Solving a System of Linear Equations Using Gaussian Elimination

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Linear Equations : Elementary Row Operations

Consider the following system of linear equations.
The solutions of this system are the unique values for x_1, x_2 and x_3 that are valid for each of the three equations.

$$x_1 + x_2 + 2x_3 = 2$$

$$2x_1 + x_2 + 3x_3 = 5$$

$$x_1 - 2x_2 + 5x_3 = 11$$

Linear Equations : Gaussian Elimination

We can express this system of equations in terms of a coefficient matrix **A**, the vector of unknown values **x**, and the solution vector **b**:

$$\mathbf{Ax} = \mathbf{b}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 11 \end{pmatrix}$$

Linear Equations

The first step in Gaussian Elimination is to construct an **Augmented Matrix**, constructed by joining **A** and **b**.

$$\begin{array}{cc} \left(\begin{array}{ccccc} 1 & 1 & 2 & : & 2 \\ 2 & 1 & 3 & : & 5 \\ 1 & -2 & 5 & : & 11 \end{array}\right) & \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \end{array}$$

(Remark : For the sake of clarity, I am going to give each row of the augment matrix a unique identifier, e.g. r_1.)

Linear Equations : Gaussian Elimination

To derive the required solution, we perform **elementary row operations** to reduce the augemented matrix to **reduced row echelon form**.

The elementary row operations are

- Adding (or subtracting) the values of one row to another.
- Multiplying each value in a row by a constant value.
- Switching two rows

Linear Equations : Gaussian Elimination

Phase 1

$$\begin{array}{cc} \left(\begin{array}{cccc} 1 & 1 & 2 & : & 2 \\ 2 & 1 & 3 & : & 5 \\ 1 & -2 & 5 & : & 11 \end{array}\right) & \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \end{array}$$

Phase 2

$$\begin{array}{cc} \left(\begin{array}{cccc} 1 & 1 & 2 & : & 2 \\ 0 & -1 & -1 & : & 1 \\ 0 & -3 & 3 & : & 9 \end{array}\right) & \begin{array}{l} r_4 = r_1 \\ r_5 = r_2 - 2r_1 \\ r_6 = r_3 - r_1 \end{array} \end{array}$$

Linear Equations : Gaussian Elimination

Phase 3

$$\left(\begin{array}{ccccc} 1 & 1 & 2 & : & 2 \\ 0 & 1 & 1 & : & -1 \\ 0 & 1 & -1 & : & -3 \end{array}\right) \quad \begin{array}{l} r_7 = r_4 \\ r_8 = -r_5 \\ r_9 = -r_6/3 \end{array}$$

Phase 4

$$\left(\begin{array}{ccccc} 1 & 1 & 2 & : & 2 \\ 0 & 1 & 1 & : & -1 \\ 0 & 0 & 1 & : & 1 \end{array}\right) \quad \begin{array}{l} r_{10} = r_7 \\ r_{11} = r_8 \\ r_{12} = r_9 - r_8 \end{array}$$