Mathematics for Computing
Functions

Contents

1 Sets of Numbers

- $\mathbb{Z}$ Set of all integers
- $\mathbb{Q}$ Set of all rational numbers
- $\mathbb{R}$ Set of all real numbers
- $\mathbb{Z}^+$ Set of all positive integers
- $\mathbb{Z}^-$ Set of all negative integers
- $\mathbb{R}^+$ Set of all positive real numbers
- $\mathbb{R}^-$ Set of all negative real numbers

2 Arrow Diagrams

- Domain
- Co-Domain
- Range

$$f(x) : \text{Domain} \to \text{Co-Domain}$$
$$f(x) : \mathbb{R} \to \mathbb{R}$$

Polynomial Functions (4.1.5)

Constants (P_0)

Linear Functions (P_1)

Quadratic Functions (P_2)

Cubic Functions (P_3)

Equality of Functions (4.1.6)

$$f(x) = g(x)$$

3 Special Mathematical Functions

3.1 Mathematical Operators

- The Square Root function
- The Floor and Ceiling functions
- The Absolute Value functions
- Root Functions
- Absolute Value Function
- Floor Function
- Ceiling Function

$$\lfloor 3.14 \rfloor = 3 \tag{1}$$
$$\lceil -4.5 \rceil = -5 \tag{2}$$
$$|-4| = 4 \tag{3}$$

For this course, only positive numbers have square roots. The square roots are positive numbers. (This statement is not strictly true. The square root of a negative number is called a complex number. However this is not part of the course).

Negative numbers can have cube roots

$$-27 = -3 \times -3 \times -3$$

$$\sqrt[3]{-27} = -3$$

4 Exponential and Logarithms

Laws of Logarithms

- Law 1 : Multiplcation of Logarithms

$$Log(a) \times Log(b) = Log(a + b)$$

- Law 2 : Division of Logarithms

$$\frac{Log(a)}{Log(b)} = Log(a - b)$$

- Law 3 : Powers of Logarithms

$$Log(a^b) = b \times Log(a)$$

4.1 Exercise

$h(x) : \mathbb{R} \to \mathbb{R}$ $g(x) : \mathbb{R} \to \mathbb{R}$

$$f(x) = sqrt(x)$$
$$g(x) = \sqrt{3}x + 2$$
$$h(x) = 2^x$$

- Is the function $h(x)$ an *onto* function?
- determine the inverse function of $h(x)$ and $g(x)$
- Simplify the following function.

$$j(x) = \log_4(h(6x))$$

4.2 Onto Functions

Definition: If every element in the co-domain of the function has an ancestor, the function is said to be "onto". An onto function has the property that the domain is equal to the co-domain.

Example 4.26 Page 53

Exponential and Logarithms

Rules

Expontials : Rules 4.18 Page 58
Logarithms : Rules 4.23 Page 61

$$log_a(a) = 1$$
$$log_a(b^c) = c \times log_a(b)$$

- $log_2(128) = 7$
- $log_2(1/4) = -2$
- $log_2(2) = 1$

$$log_a(b) = \frac{log_x(b)}{log_x(a)}$$

4.3 Logarithms

- Laws of Logarithms - Change of Base

$$\mathrm{Log}_b(x) = a$$
$$b^a = x$$
$$\mathrm{Log}_2(8) = 3$$
$$2^3 = 8$$

$$\mathrm{Log}_b(x) \times \mathrm{Log}_b(y) = \mathrm{Log}_b(x + y)$$
$$\mathrm{Log}_b(x^y) = y \times \mathrm{Log}_b(x)$$

$$\mathrm{Log}_y(x) = \frac{\mathrm{Log}_b(x)}{\mathrm{Log}_b(y)}$$

4.4 Exponents

- Rules of Exponents

$$(a^b)^c = a^{b \times c}$$

$$64^{2/3} = (4^3)^{2/3} = 4^{3 \times 2/3} = 4^2 = 16$$

$$(a^b) \times (a^c) = a^{b+c}$$
$$(3^2) \times (3^3) = 3^{2+3} = 3^5 = 243$$

Exercises

(a) Complete the following table for the functions

i) $g(x) = \log_3 x$,

ii) $h(x) = \sqrt[3]{x}$.

x	1				81	
$g(x)$		1	2			5
$h(x)$				3.00		

Express your answers to 2 decimal places only.

