Chapter 1

Session 6

Session 06: Digraphs and Relations

6A.1 In-degree and out-degree

6A.2

6A.3

Relations

6B.1 Equivalence RElations (6.2.2)

6B.2

6B.3 Relations and Cartesian Products (6.3)

- **Reflexive**:
- **Symmetric**:
- **Transitive**:
- **Anti-symmetric**:
- **Equivalence Relation**:
- **Partial Order**:
- **Order**:

Question 6

Part A : Digraphs

Suppose $A = \{1, 2, 3, 4\}$. Consider the following relation in A

$$\{(1,1),(2,2),(2,3),(3,2),(4,2),(4,4)\}$$

Draw the direct graph of A. Based on the Digraph of A discuss whether or not a relation that could be depicted by the digraph could be described as the following, justifying your answer.

(i) Symmetric

(ii) Reflexive

(iii) Transitive

(iv) Antisymmetric

Part B : Relations

Determine which of the following relations xRy are reflexive, transitive, symmetric, or antisymmetric on the following - there may be more than one characteristic. if

(i) $x = y$

(ii) $x < y$

(iii) $x^2 = y^2$

(iv) $x \geq y$

Part C : Partial Orders

Let $A = \{0, 1, 2\}$ and $R = \{(0,0),(0,1),(0,2),(1,1),(1,2),(2,2)\}$ and $S = \{(0,0),(1,1),(2,2)\}$ be 2 relations on A. Show that

(i) R is a partial order relation.

(ii) S is an equivalence relation.

1.1 Digraphs and Relatiosn

Given a flock of chickens, between any two chickens one of them is dominant. A relation, R, is defined between chicken x and chicken y as xRy if x is dominant over y. This gives what is known as a pecking order to the flock. Home Farm has 5 chickens: Amy, Beth, Carol, Daisy and Eve, with the following relations:

- Amy is dominant over Beth and Carol
- Beth is dominant over Eve and Carol

- Carol is dominant over Eve and Daisy
- Daisy is dominant over Eve, Amy and Beth
- Eve is dominant over Amy.

Chapter 2

Session 7

Session 07:Sequences and Series

7A.1 Sequences

7A.2 Induction

7A.3 Series and the Sigma Notation

Recurrence Relations(7.1.1)

$u_1 = 2$ $u_2 = u_1 + 3 = 2 + 3 = 5$ $u_3 = u_2 + 3 = 5 + 3 = 8$ Airthmetic Progression

Proof by Induction(7.2.2)

Step 1 Base case

Step 2 Induction hypothesis

Step 3 Induction step

Series and Sigma Notation(7.2.3)

Question 7

Part A : Recurrence Relations

A sequence is defined by the recurrence relations

$$x_{n+2} = 3x_{n+1} - 2x_n$$

with initial terms $x_1 = 1$ and $x_2 = 3$.

(i) Calculate x_3, x_4 and x_5, showing your workings.

(ii) Prove by induction that $x_n = 2^n - 1$ for all $n \geq 1$

Part B : Summations

Compute the following summation

$$\sum_{i=25}^{i=100} i^2 + 3i - 5)$$

Say which of the set the following numbers belong to.
If they belong to more than one of these sets, give all the sets.
$\sqrt{2}$ $\frac{3}{7}$

Section 8 Exercises

- $8^{\frac{1}{3}}$ Recall $a^{\frac{b}{c}} = a^{\frac{b}{c}}$
-
-

Question 7b

Compute the following summation

$$\sum_{i=25}^{i=100} i^2 + 3i - 5)$$

Question 7

A sequence is defined by the formula $u_n = 5n - 3$ for $n \geq 1$

$$\frac{n(n+1)(2n+1)}{6}$$

Write the following sums in the Σ notation and evaluate them

- $1^2 + 2^2 + 3^2 + \ldots + 40^2 = ?$
- $2 + 5 + 10 + \ldots + 1601 = ?$
- $2 + 8 + 18 + \ldots + 3200 = ?$

Part A : Recurrence Relations

A sequence is defined by the recurrence relations

$$x_{n+2} = 3x_{n+1} - 2x_n$$

with initial terms $x_1 = 1$ and $x_2 = 3$.

