FORMAL LANGUAGES AND AUTOMATA THEORY

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Peter Linz, An Introduction to Formal Languages and Automat, (6e), Jones & Bartlett Learning, 2016

INTRODUCTION TO THE THEORY OF COMPUTATION AND FINITE AUTOMATA

REGULAR LANGUAGES, REGULAR GRAMMARS AND PROPERTIES OF REGULAR LANGUAGES:

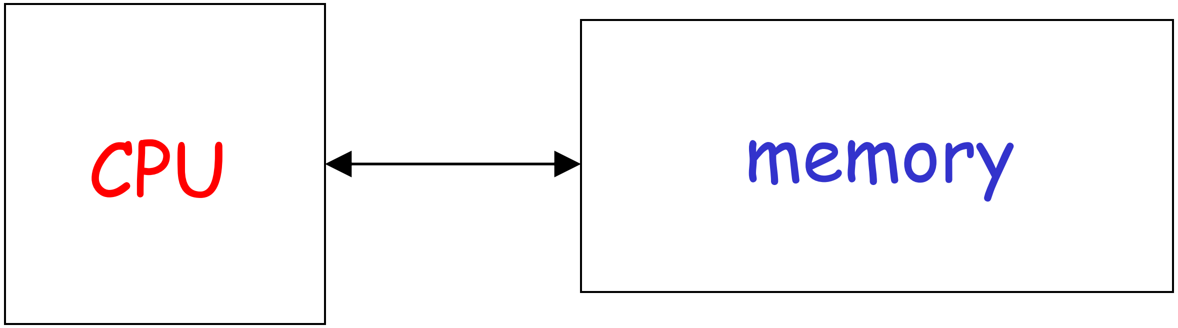
CONTEXT-FREE LANGUAGES AND SIMPLIFICATION OF CONTEXT-FREE GRAMMARS AND NORMAL FORMS

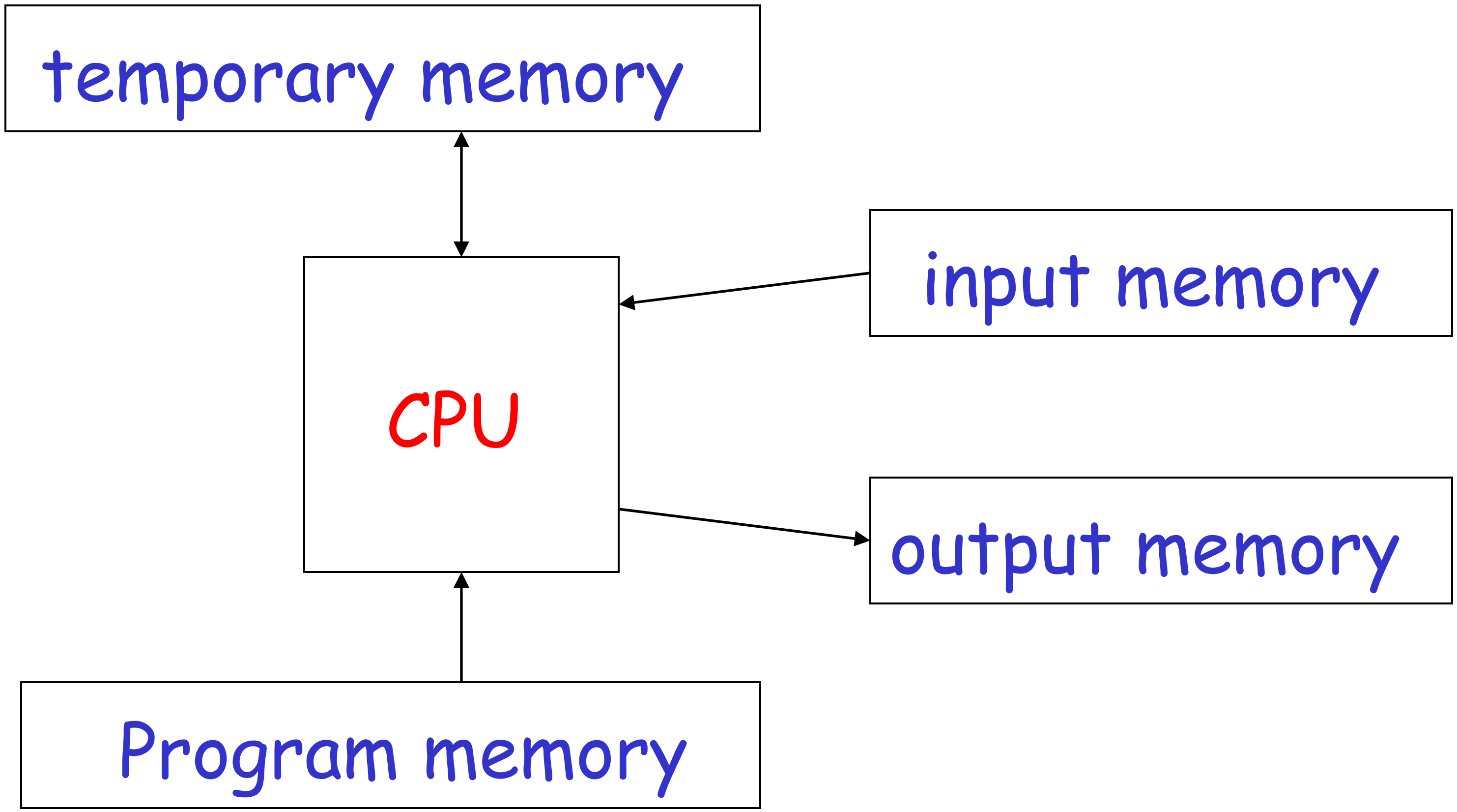
CONTEXT-FREE LANGUAGES AND SIMPLIFICATION OF CONTEXT-FREE GRAMMARS AND NORMAL FORMS

TURING MACHINES AND OTHER MODELS OF TURING MACHINES &

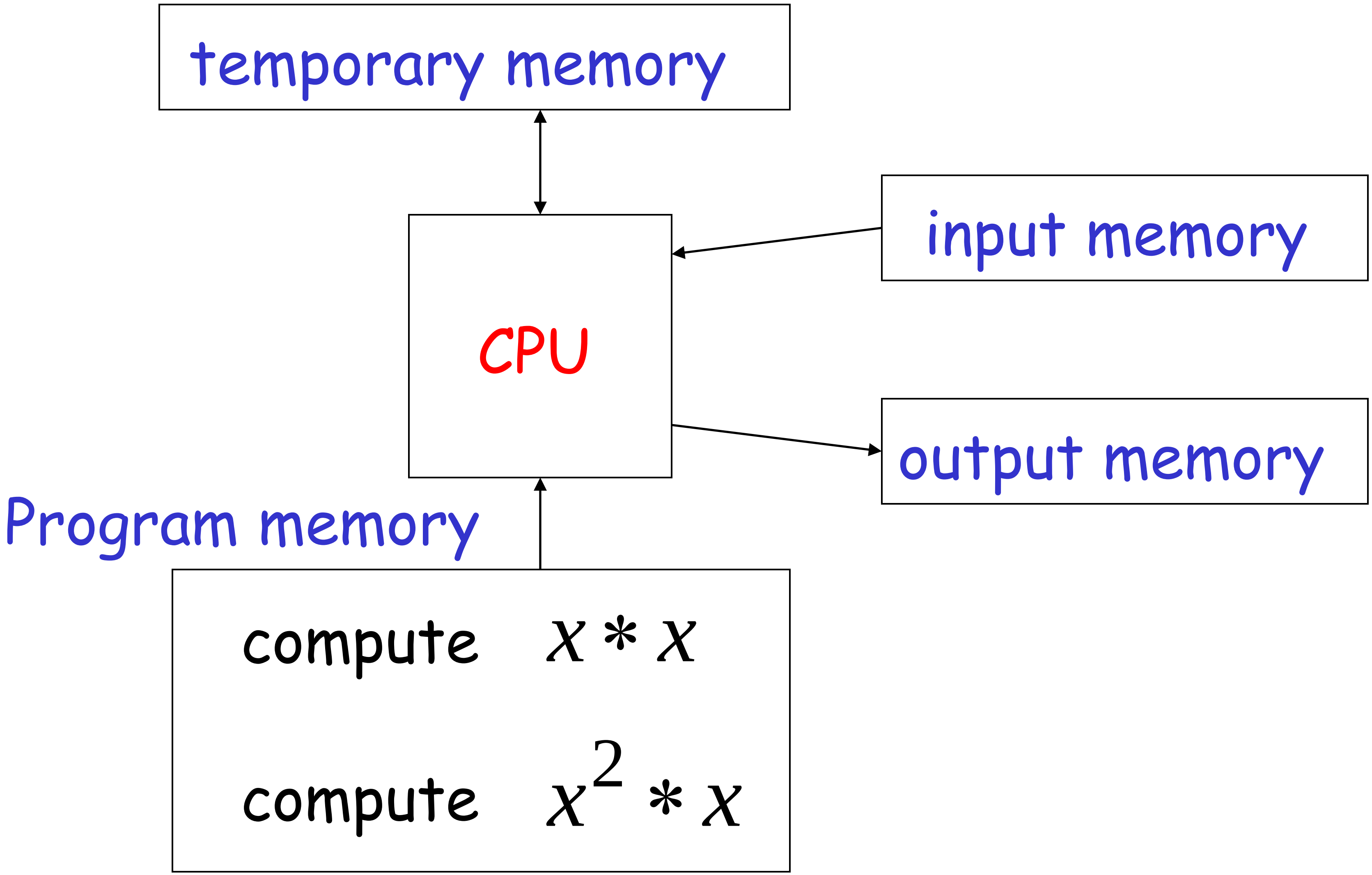
A HIERARCHY OF FORMAL LANGUAGES & AUTOMATA

Computation

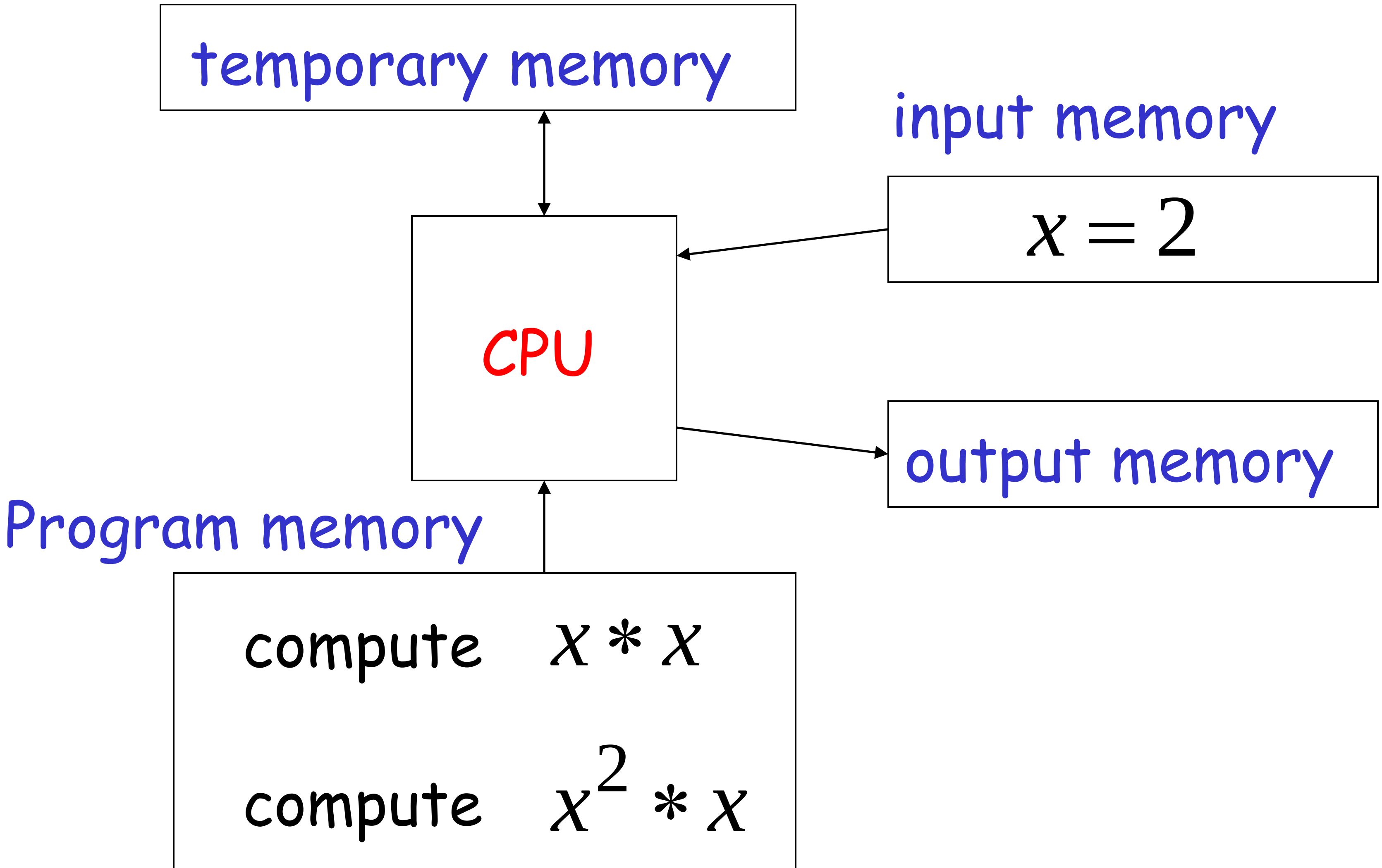



temporary memory
input memory
CPU
output memory
Program memory

Example: $f(x) = x^3$

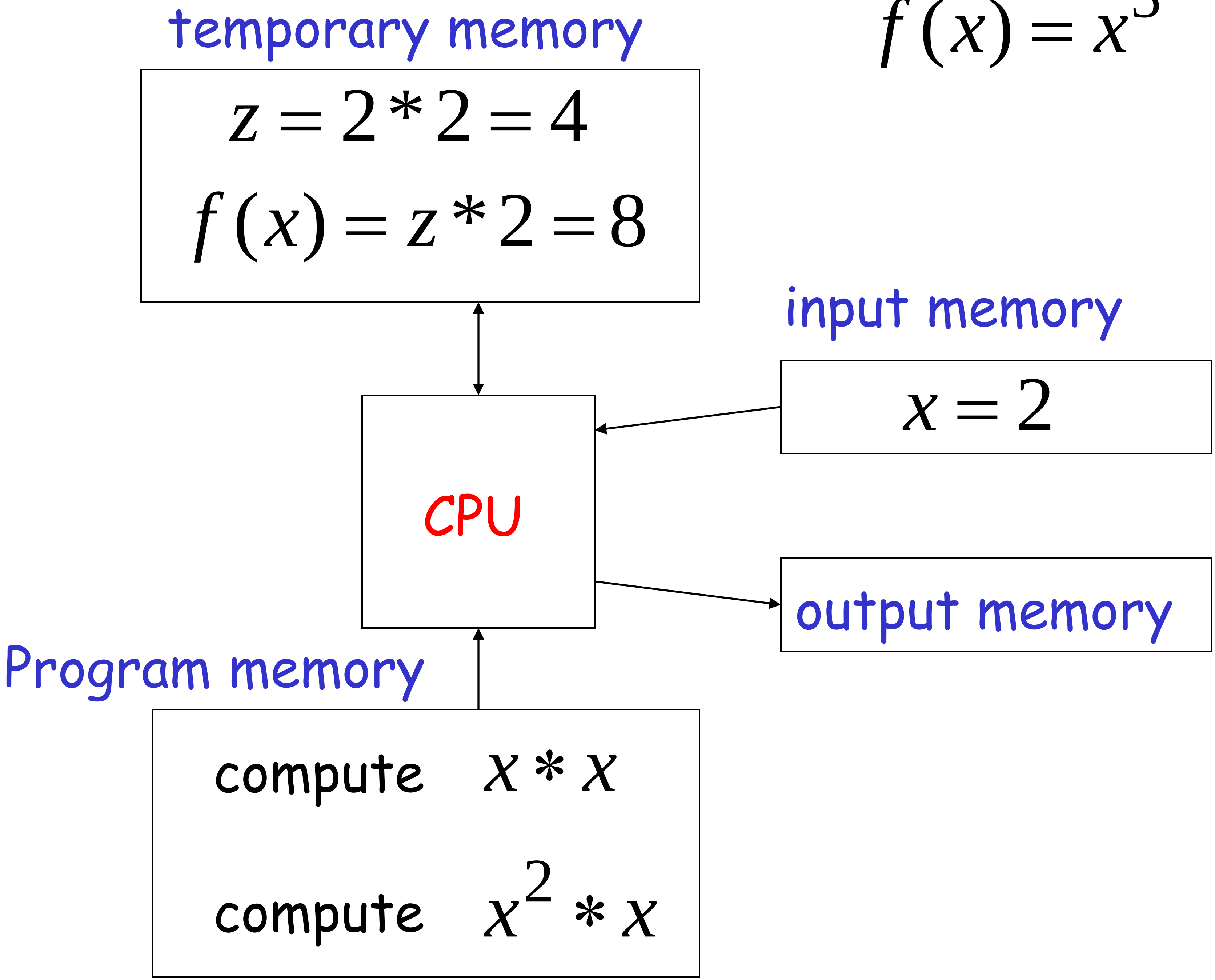


$$f(x) = x^3$$



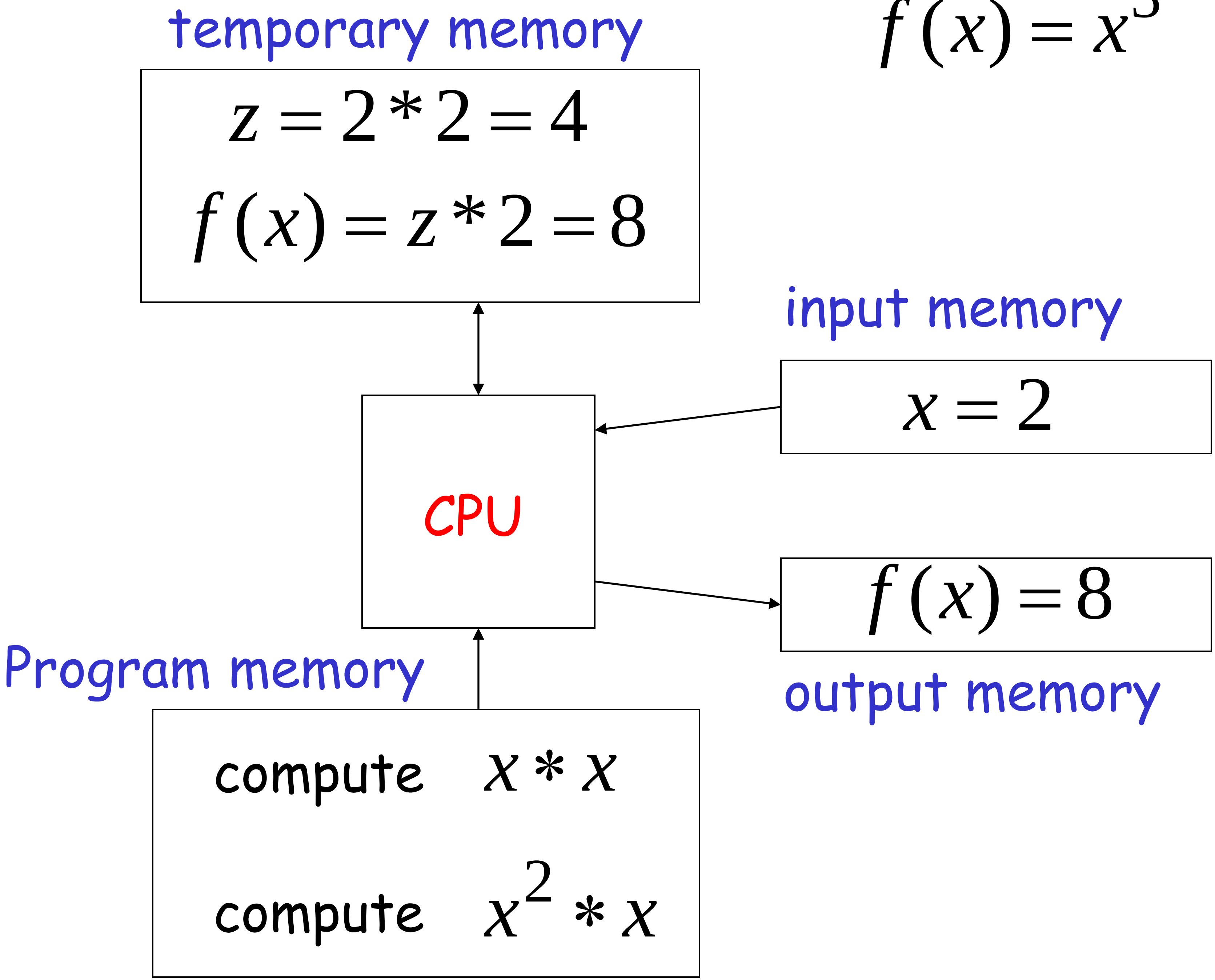
temporary memory
input memory
$x = 2$
CPU
output memory
Program memory
compute $x * x$
compute $x^2 * x$

$$f(x) = x^3$$

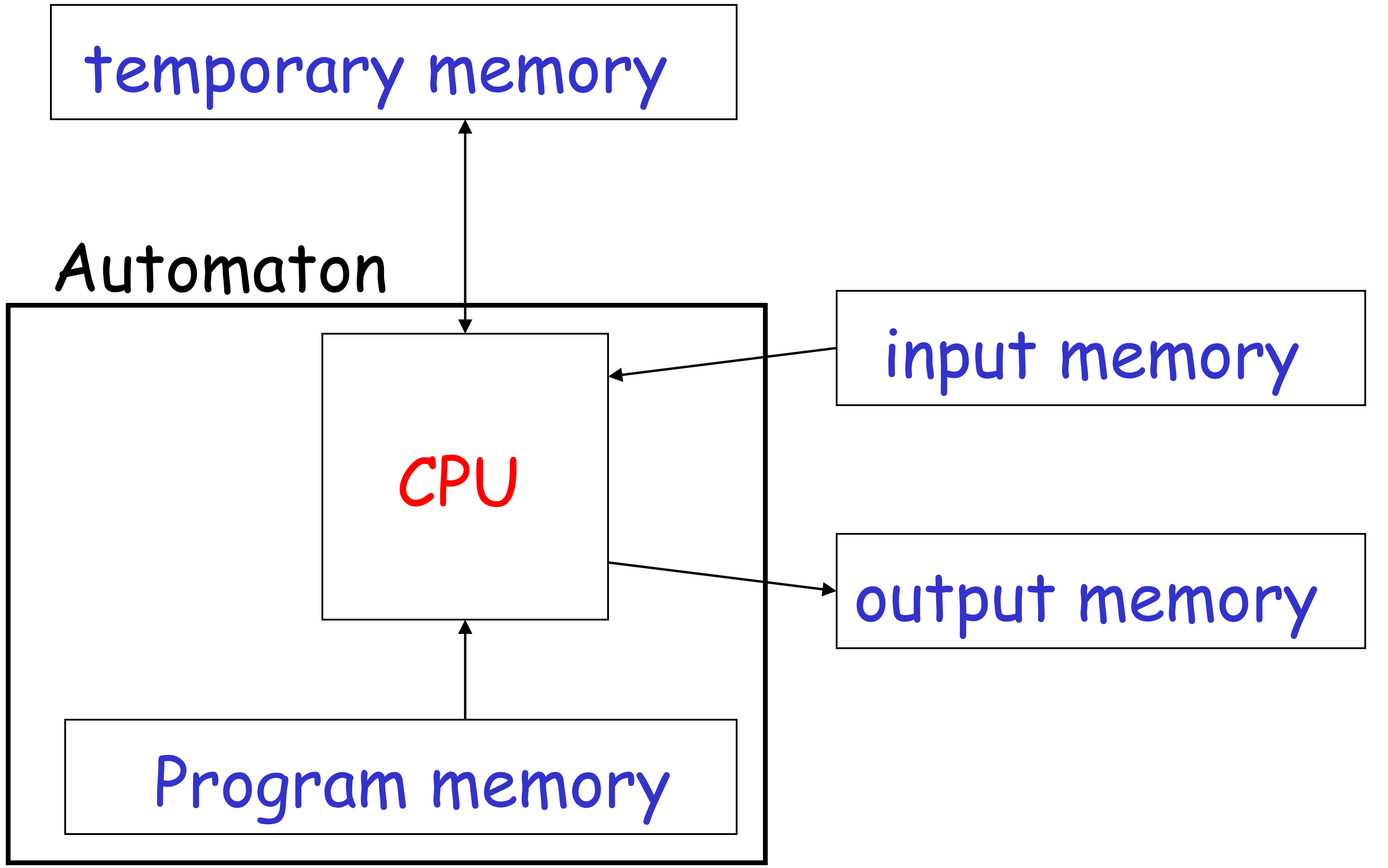


temporary memory
$z = 2*2 = 4$
$f(x) = z*2 = 8$
input memory
$x = 2$
CPU
output memory
Program memory
compute $x*x$
compute x^2*x

$$f(x) = x^3$$


temporary memory
$z = 2 * 2 = 4$
$f(x) = z * 2 = 8$
input memory
$x = 2$
CPU
$f(x) = 8$
output memory
Program memory
compute $x * x$
compute $x^2 * x$

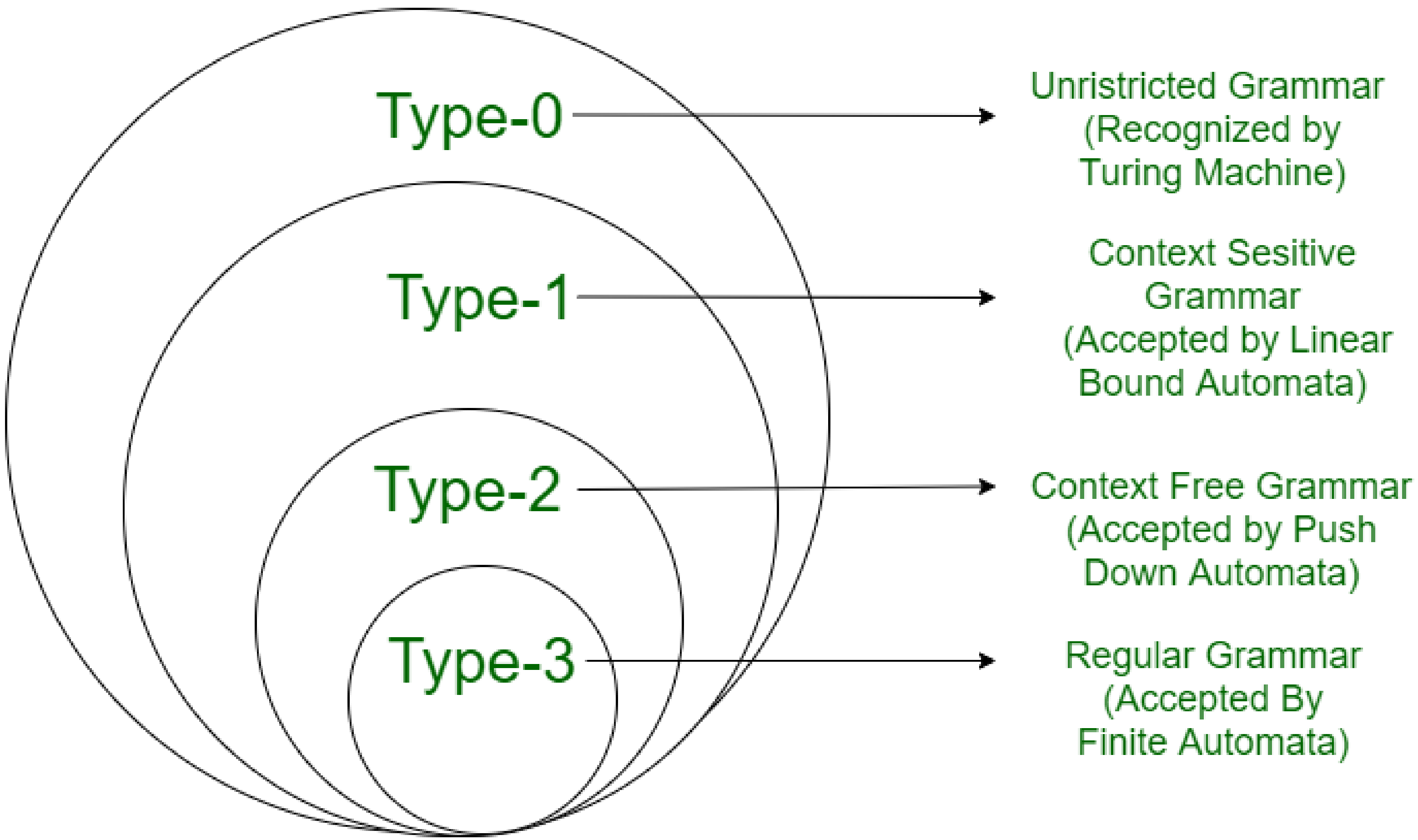
Automaton


temporary memory
Automaton
CPU
input memory
output memory
Program memory

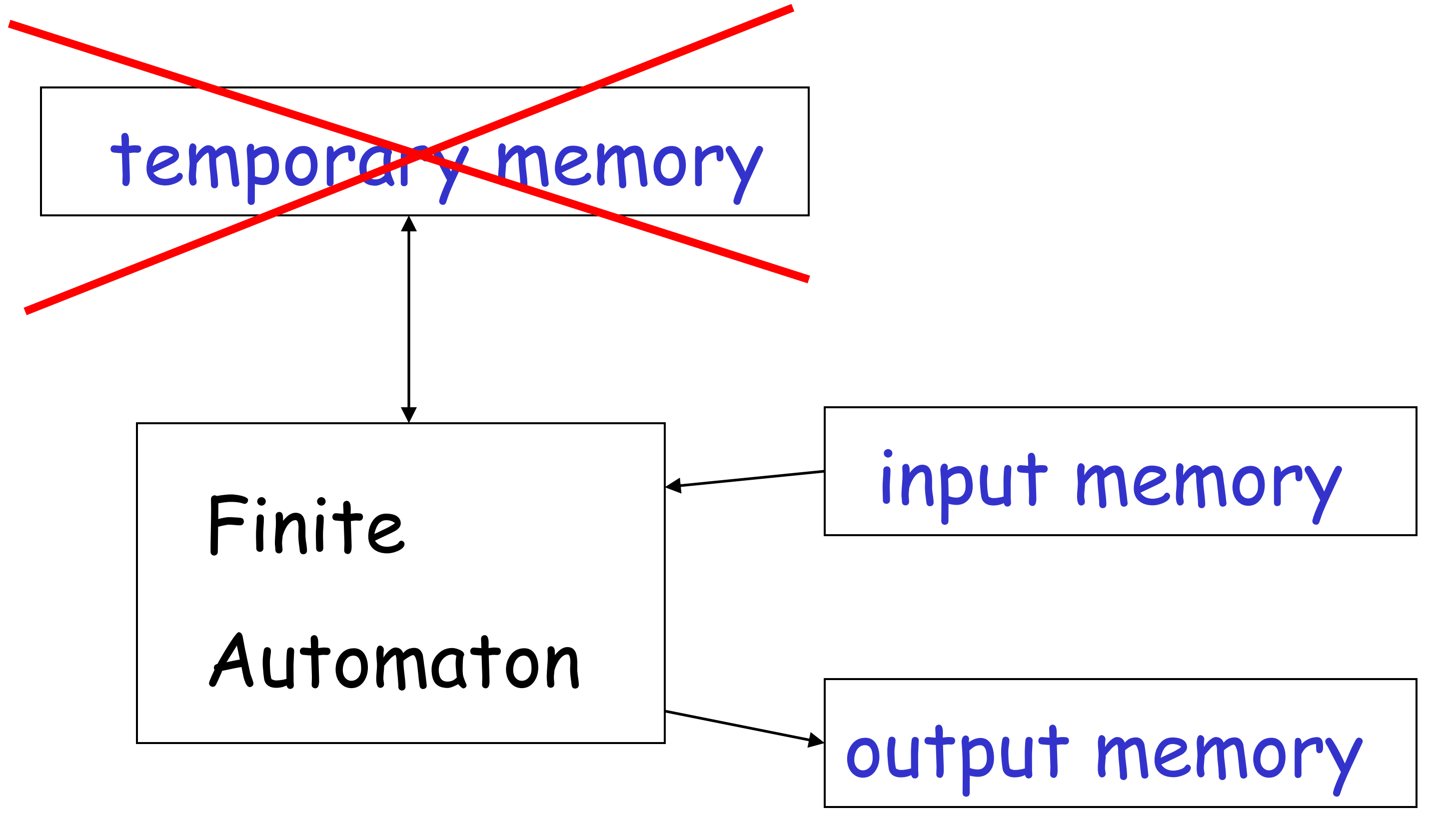
Different Kinds of Automata

Automata are distinguished by the temporary memory

- **Finite Automata:** no temporary memory
- **Pushdown Automata:** stack
- **Turing Machines:** random access memory


Type-0
Type-1
Type-2
Type-3
Unristricted Grammar
(Recognized by
Turing Machine)
Context Sesitive
Grammar
(Accepted by Linear
Bound Automata)
Context Free Grammar
(Accepted by Push
Down Automata)
Regular Grammar
(Accepted By
Finite Automata)

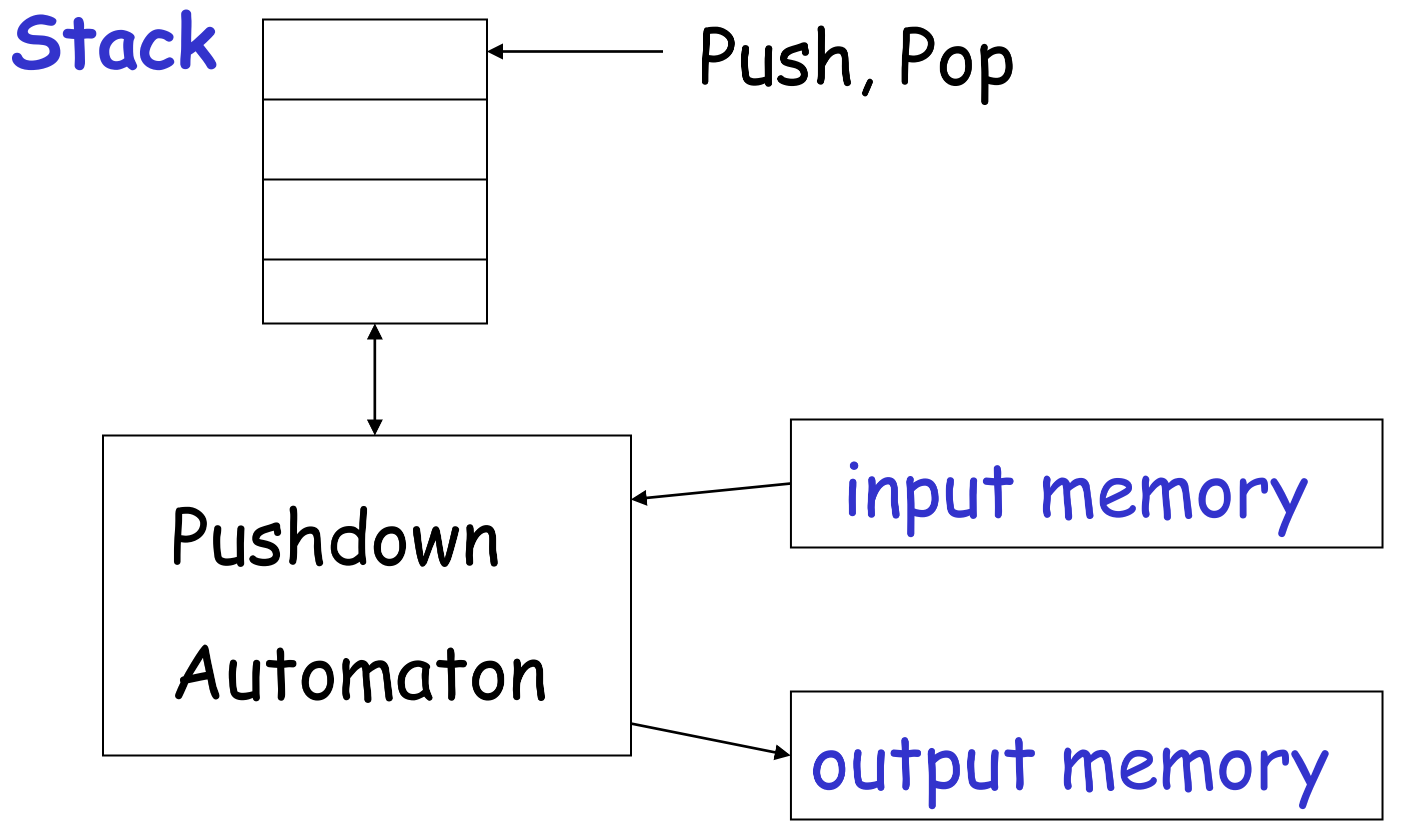
Finite Automaton



Example: Vending Machines

(small computing power)

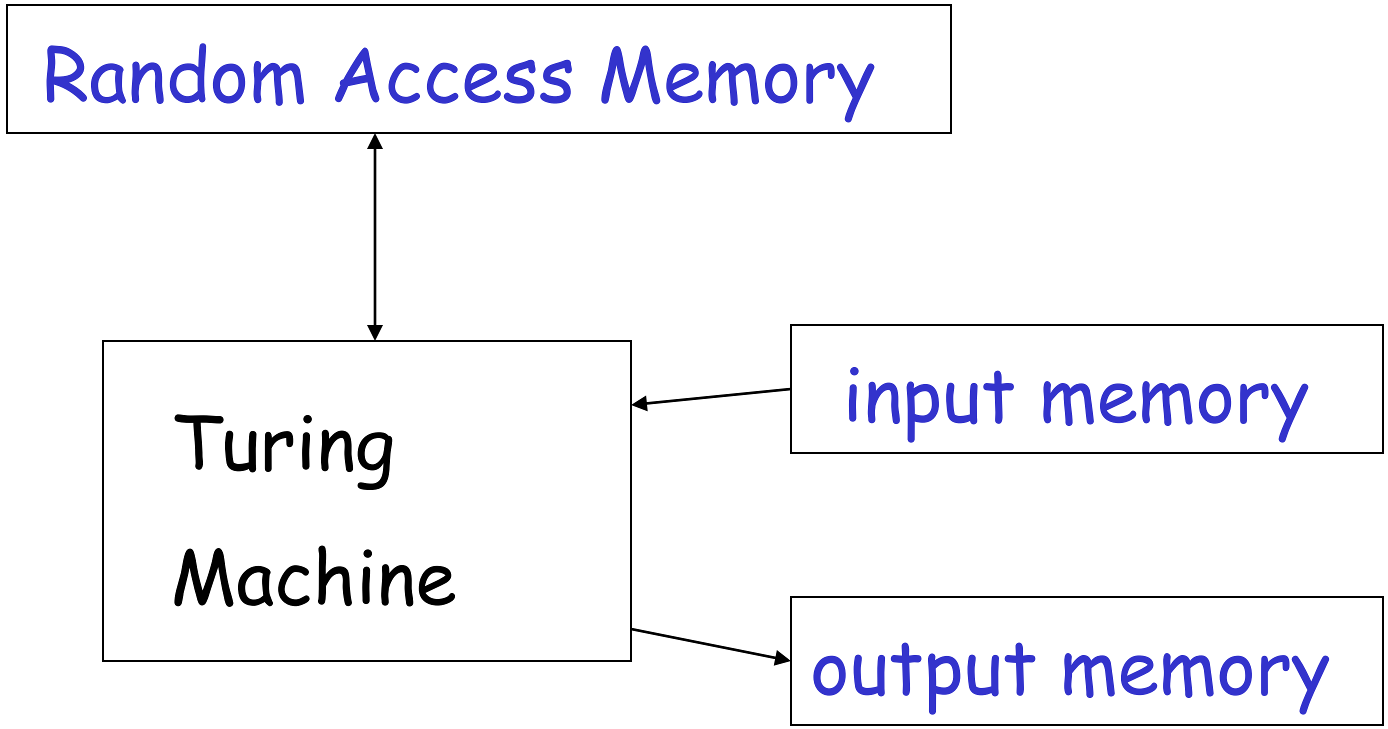
Pushdown Automaton



Example: Compilers for Programming Languages

(medium computing power)

Turing Machine



Examples: Any Algorithm

(highest computing power)

Power of Automata

Finite Automata $<$ Pushdown Automata $<$ Turing Machine

Less power $\longrightarrow$ More power

Solve more computational problems

Three Basic concepts

Languages, Grammars & Automata

A language is a set of strings

String: A sequence of letters

Examples: "cat", "dog", "house", ...

Defined over an alphabet:

$$\Sigma = \{a, b, c, \ldots, z\}$$

Alphabets and Strings

Alphabet: Finite nonempty set Σ of symbols, called the alphabet

Strings: Finite sequence of symbols from the alphabet

Strings

For example, if the alphabet is $\Sigma = \{a, b\}$, then abab & aaabbba are strings on Σ . We use lowercase letters a, b,c,… for elements of Σ & u,v,w… for string names

$$a$$

$$ab$$

$$abba$$

$$baba$$

$$aaabbbaabab$$

$$u = ab$$

$$v = bbbaaa$$

$$w = abba$$

String Operations

$w = a_1 a_2 \cdots a_n$ $\qquad$ $abba$

$v = b_1 b_2 \cdots b_m$ $\qquad$ $bbbaaa$

Concatenation

$wv = a_1 a_2 \cdots a_n b_1 b_2 \cdots b_m$ $\qquad$ $abbabbbaaa$

$$w = a_1 a_2 \cdots a_n \qquad ababaaabbb$$

Reverse

$$w^R = a_n \cdots a_2 a_1 \qquad bbbaaababa$$

String Length

$$w = a_1 a_2 \cdots a_n$$

Length: $|w| = n$

Examples:

$$|abba| = 4$$

$$|aa| = 2$$

$$|a| = 1$$

Length of Concatenation

$$|uv| = |u| + |v|$$

Example: $u = aab, \quad |u| = 3$

$v = abaab, \quad |v| = 5$

$$|uv| = |aababaab| = 8$$

$$|uv| = |u| + |v| = 3 + 5 = 8$$

Empty String

Empty string: A string with no symbols and it is denoted by λ

$$|\lambda| = 0$$

Observations:

$$\lambda w = w\lambda = w$$

$$\lambda abba = abba\lambda = abba$$

Substring

Substring of string:
a subsequence of consecutive characters

String	Substring
$\underline{ab}bab$	ab
$\underline{abba}b$	$abba$
$ab\underline{b}ab$	b
$a\underline{bbab}$	$bbab$

Prefix and Suffix

$abbab$

Prefixes	Suffixes
λ	$abbab$
a	$bbab$
ab	bab
abb	ab
$abba$	b
$abbab$	λ

$w = uv$

prefix

suffix

Another Operation

$$w^n = \underbrace{ww\cdots w}_{n}$$

Example: $(abba)^2 = abbaabba$

Definition: $w^0 = \lambda$

$$(abba)^0 = \lambda$$

The * Operation

$\Sigma*$: the set of all possible strings from alphabet Σ

$$\Sigma = \{a, b\}$$

$$\Sigma* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

The + Operation

Σ^+ : the set of all possible strings from alphabet Σ except λ

$$\Sigma = \{a, b\}$$

$$\Sigma^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

$$\Sigma^+ = \Sigma^* - \lambda$$

$$\Sigma^+ = \{a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

Languages

A language is any subset of Σ^*

Example: $\Sigma = \{a, b\}$

Languages: $\Sigma^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, \ldots\}$

$$\{\lambda\}$$

$$\{a, aa, aab\}$$

$$\{\lambda, abba, baba, aa, ab, aaaaaa\}$$

The set L = $\{a^n b^n : n \geq 0\}$ is also a language on Σ. The strings aabb and aaaabbbb are in L, but strings abb is not in L. This language is infinite

Note that:

Sets $\quad \emptyset = \{\ \} \neq \{\lambda\}$

Set size $\quad |\{\ \}| = |\emptyset| = 0$

Set size $\quad |\{\lambda\}| = 1$

String length $\quad |\lambda| = 0$

Another Example

An infinite language $L = \{a^n b^n : n \geq 0\}$

$$\left.\begin{array}{l} \lambda \\ ab \\ aabb \\ aaaaabbbbb \end{array}\right\} \in L \qquad abb \notin L$$

Operations on Languages

The usual set operations

$$\{a, ab, aaaa\} \cup \{bb, ab\} = \{a, ab, bb, aaaa\}$$

$$\{a, ab, aaaa\} \cap \{bb, ab\} = \{ab\}$$

$$\{a, ab, aaaa\} - \{bb, ab\} = \{a, aaaa\}$$

Complement: $\overline{L} = \Sigma^* - L$

$$\overline{\{a, ba\}} = \{\lambda, b, aa, ab, bb, aaa, \ldots\}$$

Reverse

Definition: $L^R = \{w^R : w \in L\}$

Examples: $\{ab, aab, baba\}^R = \{ba, baa, abab\}$

$$L = \{a^n b^n : n \geq 0\}$$

$$L^R = \{b^n a^n : n \geq 0\}$$

Concatenation

Definition: $L_1 L_2 = \{xy : x \in L_1, y \in L_2\}$

Example: $\{a, ab, ba\}\{b, aa\}$

$= \{ab, aaa, abb, abaa, bab, baaa\}$

Another Operation

Definition: $$L^n = \underbrace{LL \cdots L}_{n}$$

$$\{a,b\}^3 = \{a,b\}\{a,b\}\{a,b\} =$$
$$\{aaa, aab, aba, abb, baa, bab, bba, bbb\}$$

Special case: $L^0 = \{\lambda\}$

$$\{a, bba, aaa\}^0 = \{\lambda\}$$

More Examples

$$L = \{a^n b^n : n \geq 0\}$$

$$L^2 = \{a^n b^n a^m b^m : n, m \geq 0\}$$

$$aabbaaabbb \in L^2$$

Star-Closure (Kleene *)

Definition: $L^* = L^0 \cup L^1 \cup L^2 \cdots$

Example:

$$\{a, bb\}^* = \left\{ \begin{array}{l} \lambda, \\ a, bb, \\ aa, abb, bba, bbbb, \\ aaa, aabb, abba, abbbb, \ldots \end{array} \right\}$$

Positive Closure

Definition:
$$L^{+} = L^{1} \cup L^{2} \cup \cdots$$
$$= L^{*} - \{\lambda\}$$

$$\{a, bb\}^{+} = \left\{ \begin{array}{l} a, bb, \\ aa, abb, bba, bbbb, \\ aaa, aabb, abba, abbbb, \ldots \end{array} \right\}$$

Mathematical Preliminaries

Mathematical Preliminaries

- Sets
- Functions
- Relations
- Graphs
- Proof Techniques

SETS

A set is a collection of elements

$$A = \{1, 2, 3\}$$

$$B = \{train, bus, bicycle, airplane\}$$

We write

$$1 \in A$$

$$ship \notin B$$

Set Representations

$C = \{ a, b, c, d, e, f, g, h, i, j, k \}$

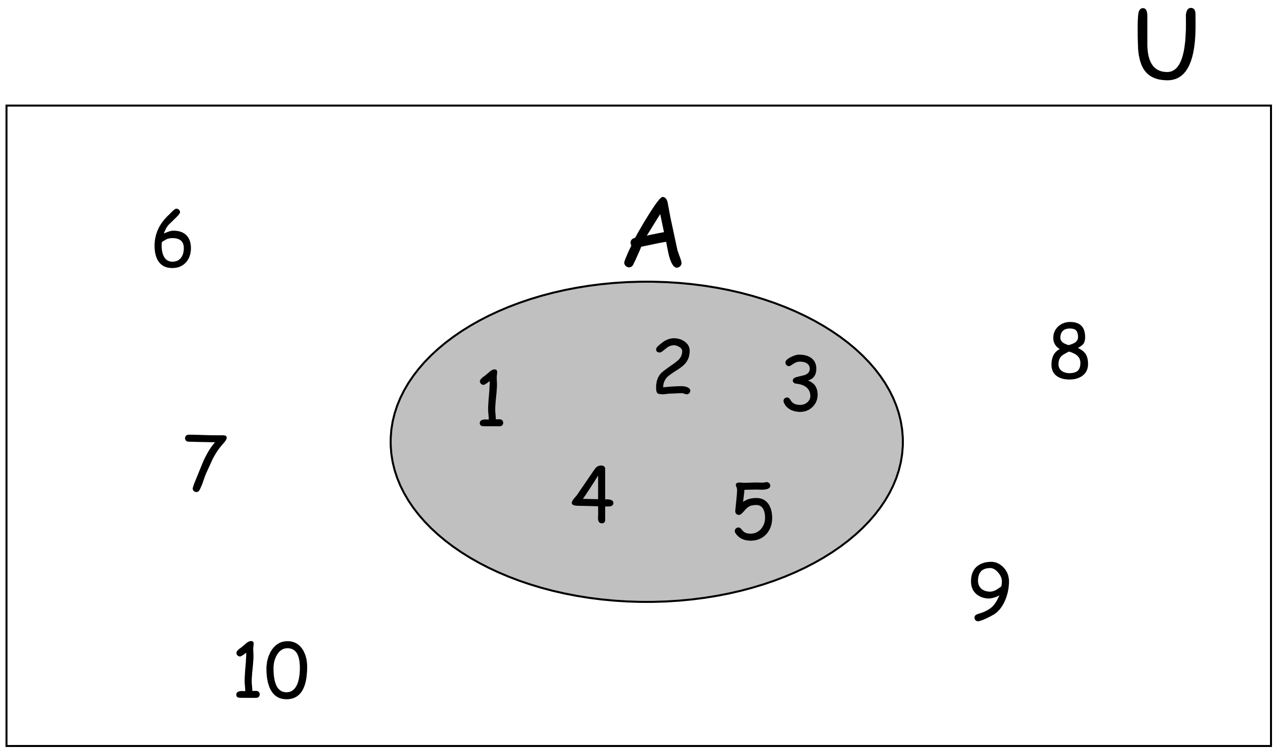
$C = \{ a, b, \ldots, k \}$ $\longrightarrow$ *finite set*

$S = \{ 2, 4, 6, \ldots \}$ $\longrightarrow$ *infinite set*

$S = \{ j : j > 0, \text{ and } j = 2k \text{ for some } k>0 \}$

$S = \{ j : j \text{ is nonnegative and even} \}$

$A = \{ 1, 2, 3, 4, 5 \}$



Universal Set: all possible elements

$U = \{ 1 , \dots , 10 \}$

Set Operations

$A = \{ 1, 2, 3 \}$ $B = \{ 2, 3, 4, 5\}$

- Union

$A \cup B = \{ 1, 2, 3, 4, 5 \}$

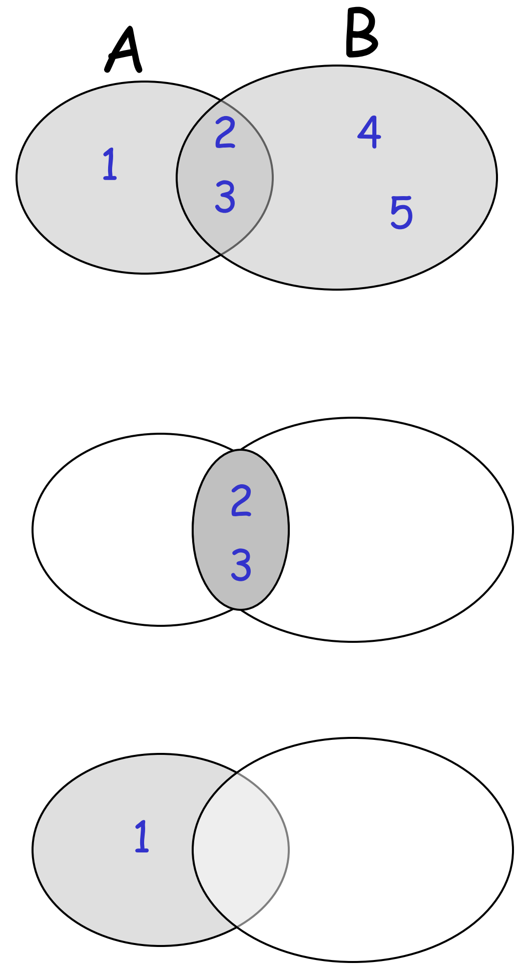
- Intersection

$A \cap B = \{ 2, 3 \}$

- Difference

$A - B = \{ 1 \}$

$B - A = \{ 4, 5 \}$

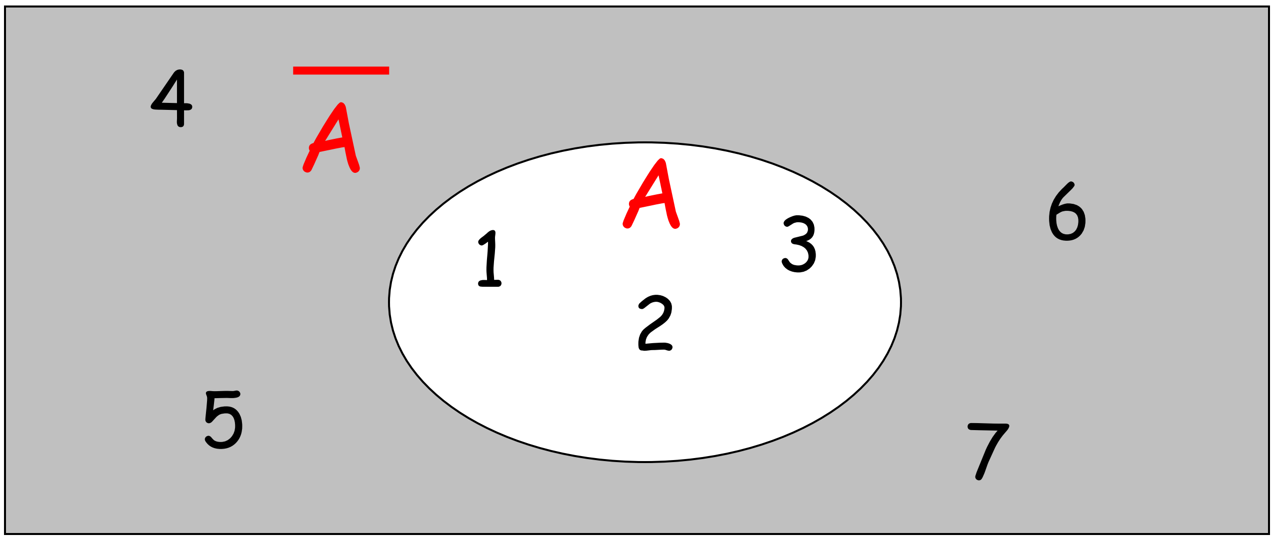


Venn diagrams

- Complement

Universal set = {1, ..., 7}

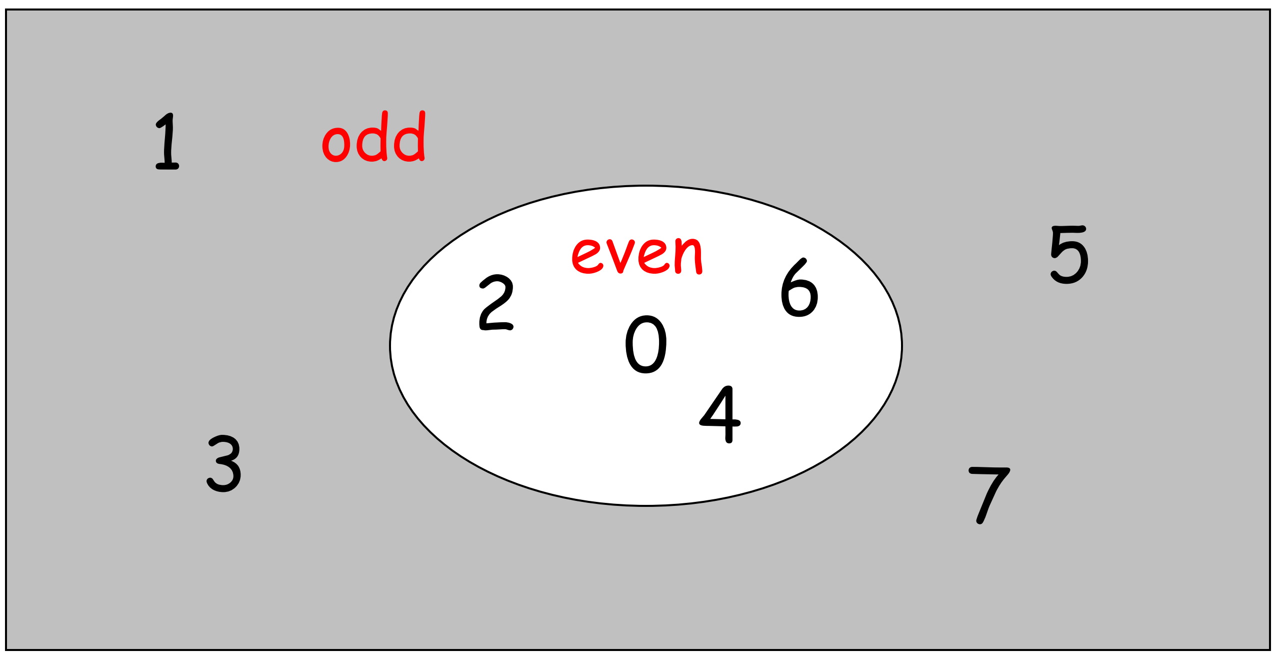
$A = \{1, 2, 3\} \Rightarrow \overline{A} = \{4, 5, 6, 7\}$



$$\overline{\overline{A}} = A$$

$$\overline{\{\text{ even integers }\}} = \{\text{ odd integers }\}$$

Integers



DeMorgan's Laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Empty, Null Set: $\emptyset$

$\emptyset = \{ \}$

$S \cup \emptyset =$

$S \cap \emptyset =$

$\overline{\emptyset} =$

$S - \emptyset =$

$\emptyset - S =$

Empty, Null Set: $\emptyset$

$\emptyset = \{\ \}$

$S \cup \emptyset = S$

$S \cap \emptyset = \emptyset$

$\overline{\emptyset}$ = Universal Set

$S - \emptyset = S$

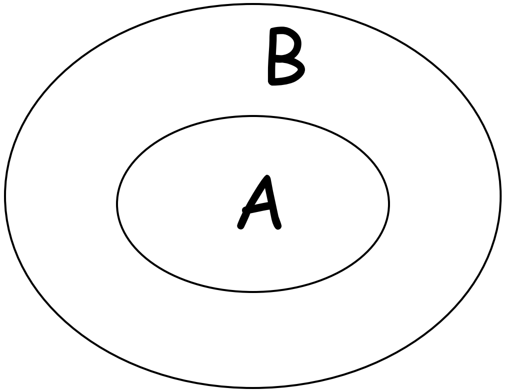
$\emptyset - S = \emptyset$

Subset

$A = \{ 1, 2, 3 \}$ $\qquad$ $B = \{ 1, 2, 3, 4, 5 \}$

$$A \subseteq B$$

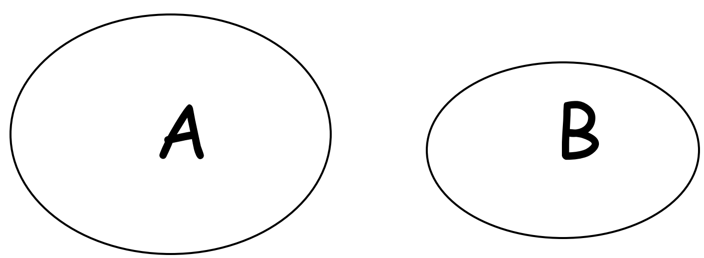
Proper Subset: $A \subset B$



Disjoint Sets

$A = \{ 1, 2, 3 \}$ $\quad\quad$ $B = \{ 5, 6 \}$

$A \cap B = \emptyset$



Set Cardinality

- For finite sets

$A = \{ 2, 5, 7 \}$

$|A| = 3$

(set size)

Powersets

A powerset is a set of sets

$S = \{ a, b, c \}$

Powerset of S = the set of all the subsets of S

$2^S = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \}$

Observation: $| 2^S | = 2^{|S|}$ $(8 = 2^3)$

Cartesian Product

$A = \{ 2, 4 \}$ $B = \{ 2, 3, 5 \}$

$A \times B = \{ (2, 2), (2, 3), (2, 5), (4, 2), (4, 3), (4, 5) \}$

$|A \times B| = |A|\,|B|$

Generalizes to more than two sets

$A \times B \times \ldots \times Z$

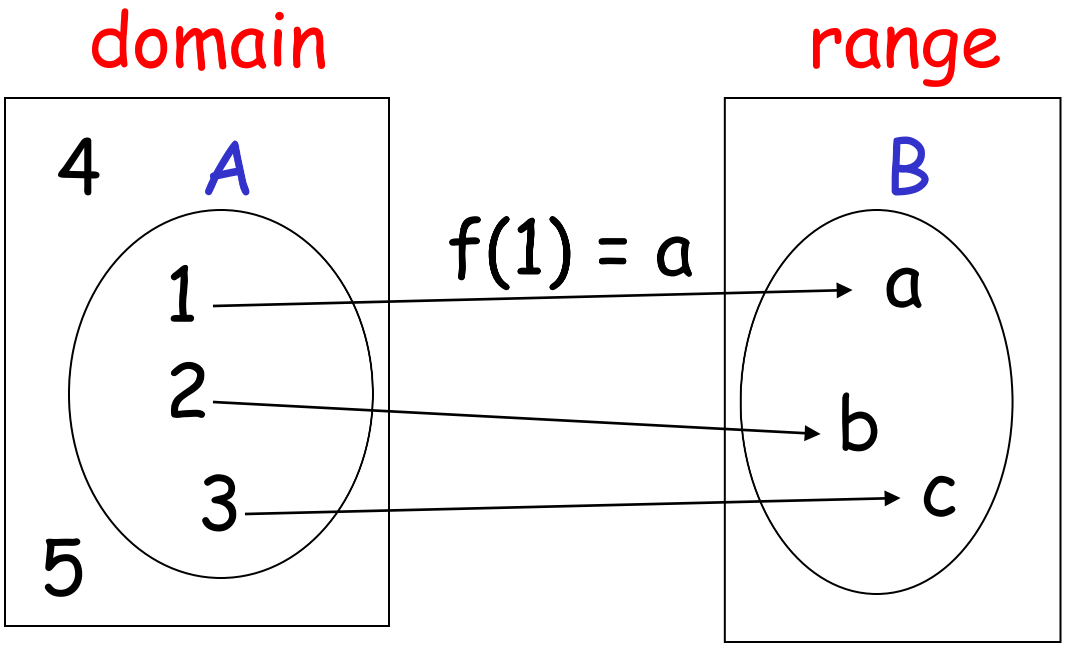
Functions and Relations

A function is a rule that assigns to elements of one set a unique element of another set.

If f denotes a function, then the first set S1 is called the domain of f, & the second set S2 is its range. We write

$$f: \quad S1 \longrightarrow S2$$

FUNCTIONS



f : A -> B

If A = domain

then f is a total function (every element of domain is associated with one element of range)

otherwise f is a partial function

RELATIONS

$R = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots\}$

$x_i \, R \, y_i$

e. g. if R = '>': $2 > 1$, $3 > 2$, $3 > 1$

Equivalence Relations

- Reflexive: $x \; R \; x$
- Symmetric: $x \; R \; y \Rightarrow y \; R \; x$
- Transitive: $x \; R \; y$ and $y \; R \; z \Rightarrow x \; R \; z$

Example: R = '='

- $x = x$
- $x = y \Rightarrow y = x$
- $x = y$ and $y = z \Rightarrow x = z$

Equivalence Classes

For equivalence relation R

equivalence class of $x = \{y : x \; R \; y\}$

Example:

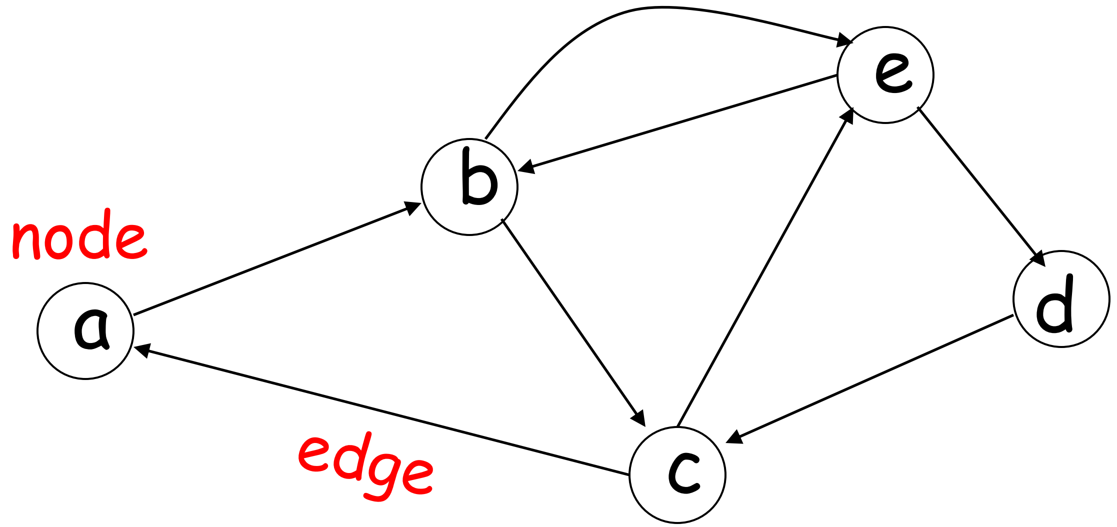
$$R = \{ (1, 1), (2, 2), (1, 2), (2, 1), (3, 3), (4, 4), (3, 4), (4, 3) \}$$

Equivalence class of $1 = \{1, 2\}$

Equivalence class of $3 = \{3, 4\}$

GRAPHS

A directed graph



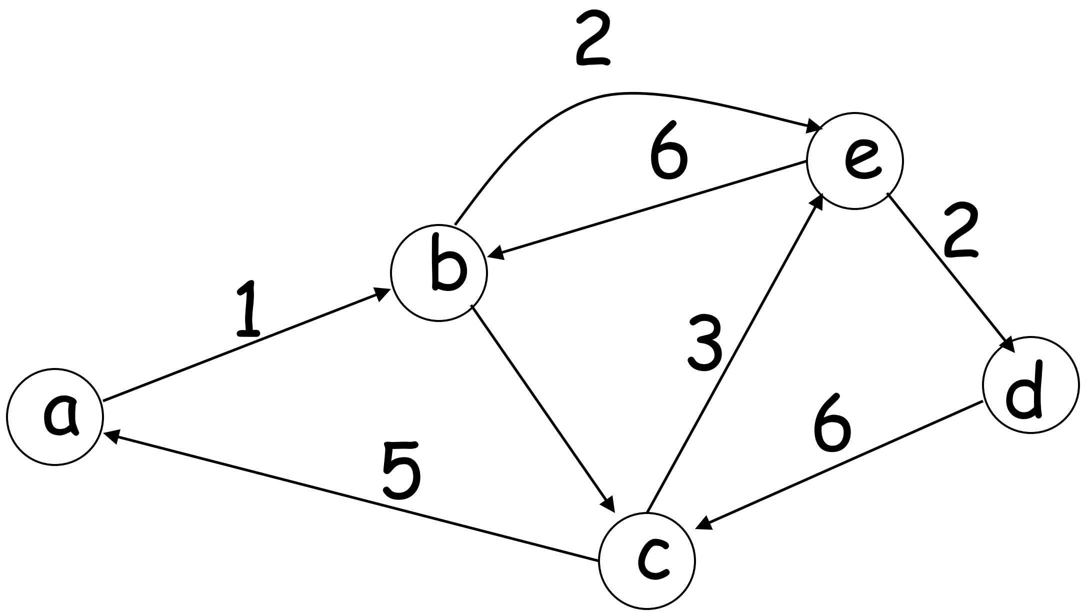
- Nodes (Vertices)

  V = { a, b, c, d, e }

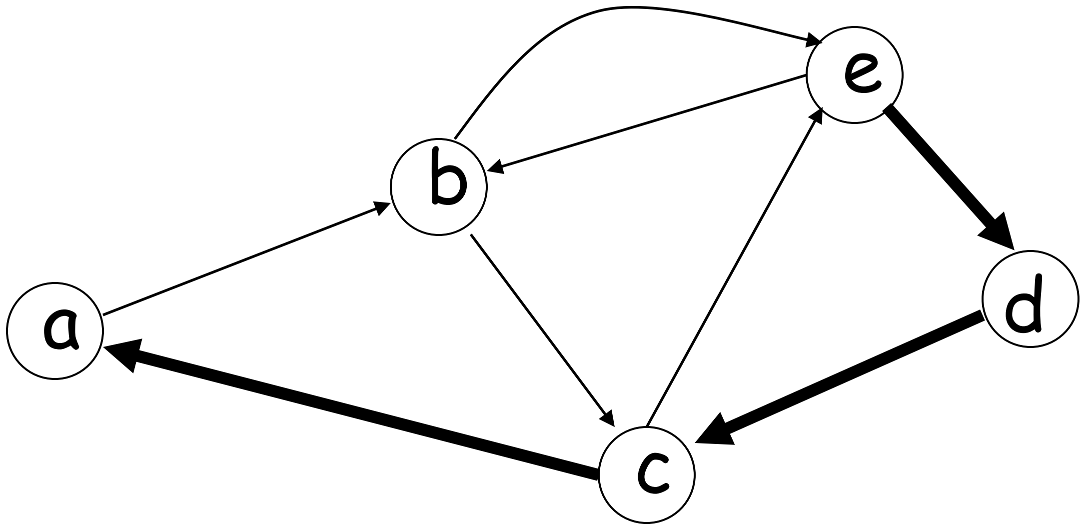
- Edges

  E = { (a,b), (b,c), (b,e),(c,a), (c,e), (d,c), (e,b), (e,d) }

Labeled Graph



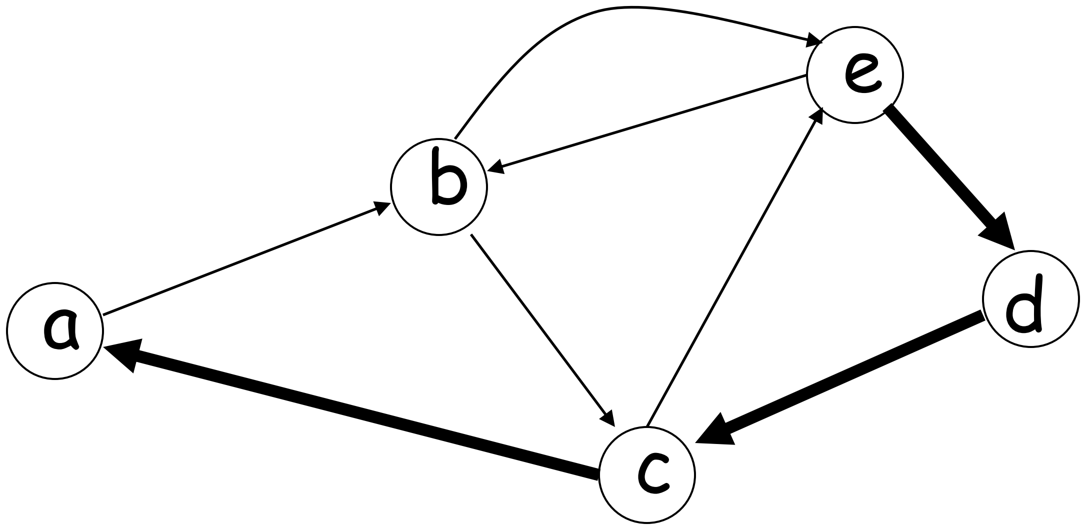
Walk



Walk is a sequence of adjacent edges

(e, d), (d, c), (c, a)

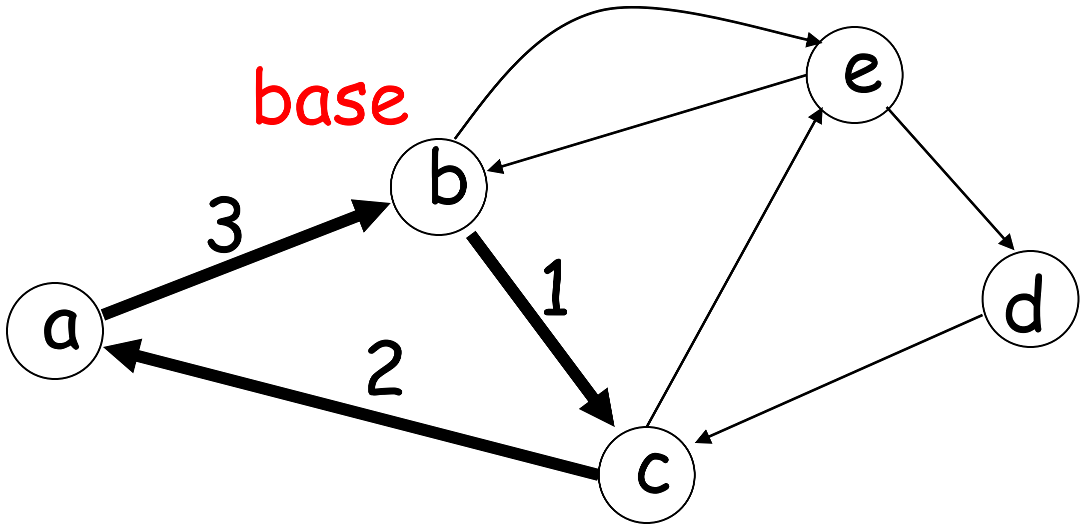
Path



Path is a walk where no edge is repeated

Simple path: no node is repeated

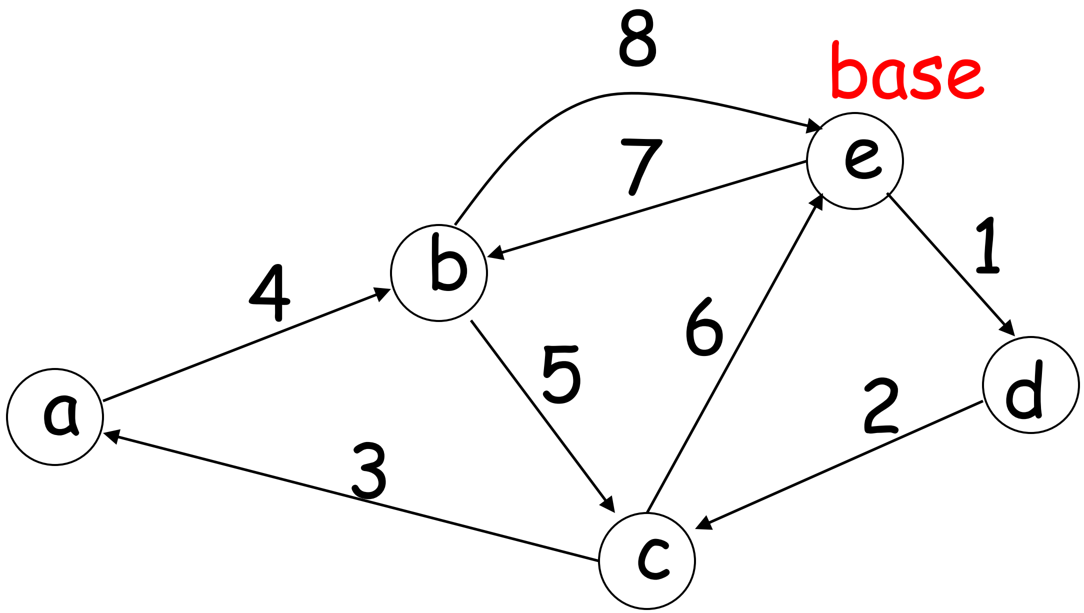
Cycle



Cycle: a walk from a node (base) to itself

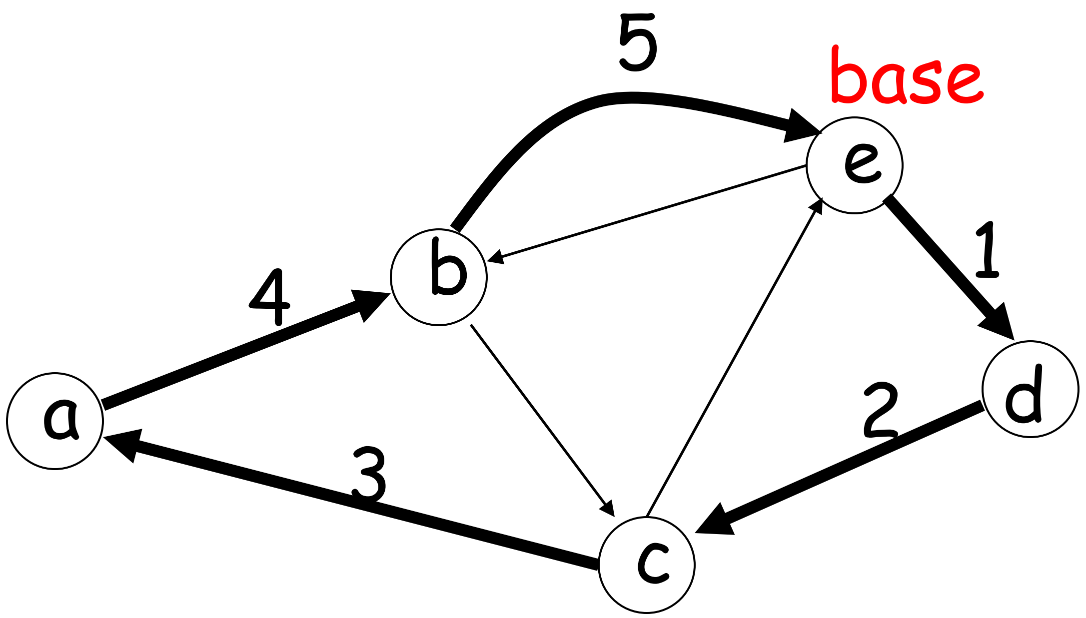
Simple cycle: only the base node is repeated

Euler Tour



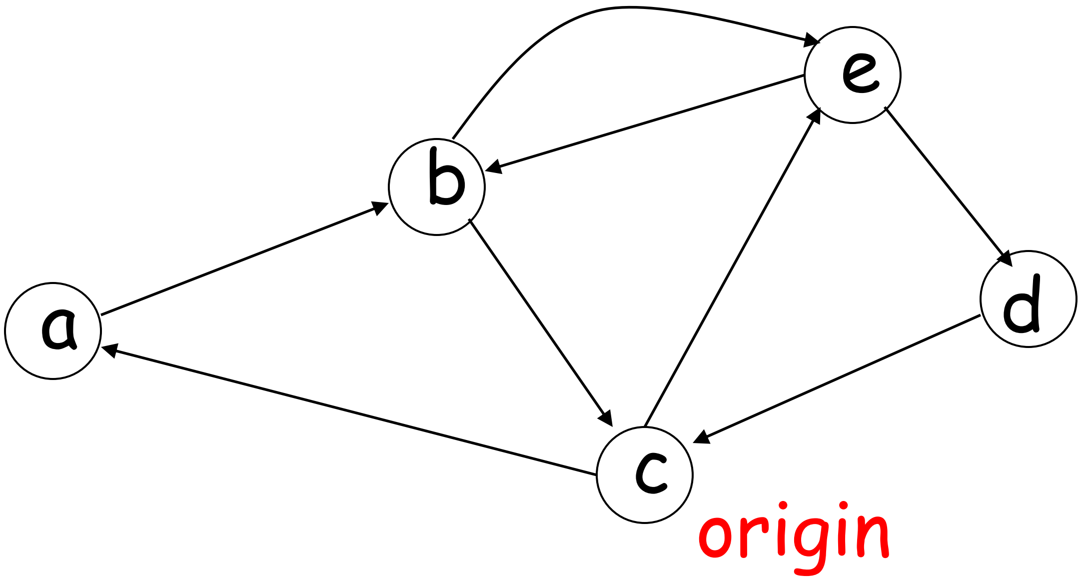
A cycle that contains each edge once

Hamiltonian Cycle

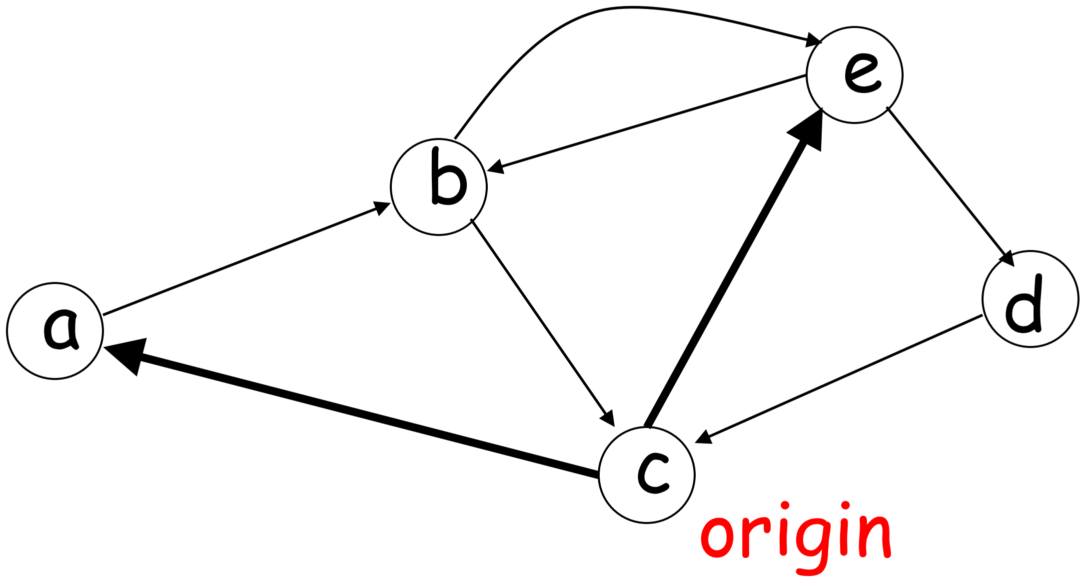


A simple cycle that contains all nodes

Finding All Simple Paths



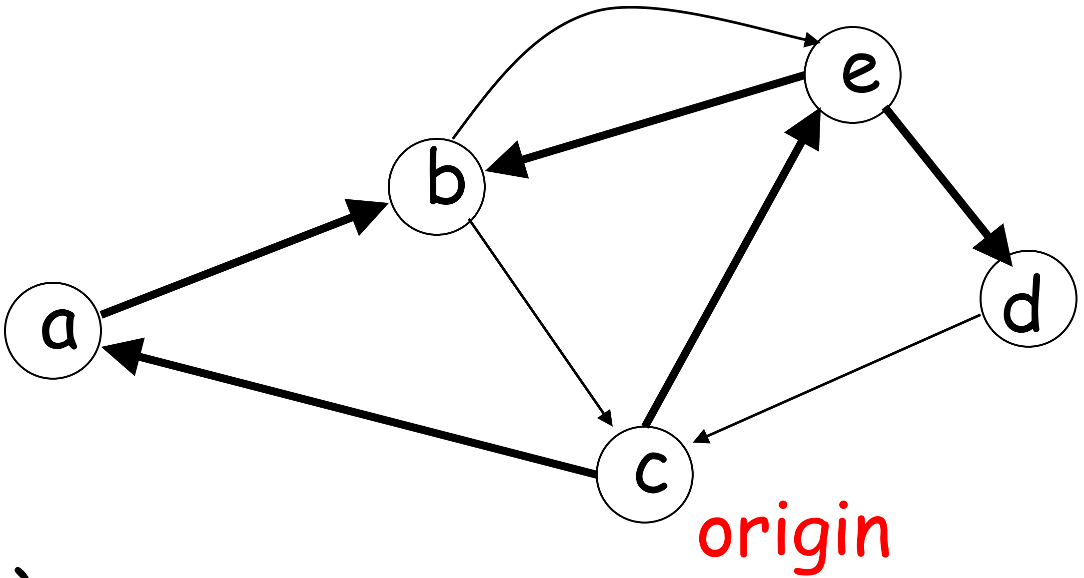
Step 1



(c, a)

