Proof by Induction

Mathematical Induction is a special way of proving things. It has only 3 steps:

Step 1. Show it is true for the first one

Step 2.

Step 3. Show that if any one is true then the next one is true Then all will be true

Proof by Induction : Example 1

- Use proof by induction to show that $3^n - 1$ is a multiple of 2, for all values of the integer n.
- Is that true? Let us find out.

Proof by Induction : Example 1

Step 1. Show it is true for n=1

- $3^1 - 1 = 3 - 1 = 2$
- Yes 2 is a multiple of 2.
- $3^1 - 1$ is true

Example 1

Step 2. Assume it is true for $n = k$ i.e. assume that $3^k - 1$ is true
This is an assumption that we treat as a fact for the rest of this exercise.

Example 1

Now, prove that $3^{k+1} - 1$ is a multiple of 2
3^{k+1} is also 3×3^k And each of these are multiples of 2

Example 1

Because: 2.3k is a multiple of 2 (you are multiplying by 2) 3k-1 is true (we said that in the assumption above) So: 3k+1-1 is true

Example 2

Example: Adding up Odd Numbers

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

1. Show it is true for n=1 1 = 1^2 is True
2. Assume it is true for n=k

$$1 + 3 + 5 + \ldots + (2k - 1) = k^2$$

Example 2

Now, prove it is true for "k+1"

$$1 + 3 + 5 + \ldots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2$$

Example 2

We know that $1 + 3 + 5 + \ldots + (2k - 1) = k^2$ (the assumption above), so we can do a replacement for all but the last term:

$$k^2 + (2(k + 1) - 1) = (k + 1)^2$$

Example 2

- Now expand all terms:

$$k^2 + 2k + 2 - 1 = k^2 + 2k + 1$$

- And simplify:

$$k^2 + 2k + 1 = k^2 + 2k + 1$$

They are the same! So it is true.

Example 2

So the following expression is true.

$$1 + 3 + 5 + \ldots + (2(k+1) - 1) = (k+1)^2$$

Proof by Induction

Prove by induction that the series $3 + 7 + 11 + \ldots$ has the sum to r terms given by S_r, where

$$S_r = 2r^2 + r.$$

Proof by Induction

Step 1
Demonstrate for $r = 1$

- We know that first term is 3
- $S_1 = 2(1)^2 + 1 = 3$

Proof by Induction

Step 2 : Make statement for $r = k$

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Proof by Induction

Step 2 : Make statement for $r = k + 1$

$$S_{k+1} = 2(k+1)^2 + (k+1)$$

- $(k+1)^2 = k^2 + 2k + 1$
- $2(k+1)^2 = 2k^2 + 4k + 2$

$$S_{k+1} = 2k^2 + 4k + 2 + (k+1) = 2k^2 + 5k + 3$$

Proof by Induction

$$S_k = 2k^2 + k$$

$$S_{k+1} = 2k^2 + 4k + 2 + (k+1) = 2k^2 + 5k + 3$$

Difference is $4k + 3$ which is also expressed as $4(k+1) - 1$

Proof By Induction

A sequence is defined by the recurrence relation

$$x_{n+2} = 3x_{n+1} - 2x_n$$

The initial terms are x_1=1 and x_2 =3.
Find the values of x_3 and x_4 showing your workings.

$$x_3 = 3x_2 - 2x_1 = 3(3) - 2(1) = 7$$

$$x_4 = 3x_3 - 2x_2 = 3(7) - 2(3) = 15$$

Proof By Induction

Prove by induction that

$$x_n = 2^n - 1 \qquad \text{for } n \geq 1$$

Proof By Induction

Step 1 Show that statement is true for $n = 1$. (N.B. $x_1 = 1$).

$$x_n = 2^n - 1$$

$$x_1 = 2^1 - 1 = 2 - 1 = 1$$

Proof By Induction

Step 2 Assume that statement is true for $n = k$.

$$x_k = 2^k - 1$$

Similarly we will assume it for $n = k - 1$.The reason for this will become obvious later on.)

$$x_{k-1} = 2^{k-1} - 1$$

Proof By Induction

Step 3 Show that statement is true for $n = k + 1$.

$$x_{k+1} = 2^{k+1} - 1$$

Re-expressing this

$$x_{k+1} = 2.2^{k} - 1$$

Proof By Induction

Recall

$$x_{k+1} = 3x_k - 2x_{k-1}$$

Looking at right hand side

- First Term: $3x_k \;=\; 3(2^k - 1) \;=\; 3.2^k - 3$
- Second Term: $2x_{k-1} \;=\; 2(2^{k-1} - 1) \;=\; 2^k - 2$

$$x_{k+1} = (3.2^k - 3) - (2^k - 2) = 2.2^k + 1$$

$$x_{k+1} = 2.2^k + 1 = 2^{k+1} + 1$$