(i) Calculate x_3, x_4 and x_5, showing your workings.

(ii) Prove by induction that $x_n = 2^n - 1$ for all $n \geq 1$

Proof By Induction

- **Base Step**
- **Induction Step**
- **Some Step**

2.0.1 Sequence and Series and Proof by Induction

$$\sum_{i=1}^{n}(n^2)$$

2.1 Proof By Induction

Prove by Induction the following expression

$$\sum_{r=1}^{r=n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

for all $n \in N_0$.

- **Step 1** : Is the proposition true for $n = 1$?
- **Step 2** : Show that if the proposition is true for $n = k$, then it is also true for $n = k + 1$.
- **Step 3** : Conclude that the proposition is true for all natural numbers greater than or equal to one.

2.1.1 Proof By Induction

Prove by induction the following expression

$$\sum_{r=1}^{r=n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

for all $n \in N_0$.

- **Step 1** : Is the proposition true for $n = 1$?

Left-hand side

$$\sum_{r=1}^{r=1} r^2 = 1^2 = 1$$

Left-hand side

$$\sum_{r=1}^{r=1} r^2 = 1^2 = \mathbf{1}$$

Right-hand side

$$\frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$$
$$= \frac{1 \times 2 \times 3}{6} = \frac{6}{6} = \mathbf{1}$$

- **Step 1** : Is the proposition true for $n = 1$?

Yes: The left-hand side and right-hand side of the equation yield the same value.

- **Step 2** : Show that if the proposition is true for $n = k$, then it is also true for $n = k + 1$.

Given:

$$\sum_{r=1}^{r=k} r^2 = \frac{k(k+1)(2k+1)}{6}$$

To Prove:

$$\sum_{r=1}^{r=k+1} r^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

2.1.2 Proof By Induction

$$\sum_{r=1}^{r=k+1} r^2 = 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$$

$$\sum_{r=1}^{r=k+1} r^2 = 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$$

$$\sum_{r=1}^{r=k+1} r^2 = \sum_{\boldsymbol{r=1}}^{\boldsymbol{r=k}} \boldsymbol{r^2} + (k+1)^2$$

$$\sum_{r=1}^{r=k+1} r^2 = \sum_{\boldsymbol{r=1}}^{\boldsymbol{r=k}} \boldsymbol{r^2} + (k+1)^2$$

$$\sum_{r=1}^{r=k+1} r^2 = \frac{k(k+1(2k+1))}{6} + (k+1)^2$$

Chapter 3

Session 8

Question 8

Part A : Spanning Trees

1. How many edges are in the spanning tree T ?
2. What is the sum of the degree sequence of T?
3. Write down all the possible degree sequences for the spanning tree T.

Part B : Binary Search Trees

Suppose a database, comprised of 30,000 internal nodes, is structured as a Binary Search Tree.

1. What is the key (number) of the Root node?
2. What are the keys of the nodes at level 1?
3. For the nodes at level1, how many subtrees are there?
4. State which nodes are in the substrees of the level 1 nodes?
5. How many nodes are the between the root (level 0) and level 7.] (Hint: use a summation theorem mentioned in session 7
6. What is the maximum number of searchs in this database?

3.1 Question 8B

Suppose a database, comprised of 30,000 internal nodes, is structured as a Binary Search Tree.

1. What is the Key of the Root node?
2. What are the keys of the nodes at level 1?

3. For the nodes at level1, how many subtrees are there?
4. State which nodes are in the substrees of the level 1 nodes?
5. How many nodes are the between the root (level 0) and level 7.] (Hint: use a summation theorem mentioned in session 7
6. What is the maximum number of searchs in this database?

Binary Search Trees

What is a Binary Search Tree

$$\lfloor \frac{\log_2}{T} \rfloor$$

Session 08:Trees

8A.1 Trees

8A.2 Spanning Trees

8A.3 Rooted trees

8A.4 Binary Search Trees

A tree is a directed graph that contains no cycles.