(c, e)

Step 2



(c, a)

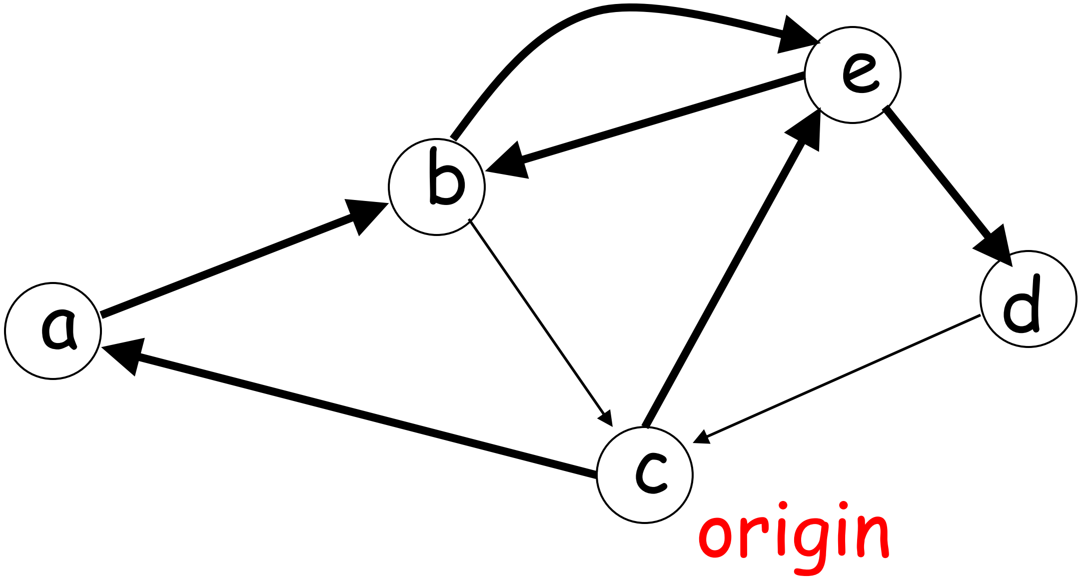
(c, a), (a, b)

(c, e)

(c, e), (e, b)

(c, e), (e, d)

Step 3



(c, a)

(c, a), (a, b)

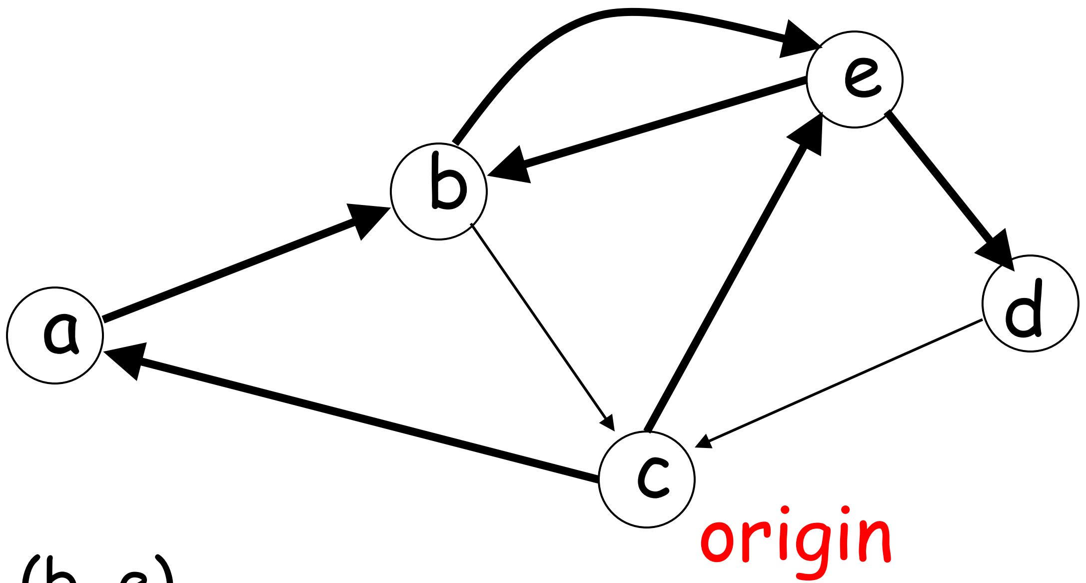
(c, a), (a, b), (b, e)

(c, e)

(c, e), (e, b)

(c, e), (e, d)

Step 4



(c, a)

(c, a), (a, b)

(c, a), (a, b), (b, e)

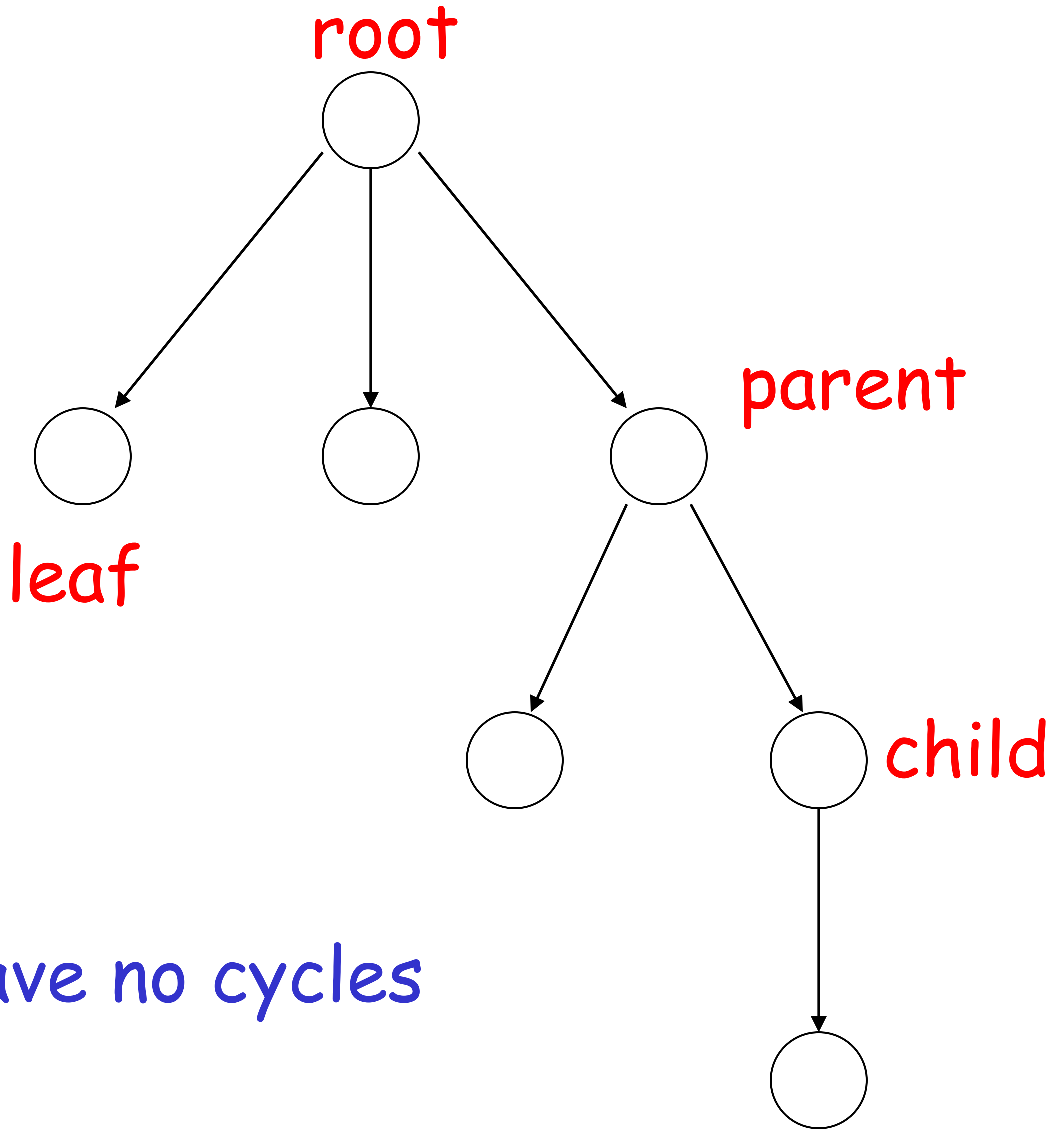
(c, a), (a, b), (b, e), (e,d)

(c, e)

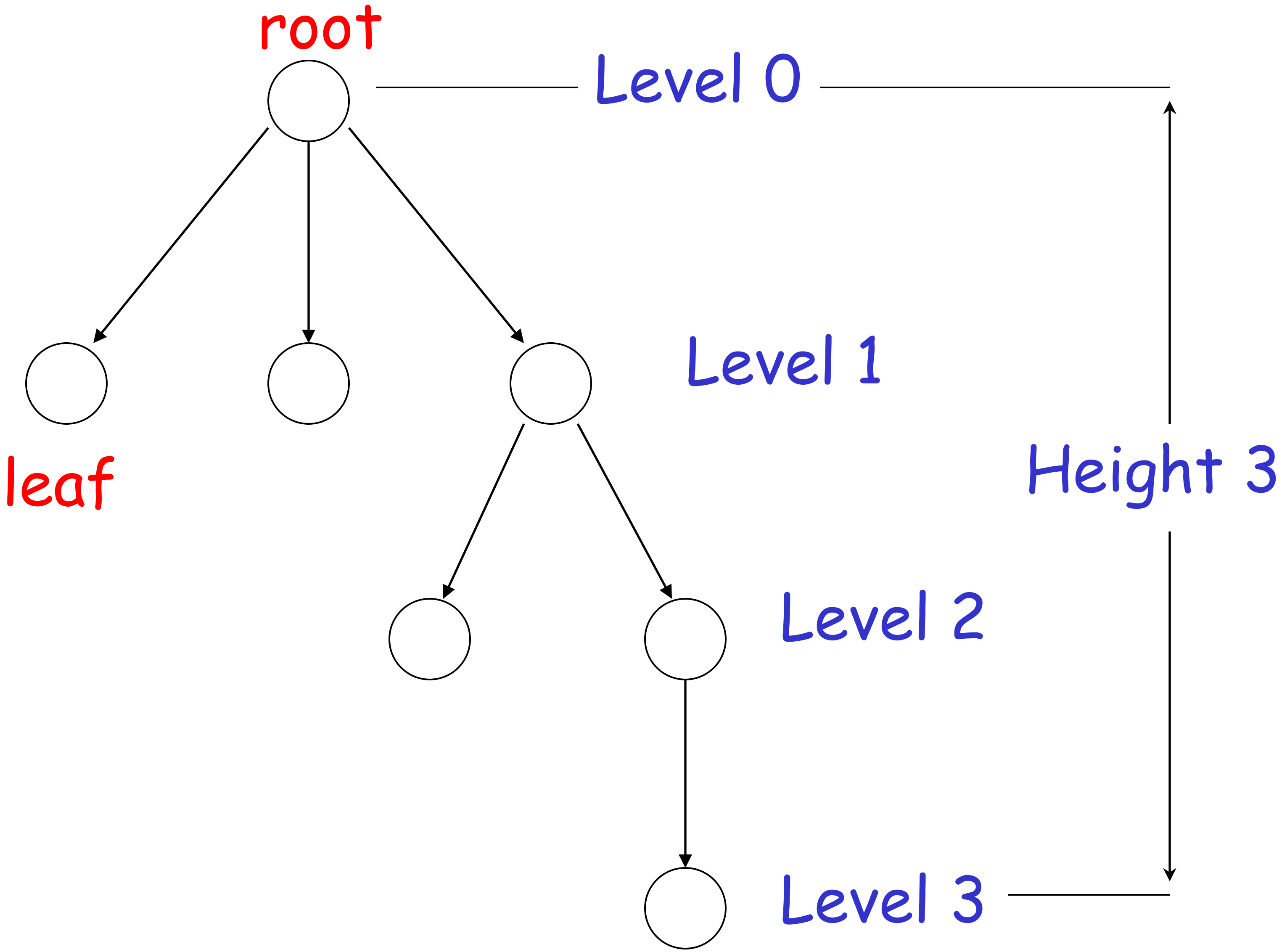
(c, e), (e, b)

(c, e), (e, d)

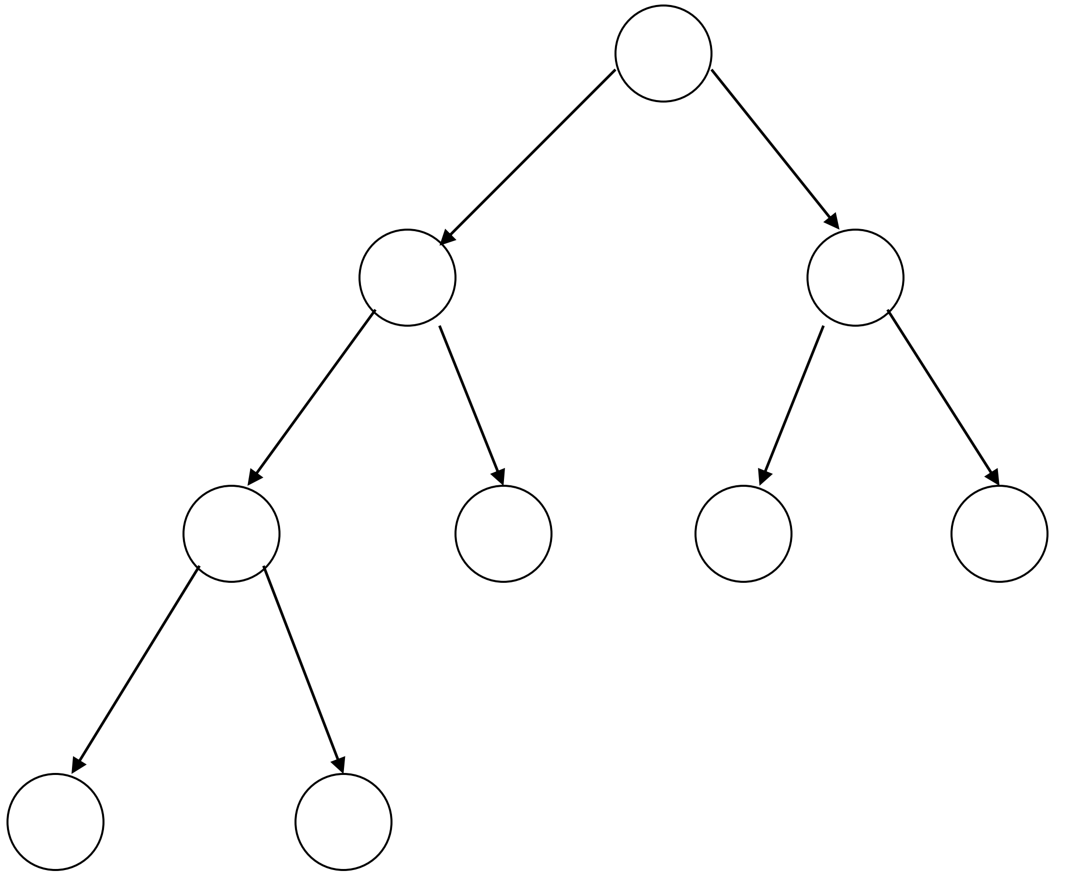
Trees



Trees have no cycles


root
Level 0
Level 1
leaf
Height 3
Level 2
Level 3

Binary Trees



PROOF TECHNIQUES

- Proof by induction
- Proof by contradiction

Induction

We have statements $P_1, P_2, P_3, \ldots$

If we know

- for some b that $P_1, P_2, \ldots, P_b$ are true
- for any $k >= b$ that

  $P_1, P_2, \ldots, P_k$ imply P_{k+1}

Then

Every P_i is true

Proof by Induction

- Inductive basis

  Find $P_1, P_2, \ldots, P_b$ which are true

- Inductive hypothesis

  Let's assume $P_1, P_2, \ldots, P_k$ are true,

  for any $k \geq b$

- Inductive step

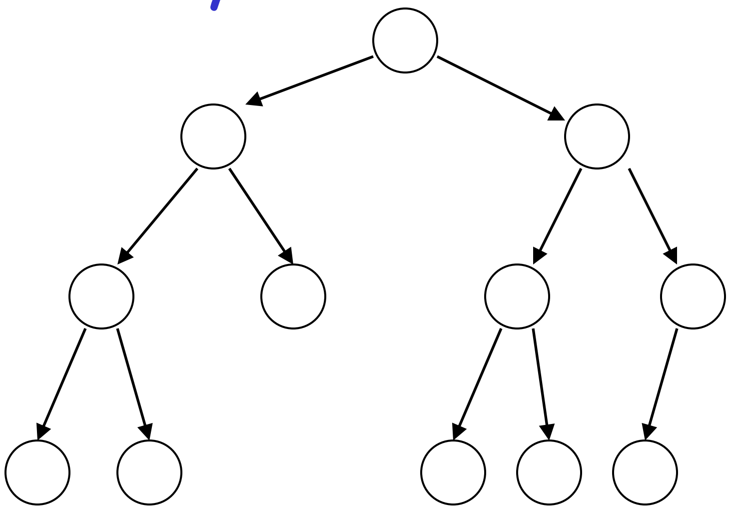
  Show that P_{k+1} is true

Example

Theorem: A binary tree of height n has at most 2^n leaves.

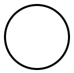
Proof by induction:

let $L(i)$ be the maximum number of leaves of any subtree at height i



We want to show: $L(i) <= 2^i$

- Inductive basis

  $L(0) = 1$ (the root node) 

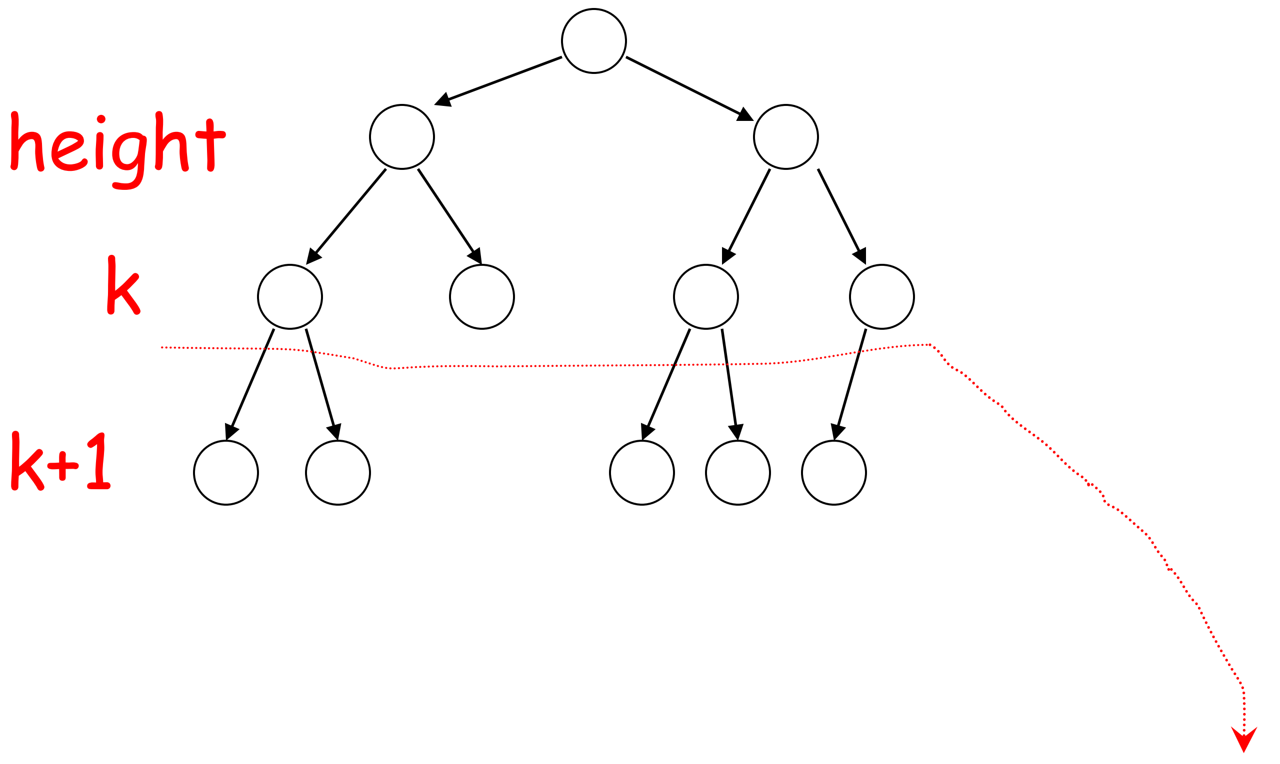
- Inductive hypothesis

  Let's assume $L(i) <= 2^i$ for all $i = 0, 1, \ldots, k$

- Induction step

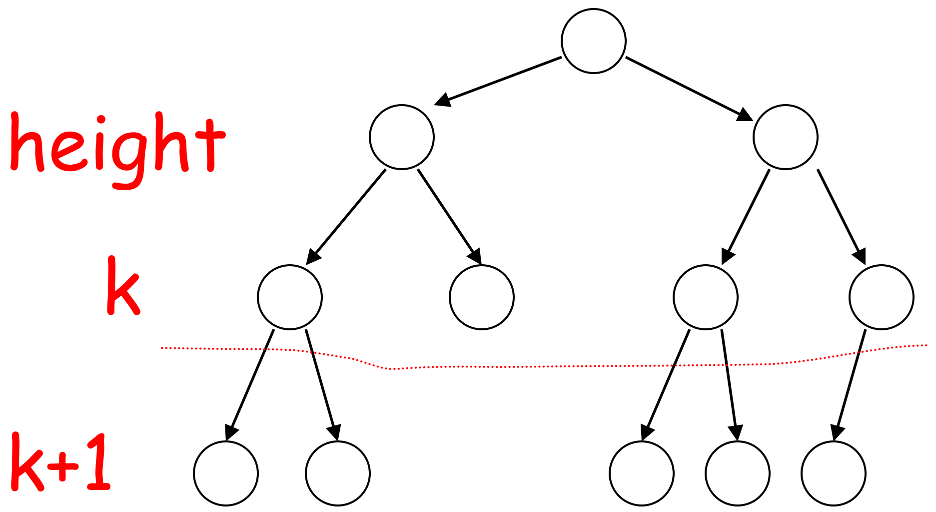
  we need to show that $L(k + 1) <= 2^{k+1}$

Induction Step



From Inductive hypothesis: $L(k) \le 2^k$

Induction Step



$L(k) <= 2^k$

$L(k+1) <= 2 * L(k) <= 2 * 2^k = 2^{k+1}$

(we add at most two nodes for every leaf of level k)

Proof by Contradiction

We want to prove that a statement P is true

- we assume that P is false
- then we arrive at an incorrect conclusion
- therefore, statement P must be true

Example

Theorem: $\sqrt{2}$ is not rational

Proof:

Assume by contradiction that it is rational

$$\sqrt{2} = n/m$$

n and m have no common factors

We will show that this is impossible

$\sqrt{2} = n/m \Rightarrow 2m^2 = n^2$

Therefore, n^2 is even $\Rightarrow$ n is even

$n = 2k$

$2m^2 = 4k^2 \Rightarrow m^2 = 2k^2 \Rightarrow$ m is even

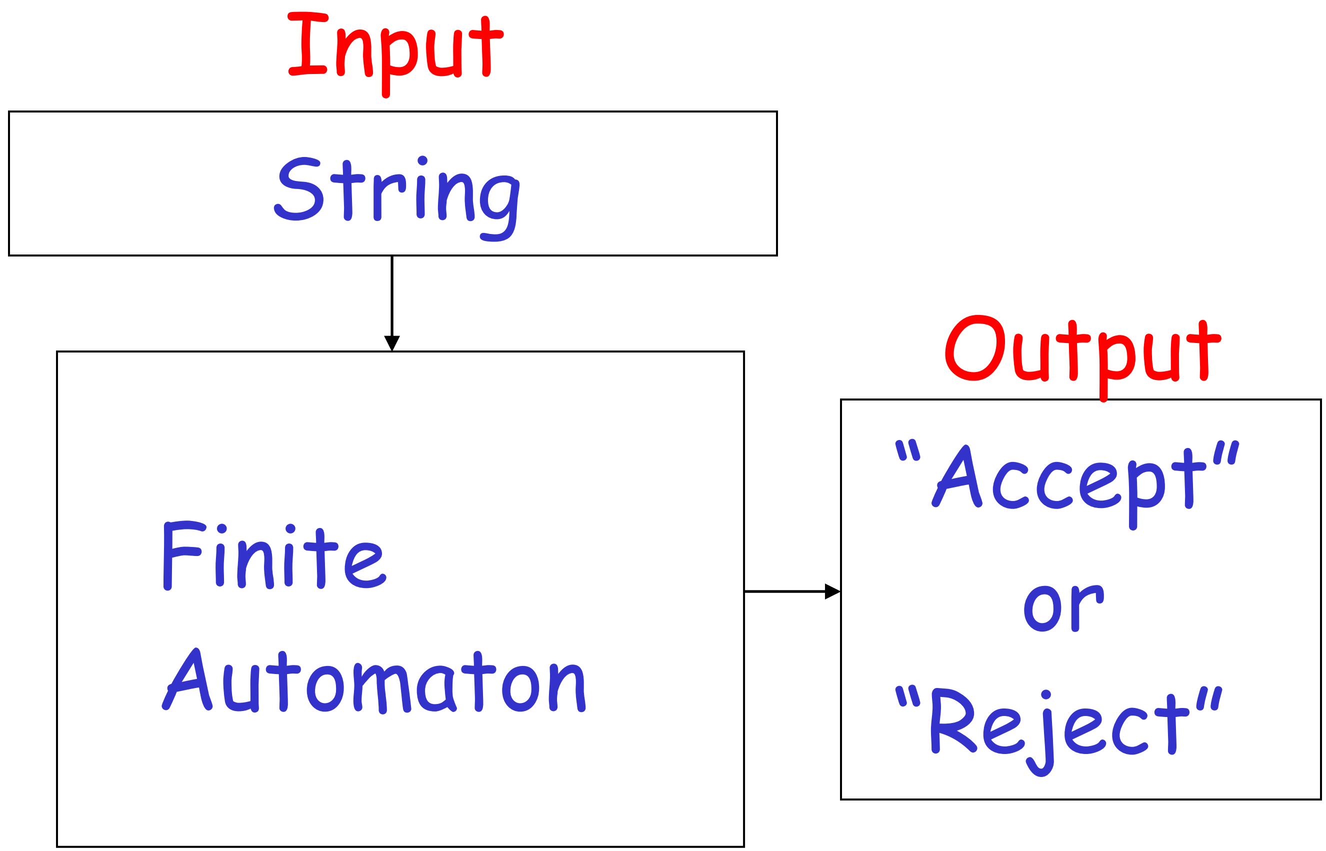
$m = 2p$

Thus, m and n have common factor 2

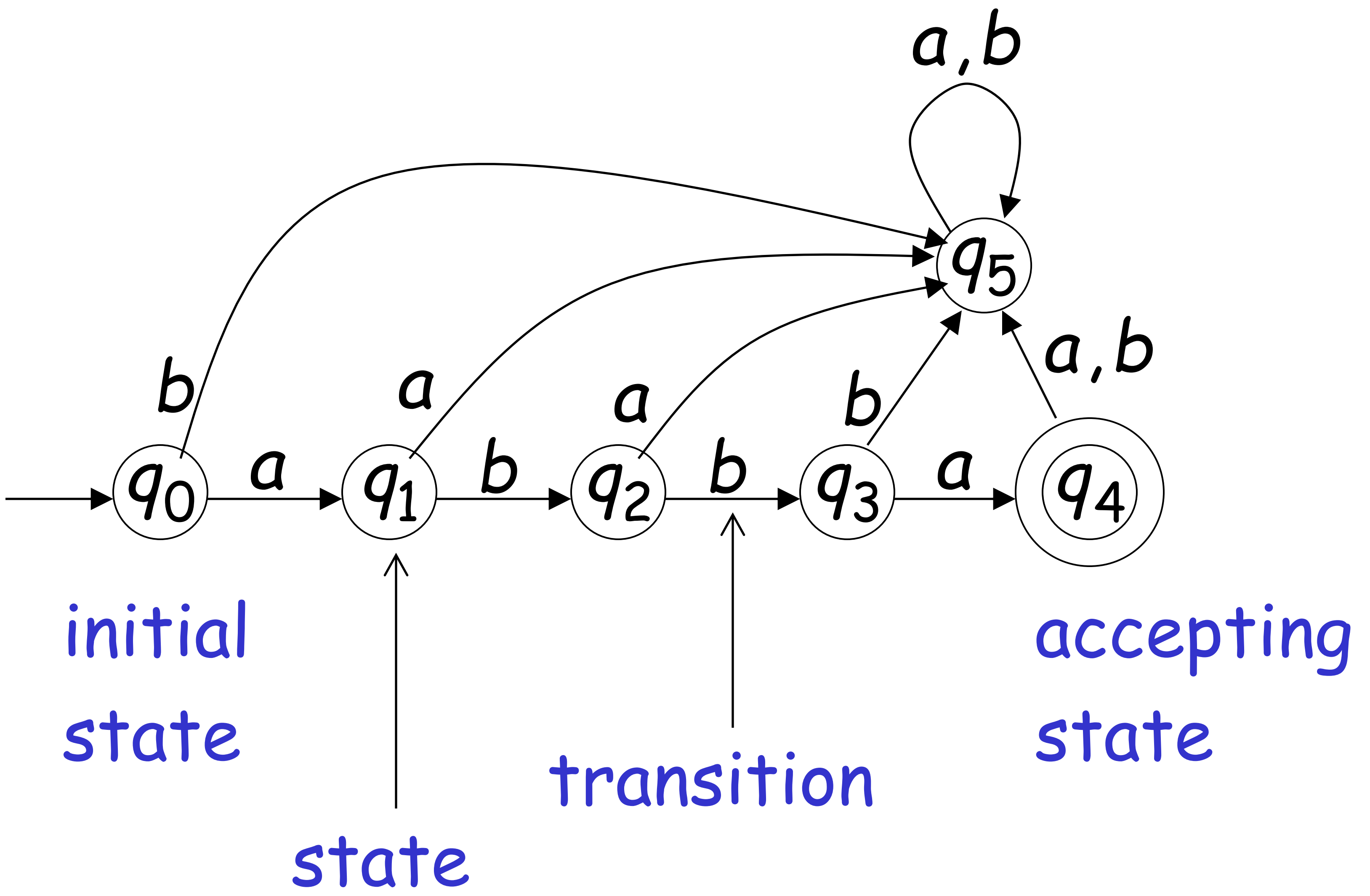
Contradiction!

Formal Languages
Finite Automata

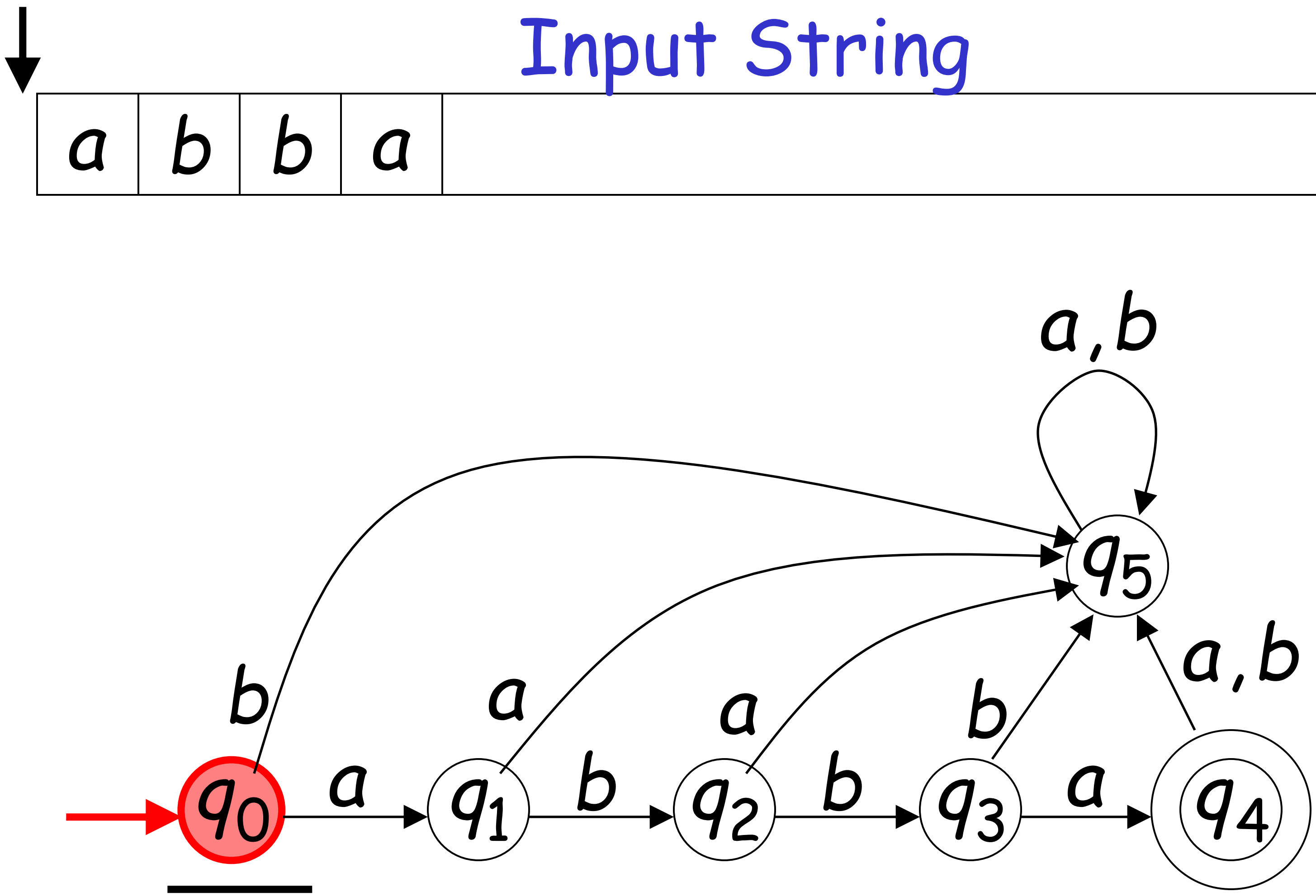
Finite Automaton



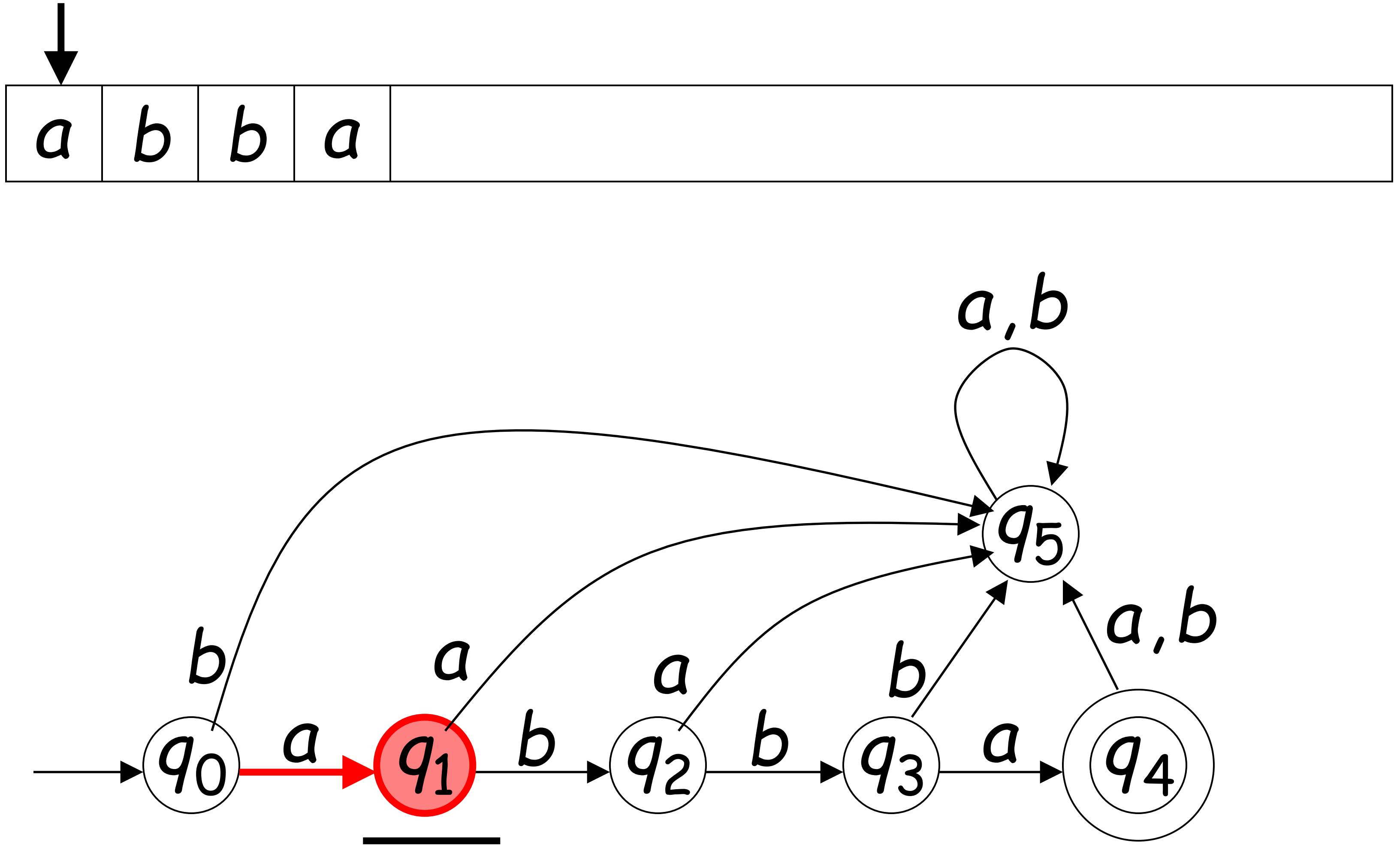
Transition Graph

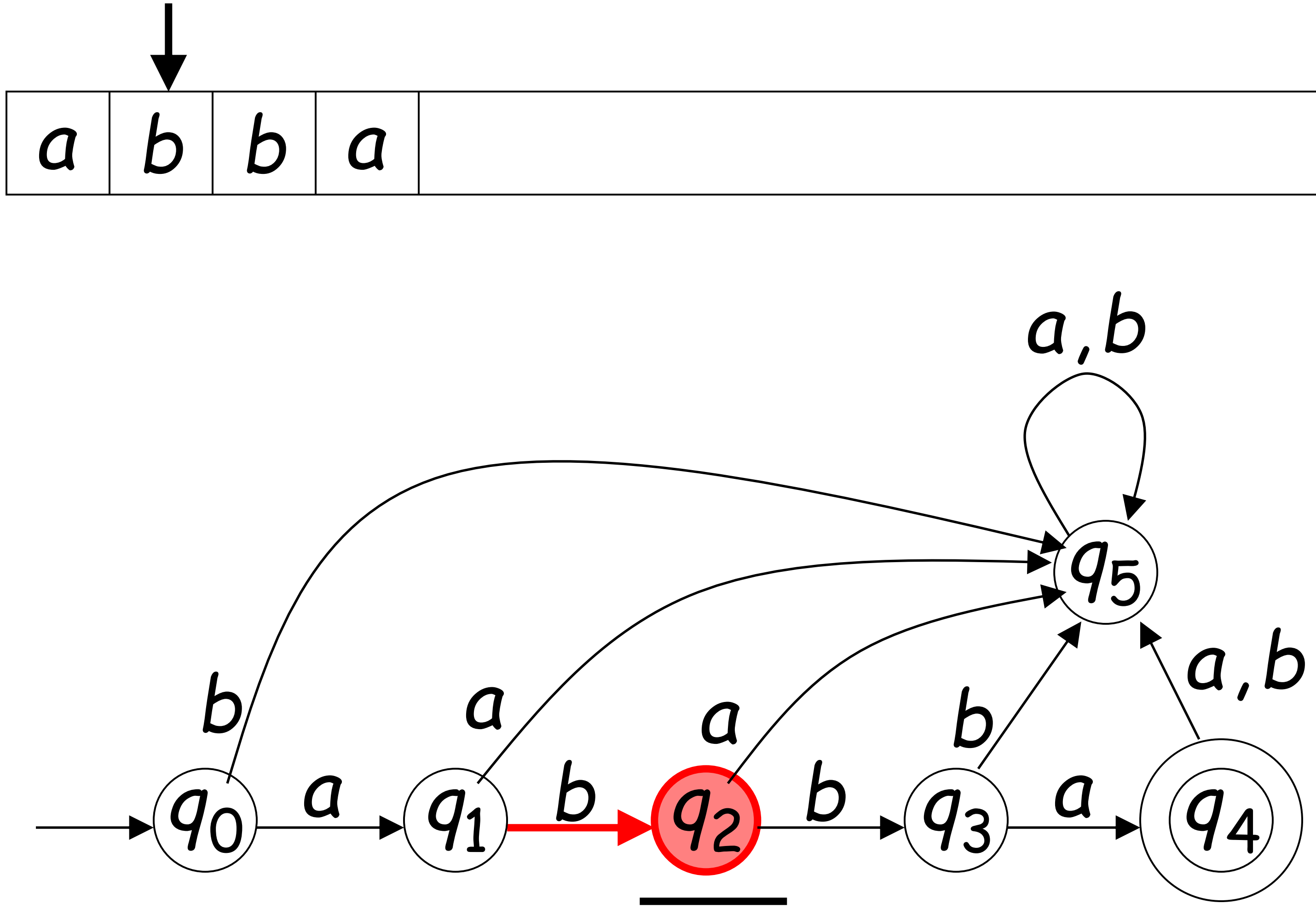


Initial Configuration

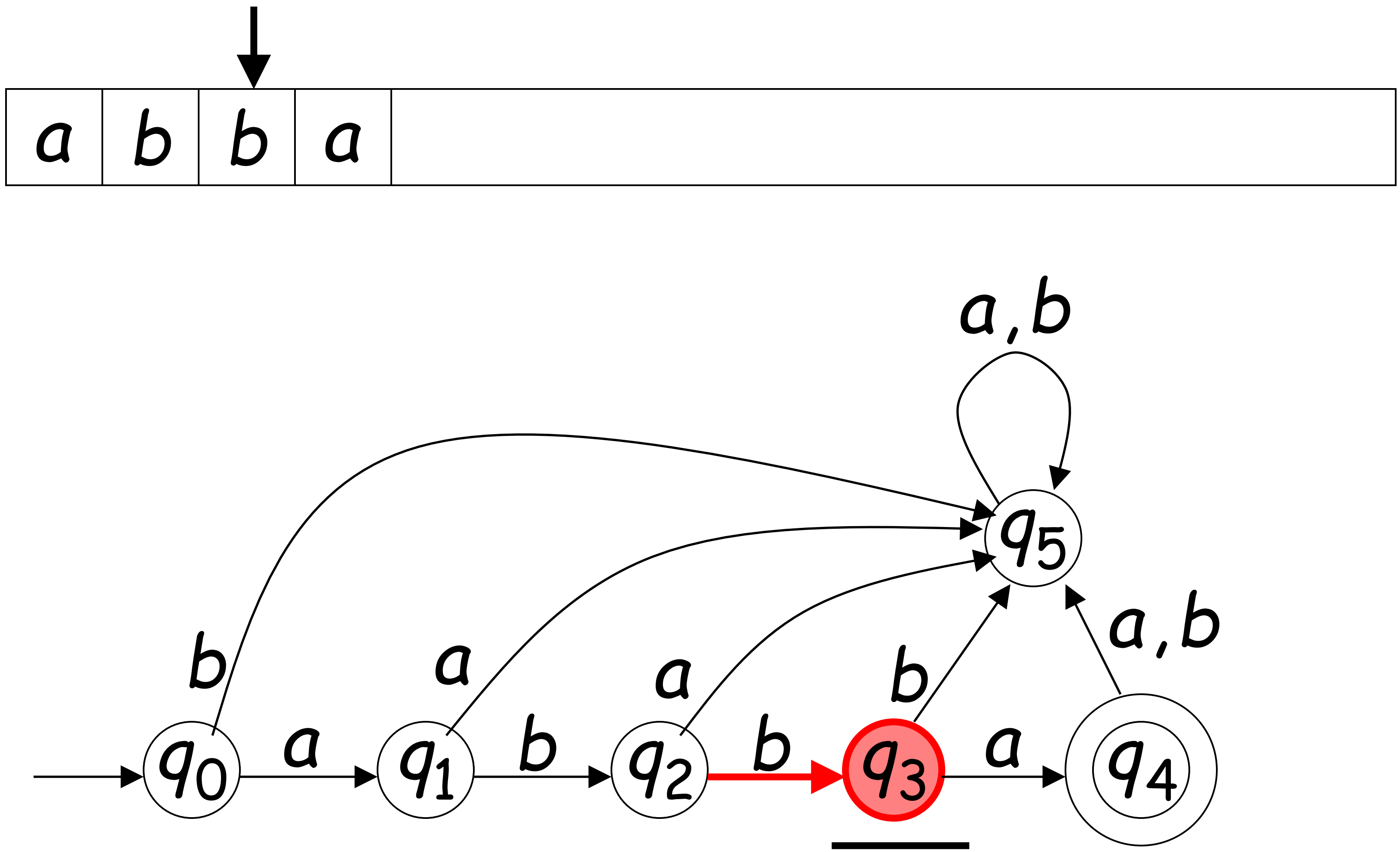


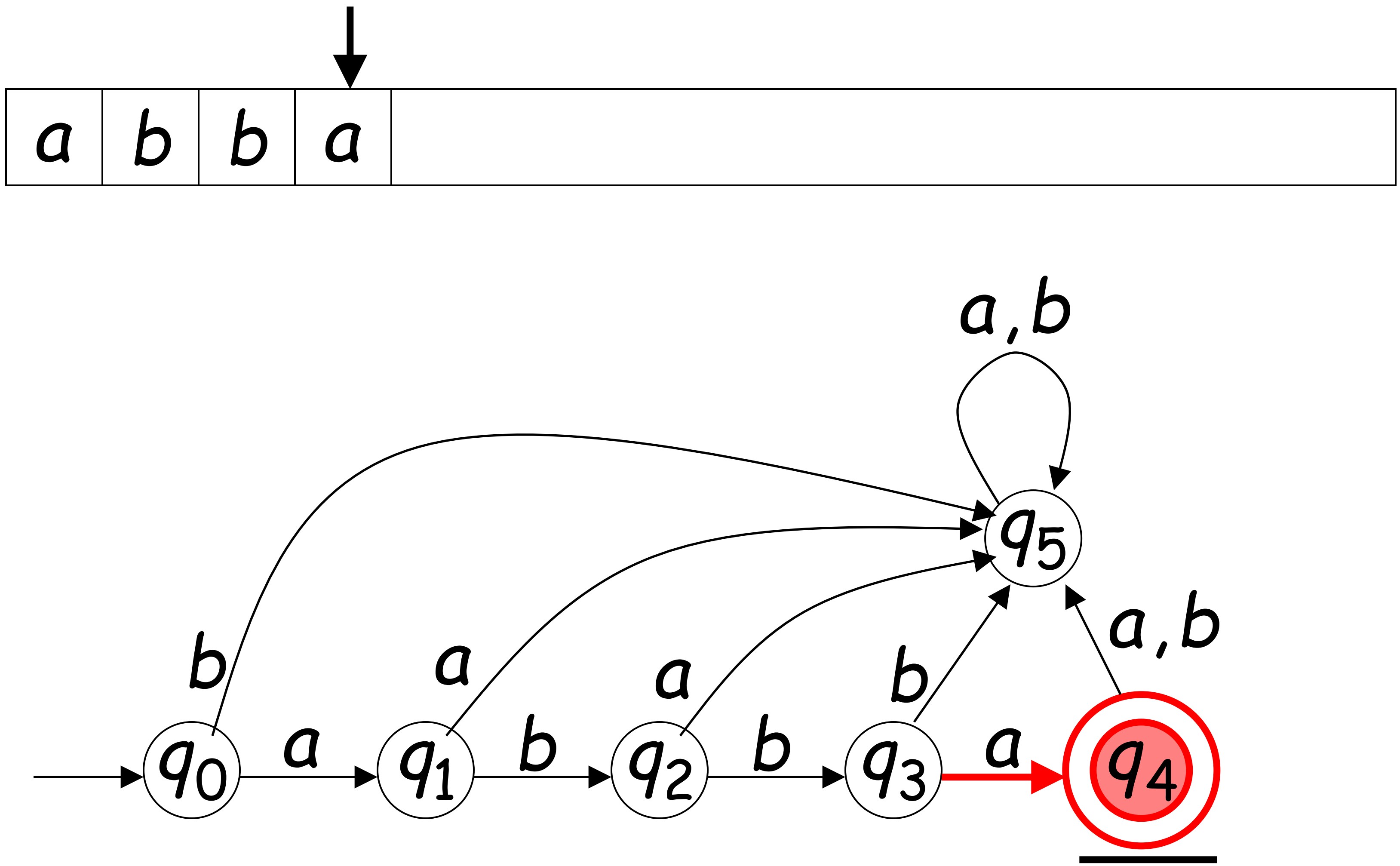
Reading the Input





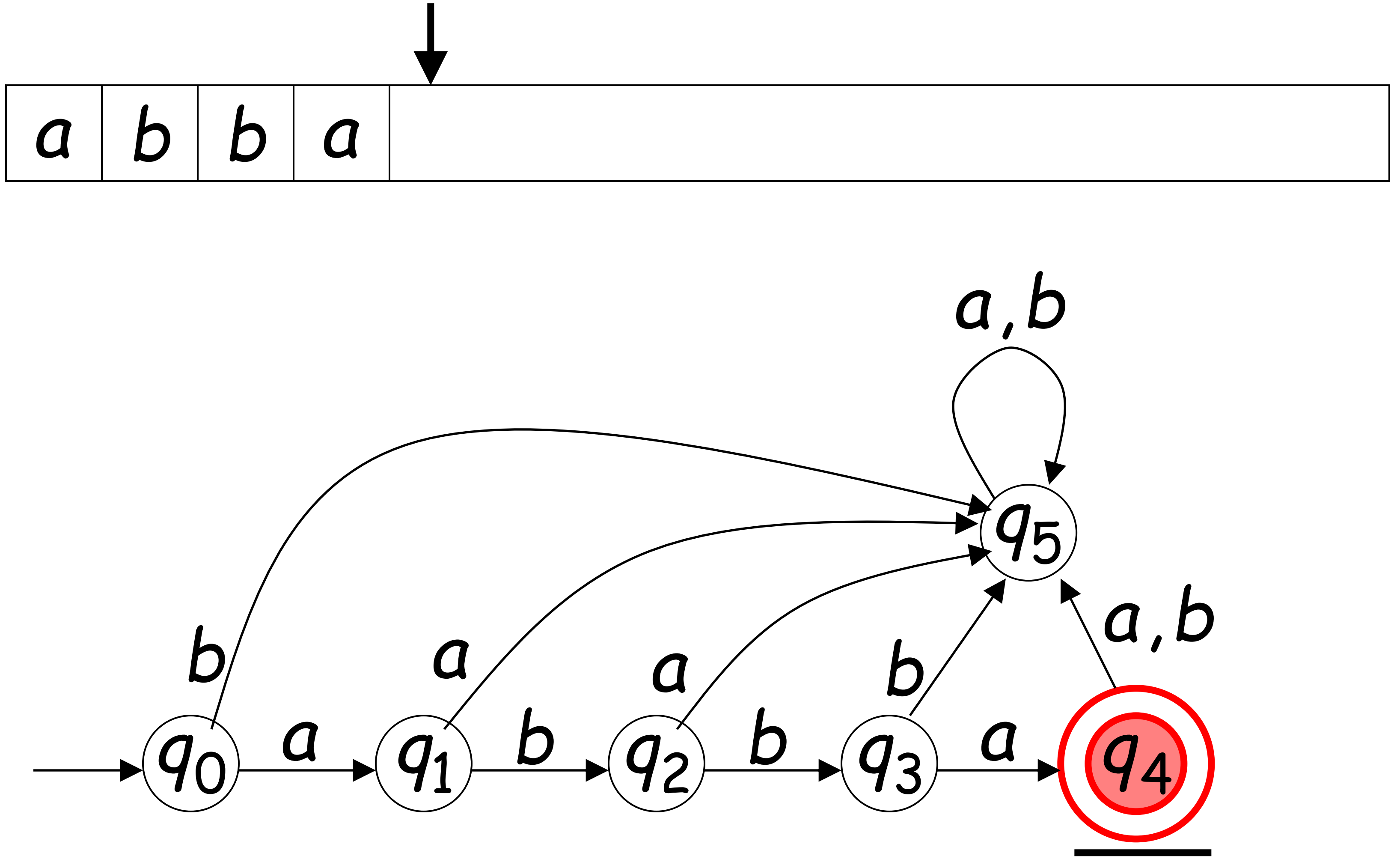
a
b
b
a
a,b
q5
b
a
a
b
a,b
q0
a
q1
b
q2
b
q3
a
q4


a
b
b
a
a,b
q_5
b
a
a
b
a,b
q_0
a
q_1
b
q_2
b
q_3
a
q_4



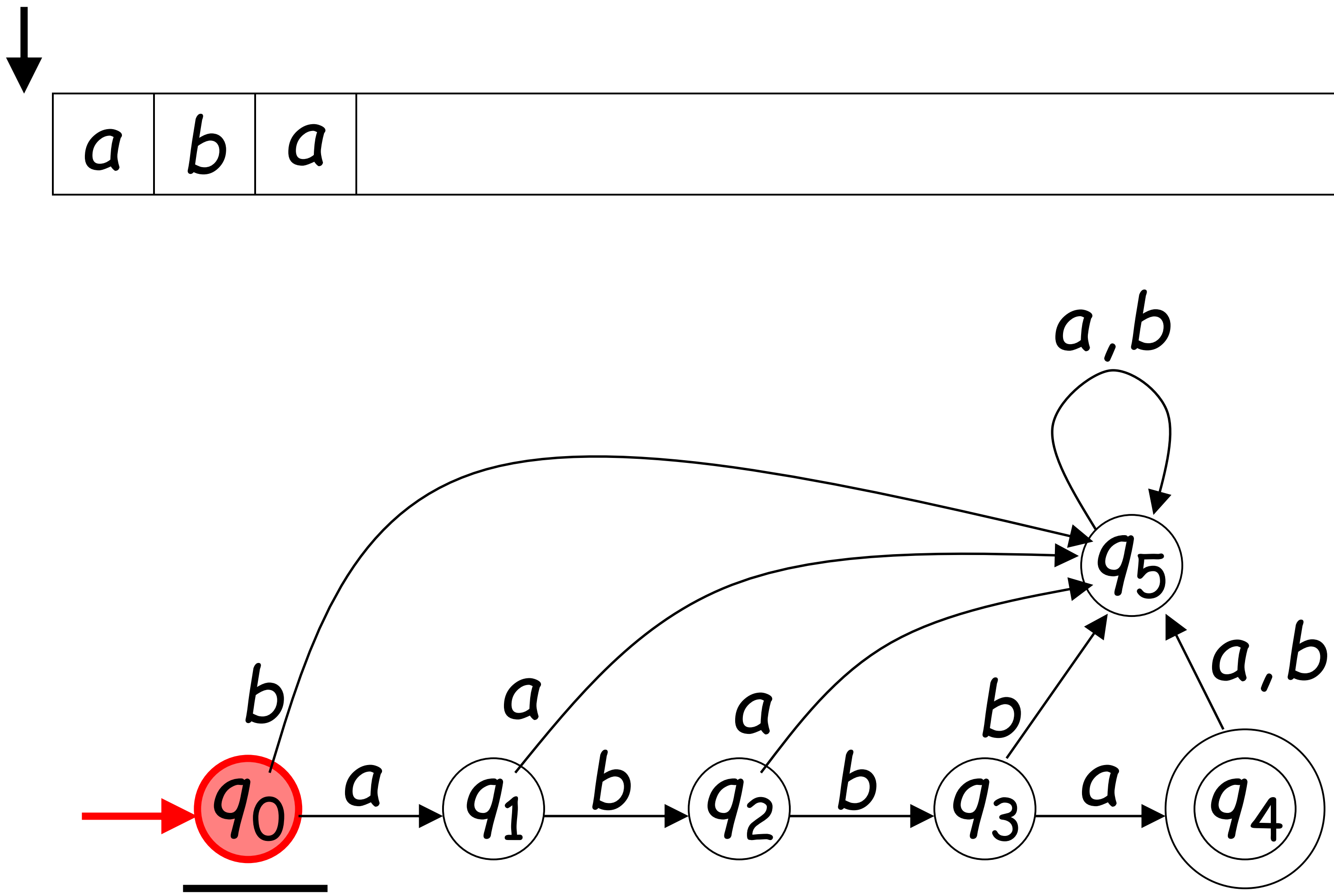
a
b
b
a
a,b
q5
b
a
a
b
a,b
q0
a
q1
b
q2
b
q3
a
q4

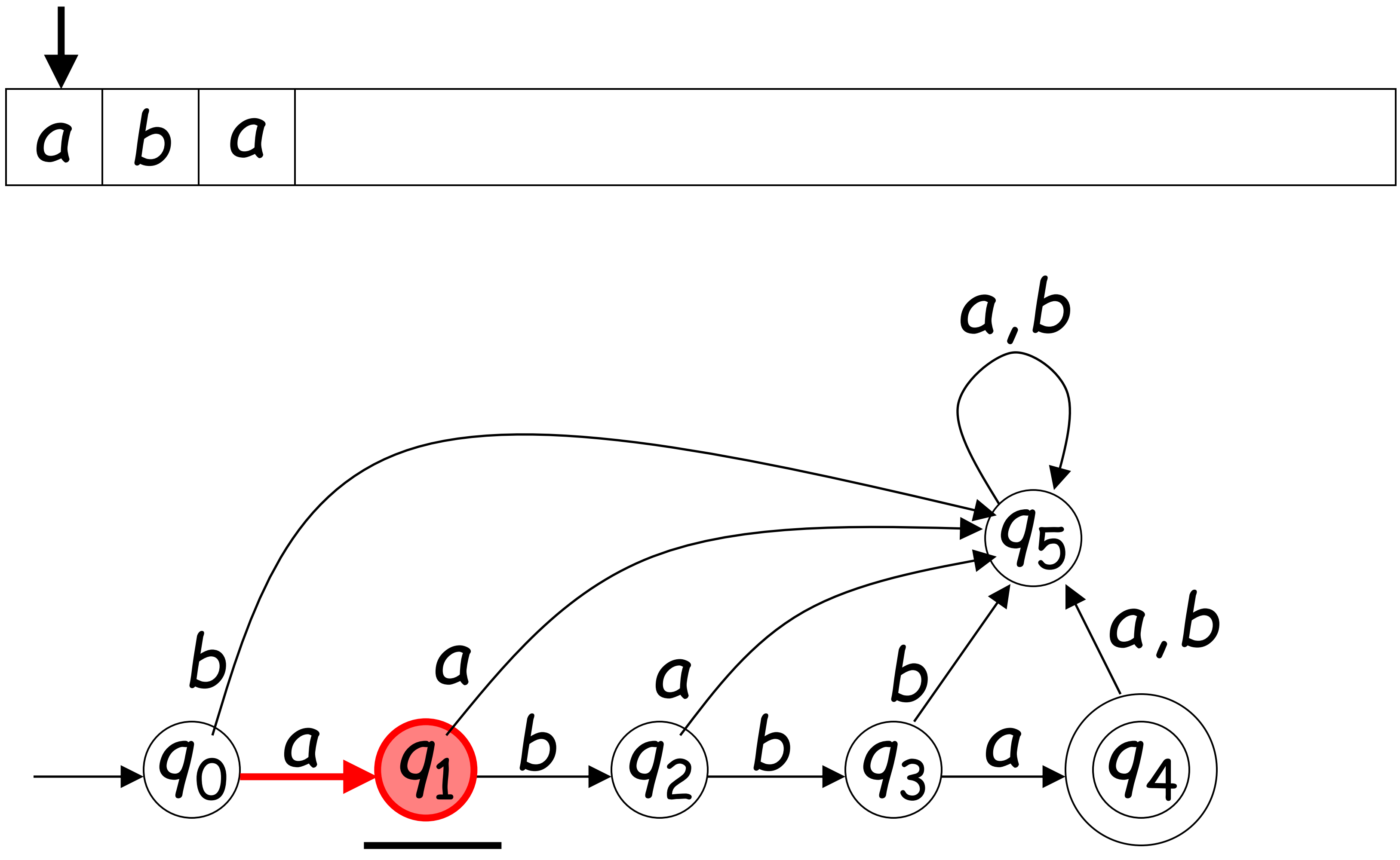
Input finished

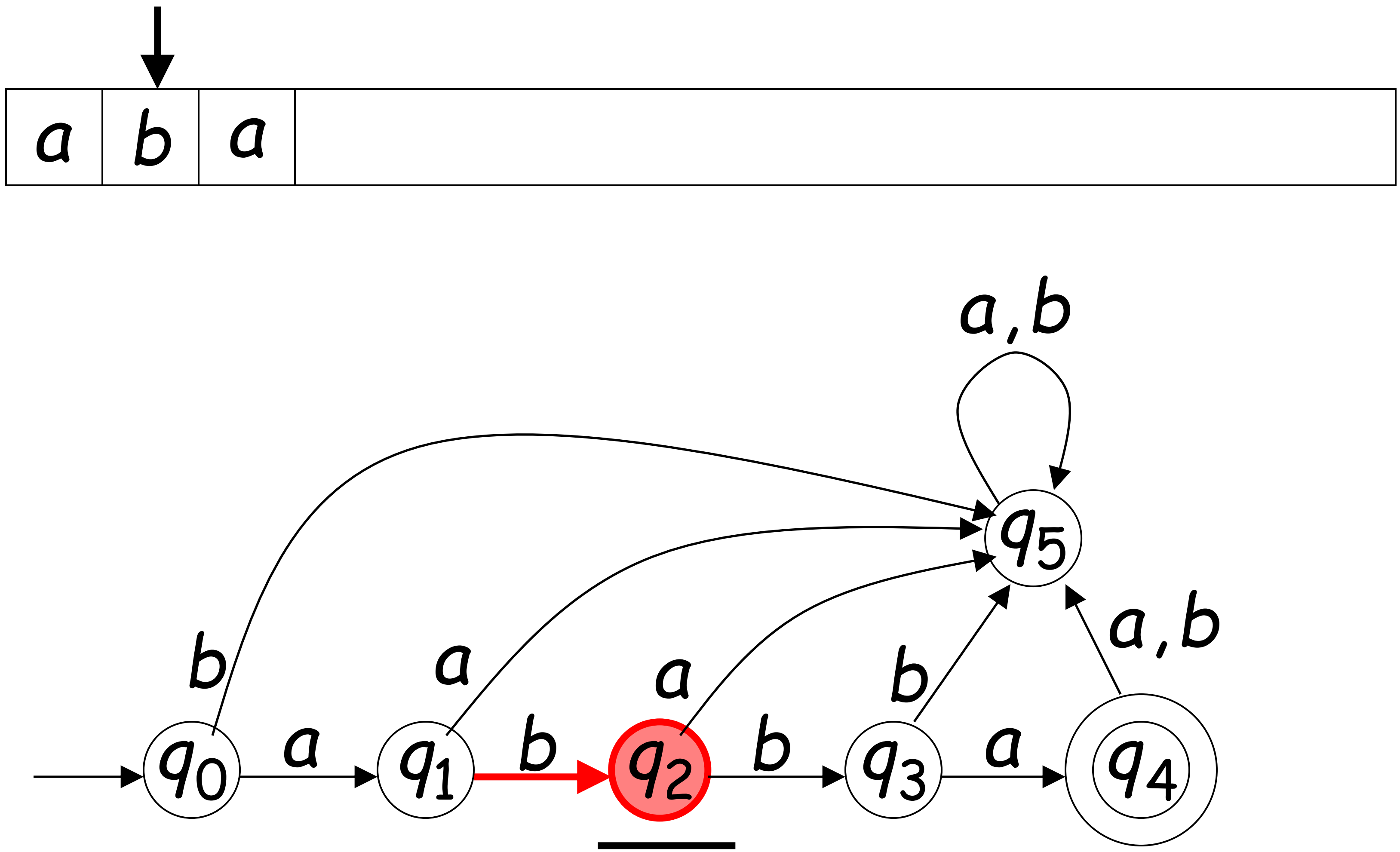


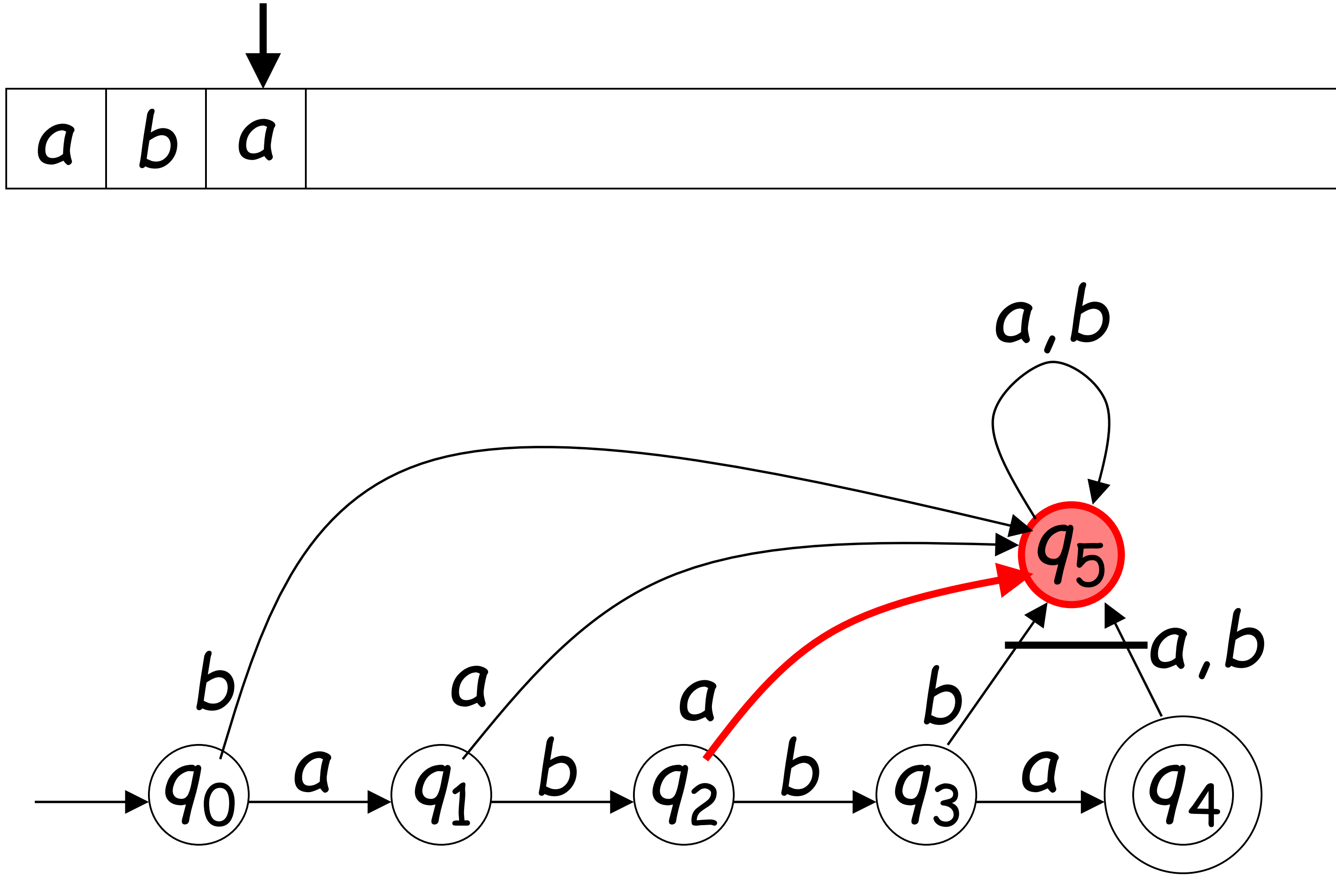
accept

Rejection

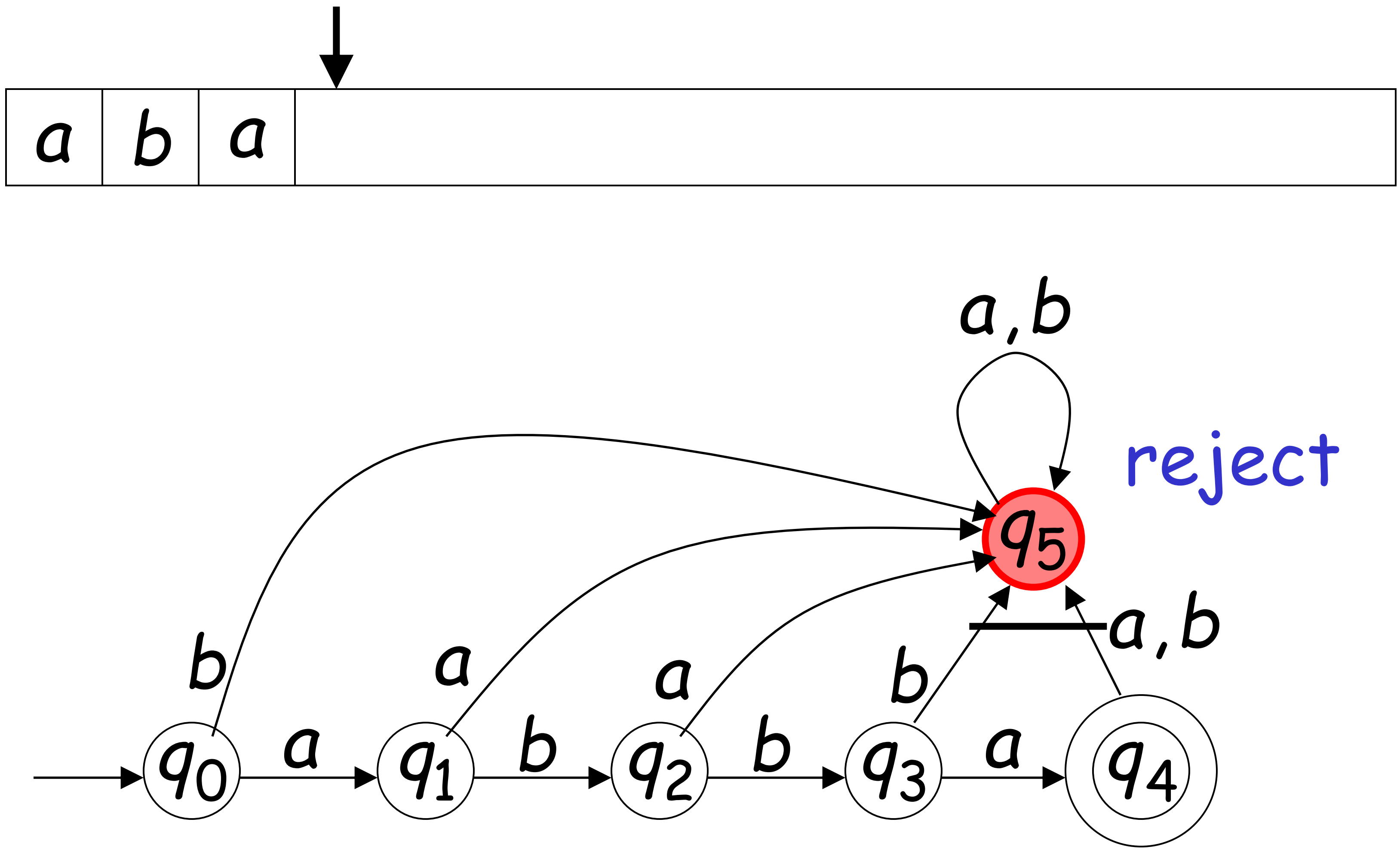



a
b
a
a,b
q5
b
a
a
b
a,b
q0
a
q1
b
q2
b
q3
a
q4

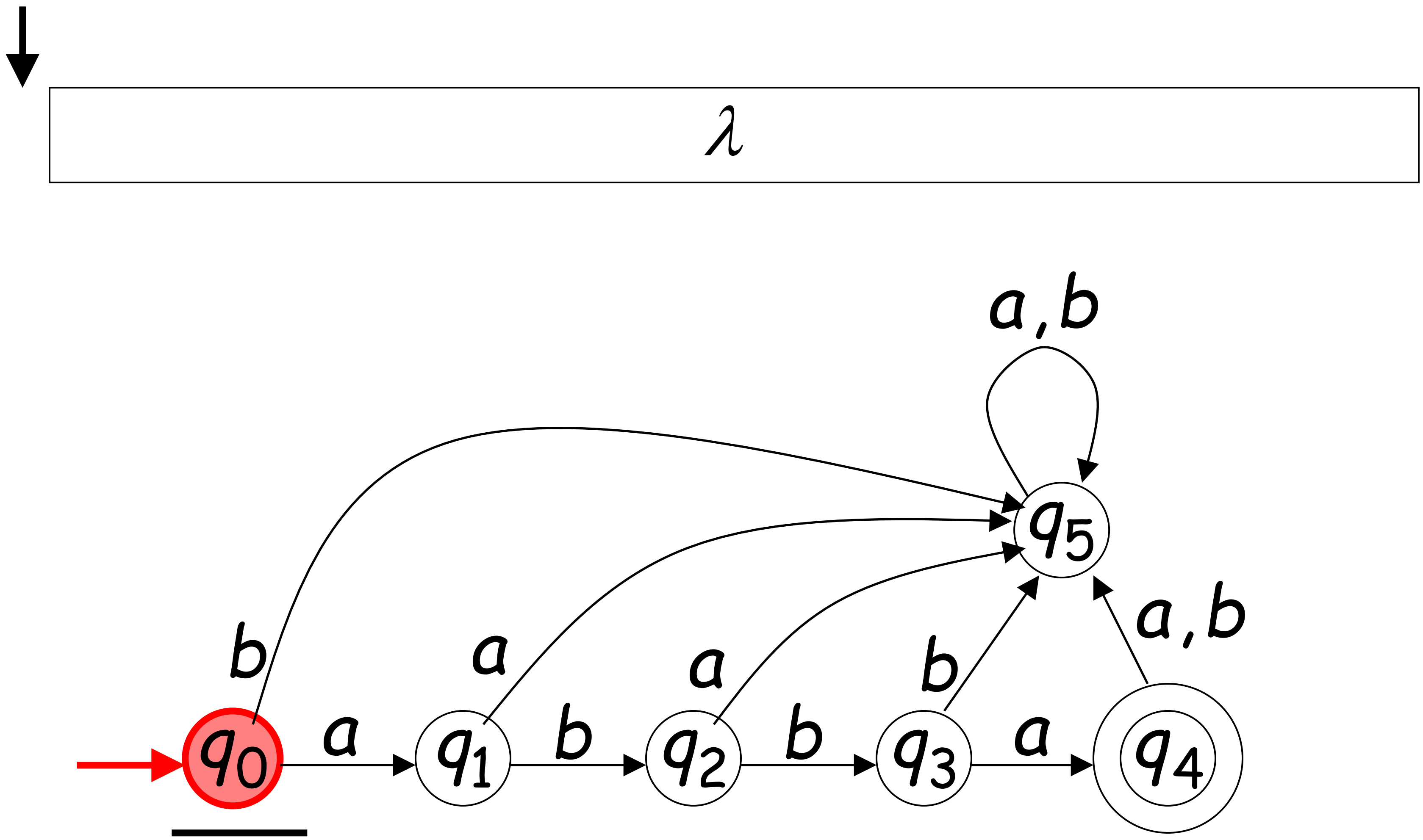

a b a
a,b
q5
b
a
a
b
a,b
a
b
b
a
q0
q1
q2
q3
q4


a
b
a
a,b
q5
b
a
a
b
a,b
q0
a
q1
b
q2
b
q3
a
q4

Input finished

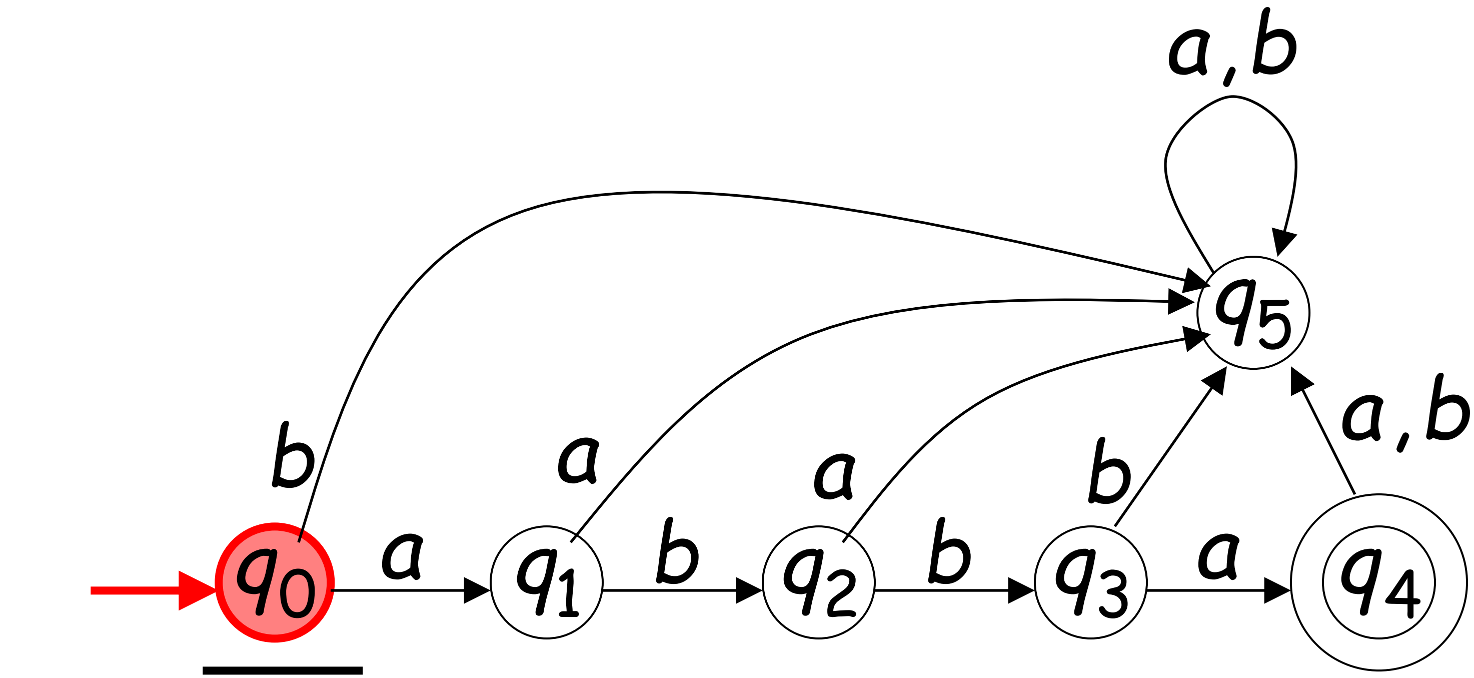


Acceptance or Rejection?

