The following sentences are not propositions.

(e) Are you coming to the Disco?

(f) Hurry up, then!

(g) My bag is heavy.

The statements (a) to (d) are either true or false, depending on the circumstances. The important thing to understand is that any two observers would agree about whether each of these statements is true or false in the same circumstances. The sentence (e) is a question and (f) is a command: these cannot be either true or false and so are not propositions. The sentence (g) is not a proposition because one observer might consider it true and another consider it false.

We shall denote propositions by lower case letters, such as p and q. We can define a **truth set** for each proposition. The truth set P for the proposition p contains all the circumstances under which p is true; its complement P' contains all the circumstances under which p is false.

Example 3.2 Suppose we toss a coin three times. Let p denote the proposition "The first toss is a head", and q denote the proposition "The third toss is a tail". The set of all possible outcomes (or results) of this experiment is

$$\mathcal{U} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

The truth set for p is

$$P = \{HHH, HHT, HTH, HTT\}$$

and the truth set for q is

$$Q = \{HHT, HTT, THT, TTT\}.$$

Tautologies and contradictions

Propositions that are always true are called **tautologies** and those that are always false are called **contradictions**.

Example 3.3 The following propositions are tautologies.

(a) The result of tossing a fair coin is either a head or a tail.

(b) A square has four sides.

(c) $(x+1)^2 = x^2 + 2x + 1$.

The following propositions are contradictions.

(d) The number x is less than 3 and greater than 10.

(e) $x = x + 1$.

Proposition (a) is true for all properly minted coins, and is an an intrinsic property of such. Similarly, proposition (b) is true for all squares because it is part of the definition of the term *square*; all definitions are tautologies. Proposition (c) is an example of an algebraic identity. An **identity** is an equation where the right side is just a rearrangement of the left side, and hence all identities are examples of tautologies. The truth set for a tautology is therefore the universal set of the objects under discussion.

There is no value of x for which either of the propositions (d) or (e) is true and hence these propositions are both examples of *contradictions*. The truth set for a contradiction is always the empty set $\emptyset$.