3.2 Question 8

1) Draw this tree 2) Construct all the isomorphic tress with 6 vertices which can be obtainbed by attaching a new vertex of degreee one to a vertex of T. 3) Explain briefly why the tree obtained in (ii) is not isomorphic to each other. 4) Construct a tree with 6 vertices which is not isomorphic to any tree you constructed in (ii)

Part b Determine the number of nodes on level 5 and level 10 Find an expression in terms of Σ and h for the number of internal nodes in such a tree. What is the smallest possible height of such a tree if there are at least 900 internal nodes.

3.3 Question 8A

1. How many edges are in the spanning tree T ?
2. What is the sum of the degree sequence of T?
3. Write down all the possible degree sequences for the spanning tree T.

3.4 Tree Definition

What properties must a graph have in order for it to be a tree? (ii) Say, with reason, whether or not it is possible to construct a tree with degree sequence 4, 3, 3, 1, 1.

Question 1

(b) Express the following hexadecimal number as a decimal number: (A32.8)16. [3] (c) Convert the following decimal number into base 2, showing all your working: $(253)_{10}$. [2] (d) Express the recurring decimal 0.4242424 . . . as a rational number in its simplest form. [2]

Question 7

Let S be a set and let R be a relation on S Explain what it means to say thet $\mathcal{R}$ is

(i) reflexive

(ii) symmetrix

(iii) anti-symmetric

(iv) Transitive

3.4.1 Question 10

(a) Given the following adjacency matrices A and B where A =
1 0 1 0 1 2 1 2 0
,B =
1 2 0 2 0 1 0 1 1
(i) Say whether or not the graphs they represent are isomorphic. (ii) Calculate A2 and A4 and say what information each gives about the graph corresponding to A. [6] (b) (i) Write down the augmented matrix for the following system of equations.

$$2x + y - z = 2$$
$$x - y + z = 4$$
$$x + 2y + 2z = 10$$

(ii) Use Gaussian elimination to solve the system. [4]

Chapter 4

Session 9

4.1 Probability and Counting

Given S is the set of all 5 digit binary strings, E is the set of a 5 digit binary strings beginning with a 1 and F is the set of all 5 digit binary strings ending with two zeroes.

(a) Find the cardinality of S, E and F.

(b) Draw a Venn diagram to show the relationship between the sets S, E and F.

(c) Show the relevant number of elements in each region of your diagram.

4.1.1 Axioms of Probability

The Axioms of Probability

- The probability of a certain event is 1.
- The probability of an impossible event is 0.
-

Session 09: Probability

9A.1 Counting Methods

9A.2 Counting using Sets

9A.3 Probability

9A.4 Independent Events

9B.1 Permutation

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

$$\binom{6}{3} = \frac{6!}{(6-3)!3!} = \frac{6!}{3! \times 3!}$$

$$\frac{6!}{3! \times 3!} = \frac{6 \times 5 \times 4 \times 3!}{3! \times 3!} = \frac{120}{6} = 120$$

- $\binom{6}{2} = 15$
- $\binom{5}{2} = 10$
- $\binom{4}{0} = 1$
- $\binom{4}{3} = 4$
- pairwise disjoint sets
- The addition principle

Theorem

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Probability

9B.2 The sample space of an experiment (S)

9B.3 The size of a sample space

9B.4 Indepedent Evcents (9.3.1)

Session 9 Probability

Binomial Coefficients

- factorials

$$n! = (n) \times (n-1) \times (n-2) \times \ldots \times 1$$

  - $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
  - $3! = 3 \times 2 \times 1$

- Zero factorial

$$0! = 1$$

The complement rule in Probability
$P(C') = 1 - P(C)$
If the probability of C is 70% then the probability of C' is 30%

Probability

Binomial Coefficients

- factorials

$$n! = (n) \times (n-1) \times (n-2) \times \ldots \times 1$$

  - $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
  - $3! = 3 \times 2 \times 1$

- Zero factorial

$$0! = 1$$

The complement rule in Probability
$P(C') = 1 - P(C)$
If the probability of C is 70% then the probability of C' is 30%

4.2 Counting

Given S is the set of all 5 digit binary strings, E is the set of a 5 digit binary strings beginning with a 1 and F is the set of all 5 digit binary strings ending with two zeroes.