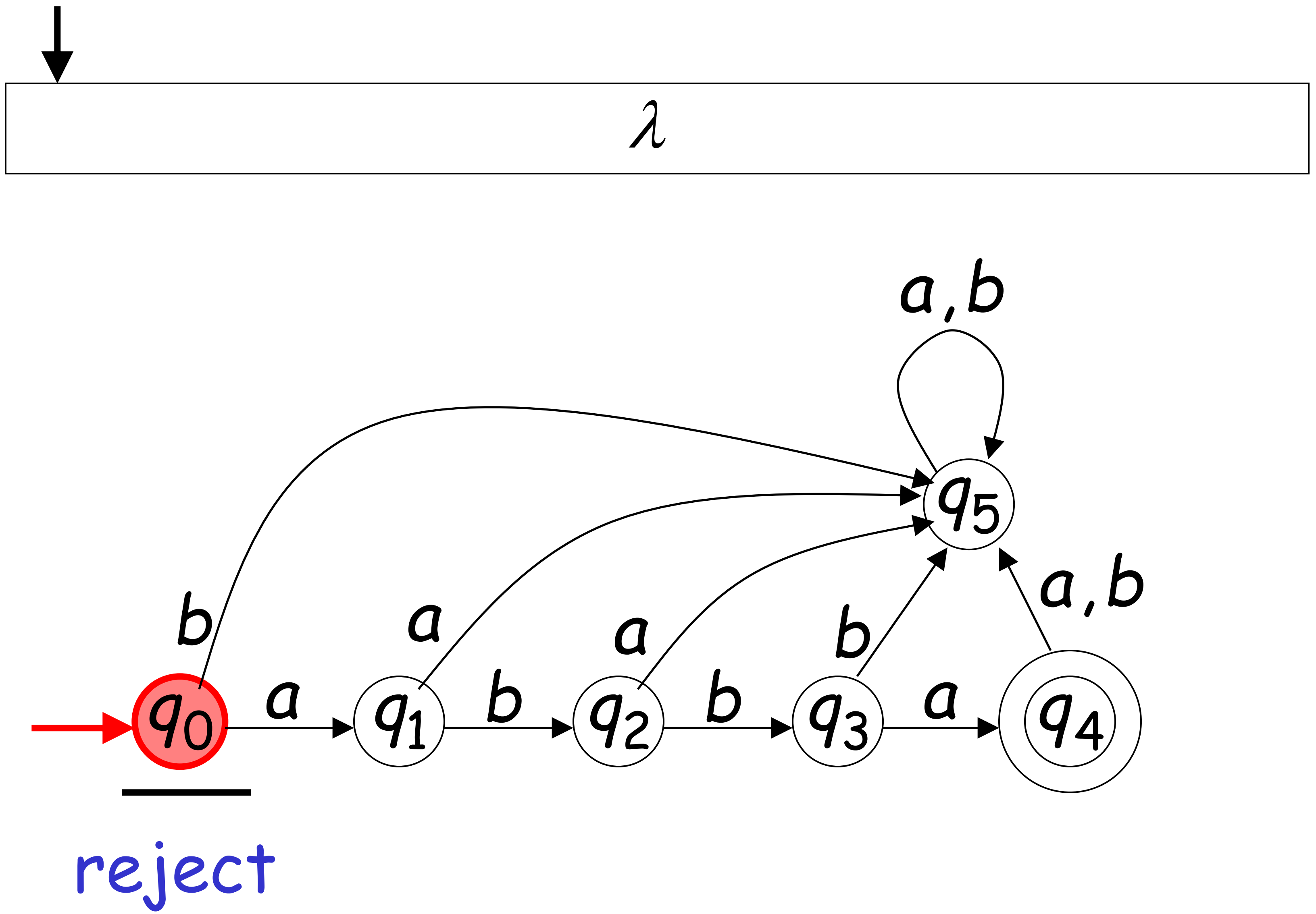


Initial State

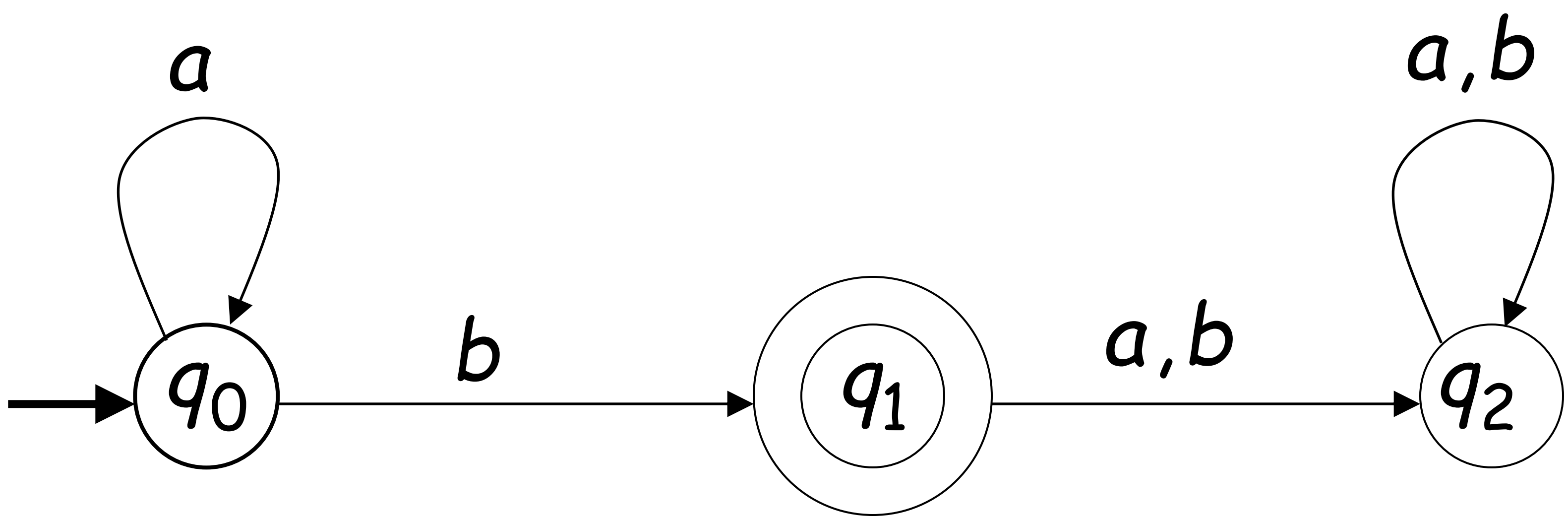
λ



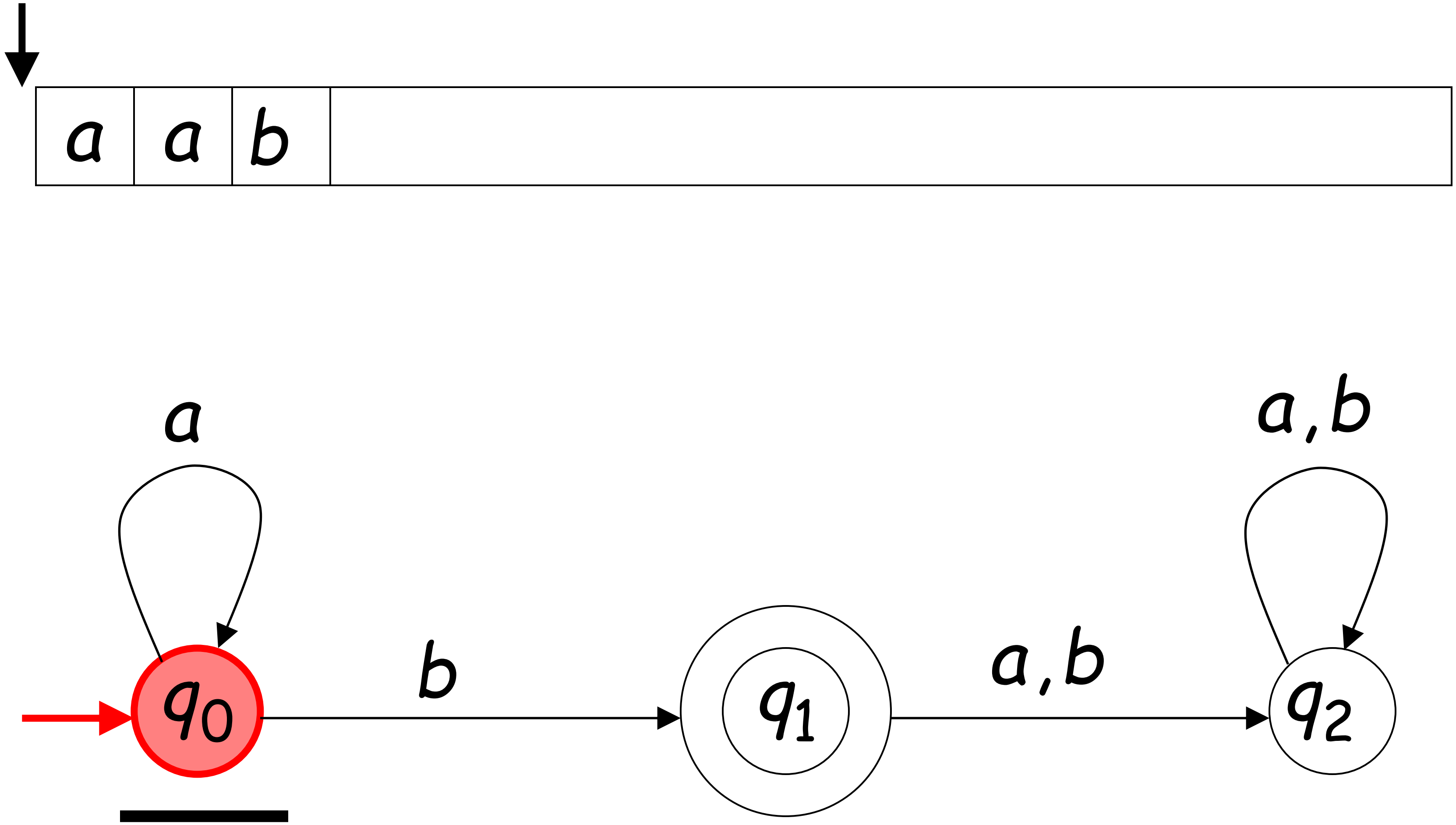
Rejection

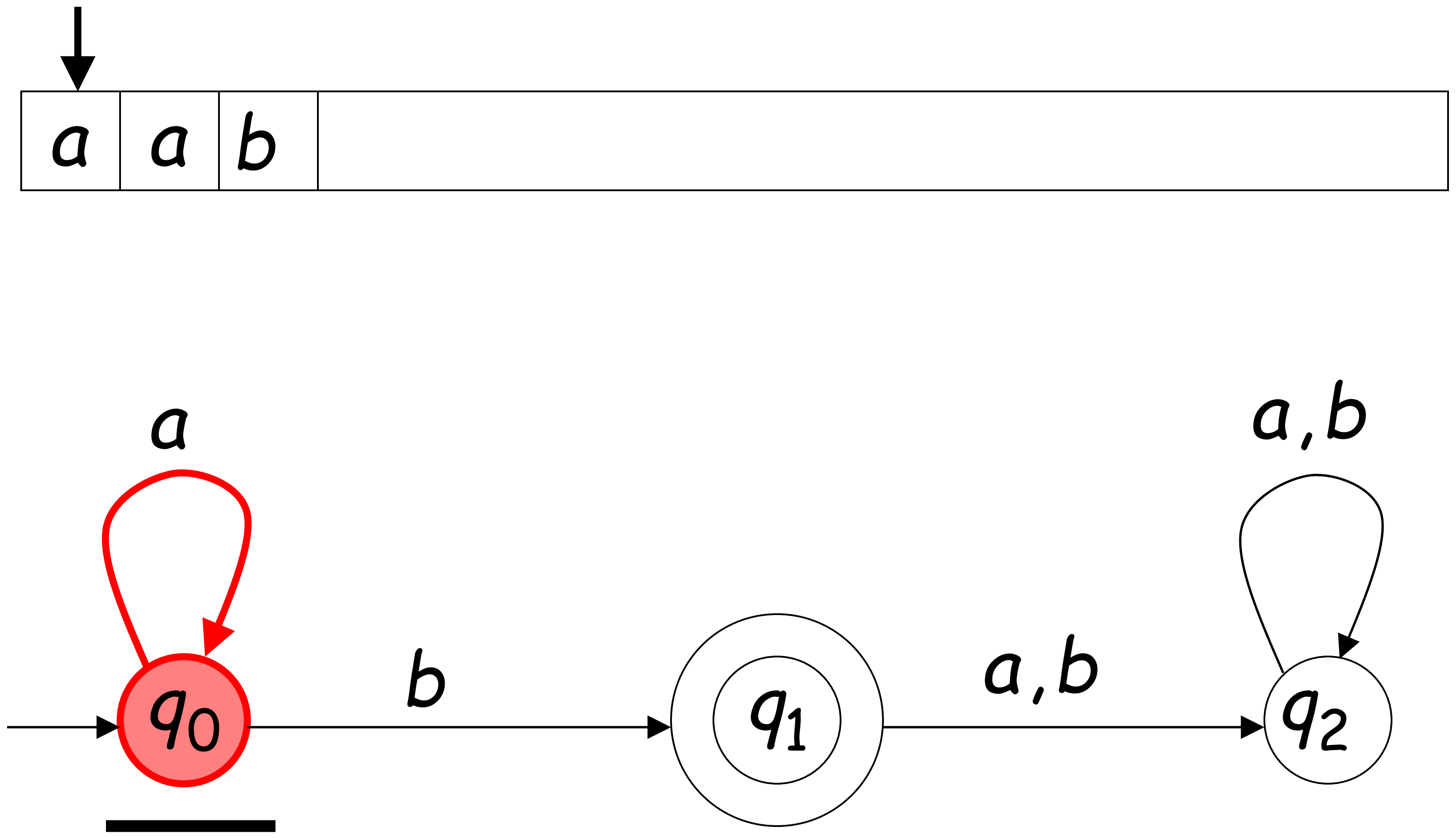


Language?

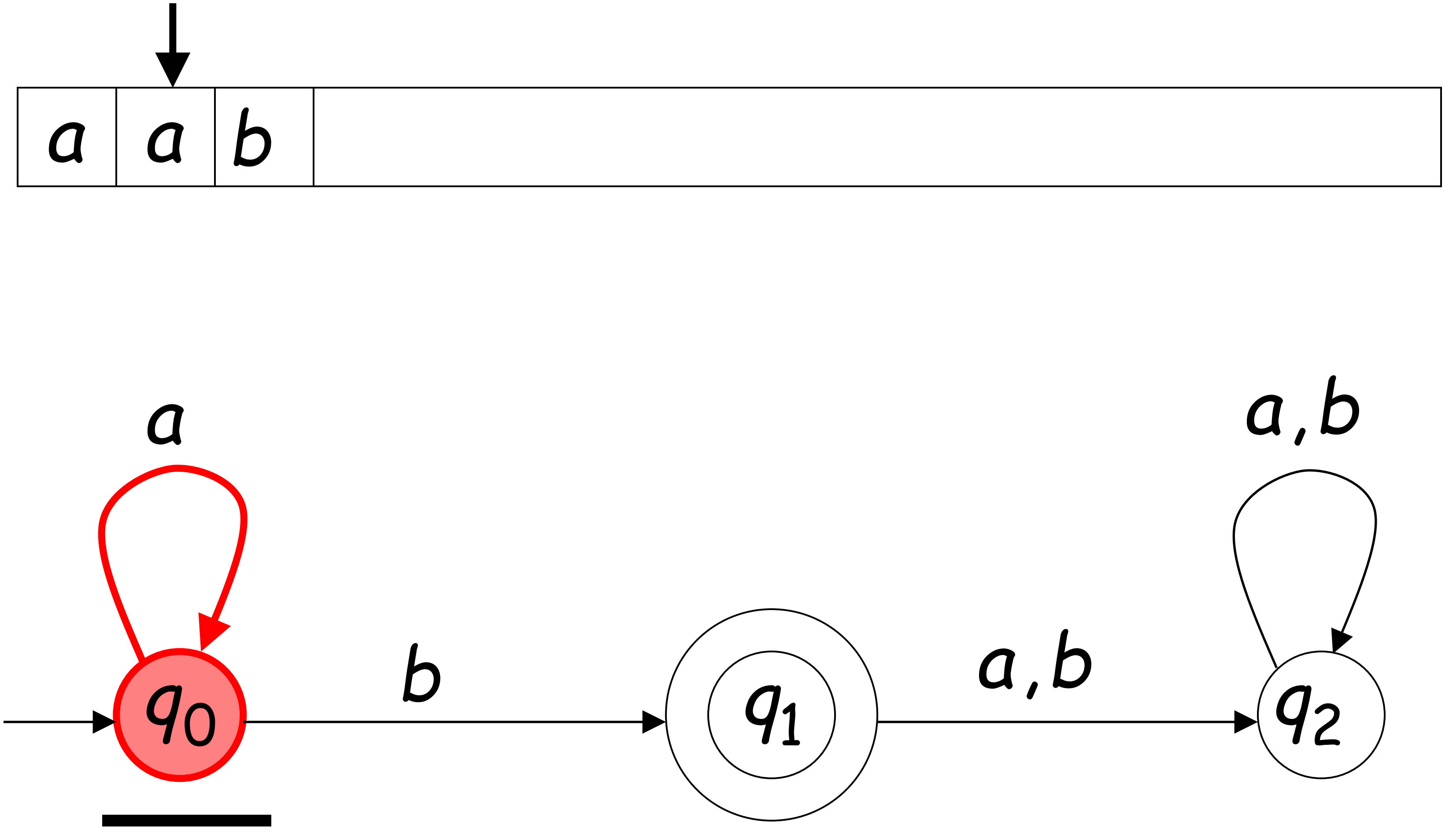


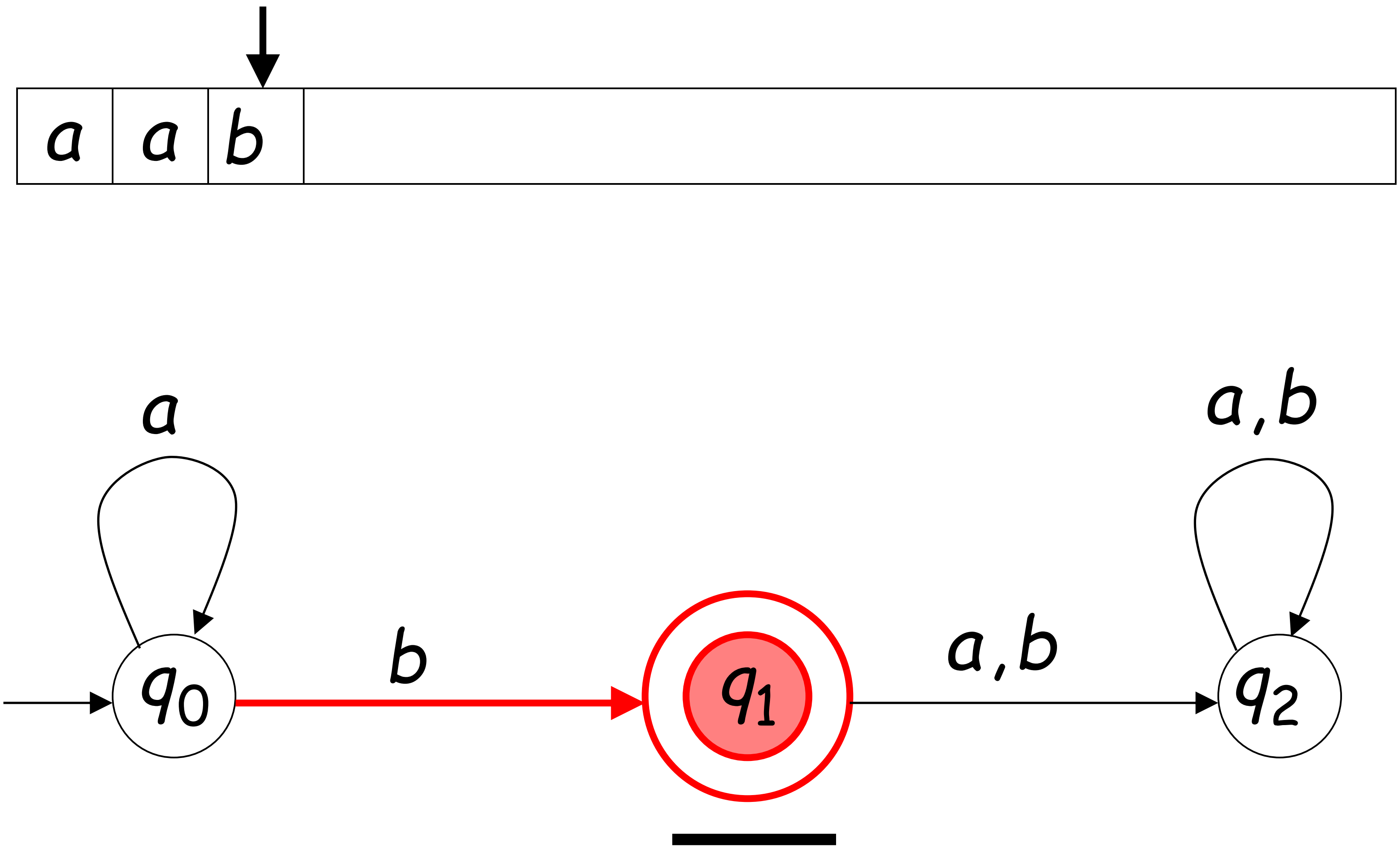
Another Example



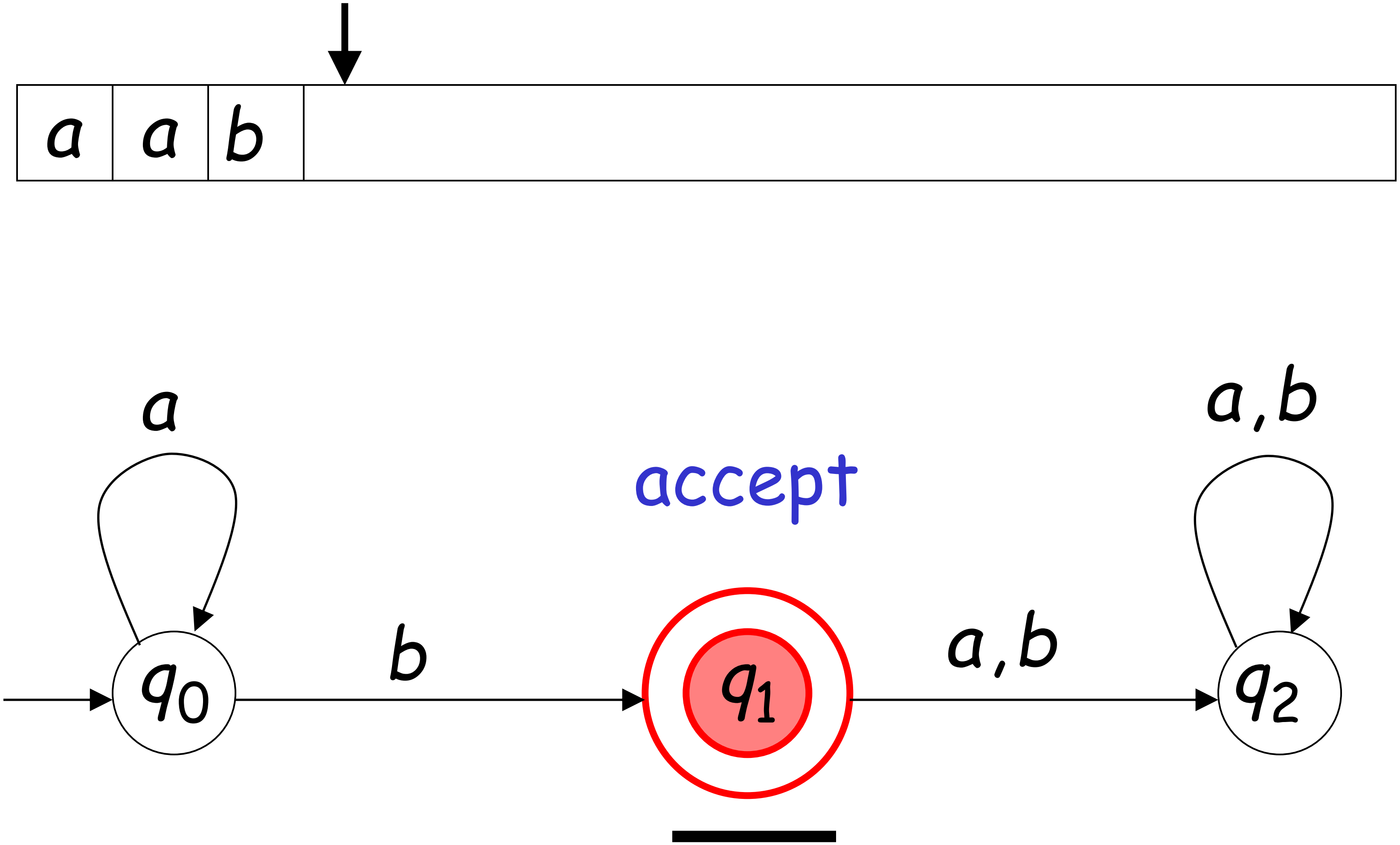


a a b
a
b
a,b
a,b
q_0
q_1
q_2

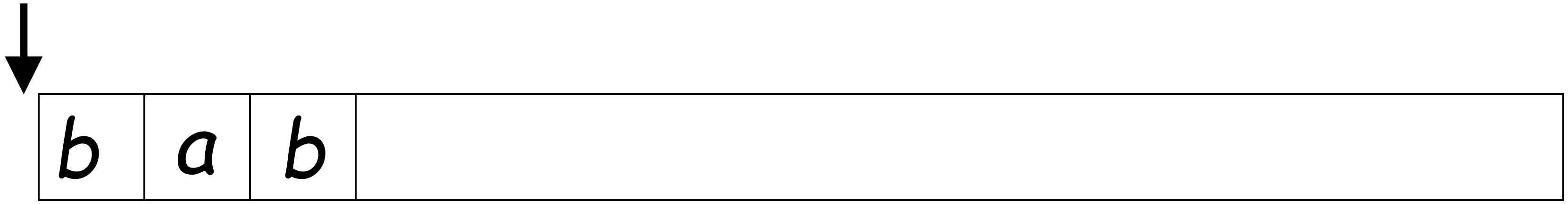

a
a
b
a
a, b
b
a, b
q_0
q_1
q_2

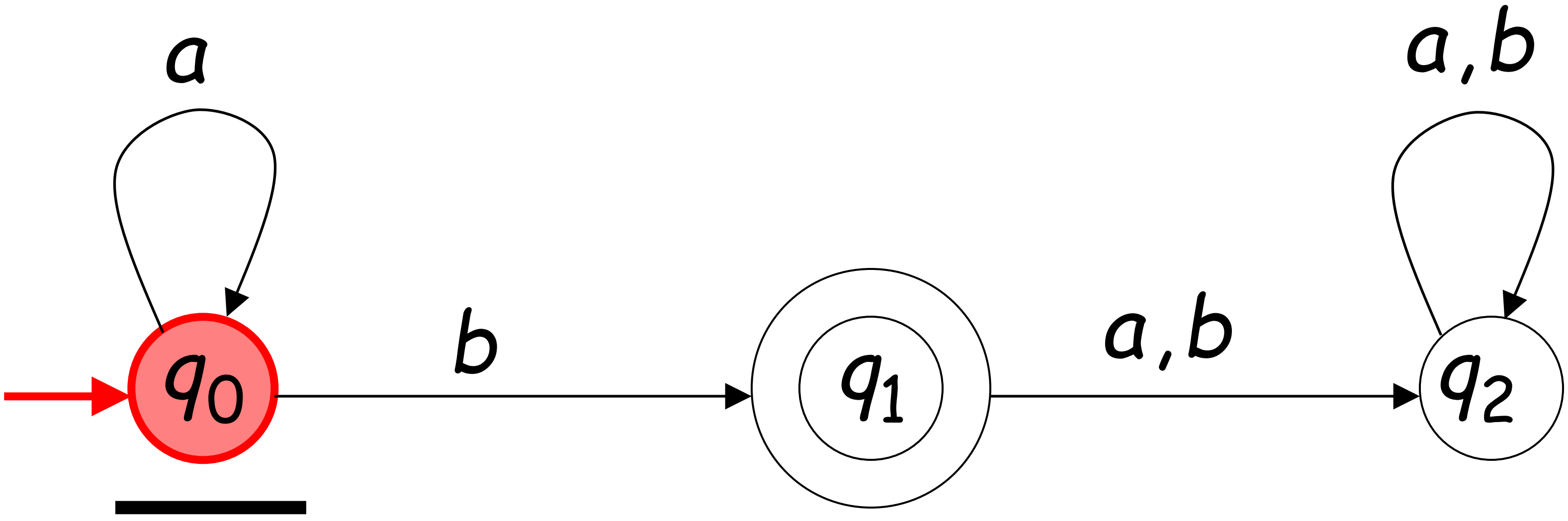

a
a
b
a
a,b
b
a,b
q_0
q_1
q_2

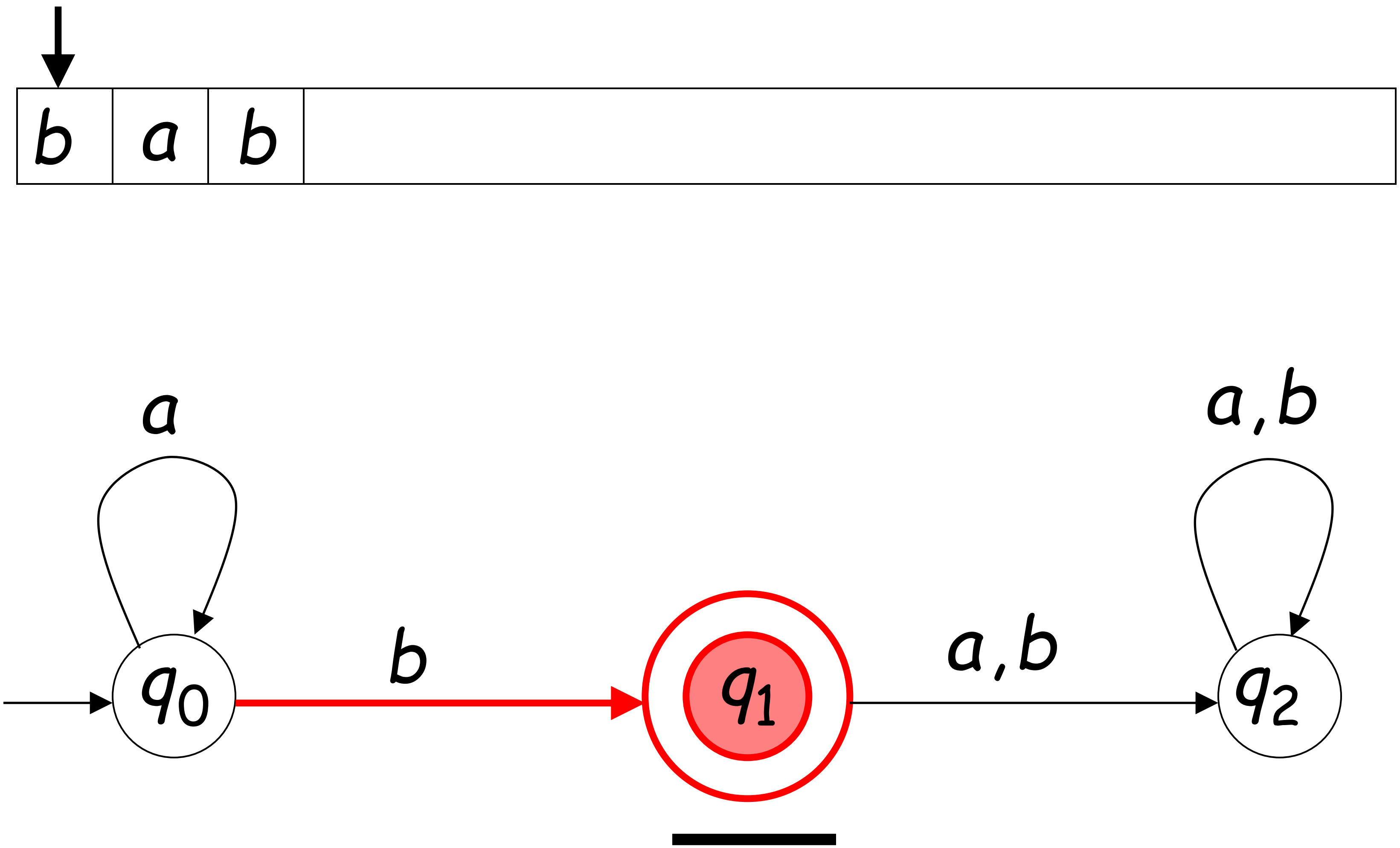
Input finished

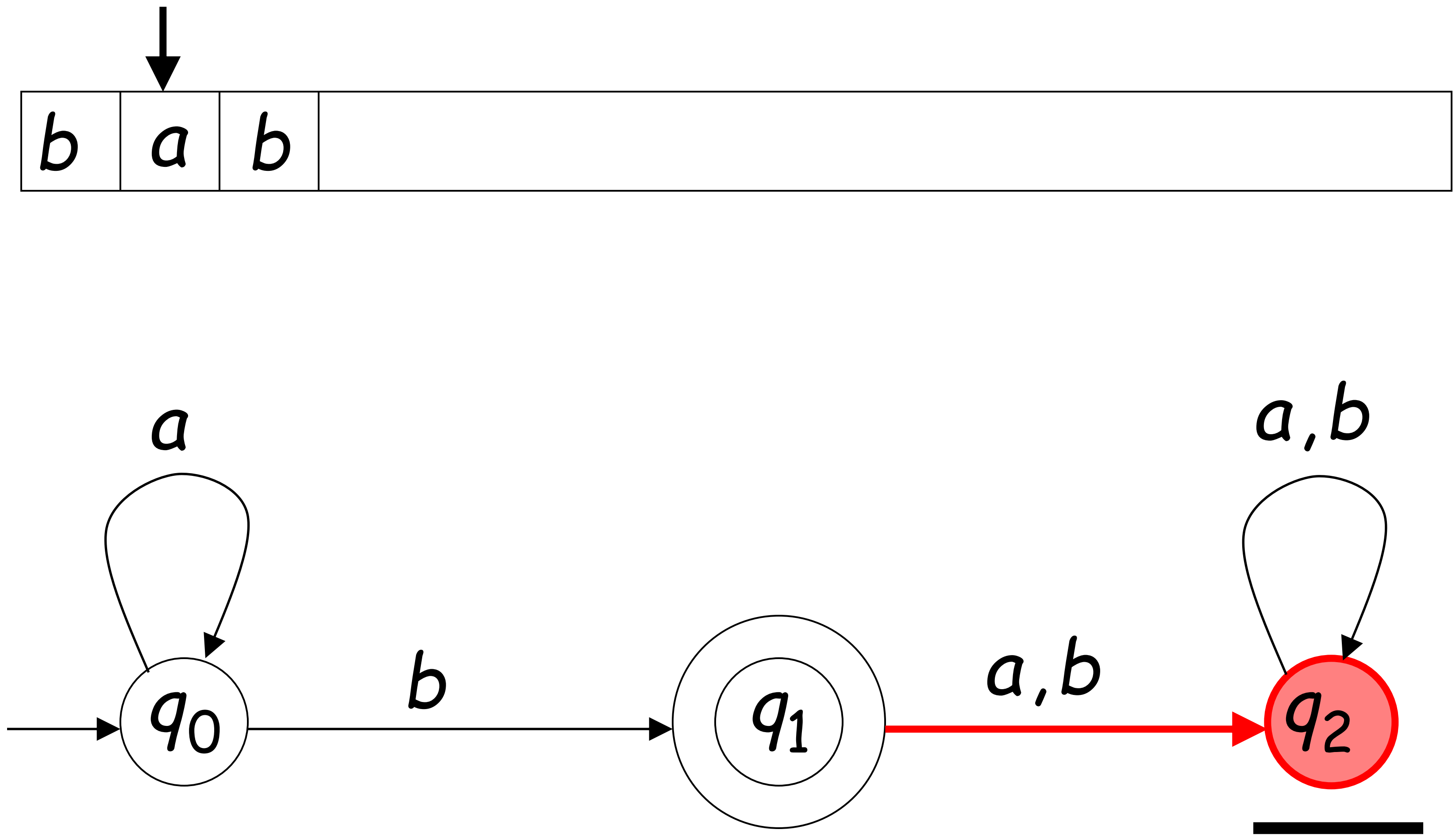


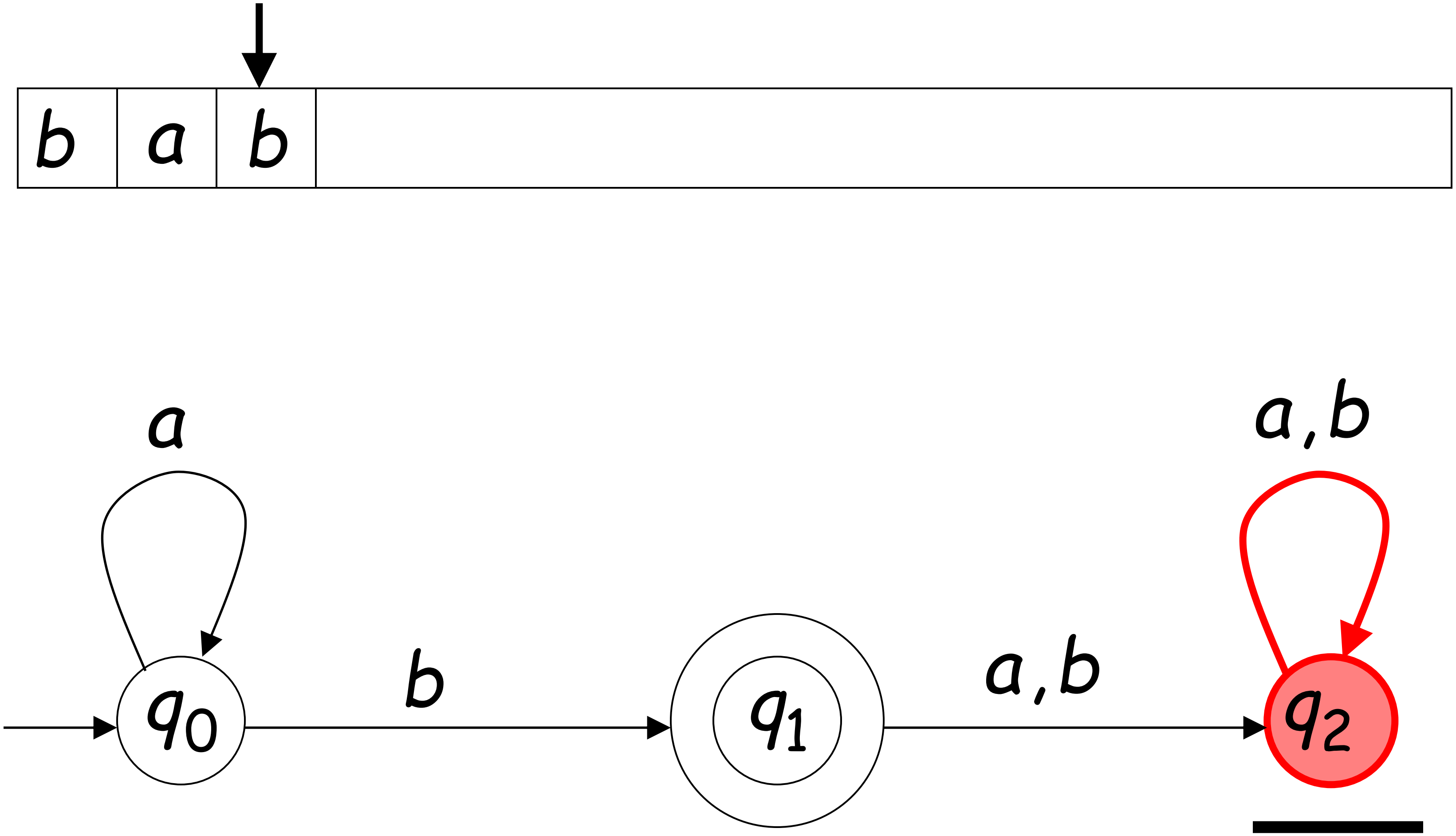
Rejection Example



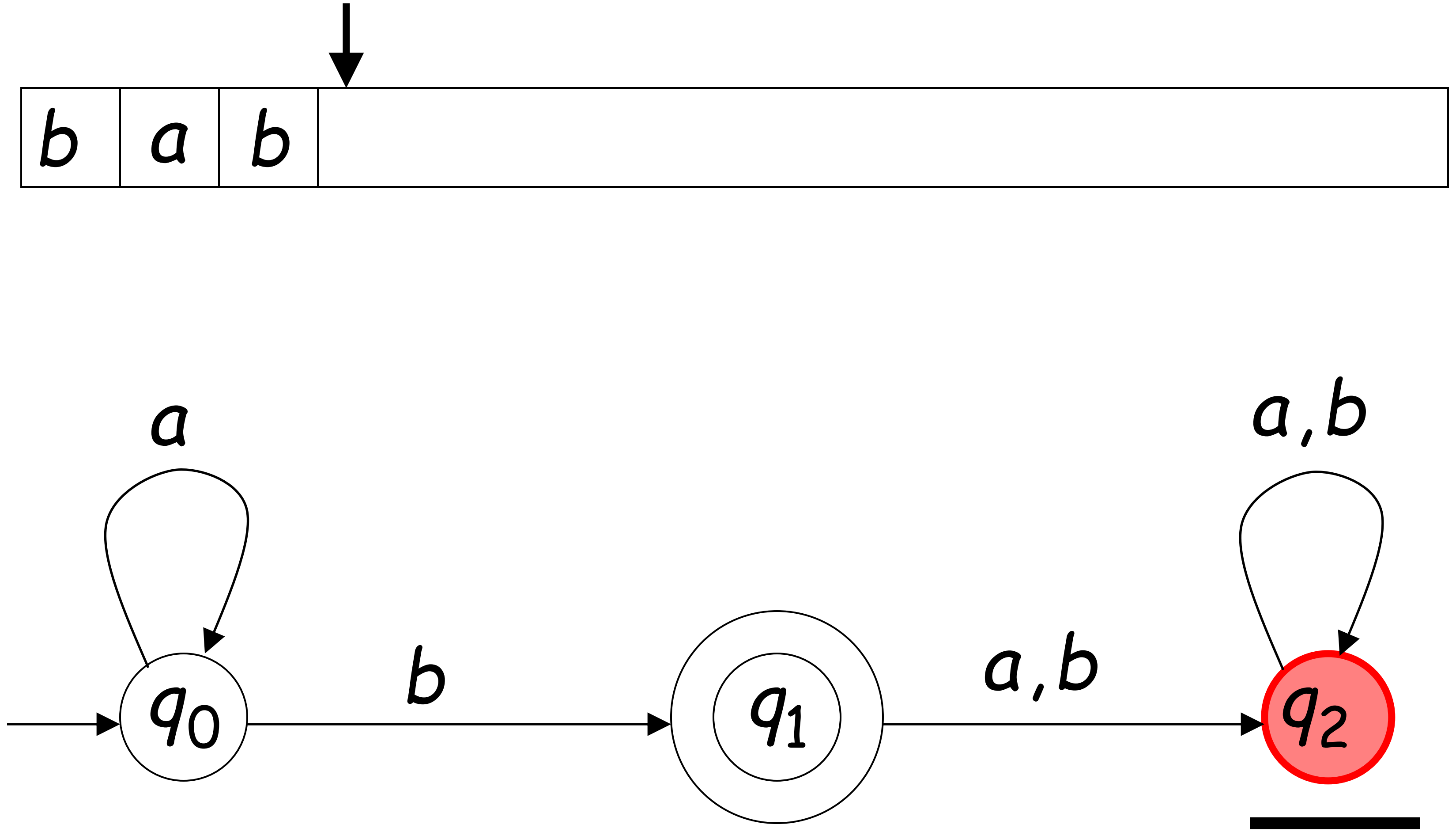



b
a
b
a
a,b
b
a,b
q0
q1
q2


b a b
a
a,b
b
a,b
q_0
q_1
q_2


b
a
b
a
a,b
b
a,b
q0
q1
q2

Input finished



reject

Languages Accepted by FAs

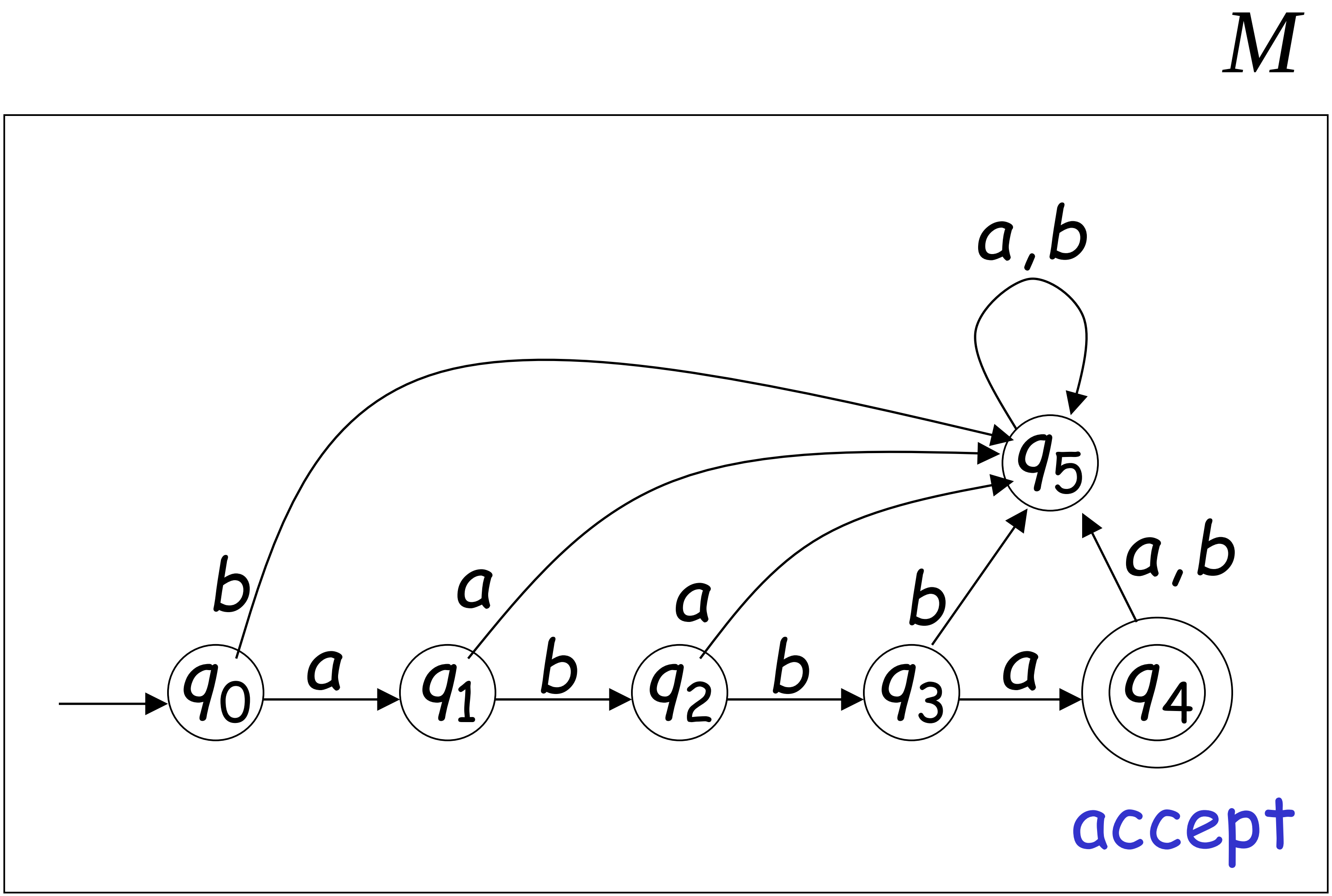
FA M

Definition:

The language $L(M)$ contains all input strings accepted by M

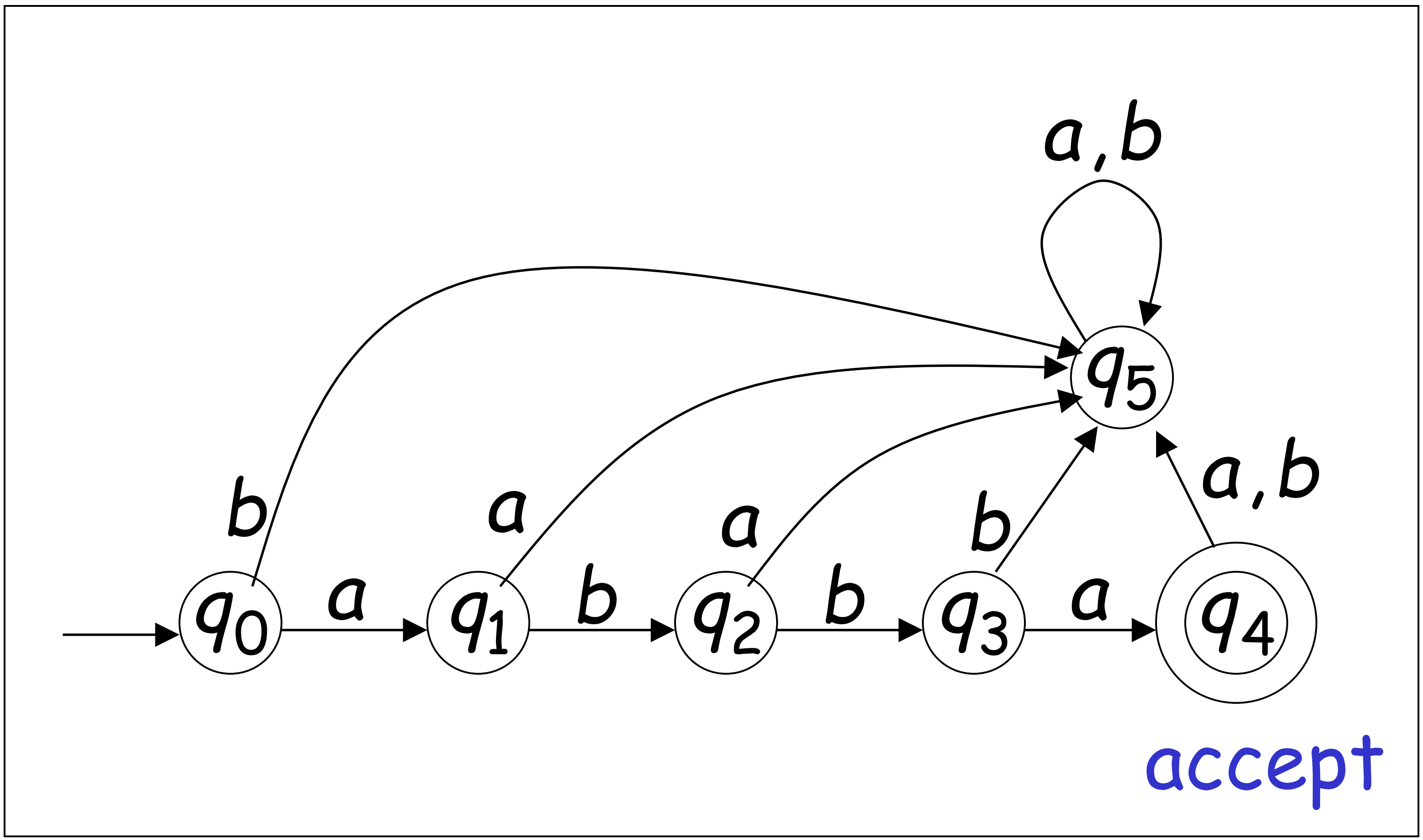
$L(M)$ = { strings that bring M to an accepting state}

Example: L(M) = ?

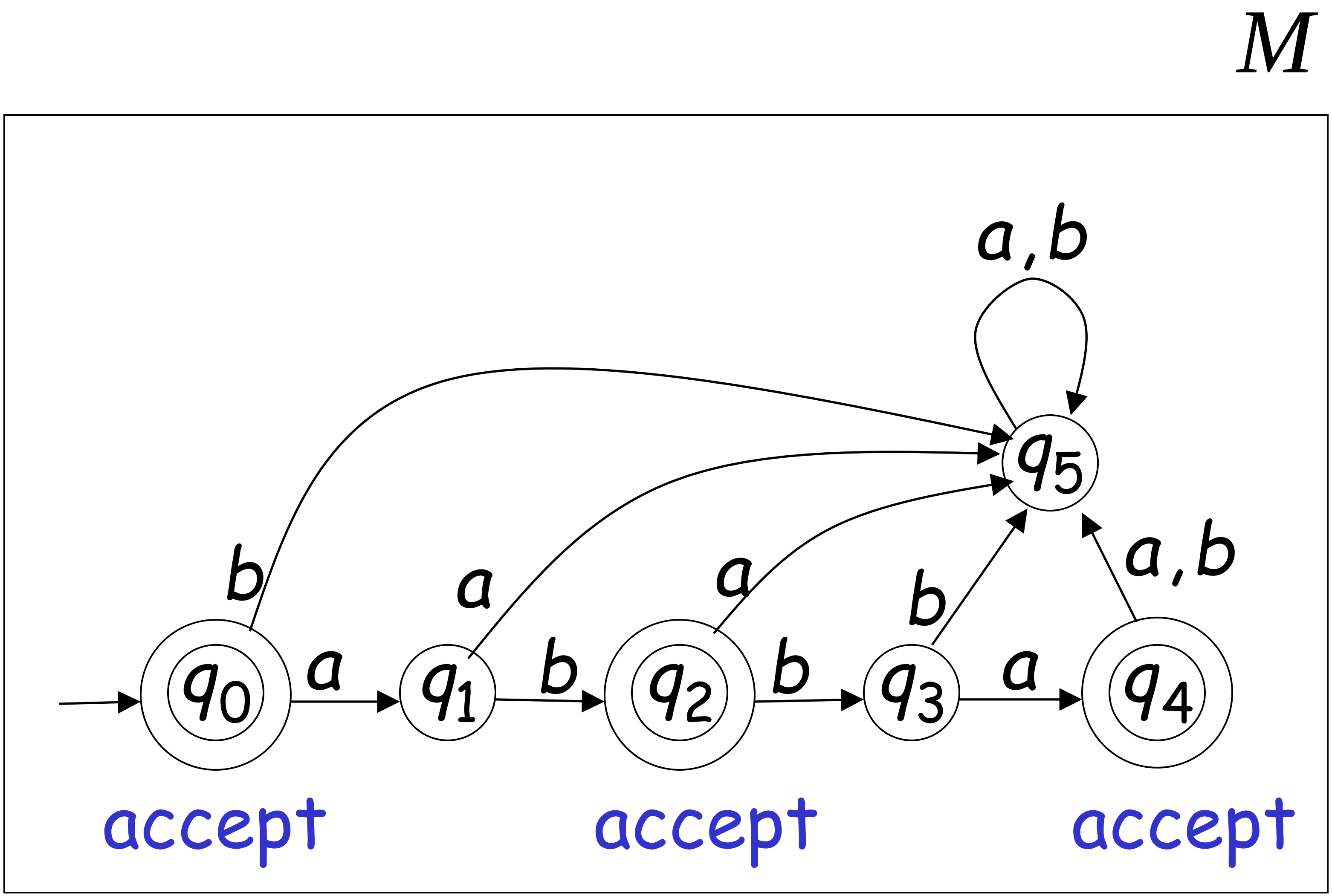


Example

M



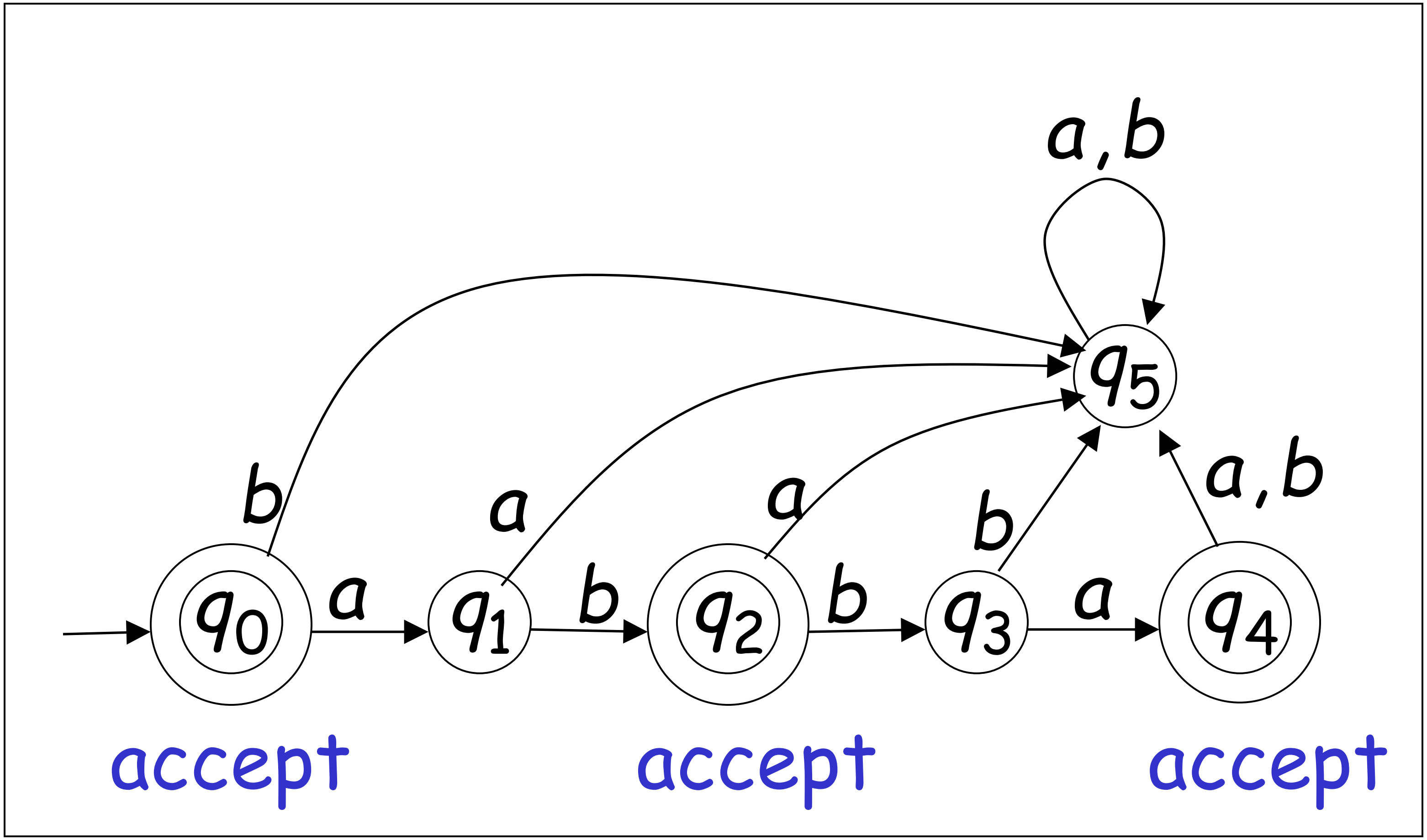
Example: L(M) = ?



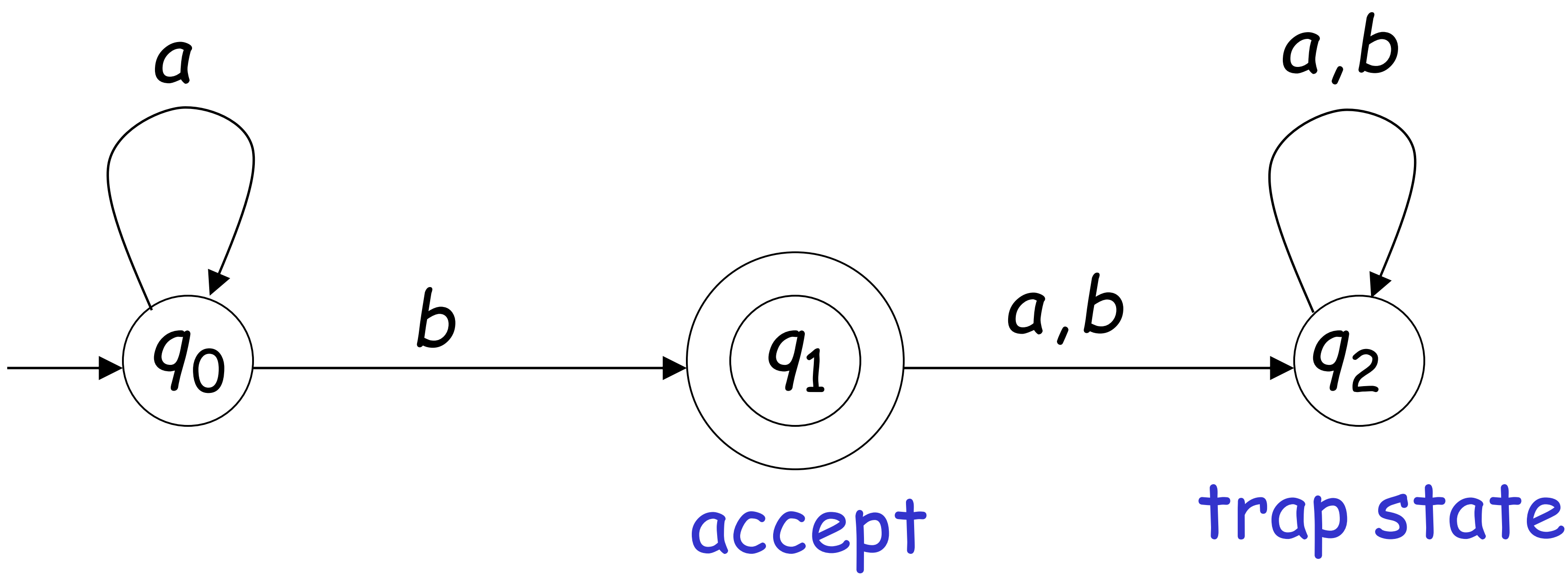
Example

$$L(M) = \{\lambda, ab, abba\}$$

M

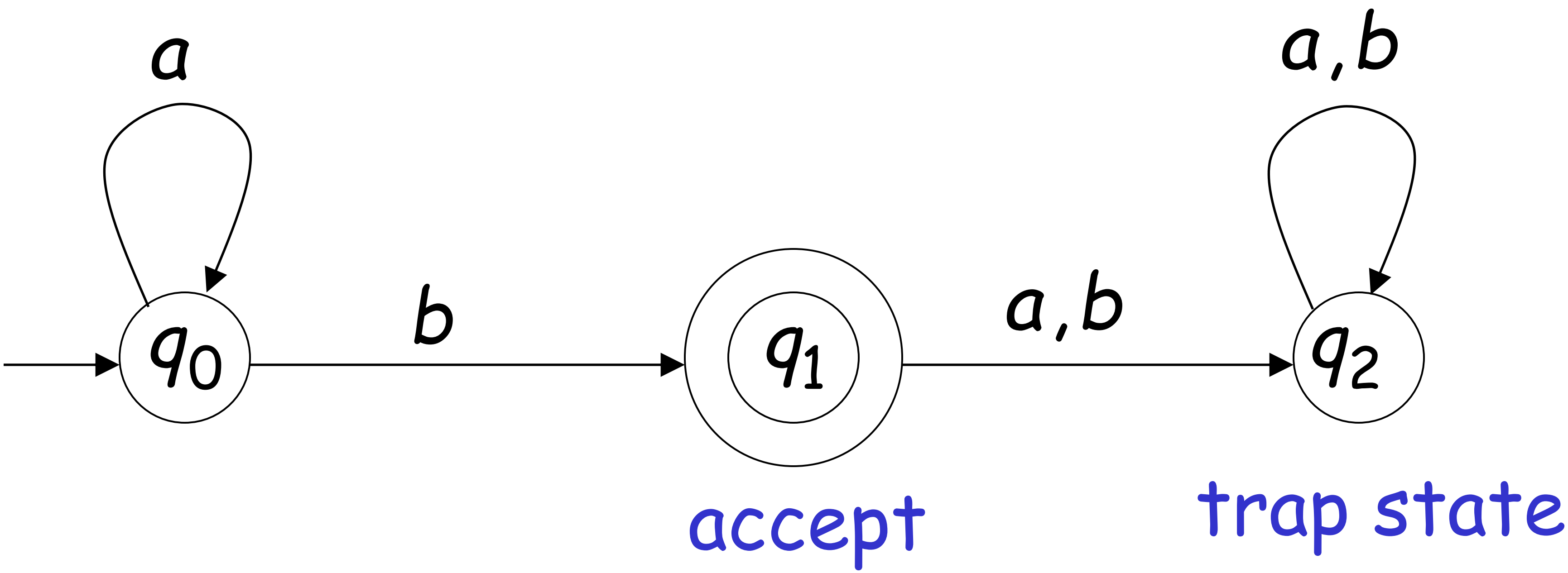


Example: L(M) = ?



Example

$$L(M) = \{a^n b : n \geq 0\}$$



Formal Definition

Finite Automaton (FA)

$$M = (Q, \Sigma, \delta, q_0, F)$$

Q : set of states

Σ : input alphabet

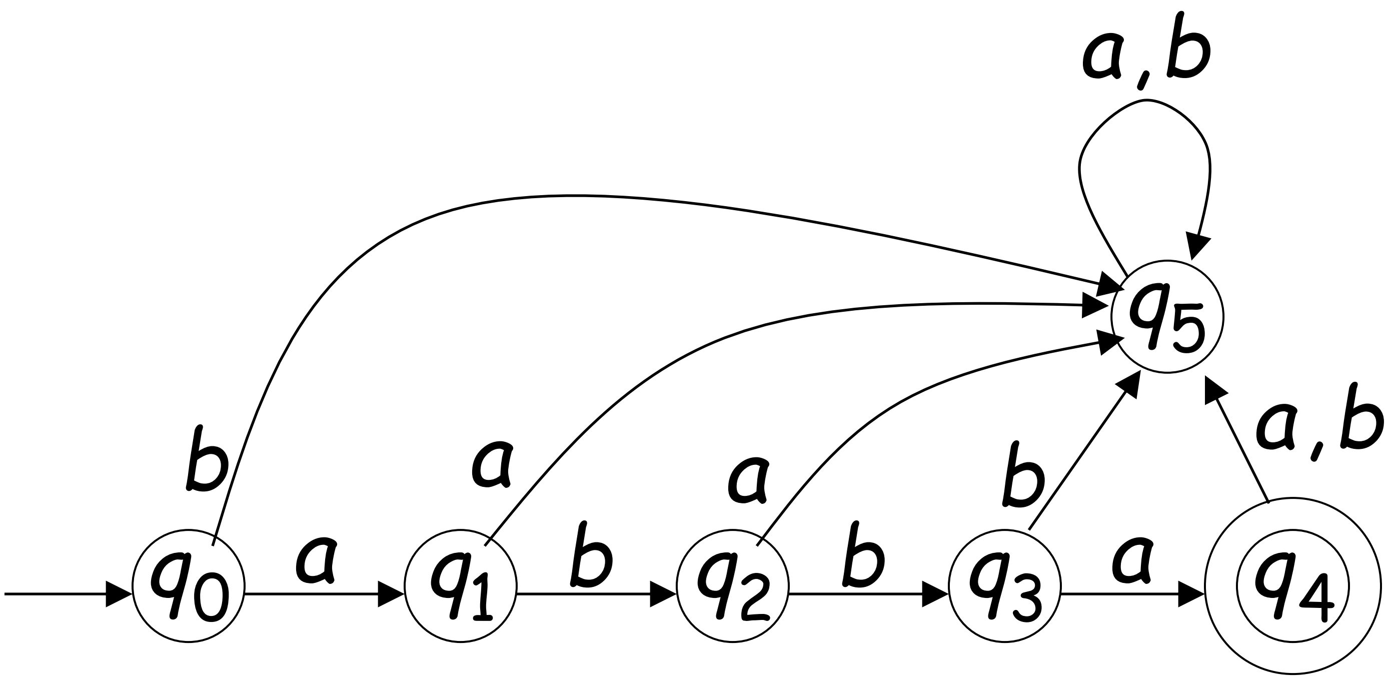
δ : transition function

q_0 : initial state

F : set of accepting states

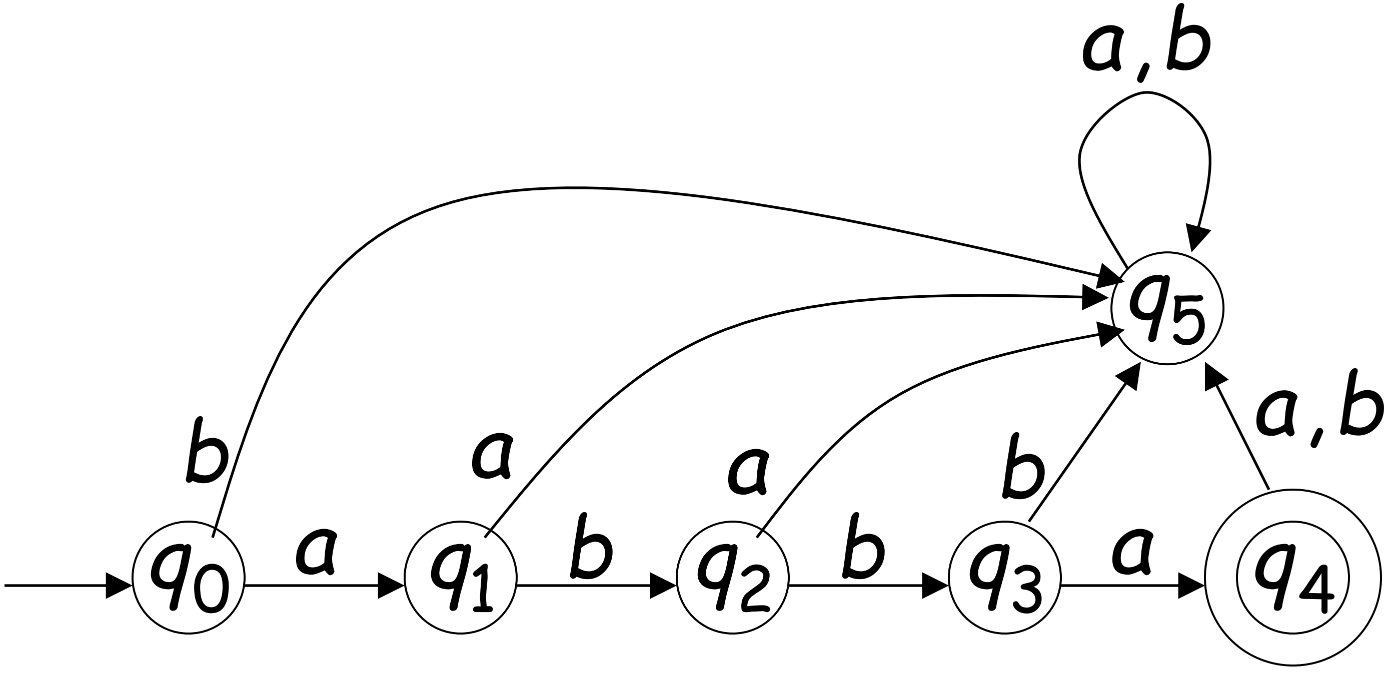
Input Alphabet Σ

$$\Sigma = \{a, b\}$$

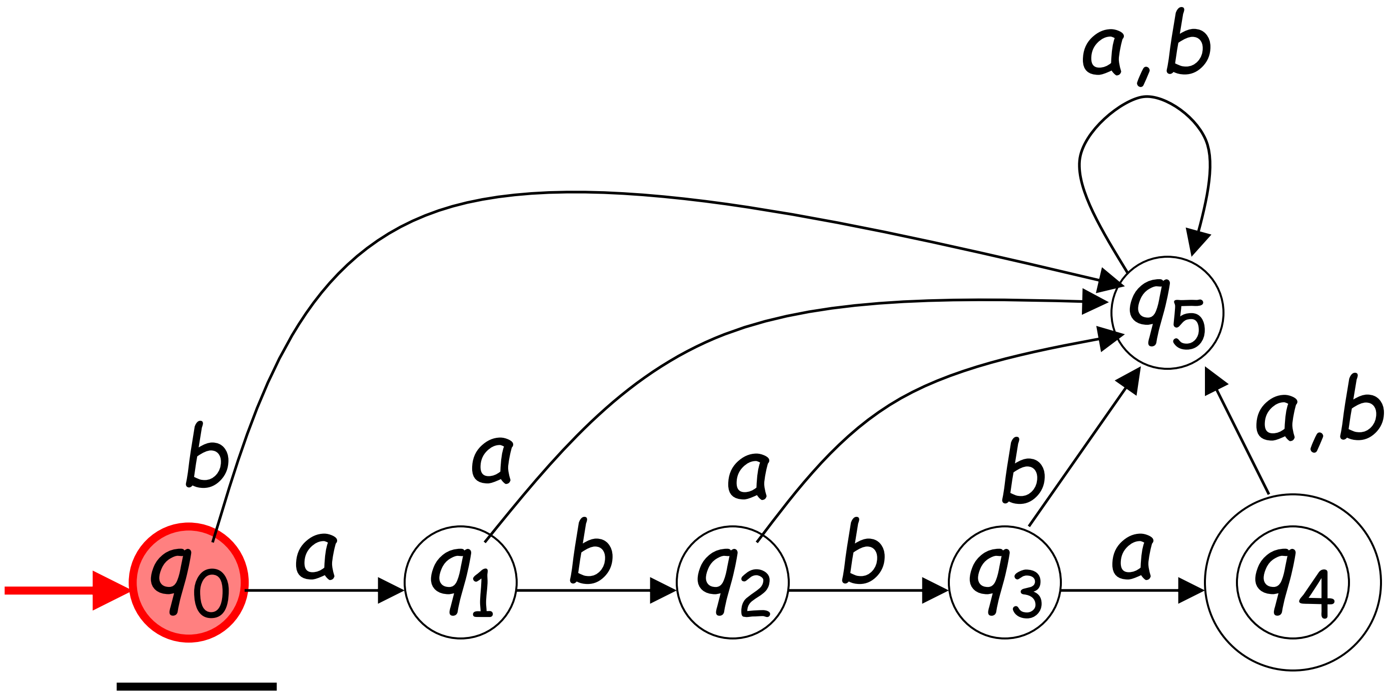


Set of States Q

$$Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$$

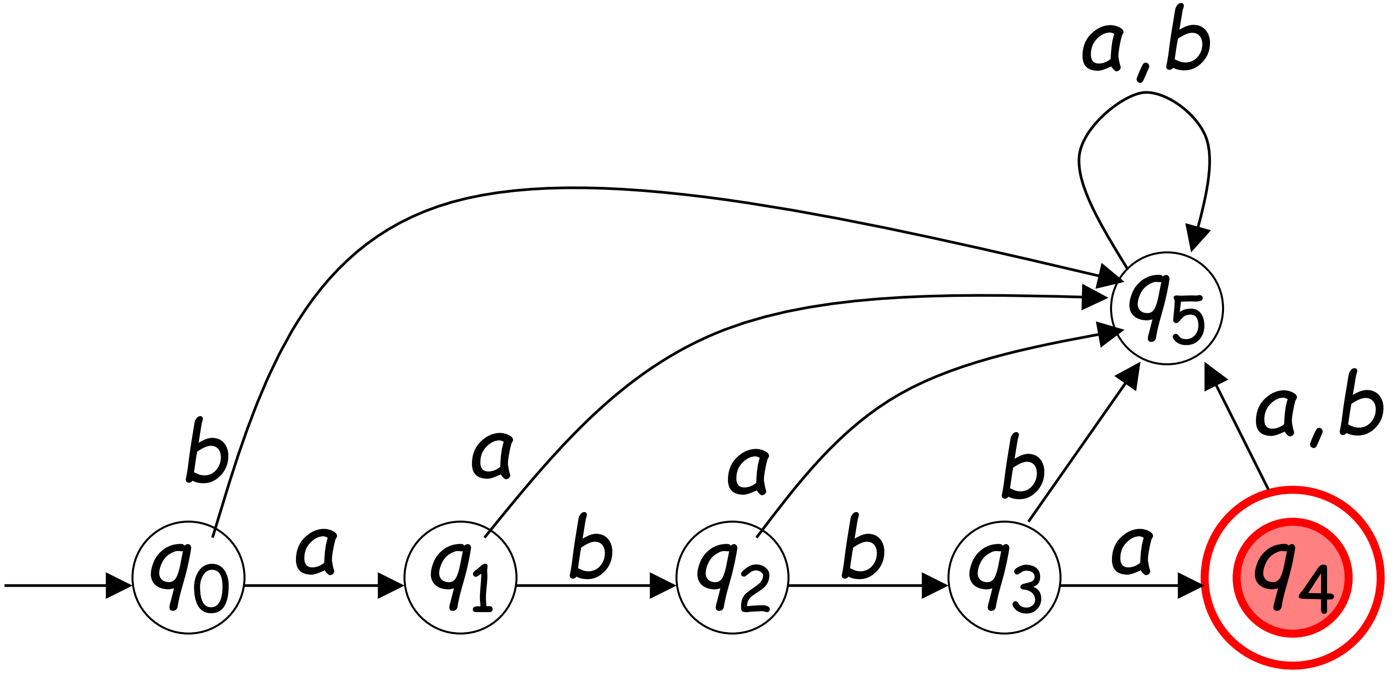


Initial State q_0



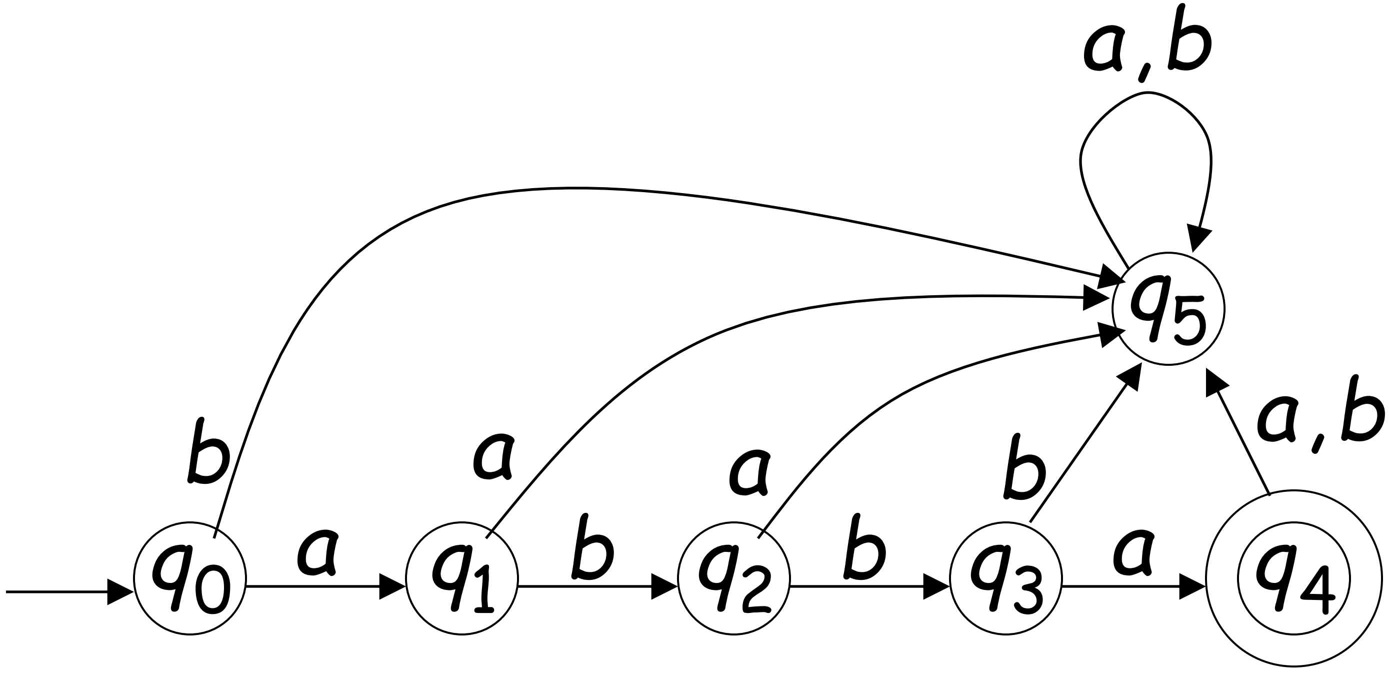
Set of Accepting States F

$$F = \{q_4\}$$

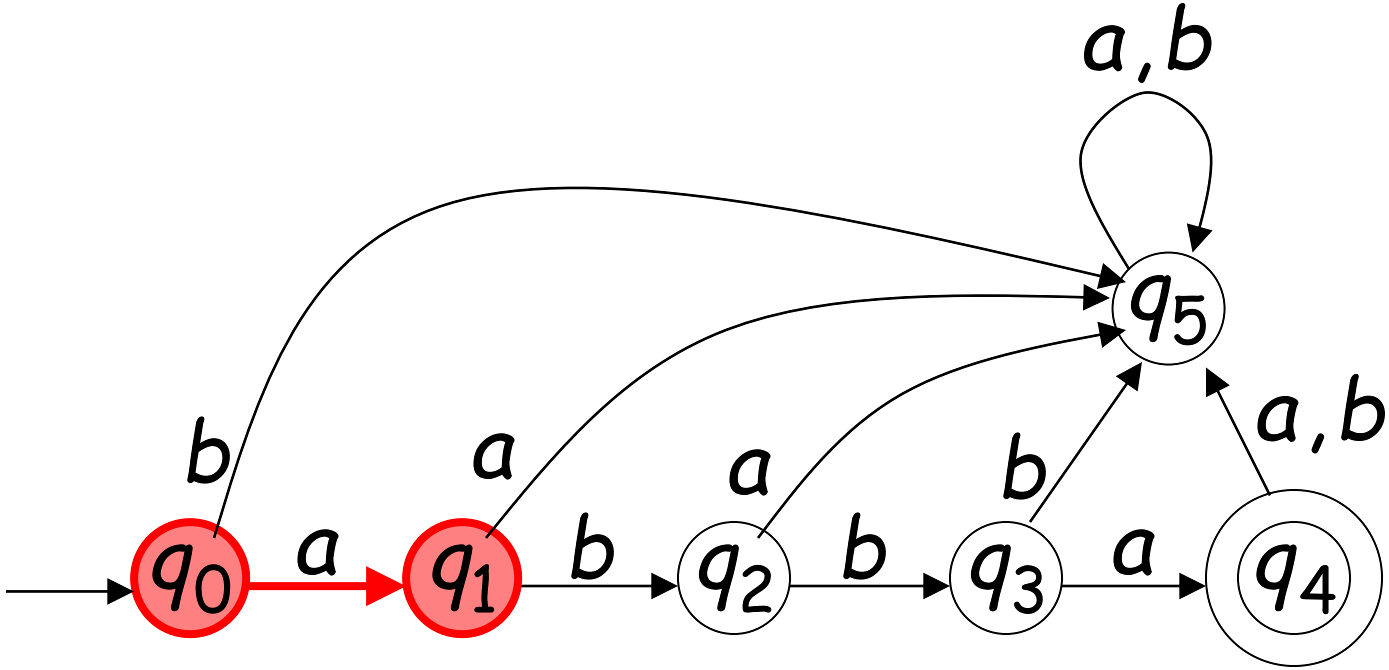


Transition Function δ

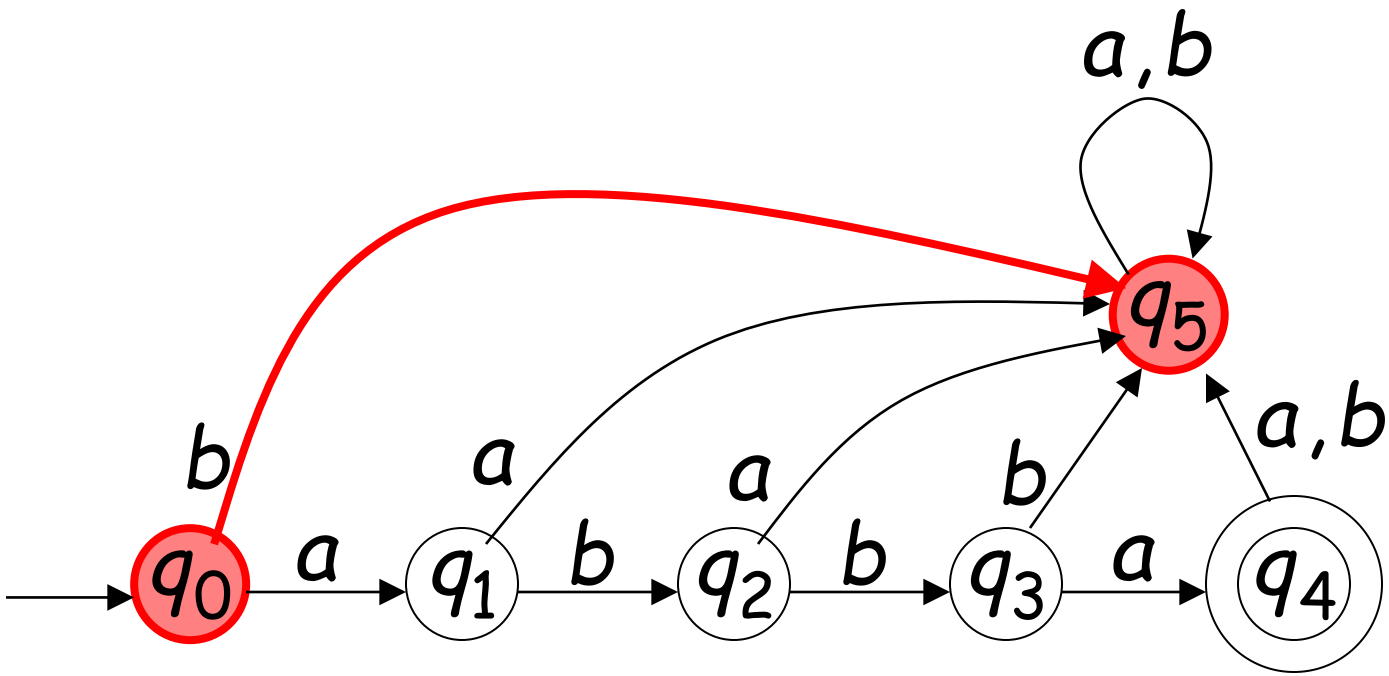
$$\delta : Q \times \Sigma \rightarrow Q$$



