Functions of One dimensional random variables

If X is a discrete random variable and $Y = H(X)$ is a continuous function of X, then Y is also a Discrete Random Variable.

Eg:

X	-1	0	1
$P(x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Suppose $Y = 3X + 1$,then pmf of Y is given by

Y	-2	1	4
$P(y)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Suppose $Y = X^2$, then pmf of $Y$is

Y	1	0
$P(y)$	$\frac{1}{2}$	$\frac{1}{2}$

Suppose X is a continuous random variable with pdf $f(x)$ and $H(X)$ is a continuous function of X. Then Y is a continuous random variable. To obtain pdf of Y we follow the following steps.

1. Obtain cdf of Y, i.e., $G(y) = P(Y \leq y)$.
2. Differentiate $G(y)$ with respect to y to get pdf of y i.e., $g(y)$.
3. Determine the range space of Y such that $g(y) > 0$.

Problems:

1. If $f(x) = \begin{cases} 2x; & 0 < x < 1 \\ 0; & Otherwise \end{cases}$, and $Y = 3X + 1$, find pdf of Y.

Soln: $G(y) = P(Y \leq y) = P(3X + 1 \leq y) = P\left(X \leq \frac{y-1}{3}\right)$

$$G(y) = \int_0^{\frac{y-1}{3}} 2x dx = \left(\frac{y-1}{3}\right)^2.$$

$$g(y) = G'(y) = \frac{2(y-1)}{9}.$$

$$0 < x < 1 \Rightarrow 0 < \frac{y-1}{3} < 1 \Rightarrow 1 < y < 4.$$

Therefore, $g(y) = \begin{cases} \frac{2(y-1)}{9}; & 1 < y < 4 \\ 0; & Otherwise \end{cases}.$

2. If $f(x) = \begin{cases} 2x; & 0 < x < 1 \\ 0; & Otherwise \end{cases}$, and $Y = e^{-X}$, find pdf of Y.

Soln: $G(y) = P(Y \leq y) = P(e^{-X} \leq y) = P\left(\log_e \frac{1}{y} \leq X\right)$

$$G(y) = \int_{\log_e \frac{1}{y}}^{1} 2x dx = 1 - \left(\log_e \frac{1}{y}\right)^2 \quad .$$

$$g(y) = G'(y) = \frac{2}{y} \log_e \frac{1}{y}.$$

$$0 < x < 1 \Rightarrow 0 < \log_e \frac{1}{y} < 1 \Rightarrow \frac{1}{e} < y < 1.$$

Therefore, $g(y) = \begin{cases} \frac{2}{y} \log_e \frac{1}{y}; & \frac{1}{e} < y < 1 \\ 0; & Otherwise \end{cases}$.

Result: Let X be a continuous random variable with pdf $f(x)$. Let $Y = X^2$. Then pdf of Y is $g(y) = \frac{1}{2\sqrt{y}}\left(f(\sqrt{y}) + f(-\sqrt{y})\right)$

Example 1: Suppose $f(x) = \begin{cases} 2xe^{-x^2}; & 0 < x < \infty \\ 0; & Otherwise \end{cases}$. Find pdf of $Y = X^2$.

Soln:

$$g(y) = \frac{1}{2\sqrt{y}}\left(f(\sqrt{y}) + f(-\sqrt{y})\right) = \frac{1}{2\sqrt{y}}\left(2\sqrt{y}e^{-y} + 0\right) = e^{-y}; 0 < x < \infty.$$

Example 2: Suppose $f(x) = \begin{cases} \frac{2}{9}(x+1); & -1 < x < 1 \\ 0; & Otherwise \end{cases}$. Find pdf of $Y = X^2$.

Soln:

$$g(y) = \frac{1}{2\sqrt{y}}\left(f(\sqrt{y}) + f(-\sqrt{y})\right) = \frac{1}{2\sqrt{y}}\left(\frac{2(\sqrt{y}+1)}{9} + \frac{2(-\sqrt{y}+1)}{9}\right) = \frac{2}{9\sqrt{y}}; 0 < x < 1.$$

Theorem: Let X be a continuous random variable with pdf $f(x)$. Suppose $Y = H(X)$ is a strictly monotone (increasing or decreasing) function of X, then pdf of Y is given by

$g(y) = f(x)\left|\frac{dx}{dy}\right|$ where $x = H^{-1}(y)$.

Example:

1. Suppose X is uniformly distributed over (0,1), find pdf of $Y = \frac{1}{X+1}$.

   Soln: We know that Y is strictly monotone.

   $f(x) = \begin{cases} 1; & 0 < x < 1 \\ 0; & Otherwise \end{cases}$.

   Note that $X = \frac{1}{Y} - 1. \Rightarrow f(x) = f\left(\frac{1}{Y} - 1\right) = 1.$

$\left|\frac{dx}{dy}\right| = \frac{1}{y^2}$.

Therefore, $g(y) = \frac{1}{y^2}; \frac{1}{2} < y < 1$.

2. If X is uniformly distributed over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, find the pdf of $Y = tanX$. (Or show that $Y = tanX$ follows Cauchy's distribution).

   Soln: Given $f(x) = \begin{cases} \frac{1}{\pi}; & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0; & Otherwise \end{cases}$.

   We know that Y is strictly monotone.

   Then $X = \tan^{-1} Y \Rightarrow f(\tan^{-1} Y) = \frac{1}{\pi}$. And $\left|\frac{dx}{dy}\right| = \frac{1}{1+y^2}$.

   Therefore, $g(y) = \frac{1}{\pi}\frac{1}{1+y^2}; \ -\infty < y < \infty$.

3. If $X \sim N(\mu, \sigma^2)$, then show that $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$ and $Y = Z^2 \sim \chi^2(1)$.

   Soln: $G(z) = P(Z \leq z) = P\left(\frac{X-\mu}{\sigma} \leq z\right) = P(\sigma z + \mu \geq x)$

   $$G(z) = F(\sigma z + \mu).$$

   $g(z) = G'(z) = F'(\sigma z + \mu)\sigma = f(\sigma z + \mu)\sigma = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sim N(0,1)$.

   $Now, g(y) = \frac{1}{2\sqrt{y}}\left(f(\sqrt{y}) + f(-\sqrt{y})\right) = \frac{1}{2\sqrt{y}}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}}\right)$

   $g(y) = \frac{1}{\sqrt{y}\sqrt{2\pi}} e^{-\frac{y}{2}}$.

   Hence, $g(y) \sim \chi^2(1)$.

   Extra Problem:

   1. A random variable X having Cauchy distribution. Show that $1/X$ also has Cauchy distribution.