0.1 Properties of Functions

The function $f : \mathbb{R} \to \mathbb{Z}$ is defined by the rule:

$$f(x) = \left\lfloor \frac{x}{2} \right\rfloor$$

0.2 Properties of Functions

The function $f : \mathbb{R} \to \mathbb{Z}$ is defined by the rule:

$$f(x) = \left\lfloor \frac{x}{2} \right\rfloor$$

What is the range of this function?

0.3 Properties of Functions

The function $f : \mathbb{R} \to \mathbb{Z}$ is defined by the rule:

$$f(x) = \left\lfloor \frac{x}{2} \right\rfloor$$

Is this function "onto" ?

0.4 Properties of Functions

The function $f : \mathbb{R} \to \mathbb{Z}$ is defined by the rule:

$$f(x) = \left\lfloor \frac{x}{2} \right\rfloor$$

Is this function "one-to-one" ?

The function $f : \mathbb{R} \to \mathbb{Z}$ is defined by the rule:

$$f(x) = \left\lfloor \frac{x}{2} \right\rfloor$$

Is this function invertible ?

0.5 Even and Odd functions

Even Functions

Then f is even if the following equation holds for all x and -x in the domain of f:

$$f(x) = f(-x),$$

Geometrically speaking, the graph face of an even function is symmetric with respect to the y-axis, meaning that its graph remains unchanged after reflection about the y-axis.

0.6 Even and Odd functions

Odd Functions
Let f(x) be a real-valued function of a real variable. Then f is odd if the following equation holds for all x and -x in the domain of f:

$$-f(x) = f(-x),$$

or

$$f(x) + f(-x) = 0.$$

0.7 Even and Odd functions

- **Important:** A function may be neither even nor odd.
- Discussion with examples on Blackboard
- Examples of Questions from Past Papers done on board

0.8 Even and Odd Functions

$F(x) = x^3$is an example of an odd function. Again, let f(x) be a real-valued function of a real variable. Then f is odd if the following equation holds for all x and -x in the domain of f:[2]

$$-f(x) = f(-x),$$

or

$$f(x) + f(-x) = 0.$$

0.9 Even and Odd Functions

Geometrically, the graph of an odd function has rotational symmetry with respect to the origin, meaning that its graph remains unchanged after rotation of 180 degrees about the origin. Examples of odd functions are x, x3, sin(x), sinh(x), and erf(x).

Absolute Value Function

Logarithms

1 Sets of Numbers

- $\mathbb{Z}$ Set of all integers
- $\mathbb{Q}$ Set of all rational numbers
- $\mathbb{R}$ Set of all real numbers
- $\mathbb{Z}^+$ Set of all positive integers
- $\mathbb{Z}^-$ Set of all negative integers
- $\mathbb{R}^+$ Set of all positive real numbers
- $\mathbb{R}^-$ Set of all negative real numbers

2 Arrow Diagrams

- Domain
- Co-Domain
- Range

$$f(x) : \text{Domain} \to \text{Co-Domain}$$
$$f(x) : \mathbb{R} \to \mathbb{R}$$

Polynomial Functions (4.1.5)

Constants (P_0)

inear Functions (P_1)

dratic Functions (P_2)

Cubic Functions (P_3)

Equality of Functions (4.1.6)

$$f(x) = g(x)$$

3 Special Mathematical Functions

3.1 Mathematical Operators

- The Square Root function
- The Floor and Ceiling functions
- The Absolute Value functions
- Root Functions
- Absolute Value Function
- Floor Function
- Ceiling Function

$$\lfloor 3.14 \rfloor = 3 \quad (1)$$
$$\lceil -4.5 \rceil = -5 \quad (2)$$
$$|-4| = 4 \quad (3)$$

For this course, only positive numbers have square roots. The square roots are positive numbers. (This statement is not strictly true. The square root of a negative number is called a complex number. However this is not part of the course).

Negative numbers can have cube roots

$$-27 = -3 \times -3 \times -3$$

$$\sqrt[3]{-27} = -3$$

4 Exponential and Logarithms

Laws of Logarithms

- Law 1 : Multiplcation of Logarithms

$$Log(a) \times Log(b) = Log(a + b)$$

- Law 2 : Division of Logarithms

$$\frac{Log(a)}{Log(b)} = Log(a - b)$$

- Law 3 : Powers of Logarithms

$$Log(a^b) = b \times Log(a)$$

4.1 Exercise

$h(x) : \mathbb{R} \to \mathbb{R}$ $g(x) : \mathbb{R} \to \mathbb{R}$

$$f(x) = sqrt(x)$$
$$g(x) = \sqrt{3}x + 2$$
$$h(x) = 2^x$$

- Is the function $h(x)$ an *onto* function?
- determine the inverse function of $h(x)$ and $g(x)$
- Simplify the following function.

$$j(x) = \log_4(h(6x))$$

4.2 Onto Functions

Definition: If every element in the co-domain of the function has an ancestor, the function is said to be "onto". An onto function has the property that the domain is equal to the co-domain.

