Chapter 6: Digraphs and Relations

Chapter 1 D ' h d R l t' Summary Digraphs; Relations, using digraphs to illustrate relations, equivalence relations, partial orders, relations and cartesian products.

Digraphs

Some problems require that we dzrect the edges of a graph, from one endpoint to the other, in order to express the relationship between the vertices A graph in which every edge has a direction assigned to ir is called a ***digraph*** (an abreviation of directed graph).

The directed edges are often called **arcs**.

Example 1.1 Suppose we want to model a plan for traflic flow in part of a city, where some streets allow trafiic flow in only one direction.

For this model, a digraph would be appropriate, We would represent each road junction by a vertex; a one-way street from junction u to junction 1;, by an arc directed from u to UQ and a street allowing traffic flow in both directions between junctions rc and y, by two arcs, one directed from x to y and the other from y to rv.

An example is given in Figure L17 where the direction of each arc is indicated by an arrow. In a digraph we define directed paths and directed cycles as in graphs, but with the added condition that arcs must be used in their correct direction,

We say a digraph D is stongly corrected if for every pair of vertices x and y of D, there is a directed path from x to y and a directed path from y to x in D. Example 1.2 The digraph in Figure 1.1 contains a direcred path P = vamp; and

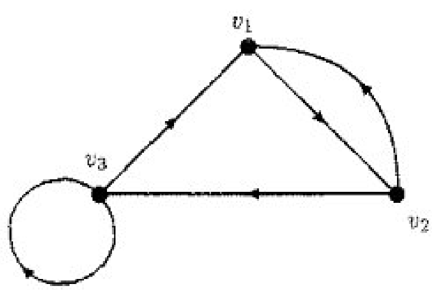

Figure 1:

a directed cycle C : ugviug. lt is strongly connected. Digraphs can be used to illustrate a variety of "one-way" relationships such as predator-prey models in ecology, the scheduling of tasks in a manufacturing process. family trees, flow diagrams for a computer program and many others.

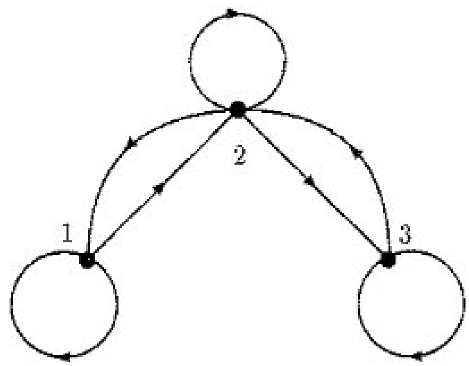

Figure 2:

Relations

Let S be a setr A relation R, on S is a rule which compares any two elements 23, $y \in S$ and tells us either that 92 is related to y or that rc is not related to y.

We write $x\mathcal{R}y$ to mean x is related to y under the relation $\mathcal{R}$". We are already familiar with many examples of relations defined in society: for example we could let S be the set of all people in London and say that two people x and y are related if x is a parent of y. We have also already seen several examples of relations in mathematics, for example "::", "¡", and "$\leq$" are all relations on the set. ofiritegers.

A different example is the following:

Example 1.3 Let $S = \{2, 3\}$. We define a relation 72 on S by saying that z is related to y if $|xy| = 1$; for all any 6 S. Thus 172,2 but 1 is not related to 3.

Using digraphs to illustrate relations

Given a relation $\mathcal{R}$ on a set S; we can model R by defining the digraph D with V(D] = S in which, for any i-wo vertices cr and yr there is an arc in D from z to y if and only if x'/'ly. We shall call D the relationship digraph corresponding to 72.

Example 1.4 Figure 1.2 gives the relationship digraph corresponding to the rela- tion defined in Example 1.3.

Conversely, given a digraph D. we can define a relation $\mathcal{R}$ on the set S : i/'(D) by saying $x\mathcal{R}y$ if and only if there is an arc in D from ar to y

Example 1.5 The digraph given in Figure 1,1 defines the relation 72 on thc sor S = viwg, M3 given by z=1Rvg, vglivg, ugI2vi, 3!?.v3, and vgiiv;.

Equivalence Relations

We can see from Example 1.5 that the definition ofa relation on at ser may be rather abitrary. Many relations which occur in practical situations however have a more "*regular structure*".

Definition 1.6 Let $\mathcal{R}$ be a relation defined on a set S. We say that $\mathcal{R}$ is:

- reflexive if for all $x \in S$, we have $x\mathcal{R}x$.
- symmetric if for all $x, y \in S$ such that $x\mathcal{R}y$, we have $y\mathcal{R}x$.
- transitive if for all $x, y, z \in S$ such that $x\mathcal{R}y$ and $y\mathcal{R}z$, we have $x\mathcal{R}z$.

In terms of the relationship digraph D 0f'R, it can be seen that:

- $\mathcal{R}$ is reflexive if all vertices of D are in a directed loop.
- $\mathcal{R}$ is symmetric if all arcs my in D are in a directed cycle of length two.
- $\mathcal{R}$ is transitive if for all directed paths of length two P : xyz of D, we have

an are xz; and for all directed cycles of length two C : mym 0f D, we have a loop 1::: and a loop yy.

Using Figure 1.2 we deduce that the relation defined in Example 1.3 is reflexive and symmetric but not transitive. Similarly, using Figure 1.1 we deduce that the relation deiined in Example 1.5 is not reliexive, symmetric or transitive.

Example 1.7 The relation $\leq$ defined on $\mathbb{Z}$ is reflexive and transitive but not symmetric. The relation $<$ defined on $\mathbb{Z}$ is transitive but not reflexive or symmetric.