OPERATIONS RESEARCH 2 - Algorithms

4.4.1 O-notation

Let X be a subset of $\mathbb{R}$ and suppose that $f : \mathbb{X} \to \mathbb{R}$ and $g : \mathbb{X} \to \mathbb{R}$ are two functions of a real variable x. If, for all large values of $x \in X$, the graph of the function f lies closer to the x-axis than the graph of some fixed positive multiple of the function g, then we say that f is *of order* g and we write "$f(x)$ is $O(g(x))$".

The distance of the point $(x, f(x))$ on the graph $y = f(x)$ from the x-axis is the absolute value of $f(x)$, which we denote by $|f(x)|$ (see Example 4.9). Thus we can express the definition of *order of a function* more formally as follows.

Definition 4.25 *Let $f : \mathbb{X} \to \mathbb{R}$ and $g : \mathbb{X} \to \mathbb{R}$ be two functions with a common domain $X \subseteq \mathbb{R}$. Then we say that f* is of order g, *written "$f(x)$ is $O(g(x))$", if we can find a positive real number M and a real number x_0 such that*

$$|f(x)| \leq M|g(x)|,$$

for all $x > x_0$.

4.4.2 Power functions

In using the O-notation, we often compare a given function f with one of the following family of functions.

Definition 4.26 *Let s be any positive rational number. Then the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = x^s$$

is called the **power function** *of x with exponent s.*

The following table compares the values of the power functions $f(x) = x^s$, where $s = 0.5, 1, 1.5, 2, 3$ (values of $f(x)$ have been entered correct to 2 decimal places).

x	0	0.5	1	2	3	4
$x^{0.5}$	0	0.71	1	1.41	1.73	2
x^1	0	0.5	1	2	3	4
$x^{1.5}$	0	0.35	1	2.83	5.20	8
x^2	0	0.25	1	4	9	16
x^3	0	0.13	1	8	27	64

The graphs of the power functions $y = x^r$, when $0 \leq x \leq 2$ and $r = 1/3, 1/2, 1, 2, 3$ are compared in Figure 4.3. They illustrate the following result.

Result 4.27 *Let r, s be any rational numbers such that $r < s$. Then when $x > 1$, we have*

$$x^r < x^s. \square$$

We see that for any pair of rational numbers r, s with $r < s$, the graph of $y = x^r$ lies closer than the graph of $y = x^s$ to the x-axis, for all values of $x > 1$. Reasoning analytically, when $x > 1$, x^r and x^s are both positive, and hence $|x^r| = x^r$ and $|x^s| = x^s$. Taking $x_0 = 1$ and $M = 1$ in the defintion of the O-notation, we have:

Result 4.28 *Let r, s be any rational numbers such that $r < s$. Then*

$$x^r \text{ is } O(x^s). \square$$

Example 4.39 Let $f(x) = x\sqrt{x}$. Then since $x\sqrt{x} = x^{1.5}$, we have

$$f(x) < x^2$$

for all $x > 1$. We can say that $f(x)$ is $O(x^2)$. $\square$