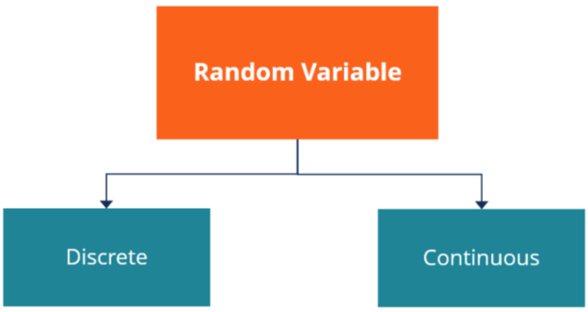
Random Variable

Definition A random variable is a **real valued function whose domain is the sample space of a random experiment.**
For example, let us consider the experiment of tossing a coin two times in succession.
The sample space of the experiment is S = {HH, HT, TH, TT}.
If X denotes the number of heads obtained, then X is a random variable and for each outcome, its value is as given below:
X(HH) = 2, X (HT) = 1, X (TH) = 1, X (TT) = 0.
More than one random variable can be defined on the same sample space. For example, let Y denote the number of heads minus the number of tails for each outcome of the above sample space S.
Then Y(HH) = 2, Y (HT) = 0, Y (TH) = 0, Y (TT) = – 2.
Thus, X and Y are two different random variables defined on the same sample space S.

Example A person plays a game of tossing a coin thrice. For each head, he is given Rs 2 by the organiser of the game and for each tail, he has to give Rs 1.50 to the organiser. Let X denote the amount gained or lost by the person. Show that X is a random variable and exhibit it as a function on the sample space of the experiment.
Solution X is a number whose values are defined on the outcomes of a random experiment. Therefore, X is a random variable.
Now, sample space of the experiment is
S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}
Then X(HHH) = Rs (2 × 3) = Rs 6
X(HHT) = X(HTH) = X(THH) = Rs (2 × 2 - 1 × 1.50) = Rs 2.50
X(HTT) = X(THT) = (TTH) = Rs (1 × 2) – (2 × 1.50) = – Re 1
and X(TTT) = Rs (3 × -1.50) = -Rs 4.50
where, minus sign shows the loss to the player. Thus, for each element of the sample space, X takes a unique value, hence, X is a function on the sample space whose range is {–1, 2.50, –4.50, 6}



1. Discrete

A discrete random variable is a (random) variable whose values take only a finite number of values. The best example of a discrete variable is a dice. Throwing a dice is a purely random event. At the same time, the dice can take only a finite number of outcomes {1, 2, 3, 4, 5, and 6}.

2. Continuous

Unlike discrete variables, continuous random variables can take on an infinite number of possible values. One of the examples of a continuous variable is the returns of stocks. The returns can take an infinite number of possible values (as percentages).

Probability Mass Function:

Let X be a discrete random variable with range $\{x_1, x_2, x_3, \ldots x_n\}$. The function $P_X(x_k) = P(X = x_k)$, for k=1,2,3,...,n is called the *probability mass function* X if it satisfies the following conditions:

1. $P(x_i) \geq 0, \forall\, i$
2. $\sum P(x_i) = 1$

Example: If X denotes the number of heads obtained. The sample space of the experiment is S = {HH, HT, TH, TT}.

X=x :	0	1	2
P(X=x)	¼	½	¼

$$\sum_{i=1}^{3} P(X = x_i) = 1$$

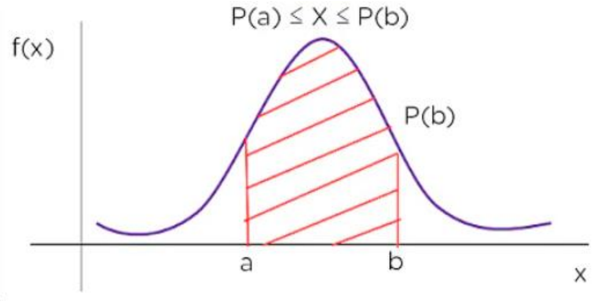
Probability Density Function:

Let X be continuous random variable if there exists a function f(x) is called the PDF of X satisfies the following Conditions.

