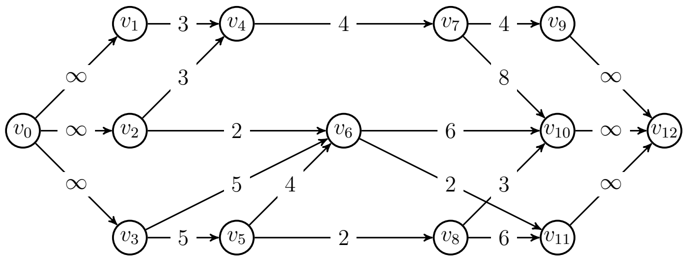
MA4016 - Engineering Mathematics 6

Problem Sheet 11: Maximal Flow and Turing machines

(April 23, 2010)

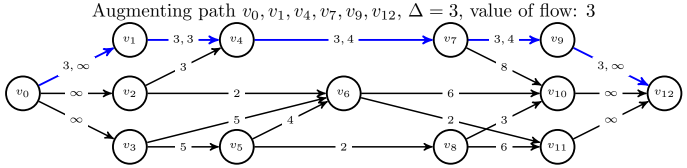
1. Find a maximal flow and a minimal cut in the following pumping network.



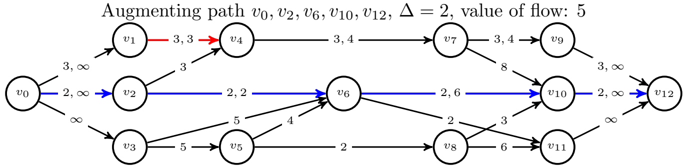
Find a maximal flow after changing the following labels: $(v_6, v_{11}) = 4$, $(v_4, v_7) = 8$.

A minimal cut is $P = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, $P' = \{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ with $C(P, P') = 14$. We use the Ford-Fulkerson algorithm and increase the flow according to the following pictures. On edges with no flow given assume a flow of 0.

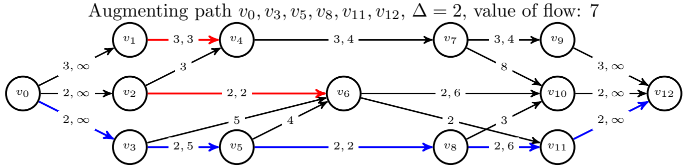
Augmenting path $v_0, v_1, v_4, v_7, v_9, v_{12}$, $\Delta = 3$, value of flow: 3



Augmenting path $v_0, v_2, v_6, v_{10}, v_{12}$, $\Delta = 2$, value of flow: 5



Augmenting path $v_0, v_3, v_5, v_8, v_{11}, v_{12}$, $\Delta = 2$, value of flow: 7



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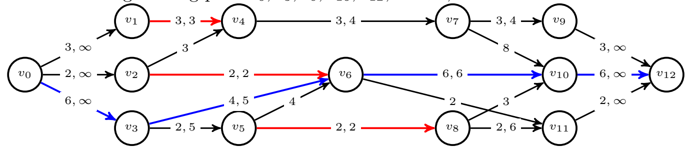
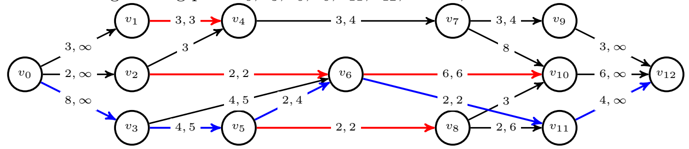
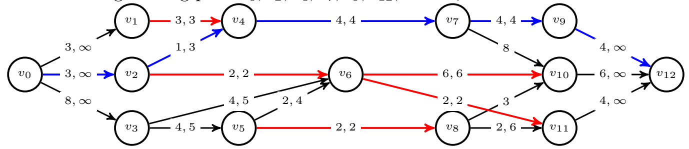
Augmenting path $v_0, v_3, v_6, v_{10}, v_{12}$, $\Delta = 4$, value of flow: 11



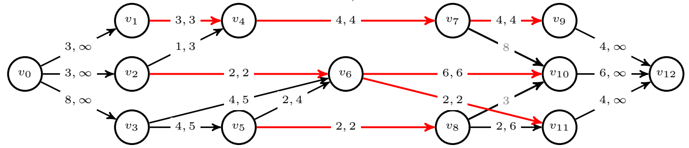
Augmenting path $v_0, v_3, v_5, v_6, v_{11}, v_{12}$, $\Delta = 2$, value of flow: 13