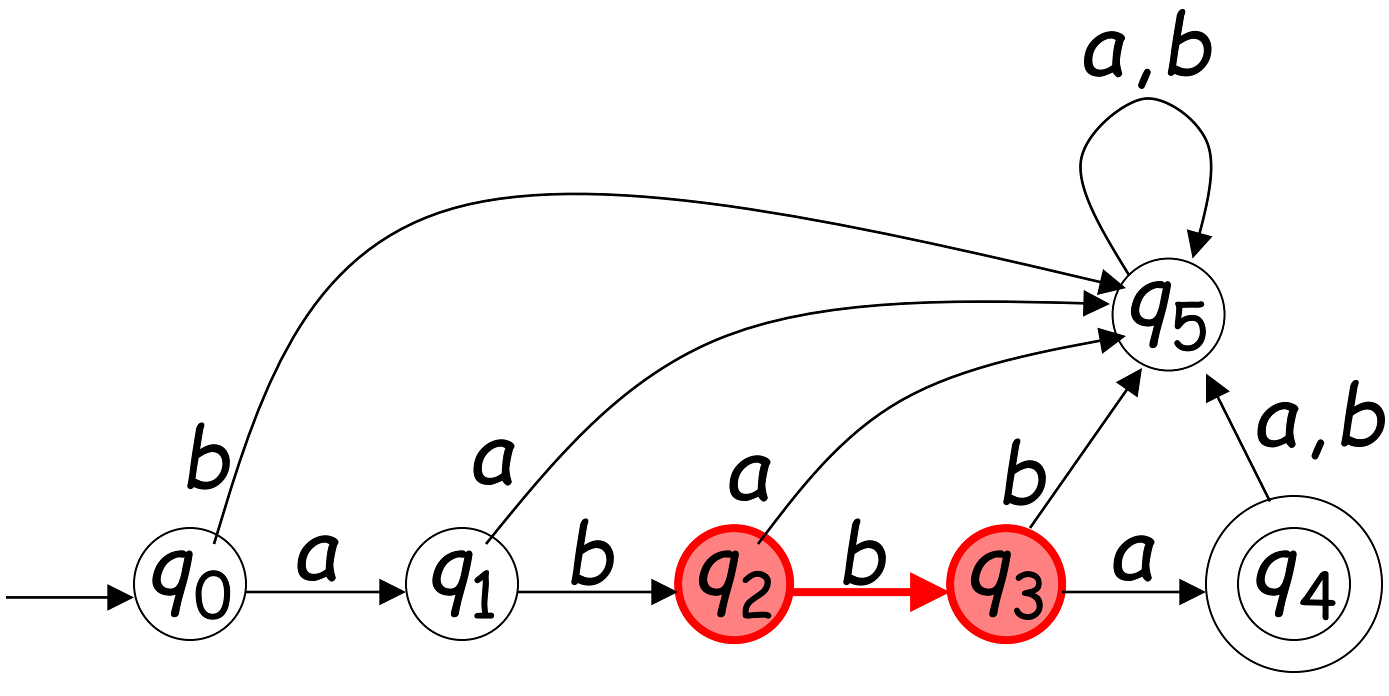
$$\delta(q_0, a) = q_1$$



$$\delta(q_0, b) = q_5$$

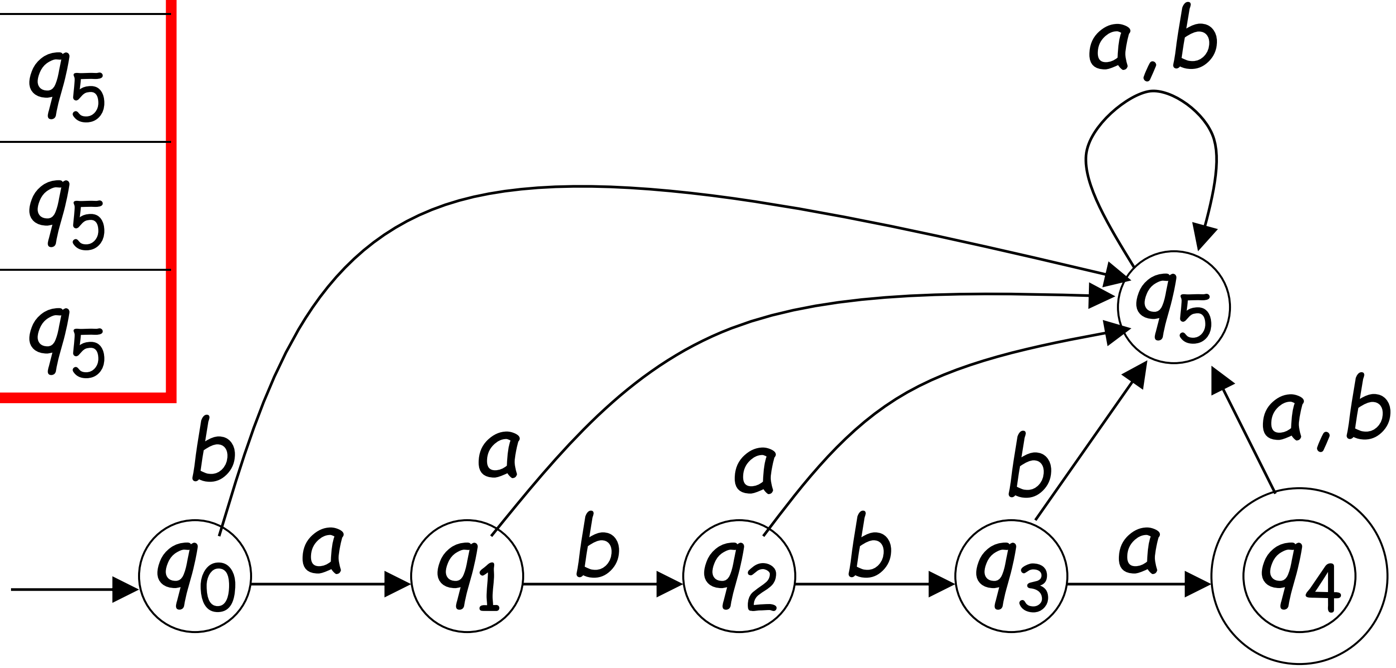


$$\delta(q_2, b) = q_3$$



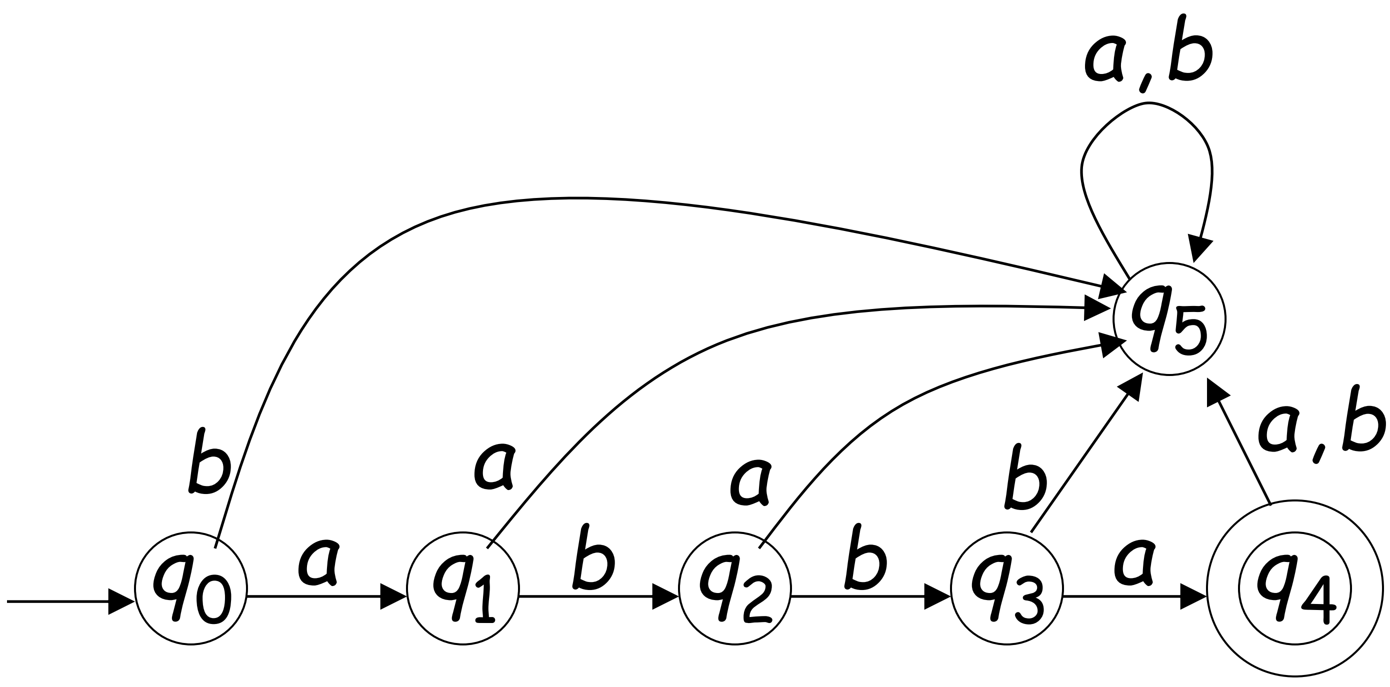
Transition Function δ

δ	a	b
q_0	q_1	q_5
q_1	q_5	q_2
q_2	q_5	q_3
q_3	q_4	q_5
q_4	q_5	q_5
q_5	q_5	q_5

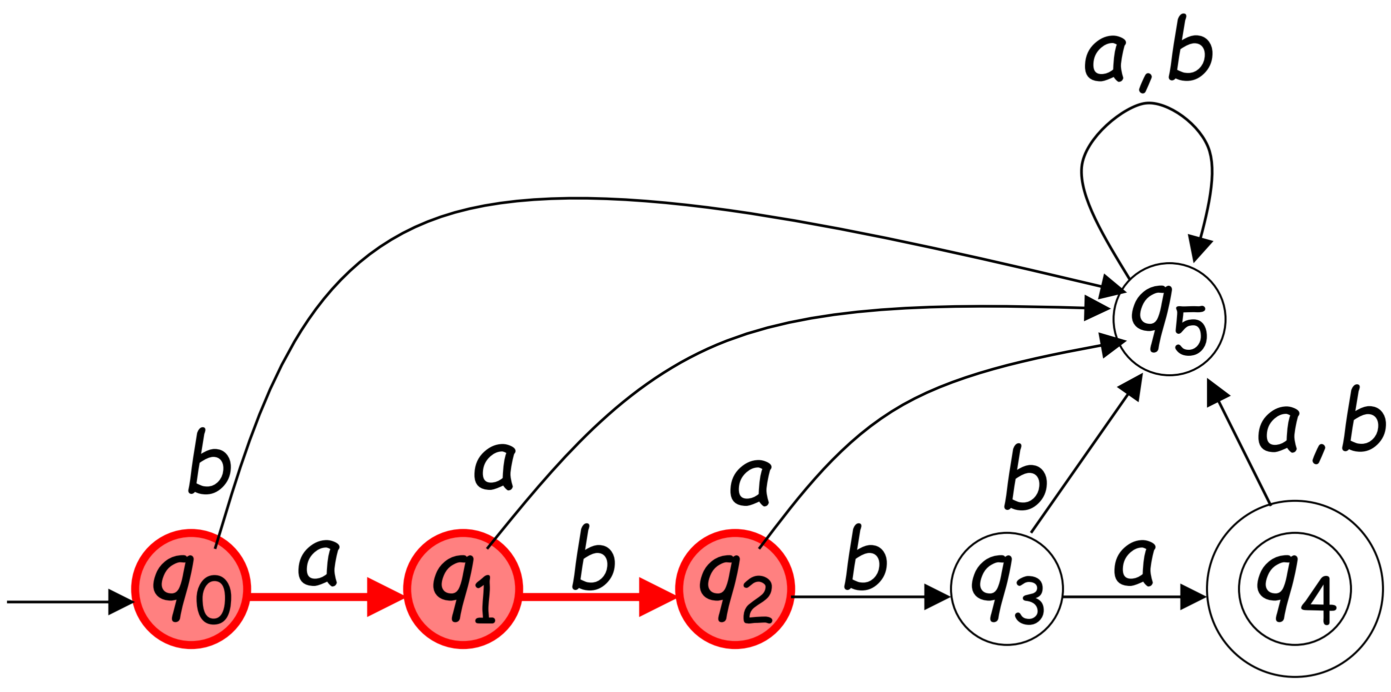


Extended Transition Function $\delta *$

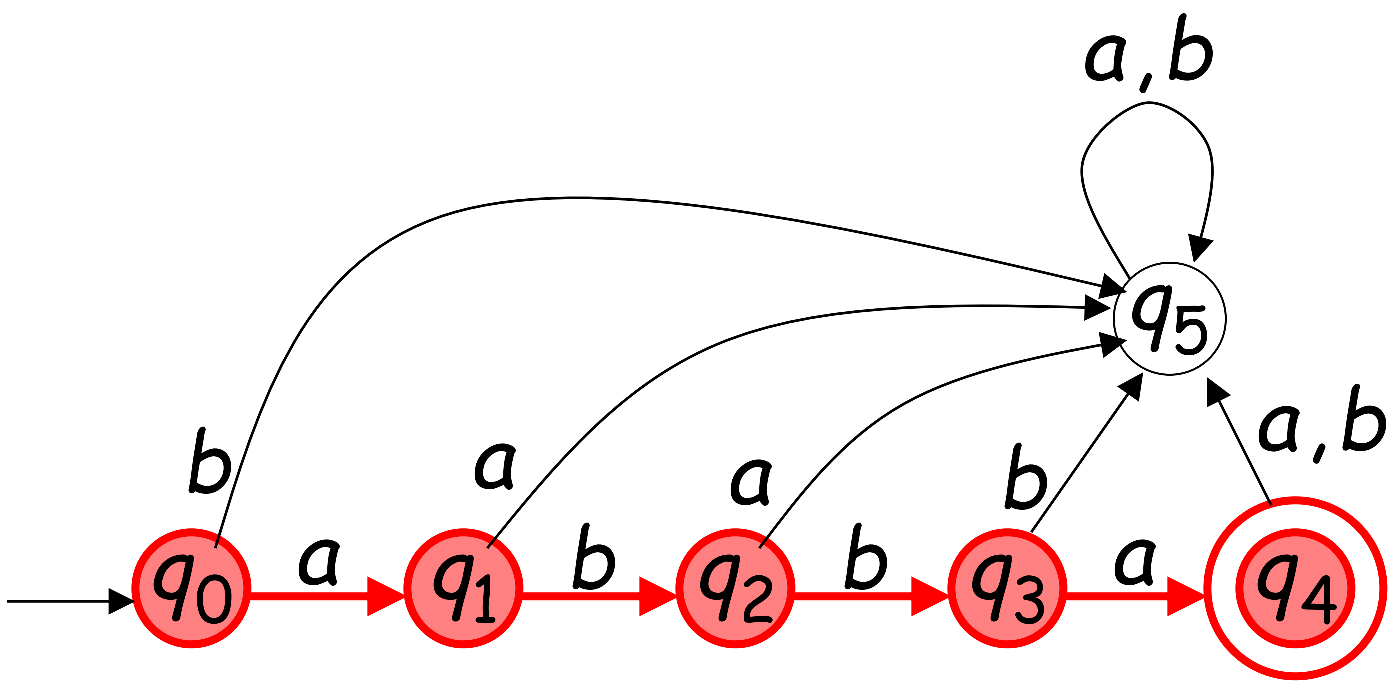
$$\delta * : Q \times \Sigma * \rightarrow Q$$



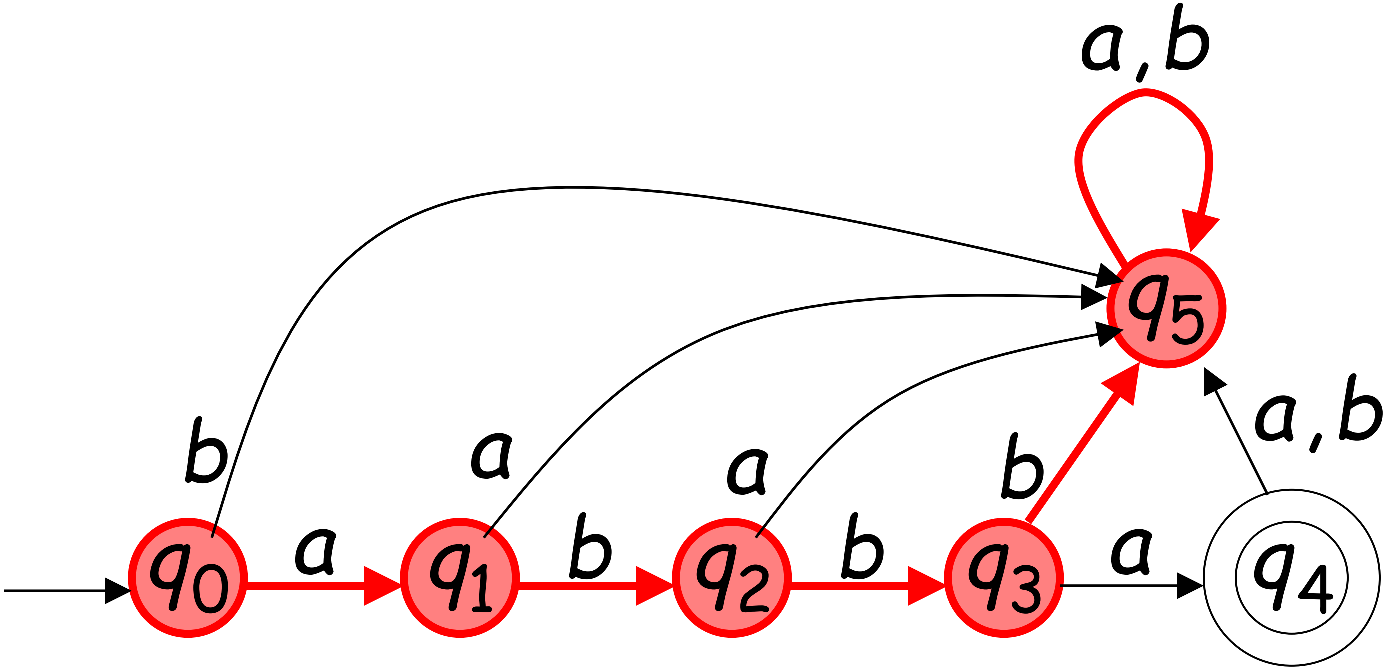
$$\delta^*(q_0, ab) = q_2$$



$$\delta^*(q_0, abba) = q_4$$

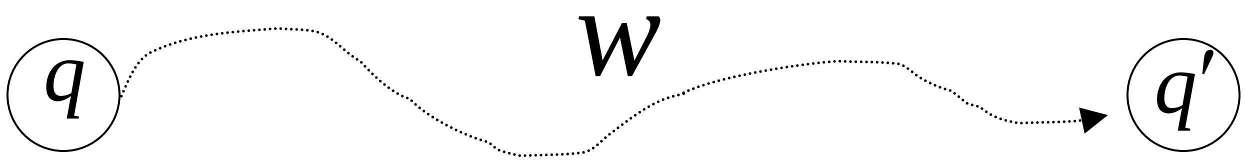


$$\delta^*(q_0, abbbaa) = q_5$$

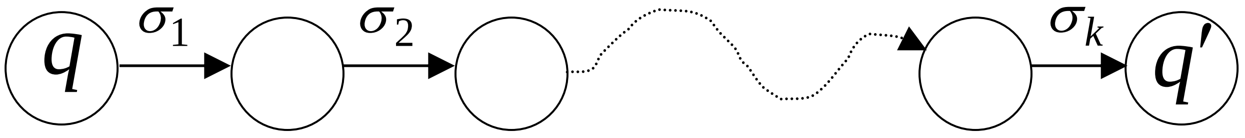


Observation: if there is a walk from q to q' with label w then

$$\delta^*(q, w) = q'$$

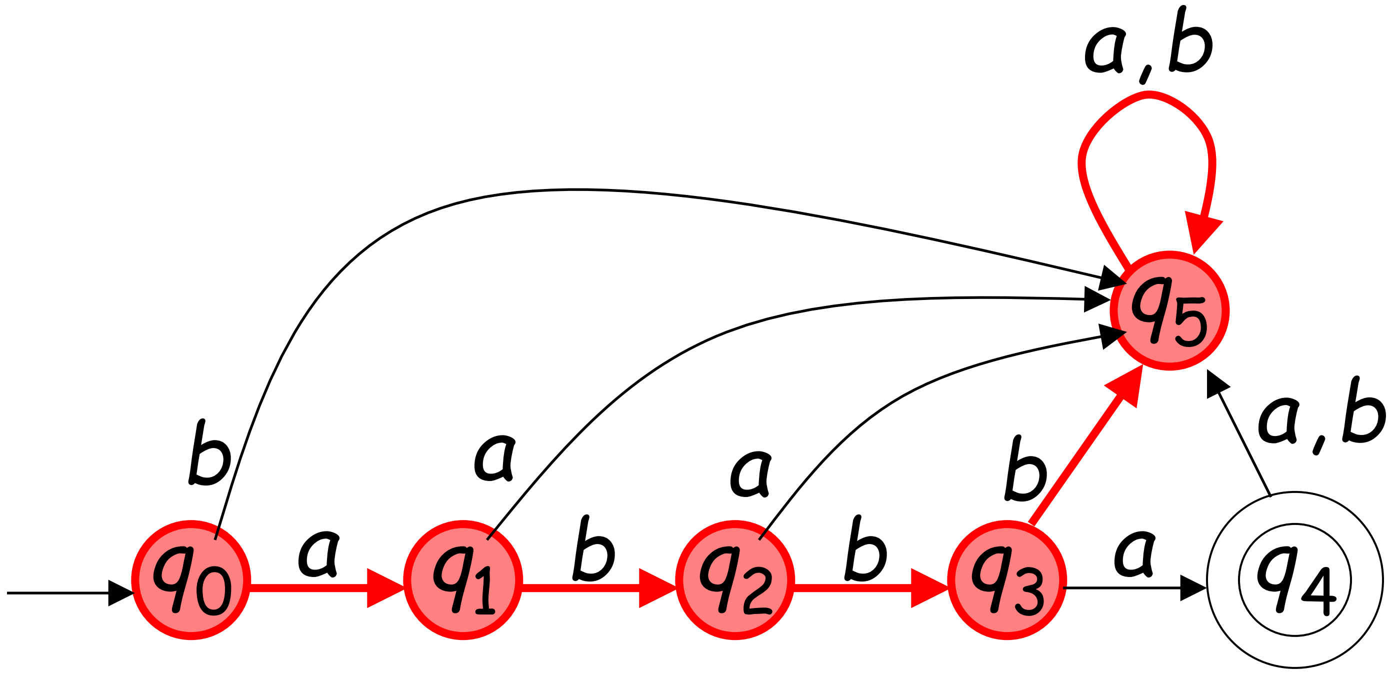


$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$



Example: There is a walk from q_0 to q_5 with label $abbbaa$

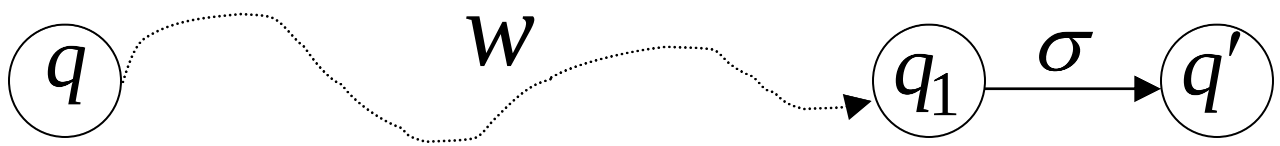
$$\delta^*(q_0, abbbaa) = q_5$$



Recursive Definition

$$\delta^*(q,\lambda) = q$$

$$\delta^*(q,w\sigma) = \delta(\delta^*(q,w),\sigma)$$



$$\left.\begin{array}{l}\delta^*(q,w\sigma) = q' \\ \\ \delta(q_1,\sigma) = q'\end{array}\right\} \Rightarrow \delta^*(q,w\sigma) = \delta(q_1,\sigma)$$

$$\left.\begin{array}{l}\delta^*(q,w\sigma) = \delta(q_1,\sigma) \\ \\ \delta^*(q,w) = q_1\end{array}\right\} \Rightarrow \delta^*(q,w\sigma) = \delta(\delta^*(q,w),\sigma)$$

$$\delta^*(q_0, ab) =$$

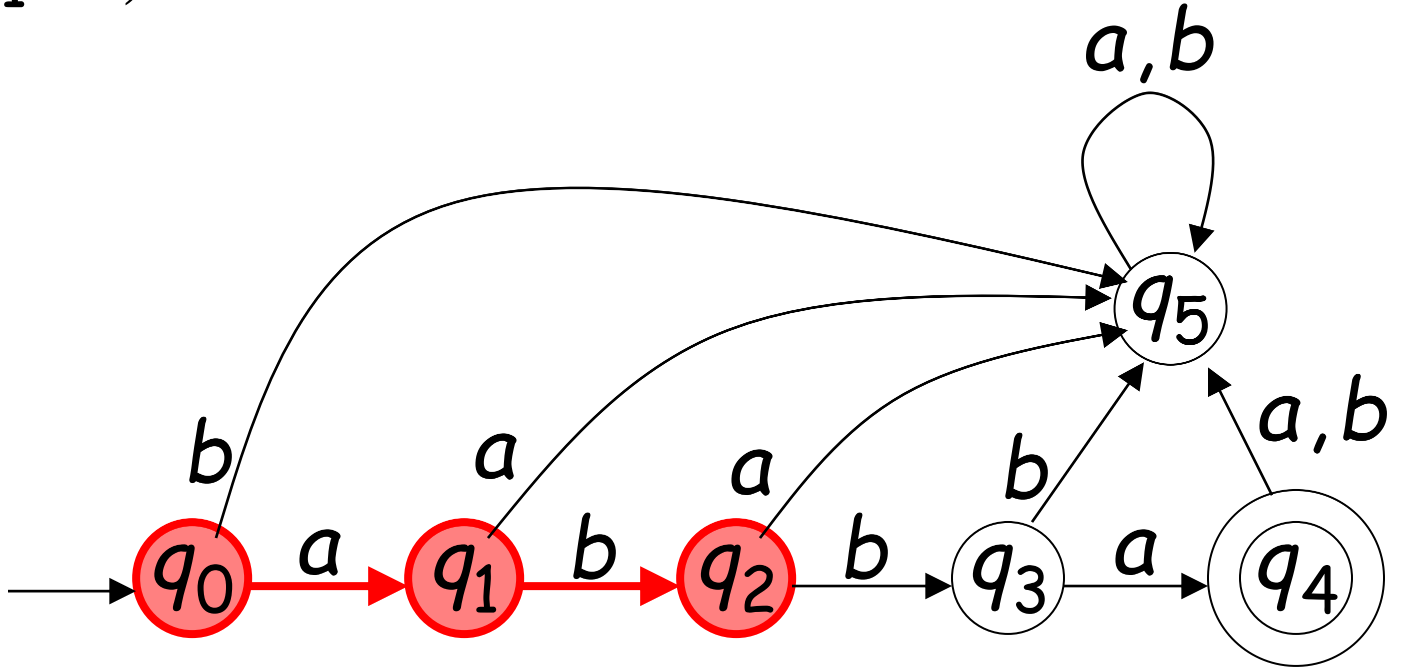
$$\delta(\delta^*(q_0, a), b) =$$

$$\delta(\delta(\delta^*(q_0, \lambda), a), b) =$$

$$\delta(\delta(q_0, a), b) =$$

$$\delta(q_1, b) =$$

$$q_2$$

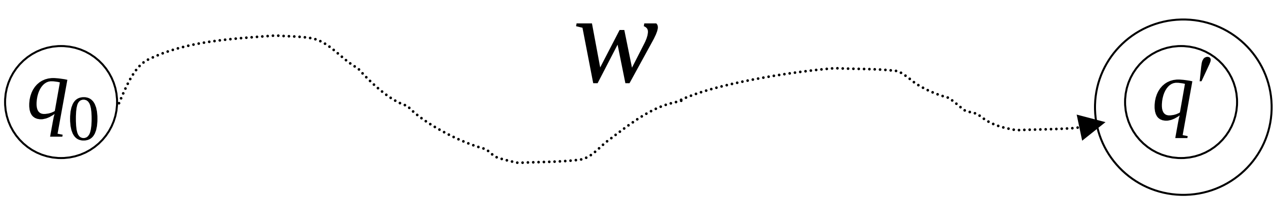


Language Accepted by FAs

For a FA $M = (Q, \Sigma, \delta, q_0, F)$

Language accepted by M :

$$L(M) = \{w \in \Sigma^* : \delta^*(q_0, w) \in F\}$$

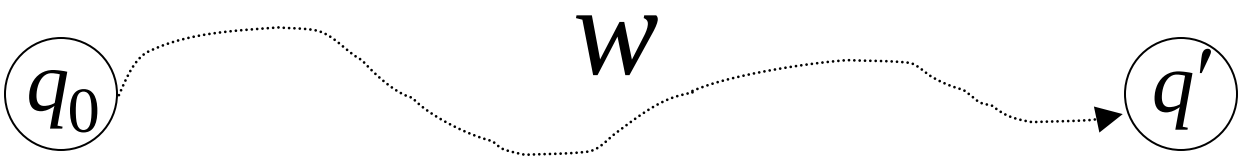


$q' \in F$

Observation

Language rejected by M :

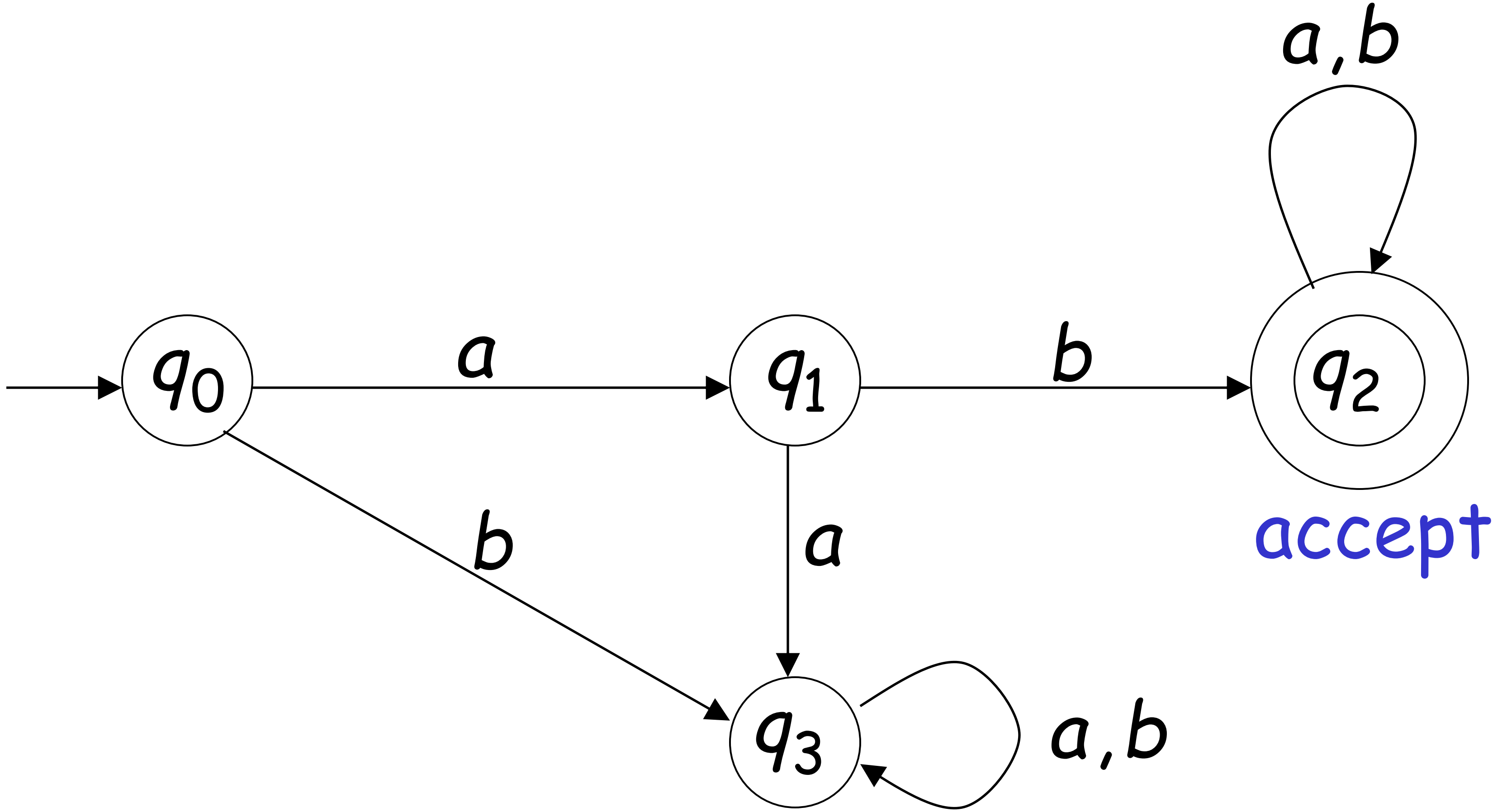
$$\overline{L(M)} = \{ w \in \Sigma^* : \delta^*(q_0, w) \notin F \}$$



$q' \notin F$

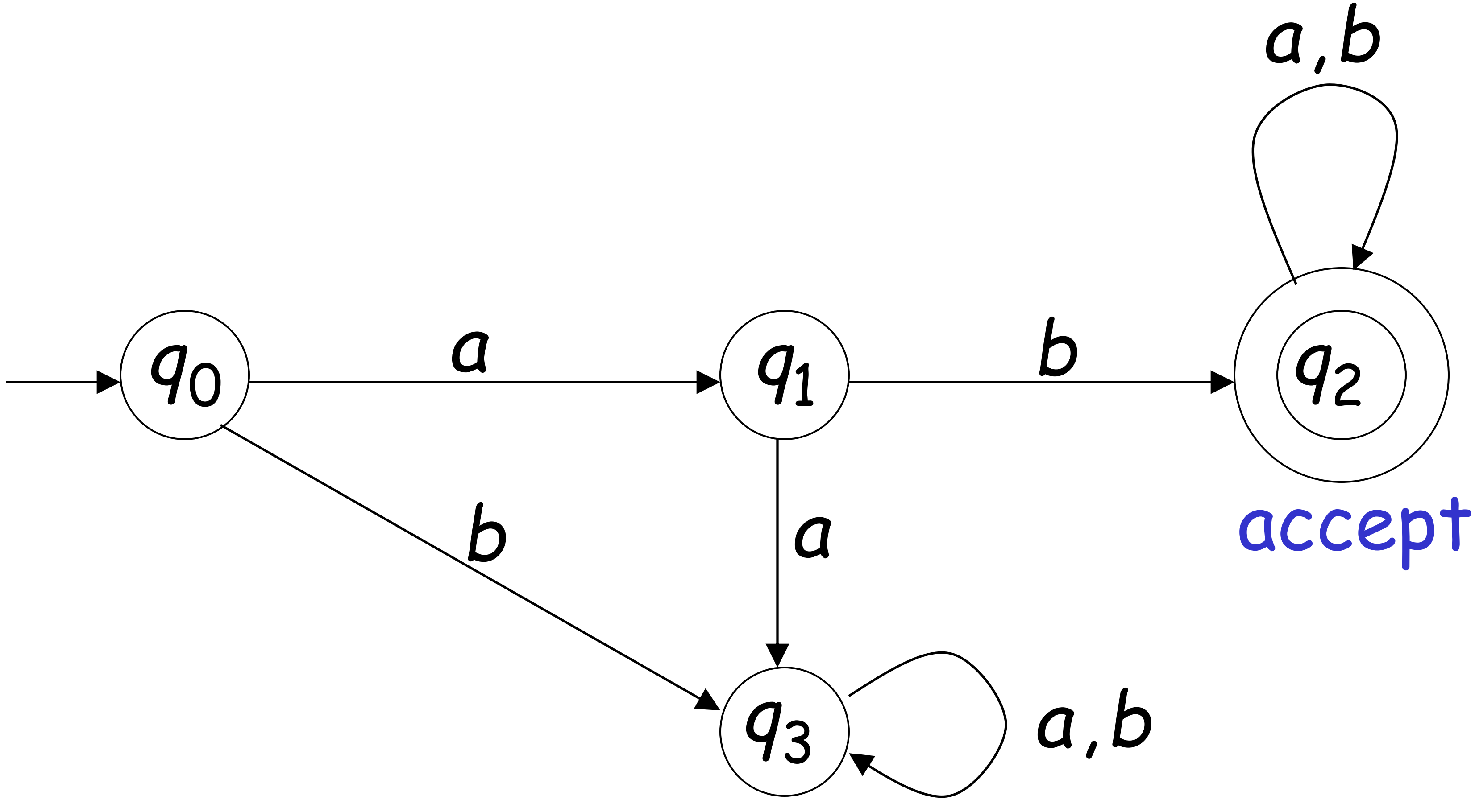
L(M) ?

ab



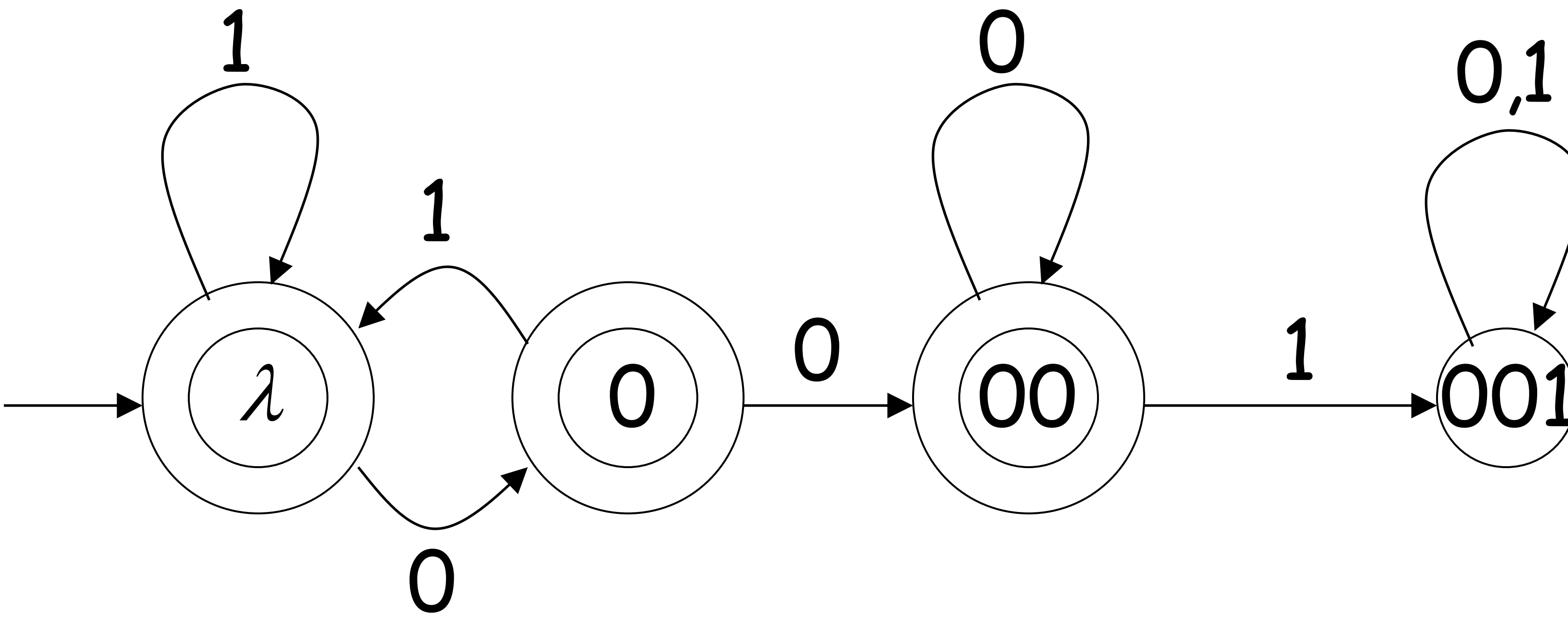
Example

$L(M) = \{$ all strings with prefix ab $\}$



Try-Starting with a and ending with b

L(M)?

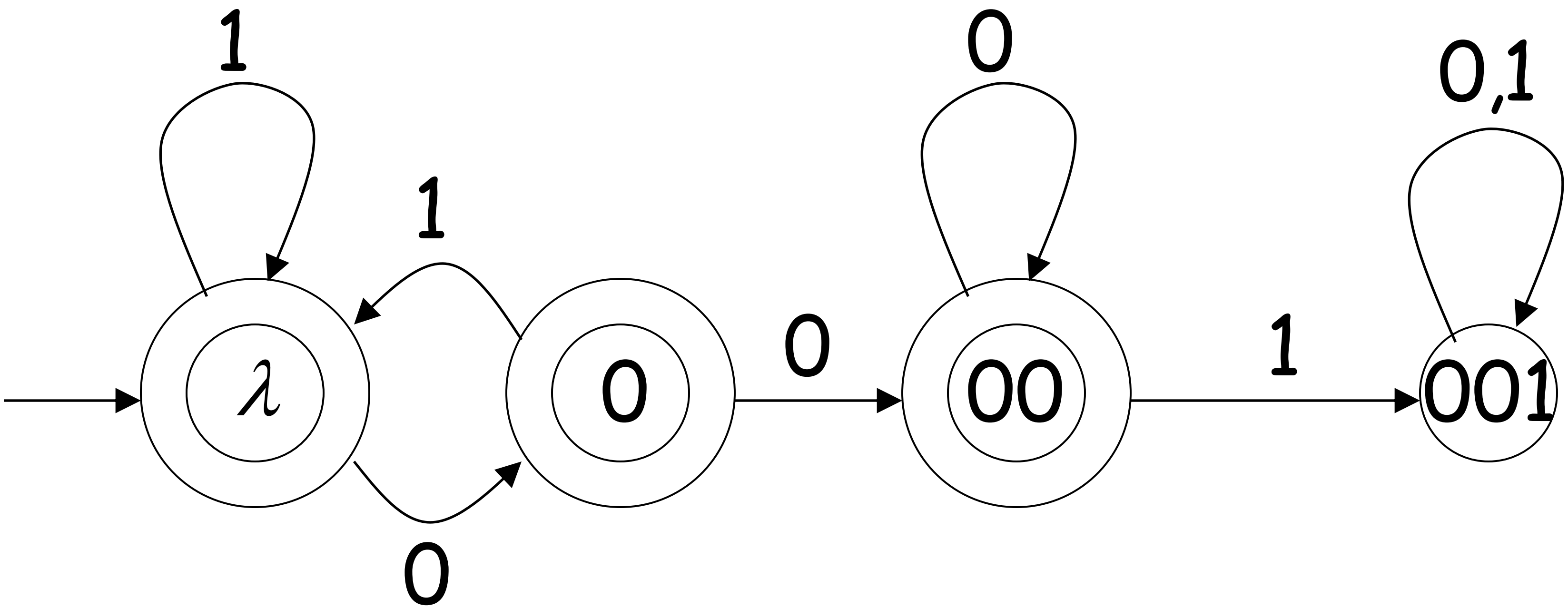


Example

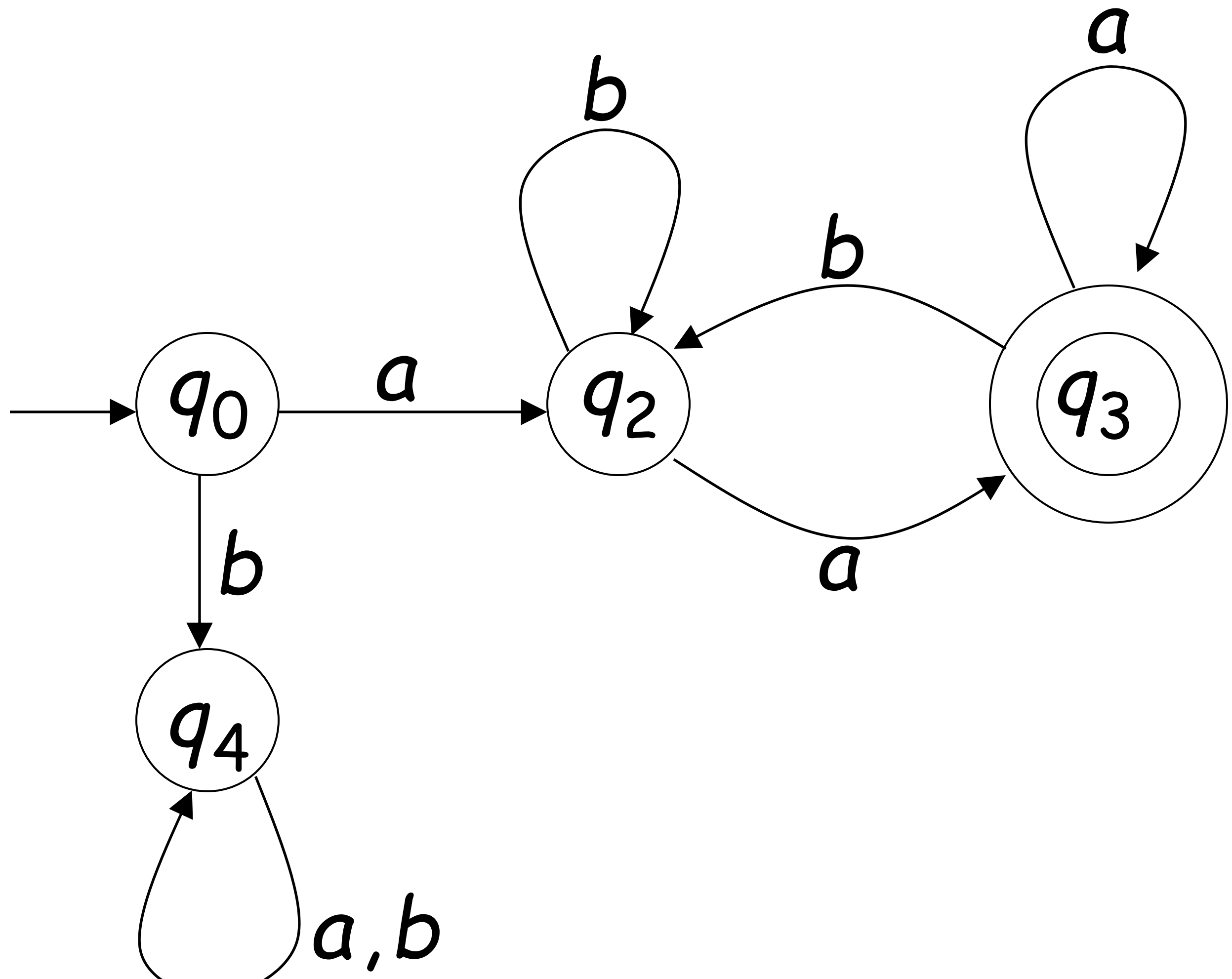
$L(M)$ = { all strings without substring 001 }

Example

$L(M) =$ { all strings without substring 001 }

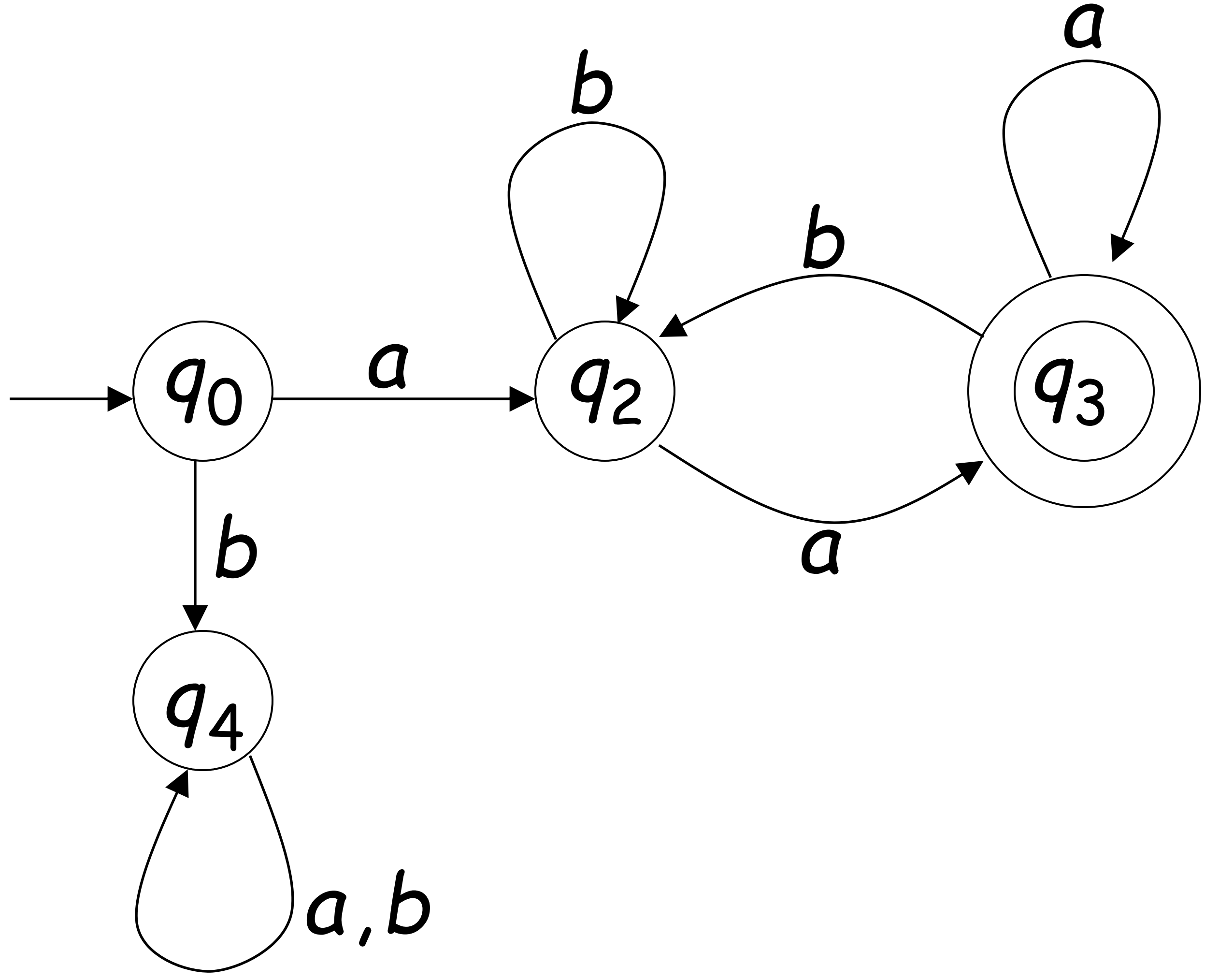


L(M) ?



Example

$$L(M) = \{awa : w \in \{a,b\}^*\}$$



Regular Languages

Definition:

A language L is regular if there is FA M such that $L = L(M)$

Observation:

All languages accepted by FAs form the family of regular languages

Examples of regular languages:

$\{abba\}$ $\{\lambda, ab, abba\}$

$\{awa : w \in \{a,b\}^*\}$ $\{a^n b : n \geq 0\}$

{ all strings with prefix ab }

{ all strings without substring 001 }

There exist automata that accept these Languages (see previous slides).

There exist languages which are <u>not</u> Regular:

Example: $L = \{a^n b^n : n \geq 0\}$

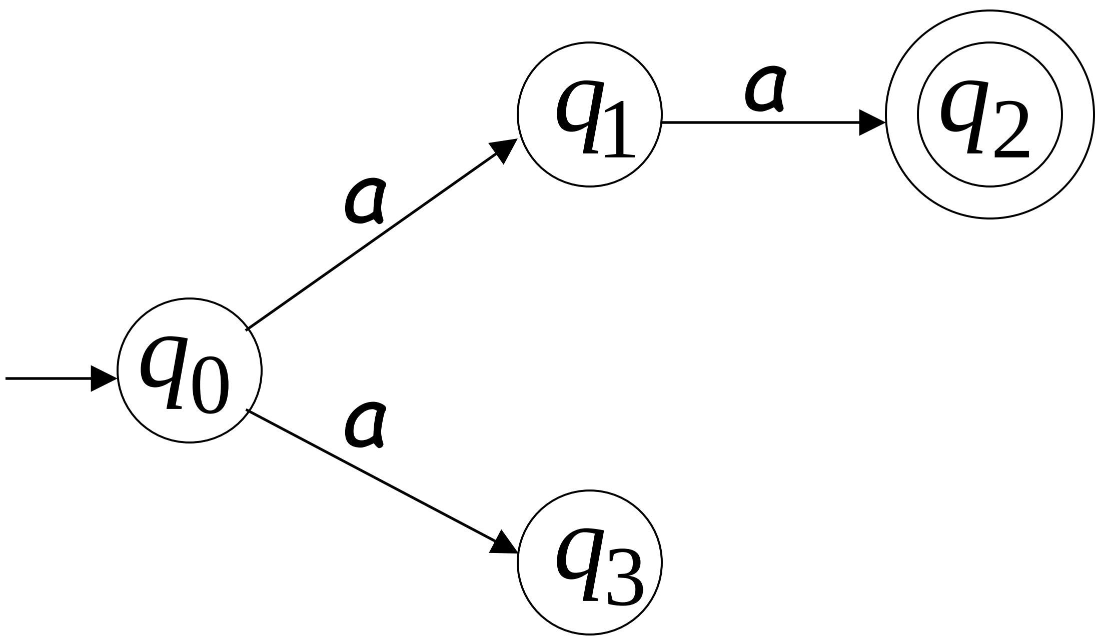
There is no FA that accepts such a language

Formal Languages

Non-Deterministic Automata

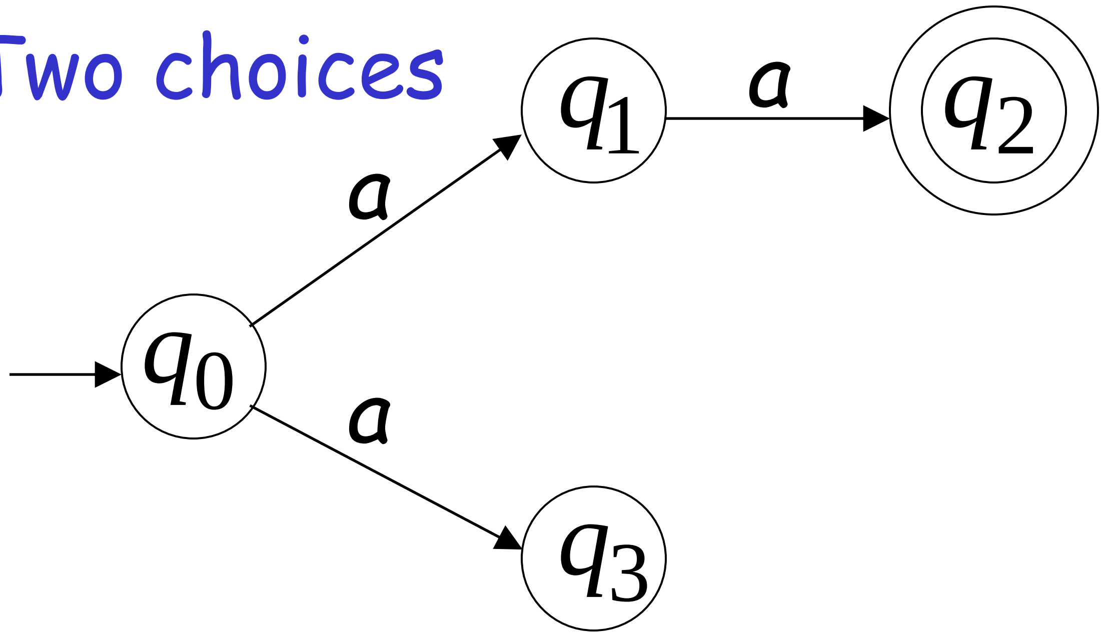
Nondeterministic Finite Automaton (NFA)

Alphabet = $\{a\}$

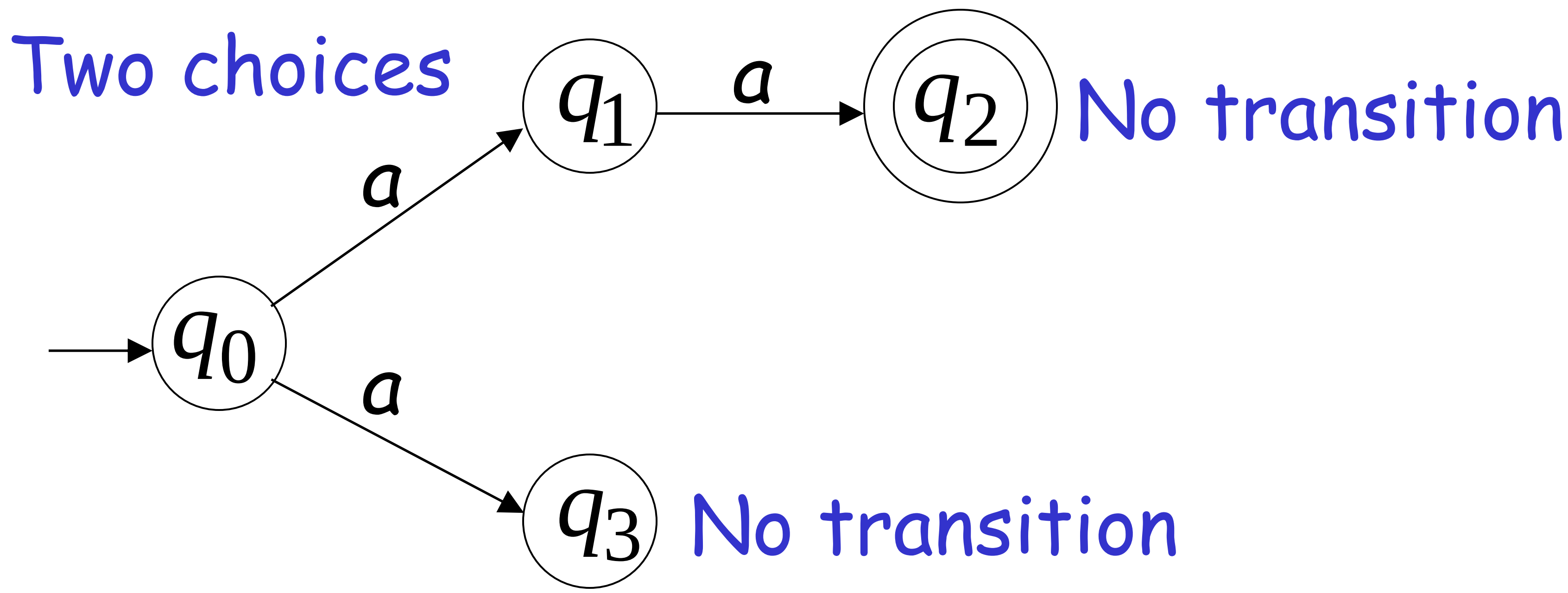


Alphabet = $\{a\}$

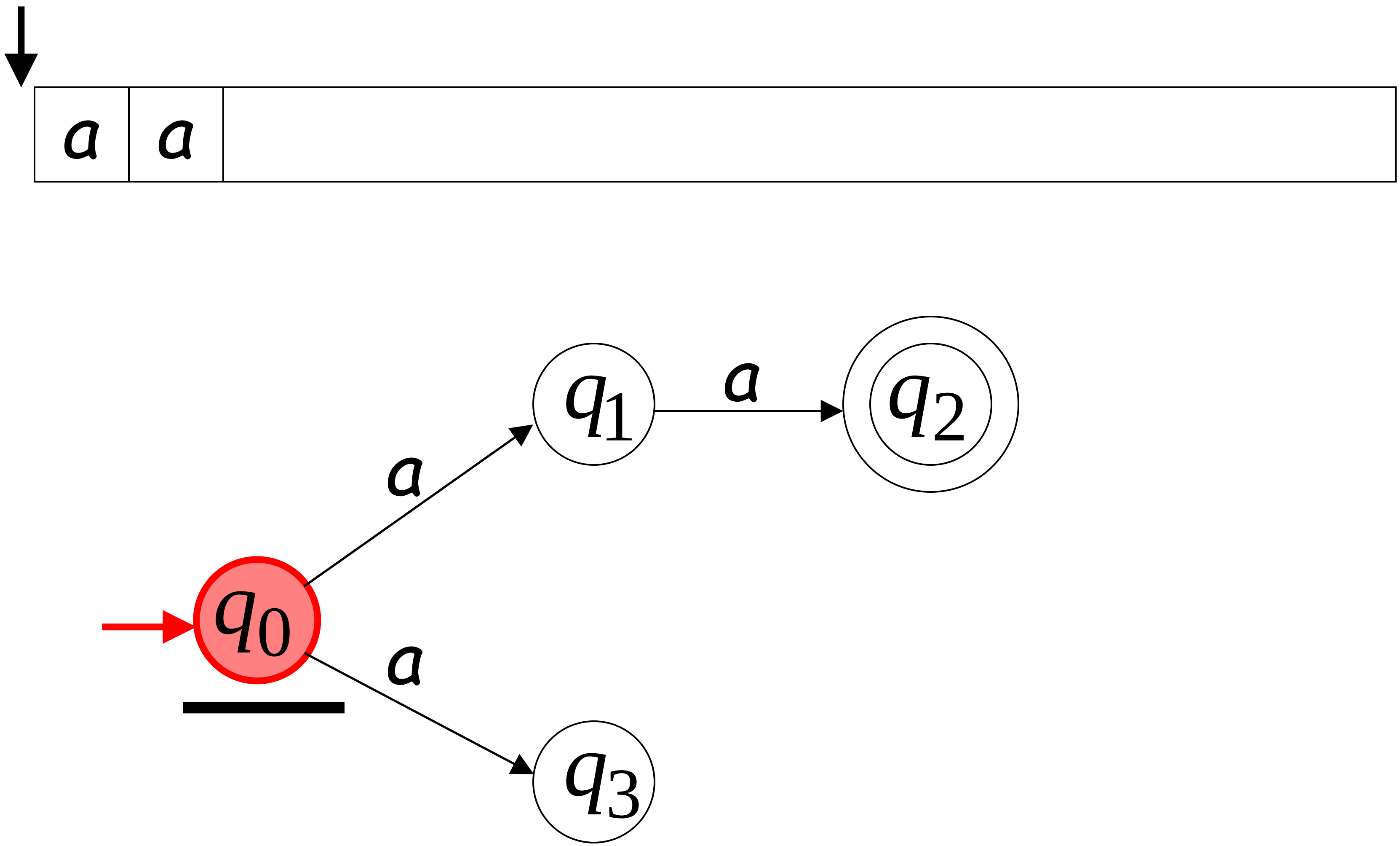
Two choices



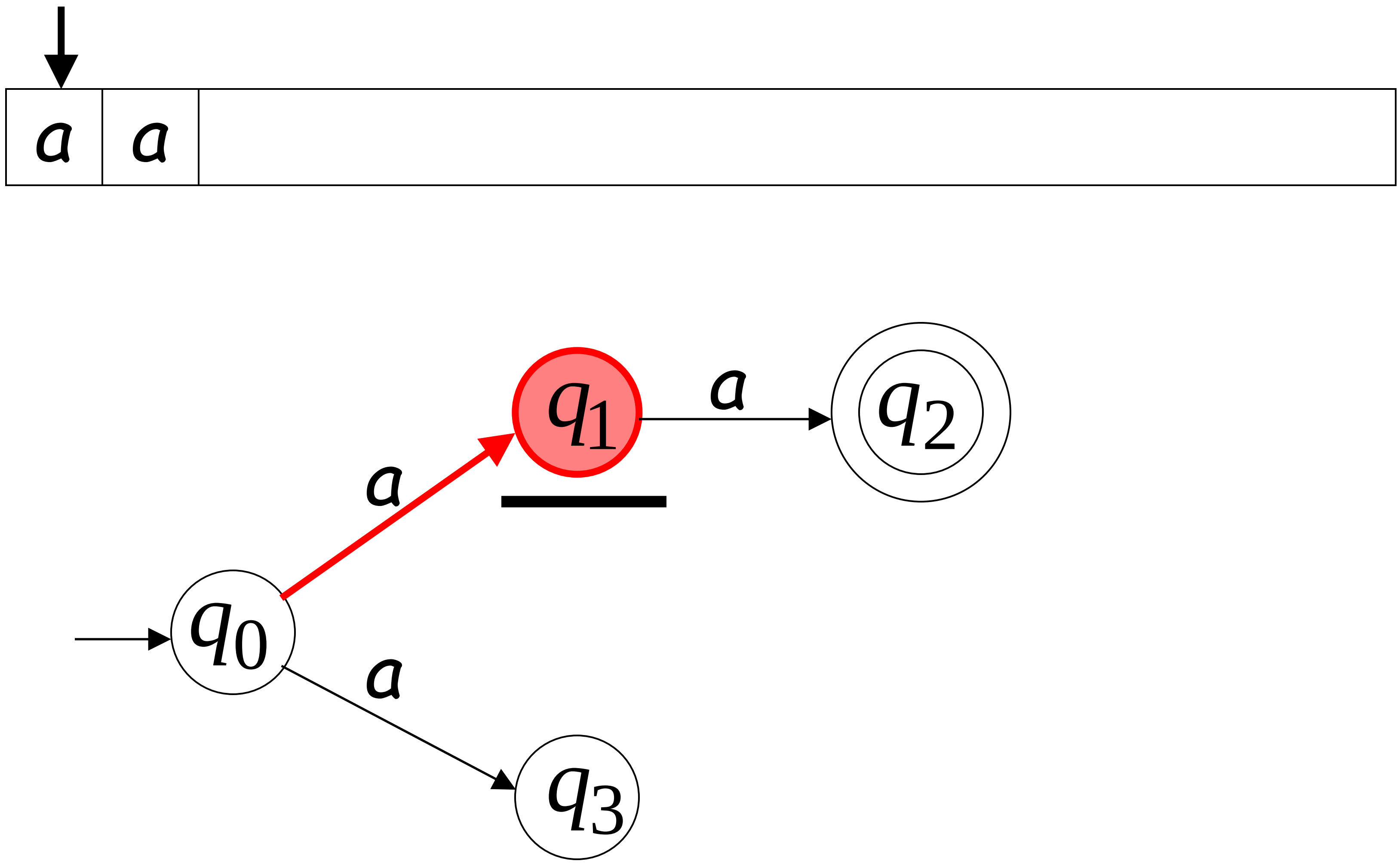
Alphabet = $\{a\}$



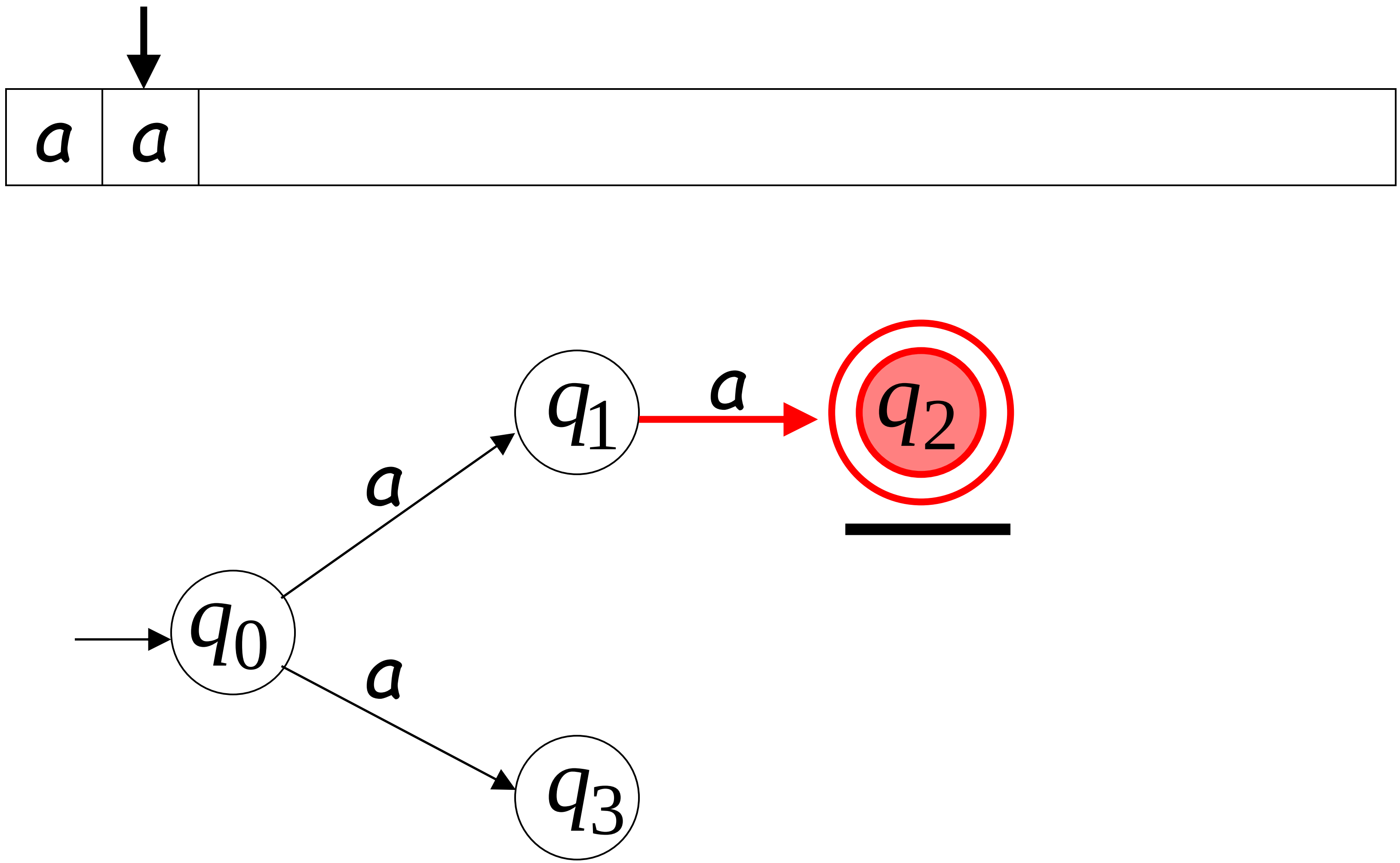
First Choice



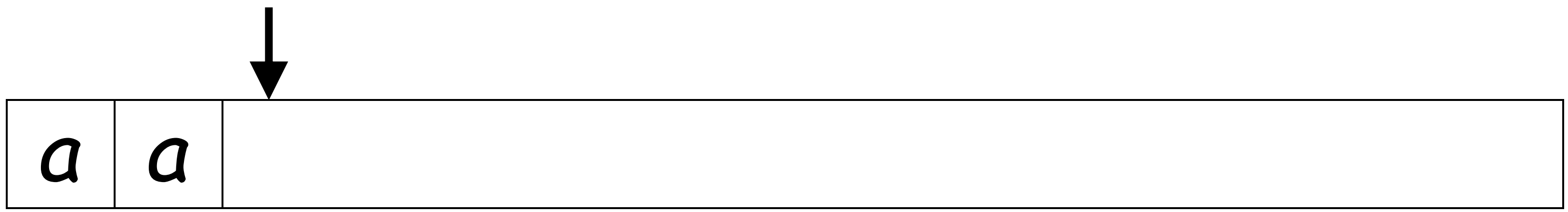
First Choice



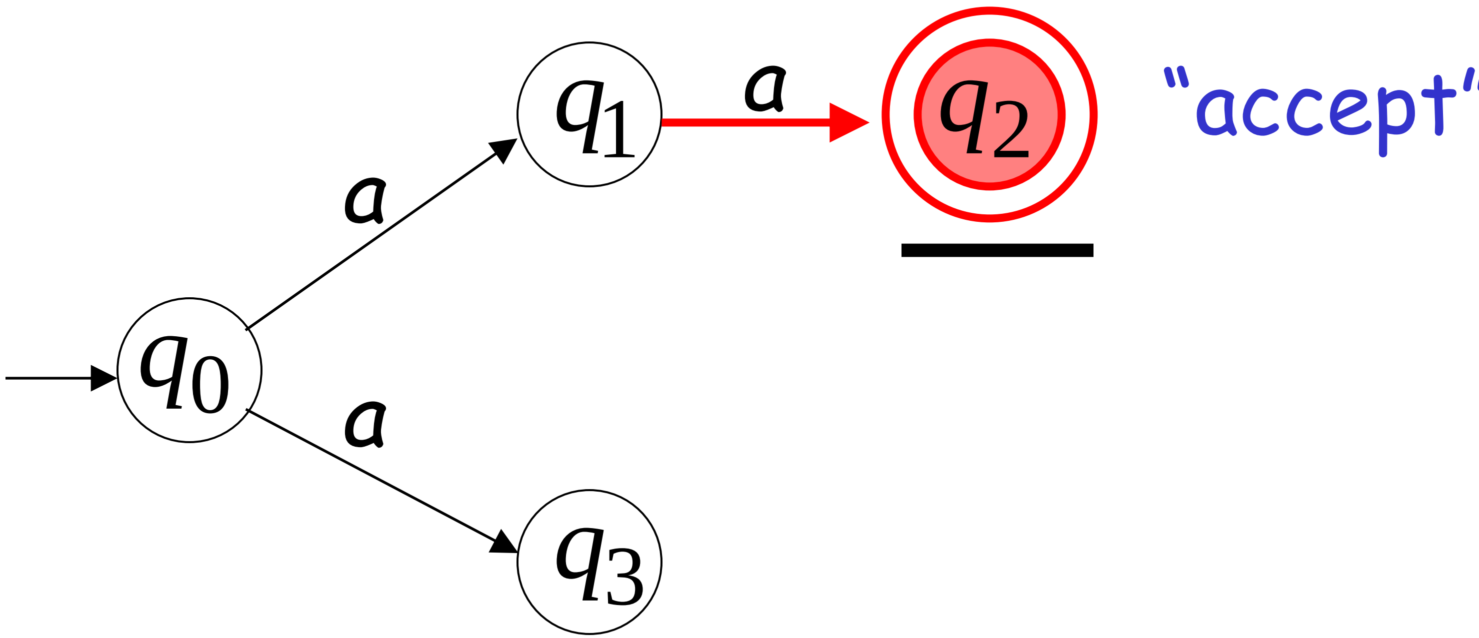
First Choice



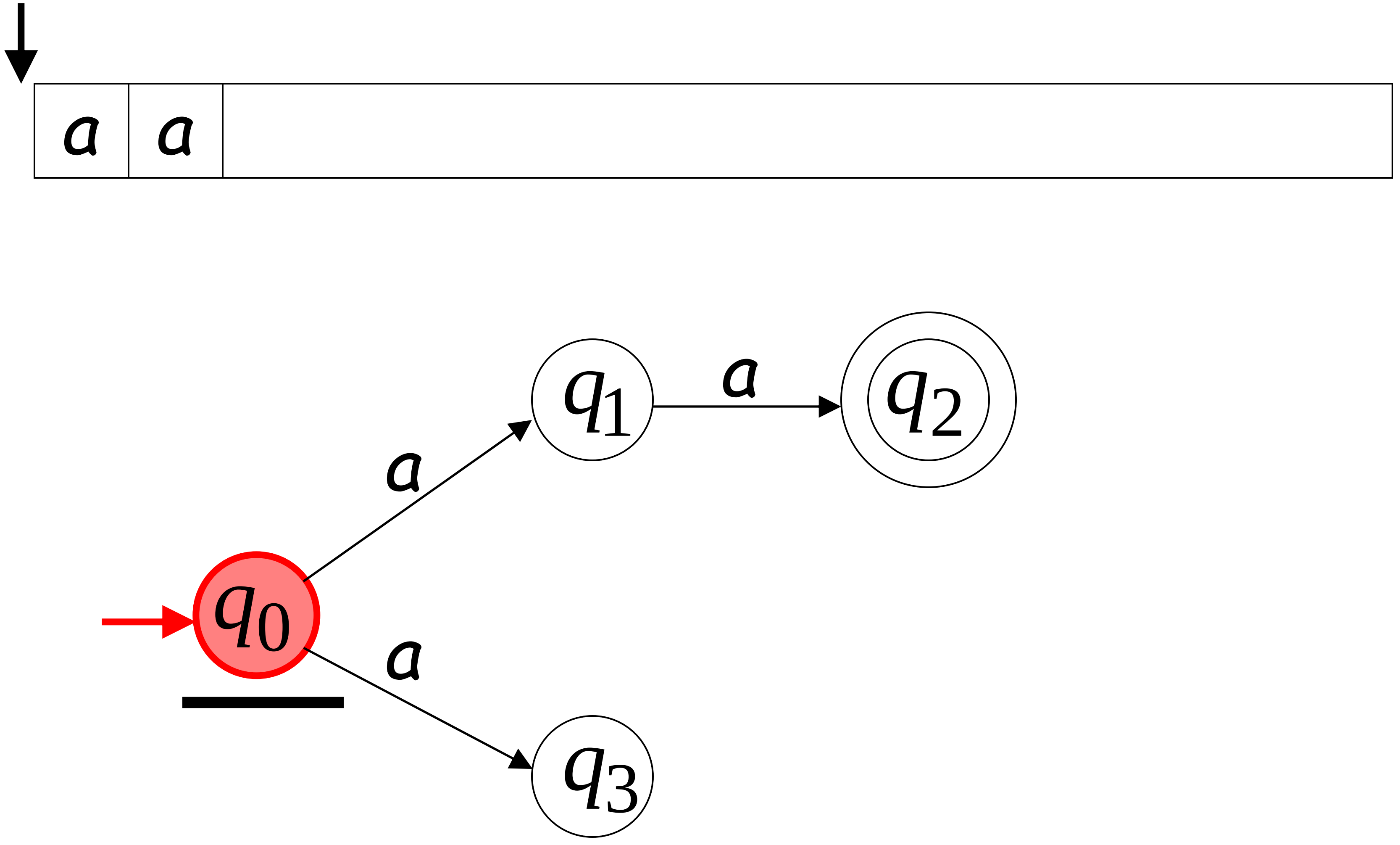
First Choice



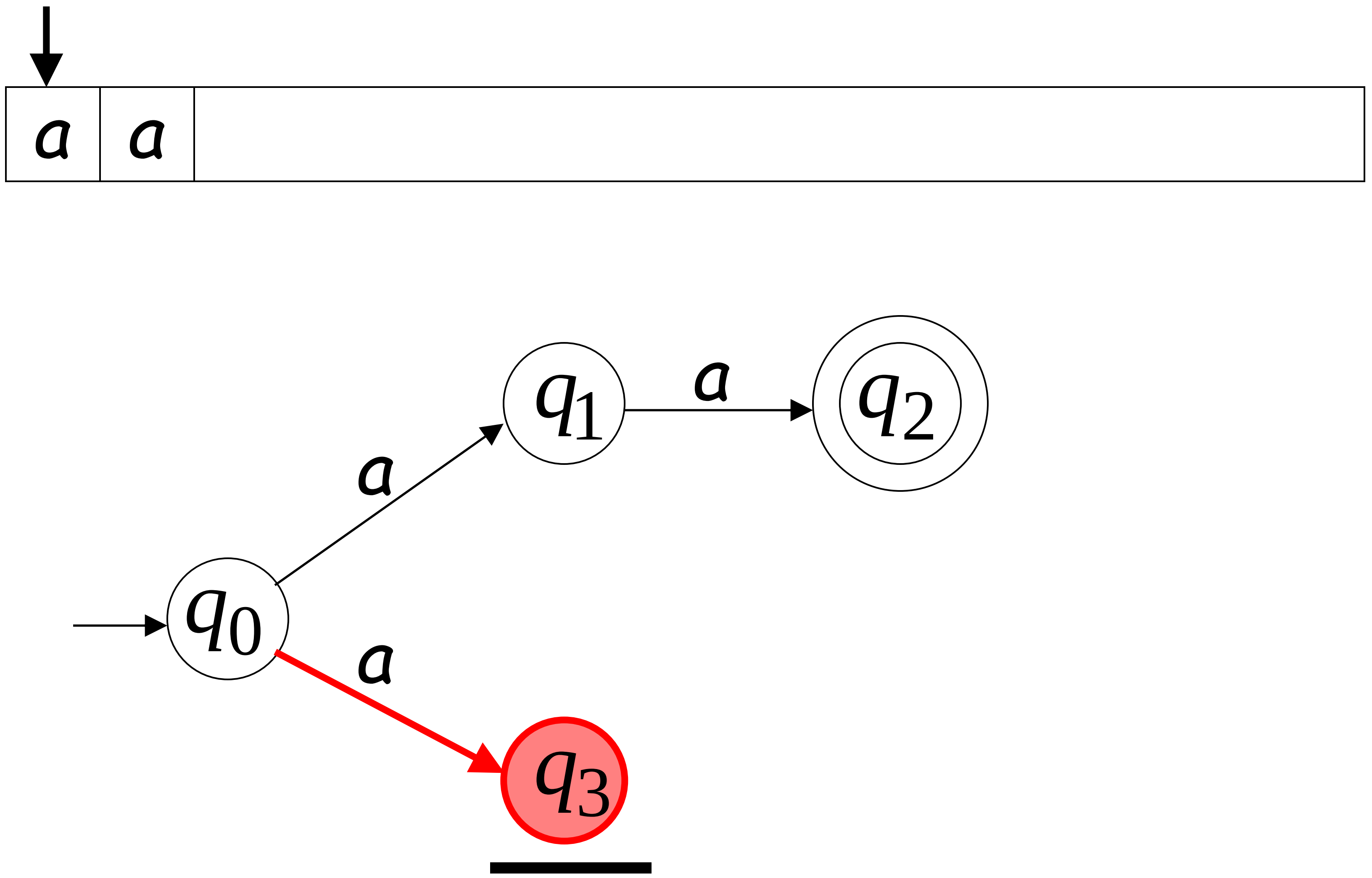
All input is consumed



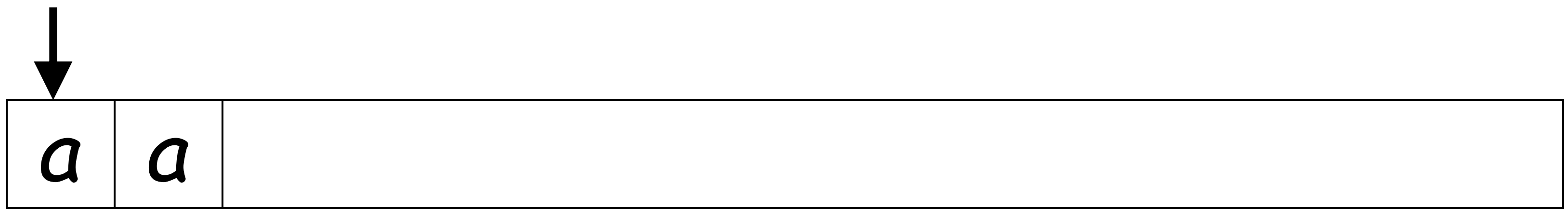
Second Choice

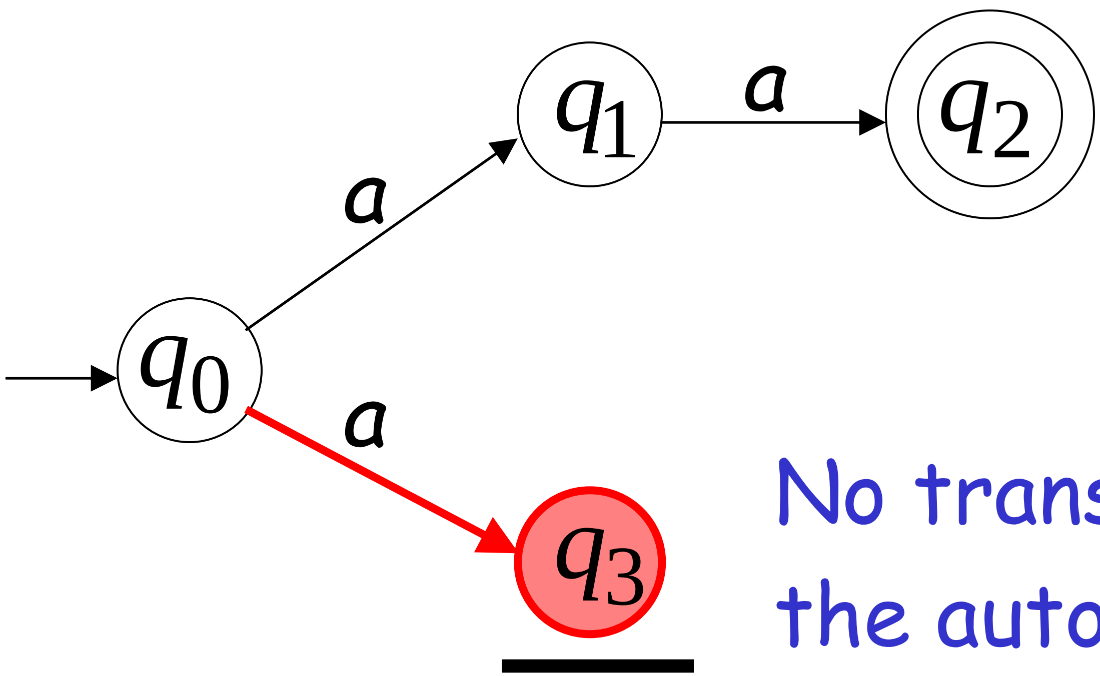


Second Choice



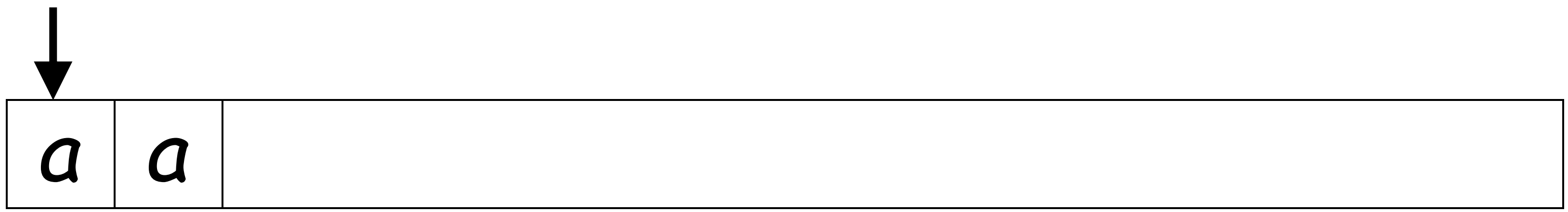
Second Choice



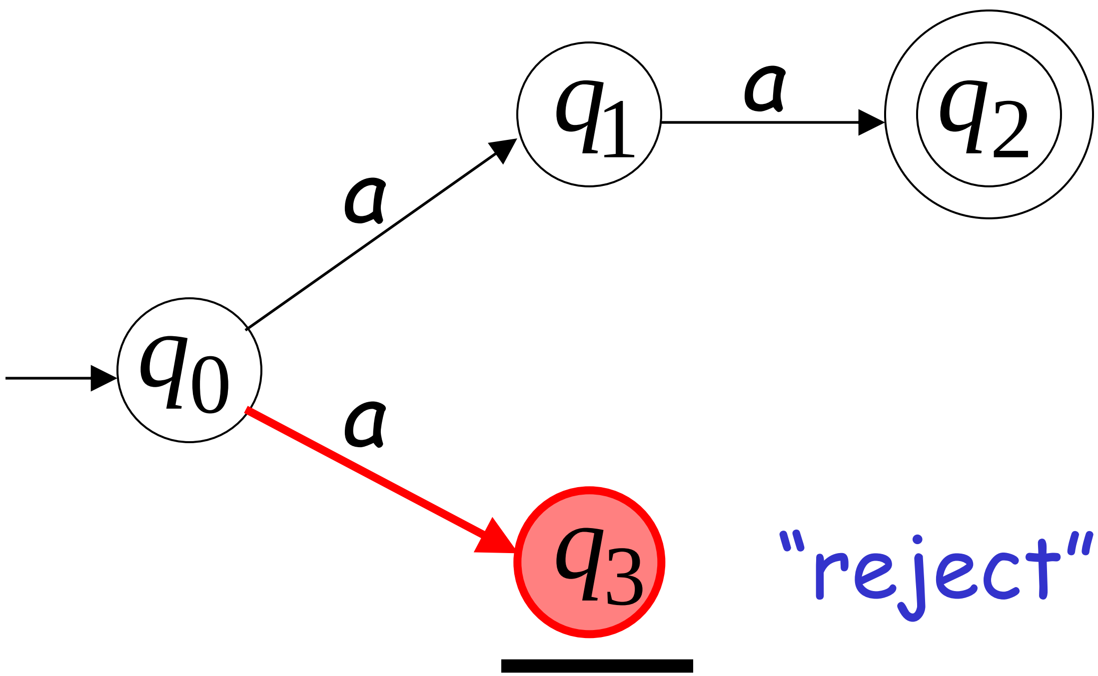


No transition:
the automaton hangs

Second Choice



Input cannot be consumed



"reject"