5 *One-to-One* Functions and *Onto* Functions

5.1 Invertible Functions

- One-to-One Function
- Onto Function

Onto Functions : Range and Co-Domain are equivalent

5.2 Inverting a Function

- You are given $f(x)$ in terms of x
- Re-arrange the equation so that x is given in terms of $f(x)$
- Replace x with $f^{-1}(x)$ and $f(x)$ with x

5.2.1 Example

- Determine the inverse function of $f(x)$. Re-arrange the equation so that x is given in terms of $f(x)$

$$f(x) : \mathbb{R} \to \mathbb{R} \ f(x) = \sqrt{x+1}$$

- Square both sides of the equation.

$$[f(x)]^2 = x + 1$$

- Subtract 1 from both sides of the equation. We have the equation written in terms of x.

$$f(x)^2 - 1 = x$$

- Replace x with $f^{-1}(x)$ and $f(x)$ with x

$$x^2 - 1 = f^{-1}(x)$$

- Re-arrange equation and specify domain and co-domain.

$$f(x) : \mathbb{R} \to \mathbb{R} \ f^{-1}(x) = x^2 - 1$$

6 Big O-Notation

(b) Let S be the set of all 4 bit binary strings.

The function $f : S \to \mathbb{Z}$ is defined by the rule:

$$f(x) = \text{the number of zeros in x}$$

for each binary string $x \in S$.
Find:

1. the number of elements in the domain
2. $f(1000)$
3. the set of pre-images of 1
4. the range of f.

(c)

4.a $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$

4.b

4.c

7 Section 4 Functions

7.1 Invertible Functions

A function is invertible if it fulfils two criteria

- The function is ***onto***,
- The function is ***one-to-one***.

State the conditions to be satisfied by a function $f : X \leftarrow Y$ for it to have an inverse function $f^{-1} : Y \leftarrow X$.
$\lceil \frac{x^2+1}{4} \rceil$ where $f : A \rightarrow \mathbf{Z}$

(i) Find $f(4)$ and the ancestors of 3.

(ii) Find the range of f.

(iii) Is f invertible? Justify your answer

Given $f : \mathbf{R} \rightarrow \mathbf{R}$ where f(x) =3x-1,define fully the inverse of the function f ,i.e.f^{-1}. State the value of $f^{-1}(2)$

7.2 Precision Functions

- Absolute Value Function $|x|$
- Ceiling Function $\lceil x \rceil$
- Floor Function $\lfloor x \rfloor$

Question1.2: State the range and domain of the following function

$$F(x) = \lfloor x - 1 \rfloor$$

7.3 Powers

$$2^4 = 2 \times 2 \times 2 \times 2 = 16$$

$$5^3 = 5 \times 5 \times 5 = 125$$

7.3.1 Special Cases

Anything to the power of zero is always 1

$$X^0 = 1 \text{ for all values of X}$$

Sometimes the power is a negative number.

$$X^{-Y} = \frac{1}{X^Y}$$

Example

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

7.4 Exponentials Functions

$$e^a \times e^b = e^{a+b}$$

$$(e^a)^b = e^{ab}$$

7.5 Logarithmic Functions

7.5.1 Laws for Logarithms

The following laws are very useful for working with logarithms.

1. $\log_b(X) + \log_b(Y) = \log_b(X \times Y)$
2. $\log_b(X)$ - $\log_b(Y) = \log_b(X/Y)$
3. $\log_b(X^Y)$= $Y \log_b(X)$

Question1.3 Compute the Logarithm of the following

- $\log_2(8)$
- $\log_2(\sqrt{128})$
- $\log_2(64)$
- $\log_5(125) + \log_3(729)$
- $\log_2(64/4)$

- $a^x = y \; log_a(y) = x$
- $e^x = y \; ln(y) = x$
- $log_a(x \times y) = log_a(x) + log_a(y)$
- $log_a(\frac{x}{y}) = log_a(x) - log_a(y)$
- $log_a(\frac{1}{x}) = -log_a(x)$
- $log_a(a) = 1$
- $log_a(1) = 0$
- $\lceil x \rceil$
- $\lfloor x \rfloor$

Sample value x	Floor $\lfloor x \rfloor$	Ceiling $\lceil x \rceil$	Fractional part $\{x\}$
$12/5 = 2.4$	2	3	$2/5 = 0.4$
2.7	2	3	0.7
-2.7	-3	-2	0.3
-2	-2	-2	0