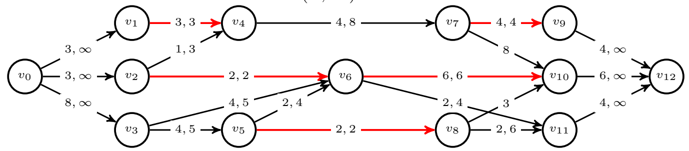
Augmenting path $v_0, v_2, v_4, v_7, v_9, v_{12}$, $\Delta = 1$, value of flow: 14



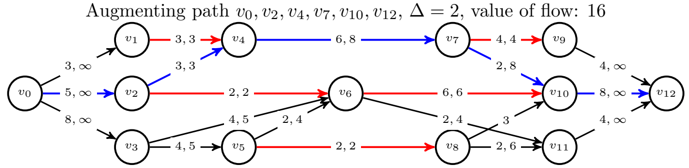
maximal flow, value of 14



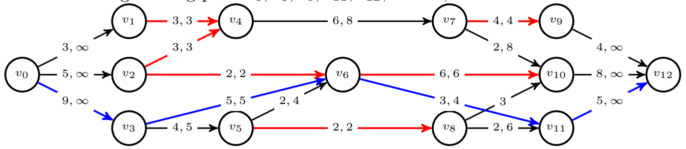
After change of labels the flow in the previous network is still a flow in the new network. We use it as initial flow. A new minimal cut is $P = \{v_0, v_1, v_2, v_3, v_5, v_6\}$, $P' = \{v_4, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ with $C(P, P') = 18$.



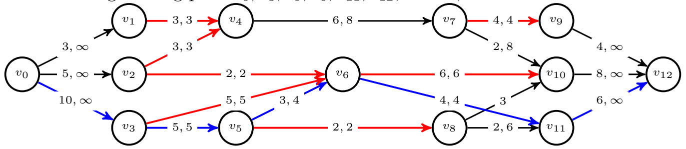
Augmenting path $v_0, v_2, v_4, v_7, v_{10}, v_{12}$, $\Delta = 2$, value of flow: 16



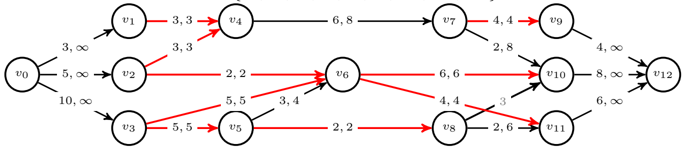
Augmenting path $v_0, v_3, v_6, v_{11}, v_{12}$, $\Delta = 1$, value of flow: 17



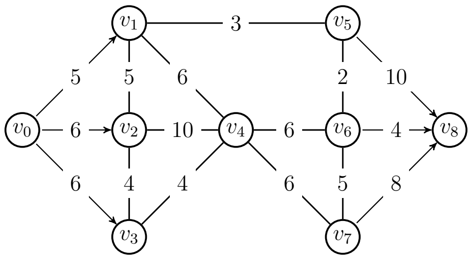
Augmenting path $v_0, v_3, v_5, v_6, v_{11}, v_{12}$, $\Delta = 1$, value of flow: 18



maximal flow with value 18, another minimal cut is $P = \{v_0, v_1, v_2, v_3\}$ and $P' = \{v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$



2. We want to maximise the flow from v_0 to v_8. The flow between two vertices, neither of which is v_0 or v_8, can be in either direction. Model this as system as a network and find a maximal flow and a minimal cut.



We replace each undirected labelled edge by two directed edges pointing in opposite directions with the same label. In this network graph we use the Ford-Fulkerson algorithm to find a maximal flow. A minimal cut is given by $P = \{v_0, v_1, v_2, v_3, v_4\}$ and $P' = \{v_5, v_6, v_7, v_8\}$ with $C(P, P') = 15$. Applying the algorithm gives the maximum flow in the following picture.