(a) Find the cardinality of S, E and F.

(b) Draw a Venn diagram to show the relationship between the sets S, E and F.

(c) Show the relevant number of elements in each region of your diagram.

Probability: Binomial Coefficients

- factorials

$$n! = (n) \times (n-1) \times (n-2) \times \ldots \times 1$$

  - $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
  - $3! = 3 \times 2 \times 1$

- Zero factorial

$$0! = 1$$

The complement rule in Probability
$P(C') = 1 - P(C)$
If the probability of C is 70% then the probability of C' is 30%
Basic Operations with Matrices Basic Operations with Matrices

- Addition of Matrices
- Transpose of a Matrix
- Adding and Subtracting Matrices
- Scalar Multiplication

Matrix Multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} u & v \\ w & x \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} u & v \\ w & x \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix}$$

Matrix Multiplication

$$\begin{pmatrix} \underline{\boldsymbol{a}} & \underline{\boldsymbol{b}} \\ c & d \end{pmatrix} \times \begin{pmatrix} u & v \\ w & x \end{pmatrix}$$

Matrix Multiplication

$$\begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ c & d \end{pmatrix} \times \begin{pmatrix} \underline{u} & v \\ \underline{w} & x \end{pmatrix}$$

Matrix Multiplication

$$\begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ c & d \end{pmatrix} \times \begin{pmatrix} u & v \\ w & x \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{a}u + \boldsymbol{b}w & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{pmatrix}$$

Transpose of a Matrix

- The transpose of a matrix is transformation of that matrix when all the rows are arranged into columns and columns arranged by rows.
- The transpose of a matrix A is usually denoted A^T or A'.
- The relevant `R` function is `t()`.

Sample Space (2) Consider a random experiment in which a coin is tossed once, and a number between 1 and 4 is selected at random. Write out the sample space S for this experiment.

$$S = \{(H,1),(H,2),(H,3),(H,4),(T,1),(T,2),(T,3),(T,4)\}$$

(H and T denoted 'Heads' and 'Tails' respectively.)

Contingency Tables Suppose there are 100 students in a first year college intake.

- 44 are male and are studying computer science,
- 18 are male and studying statistics
- 16 are female and studying computer science,
- 22 are female and studying statistics.

We assign the names M, F, C and S to the events that a student, randomly selected from this group, is male, female, studying computer science, and studying statistics respectively. Contingency Tables The most effective way to handle this data is to draw up a table. We call this a ***contingency table***.

A contingency table is a table in which all possible events (or outcomes) for one variable are listed as row headings, all possible events for a second variable are listed as column headings, and the value entered in each cell of the table is the frequency of each joint occurrence.

	C	S	Total
M	44	18	62
F	16	22	38
Total	60	40	100

Contingency Tables It is now easy to deduce the probabilities of the respective events, by looking at the totals for each row and column.

- P(C) = 60/100 = 0.60
- P(S) = 40/100 = 0.40
- P(M) = 62/100 = 0.62
- P(F) = 38/100 = 0.38

Remark:
The information we were originally given can also be expressed as:

- $P(C \cap M) = 44/100 = 0.44$

- $P(C \cap F) = 16/100 = 0.16$
- $P(S \cap M) = 18/100 = 0.18$
- $P(S \cap F) = 22/100 = 0.22$

Conditional Probability (1)

The definition of conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- $P(B)$ Probability of event B.

of event A.

- $P(A|B)$ Probability of event A given that B has occurred.
- $P(A \cap B)$ Joint Probability of event A and event B.
- Will be given tomorrow.

Conditional Probabilities (2)

Compute the following:

1. $P(C|M)$: Probability that a student is a computer science student, given that he is male.
2. $P(S|M)$: Probability that a student studies statistics, given that he is male.

3. $P(F|S)$: Probability that a student is female, given that she studies statistics.

