MA4016 - Engineering Mathematics 6

Solution Sheet 1 (February 04, 2010)

1. Proof by contraposition: Assume not (m is even or n is even).

2. Proof by contradiction: Assume that at most 2 days of the 25 have the same month. We have 12 different month and therefore only 24 days fulfil this assumption.

3. Existence proof: $3 = 1 + 2$
Uniqueness proof: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \overset{!}{=} n + 1$ gives as only positive solution $n = 2$ and therefore the number $n + 1 = 3$.

4. Existence proof:

8gl jug	5gl jug	3gl jug
8	0	0
3	5	0
3	2	3
6	2	0
6	0	2
1	5	2
1	4	3

5. Proof by mathematical induction: IS:

$$\sum_{i=1}^{k+1} i^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = (k+1)^2\left(\frac{k^2+4k+4}{4}\right) = \left(\frac{(k+1)(k+2)}{2}\right)^2.$$

6. The sequence of numbers $c_n = \sum_{i=1}^{n} 1/2^i$ leads to the conjecture $\sum_{i=1}^{n} 1/2^i = 1 - 2^{-n}$.
Proof with mathematical induction.
Alternatively the formula for the geometric sum can be applied.

7. Proof by mathematical induction: IS:

$$(1+x)^{k+1} = (1+x)(1+x)^k \geq (1+x)(1+kx) = 1 + (k+1)x + \underbrace{kx^2}_{\geq 0} \geq 1 + (k+1)x$$

8. We check $n = 1, 2, 3, 4, 5$ and see $n = 1$ works and $n \geq 4$ too. The later can be proved with mathematical induction.
IS:

$$(k+1)! = (k+1)k! \geq (k+1)k^2 = (k+1)^2\frac{k^2}{k+1} > (k+1)^2$$

where we used

$$\frac{k^2}{k+1} = \frac{k^2-1}{k+1} + \frac{1}{k+1} = \underbrace{k-1}_{\geq 3} + \underbrace{\frac{1}{k+1}}_{\geq 0} \geq 3 > 1.$$

Thus the inequality is true for $n = 1$ and $n \geq 4$.
Alternatively an direct proof instead of the induction can be used, starting with

$$0 < n^2 - 4n + 2 = (n-1)(n-2) - n \text{ for } n \geq 4.$$

9. Mathematical induction: IS:

$$(k+1)^3+2(k+1) = k^3+3k^2+3k+1+2k+2 = k^3+2k+3(k^2+k+1) = 3\cdot(m+k^2+k+1).$$

Direct proof: Use the three distinct cases $n = 3m$, $n = 3m + 1$ and $n = 3m + 2$ with m positive integer and prove seperately.
Alternatively use the equivalence

$$\text{“3 divides } n^3 + 2n\text{”} \equiv \text{“3 divides } n^3 - n = (n-1)n(n+1).$$