Example 4.26 Page 53

Exponential and Logarithms

Rules

Expontials : Rules 4.18 Page 58
Logarithms : Rules 4.23 Page 61

$$log_a(a) = 1$$
$$log_a(b^c) = c \times log_a(b)$$

- $log_2(128) = 7$
- $log_2(1/4) = -2$
- $log_2(2) = 1$

$$log_a(b) = \frac{log_x(b)}{log_x(a)}$$

4.3 Logarithms

- Laws of Logarithms - Change of Base

$$\begin{gathered} \mathrm{Log}_b(x) = a \\ b^a = x \\ \mathrm{Log}_2(8) = 3 \\ 2^3 = 8 \end{gathered}$$

$$\begin{gathered} \mathrm{Log}_b(x) \times \mathrm{Log}_b(y) = \mathrm{Log}_b(x+y) \\ \mathrm{Log}_b(x^y) = y \times \mathrm{Log}_b(x) \end{gathered}$$

$$\mathrm{Log}_y(x) = \frac{\mathrm{Log}_b(x)}{\mathrm{Log}_b(y)}$$

4.4 Exponents

- Rules of Exponents

$$(a^b)^c = a^{b \times c}$$

$$64^{2/3} = (4^3)^{2/3} = 4^{3 \times 2/3} = 4^2 = 16$$

$$\begin{gathered} (a^b) \times (a^c) = a^{b+c} \\ (3^2) \times (3^3) = 3^{2+3} = 3^5 = 243 \end{gathered}$$

Exercises

(a) Complete the following table for the functions

i) $g(x) = \log_3 x$,
ii) $h(x) = \sqrt[3]{x}$.

x	1				81	
$g(x)$		1	2			5
$h(x)$				3.00		

Express your answers to 2 decimal places only.

5 *One-to-One* Functions and *Onto* Functions

5.1 Invertible Functions

- One-to-One Function
- Onto Function

Onto Functions : Range and Co-Domain are equivalent

5.2 Inverting a Function

- You are given $f(x)$ in terms of x
- Re-arrange the equation so that x is given in terms of $f(x)$
- Replace x with $f^{-1}(x)$ and $f(x)$ with x

5.2.1 Example

- Determine the inverse function of $f(x)$. Re-arrange the equation so that x is given in terms of $f(x)$

$$f(x) : \mathbb{R} \rightarrow \mathbb{R} \ f(x) = \sqrt{x+1}$$

- Square both sides of the equation.

$$[f(x)]^2 = x + 1$$

- Subtract 1 from both sides of the equation. We have the equation written in terms of x.

$$f(x)^2 - 1 = x$$

- Replace x with $f^{-1}(x)$ and $f(x)$ with x

$$x^2 - 1 = f^{-1}(x)$$

- Re-arrange equation and specify domain and co-domain.

$$f(x) : \mathbb{R} \rightarrow \mathbb{R} \ f^{-1}(x) = x^2 - 1$$

6 Big O-Notation

(b) Let S be the set of all 4 bit binary strings.

The function $f : S \to \mathbb{Z}$ is defined by the rule:

$$f(x) = \text{the number of zeros in x}$$

for each binary string $x \in S$.
Find:

1. the number of elements in the domain
2. $f(1000)$
3. the set of pre-images of 1
4. the range of f.

(c)

4.a $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$

4.b

4.c

7 Section 4 Functions

7.1 Invertible Functions

A function is invertible if it fulfils two criteria

- The function is ***onto***,
- The function is ***one-to-one***.

State the conditions to be satisfied by a function $f : X \leftarrow Y$ for it to have an inverse function $f^{-1} : Y \leftarrow X$.

$\lceil \frac{x^2+1}{4} \rceil$ where $f : A \rightarrow \mathbf{Z}$

(i) Find $f(4)$ and the ancestors of 3.

(ii) Find the range of f.