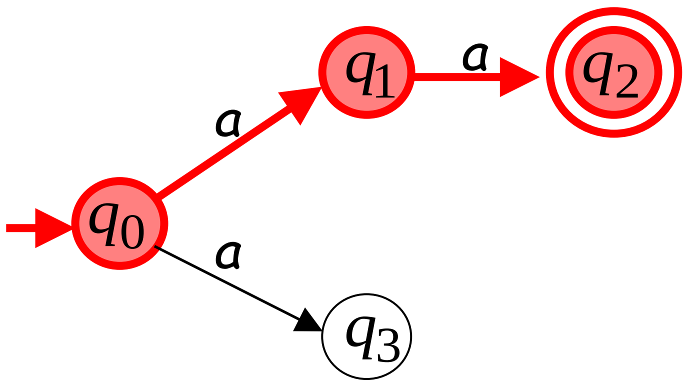
An NFA accepts a string:
when there is a computation of the NFA
that accepts the string

There is a computation:
all the input is consumed and the automaton
is in an accepting state

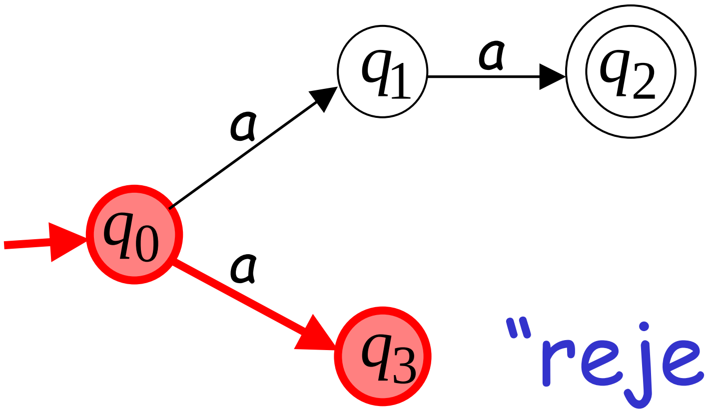
Example

aa is accepted by the NFA:

“accept”

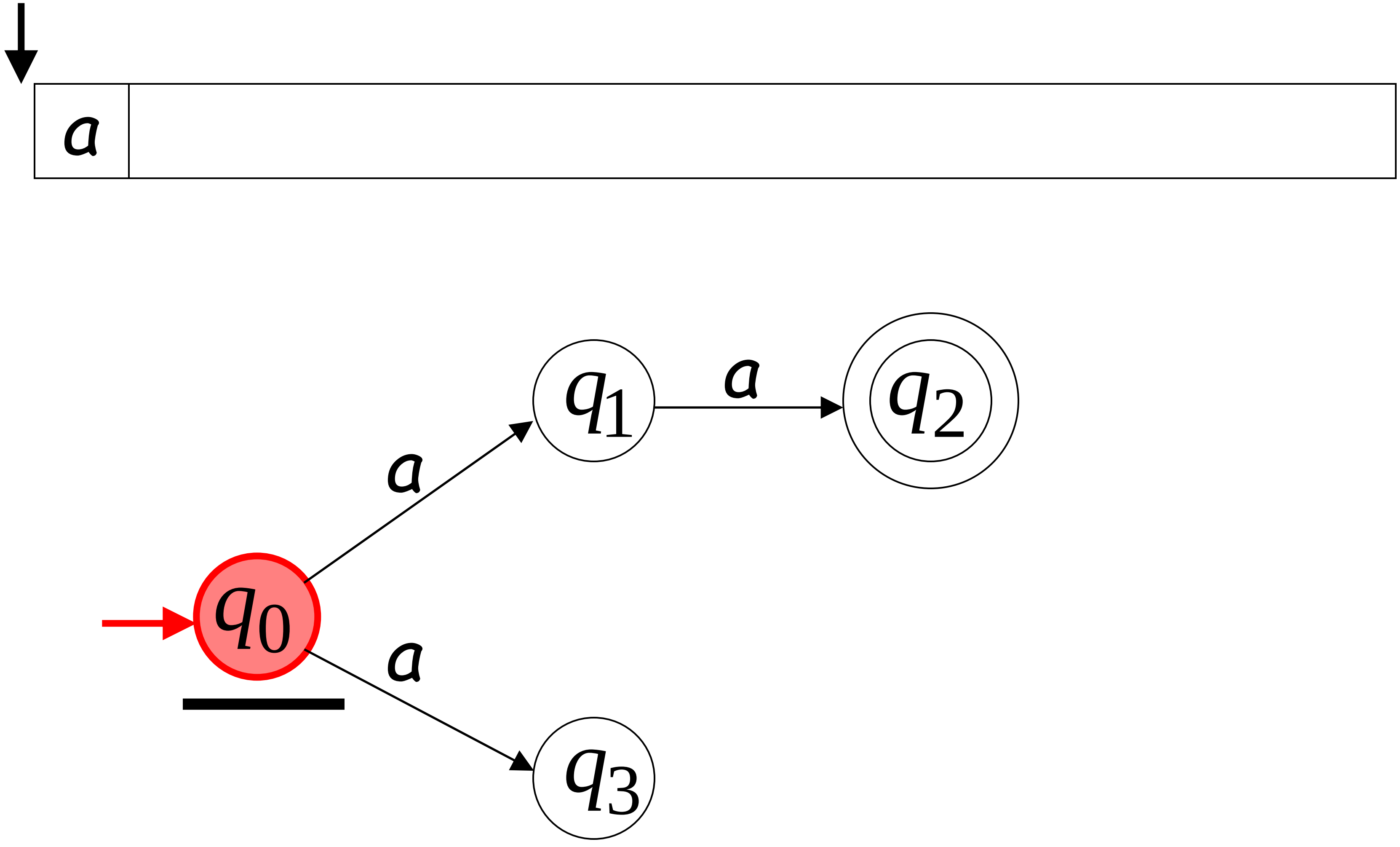


because this computation accepts aa

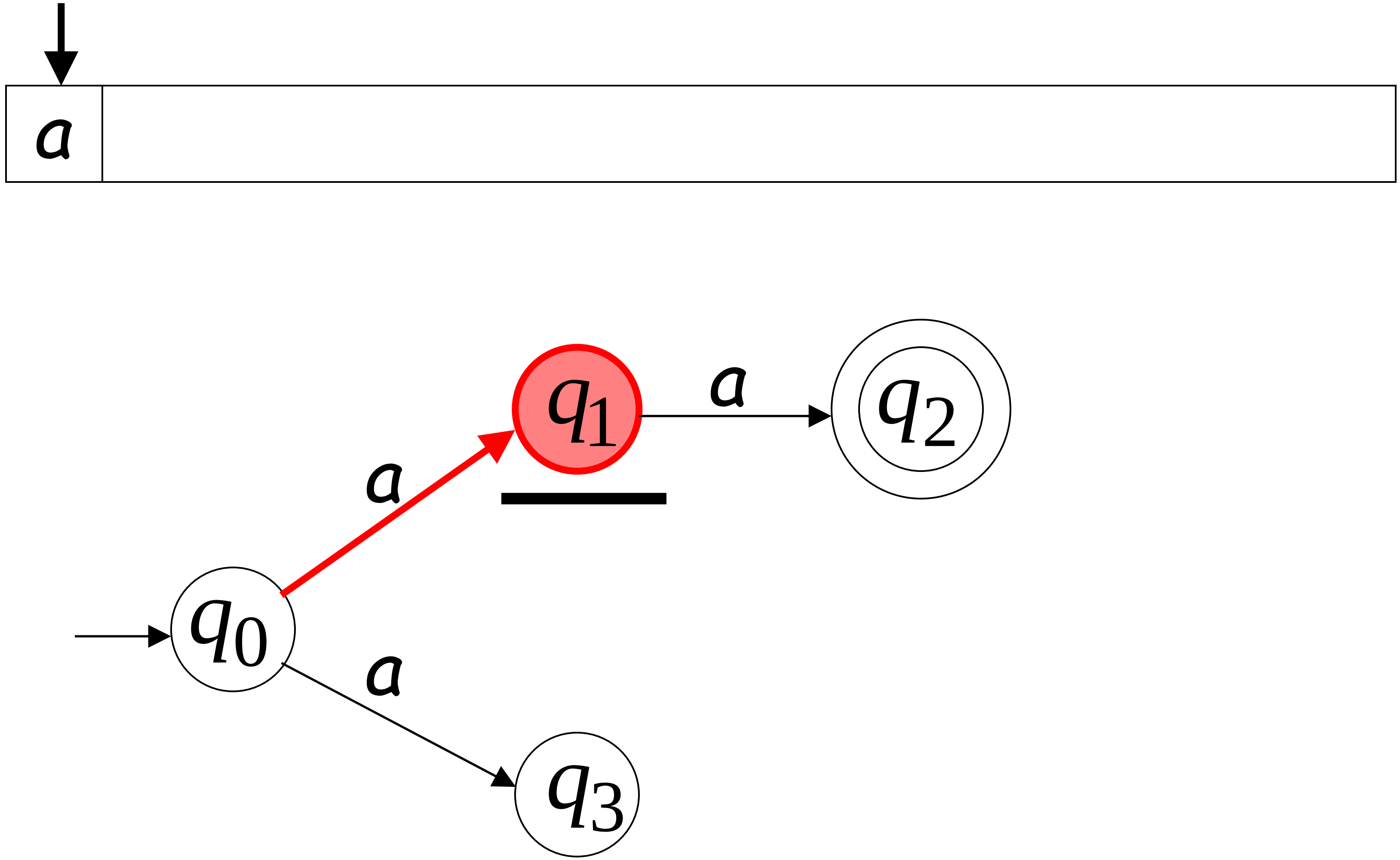


“reject”

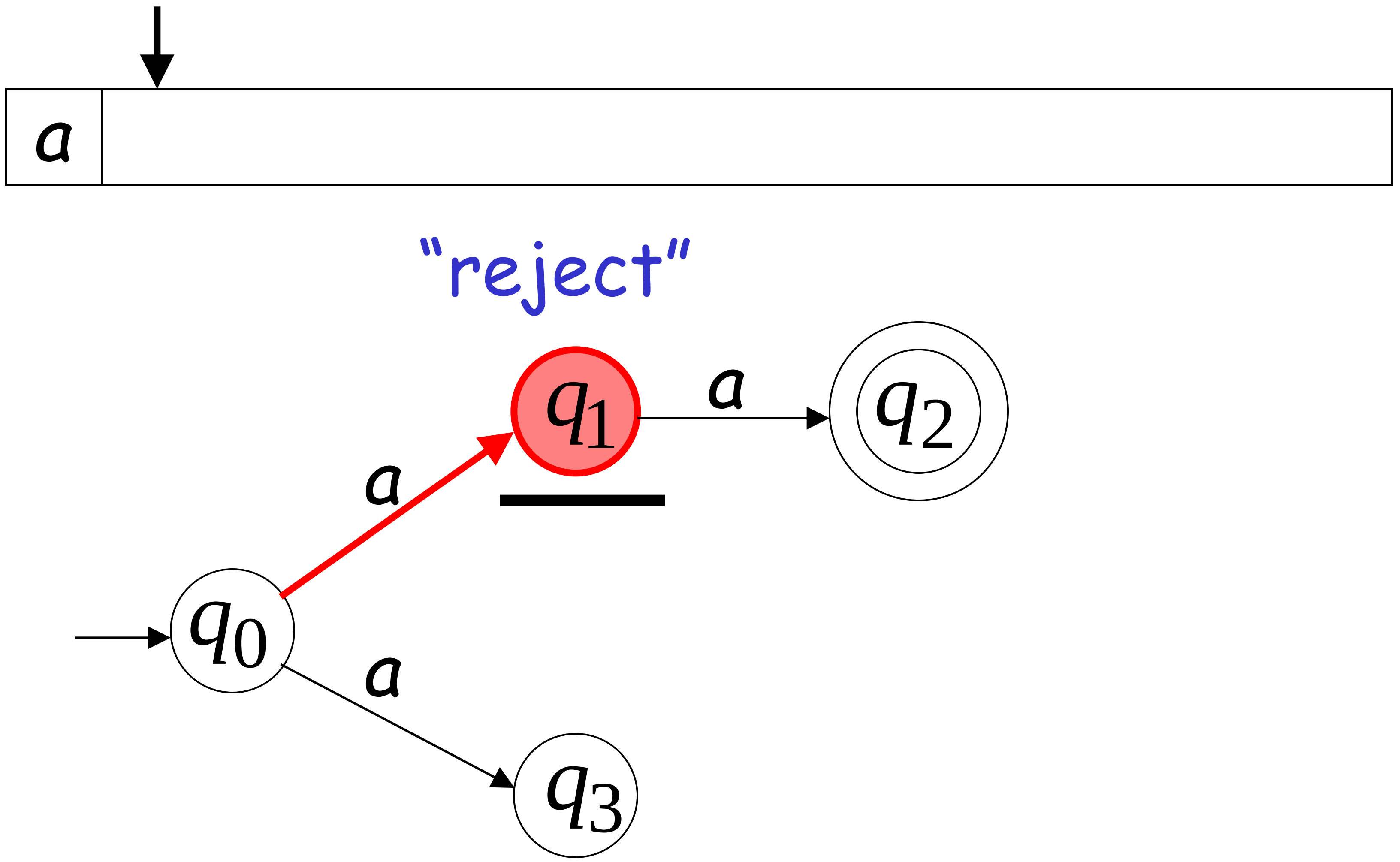
Rejection example



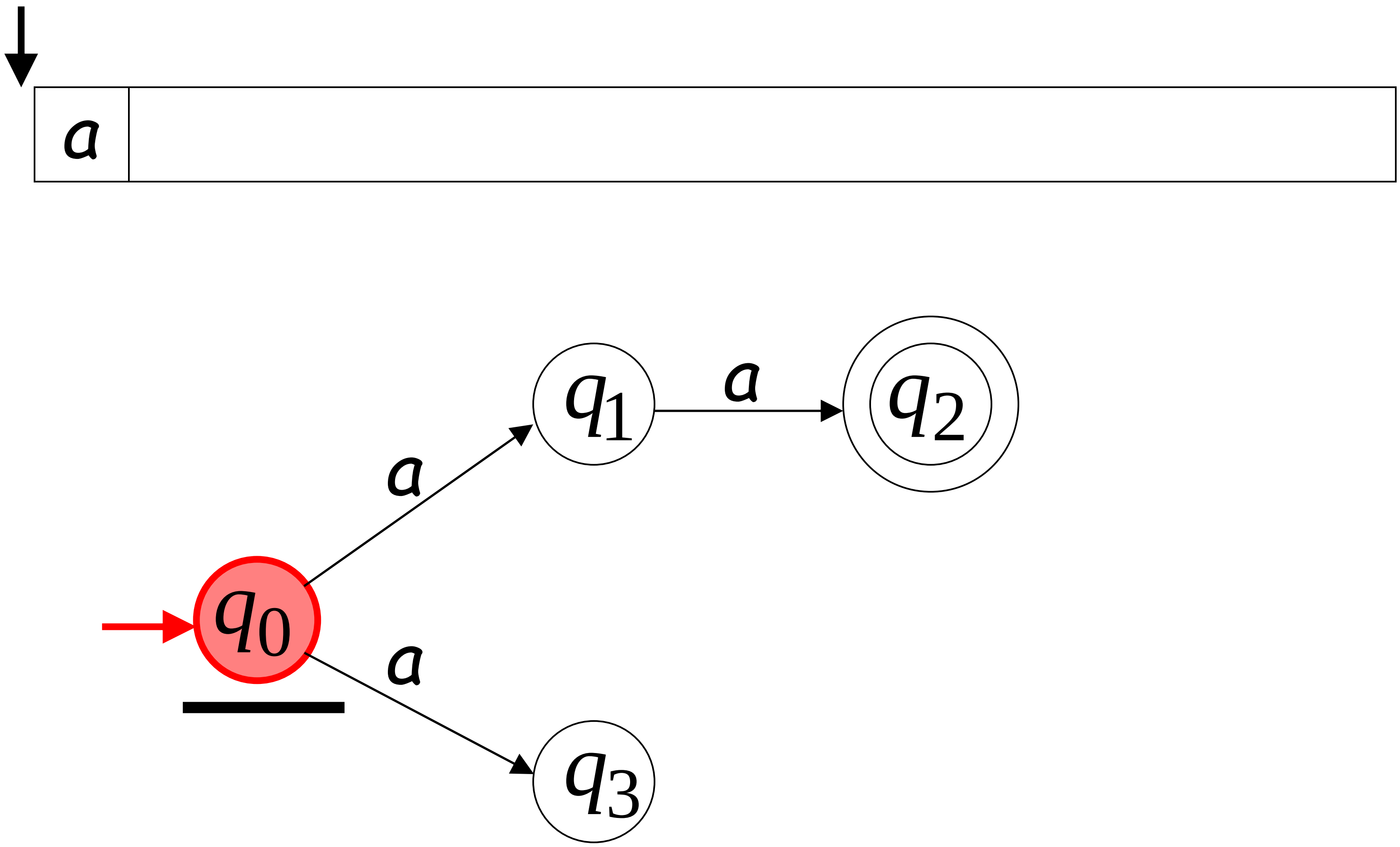
First Choice



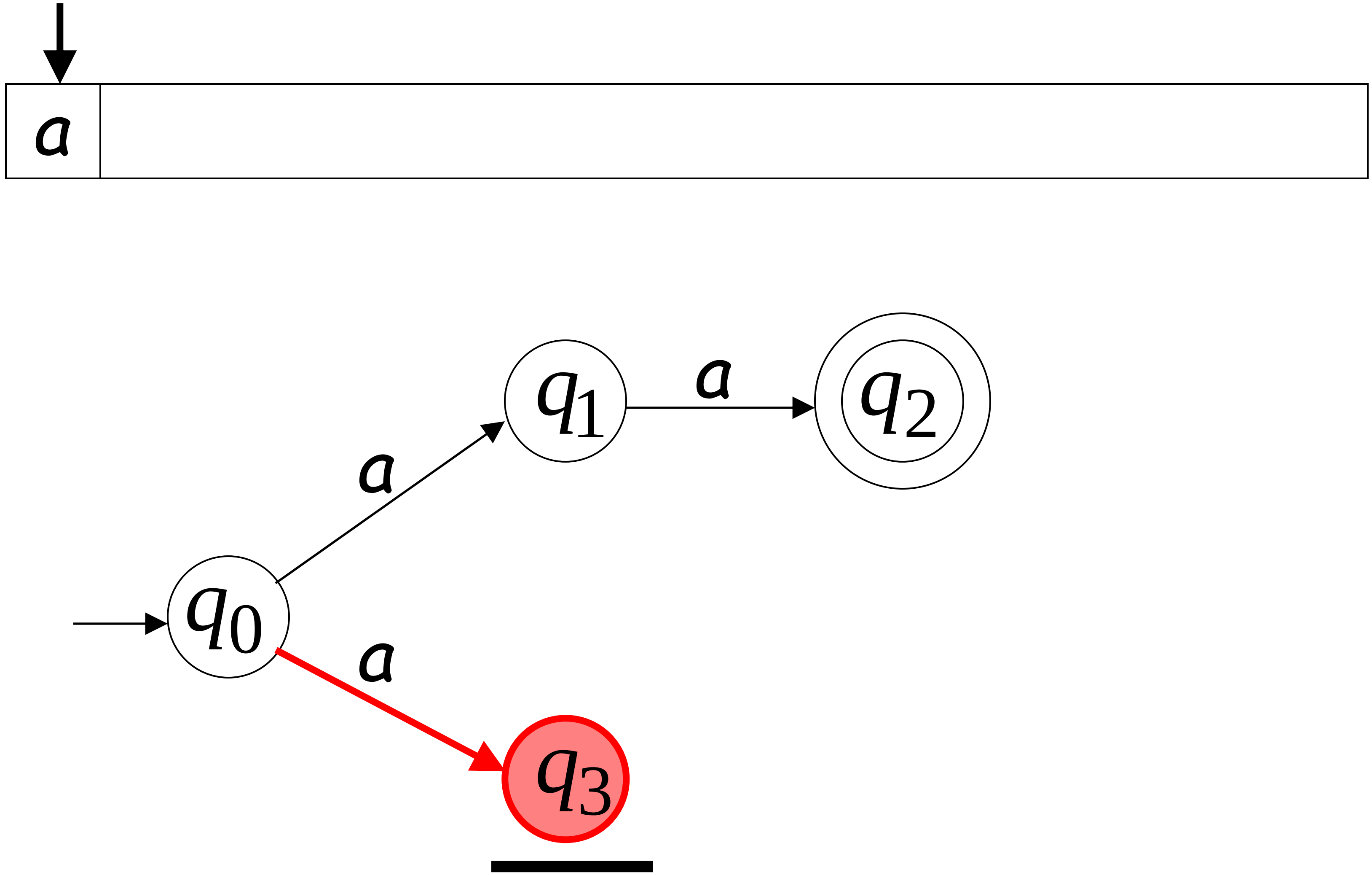
First Choice



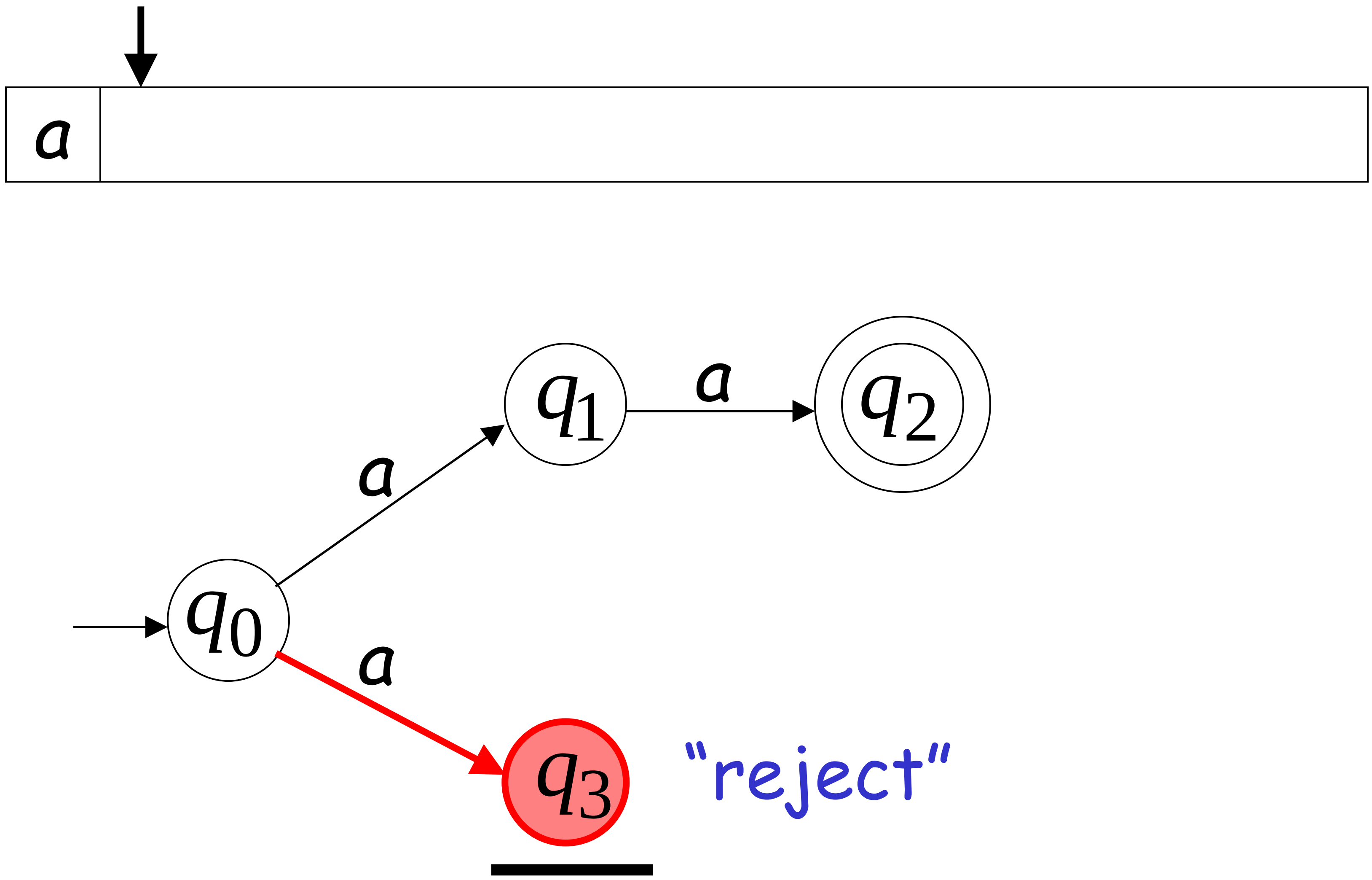
Second Choice



Second Choice



Second Choice



An NFA rejects a string:

when there is no computation of the NFA that accepts the string.

For each computation:

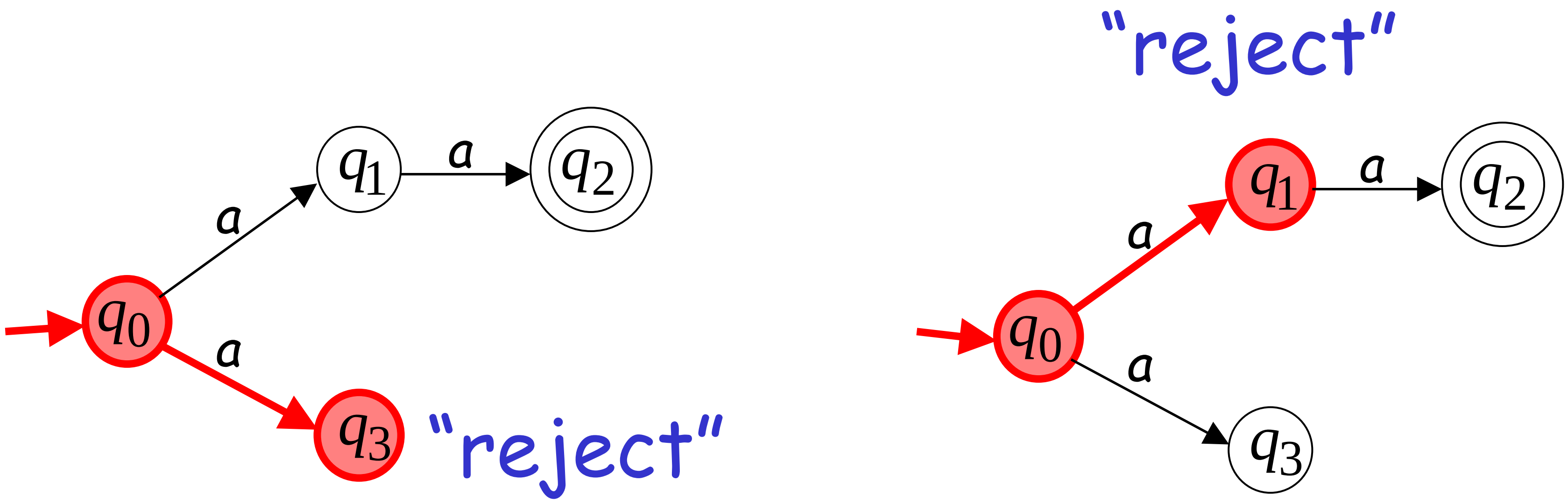
- All the input is consumed and the automaton is in a non final state

OR

- The input cannot be consumed

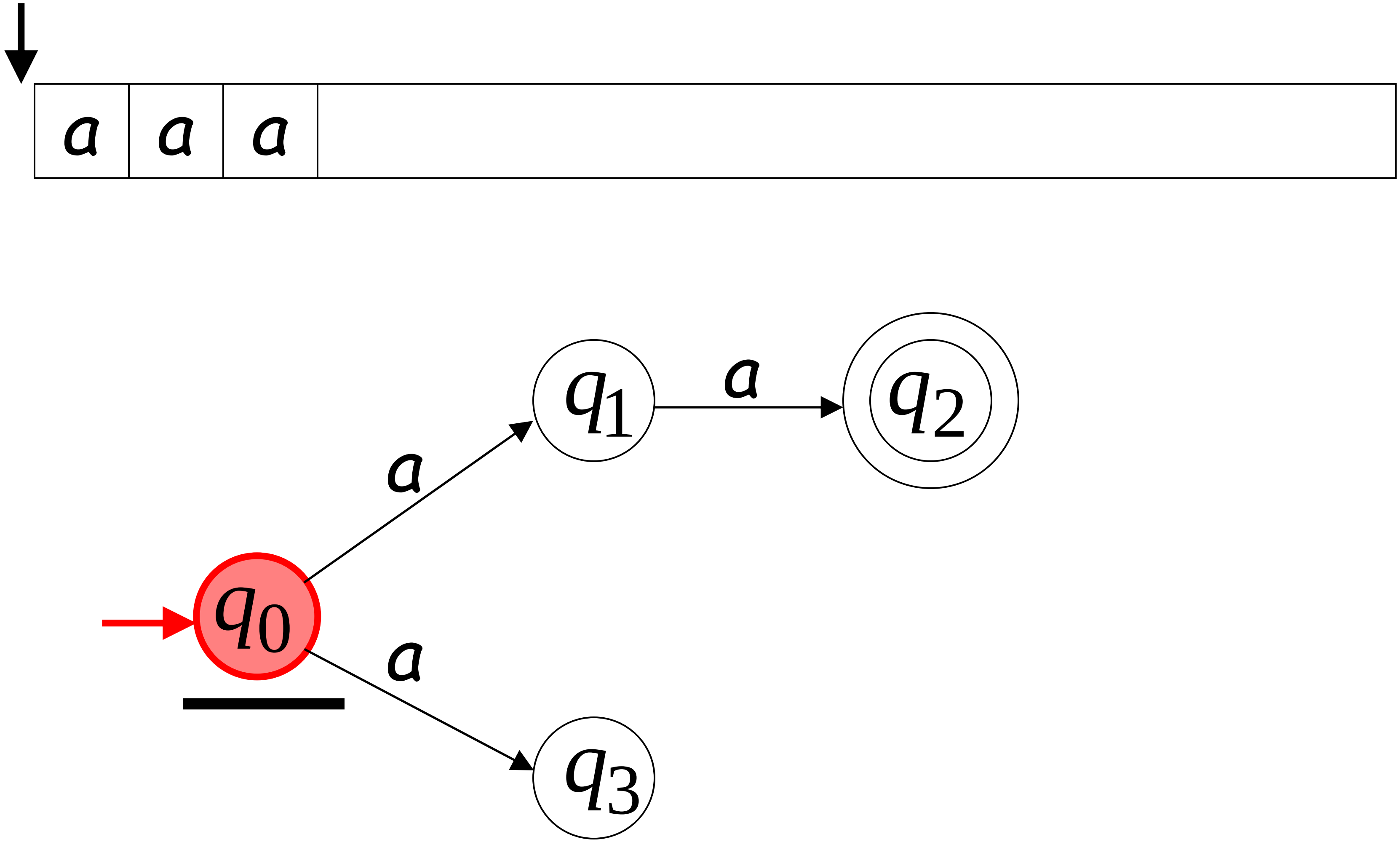
Example

a is rejected by the NFA:

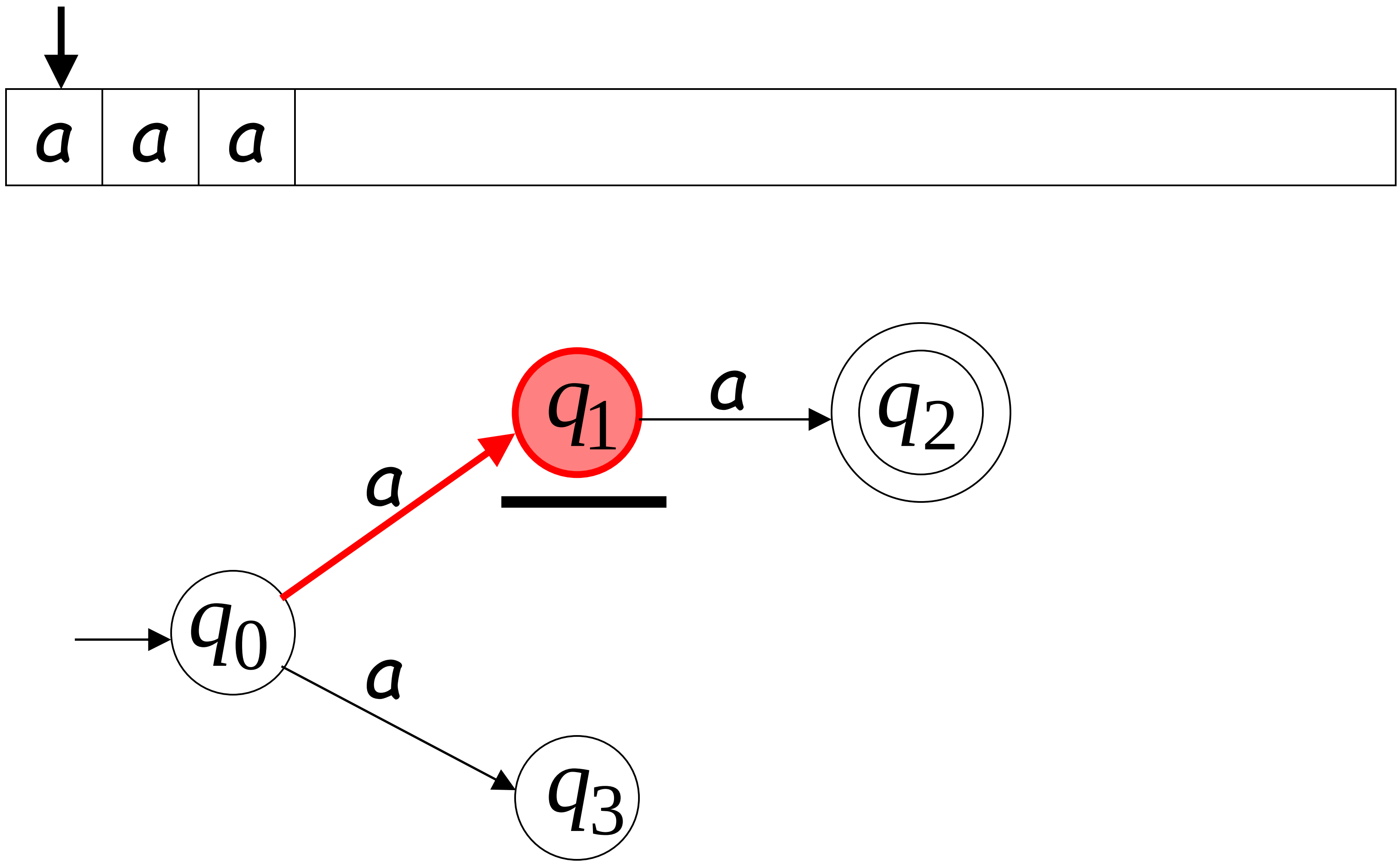


All possible computations lead to rejection

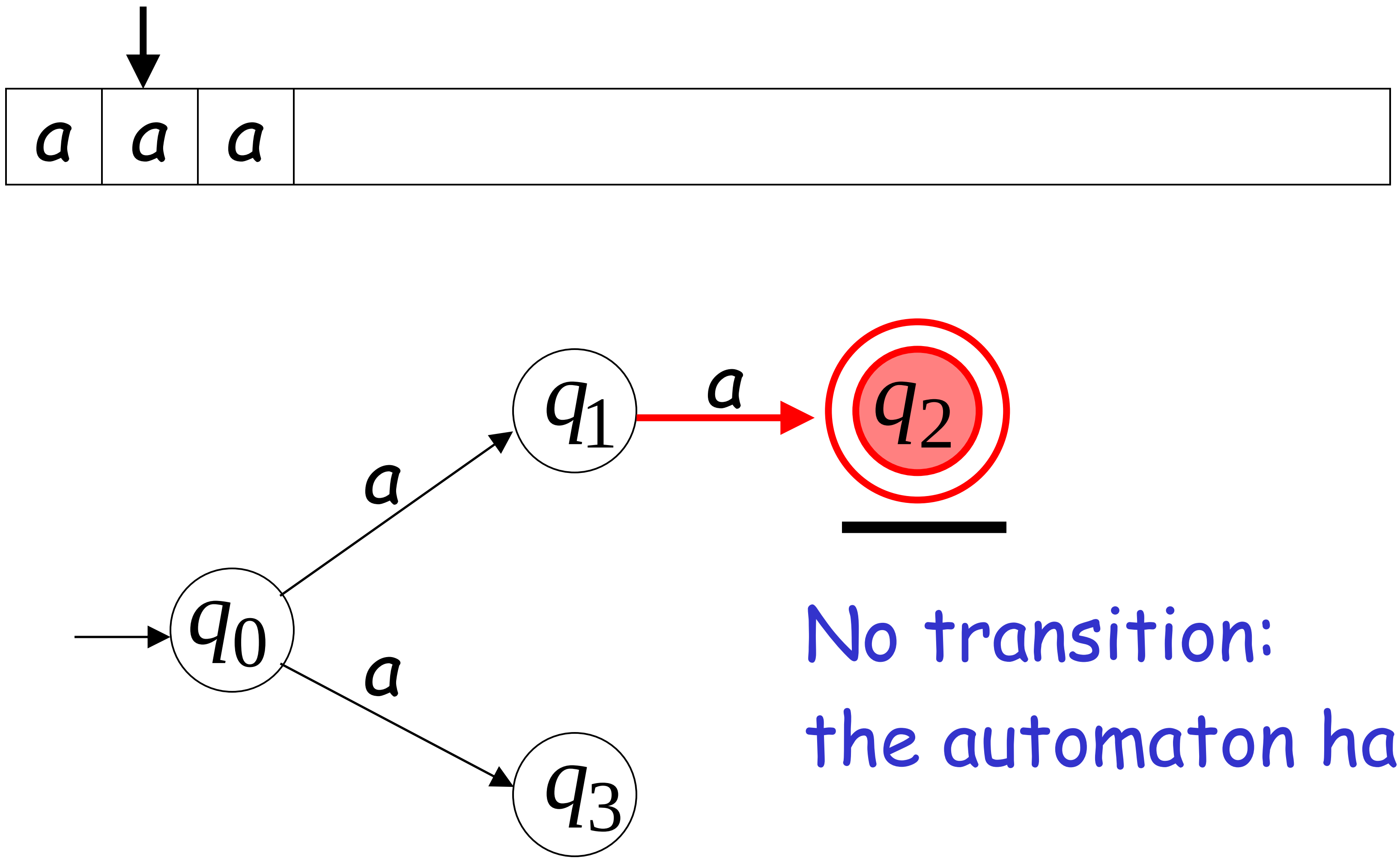
Rejection example



First Choice

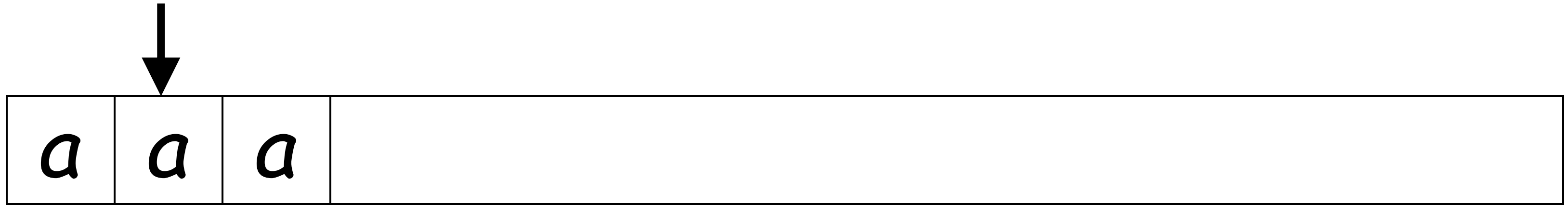


First Choice

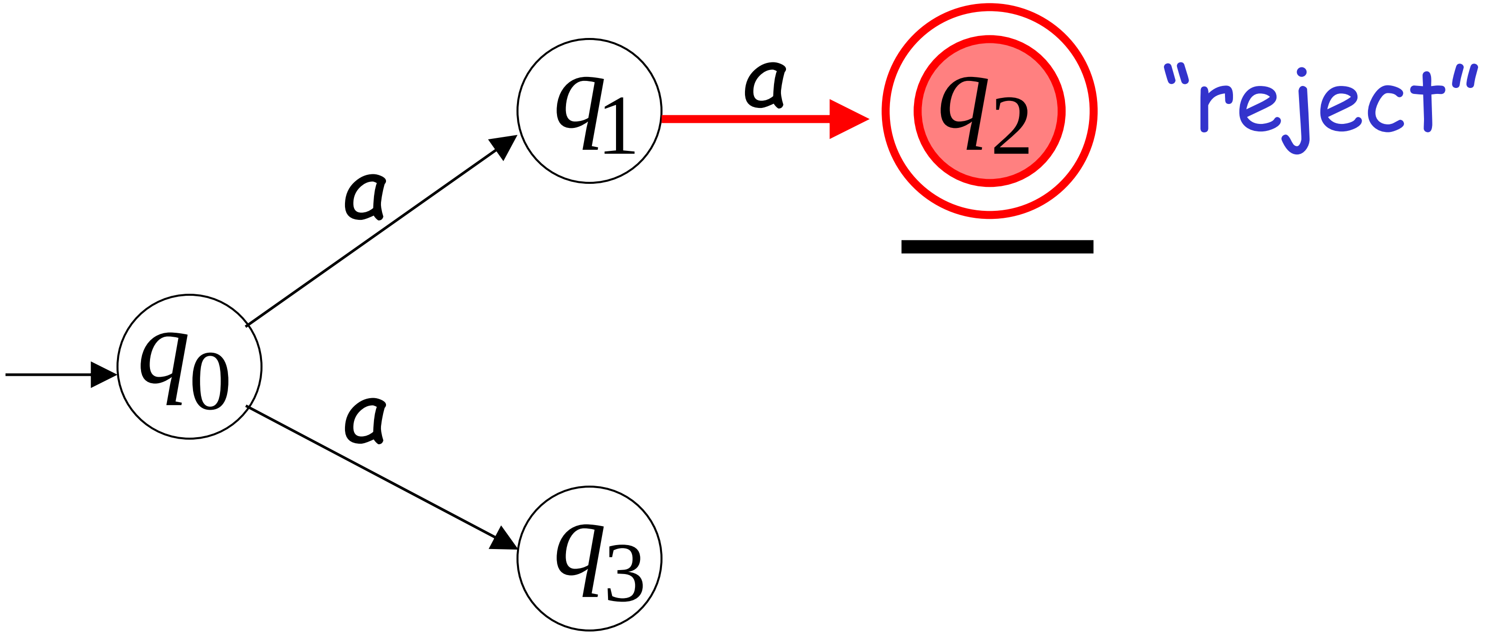


No transition:
the automaton hangs

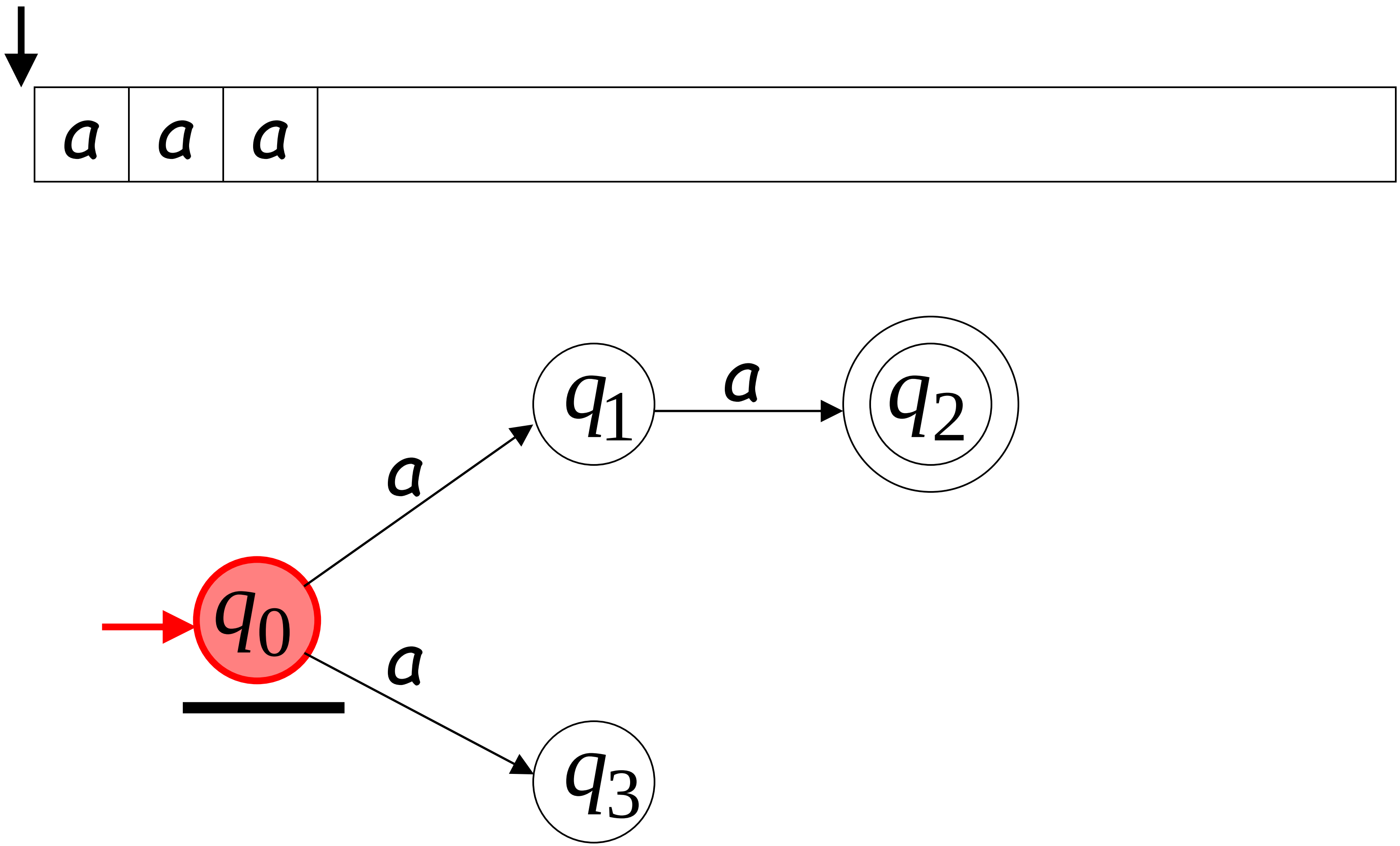
First Choice



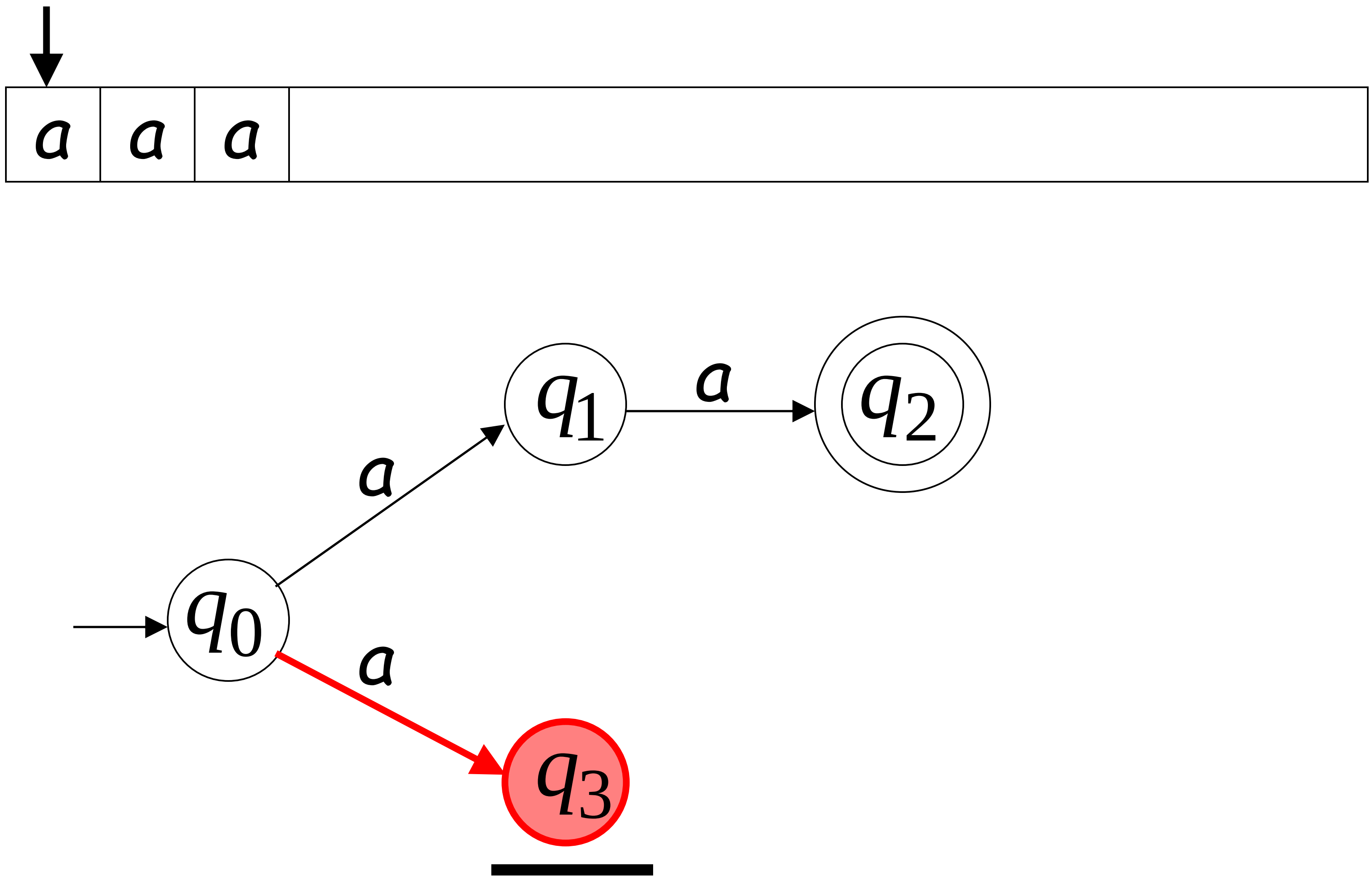
Input cannot be consumed



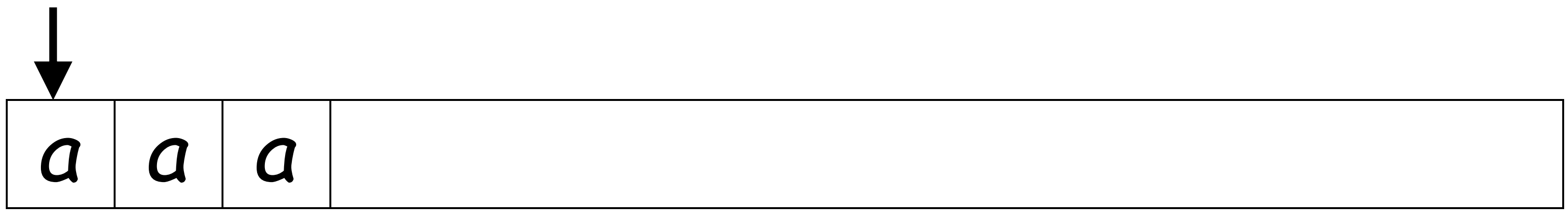
Second Choice

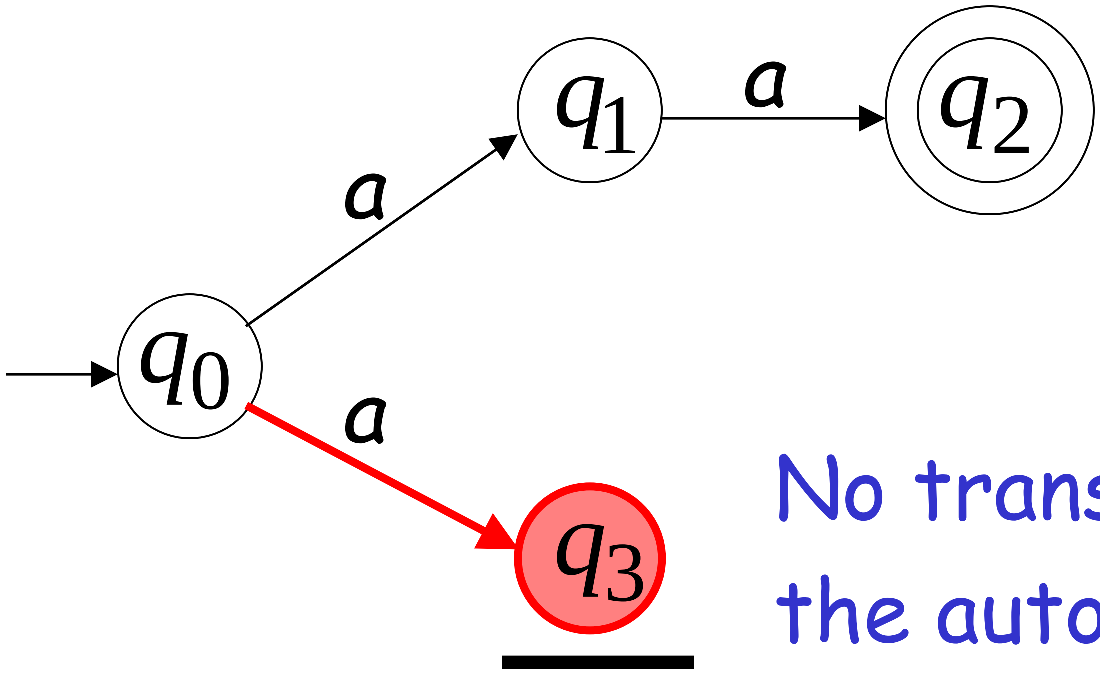


Second Choice



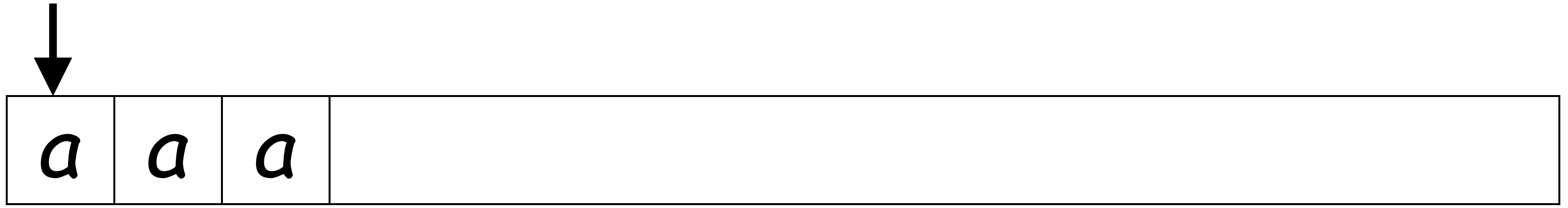
Second Choice



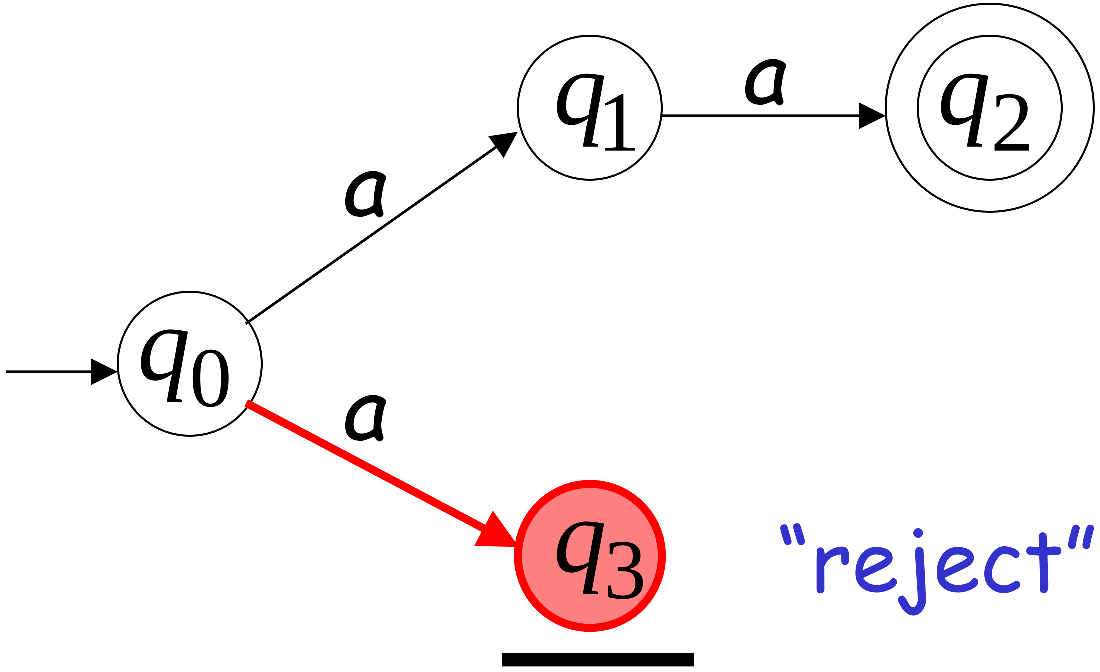


No transition:
the automaton hangs

Second Choice

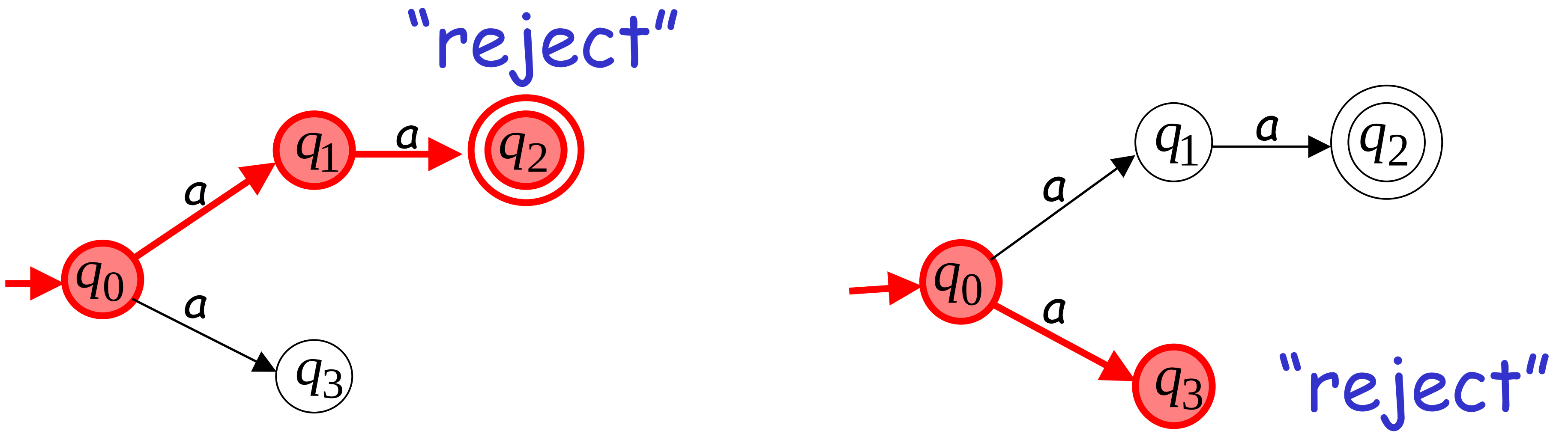


Input cannot be consumed



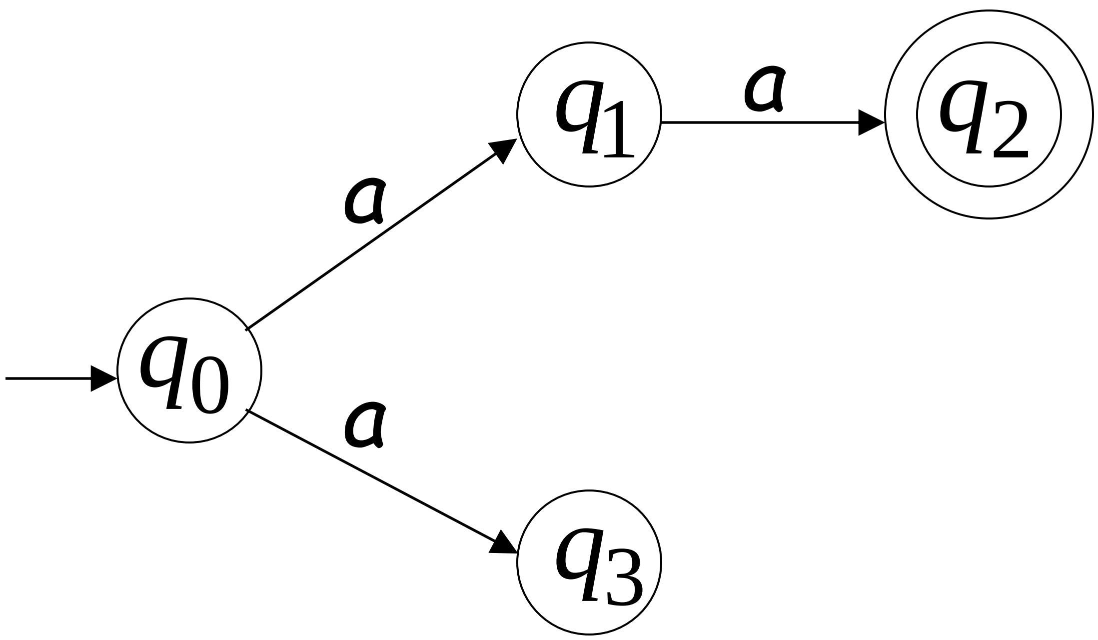
"reject"

aaa is rejected by the NFA:

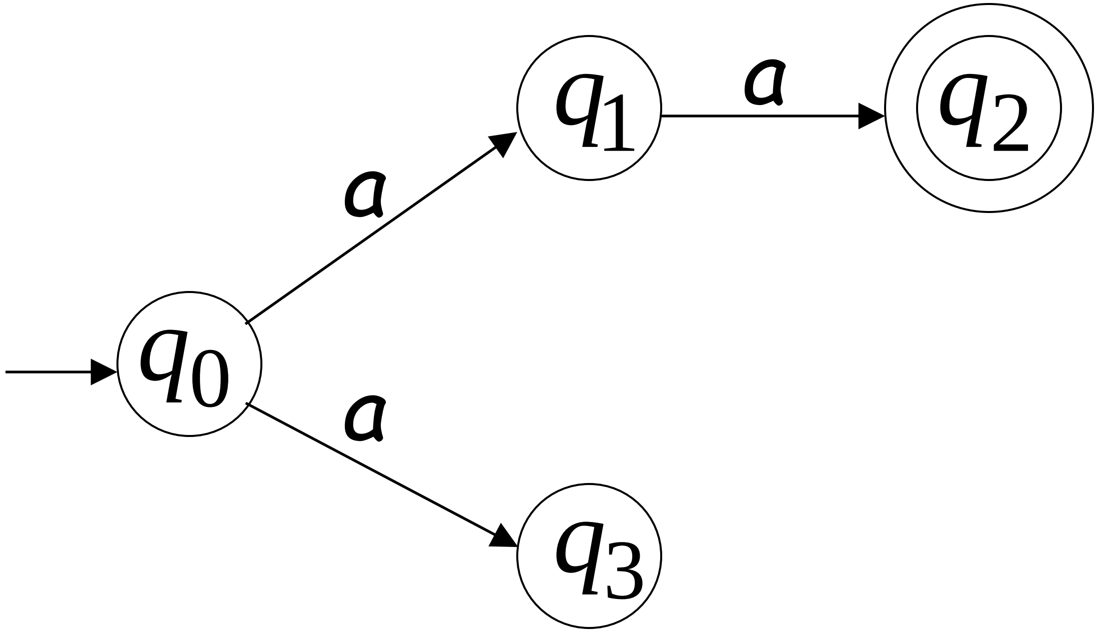


All possible computations lead to rejection

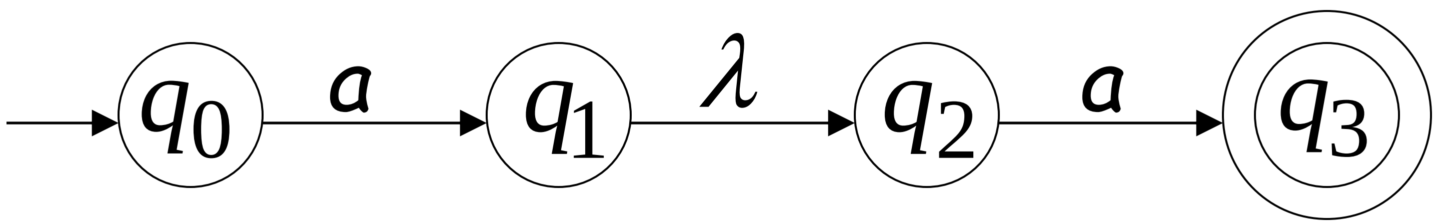
L(M)?

