Expectation and Variance

Given a random variable, we often compute the expectation and variance, two important summary statistics. The expectation describes the average value and the variance describes the spread (amount of variability) around the expectation.

Let X be a random variable whose possible values $x_1, x_2, x_3, \ldots, x_n$ occur with probabilities $p_1, p_2, p_3, \ldots, p_n$, respectively. The mean of X, denoted by μ, is the number $\sum x_i p_i$ i.e. the **mean of X is the weighted average of the possible values of X**, each value being weighted by its probability with which it occurs. The mean of a random variable X is also called the expectation of X.
Thus, $\boldsymbol{\mu = E(X) = x_1 p_1 + x_2 p_2 + \ldots + x_n p_n}$.

Definition: Let X be a **continuous** random variable with p.d.f. $f_X(x)$. The expected value of X is

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

Definition: Let X be a **discrete** random variable with probability function $f_X(x)$. The expected value of X is

$$\mathbb{E}(X) = \sum_x x f_X(x) = \sum_x x\mathbb{P}(X = x).$$

Properties of Expectation:

1. E(c) = c
   i.e E(c) = $\sum cP(X = c) = c.1 = c$
2. E(aX+b)= $\sum (aX + b)P(X = x) = a\sum xP(X = x) + b\sum P(X = x)$
   = aE(x)+b.1= aE(X)+b
3. E(E(X))= E(X)

Example Let a pair of dice be thrown and the random variable X be the sum of the numbers that appear on the two dice. Find the mean or expectation of X.
Solution The sample space of the experiment consists of 36 elementary events in the form of ordered pairs (x_i , y_i), where x_i = 1, 2, 3, 4, 5, 6 and y_i = 1, 2, 3, 4, 5, 6. The random variable X i.e. the sum of the numbers on the two dice takes the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 or 12.

X or x_i	2	3	4	5	6	7	8	9	10	11	12
P(X) or p_i	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Now $\mu = 7$ Thus, the mean of the sum of the numbers that appear on throwing two fair dice is 7.

Variance

The mean of the random variable does not give us information about the variability in the values of the random variable. In fact, if the variance is small, then the values of the random variable are close to the mean. Also random variables with different probability distributions can have equal means.

$$s_x = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$$

$$\sigma_x^2 = \mathrm{Var}(X) = \sum_{i=1}^{n}(x_i - \mu)^2 \, p(x_i)$$

$$= \sum_{i=1}^{n} x_i^2 \, p(x_i) + \sum_{i=1}^{n} \mu^2 \, p(x_i) - \sum_{i=1}^{n} 2\mu x_i \, p(x_i)$$

$$= \sum_{i=1}^{n} x_i^2 \, p(x_i) + \mu^2 \sum_{i=1}^{n} p(x_i) - 2\mu \sum_{i=1}^{n} x_i \, p(x_i)$$

$$= \sum_{i=1}^{n} x_i^2 \, p(x_i) + \mu^2 - 2\mu^2 \left[\text{since} \sum_{i=1}^{n} p\,(x_i) = 1 \text{ and } \mu = \sum_{i=1}^{n} x_i \, p(x_i)\right]$$

$$= \sum_{i=1}^{n} x_i^2 \, p(x_i) - \mu^2$$

$$\mathrm{Var}\,(X) = \sum_{i=1}^{n} x_i^2 \, p(x_i) - \left(\sum_{i=1}^{n} x_i \, p(x_i)\right)^2$$

$$\mathrm{Var}\,(X) = E(X^2) - [E(X)]^2, \text{ where } E(X^2) = \sum_{i=1}^{n} x_i^2 \, p(x_i)$$

Example Find the variance of the number obtained on a throw of an unbiased die.

Solution The sample space of the experiment is S = {1, 2, 3, 4, 5, 6}.