(iii) Is f invertible? Justify your answer

Given $f : \mathbf{R} \rightarrow \mathbf{R}$ where f(x) =3x-1,define fully the inverse of the function f ,i.e.f^{-1}. State the value of $f^{-1}(2)$

7.2 Precision Functions

- Absolute Value Function $|x|$
- Ceiling Function $\lceil x \rceil$
- Floor Function $\lfloor x \rfloor$

Question1.2: State the range and domain of the following function

$$F(x) = \lfloor x - 1 \rfloor$$

7.3 Powers

$$2^4 = 2 \times 2 \times 2 \times 2 = 16$$

$$5^3 = 5 \times 5 \times 5 = 125$$

7.3.1 Special Cases

Anything to the power of zero is always 1

$$X^0 = 1 \text{ for all values of X}$$

Sometimes the power is a negative number.

$$X^{-Y} = \frac{1}{X^Y}$$

Example

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

7.4 Exponentials Functions

$$e^a \times e^b = e^{a+b}$$

$$(e^a)^b = e^{ab}$$

7.5 Logarithmic Functions

7.5.1 Laws for Logarithms

The following laws are very useful for working with logarithms.

1. $\log_b(X) + \log_b(Y) = \log_b(X \times Y)$
2. $\log_b(X)$ - $\log_b(Y) = \log_b(X/Y)$
3. $\log_b(X^Y)$= $Y\log_b(X)$

Question1.3 Compute the Logarithm of the following

- $\log_2(8)$
- $\log_2(\sqrt{128})$
- $\log_2(64)$
- $\log_5(125) + \log_3(729)$
- $\log_2(64/4)$
- $a^x = y \ log_a(y) = x$
- $e^x = y \ ln(y) = x$
- $log_a(x \times y) = log_a(x) + log_a(y)$
- $log_a(\frac{x}{y}) = log_a(x) - log_a(y)$
- $log_a(\frac{1}{x}) = -log_a(x)$
- $log_a(a) = 1$
- $log_a(1) = 0$
- $\lceil x \rceil$
- $\lfloor x \rfloor$

Sample value x	Floor $\lfloor x \rfloor$	Ceiling $\lceil x \rceil$	Fractional part $\{x\}$
$12/5 = 2.4$	2	3	$2/5 = 0.4$
2.7	2	3	0.7
-2.7	-3	-2	0.3
-2	-2	-2	0

Topic 1 : Special Functions

7.6 Special Functions

- Absolute Value Function
- The Sign Function
- Floor and Ceiling Functions
- Hyperbolic Functions

7.7 Absolute Value Function

Absolute Value Function

- The absolute value (or modulus) $|x|$ of a real number x is the non-negative value of x without regard to its sign.

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

7.8 Topic 1 : Absolute Value Function

- For a positive x, $|x| = x$
- For a negative x (in which case x is positive) $|x| = -x$
- The absolute value of 0 is 0: $|0| = 0$.
- For example, the absolute value of 4 is 4, and the absolute value of -4 is also 4.
- IMPORTANT: The input to this function is any real number. The output of this function will always be a positive real numbers.

7.9 Topic 1 : Sign Function

Sign Function

- The sign function $sng(x)$ of a real number x is a signed value of absolute value of 1, dependent on the sign of x.
- IMPORTANT: The input to this function is any real number. The output of this function will always be either 1 or -1.

$$sgn(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0. \end{cases}$$

7.10 Topic 1 : Floor and Ceiling Functions

- The floor and ceiling functions map a real number to the largest previous or the smallest following integer, respectively.
- More precisely,

  $$floor(x) = \lfloor x \rfloor$$

  is the largest integer not greater than x and

  $$ceiling(x) = \lceil x \rceil$$

  is the smallest integer not less than x.

7.11 Topic 1 : Floor and Ceiling Functions

Examples

$$\lfloor 3.14 \rfloor = 3 \tag{4}$$

$$\lceil -4.5 \rceil = -5 \tag{5}$$

$$|-4| = 4 \tag{6}$$

Remark: Input to the floor and ceiling function can be any really number, but outputs are always integers.

7.12 Floor and Ceiling Functions

The floor and ceiling functions map a real number to the largest previous or the smallest following integer, respectively. More precisely, floor(x) = $\lfloor x \rfloor$ is the largest integer **not greater** than x and ceiling(x) = $\lceil x \rceil$ is the smallest integer **not less** than x.

7.13 Floor and Ceiling Functions

- $\lceil x \rceil$: Ceiling function
- $\lfloor x \rfloor$: Floor Function
- $\{x\}$: Fractional Part of a number

$$\{x\} = x - \lfloor x \rfloor$$