1. $f(x) \geq 0$.
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

Note:

1. $P(a \leq x \leq b) = \int_a^b f(x)dx$
2. $P(x \geq a) = \int_a^{\infty} f(x)dx$
3. $P(x \leq a) = \int_{-\infty}^{a} f(x)dx$



Cumulative Distribution Function:

The *cumulative distribution function* (CDF) $F_X(x)$ describes the probability that a random variable *X* with a given probability distribution will be found at a value less than or equal to *x*, *i.e F(X)=P(X ≤x).*

1. If X is a discrete random variable

$$F(x) = \sum_{x_j \le x} P(X = x_j)$$

2. If X is a continuous random variable with PDF "f"

$$F_X(x) = P\left[X \le x\right] = \int_{-\infty}^{x} f_X(u)du$$

Note:

1. F(-∞)= 0 and F(∞)= 1
2. If X is a continuous random variable, with PDF "f" and CDF 'F' then $f(x)= d/dx(F(x))$
3. Let X is a discrete random variable and F is a CFD of X. $P(X = x_j) = F(x_j) - F(x_{j-1})$

Problems:

1. Suppose a random variable X assumes the value 0,1,2 with the probabilities 1/3, 1/6 and ½ respectively. Find the cumulative distribution function.

Ans:

X	0	1	2
P(X)	1/3	1/6	1/2

F(X)= P(X=x)

x<0	F(x)=0
$0 \le x < 1$	F(x)=0+1/3=1/3
$1 \le x < 2$	F(x)= 0+1/3+1/6=1/2
$x \ge 2$	F(x)= 0+1/3+1/6+1/2=1

Therefore,

$$F(x)=\begin{cases} 0 & x<0 \\ \frac{1}{3} & 0 \le x < 1 \\ \frac{1}{2} & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

2. Suppose X is a continuous random variable with PDF

$f(x)= \begin{cases} 2x & 0 < x < 1 \\ 0 & elsewhere \end{cases}$ Find CDF.

Ans: 0, x^2 and 1.

3. Let X be a continuous random variable with probability density function f(x) is defined by;

$$f(x)=\begin{cases} ax & 0 \le x \le 1 \\ a & 1 \le x \le 2 \\ -ax+3a & 2 \le x \le 3 \\ 0 & elsewhere \end{cases}$$

Then find (i)Value of "a" (ii) CDF of X

(iii) P (1\3≤ $x \le 2\backslash 3$), P(X≥ $1\backslash 2$), P($X \le 1\backslash 4$)

4. A coin is known to come up 3 times head as often as tail. The coin is tossed 3 times. Let X denotes the number of heads that appears write the probability distribution of X and find CDF of X.

5. A random variable X has the probability distribution.

X	0	1	2	3	4	5	6	7
P(X=x)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

Find (i) k (ii)Find the smallest value of λ *for which* P ($x \le \lambda$)> ½.

6. A random variable X has the probability function P(X=k)= $\frac{c}{2^k}$, k=0,1,2,…,n

Find (i) Find C (ii) P(X≥ 5) (iii) P(X≤ $\frac{1}{2}$)

7. Let X be a continuous random variable with pdf

$$f(x)=\begin{cases} kx^4 & 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

Find i) k ii) P(1/4 < x < ¾) iii) P(x>1/2) iv) P(x < 1/8)

8. The diameter of an electric cable X is assumed to be a continuoius random variable with pdf F(x)= 6x(1-x) 0≤ $x \le 1$; otherwise 0.
   i) Check f(x) is a pdf or not
   ii) Obtain an expression for cdf of x
   iii) Compute P(x≤ $1/2$ / 1/3 <x< 2/3)
   iv) Determine b so that P(x<b)= 2P(X≥ b)

9. $f(x)=\begin{cases} 0 & x \le 0 \\ 1-e^{-x} & x > 0 \end{cases}$ is a cdf of X find the pdf of X.