Conditional Probabilities (3)

Part 1) Probability that a student is a computer science student, given that he is male.

$$P(C|M) = \frac{P(C \cap M)}{P(M)} = \frac{0.44}{0.62} = 0.71$$

Part 2) Probability that a student studies statistics, given that he is male.

$$P(S|M) = \frac{P(S \cap M)}{P(M)} = \frac{0.18}{0.62} = 0.29$$

Conditional Probabilities (4)

Part 3) Probability that a student is female, given that she studies statistics.

$$P(F|S) = \frac{P(F \cap S)}{P(S)} = \frac{0.22}{0.40} = 0.55$$

Bayes' Theorem Bayes' Theorem is a result that allows new information to be used to update the conditional probability of an event.

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}$$

Use this theorem to compue $P(S|F)$, the probability that a student studies statistics, given that she is female.

$$P(S|F) = \frac{P(F|S) \times P(S)}{P(F)} = \frac{0.55 \times 0.40}{0.38} = 0.578$$

Independent Events

- Suppose that a man and a woman each have a pack of 52 playing cards.
- Each draws a card from his/her pack. Find the probability that they each draw a Queen.
- We define the events:
  - A = probability that man draws a Queen = 4/52 = 1/13
  - B = probability that woman draws a Queen = 1/13
- Clearly events A and B are independent so:

$$P(A \cap B) = 1/13 \times 1/13 = 0.005917$$

Expected Value and Variance of a Random Variable

The probability distribution of a discrete random variable is be tabulated as follows

x_i	1	2	3	4	5	6
$p(x_i)$	2/8	1/8	1/8	1/8	c	1/8

- What is the value of c?
- What is expected value and variance of the outcomes?

Expected value(1)

- Necessarily $C = 0.25 = 2/8$.

- We must compute $E(X)$ as follows

$$E(X) = \sum x_i p(x_i)$$

- That formula is **not** given in the formulae.

$E(X) = (1 \times \frac{2}{8}) + (2 \times \frac{1}{8}) + \ldots + (5 \times \frac{2}{8}) + (6 \times \frac{1}{8})$
$E(X) = 27/8 = 3.375$ Variance(1)

- The formula for computing the variance of a discrete random variable

$$V(X) = E(X^2) - E(X)^2$$

- This is not given in the formulae for tomorrow's exam.

- We must compute $E(X^2)$

x_i	1	2	3	4	5	6
x_i^2	1	4	9	16	25	36
$p(x_i)$	2/8	1/8	1/8	1/8	2/8	1/8

Variance (2)

- $E(X^2) = (1 \times \frac{2}{8}) + (4 \times \frac{1}{8}) + \ldots + (25 \times \frac{2}{8}) + (36 \times \frac{1}{8})$
- $E(X^2) = \frac{117}{8} = 14.625$
- $V(X) = E(X^2) - E(X)^2 = 14.625 - (3.375)^2 = 3.2344$

Combinations (1) Combinations formula

$$^nC_k = \frac{n!}{k! \times (n-k)!}$$

- Remark $n! = n \times (n-1)!$
- $0! = 1$

Combinations (2) Show that

$$^nC_0 = 1$$

Solution:

$$^nC_0 = \frac{n!}{0! \times (n-0)!} = \frac{n!}{n!} = 1$$

Combinations (3) Show that

$$^{n}C_1 = n$$

Solution:

$$^{n}C_1 = \frac{n!}{1! \times (n-1)!} = \frac{n \times (n-1)!}{(n-1)!} = n$$

Combinations (4) Compute $^{7}C_2$

Solution:

$$^{7}C_2 = \frac{7!}{2! \times (7-2)!} = \frac{7 \times 6 \times 5!}{2! \times 5!} = \frac{42}{2} = 21$$

Combinations (5) Compute $^{11}C_1$

Solution:

$$^{11}C_1 = \frac{11!}{1! \times 10!} = \frac{11 \times 10!}{1 \times 10!} = 11$$