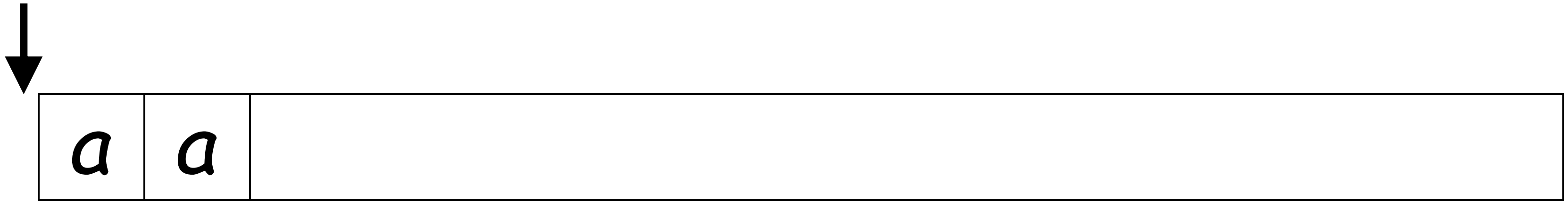


Language accepted: $L = \{aa\}$

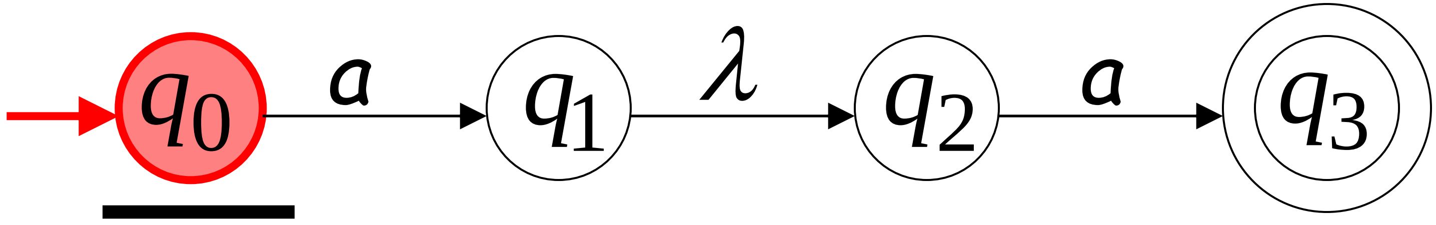


Lambda Transitions

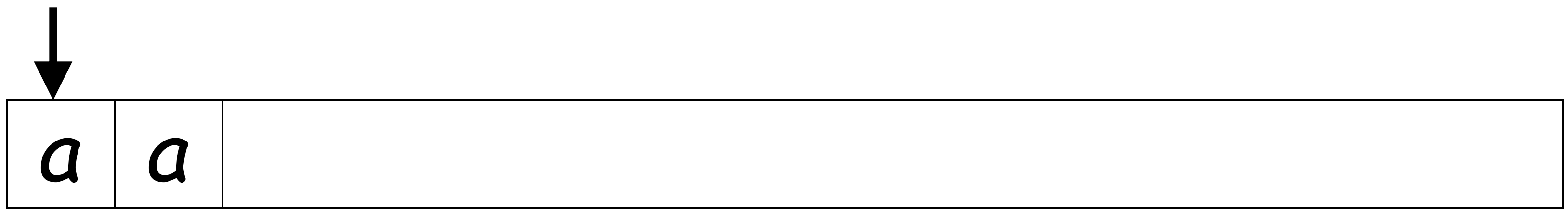




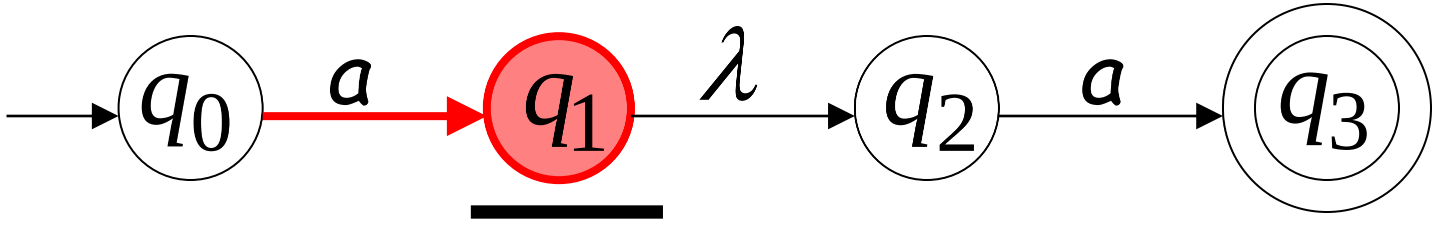
a
a



q_0
a
q_1
λ
q_2
a
q_3

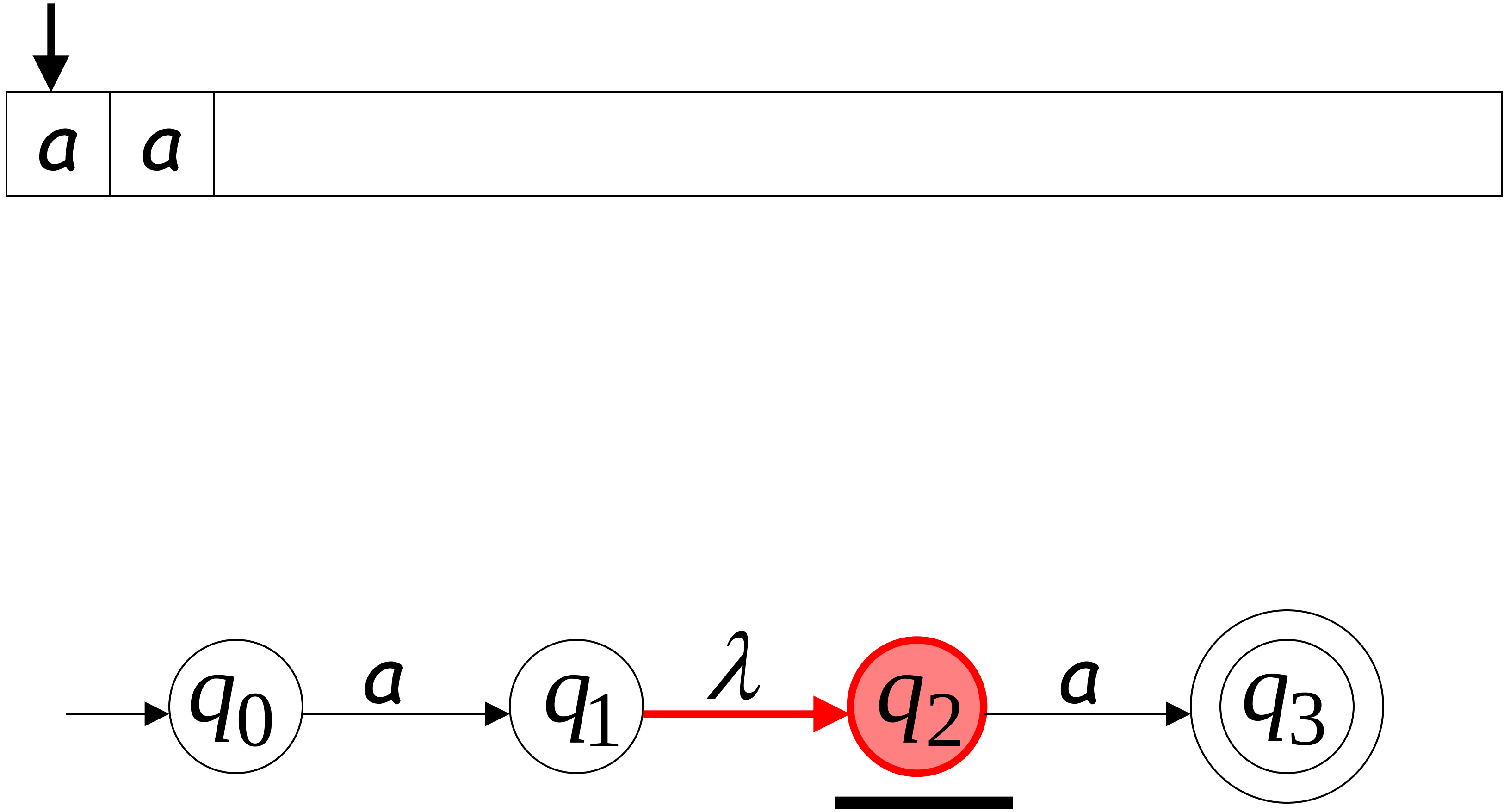


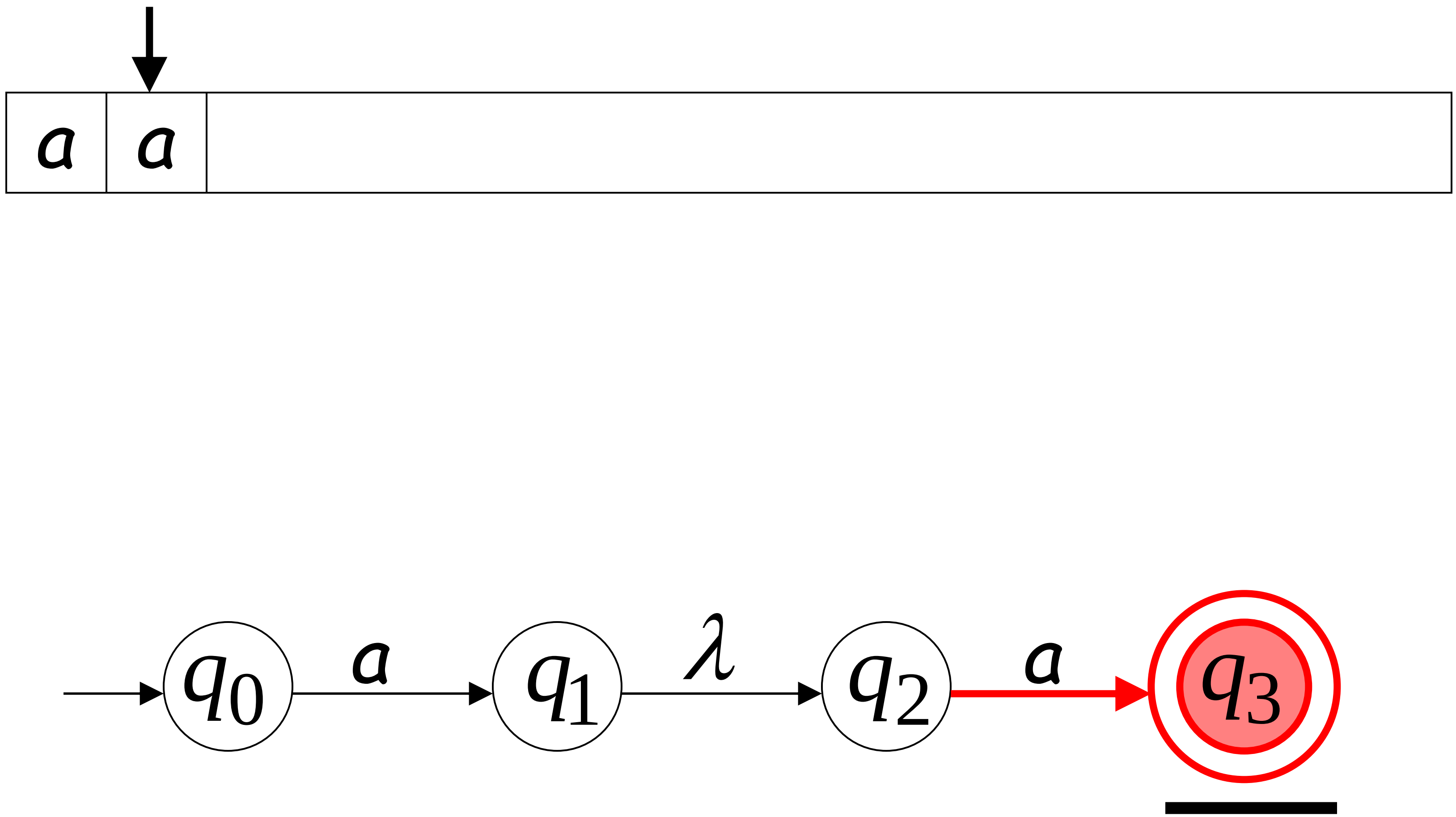
a
a



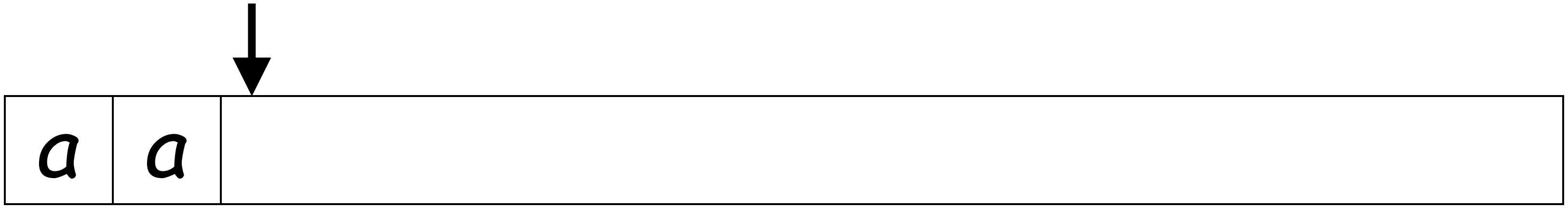
q_0
a
q_1
λ
q_2
a
q_3

(read head does not move)

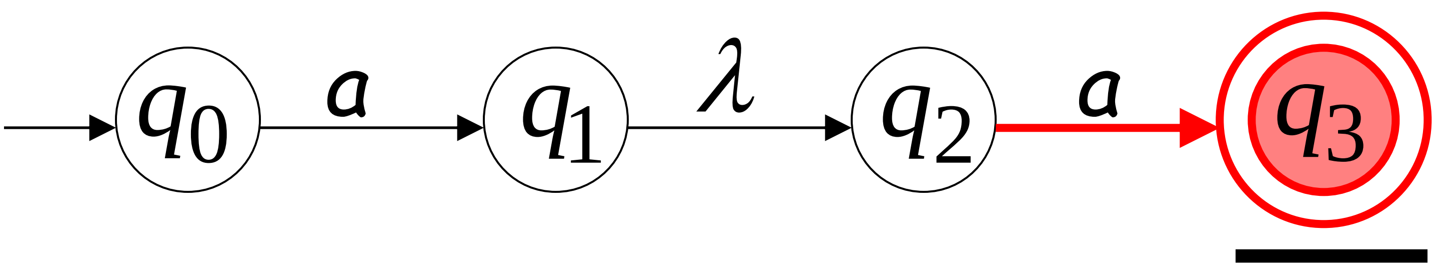



a
a
q0
a
q1
λ
q2
a
q3

all input is consumed

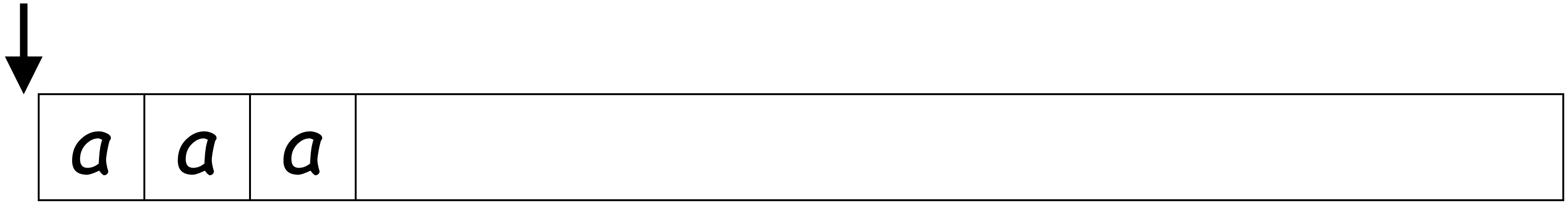


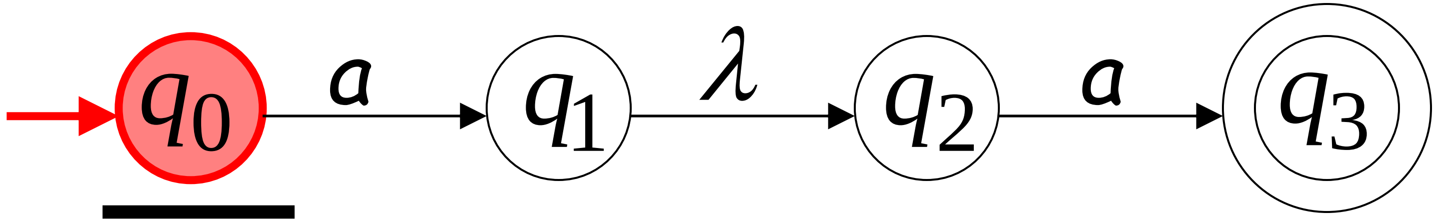
"accept"

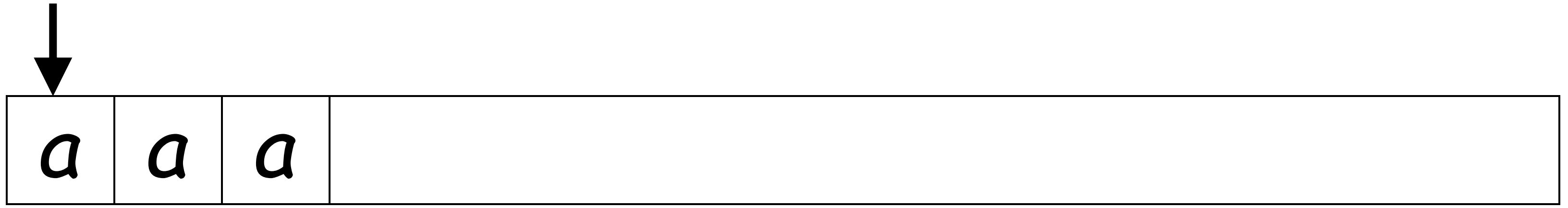


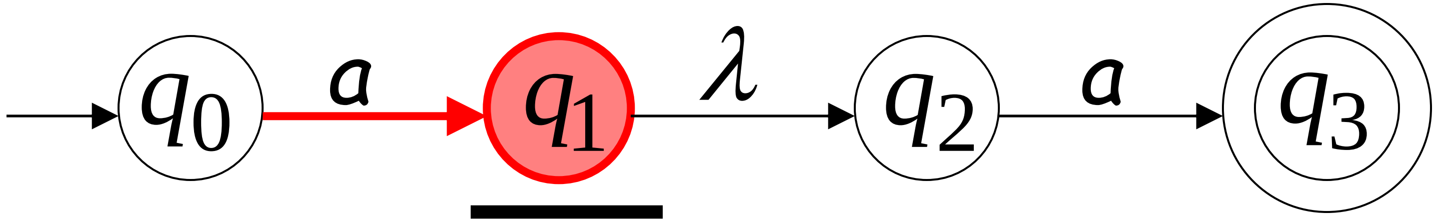
String aa is accepted

Rejection Example

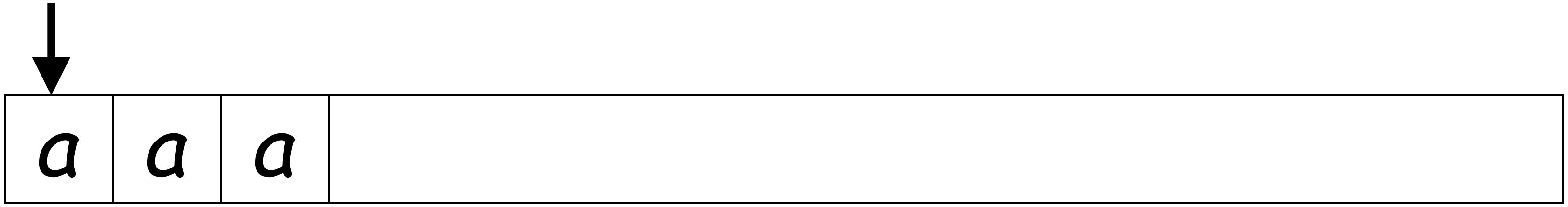


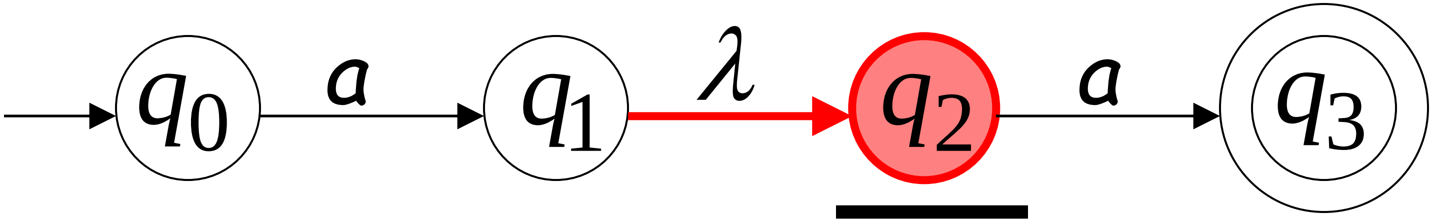


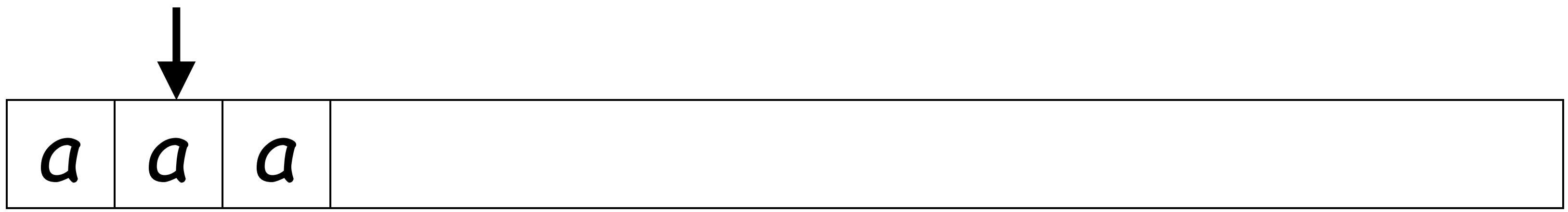

a a a

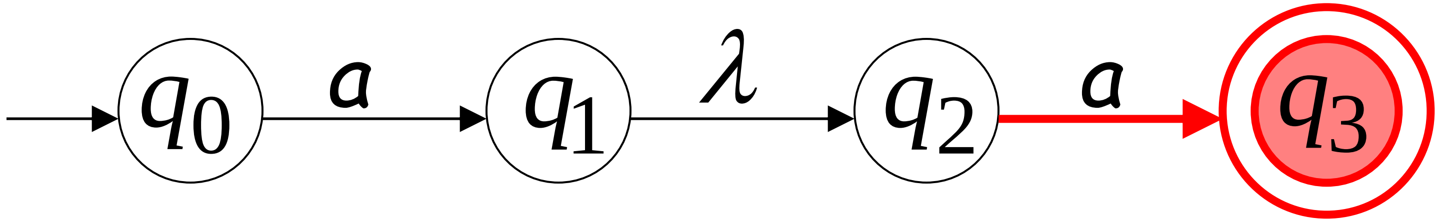

q_0
a
q_1
λ
q_2
a
q_3

(read head doesn't move)



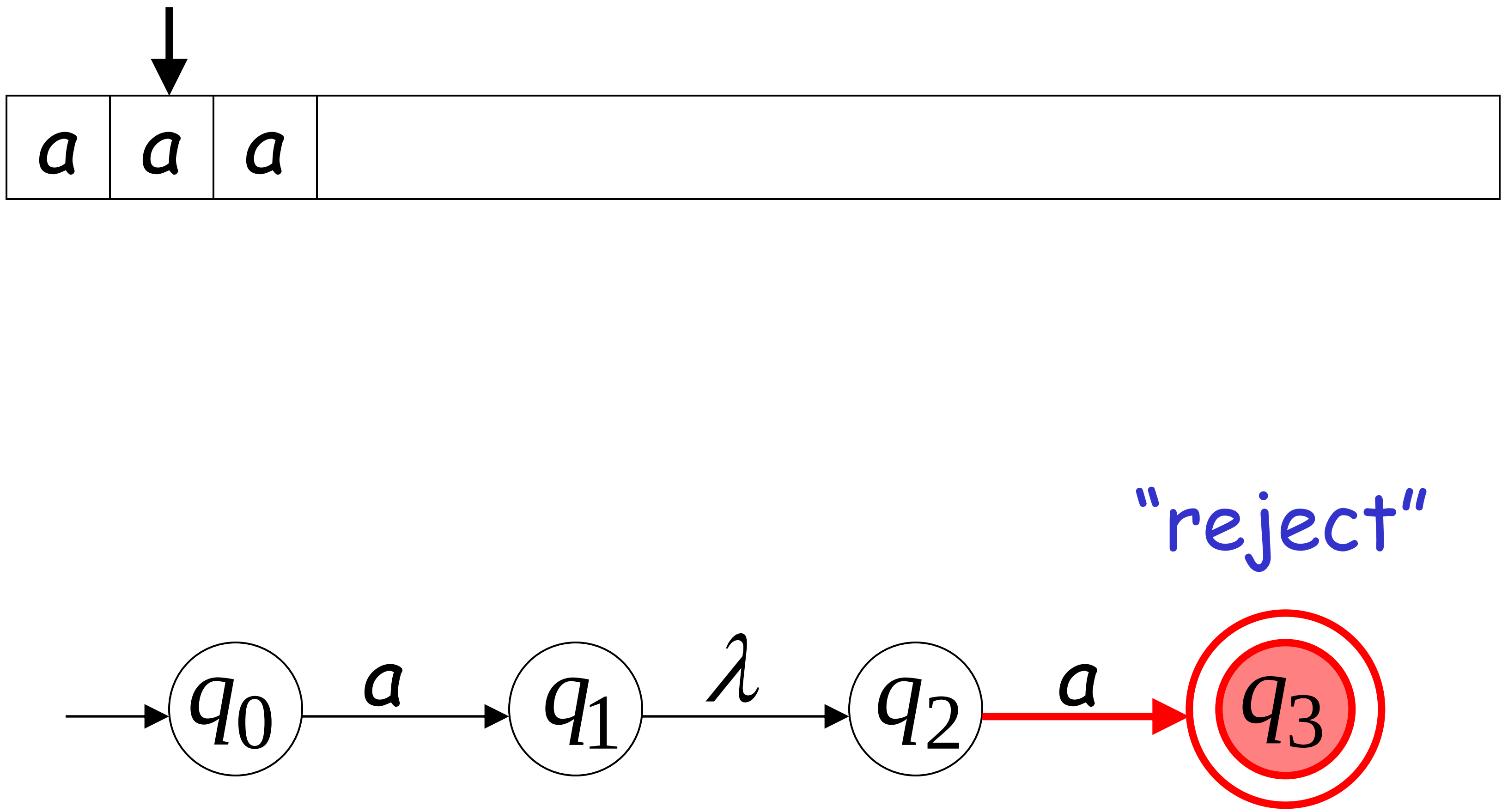






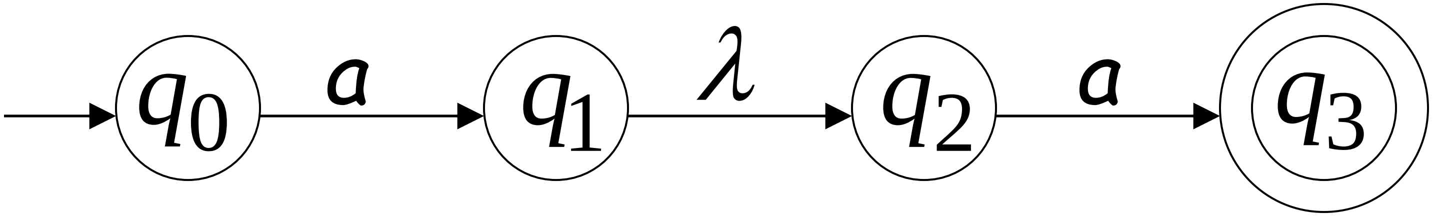
No transition:
the automaton hangs

Input cannot be consumed

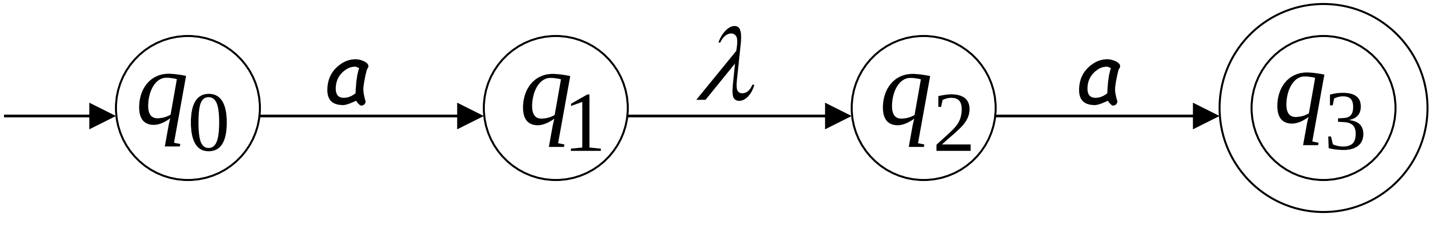


String aaa is rejected

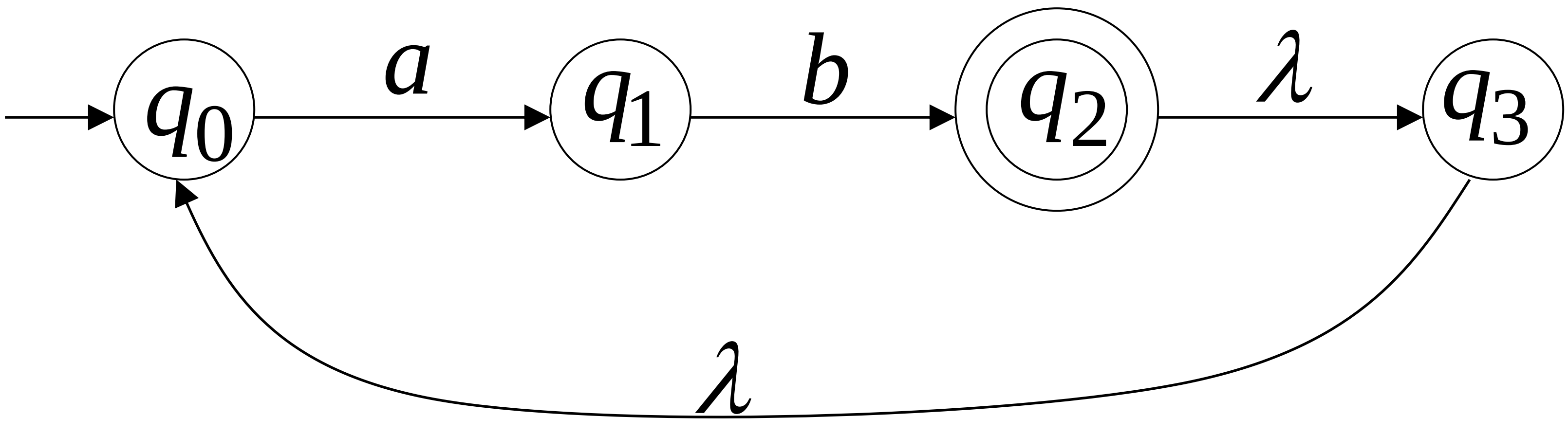
$L(M)$?

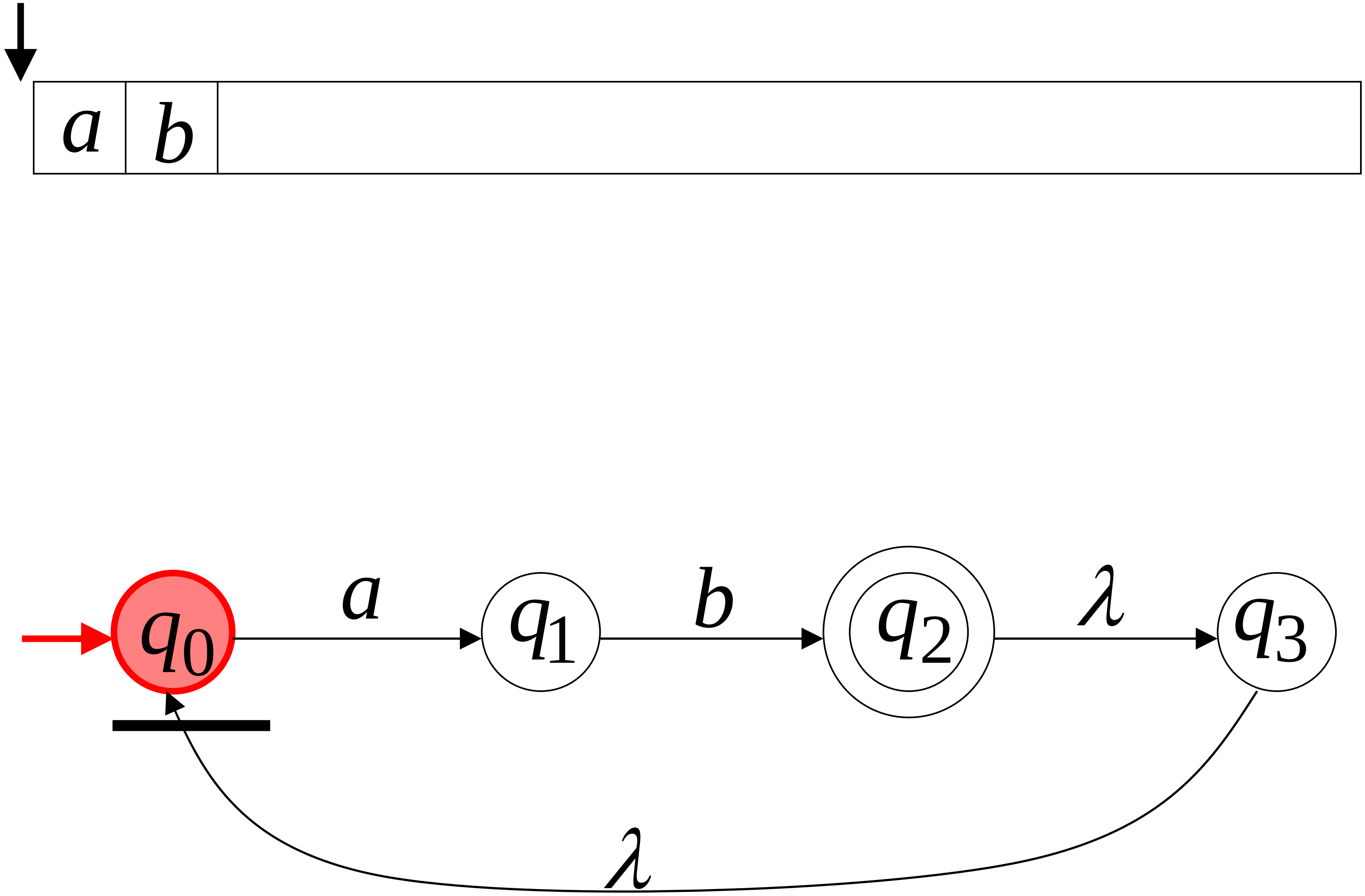


Language accepted: $L = \{aa\}$

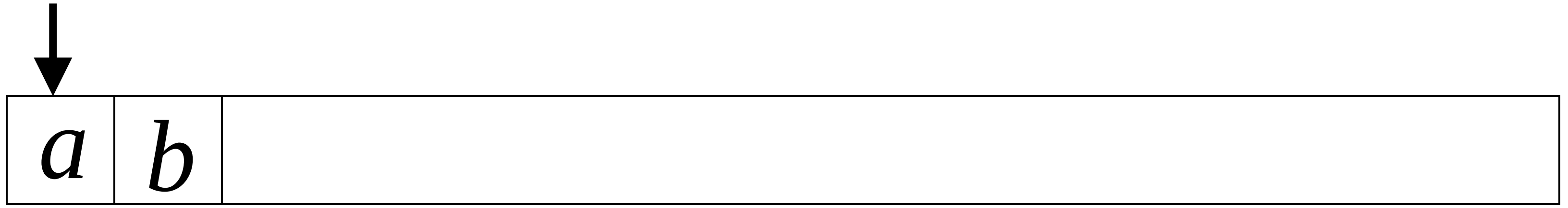


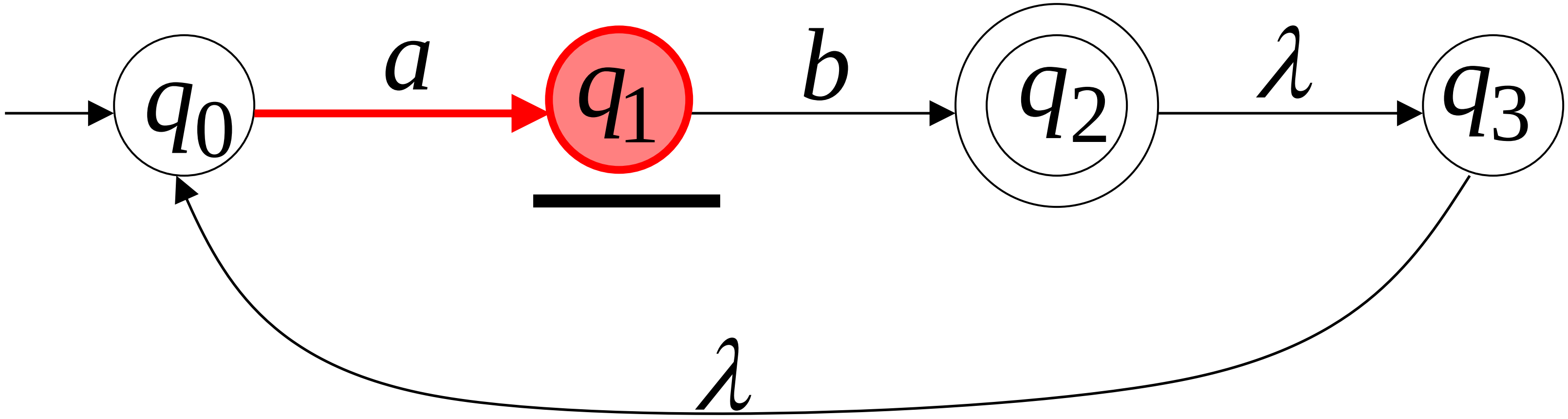
Another NFA Example: L(M)?

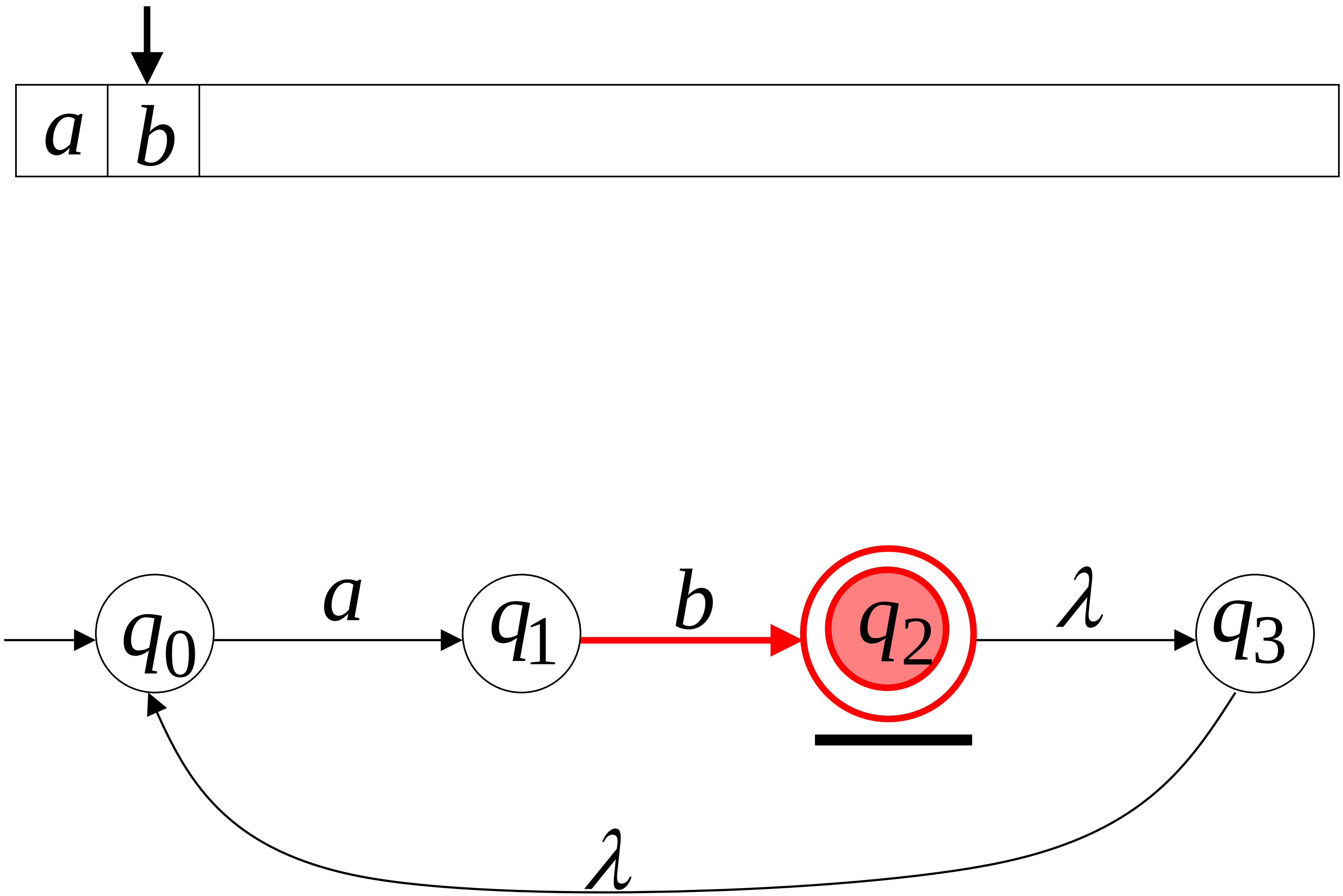




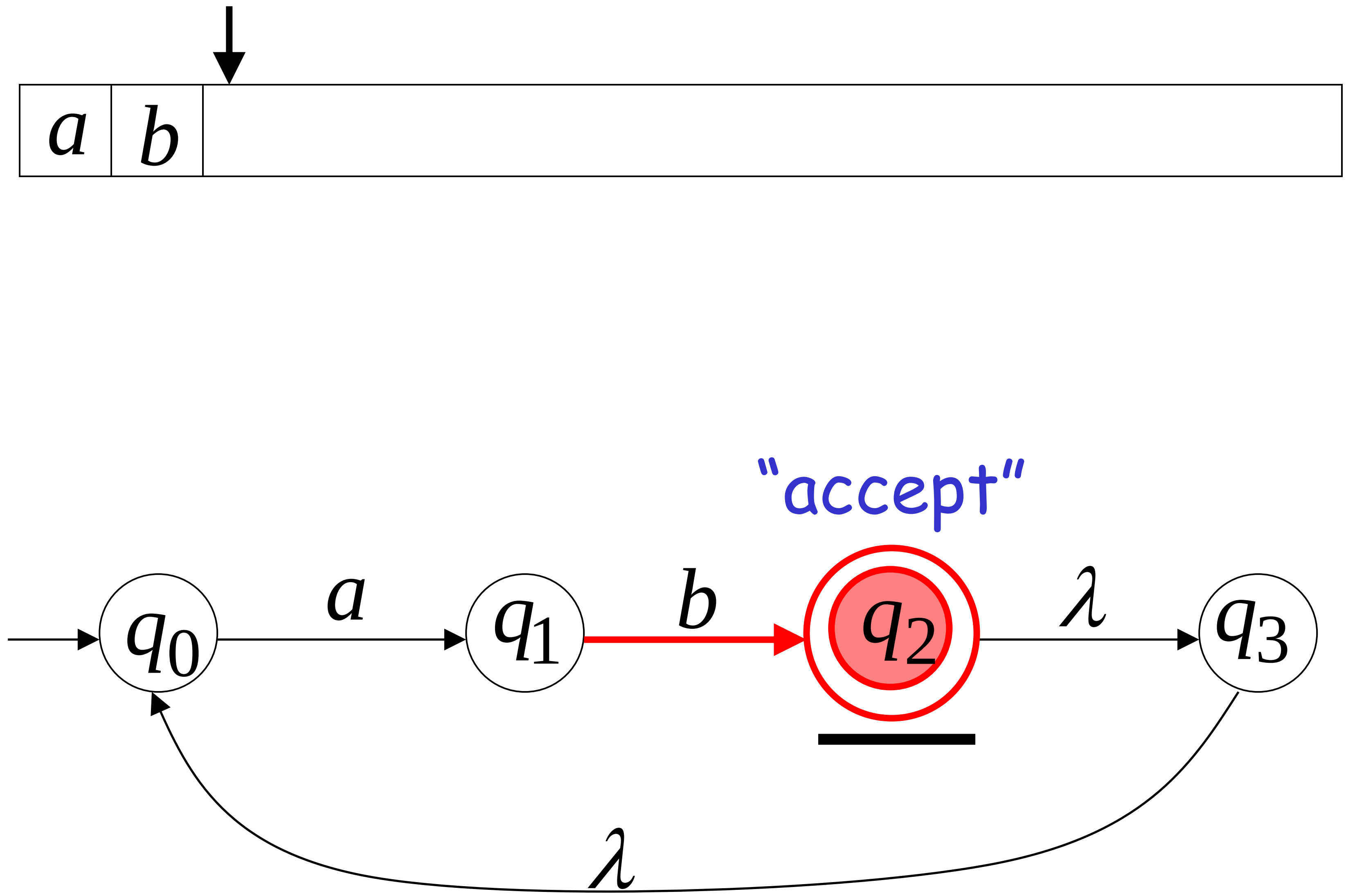
a
b
a
q0
b
q1
q2
λ
q3
λ


a
b
a
b
q0
q1
q2
λ
q3
λ



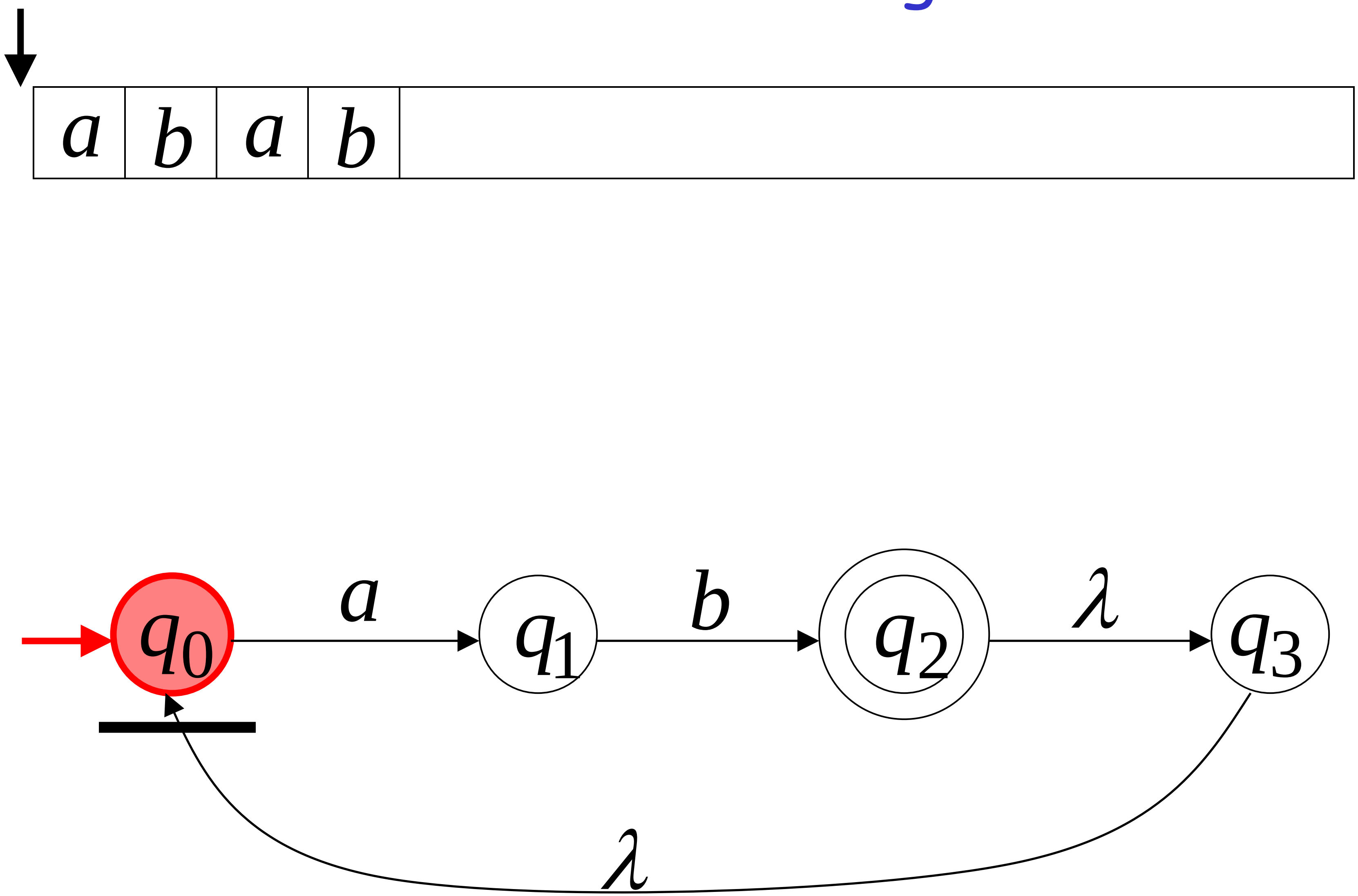


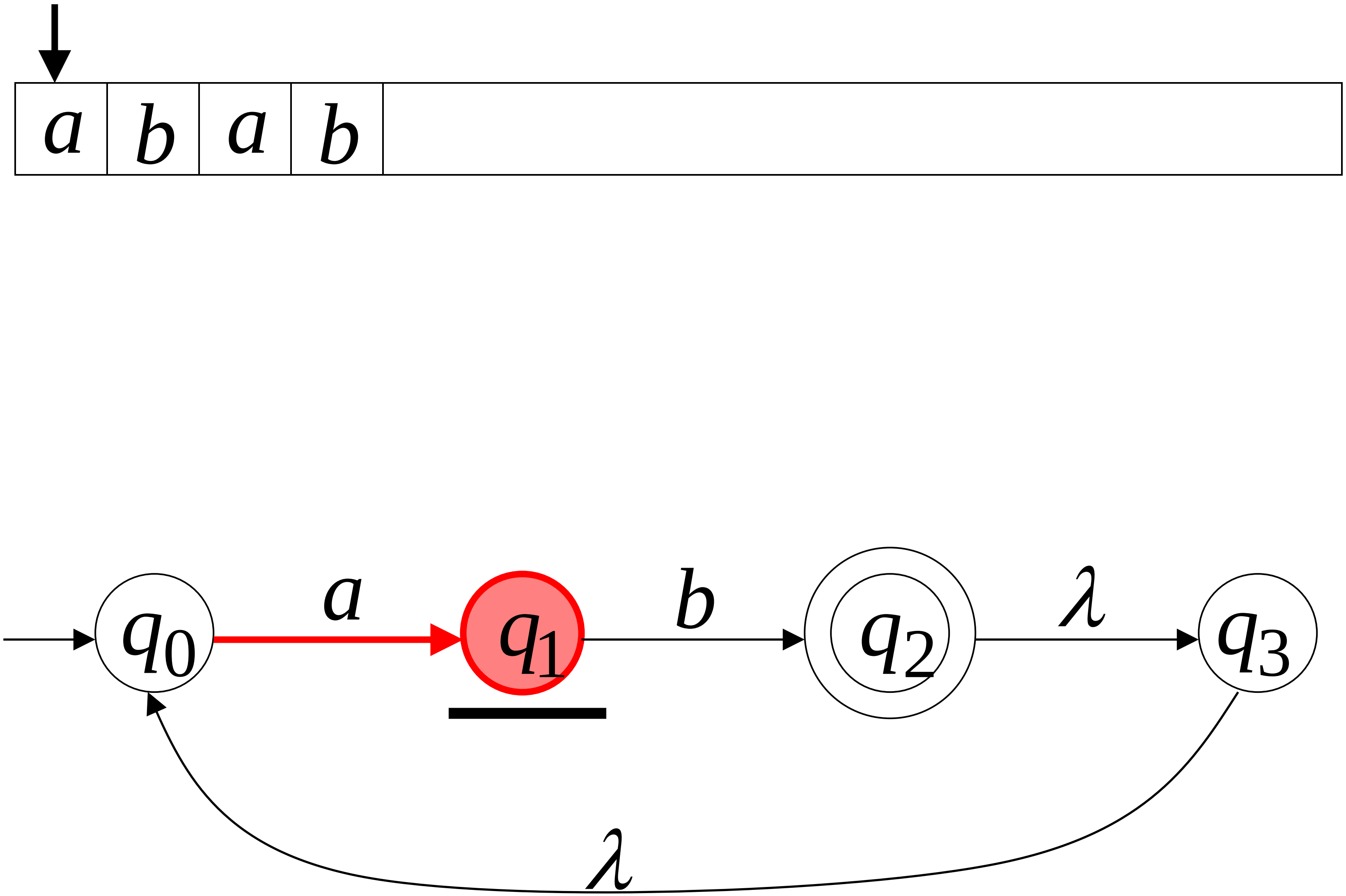
a
b
q_0
a
q_1
b
q_2
λ
q_3
λ

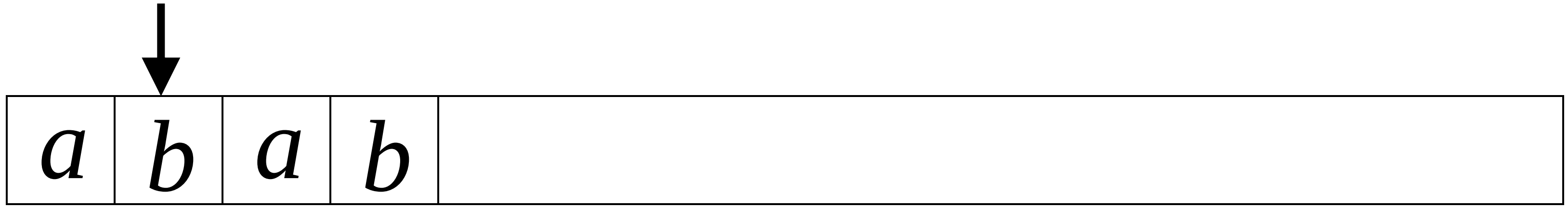


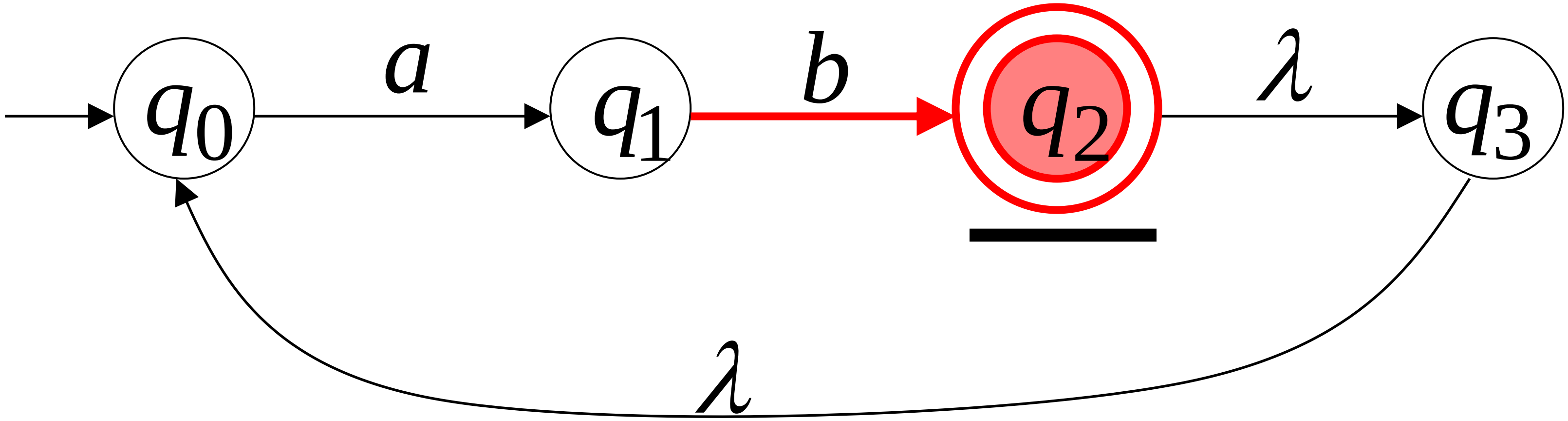
a
b
"accept"
a
b
λ
λ
q_0
q_1
q_2
q_3

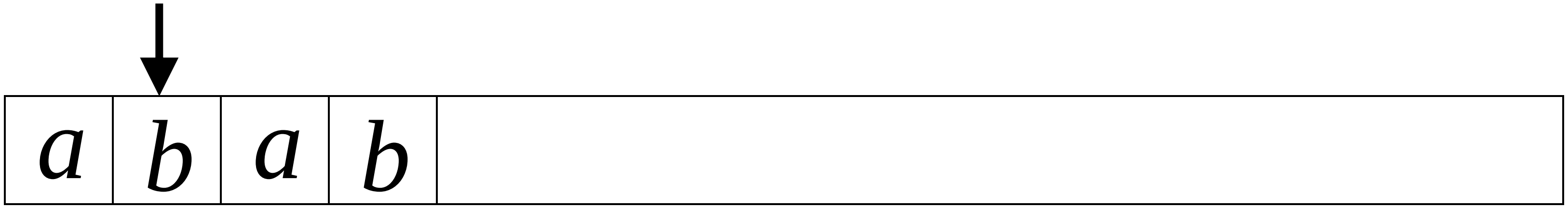
Another String

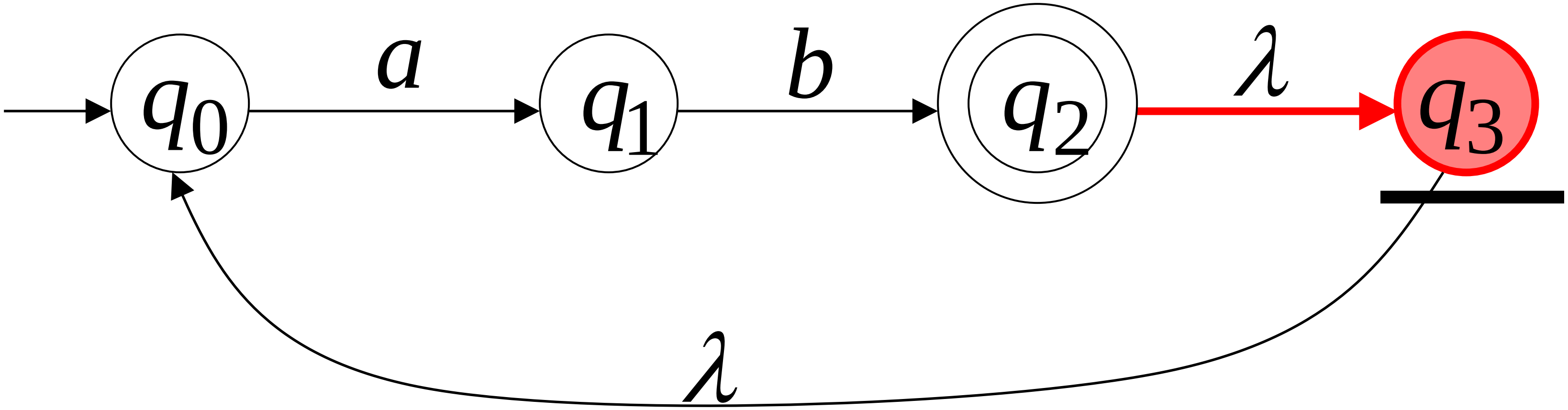


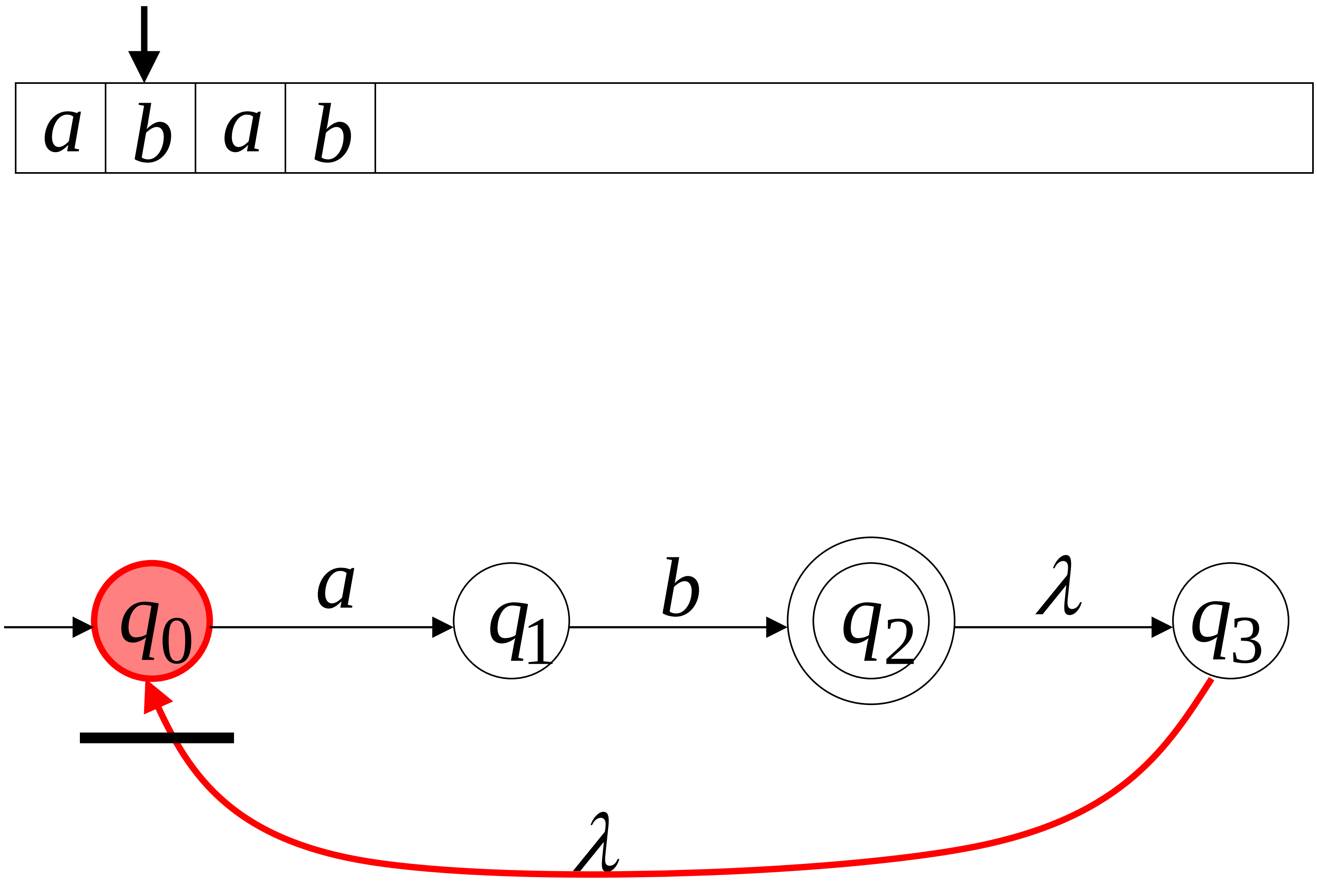

a
b
a
b
q0
a
q1
b
q2
λ
q3
λ

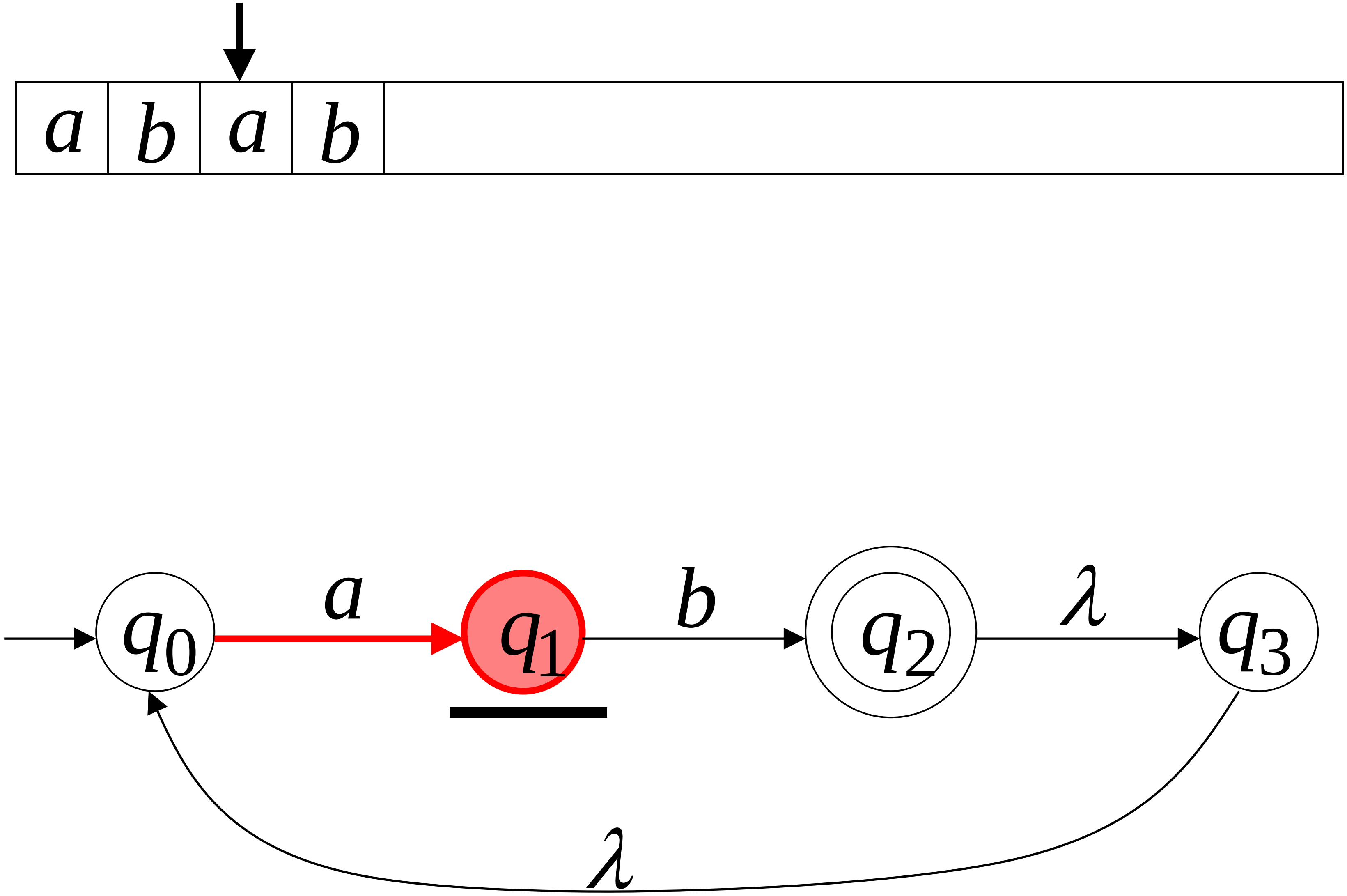

a
b
a
b

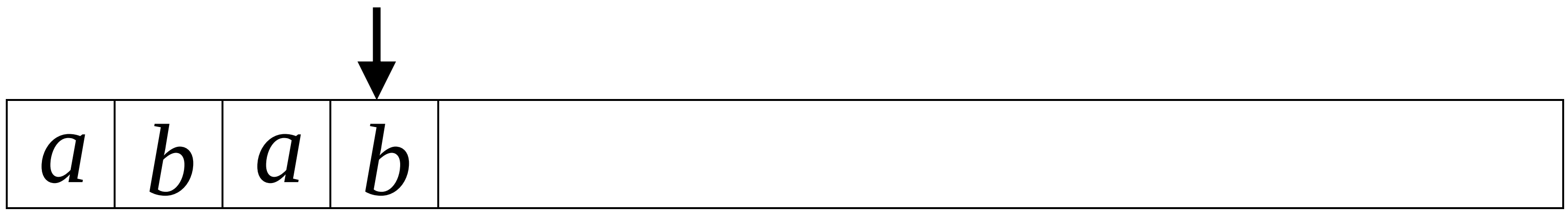

q_0
a
q_1
b
q_2
λ
q_3
λ

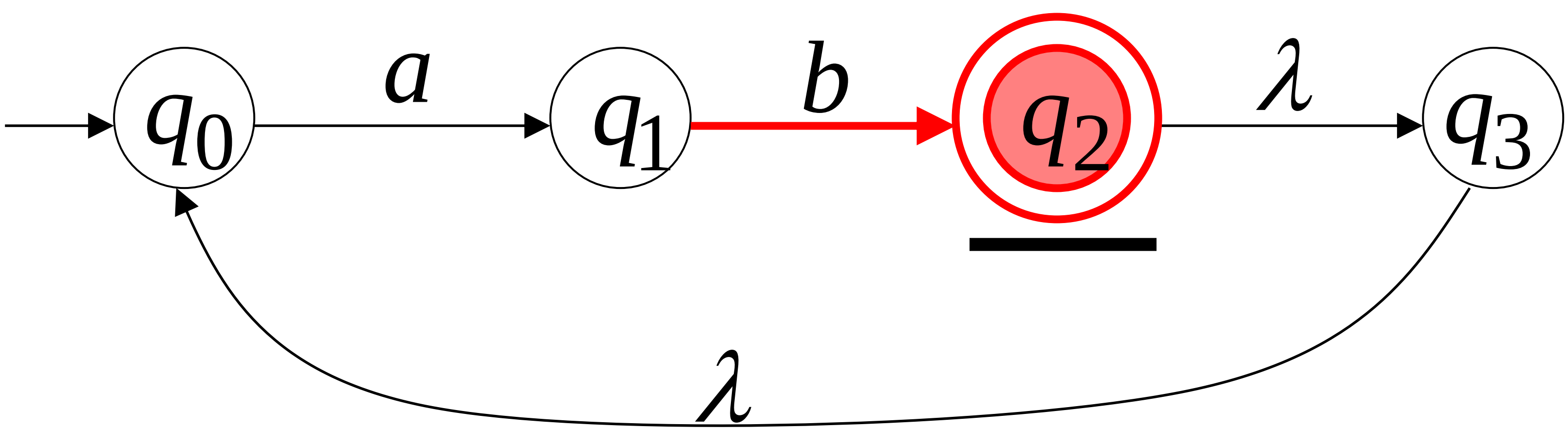

a
b
a
b

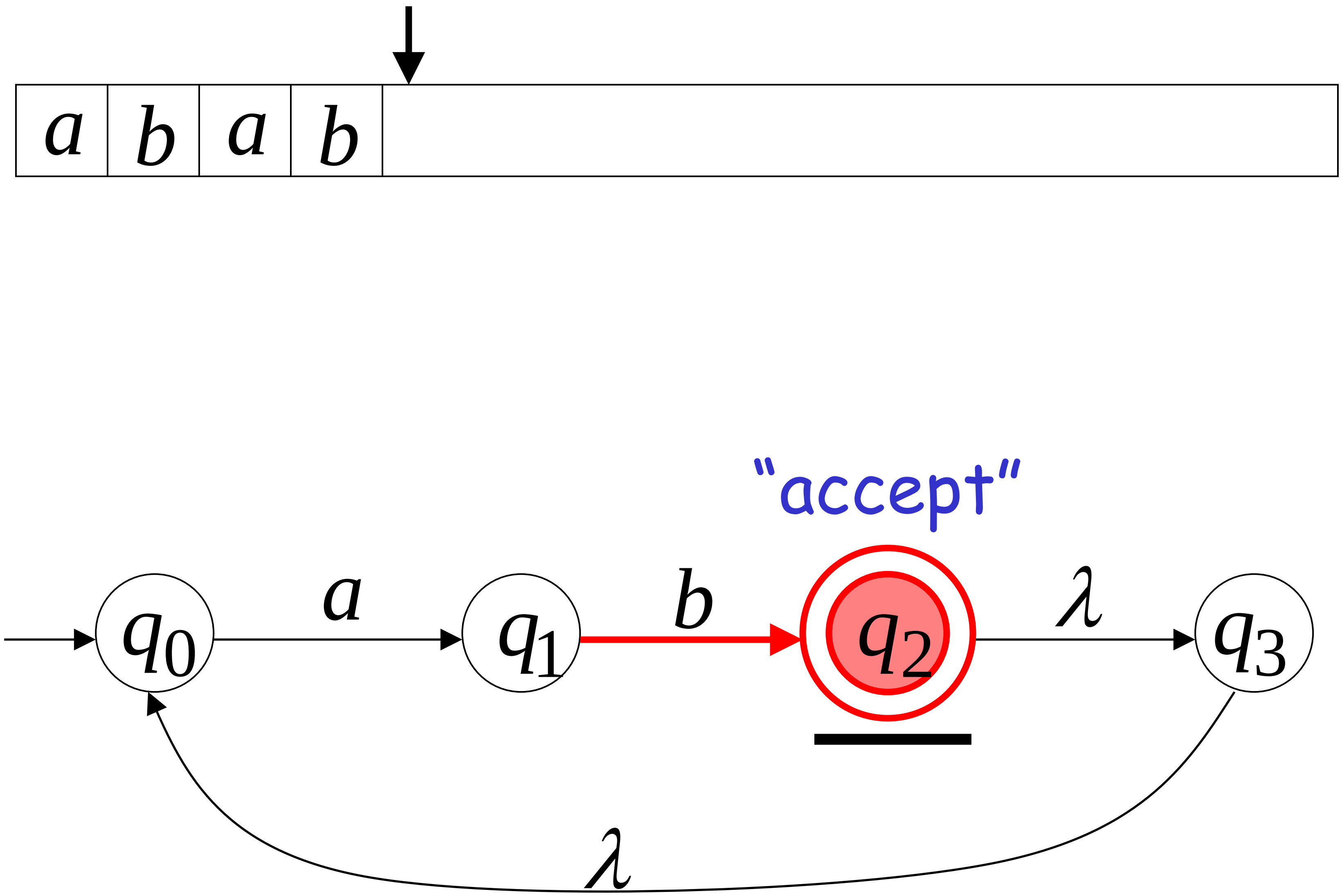

q_0
a
q_1
b
q_2
λ
q_3
λ


a
b
a
b
a
b
λ
λ


a
b
a
b
a
b
λ
q0
q1
q2
q3
λ


a
b
a
b

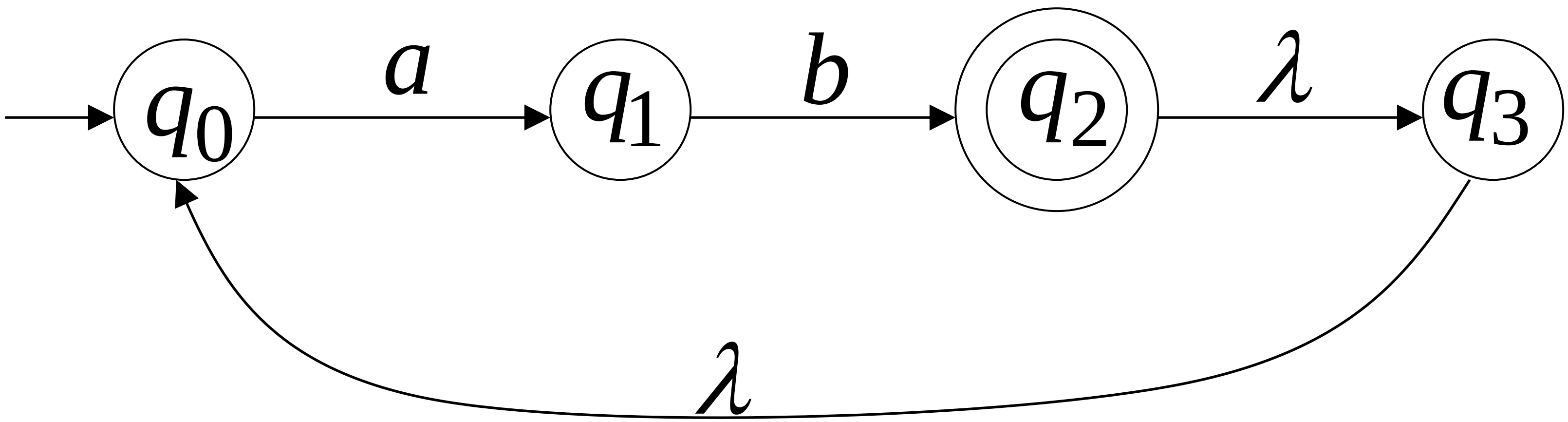

q_0
a
q_1
b
q_2
λ
q_3
λ


a
b
a
b
"accept"
a
b
λ
λ
q_0
q_1
q_2
q_3

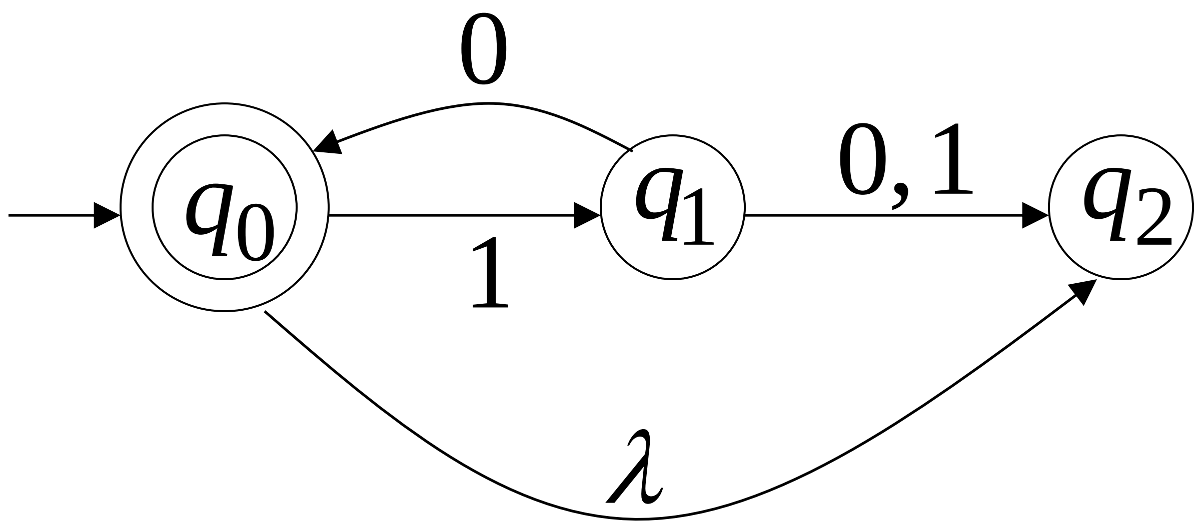
Language accepted

$$L = \{ab,\ abab,\ ababab,\ ...\}$$

$$= \{ab\}^{+}$$



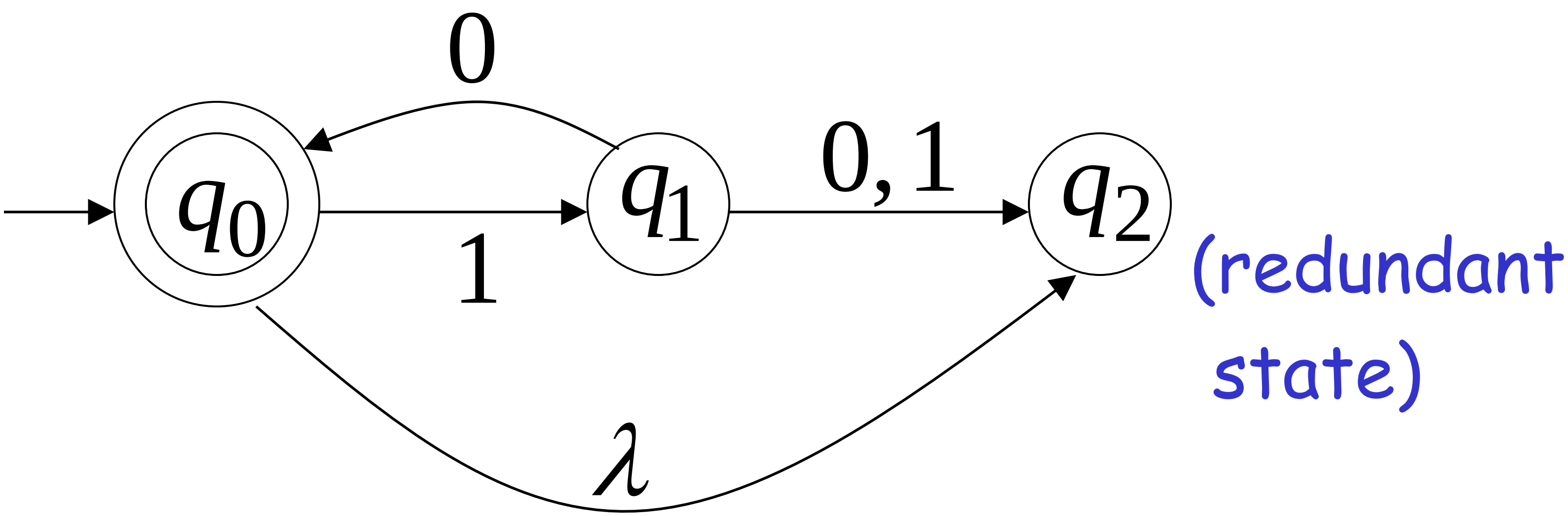
Another NFA Example: L(M)?



Language accepted

$$L(M) = \{\lambda,\ 10,\ 1010,\ 101010,\ \ldots\}$$

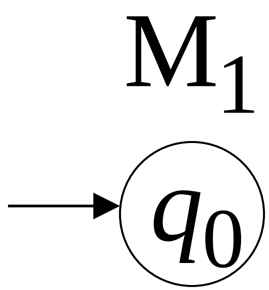
$$= \{10\}*$$

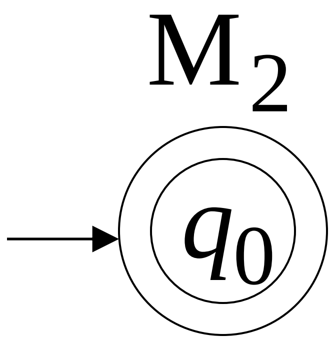


Remarks:

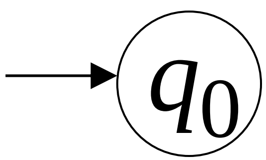
- The λ symbol never appears on the input tape

• Simple automata: Languages?

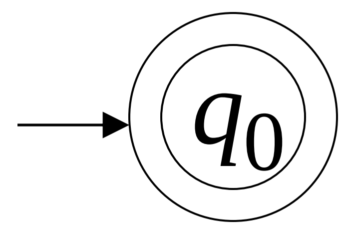




M_1



M_2

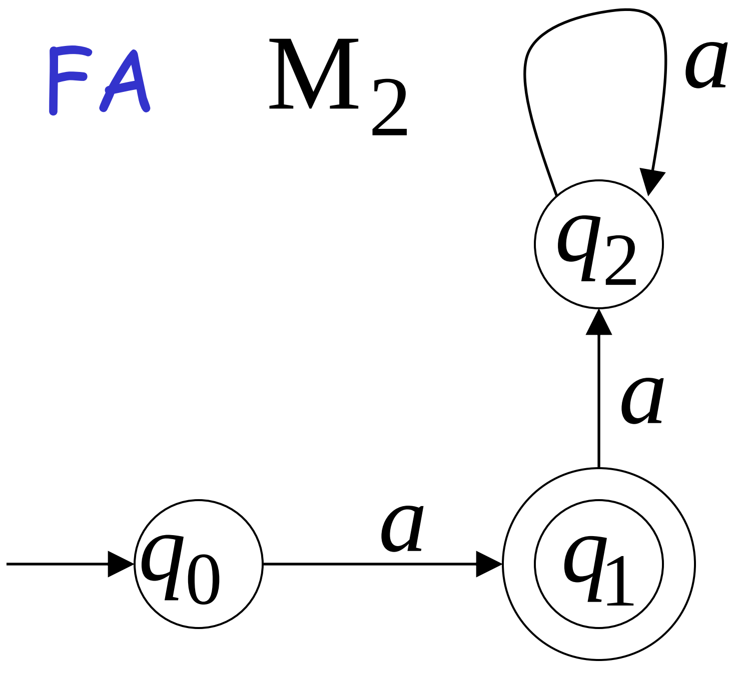


$L(M_1) = \{\}$

$L(M_2) = \{\lambda\}$

λ-transition in deterministic automata?

