MA4016 Problem Sheet 4: Divide and Conquer

1. Consider the functions

$$f(n) = \quad n^2, \quad n\log_2 n, \quad n^{2.5}, \quad 1.1^n$$

. Which of the following is true:

$$\begin{array}{rcllll} f(n) & = & O(n), & O(n^2), & O(n^5), & O(n\log_{10} n) \\ & = & \Omega(n), & \Omega(n^2), & \Omega(n^5), & \Omega(n\log_{10} n) \\ & = & \Theta(n), & \Theta(n^2), & \Theta(n^5), & \Theta(n\log_{10} n) \end{array}$$

2. For each of the following recurrence relations, give an expression for $T(n)$ if the Master theorem applies, or state why the theorem does not apply.

(1) $T(n) = 2T(n/2) + n\log n$
(2) $T(n) = 8T(n/4) + n$
(3) $T(n) = 2T(n/4) + n^{0.6}$
(4) $T(n) = \frac{1}{2}T(n/2) + n^2$
(5) $T(n) = 3T(n/3) + n/2$
(6) $T(n) = 4T(n/2) + \log n$

3. Write down a recurrence relation for the number of comparisons required to sort a list of n numbers using Merge Sort. Use the Master theorem to find an expression for this number.

4. The basis of the *Strassen* algorithm (SA) for matrix multiplication is as follows: To multiply the (square block) 2×2 matrices

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \qquad \left(= \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}, \quad \text{say} \right)$$

compute

$$\begin{aligned} p_1 &= (x_{11} + x_{22})(y_{11} + y_{22}) \\ p_2 &= (x_{21} + x_{22})y_{11} \\ p_3 &= x_{11}(y_{12} - y_{22}) \\ p_4 &= (-x_{11} + x_{21})(y_{11} + y_{12}) \\ p_5 &= (x_{11} + x_{12})y_{22} \\ p_6 &= x_{22}(-y_{11} + y_{21}) \\ p_7 &= (x_{12} - x_{22})(y_{21} + y_{22}) \end{aligned}$$

Then

$$\begin{aligned} z_{11} &= p_1 + p_6 - p_5 + p_7 \\ z_{12} &= p_3 + p_5 \\ z_{21} &= p_2 + p_6 \\ z_{22} &= p_1 - p_2 + p_3 + p_4 \end{aligned}$$

(a) Verify that the algorithm works for 2×2 matrices.

(b) Use SA to compute the following:

$$\begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 2 \\ 4 & 5 & 4 \\ 2 & -2 & 3 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 0 & 1 & -2 \\ 1 & 1 & -1 \end{pmatrix}$$

(c) Use the Master theorem to"solve" the following recurrence relation which gives the number of additions used by SA to multiply two $n \times n$ matrices:

$$a_n = 18n^2 + 7a_{n/2}, \qquad a_1 = 0$$

5. *Fibonacci* (Minimum) Search can be used effectively to find the minimum of an unimodal or "one-hump" function described in tabular form.
Let F_k represent the k-th *Fibonacci* number (The complete *Fibonacci* sequence is described by $F_{k+2} = F_{k+1} + F_k, \quad k \geq 0 \;\; F_0 = 0, F_1 = 1$).
To find the minimum of a unimodal function with $n = F_m$ values , proceed as follows:

(a) Set $k = m$.

(b) If $k \leq 3$, look up all and chose minimum.

(c) Look up values in positions F_{k-2} and F_{k-1}. Denote them by V_{Left} and V_{Right} respectively.

(d) If $V_{Left} < V_{Right}$, discard values from positions $F_{k-1} + 1$ to F_k. Set $k = k - 1$ and go to (b).

(e) If $V_{Left} > V_{Right}$, discard values from positions 1 to F_{k-2}. Renumber the remaining positions from 1 to F_{k-1}, set $k = k - 1$ and go to (b).

(f) If $V_{Left} = V_{Right}$, discard values from positions 1 to F_{k-2} and from positions $F_{k-1}+1$ to F_k. Renumber the remaining positions from 1 to F_{k-3}, set $k = k-3$ and go to (b).[1]

Answer the following:

(i) Detail how the algorithm works on the function $f(x) = |x - 360|$ with 987 values with position indexed by x.

(ii) How many table look ups are needed in the worst case when $n = F_m$?

(iii) How is the above algorithm modified if n is not a *Fibonacci* number.

(iv) Is this a "divide and conquer" algorithm ? If so, write down a recurrence for the number of look ups. Does the Master theorem say anything about this recurrence ?

[1]NB: There is an implicit assumption about the shape of the unimodal function being used here. What is it?