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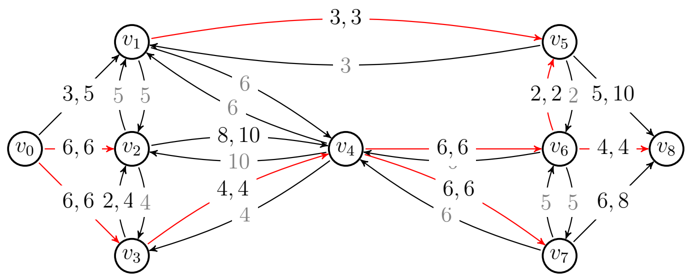
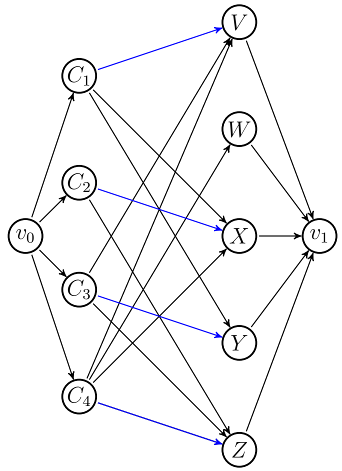
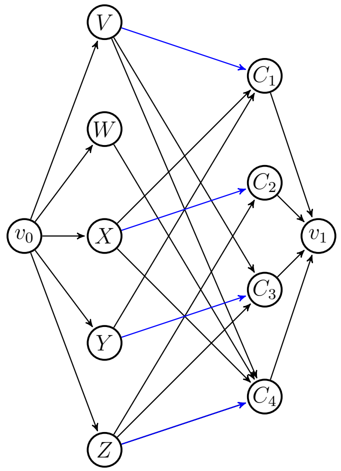


3. Five students, V, W, X, Y, and Z, are members of four committees, C_1, C_2, C_3, and C_4. The members of C_1 are V, X, and Y; the members of C_2 are X and Z; the members of C_3 are V, Y, and Z; and the members of C_4 are V, W, X, and Z. Each committee is to send a representative to the administration. No student can represent more than one committee.
Model this situation in a matching network and find a maximal matching. Is this a complete matching?
The corresponding matching network with oriented edges from the committees to the students and a complete matching in blue is given in the figure to the left. If we model the problem with oriented edges from the students to the committees, we find a maximal but not complete matching, see figure to the right. It is the same matching, so "complete" depends on the model.





4. Construct a Turing machine with tape symbols 0, 1, and B, when given a bit string as input, adds a 1 to the end of the bit string and does not change any of the other symbols on the tape.

add 1	0	1	B
s_0	s_0 0 R	s_0 1 R	s_1 1 R

5. Construct a Turing machine with tape symbols 0, 1, and B, when given a bit string as input, replaces the first 0 with a 1 and does not change any of the other symbols on the tape.

first 0 $\to$ 1	0	1	B
s_0	s_1 1 R	s_0 1 R	s_1 B R

6. Construct a Turing machine that recognises the set of all bit strings that end with a 0.

**0	0	1	B
s_0	s_0 0 R	s_0 1 R	s_1 B L
s_1	s_2 0 R		

s_2 is final state, s_1 is non final state. String is recognised as ending with 0 if Turing machine stops in state s_2.

7. Construct a Turing machine that computes the function $f(n) = n + 2$ for all nonnegative integers n.
We represent $n \geq 0$ in the unary system by $n + 1$ 1's on the tape and append two more 1's to the string.

add 2	1	B
s_0	s_0 1 R	s_1 1 R
s_1		s_2 1 R

8. Construct a Modulo-4 Machine, i.e. a *Turing* Machine which takes as tape input a string of symbols representing an integer and produces as tape output the string of symbols representing the remainder after division by 4 of the integer.
Illustrate the operation of the Modulo-4 Machine on the input string representing the number *seven*.

First let us use the unary system—encoding of n with $n+1$ 1's on the tape. We have to check, whether 5=4+1 1's are on the tape. If not, we are finished. If they are on the tape, we have to erase the left four 1's (idea: n mod 4 =$(n-4)$ mod 4).
We get a more compact Turing machine if we use a different encoding of n. Here a suitable one is the representation of n to the basis 4, e.g. $4 = (10)_4$, $7 = (13)_4$ ... Our alphabet is 0, 1, 2, 3, B. We have to check for two non-blank symbols on the tape and erase the left one (idea: n mod 4=$(n-4k)$ mod 4 for integers k).

