Chapter 6: Digraphs and Relations

Example 1.19 Let X be the set of all students registered at a college and Y be the set or" atl courses being taught at the college. Then the college database keeps a record of the relationship in which a student z is related to a course y if 1: is registered for y. We can model a relation 'R between sets X and Y by a digraph D in much the same way as we did previously for relations on a. single set: We put l/(D) = X U Y and draw an arc from a vertex m 5 X to a vertex y E Y if x7?.y. Example 1.20 Let $X = \{1, 2, 3\}$ and $Y = \{b, c\}$. Define R by lR.b, 1Rc, 272::, and 3I?.c. Then the relationship digraph for 'R is shown in Figure 1.3. There is at strong similarity between Figure 1.3 and the figures drawn in Volume 1, Chapter 4 to illustrate functions, Indeed we can view a function $f : X \to Y$ as being a relation in which each a.- E X is related to a unique element of Y. We write f(:c) to represent the unique element of Y which is related to m.

Cartesian Product

We close this chapter by introducing one more mathematical structure which can be used to define a relation.

Definition 1.21 Let X and Y be sets. Then the **Cartesian product** $X \times Y$ is the set whoose elements are all ordered pairs of elements (1, y) where $x \in X$ and $y \in Y$. **Example 1.22** Let $X : \{1, 2, 3\}$ and $Y : \{b, c\}$. 10 CHAPTER 1. DIGRAPHS AND RELATIONS Given a relationship R between X and Y, we can use the cartesian product of X and Y to help ns define 'R,: we simply give the subset of X x Y containing all ordered pairs (m,y) for which z7y. This is equivalent. so listing the ordered pairs corresponding to the arcs in the relationship digraph of R. Example 1.23 The relation 72 in Example 1.20 is defined by the subset $\{(1, b), (1, c), (2, c), (3, c)\}$ of $X \times Y$. Example 1.24 In Example 1.3 we have $X = \{l, 2, 3\}$ = Y, and $\mathcal{R}$ is defined by the subset

$$\{(1, 1), (1.2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$$

of $X \times X$. Definition 1.25 We can generalize the idea of an ordered pair to an ordered n-tuple, $(x_1, x_2, x_3, \ldots, x_n)$, and the cartesian product of two sets to the Cartesian product of n sets, for any $n \in \mathbb{Z}^+$. When each of the sets in a cartesian product is the same, as in Example 1.24, we often use an abbreviated notation: we denote $X \times X$ by X2 and in general, we denote the set $X \times X$ x . .. x X, by X^n, where there are ri Xs altogether in the product. Example 1.26 An n-bit binary string is an example of an ordered ntuple, even though we write it without the brackets and without commas between the entries. Lei B : 0.1 the set of bits).

Then the set of all nbit binary strings is cz;a2...¡2,r ::1] E B,agB..,..a,, E B, and this set. can be denoted by B^n. .

Exercise 2 For each of the relations given in questions l, 2 and 3:

(a) Draw the relationship digraph.

(b) Determine when the relation is either refiexive, symrneric, transitive or antir symmetric. For the cases when one of these properties does not hold, justify your answer by giving an example to show that it does not hold.

(c) Determine which of the rekations is an equivalence relation? For the case when it is an equivalence relation, calculate the distinct equivalence classes and verify that they give a partition of S.

(d) Determine which of the relations is a partial order or an order.

Q1 Let S : 0,1,2,3. Define a relation R; between the elements of S by ":c is related to y if the product my is even. Q2 Let S : 0,1,2,2. Define a relation R2 between the elements of S by "z is related to y if 0: y E 0,3, -3". Q3 Let S : l,1,2, I, 2,3, 1,2,4. Denne a relation 728 between the ele- ments of S by "X is related to Y ifX Q Y . [B0 Q4 Let S be a set and R be a relation on S. Explain what it means to say that R is

(a) reflexive,

(b) symmetric,

(c) transitive,

(d) anti-symmetric,

(e) an equivalence relation,

(f) a partial order,

(g) an order.

Q5 Lgt X .: 0,1,2 and $Y = \{3, 4\}$. Define a relation 72, from X to Y by 1 is related to y ifs =y3",for so EX and y EY. (a) Draw the relationship digraph corresponding to 72. (bl Determine the set X ¿¡ Y. Determine the subset of X ¿¡ Y corresponding to 72. [cz Is 72 a function from X to Y'? Justify your answer.