Chapter 5

Session 10

Systems of Linear Equation

5.1 Matrices

What are the dimensions of the following matrix

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \left(\begin{matrix} (a_1 \times c_1) + (a_2 \times c_2) & (a_1 \times d_1) + (a_2 \times \\ (b_1 \times c_1) + (b_2 \times c_2) & (b_1 \times d_1) + (b_2 \times \end{matrix} \right.$$

$$\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} (1 \times 1) + (3 \times 4) & (1 \times 2) + (3 \times 1) \\ (0 \times 4) + (2 \times 4) & (0 \times 2) + (2 \times 1) \end{pmatrix} = \begin{pmatrix} 14 & 5 \\ 8 & 2 \end{pmatrix}$$

$$\left(\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix} \begin{matrix} 1 & 3 \\ 0 & 2 \end{matrix} \right) =?$$

Reduced Echelon Form

-

-
-

Session 10: Matrices and Systems of Equations

10A.1 Dimensions of a Matrix

10A.2 Matrix Multiplication

10A.3 Matrix Calculations

10A.4

10B.1 Systems of Equations

10B.2 Expression Systems of Equations as Matrices

10B.3 Augmented Matrices

10B.4 Guassian Elimination

Question 10

Session 10: Matrices and Systems of Equations

10A.1 Dimensions of a Matrix

10A.2 Matrix Multiplication

10A.3 Matrix Calculations

10A.4

10B.1 Systems of Equations

10B.2 Expression Systems of Equations as Matrices

10B.3 Augmented Matrices

10B.4 Guassian Elimination

Question 10A

Say what information the first row of the matrix contains. Find the number of edges of G.

Write doen the augmented matrix for the following system of equations. x+y+2z=7 2x+y+3z =11 x-27+5z=4

Use Gaussian elimination to solve the system.

Part B : Summations

Compute the following summation

$$\sum_{i=25}^{i=100} i^2 + 3i - 5)$$

5.2 Matrices

What are the dimensions of the following matrix

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} (a_1 \times c_1) + (a_2 \times c_2) & (a_1 \times d_1) + (a_2 \times d_2) \\ (b_1 \times c_1) + (b_2 \times c_2) & (b_1 \times d_1) + (b_2 \times d_2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} (1 \times 1) + (3 \times 4) & (1 \times 2) + (3 \times 1) \\ (0 \times 4) + (2 \times 4) & (0 \times 2) + (2 \times 1) \end{pmatrix} = \begin{pmatrix} 14 & 5 \\ 8 & 2 \end{pmatrix}$$

$$\left(\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix} \begin{matrix} 1 & 3 \\ 0 & 2 \end{matrix} \right) =?$$

Question 1

(i) (1 Mark) When is the positive integer p said to be a prime number?

(ii) (2 Marks) Express the following hexadecimal number as a decimal number, and as a binary number:

$$(A32.8)_{16}$$

(iii) (2 Marks) Convert the following decimal number into base 2, showing all your working: $(253)_{10}$.

(iv) (2 Marks) Covert the decimal integer $(407)_{10}$ to binary notation.

(v) (2 Marks) Showing your working, express the following number

$$1.024024024024\ldots$$

as a ration number in its simplest form.

(vi) (1 Mark) Compute the following $101101_2 + 1101_2$

Question 2

Let A and B and C be subsets of a universal set U.

(a) (1 Mark) Draw a labelled Venn diagram depicting A,B,C in such a way that they divide U into 8 disjoint regions. [1]

(b) (3 Marks) The subset $X \subseteq U$ is defined by the following membership table below. Shade the region X on your diagram. Describe the region you have shaded in set notation as simply as you can.

A	B	C	X
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

(c) (3 Marks) The subset $Y \subseteq U$ is defined as $Y = A \cup (C - B)$. Construct the membership table for Y.

(d) (3 Marks) For each of the following statements say whether it is true or false, justifying your answer, using the Venn diagram you drew earlier.

(i) $Y \subseteq X$
(ii) $Y' \subseteq X'$
(iii) $Y - X = A \cap B \cap C$.

Question 3

(a) Let n be an element of the set $\{10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$, and p and q be the propositions:

$$p : n \text{ is odd}, q : n < 15$$

. Draw up truth tables for the following statements and find the values of n for which they are true:

(i) $p \vee \neg q$

(ii) $\neg p \wedge q$

(b) Use truth tables to find a statement that is logically equivalent to $\neg p \rightarrow q$.