Let X denote the number obtained on the throw. Then X is a random variable which can take values 1, 2, 3, 4, 5, or 6.

$$E(X^2) = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6} = \frac{91}{6}$$

$$\mathrm{Var}\,(X) = E\,(X^2) - (E(X))^2$$

$$= \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{91}{6} - \frac{441}{36} \quad \frac{35}{12}$$

Properties of Variance:

1. **V(c) =0**
   i.e V(c) = E(c^2) –[E(c)]^2= $c^2 - [c]^2 = 0$
2. **V(aX+b) = $a^2 V(X)$**
   WKT, V(X)= E(X^2) –[E(X)]^2
   V(aX+b)= E({aX + b}^{2}) –[E(aX + b)]^2
   = E($a^2x^2 + b^2 + 2abX$) –{aE(X) + b}^{2}
   = $a^2E(X^2) + b^2 + 2abE(X) - [a^2\{E(X)\}^2 + b^2 + 2abE(X)]$
   = a^2\{E(X^2) –[E(X)]^2\} – 0
   = $a^2V(X)$

Example: Let X be a continuous random variable with p.d.f.

$$f_X(x) = \begin{cases} 2x^{-2} & \text{for } 1 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathbb{E}(X)$ and $\text{Var}(X)$.

$$\begin{aligned} E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_1^2 x \times 2x^{-2}\, dx \quad &= \int_1^2 2x^{-1}\, dx \\ &= \Big[2\log(x)\Big]_1^2 \\ &= 2\log(2) - 2\log(1) \\ &= 2\log(2). \end{aligned}$$

For Var(X), we use

$$\text{Var}(X) = \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2 .$$

Now

$$\begin{aligned} E(X^2) = \int_{-\infty}^{\infty} x^2\, f_X(x)\, dx = \int_1^2 x^2 \times 2x^{-2}\, dx \quad &= \int_1^2 2\, dx \\ &= \Big[2x\Big]_1^2 \\ &= 2 \times 2 - 2 \times 1 \\ &= 2. \end{aligned}$$

Thus

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2 \\ &= 2 - \{2\log(2)\}^2 \\ &= 0.0782. \end{aligned}$$

Mean, Median and MODE

If X is A CRV then median M, $\int_{-\infty}^{M} f(x)dx = \int_{M}^{\infty} f(x)\,dx = \frac{1}{2}$
And MODE of X for which f(x) is maximum, $f^{|} = 0$ and $f^{||} < 0$.

Problems:

1. Find mean, Median and mode also variance of a random variable X having pdf f(x)= $\begin{cases} 6x(1-x) & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$.
   Ans: E(x)= ½ , E(x^2)= 3/10, V(x)= 1/20, M=½, MODE= ½

2. Find pdf, mean, Median and mode also variance of a random variable X having cdf F(x)= $\begin{cases} 1-e^{-x}-xe^{-x} & x \ge 0 \\ 0 & otherwise \end{cases}$.
   Ans: E(x)= 2 , E(x^2)= 6, V(x)= 2, M= 5/3, MODE=1

   Formulas: MODE= 3M-2E(X) AND $\int UV$ PARTS: $\int UV = U \int V - \int U' \int V$

3. If $F(x) = \begin{cases} -e^{-\frac{x^2}{2}} & x > 0 \\ 0 & x \le 0 \end{cases}$ then find V(X).

   Solution:

   The pdf of X is $f(x) = \begin{cases} xe^{-\frac{x^2}{2}} & x > 0 \\ 0 & x \le 0 \end{cases}$,
   E(X) = $\int_0^\infty x^2 \; e^{-\frac{x^2}{2}} dx = \sqrt{2} \int_0^\infty \sqrt{t}\, e^{-t}\, dt$ using $\frac{x^2}{2} = t,\; x\,dx = dt$

   WKT, $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$ and $\Gamma(n+1) = n\,\Gamma(n)$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

   E(X)= $\sqrt{2}\,\Gamma(3/2) = \sqrt{2}\,.\,\frac{1}{2}.\,\sqrt{\pi} = \frac{\sqrt{\pi}}{\sqrt{2}}$

   E(X^2)= $\int_0^\infty x^3 \; e^{-\frac{x^2}{2}} dx = 2\int_0^\infty te^{-t}\, dt$ using $\frac{x^2}{2}$ =t
   = 2 $\Gamma(2)$ =2

   V(X) = 2 - $\left(\sqrt{\pi}\frac{1}{\sqrt{2}}\right)^2 = 4 - \frac{\pi}{2}$