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Turing machine using the encoding with $n+1$ 1's for n.

mod 4	1	B	
s_0	s_1 1 R		check for 5 1's
s_1	s_2 1 R		
s_2	s_3 1 R		
s_3	s_4 1 R		
s_4	s_5 1 L		
s_5	s_6 B L		delete four leftmost 1's
s_6	s_7 B L		
s_7	s_8 B L		
s_8	s_9 B R		
s_9		s_{10} B R	go to new start of string
s_{10}		s_{11} B R	and restart Turing machine
s_{11}		s_0 B R	

First Turing machine applied to the encoding $7 = 11111111$

tape	state	rule
...B $\boxed{1}$ 1 1 1 1 1 1 1 B ...	s_0	$(s_0, 1, s_1, 1,\mathrm{R})$
...B 1 $\boxed{1}$ 1 1 1 1 1 1 B ...	s_1	$(s_1, 1, s_2, 1,\mathrm{R})$
...B 1 1 $\boxed{1}$ 1 1 1 1 1 B ...	s_2	$(s_2, 1, s_3, 1,\mathrm{R})$
...B 1 1 1 $\boxed{1}$ 1 1 1 1 B ...	s_3	$(s_3, 1, s_4, 1,\mathrm{R})$
...B 1 1 1 1 $\boxed{1}$ 1 1 1 B ...	s_4	$(s_4, 1, s_5, 1,\mathrm{L})$
...B 1 1 1 $\boxed{1}$ 1 1 1 1 B ...	s_5	$(s_5, 1, s_6,\mathrm{B},\mathrm{L})$
...B 1 1 $\boxed{1}$ B 1 1 1 1 B ...	s_6	$(s_6, 1, s_7,\mathrm{B},\mathrm{L})$
...B 1 $\boxed{1}$ B B 1 1 1 1 B ...	s_7	$(s_7, 1, s_8,\mathrm{B},\mathrm{L})$
...B $\boxed{1}$ B B B 1 1 1 1 B ...	s_8	$(s_8, 1, s_9,\mathrm{B},\mathrm{R})$
...B B $\boxed{B}$ B B 1 1 1 1 B ...	s_9	$(s_9,\mathrm{B}, s_{10},\mathrm{B},\mathrm{R})$
...B B B $\boxed{B}$ B 1 1 1 1 B ...	s_{10}	$(s_{10},\mathrm{B}, s_{11},\mathrm{B},\mathrm{R})$
...B B B B $\boxed{B}$ 1 1 1 1 B ...	s_{11}	$(s_{11},\mathrm{B}, s_0,\mathrm{B},\mathrm{R})$
...B B B B B $\boxed{1}$ 1 1 1 B ...	s_0	$(s_0, 1, s_1, 1,\mathrm{R})$
...B B B B B 1 $\boxed{1}$ 1 1 B ...	s_1	$(s_1, 1, s_2, 1,\mathrm{R})$
...B B B B B 1 1 $\boxed{1}$ 1 B ...	s_2	$(s_2, 1, s_3, 1,\mathrm{R})$
...B B B B B 1 1 1 $\boxed{1}$ B ...	s_3	$(s_3, 1, s_4, 1,\mathrm{R})$
...B B B B B 1 1 1 1 $\boxed{B}$ ...	s_4	no rule $\to$ stop

Turing machine using the encoding of n to the basis 4.

mod 4	0	1	2	3	B	
s_0	s_1 0 R	s_1 1 R	s_1 2 R	s_3 3 R		check for two non-blank symbols
s_1	s_2 0 L	s_2 1 L	s_2 2 L	s_2 3 L		
s_2	s_0 B R	s_0 B R	s_0 B R	s_0 B R		delete left one and restart

Second Turing machine applied to the encoding $7 = (13)_4$

tape	state	rule
...B $\boxed{1}$ 3 B ...	s_0	$(s_0, 1, s_1, 1,\mathrm{R})$
...B 1 $\boxed{3}$ B ...	s_1	$(s_1, 3, s_2, 3,\mathrm{L})$
...B $\boxed{1}$ 3 B ...	s_2	$(s_2, 1, s_0, B,\mathrm{R})$
...B B $\boxed{3}$ B ...	s_0	$(s_0, 3, s_1, 3,\mathrm{R})$
...B B 3 $\boxed{B}$ ...	s_1	no rule $\to$ stop