(i)

Question 4

Let S be the set of all 4 bit binary strings. The function $f : S \rightarrow Z$ is defined by the rule:

$$f(x) = \text{ the number of zeros in x}$$

for each binary string $x \in S$. Find:

(a) (4 Marks) Answer the following questions

(i) the number of elements in the domain

(ii) f(1010)

(iii) the set of pre-images of 1

(iv) the range of f.

(b) (2 Marks) Decide whether the function f, as defined above, has either the one to one or the onto property, justifying your answers.

(c) (2 Marks) State the condition to be satisfied by a function $f : X \rightarrow Y$ for it to have an inverse function $f^{-1} : Y \rightarrow X$.

(d) (2 Marks) Define the inverse functions for each of the following:

Question 6

Given a flock of chickens, between any two chickens one of them is dominant. A relation, R, is defined between chicken x and chicken y as xRy if x is dominant over y. This gives what is known as a pecking order to the flock. Home Farm has 5 chickens: Amy, Beth, Carol, Daisy and Eve, with the following relations:

- Amy is dominant over Beth and Carol
- Beth is dominant over Eve and Carol
- Carol is dominant over Eve and Daisy
- Daisy is dominant over Eve, Amy and Beth
- Eve is dominant over Amy.

Question 6

Let $A = \{0, 1, 2\}$ and $R = \{(0,0), (0,1), (0,2), (1,1), (1,2), (2,2)\}$ and $S = \{(0,0), (1,1), (2,2)\}$ be 2 relations on A. Show that

(i) R is a partial order relation.

(ii) S is an equivalence relation.

Let S be a set and let R be a relation on S Explain what it means to say thet $\mathcal{R}$ is

(i) reflexive

(ii) symmetrix

(iii) anti-symmetric

(iv) Transitive

Question 7

Let the sequence un be defined by the recurrence relation

$$u_{n+1} = u_n + 2n, \text{ for n = 1, 2, 3, ...}$$

and let $u_1 = 1$.

Question 8

(Part B :Binary Search Trees - 5 Marks)
Suppose a database, comprised of 30,000 internal nodes, is structured as a Binary Search Tree.

(i) What is the Key of the Root node?

(ii) What are the keys of the nodes at level 1?

(iii) For the nodes at level 1, how many subtrees are there?

(iv) How many nodes are the between the root (level 0) and level 4.]

(v) What is the maximum number of searchs in this database?

Question 9

Given S is the set of all 5 digit binary strings, E is the set of a 5 digit binary strings beginning with a 1 and F is the set of all 5 digit binary strings ending with two zeroes.

(a) Find the cardinality of S, E and F.

(b) Draw a Venn diagram to show the relationship between the sets S, E and F. Show the relevant number of elements in each region of your diagram.

- A college teaches courses in the following subjects areas: mathematics, computing and statistics.
- Students in the college may choose their courses from these three subject areas.
- Students are not obliged to take courses from these three subject areas, and may instead take courses in other subject areas.
- Let the subject areas be represented by the letters **M** for mathematics, **C** for computing and **S** for statistics.
- Draw a labelled Venn diagram showing the areas **M**, **C**, and **S** in such a way as to represent the students studying at the college.
- On your diagram show the number of students studying in each region of the Venn diagram.
  - Currently 600 students are enrolled in the college.
  - 300 students are taking mathematics courses.

- 120 student are taking statistics courses.
- 380 students are taking computing courses.
- 40 students study courses from all three subject areas.
- 200 mathematics students are taking computing courses as well.
- 60 computing students are also takings statistics courses.
- 70 statistics students are also taking mathematics course.

(i) How many students study none of these courses at all?

(ii) How many students are taking mathematics courses but not computing or statistics courses.

(iii) How many students study courses from precisely two of these subject areas?

Question 10

Part B : Gaussian Elimination - 5 Marks

(i) Say whether or not the graphs they represent are isomorphic.

(ii) Calculate A^2 and A^4 and say what information each gives about the graph corresponding to A. [6]

(i) Write down the augmented matrix for the following system of equations.

$$2x + y - z = 2$$

$$x - y + z = 4$$

$$x + 2y + 2z = 10$$

(ii) Use Gaussian elimination to solve the system.