UNIVERSITY *of* LIMERICK

OLLSCOIL LUIMNIGH

College of Informatics and Electronics

MID TERM ASSESSMENT PAPER

MODULE CODE: MA4016 SEMESTER: Spring 2009

MODULE TITLE: Engineering Mathematics 6 DURATION OF EXAMINATION: 45 minutes

LECTURER: Dr. M. Burke PERCENTAGE OF TOTAL MARKS: 20 %

EXTERNAL EXAMINER: Prof. J. Flavin

INSTRUCTIONS TO CANDIDATES: **Answer both questions. Each question is worth 10 marks. Each part of Question 2 carries equal marks. Use the Answer Sheet provided for Question 2.**

ANSWER SHEET

STUDENT'S NAME:
STUDENT'S ID NUMBER:

For each part of Question 2, place an "**X**" in the box of your choice.

Question	a	b	c	d	e	Do not write in this column
(i)	**X**					
(ii)		**X**				
(iii)		**X**				
(iv)			**X**			
(v)				**X**		

1. Solve the system of difference equations

$$\begin{aligned} x_{n+1} &= 2x_n + y_n, & x_0 = 2 \\ y_{n+1} &= x_n + 2y_n + 1, & y_0 = 1 \end{aligned}$$

2. (i) Algorithm A1 solves a problem of size n using $\Theta(n^5)$ operations, while algorithm A2 solves the same problem with $\Theta(2^n)$ operations. Which algorithm is the more efficient of the two in terms of operations used for large n ?
(a) A1 (b) A2 (c) Either one
(d) It depends on n (e) Not computable from information given

(ii) The standard polynomial-time algorithm to compute the determinant of a $n \times n$ matrix requires a number of scalar multiplications which is $\Theta(f(n))$ where $f(n) =$
(a) n (b) n^3 (c) n^4
(d) $n!$ (e) Not computable from information given

(iii) The 2-state *Markov* Chain:

	↓	S (current)	T (current)
next	S	1/4	3/4
next	T	3/4	1/4

has equilibrium probability $\lim_{n\to\infty} p_S(n) =$

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$
(d) 1 (e) Not computable from information given

(iv) For the *Markov* Chain shown in Fig 1, the probability of arriving at state C at time $n+1$ is given by $p_C(n+1) =$

$$(a)\ \frac{1}{3}p_B(n) + p_C(n) \quad (b)\ \frac{2}{3}p_B(n) + \frac{1}{2}p_C(n) \quad (c)\ \frac{1}{3}p_B(n) + \frac{1}{3}p_C(n)$$

$$(d)\ 1 - [p_A(n) + p_B(n)] \quad (e)\ \frac{1}{3}p_C(n)$$

(v) The number of operations used in a particular divide and conquer algorithm satisfies the recurrence $T(n) = 4T(n/2) + n^2$. Its asymptotic solution is $T(n) = \Theta(g(n))$ where $g(n) =$
(a) 1 (b) n (c) n^2
(d) $n^2 \log n$ (e) Not computable using the Master theorem

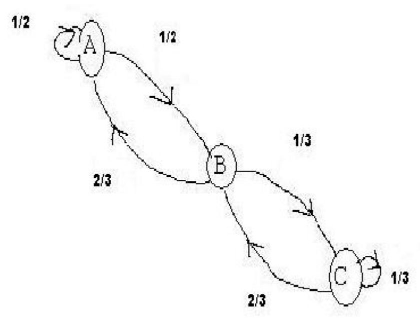

Figure 1: Markov Chains of Question 2 (iv)

Question 1 Solution

Either - convert to 2nd order difference equation:

$$x_{n+2} = 2x_{n+1} + y_{n+1} = 2x_{n+1} + x_n + 2y_n + 1 = 2x_{n+1} + x_n + 2[x_{n+1} - 2x_n] + 1$$

$$\Rightarrow x_{n+2} - 4x_{n+1} + 3x_n = 1, \qquad x_0 = 2, x_1 = 2x_0 + y_0 = 5$$

Homogeneous Equation:

$$x^h_{n+2} - 4x^h_{n+1} + 3x^h_n = 0$$

has characteristic equation

$$\begin{aligned} C^2 - 4C + 3 &= 0 \\ \Rightarrow C &= 1 \text{ or } 3 \\ \Rightarrow x^h_n &= A1^n + B3^n \\ &= A + B3^n \end{aligned}$$

Particular Solution: (Since the forcing function is a solution of the homogeneous equation, it is necessary to multiply the "original guess" by the appropriate power of n). In this case $x^p_n = Dn$. The (original) difference equation becomes

$$\begin{aligned} D(n+2) - 4D(n+1) + 3Dn &= 1 \\ \Rightarrow D[(1-4+3)n + (2-4) &= 1 \\ \Rightarrow D &= -\frac{1}{2} \end{aligned}$$

The general solution is thus $x_n = A + B3^n - \frac{1}{2}n$. The initial conditions (IC) give

$$\begin{aligned} 2 = x_0 &= A + B \\ 5 = x_1 &= A + 3B - \frac{1}{2} \\ \Rightarrow A &= \frac{1}{4} \\ B &= \frac{7}{4} \end{aligned}$$

Hence

$$\begin{aligned} x_n &= \frac{1}{4} + \frac{7}{4}(3^n) - \frac{1}{2}n \\ y_n = x_{n+1} - 2x_n &= -\frac{3}{4} + \frac{7}{4}(3^n) + \frac{1}{2}n \end{aligned}$$

Or - deal with as a system of equations:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

has solution

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=0}^{n-1} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{n-1-j} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The system matrix is diagonalisable:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1+3^n & -1+3^n \\ -1+3^n & 1+3^n \end{pmatrix}$$

so

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} 1+3(3^n) \\ -1+3(3^n) \end{pmatrix} + \frac{1}{2} \sum_{j=0}^{n-1} \begin{pmatrix} -1+3^{n-1-j} \\ 1+3^{n-1-j} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1+3(3^n) \\ -1+3(3^n) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -n+\frac{1-3^n}{1-3} \\ n+\frac{1-3^n}{1-3} \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 1+7(3^n)-2n \\ -3+7(3^n)+2n \end{pmatrix} \end{aligned}$$