dr. S. Franz



3. Suppose that the function f satisfies the recurrence relation

$$f(n) = 2f(\sqrt{n}) + \log_2 n$$

whenever n is a perfect square greater than 1 and f(2) = 1.

a) Find f(16).

$$f(16) = 2f(4) + 4 = 4f(2) + 4 + 4 = 12$$

**b)** Find a big- $\mathcal{O}$  estimate for f(n).

Hint: Make the substitution  $m = \log_2 n$ .

$$f(2^m) = 2f(2^{m-1}) + m$$
. Let  $\tilde{g}(m) = f(2^m)$ . Then

$$\tilde{g}(m) = 2\tilde{g}(m/2) + m$$

and Master Theorem (a = 2, b = 2, g(m) = m, case II) gives

$$\tilde{g}(m) = \Theta(m \log m)$$

with  $m = \log_2 n$  follows

$$f(n) = f(2^m) = \tilde{g}(m) = \Theta(m \log m) = \Theta(\log n \cdot \log \log n).$$

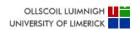
4. An efficient algorithm for evaluating polynomials is called Horner's method. It uses the representation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = (((((a_n)x + a_{n-1})x + a_{n-2})x + \dots)x + a_1)x + a_0.$$

Write an iterative pseudo code algorithm to compute the value of a polynomial given by  $a_0, a_1, \ldots, a_n$  at x. How many multiplications and additions are used to evaluate a polynomial of degree n at a position x?

$$\begin{array}{lll} procedure & horner\,(x\,,a\_0\,,a\_1\,,\dots\,,a\_n\colon \,real\,)\\ y\!:=\!a\_n & for & i\!:=\!n\!-\!1 & downto & 0\\ & y\!:=\!y\!*x\!+\!a\_i & end \\ & return\,(y) & end \end{array}$$

Multiplications: n, Additions: n. Standard algorithm (represented by left side) uses 2n-1 multiplications and n additions.



 $\begin{tabular}{ll} \bf 5. & What sequences of pseudorandom numbers is generated using the linear congruential generator \\ \end{tabular}$ 

$$x_{n+1} = (4x_n + 1) \mod 7$$
, with seeds  $x_0 = 1, 2$  and 3?

Explain this behaviour.

For  $x_0 = 1$  we obtain sequence  $\overline{1,5,0}$ , for  $x_0 = 2$  the sequence  $\overline{2}$  and for  $x_0 = 3$  the sequence  $\overline{3,6,4}$ , with  $\overline{\{a\}}$  indicating the subsequence  $\{a\}$  that repeats itself. We have three different sequences with period less then m = 7. The reason is, that a - 1 = 3 is not divisible by m = 7 as needed for maximal period.