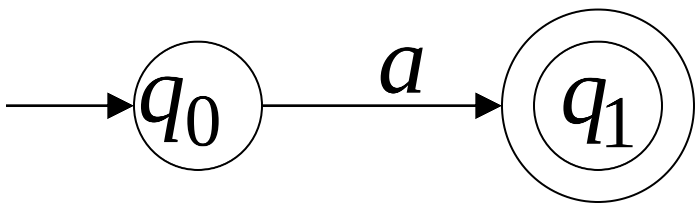
- NFAs are interesting because we can express languages easier than FAs



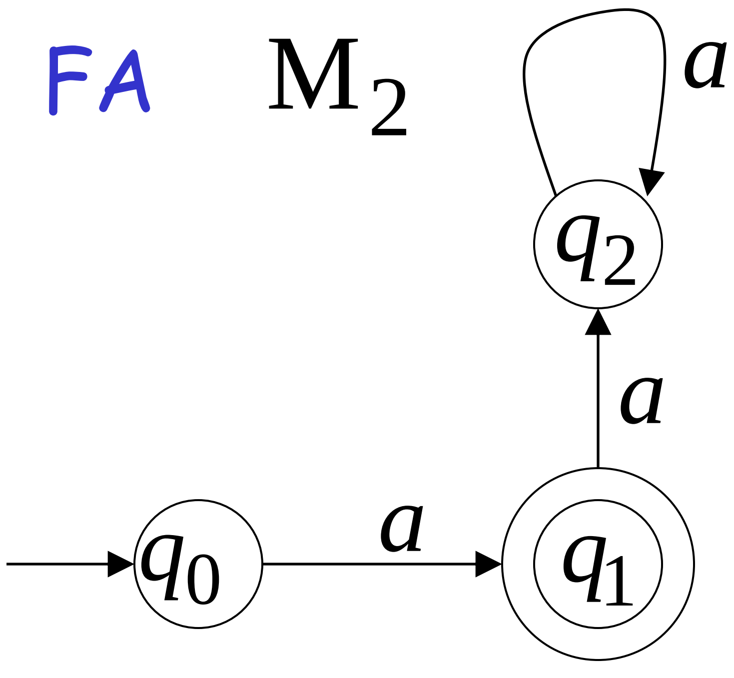
$$L(M_2) = \{a\}$$

- NFAs are interesting because we can express languages easier than FAs

NFA M_1



FA M_2



$L(M_1) = \{a\}$ $L(M_2) = \{a\}$

Formal Definition of NFAs

$$M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$

$Q:$ Set of states, i.e. $\{q_0, q_1, q_2\}$

$\Sigma:$ Input alphabet, i.e. $\{a, b\}$

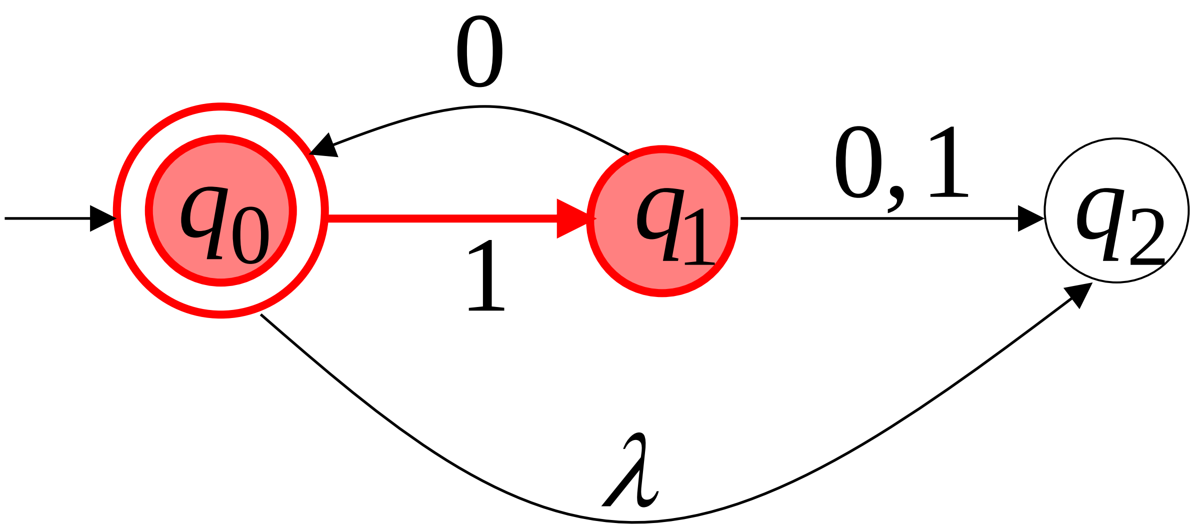
$\delta:$ Transition function

$q_0:$ Initial state

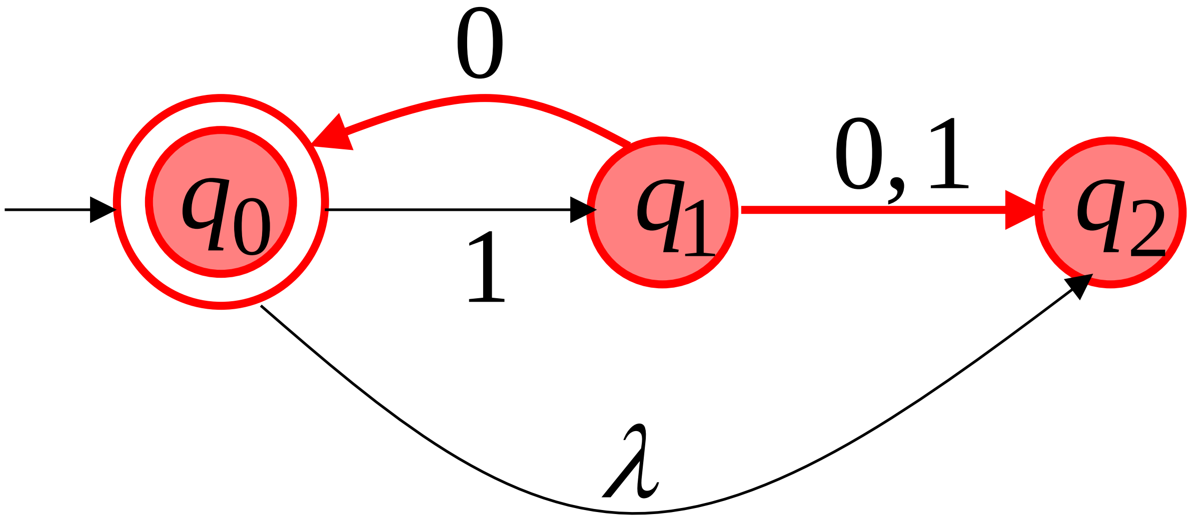
$F:$ Accepting states

Transition Function δ

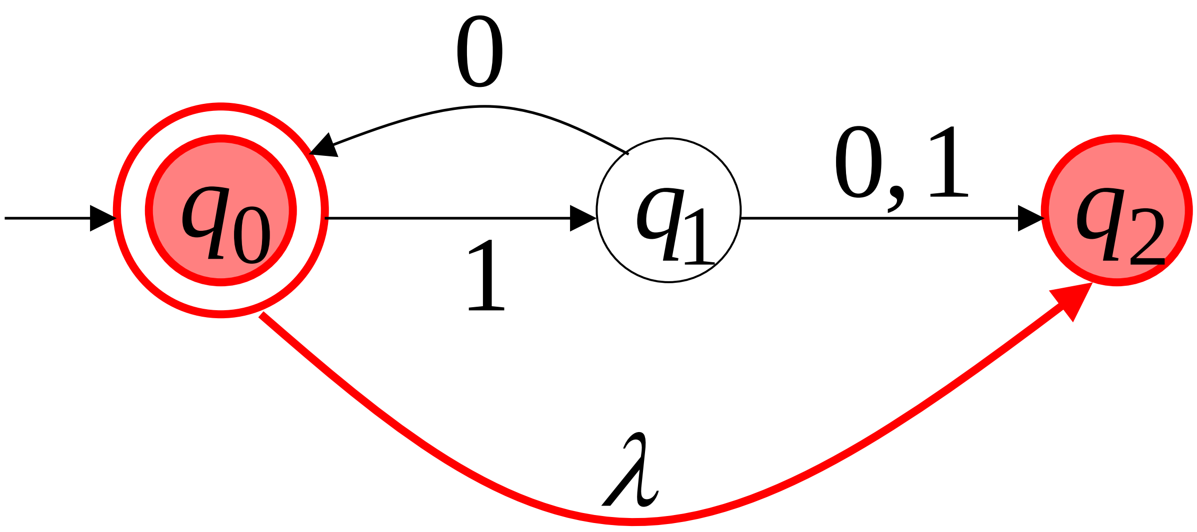
$$\delta(q_0, 1) = \{q_1\}$$



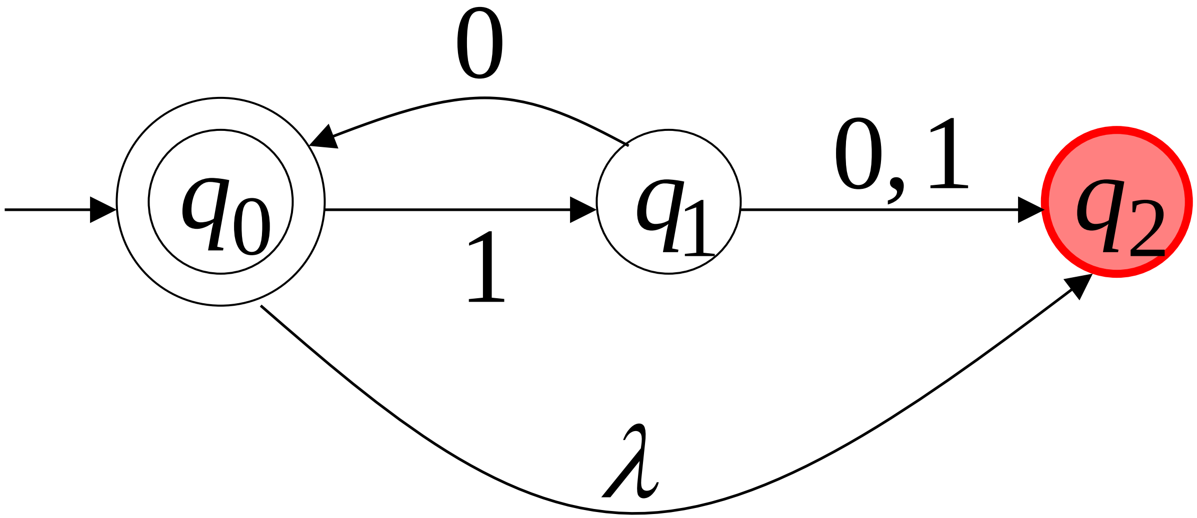
$$\delta(q_1, 0) = \{q_0, q_2\}$$



$$\delta(q_0, \lambda) = \{q_0, q_2\}$$

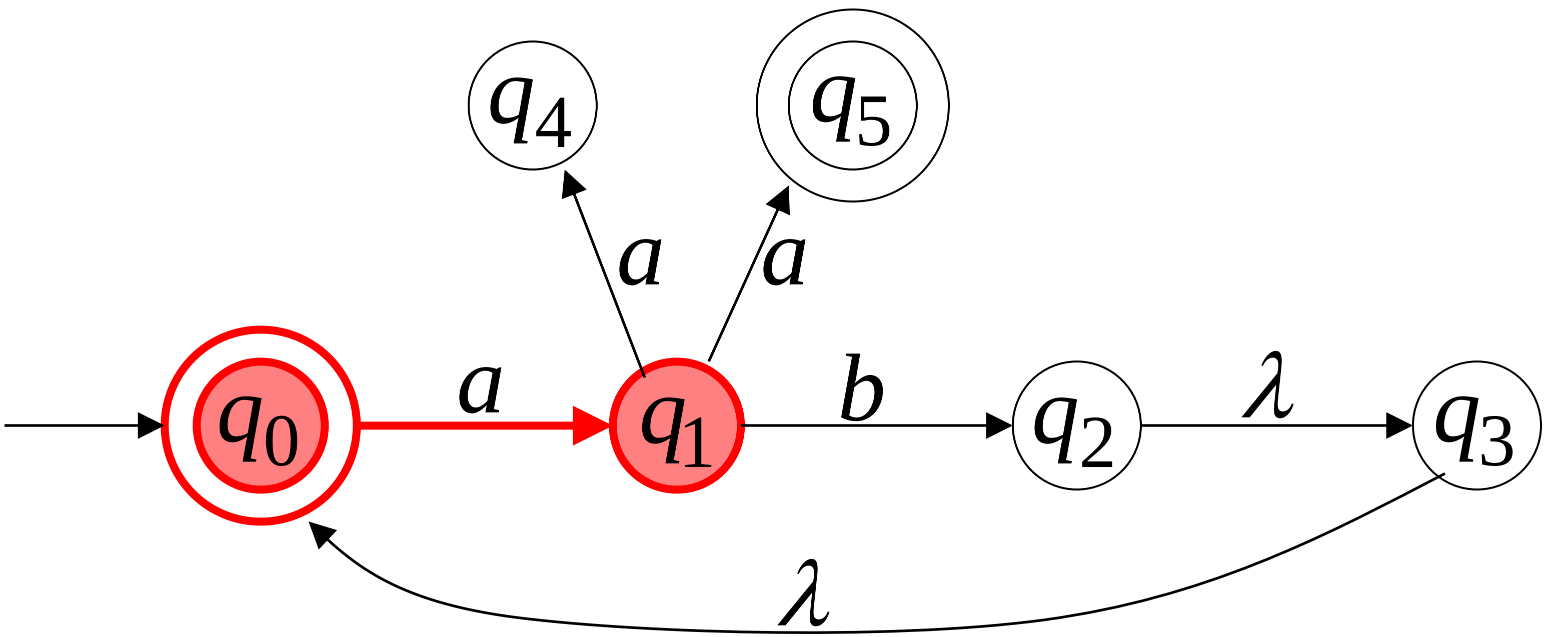


$$\delta(q_2, 1) = \emptyset$$

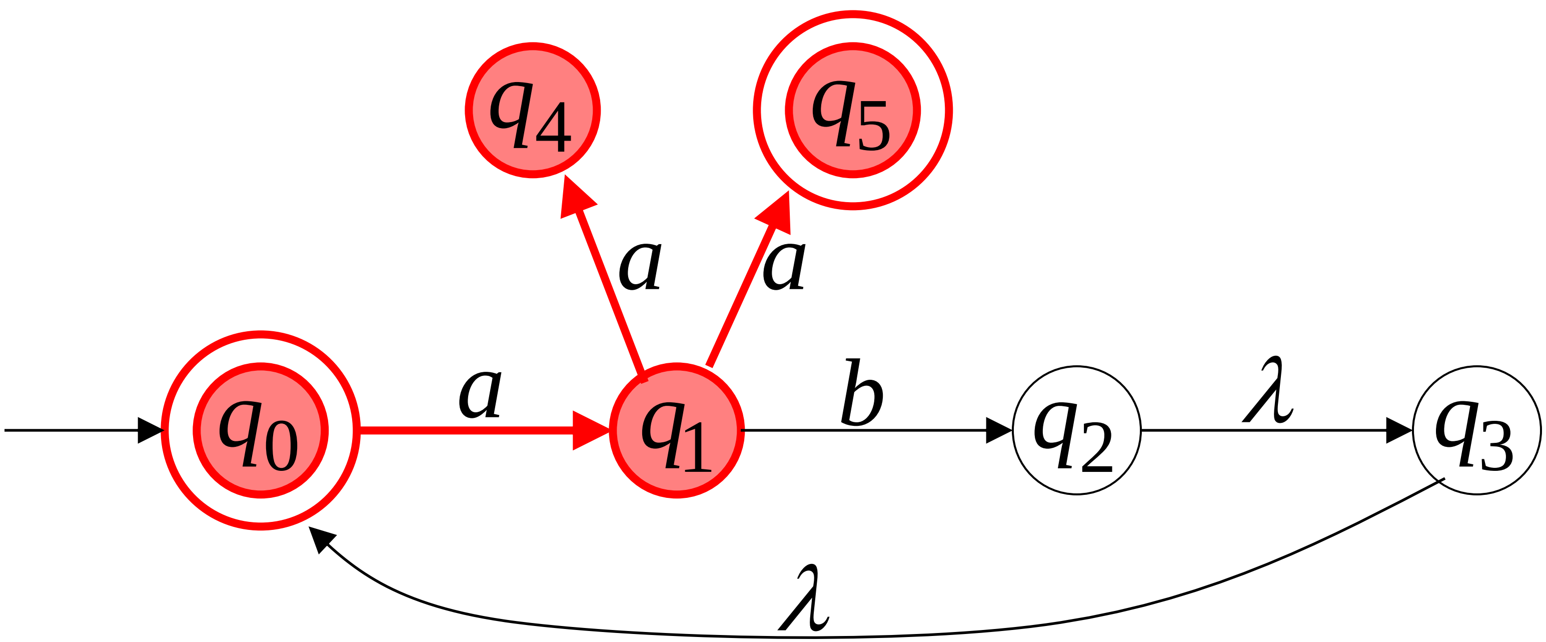


Extended Transition Function δ^*

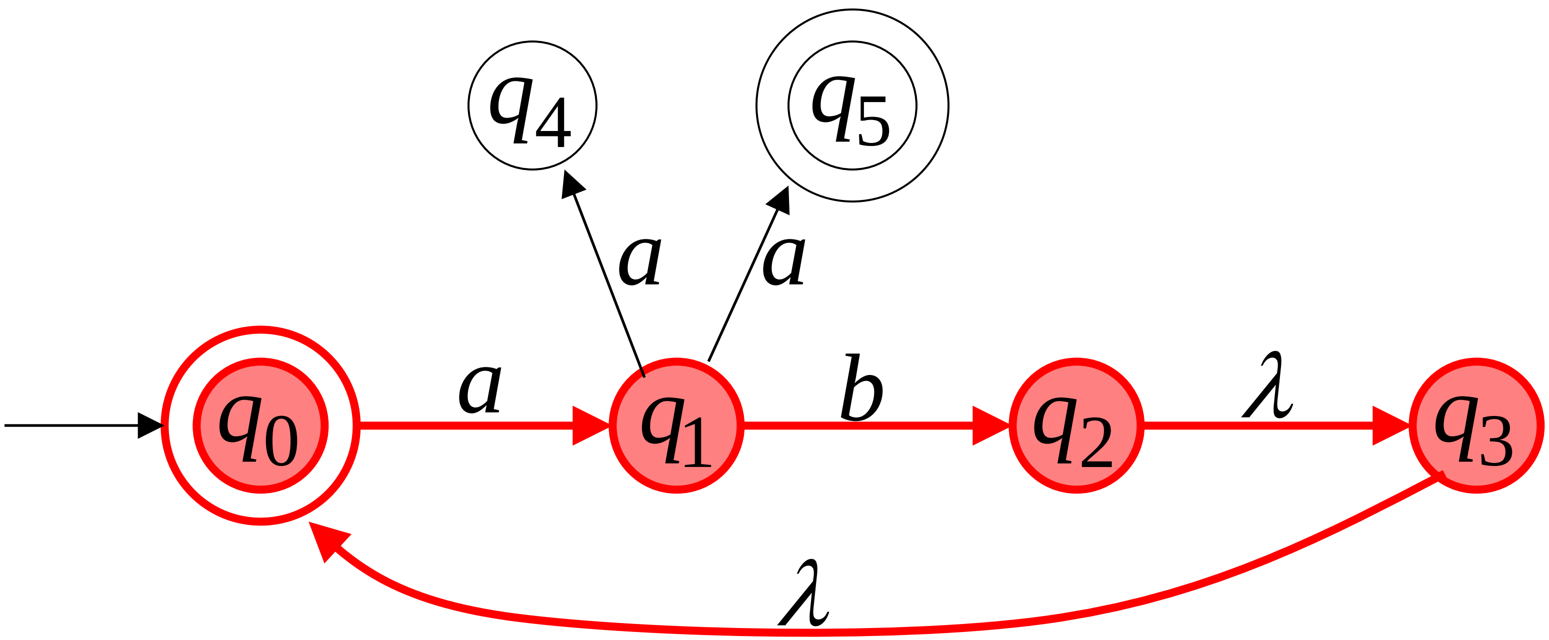
$$\delta^*(q_0, a) = \{q_1\}$$



$$\delta^*(q_0, aa) = \{q_4, q_5\}$$

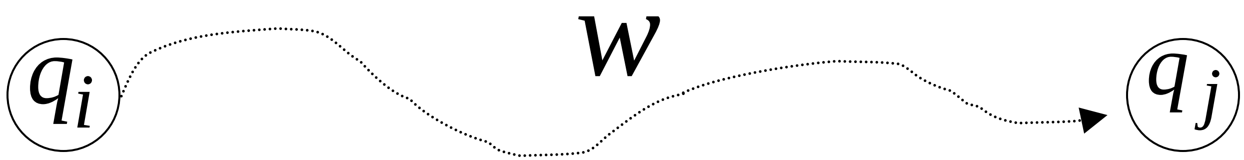


$$\delta^*(q_0, ab) = \{q_2, q_3, q_0\}$$

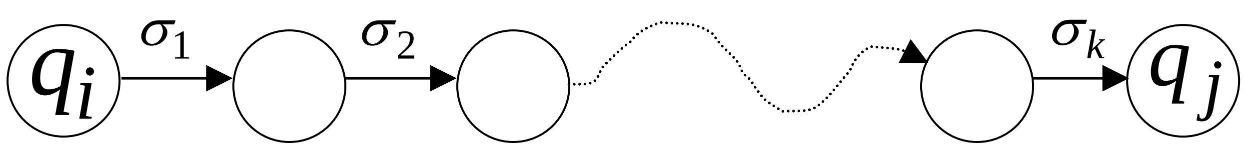


Formally

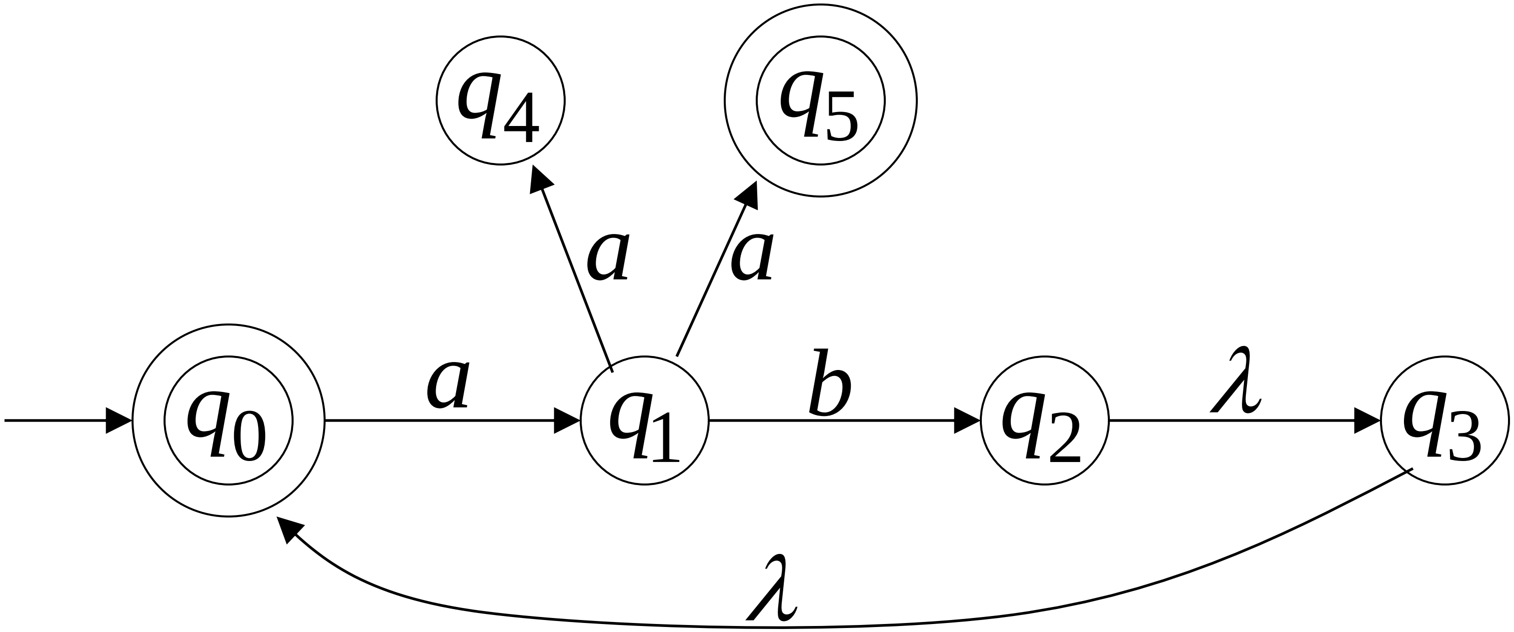
$q_j \in \delta^*(q_i, w)$: there is a walk from q_i to q_j with label w



$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$

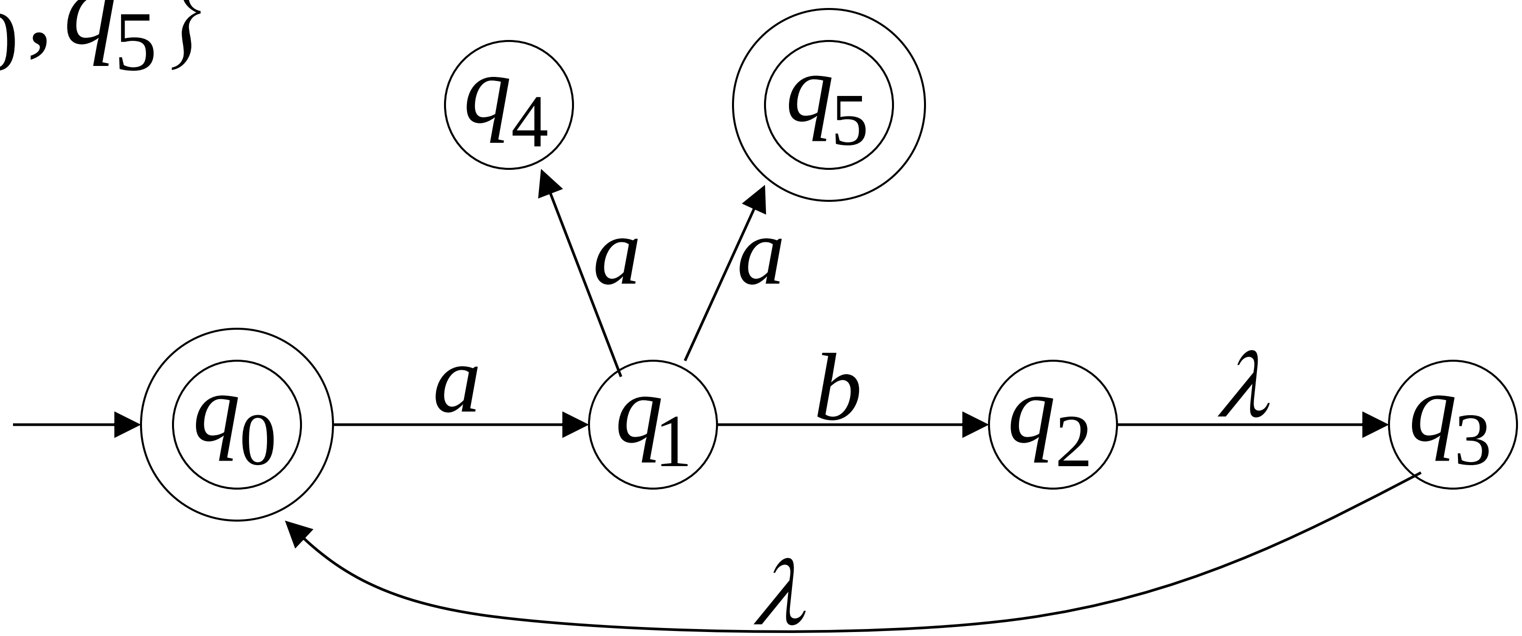


L(M)?



The Language of an NFA M

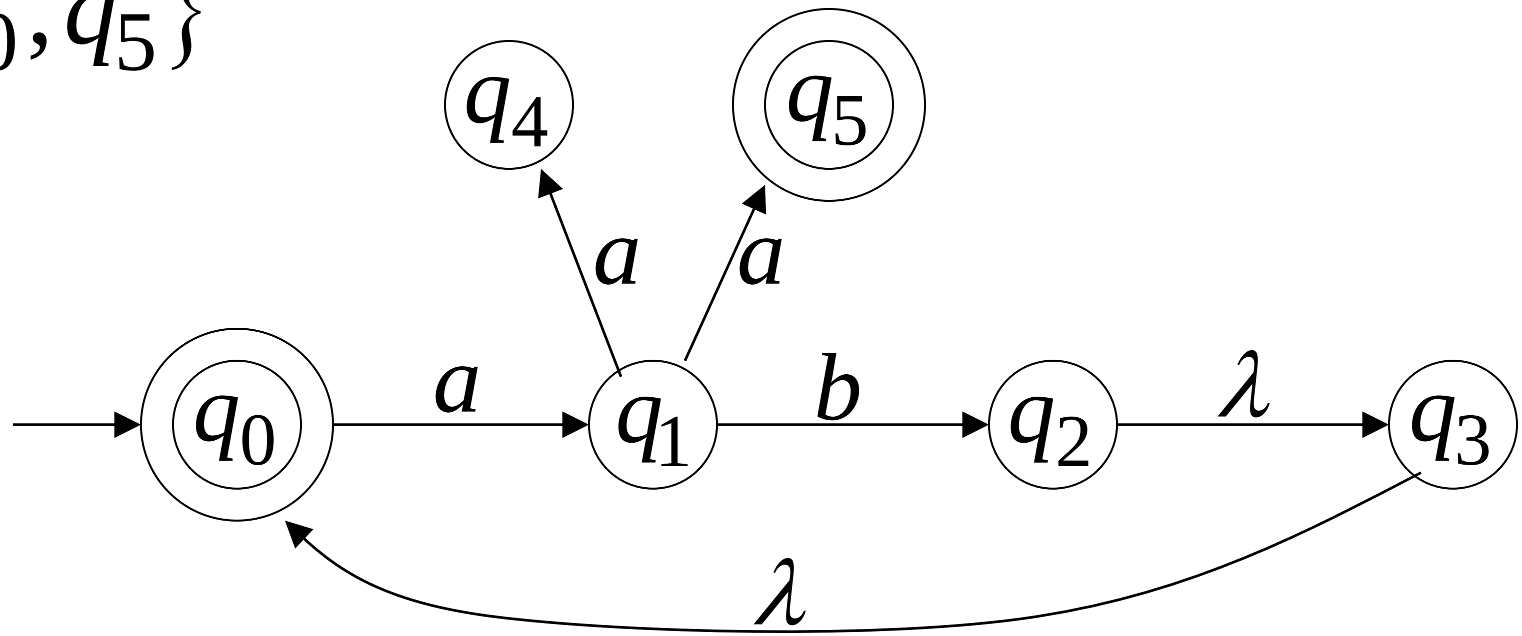
$F = \{q_0, q_5\}$



$$\delta^*(q_0, aa) = \{q_4, \underline{q_5}\} \qquad aa \in L(M)$$

$$\searrow \in F$$

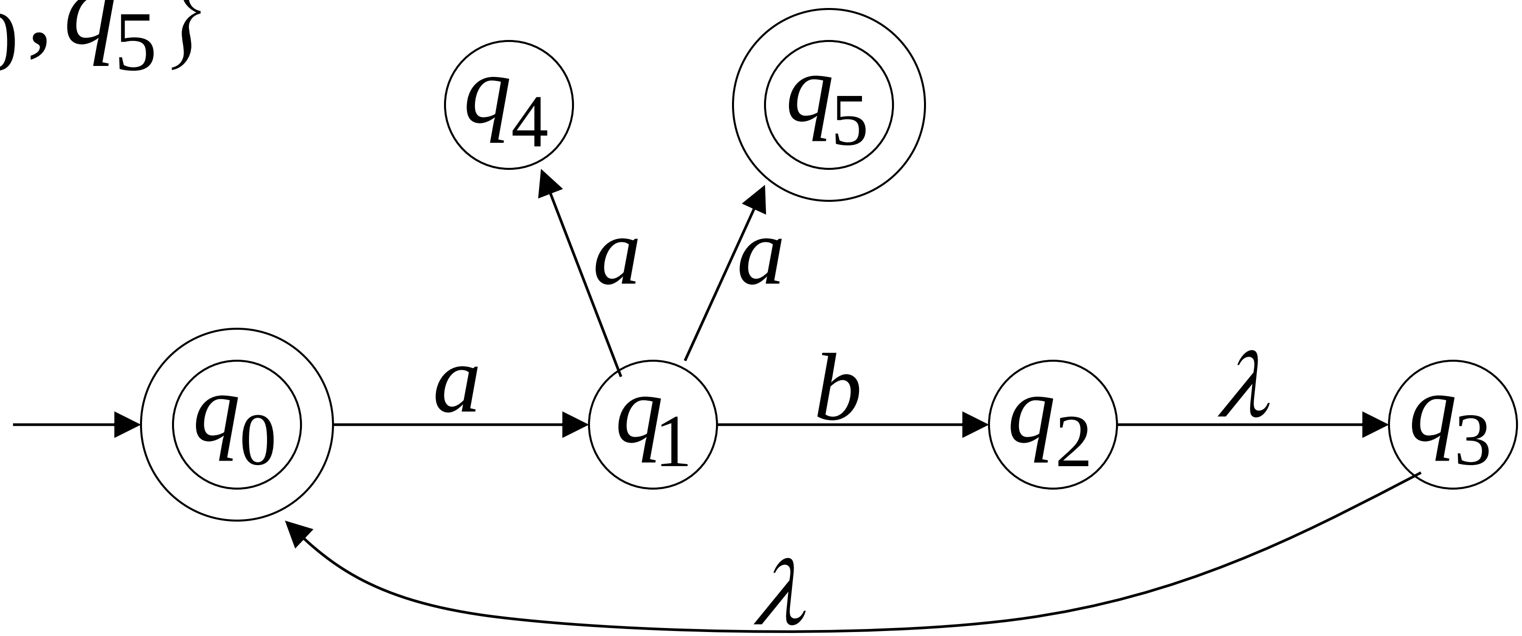
$$F = \{q_0, q_5\}$$



$$\delta^*(q_0, ab) = \{q_2, q_3, \underline{q_0}\} \qquad ab \in L(M)$$

$$\searrow \in F$$

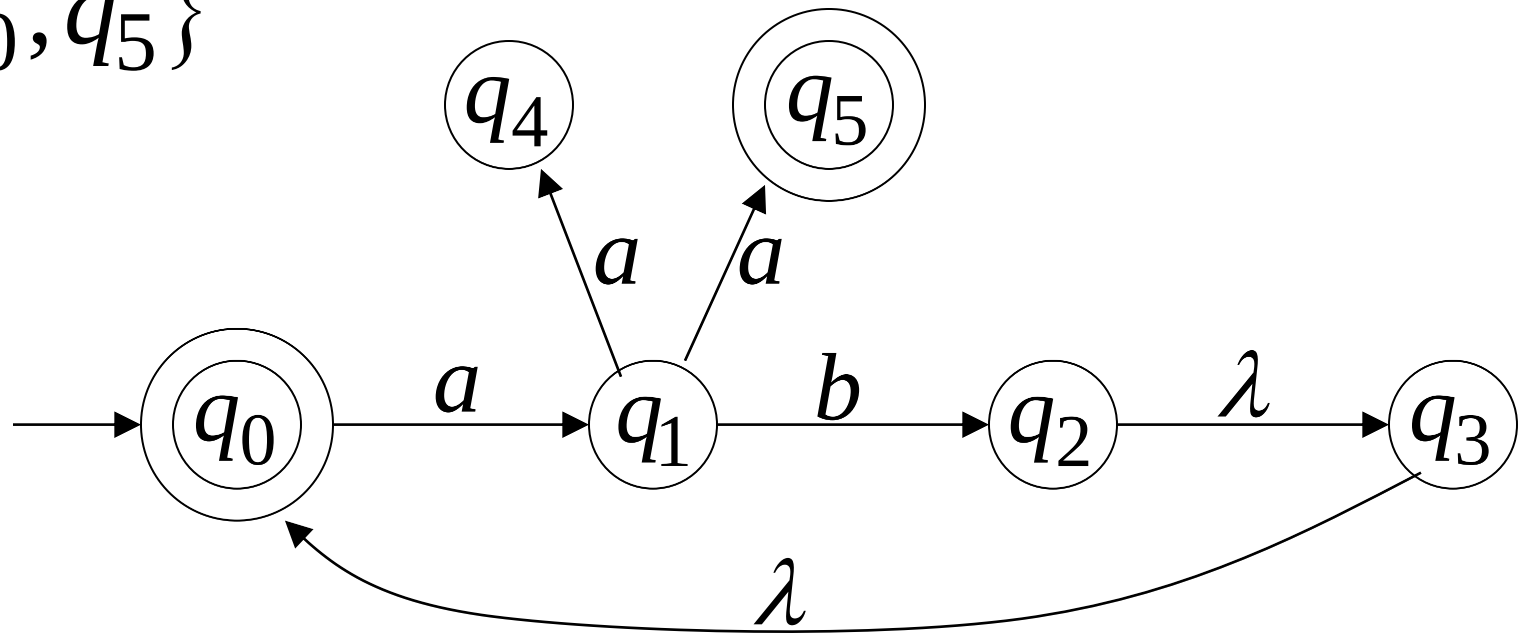
$F = \{q_0, q_5\}$



$\delta^*(q_0, abaa) = \{q_4, \underline{q_5}\}$ $\qquad$ $aaba \in L(M)$

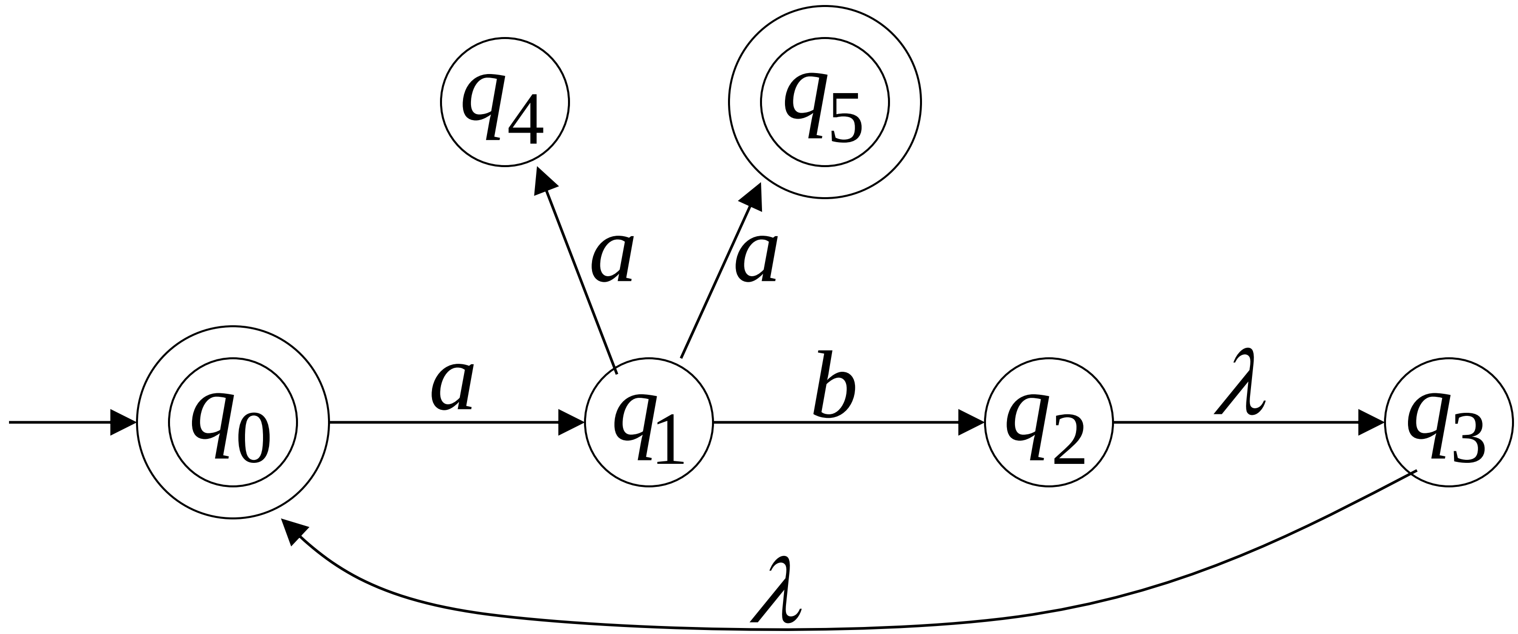
$\rightarrow \in F$

$$F = \{q_0, q_5\}$$



$$\delta^*(q_0, aba) = \{q_1\} \qquad aba \notin L(M)$$

$$\notin F$$



$$L(M) = \{\lambda\} \cup \{ab\}^* \{aa\}$$

Formally

The language accepted by NFA M is:

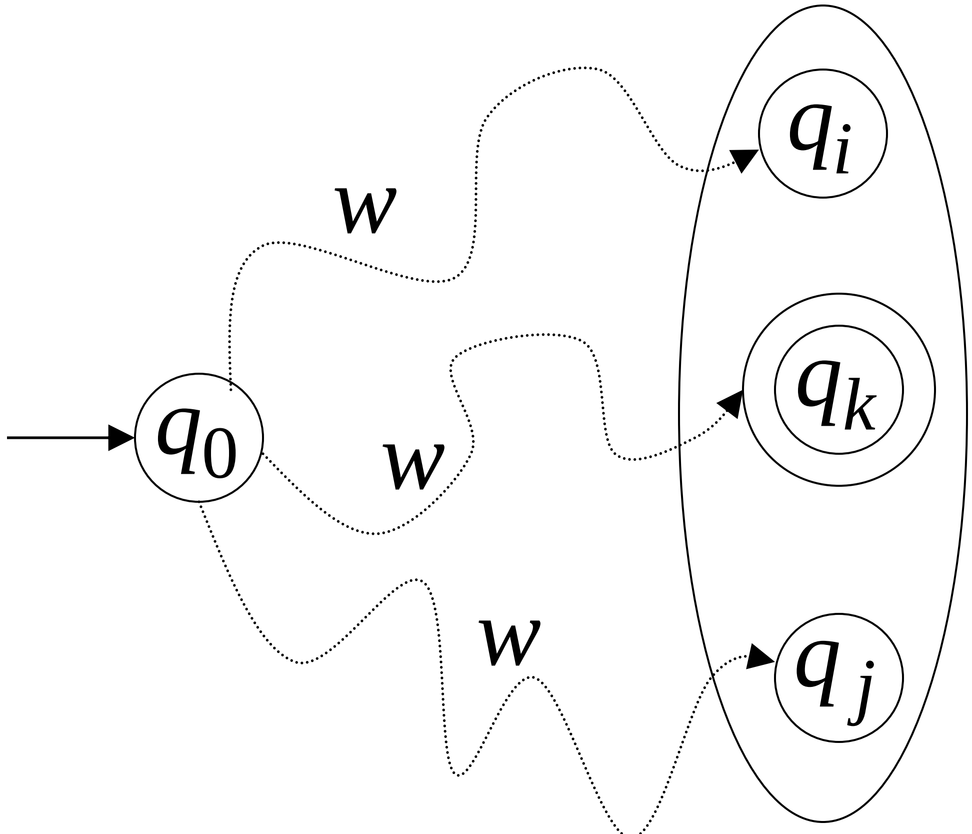
$$L(M) = \{w_1, w_2, w_3, \ldots\}$$

where $\delta^*(q_0, w_m) = \{q_i, q_j, \ldots, q_k, \ldots\}$

and there is some $q_k \in F$ (accepting state)

$$w \in L(M)$$

$$\delta^*(q_0, w)$$



$$q_k \in F$$

Formal Languages

NFAs Accept the Regular Languages

Equivalence of Machines

Definition:

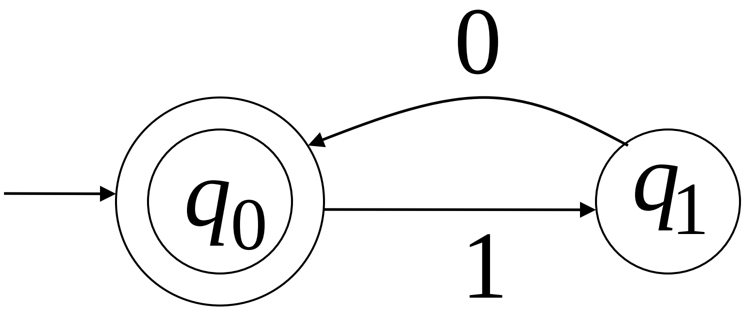
Machine M_1 is equivalent to machine M_2

if $L(M_1) = L(M_2)$

Example of equivalent machines

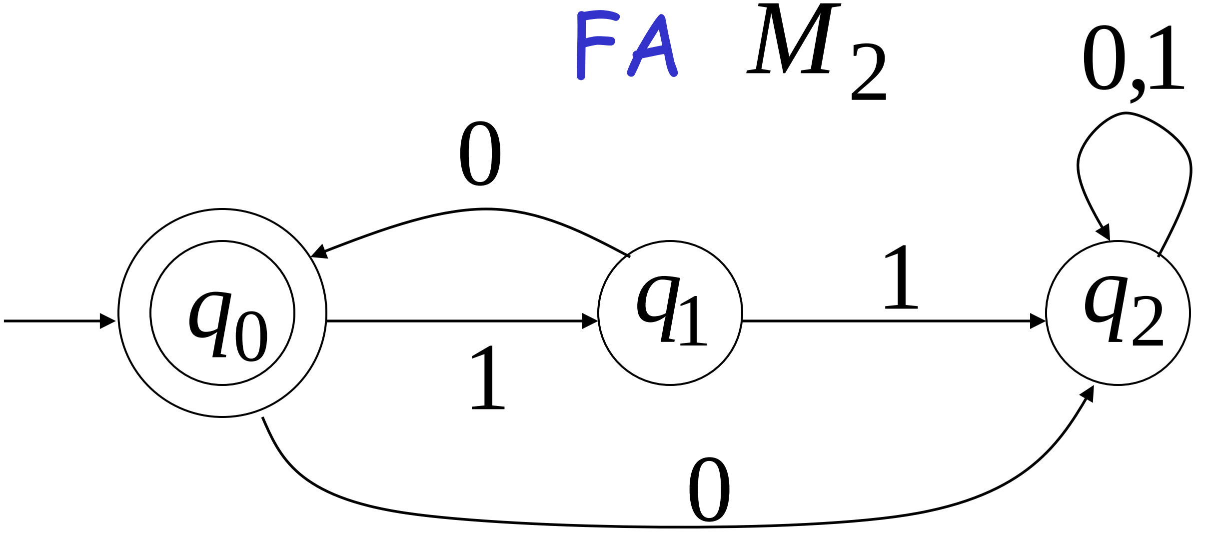
NFA M_1

$$L(M_1) = \{10\}^*$$



FA M_2

$$L(M_2) = \{10\}^*$$



We will prove:

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} = \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Languages
accepted
by FAs

NFAs and FAs have the same computation power

We will show:

$$\left\{\begin{array}{l}\text{Languages}\\ \text{accepted}\\ \text{by NFAs}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

$$\left\{\begin{array}{l}\text{Languages}\\ \text{accepted}\\ \text{by NFAs}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

Proof-Step 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Proof?

Proof-Step 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Proof: Every FA is trivially an NFA

⇓

Any language L accepted by a FA is also accepted by an NFA

Proof-Step 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

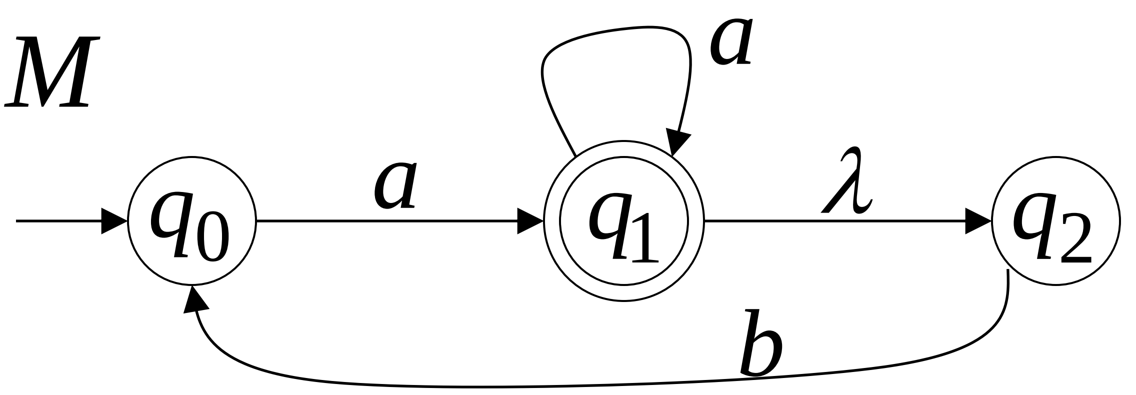
Proof: Any NFA can be converted to an equivalent FA

$\Downarrow$

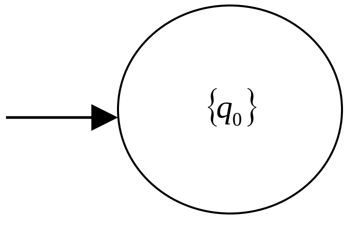
Any language L accepted by an NFA is also accepted by a FA

Convert NFA to FA

NFA M

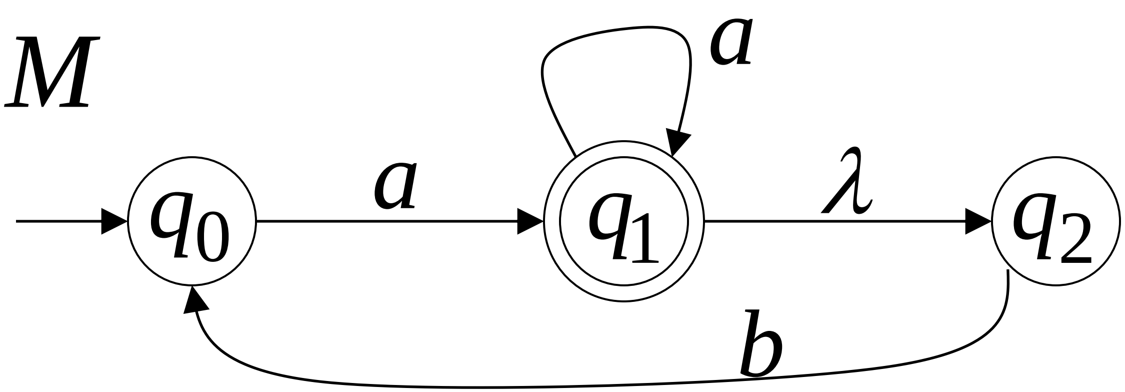


FA M'

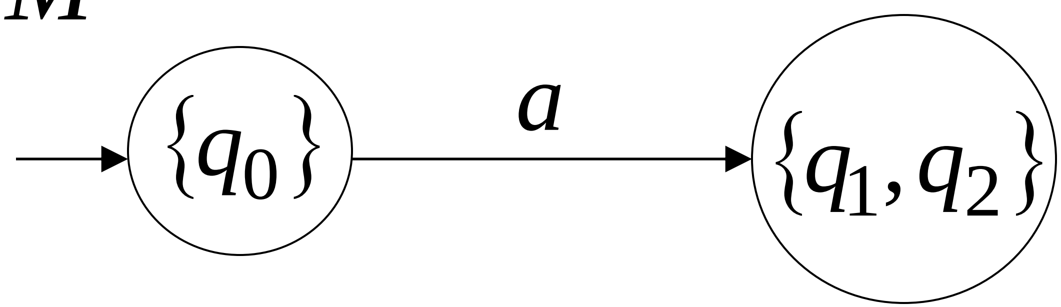


Convert NFA to FA

NFA M

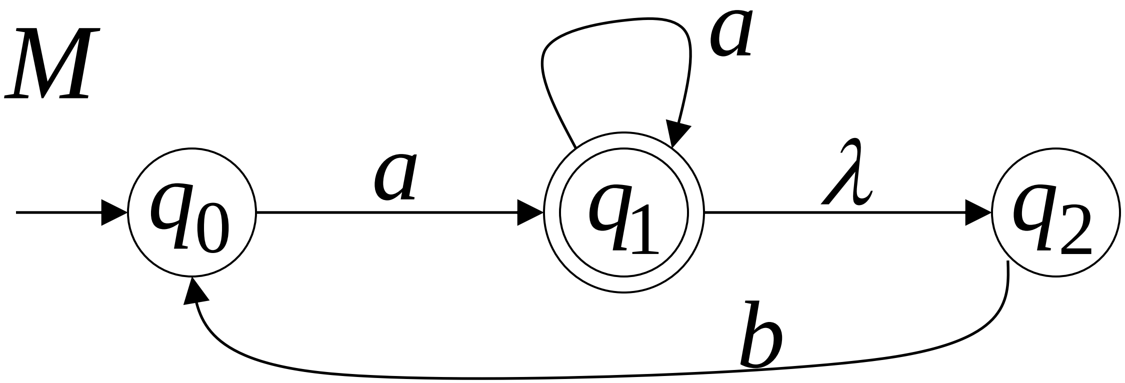


FA M'

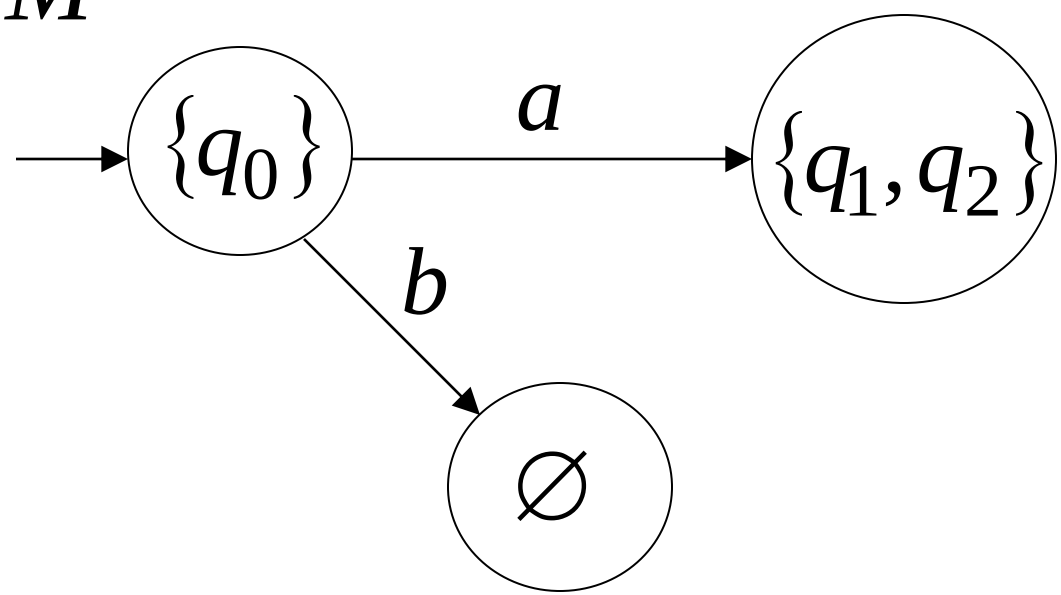


Convert NFA to FA

NFA M

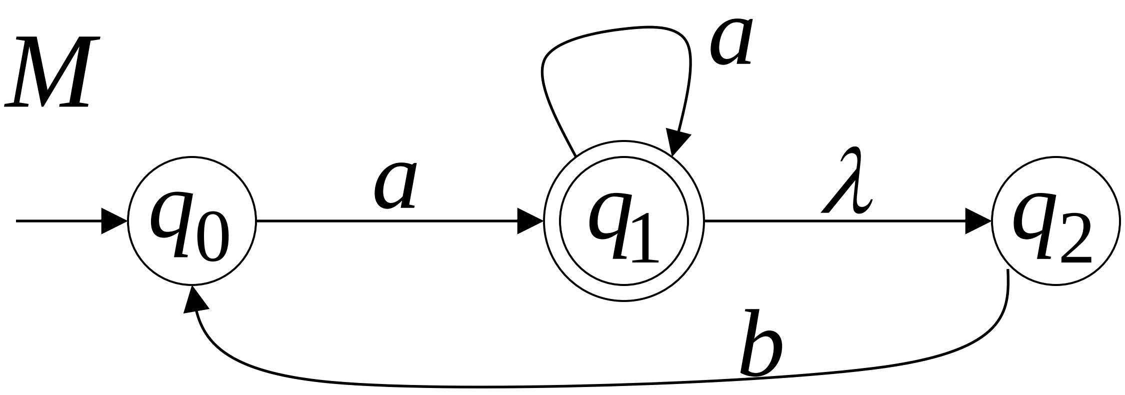


FA M'

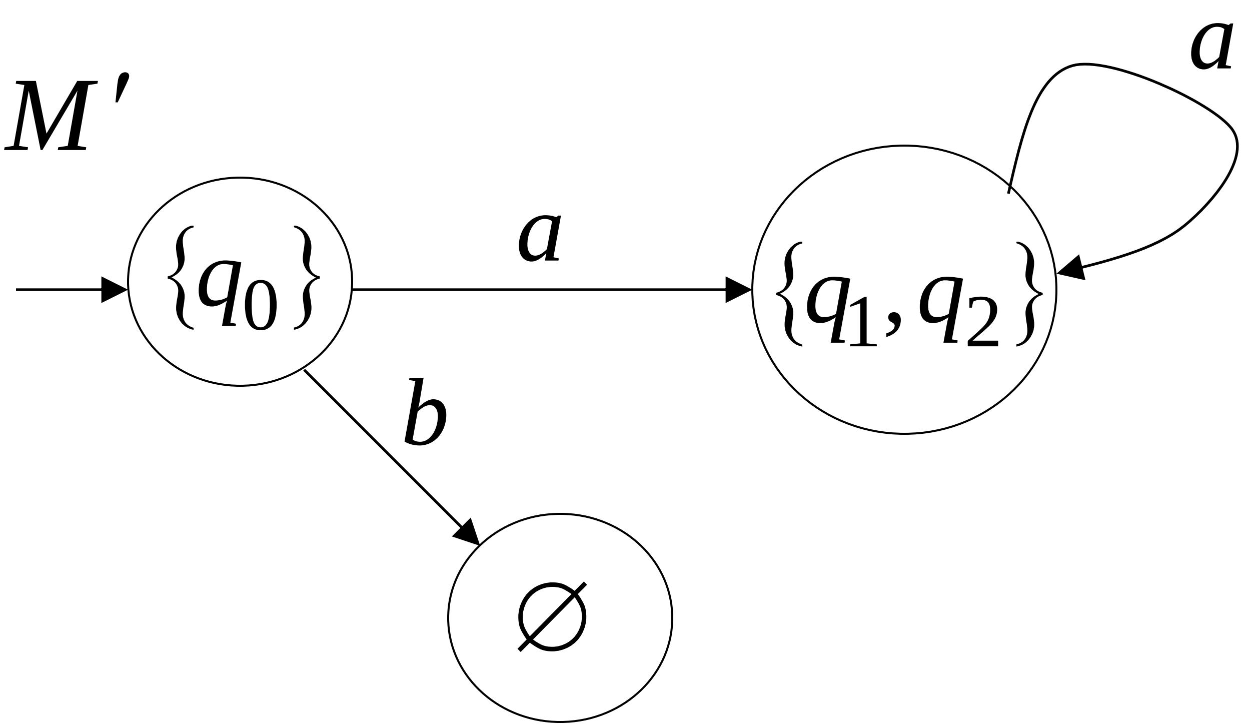


Convert NFA to FA

NFA M

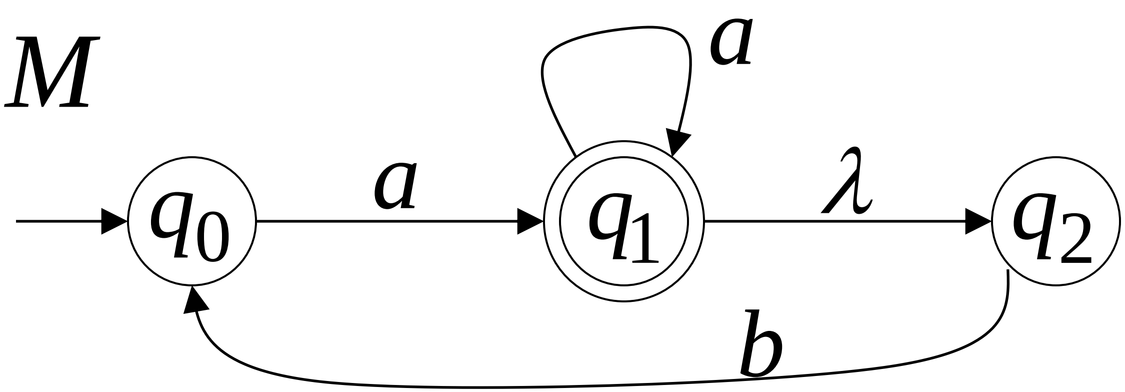


FA M'

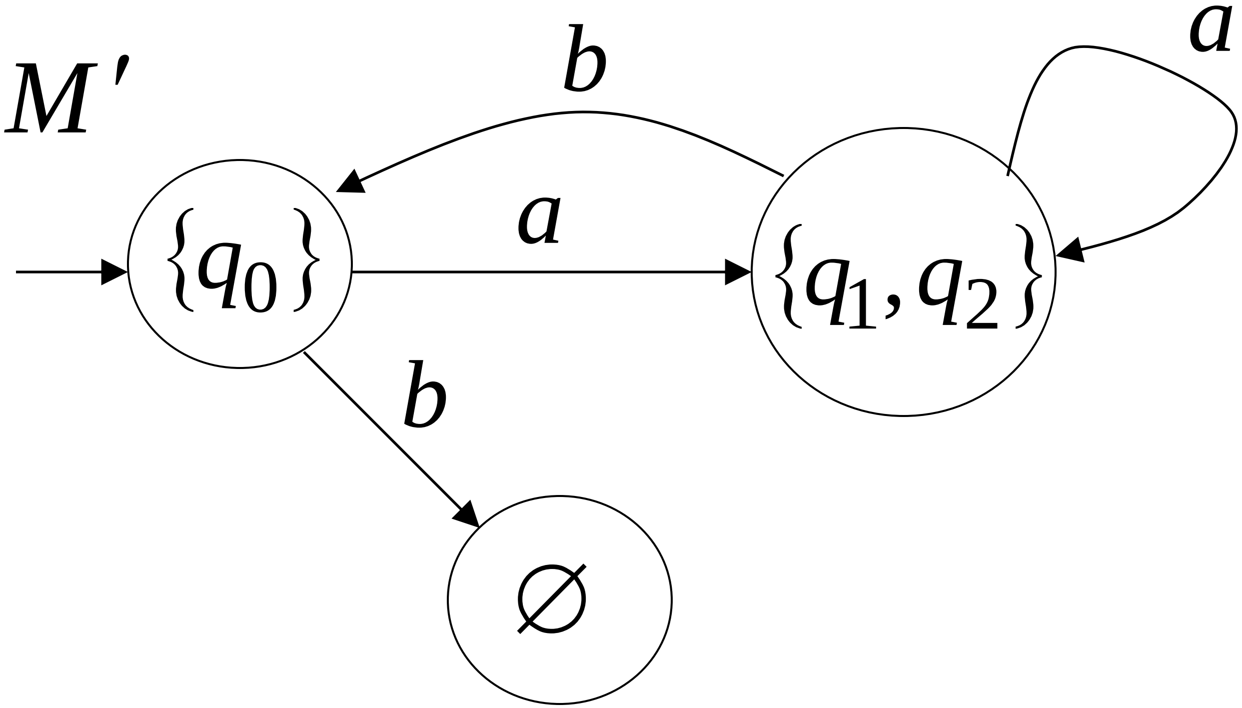


Convert NFA to FA

NFA M

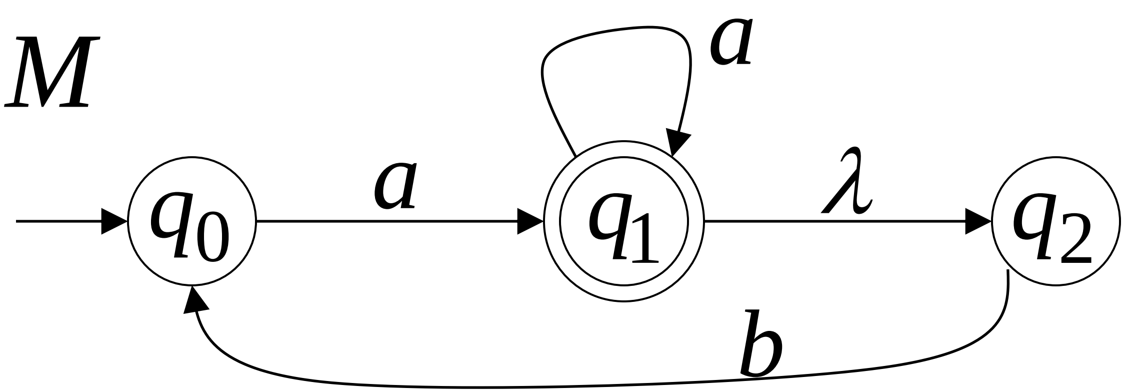


FA M'

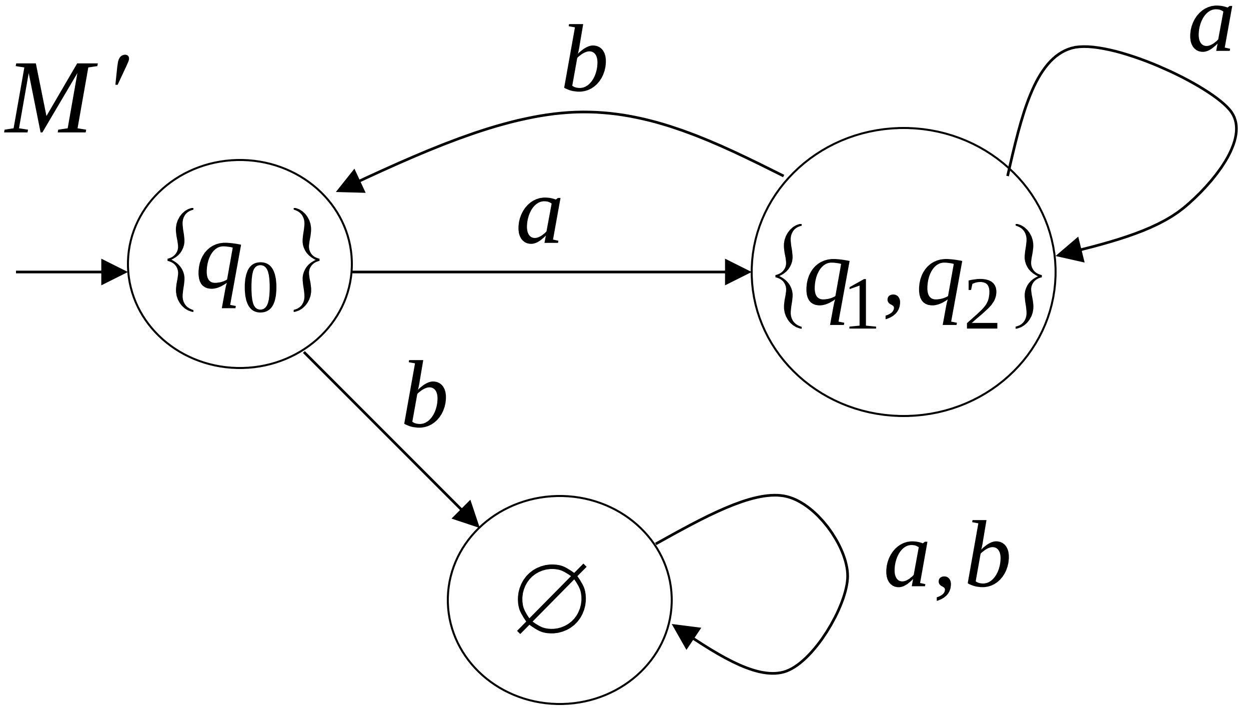


Convert NFA to FA

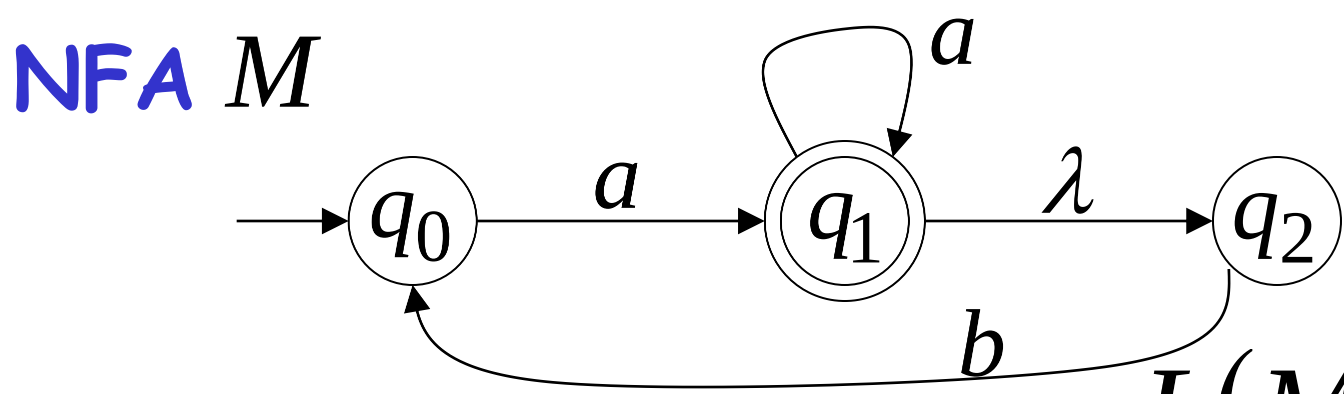
NFA M



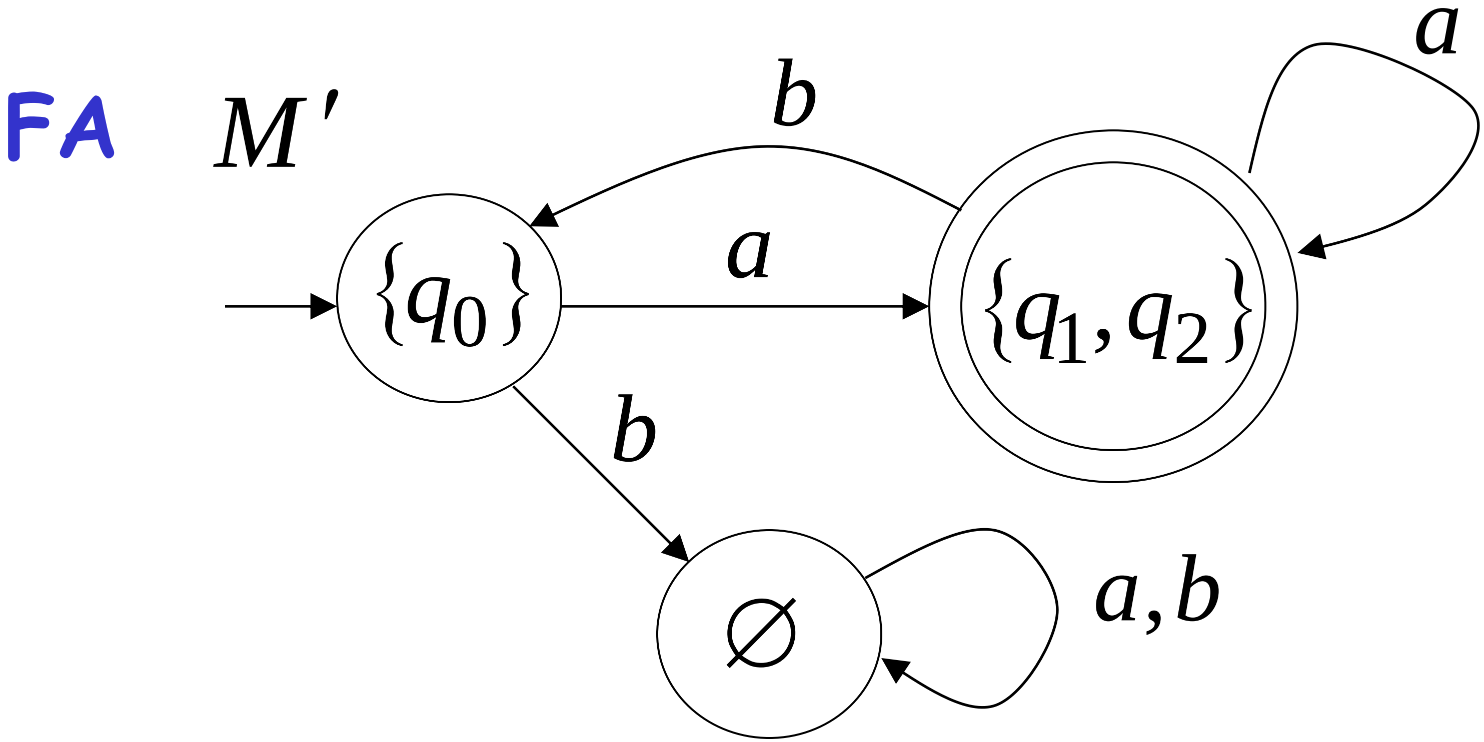
FA M'



Convert NFA to FA



$$L(M) = L(M')$$



NFA to FA Conversion

We are given an NFA M

We want to convert it
to an equivalent FA M'

With $L(M) = L(M')$

What we need to construct

Finite Automaton (FA)

$$M = (Q, \Sigma, \delta, q_0, F)$$

Q : set of states

Σ : input alphabet

δ : transition function

q_0 : initial state

F : set of accepting states

If the NFA has states

$$q_0, q_1, q_2, \ldots$$

the FA has states in the power set

$$\emptyset, \{q_0\}, \{q_1\}, \{q_1, q_2\}, \{q_3, q_4, q_7\}, \ldots.$$

Procedure NFA to FA

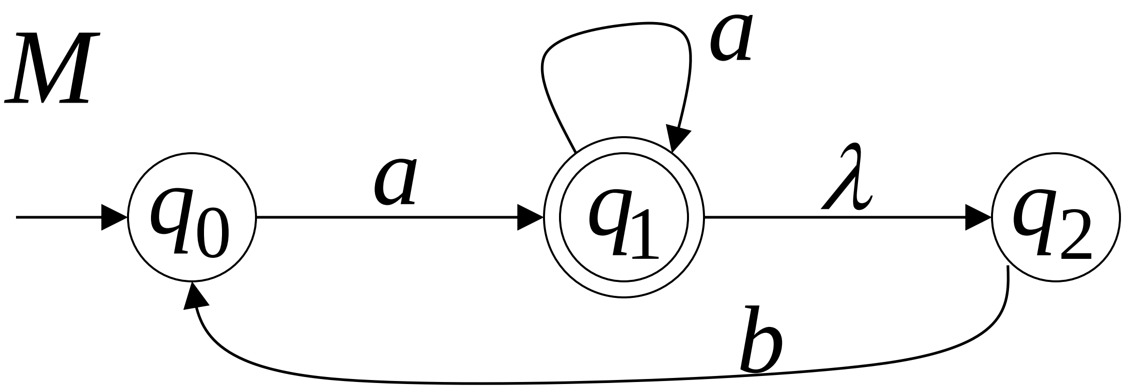
1. Initial state of NFA: q_0

$\Downarrow$

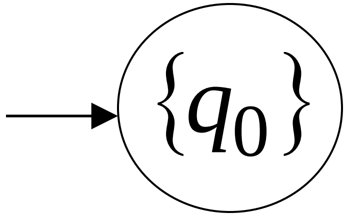
Initial state of FA: $\{q_0\}$

Example

NFA M



FA M'



Procedure NFA to FA

2. For every FA's state $\{q_i, q_j, \ldots, q_m\}$

Compute in the NFA

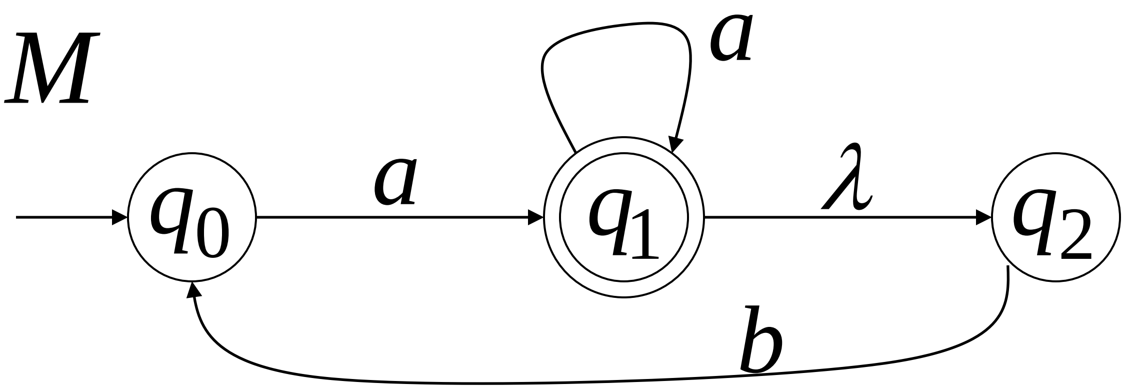
$$\left.\begin{array}{l} \delta^*(q_i, a), \\ \delta^*(q_j, a), \\ \ldots \end{array}\right\} = \{q'_i, q'_j, \ldots, q'_m\}$$

Add transition to FA

$$\delta(\{q_i, q_j, \ldots, q_m\},\ a) = \{q'_i, q'_j, \ldots, q'_m\}$$

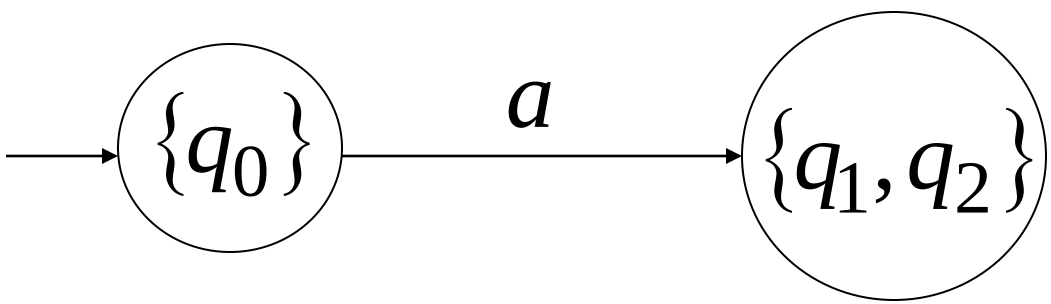
Example

NFA M



$$\delta^*(q_0, a) = \{q_1, q_2\}$$

FA M'



$$\delta(\{q_0\}, a) = \{q_1, q_2\}$$

Procedure NFA to FA

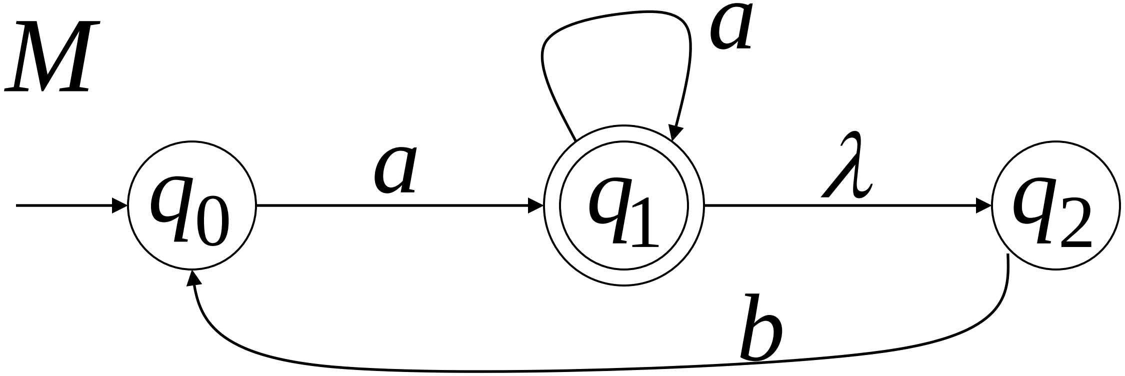
Repeat Step 2 for all letters in alphabet,

until

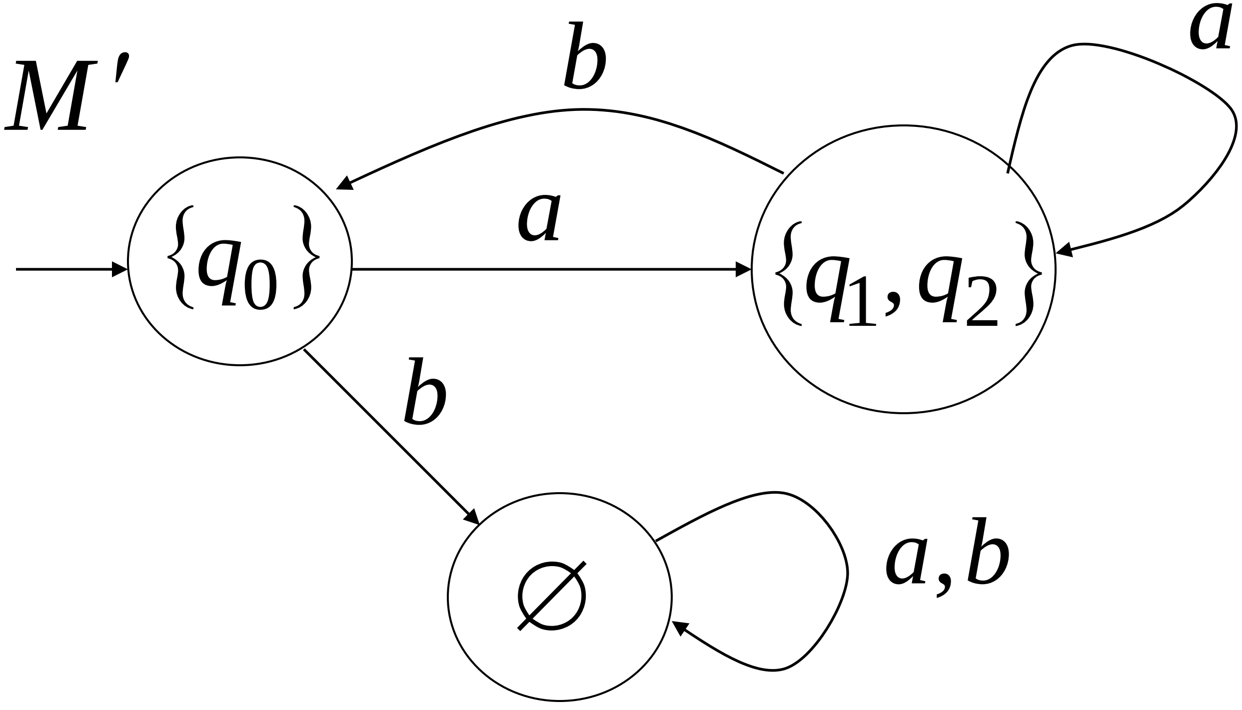
no more transitions can be added.

Example

NFA M



FA M'



Done?

