Ratio Test

In mathematics, the ratio test is a test (or "criterion") for the convergence of a series

$$\sum_{n=1}^{\infty} a_n,$$

where each term is a real or complex number and a_n is nonzero when n is large.

The test was first published by Jean le Rond d'Alembert and is sometimes known as d'Alembert's ratio test or as the Cauchy ratio test.

Motivation

Given the following geometric series:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

The quotient $a_{n+1}/a_n = (1/2)^{n+1}/(1/2)^n$ of any two adjacent terms is $1/2$. The sum of the first m terms is given by:

$$1 - \frac{1}{2^m}.$$

As m increases, this converges to 1, so the sum of the series is 1. On the other hand given this geometric series:

$$\sum_{n=1}^{\infty} 2^n = 2 + 4 + 8 + \cdots$$

- The quotient a_{n+1}/a_n of any two adjacent terms is 2.
- The sum of the first m terms is given by $2^{m+1} - 2$, which increases without bound as m increases, so this series diverges.
- More generally, the sum of the first m terms of the geometric series $\sum_{n=1}^{\infty} r^n$ is given by:

$$\sum_{n=1}^{m} r^n = \frac{r}{r-1}(r^m - 1).$$

Whether this converges or diverges as m increases depends on whether r, the quotient of any two adjacent terms, is less than or greater than 1. Now consider the series:

$$\sum_{n=1}^{\infty} \frac{n+1}{n} \left(\frac{1}{2}\right)^n = \frac{2}{1} \cdot \frac{1}{2} + \frac{3}{2} \cdot \frac{1}{4} + \frac{4}{3} \cdot \frac{1}{8} + \cdots$$

This is similar to the first convergent sequence above, except that now the ratio of two terms is not fixed at exactly $1/2$:

$$\left(\frac{n+1}{n}\left(\frac{1}{2}\right)^{n}\right) \Big/ \left(\frac{n}{n-1}\left(\frac{1}{2}\right)^{n-1}\right) = \frac{n^2-1}{2n^2} = \frac{1}{2} - \frac{1}{2n^2}.$$

- However, as n increases, the ratio still tends in the limit towards the same constant $1/2$.
- The ratio test generalizes the simple test for geometric series to more complex series like this one where the quotient of two terms is not fixed, but in the limit tends towards a fixed value.
- The rules are similar: if the quotient approaches a value less than one, the series converges, whereas if it approaches a value greater than one, the series diverges.

The Ratio Test

The usual form of the test makes use of the limit

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| .$$

The ratio test states that:

- if $L < 1$ then the series converges absolutely;
- if $L > 1$ then the series does not converge;
- if $L = 1$ or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.