Procedure NFA to FA

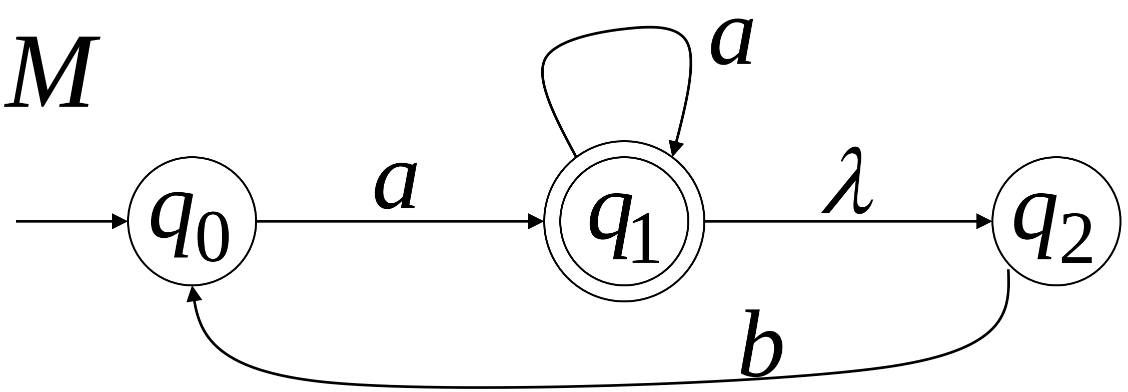
3. For any FA state $\{q_i, q_j, \ldots, q_m\}$

If q_j is accepting state in NFA

Then, $\{q_i, q_j, \ldots, q_m\}$
is accepting state in FA

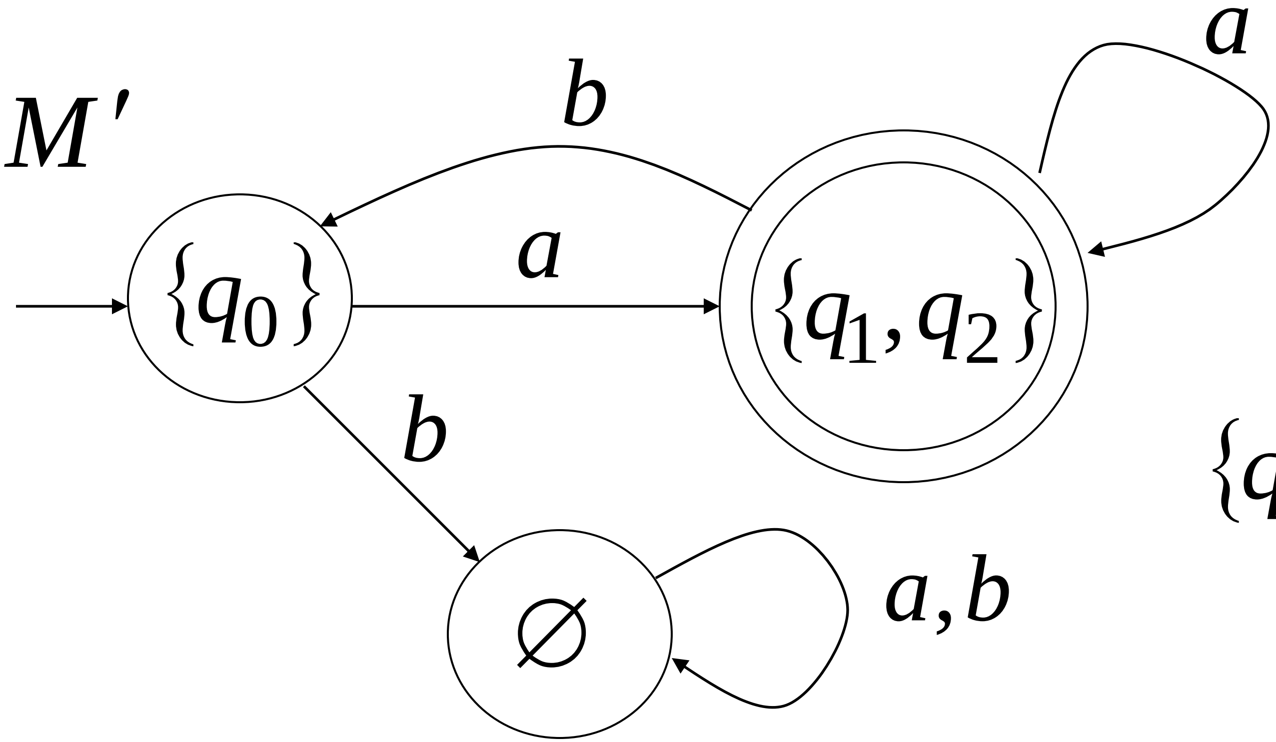
Example

NFA M



$q_1 \in F$

FA M'



$\{q_1, q_2\} \in F'$

Theorem

Take NFA M

Apply procedure to obtain FA M'

Then M and M' are equivalent :

$$L(M) = L(M')$$

Proof

$$L(M) = L(M')$$

$$\Updownarrow$$

$$L(M) \subseteq L(M') \quad \text{AND} \quad L(M) \supseteq L(M')$$

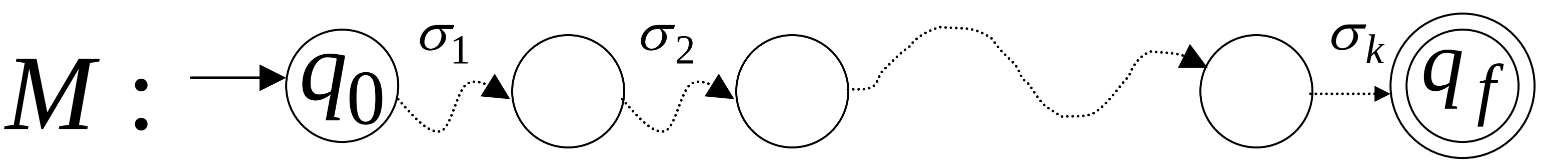
First we show: $L(M) \subseteq L(M')$

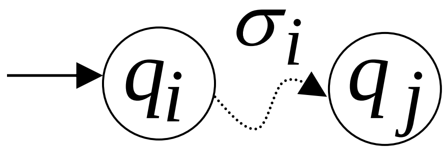
Take arbitrary: $w \in L(M)$

We will prove: $w \in L(M')$

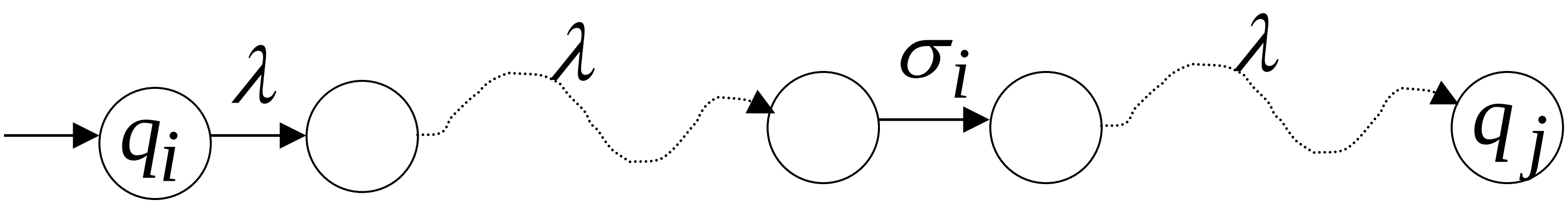
$$w \in L(M)$$

$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$



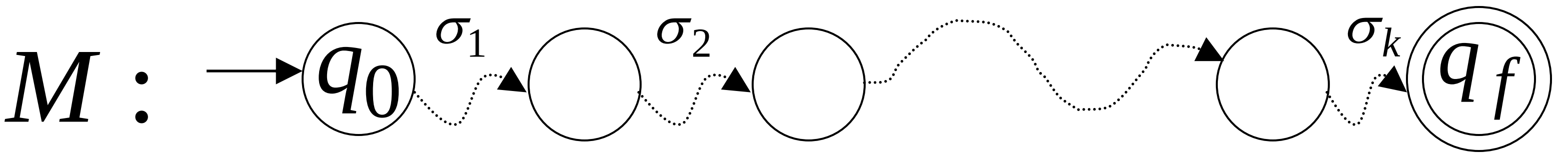


denotes

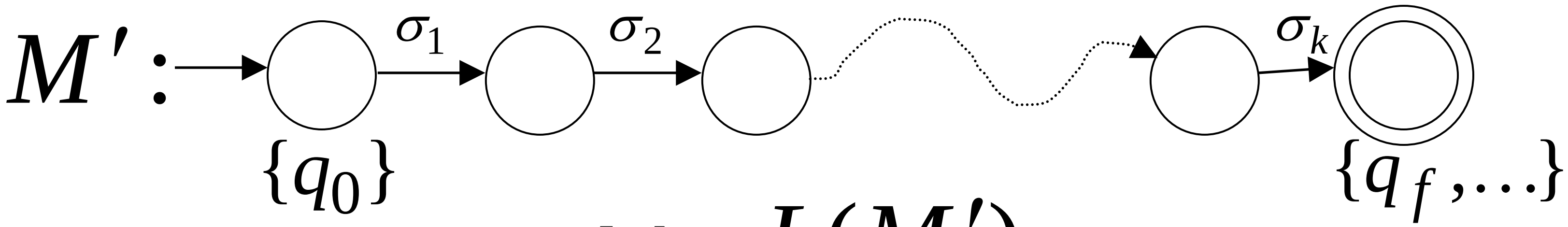


We will show that if $w \in L(M)$

$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$



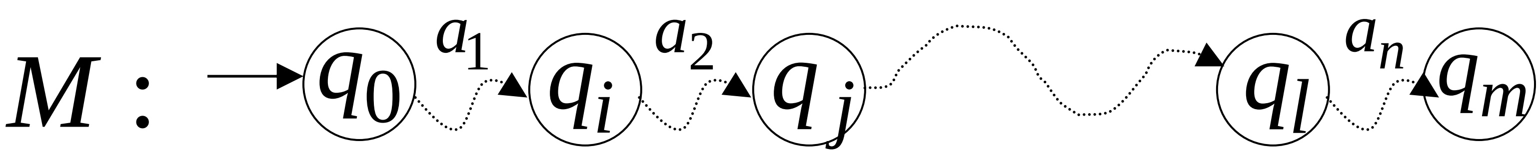
then



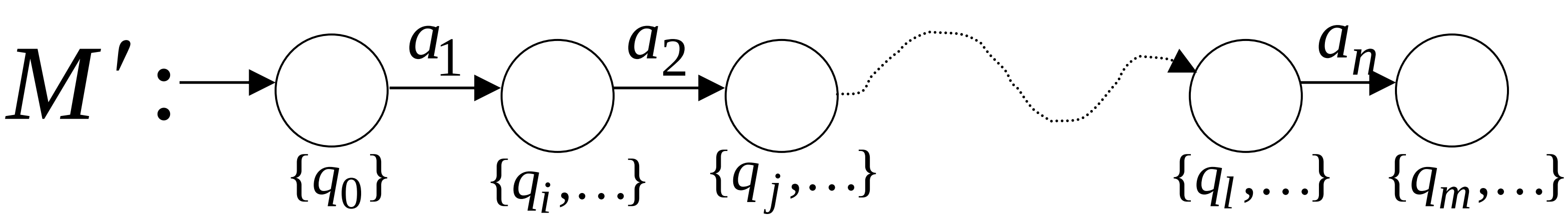
$$w \in L(M')$$

More generally, we will show that if in M:

(arbitrary string) $v = a_1 a_2 \cdots a_n$



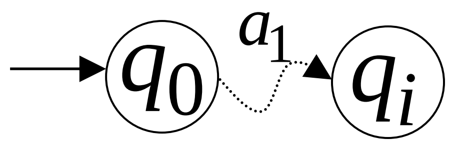
then



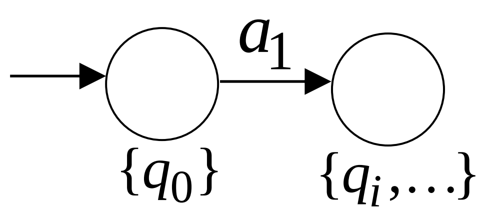
Proof by induction on $|v|$

Induction Basis: $v = a_1$

$M:$



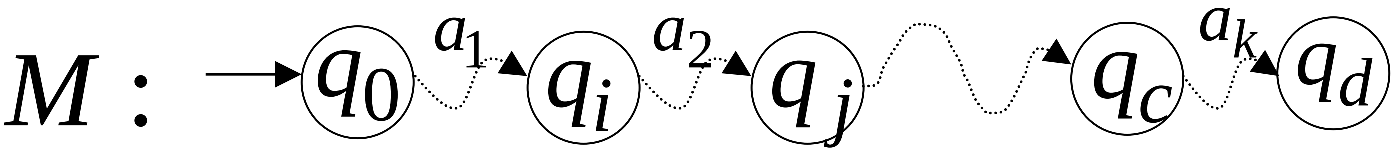
$M':$

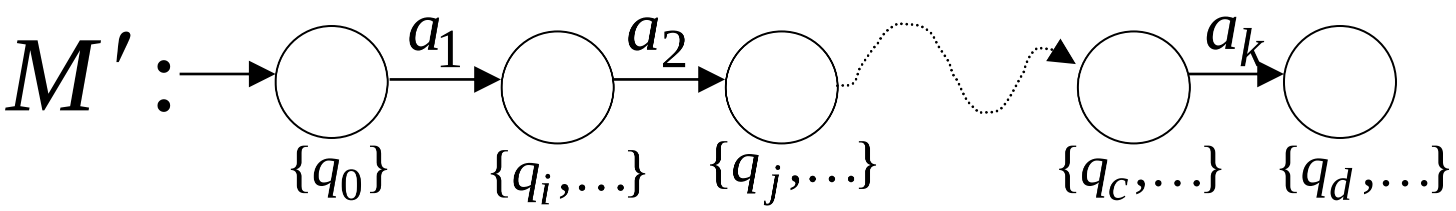


Is true by construction of M':

Induction hypothesis: $1 \le |v| \le k$

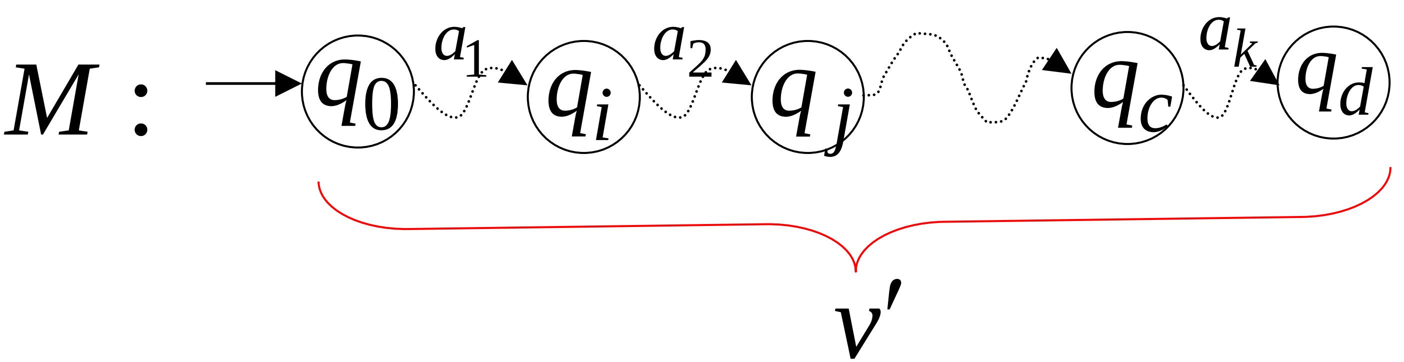
$$v = a_1 a_2 \cdots a_k$$

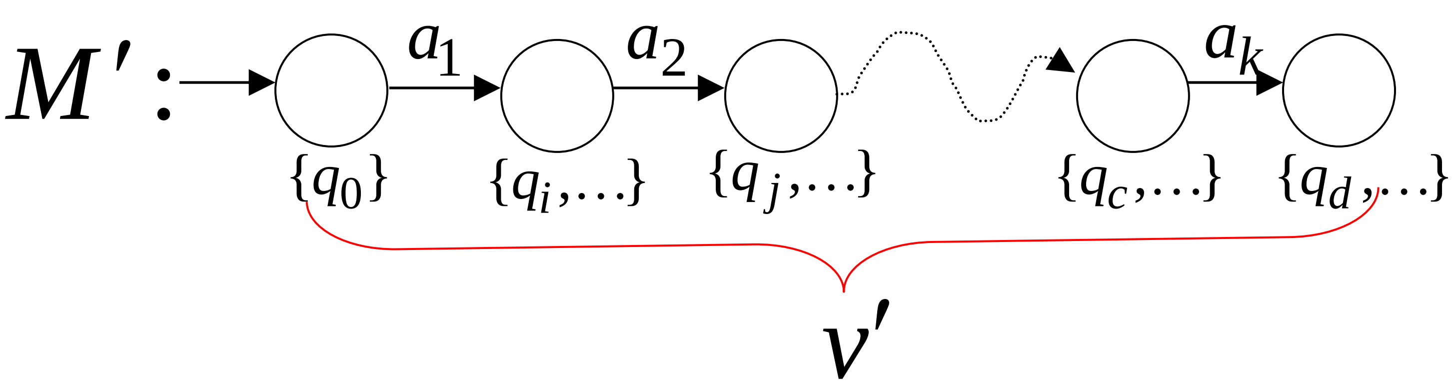




Induction Step: $|v| = k+1$

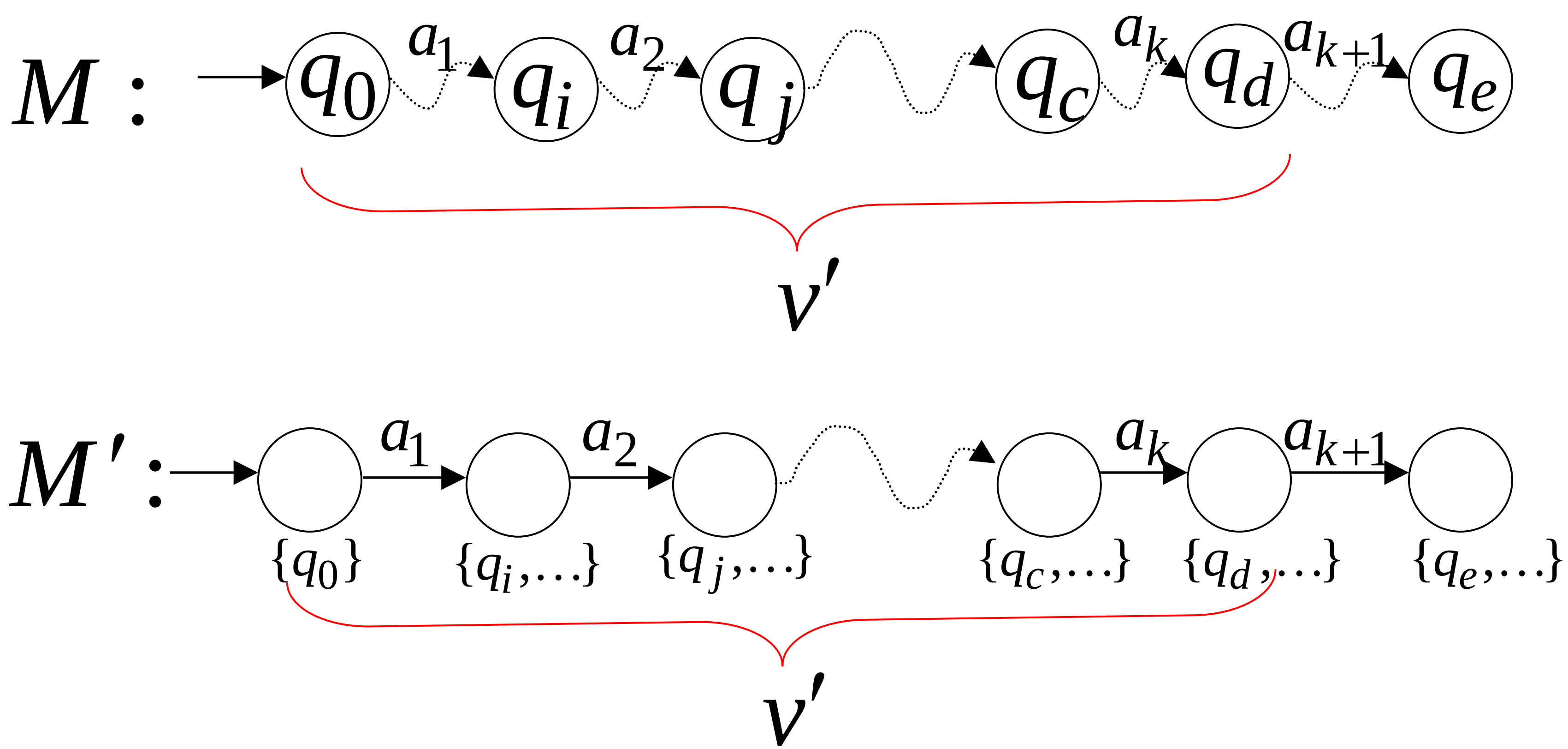
$$v = \underbrace{a_1 a_2 \cdots a_k}_{v'} a_{k+1} = v' a_{k+1}$$





Induction Step: $|v| = k+1$

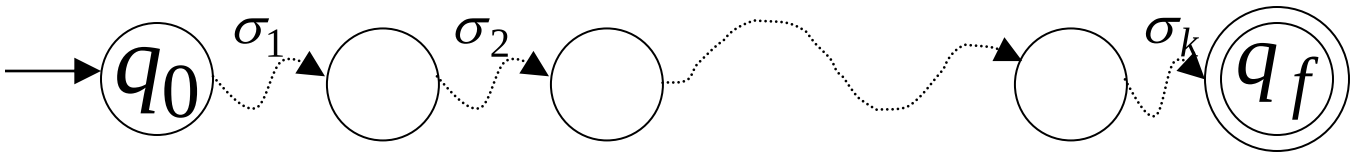
$$v = \underbrace{a_1 a_2 \cdots a_k}_{v'} a_{k+1} = v' a_{k+1}$$



Therefore if $w \in L(M)$

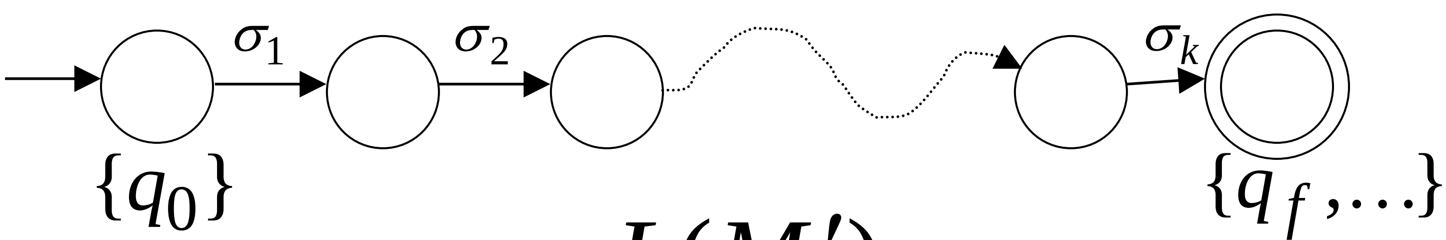
$w = \sigma_1 \sigma_2 \cdots \sigma_k$

$M :$



then

$M' :$



$w \in L(M')$

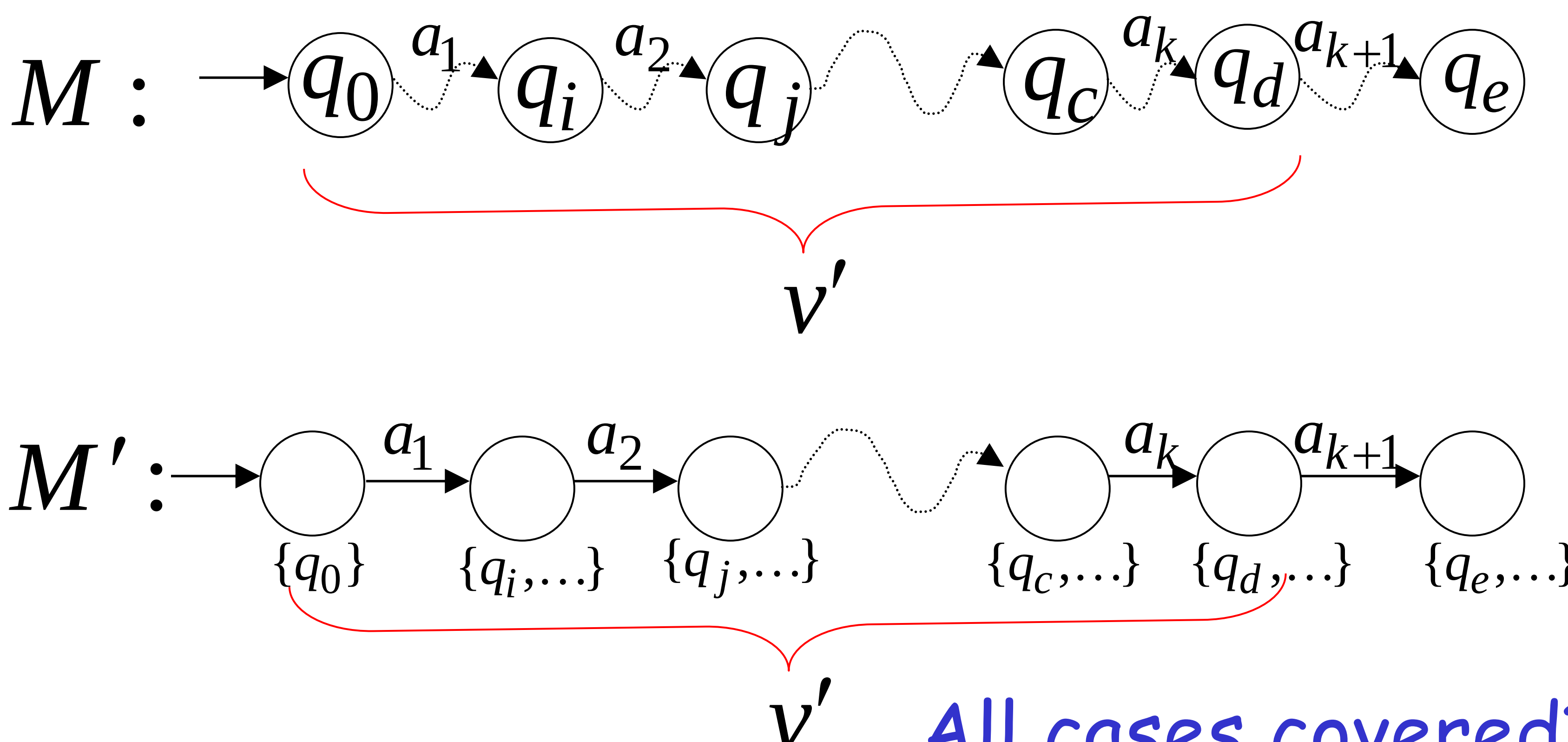
We have shown: $L(M) \subseteq L(M')$

We also need to show: $L(M) \supseteq L(M')$

(proof is similar)

Induction Step: $|v| = k+1$

$$v = \underbrace{a_1 a_2 \cdots a_k}_{v'} a_{k+1} = v' a_{k+1}$$



All cases covered?

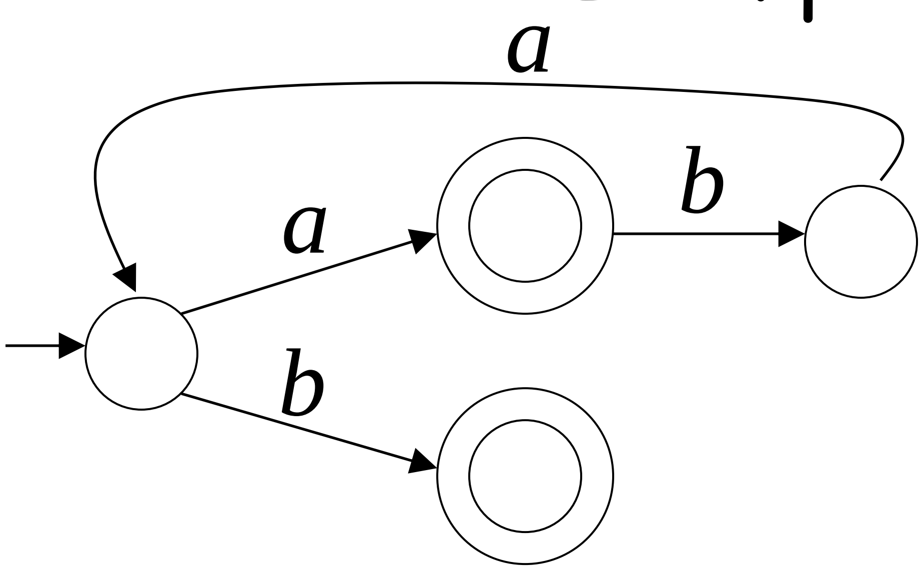
Single Accepting State for NFAs

Any NFA can be converted

to an equivalent NFA

with a single accepting state

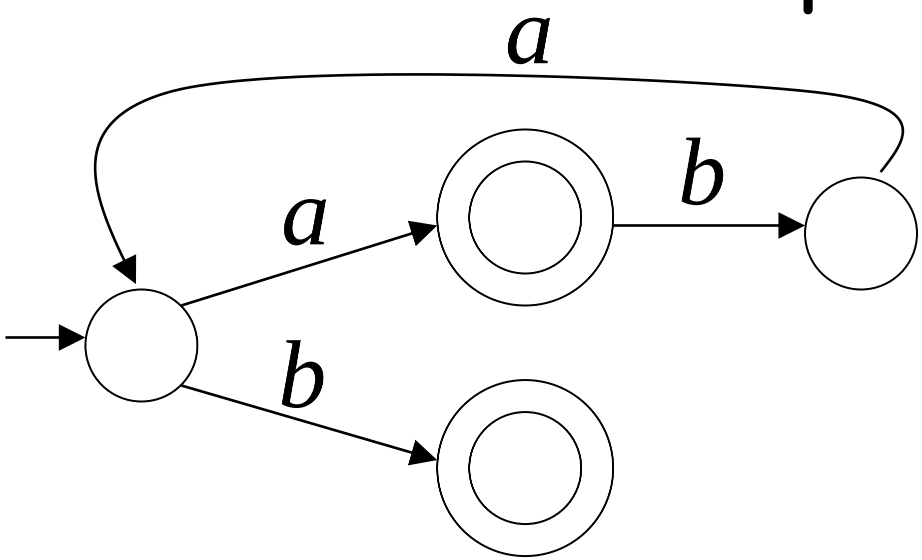
Example



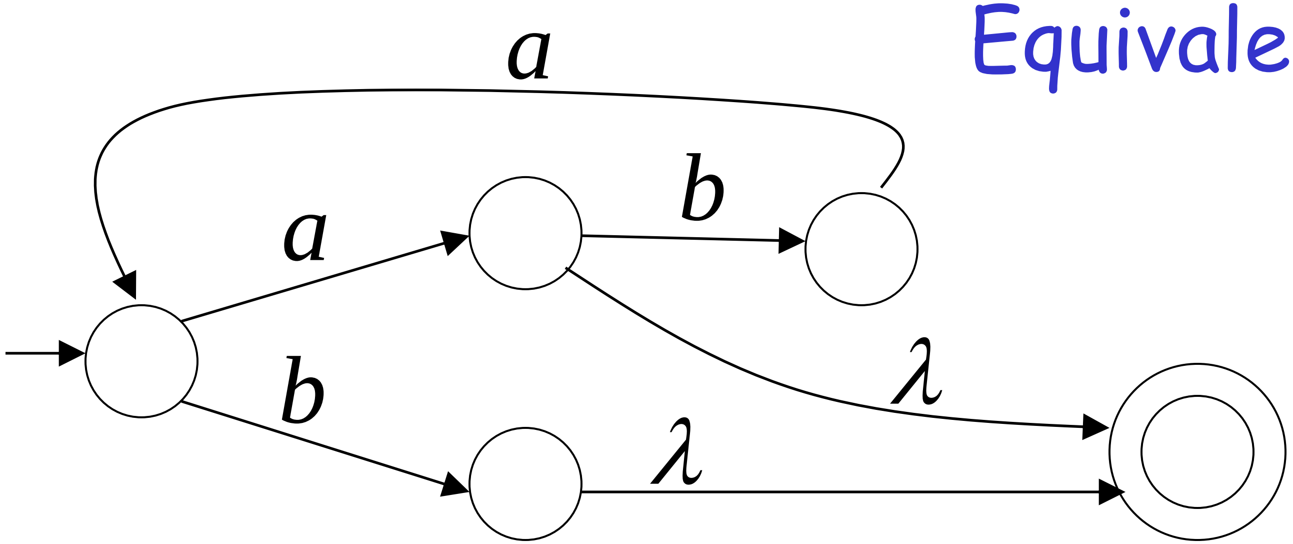
NFA

Equivalent NFA
with single
accepting state?

Example



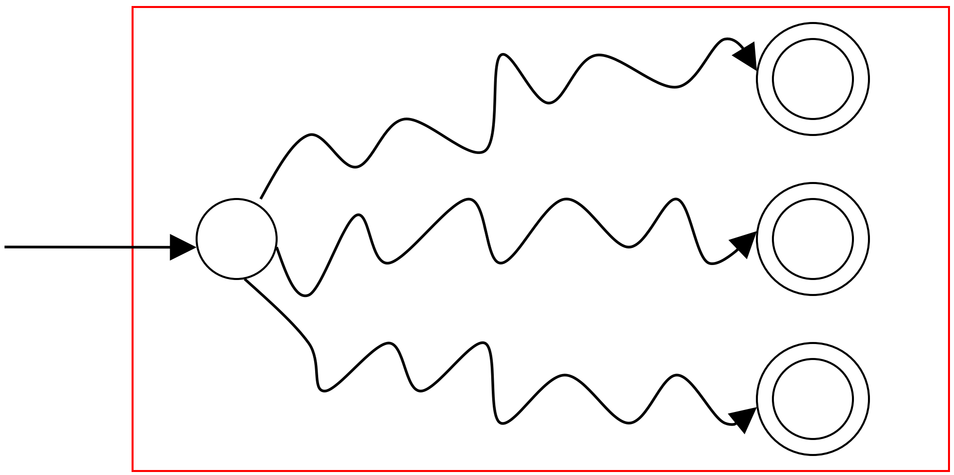
NFA



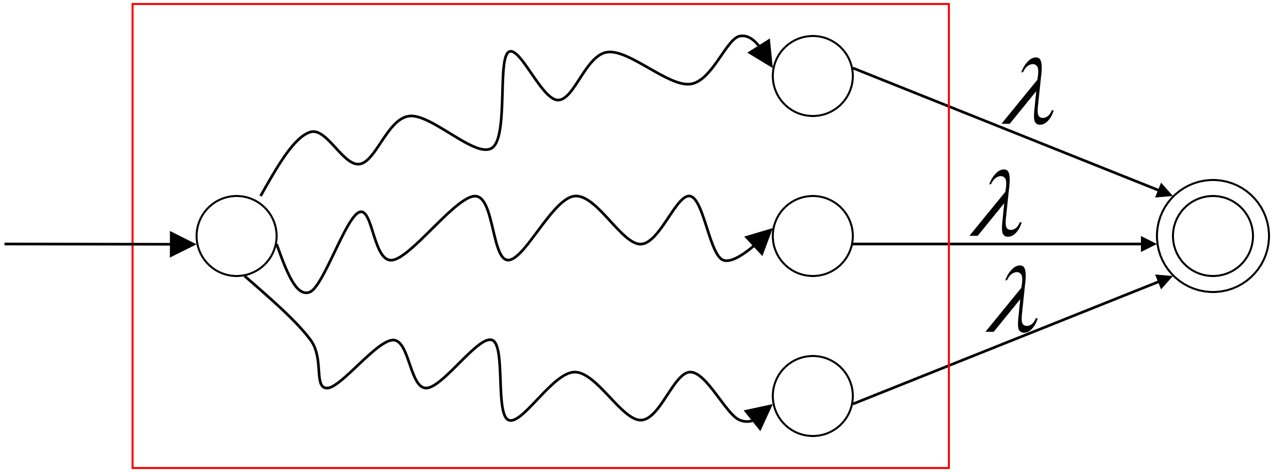
Equivalent NFA

In General

NFA



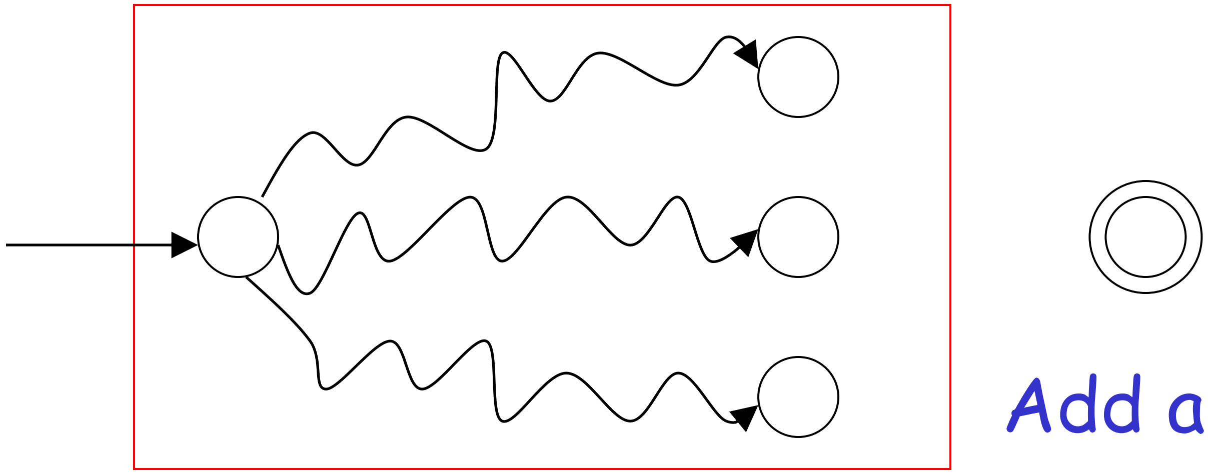
Equivalent NFA



Single accepting state

Extreme Case

NFA without accepting state



Add an accepting state without transitions

Properties of Regular Languages

For regular languages L_1 and L_2
we will prove that:

$$\left.\begin{array}{rl} \text{Union:} & L_1 \cup L_2 \\ \text{Concatenation:} & L_1 L_2 \\ \text{Star:} & L_1 * \\ \text{Reversal:} & L_1^R \\ \text{Complement:} & \overline{L_1} \\ \text{Intersection:} & L_1 \cap L_2 \end{array}\right\}$$

Are regular Languages

We say: Regular languages are **closed under**

Union: $L_1 \cup L_2$

Concatenation: $L_1 L_2$

Star: $L_1 *$

Reversal: L_1^R

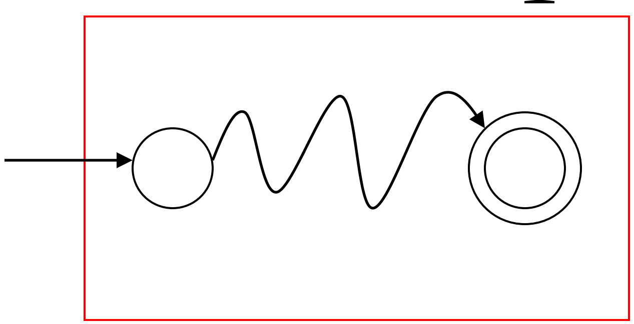
Complement: $\overline{L_1}$

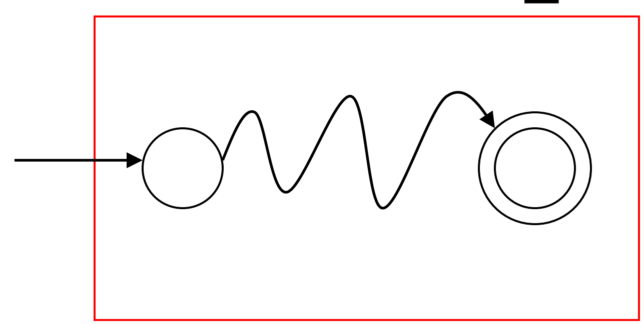
Intersection: $L_1 \cap L_2$

Regular language L_1 Regular language L_2

$L(M_1) = L_1$ $L(M_2) = L_2$

NFA M_1 NFA M_2





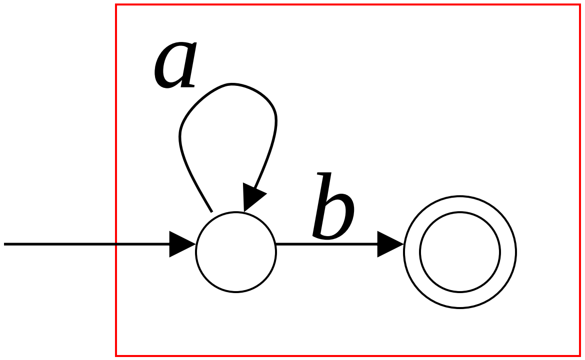
Single accepting state Single accepting state

Example

$$n \geq 0$$

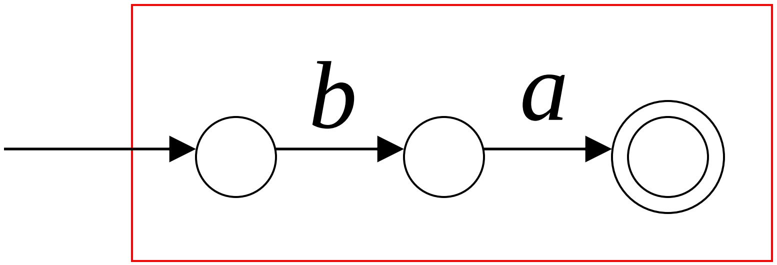
$$L_1 = \{a^n b\}$$

M_1



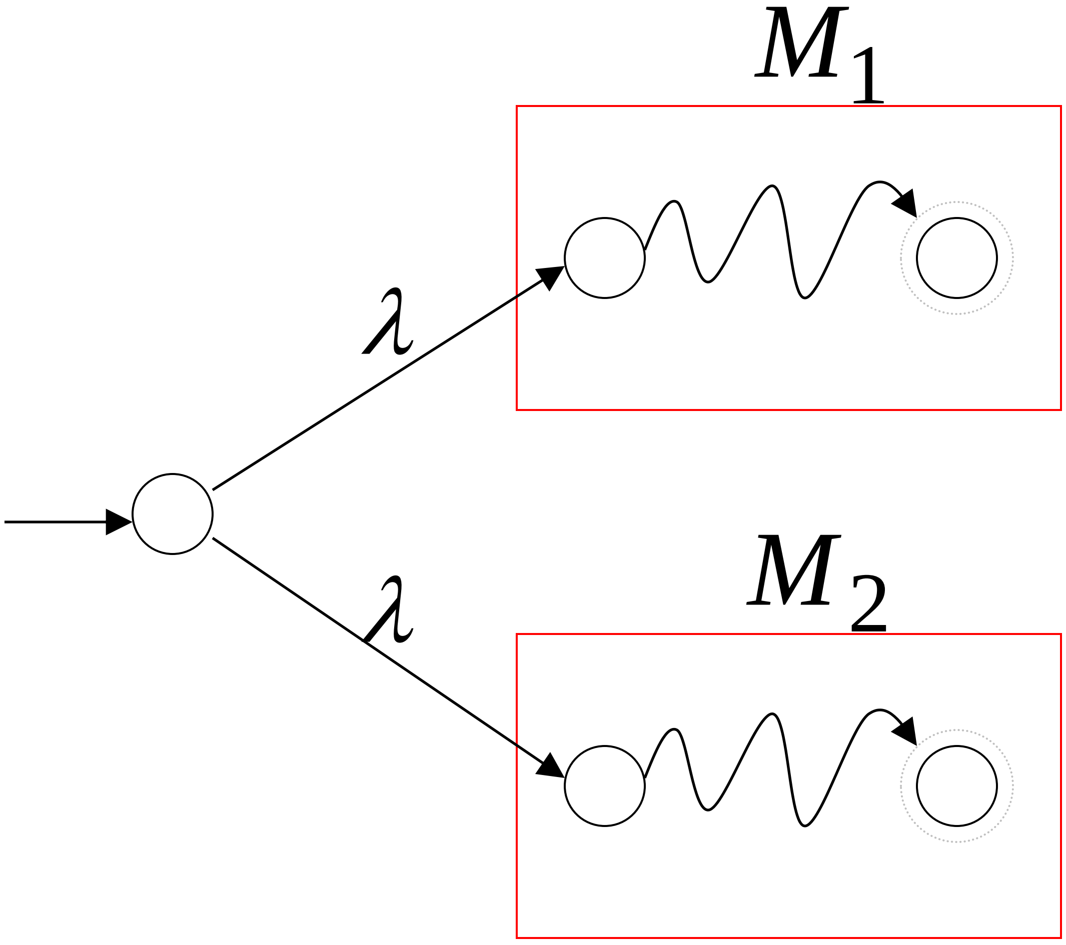
$$L_2 = \{ba\}$$

M_2



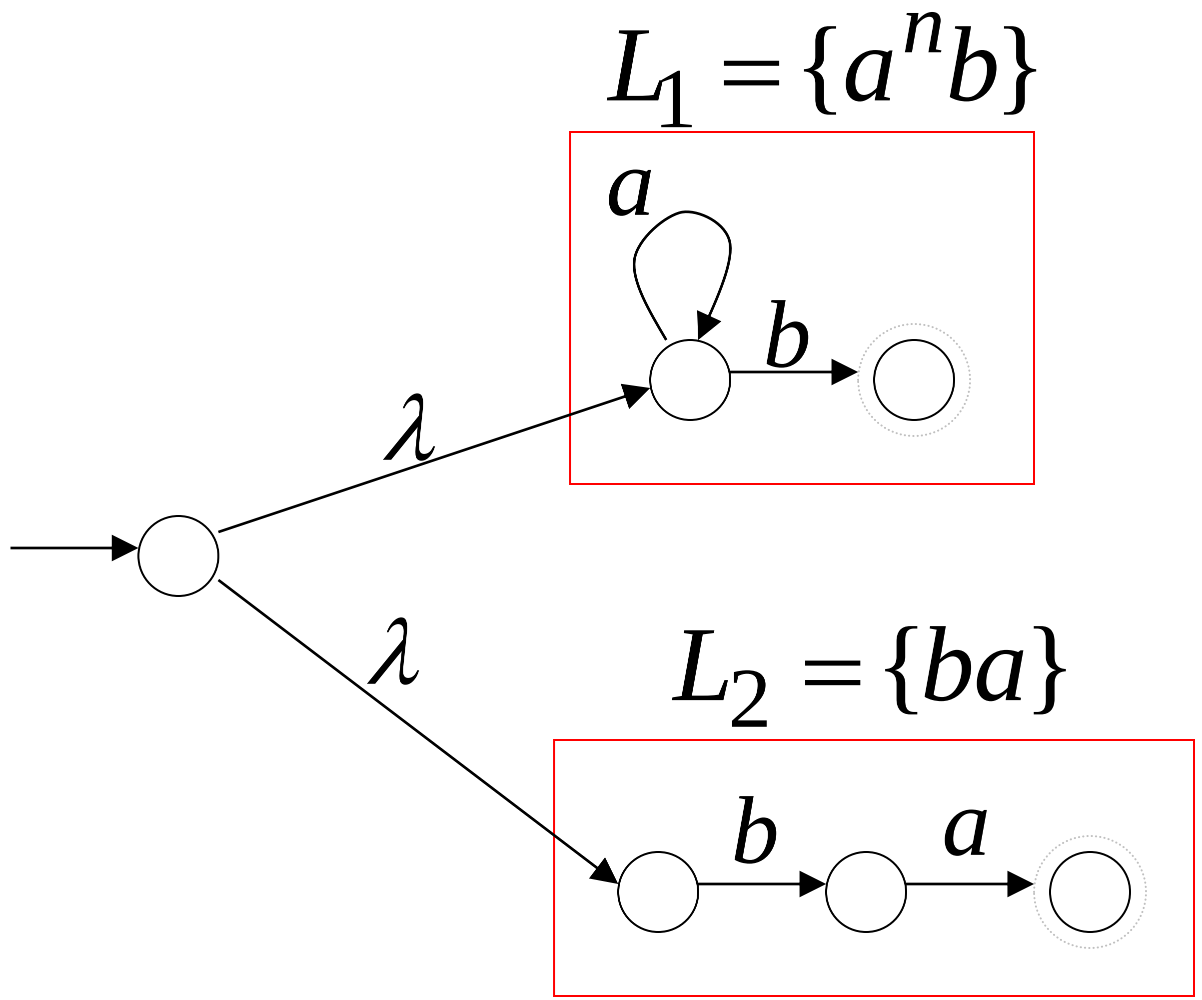
Union

NFA for $L_1 \cup L_2$



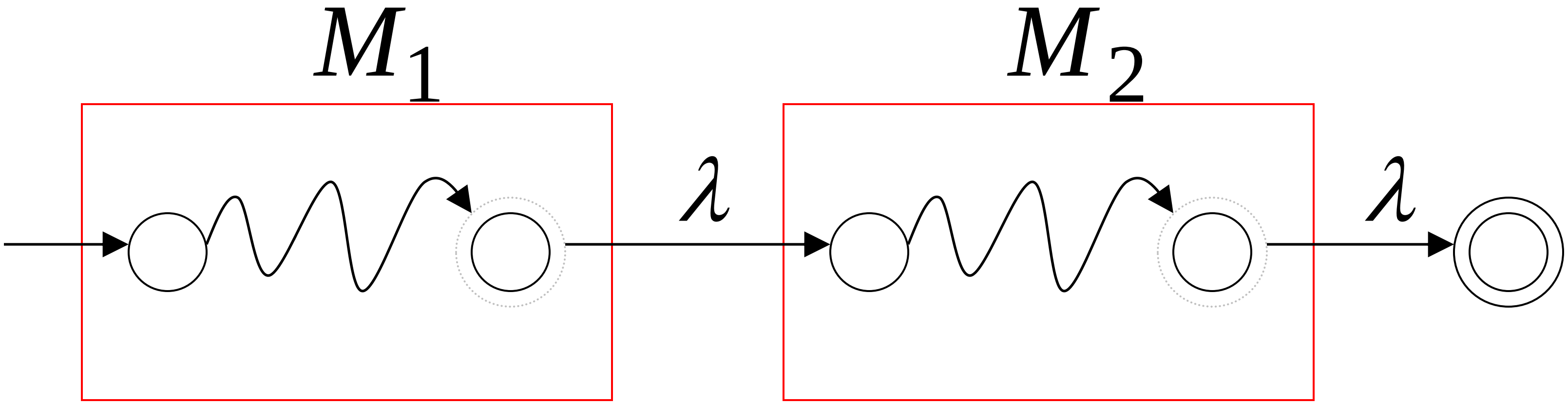
Example

NFA for $L_1 \cup L_2 = \{a^n b\} \cup \{ba\}$



Concatenation

NFA for $L_1 L_2$

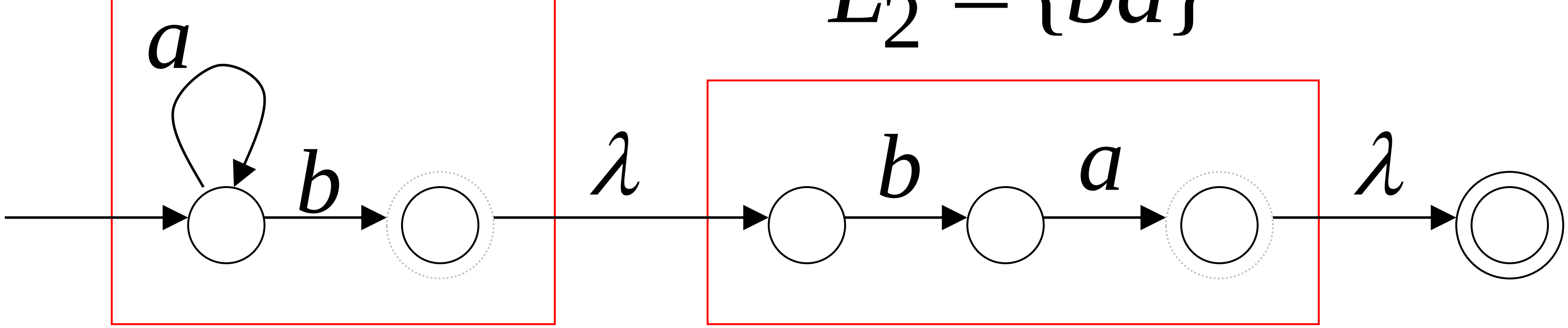


Example

NFA for $L_1 L_2 = \{a^n b\}\{ba\} = \{a^n bba\}$

$L_1 = \{a^n b\}$

$L_2 = \{ba\}$



How do we construct automata for the remaining operations?

Union: $L_1 \cup L_2$

Concatenation: $L_1 L_2$

Star: $L_1{}^*$

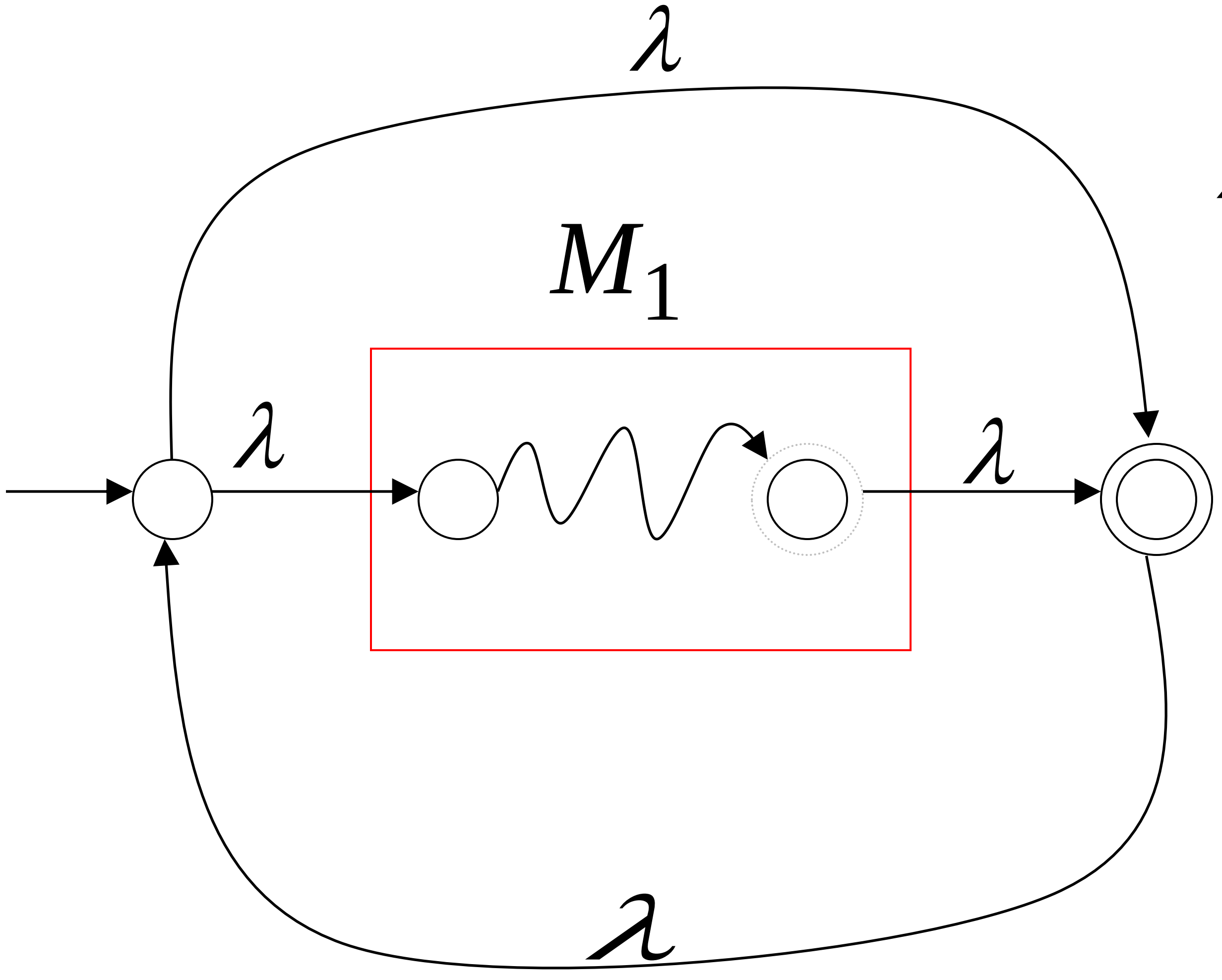
Reversal: $L_1{}^R$

Complement: $\overline{L_1}$

Intersection: $L_1 \cap L_2$

Star Operation

NFA for $L_1{}^*$



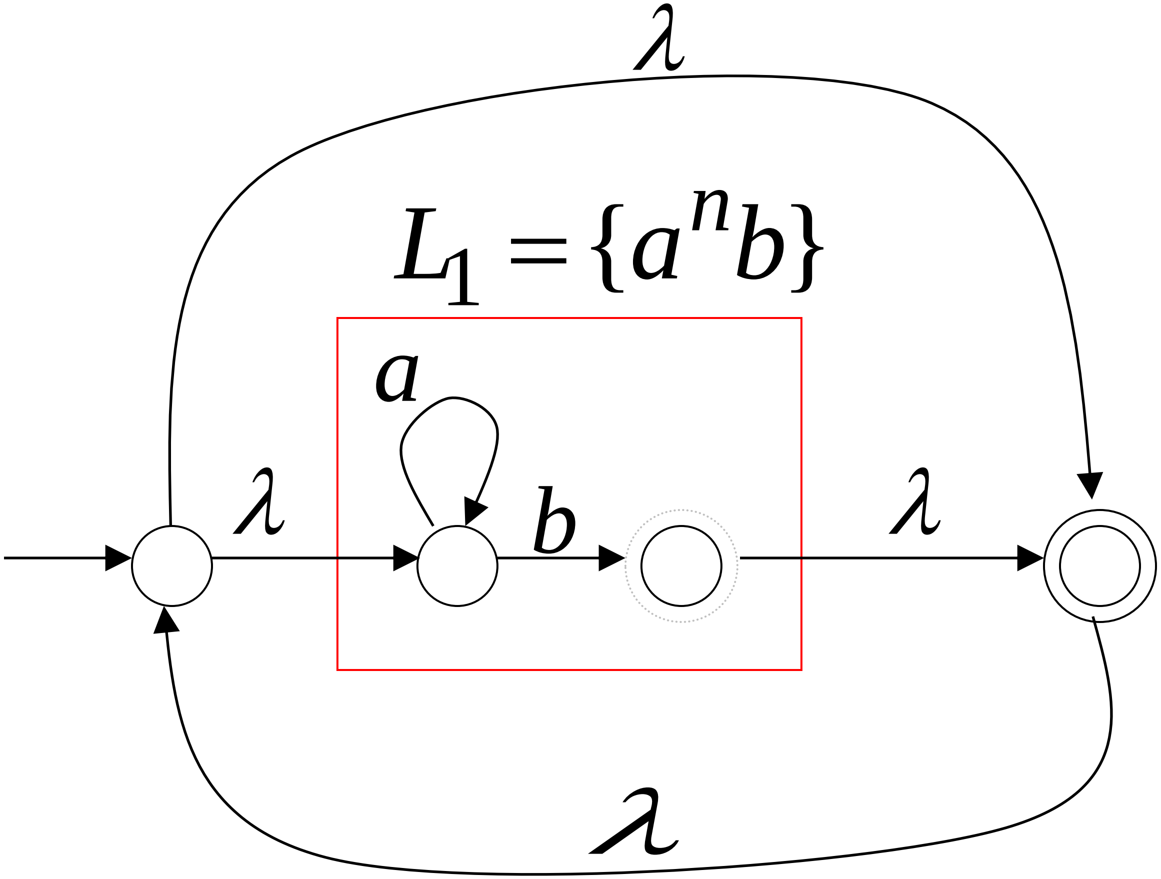
$\lambda \in L_1{}^*$

Example

NFA for $L_1{}^* = \{a^n b\}^*$

$w = w_1 w_2 \cdots w_k$

$w_i \in L_1$



Reverse

NFA for L_1^R

L_1 $\quad M_1$ $\qquad\qquad M_1'$

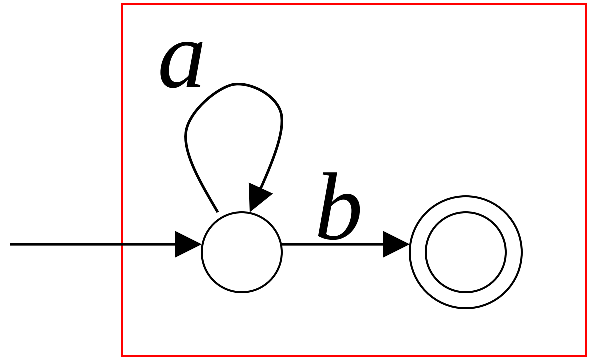
1. Reverse all transitions

2. Make initial state accepting state and vice versa

Example

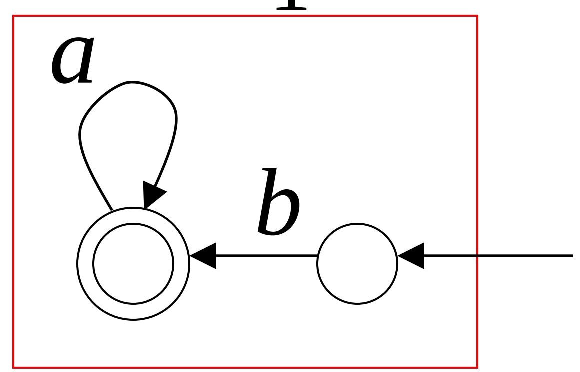
$$L_1 = \{a^n b\}$$

M_1

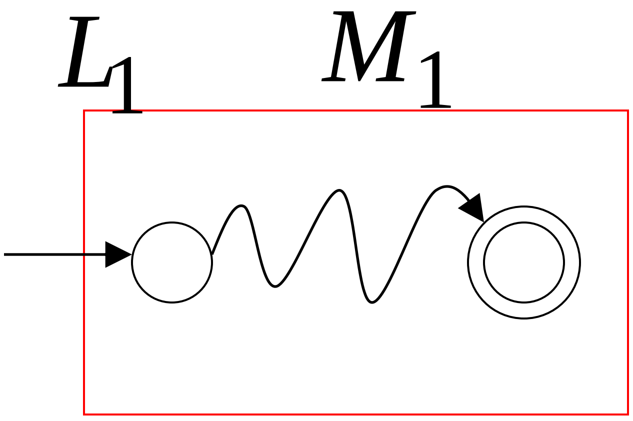


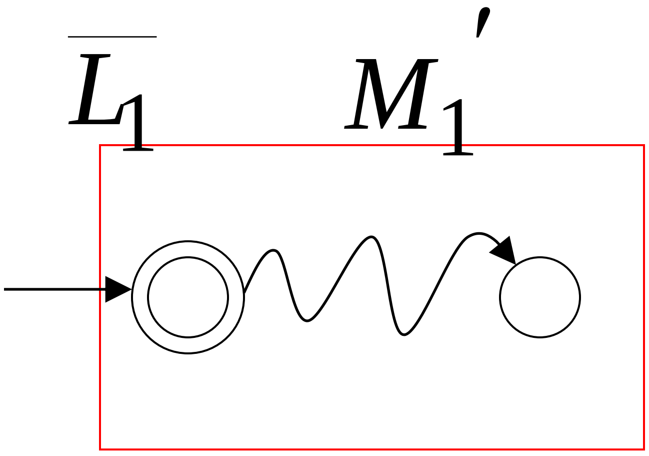
$$L_1^R = \{b a^n\}$$

M_1'



Complement





1. Take the **FA** that accepts L_1

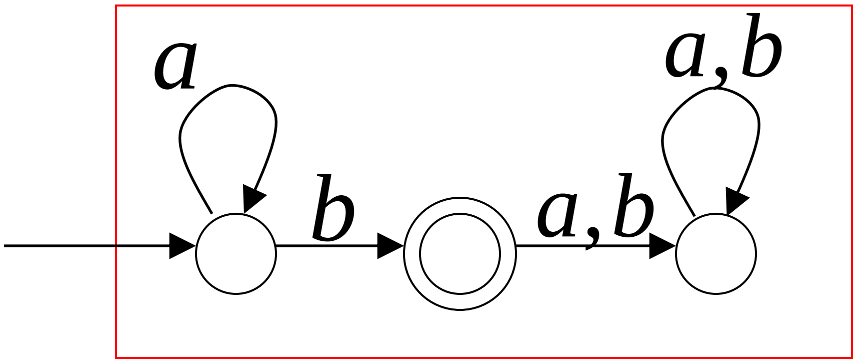
2. Make final states non-final, and vice-versa

Why not NFA?

Example

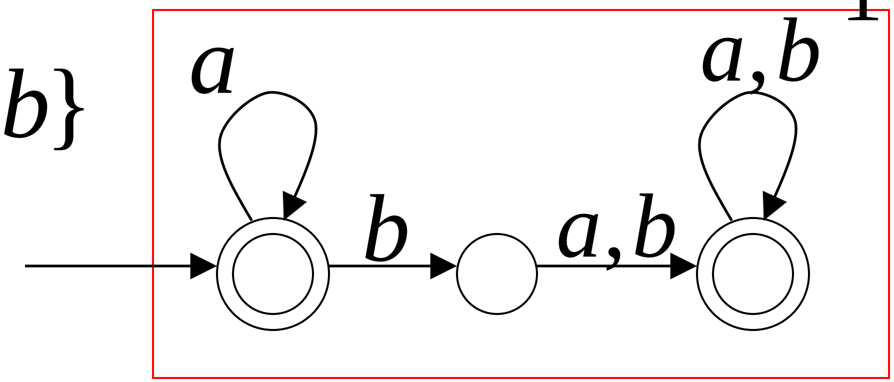
$$M_1$$

$$L_1 = \{a^n b\}$$



$$M_1'$$

$$\overline{L_1} = \{a,b\}^* - \{a^n b\}$$



Intersection

L_1 regular

L_2 regular

We show

$L_1 \cap L_2$

regular

DeMorgan's Law: $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$

$L_1 \, , \;\; L_2$ regular

$\Rightarrow$ $\overline{L_1} \, , \;\; \overline{L_2}$ regular

$\Rightarrow$ $\overline{L_1} \cup \overline{L_2}$ regular

$\Rightarrow$ $\overline{\overline{L_1} \cup \overline{L_2}}$ regular

$\Rightarrow$ $L_1 \cap L_2$ regular

Example

$L_1 = \{a^n b\}$ regular

$L_2 = \{ab, ba\}$ regular

$\Rightarrow$ $L_1 \cap L_2 = \{ab\}$

regular

Another Proof for Intersection Closure

Machine M_1

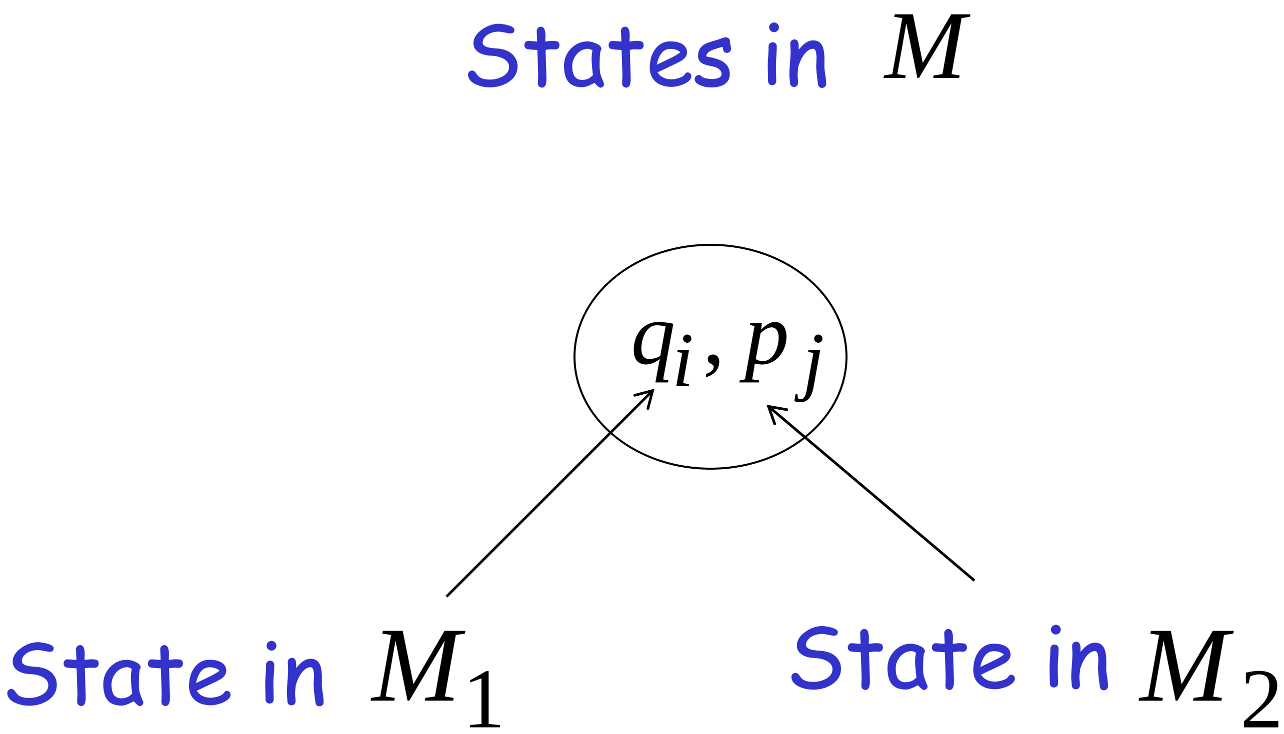
FA for L_1

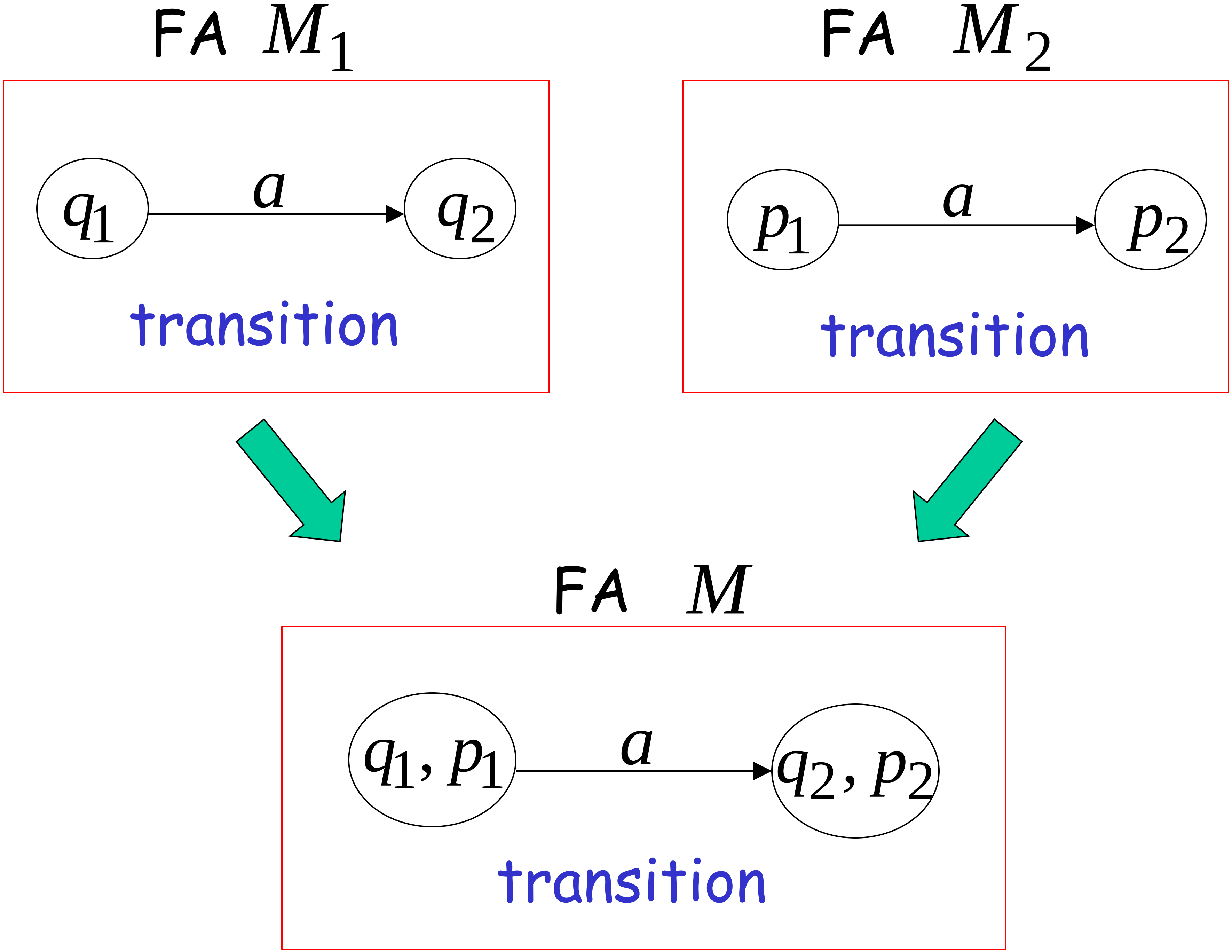
Machine M_2

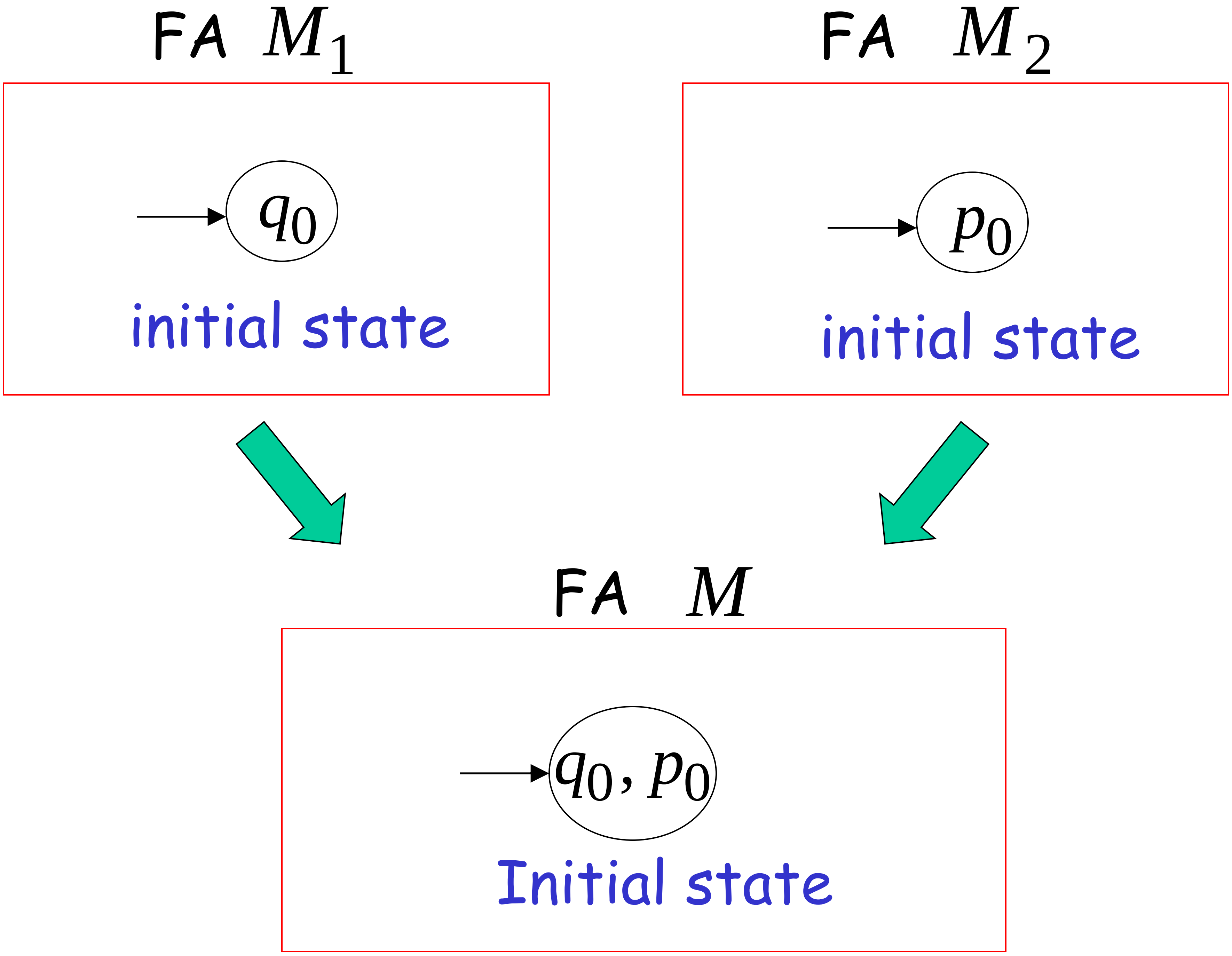
FA for L_2

Construct a new FA M that accepts $L_1 \cap L_2$

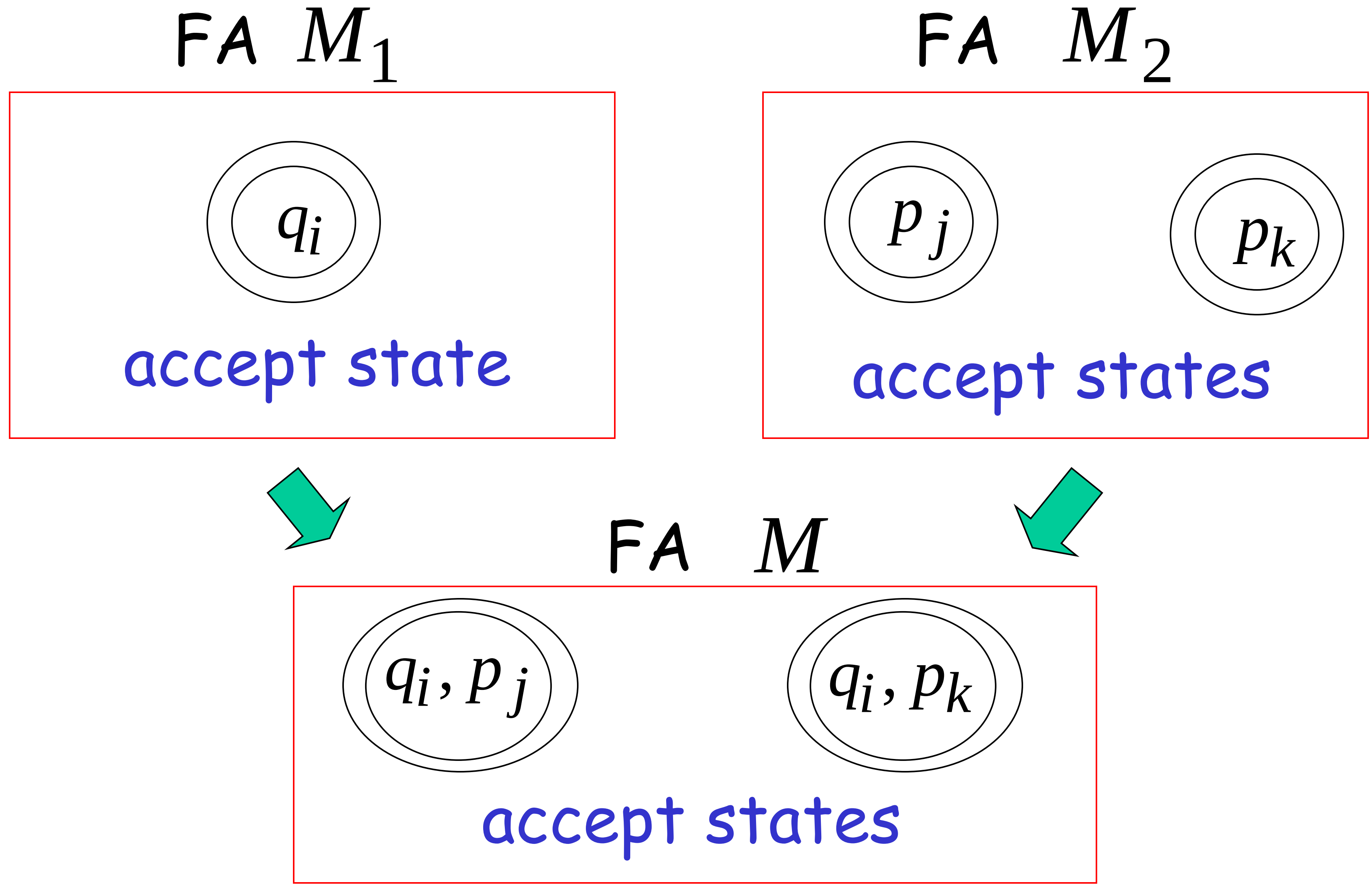
M simulates in parallel M_1 and M_2


States in M
q_i, p_j
State in M_1
State in M_2


FA M_1
q_1
a
q_2
transition
FA M_2
p_1
a
p_2
transition
FA M
q_1, p_1
a
q_2, p_2
transition



FA M_1
q_0
initial state
FA M_2
p_0
initial state
FA M
q_0, p_0
Initial state



Both constituents must be accepting states

M simulates in parallel M_1 and M_2

M accepts string w if and only if

M_1 accepts string w and

M_2 accepts string w

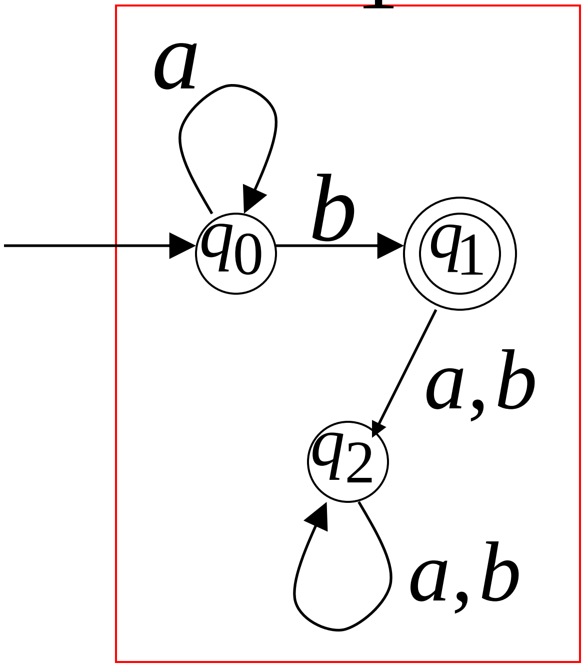
$$L(M) = L(M_1) \cap L(M_2)$$

Example:

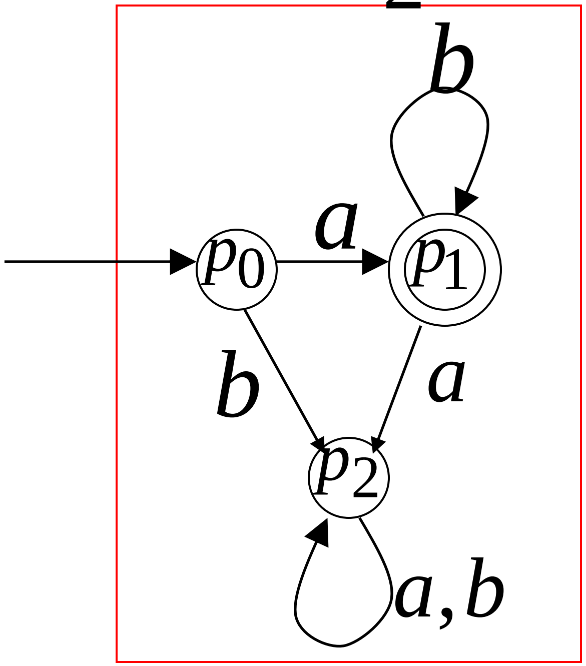
$$L_1 = \{a^n b\} \quad n \geq 0$$

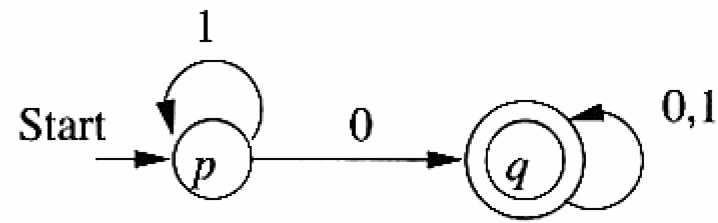
$$L_2 = \{ab^m\} \quad m \geq 0$$

M_1

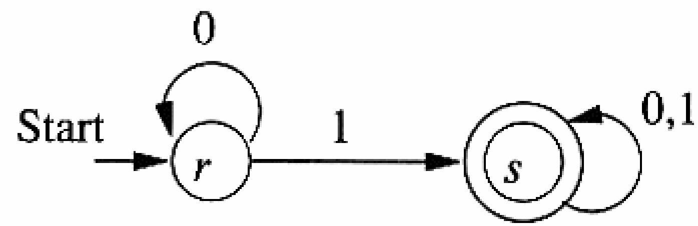


M_2

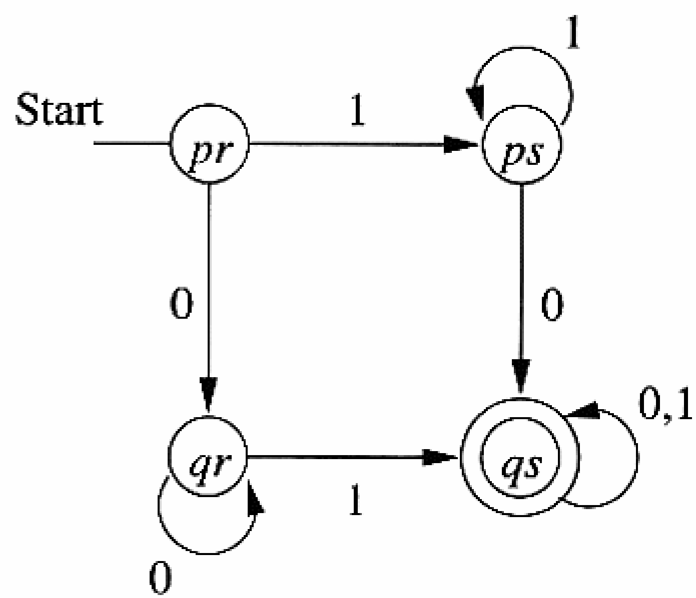




(a)



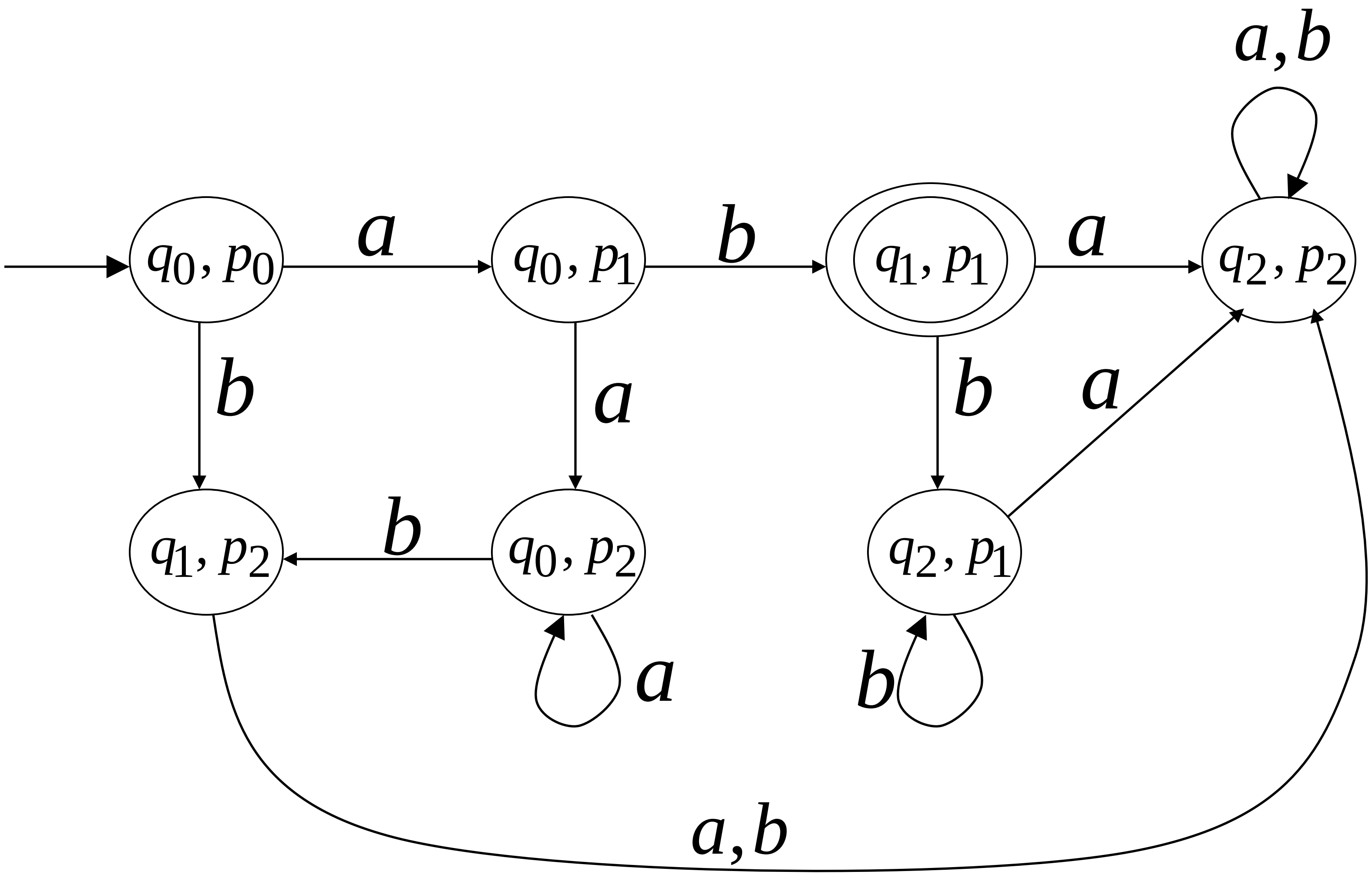
(b)



Construct machine for intersection

Automaton for intersection

$$L = \{a^n b\} \cap \{ab^n\} = \{ab\}$$



Note how easy it was to prove closure under union, star, concatenation with NFAs. Would be much harder with DFAs.