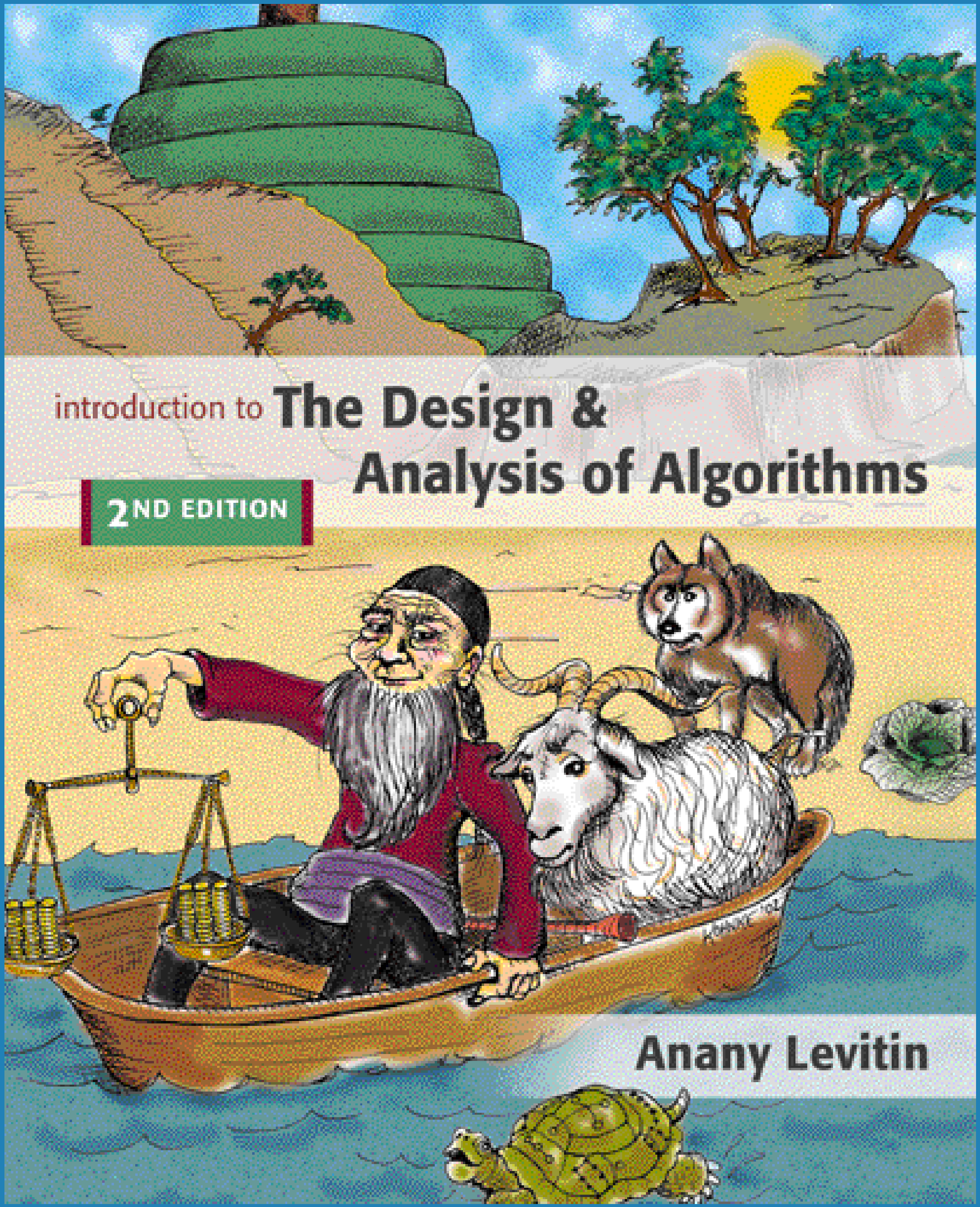
Chapter 8

Dynamic Programming



Dynamic Programming

Dynamic Programming **is a general algorithm design technique for solving problems defined by or formulated as recurrences with overlapping subinstances**

- **Invented by American mathematician Richard Bellman in the 1950s to solve optimization problems and**

- **"Programming" here means "planning"**

- **Main idea:**
  - **set up a recurrence relating a solution to a given problem with solutions to its smaller subproblems of the same type**
  - **solve smaller instances once**
  - **record solutions in a table**
  - **extract solution to the initial instance from that table**

Example: Fibonacci numbers

- Recall definition of Fibonacci numbers:

$$F(n) = F(n\text{-}1) + F(n\text{-}2)$$
$$F(0) = 0$$
$$F(1) = 1$$

- Computing the n^{th} Fibonacci number recursively (top-down):

$$F(n)$$

$$F(n\text{-}1) \quad + \quad F(n\text{-}2)$$

$$F(n\text{-}2) \quad + \quad F(n\text{-}3) \qquad F(n\text{-}3) \quad + \quad F(n\text{-}4)$$

...

Example: Fibonacci numbers (cont.)

Computing the n^{th} Fibonacci number using bottom-up iteration and recording results:

$F(0) = 0$
$F(1) = 1$
$F(2) = 1+0 = 1$
…
$F(n\text{-}2) =$
$F(n\text{-}1) =$
$F(n) = F(n\text{-}1) + F(n\text{-}2)$

0	1	1	. . .	$F(n\text{-}2)$	$F(n\text{-}1)$	$F(n)$

Efficiency:
- time n
- space n

Examples of DP algorithms

- **Computing a binomial coefficient**

- **Warshall's algorithm for transitive closure**

- **Floyd's algorithm for all-pairs shortest paths**

•Some instances of difficult discrete optimization problems:
  - **traveling salesman**
  - **knapsack**

Computing a binomial coefficient by DP

Computing a binomial coefficient is a standard example of applying dynamic programming to a nonoptimization problem. You may recall from your studies of elementary combinatorics that the ***binomial coefficient***, denoted $C(n, k)$ or $\binom{n}{k}$, is the number of combinations (subsets) of k elements from an n-element set $(0 \le k \le n)$. The name "binomial coefficients" comes from the participation of these numbers in the binomial formula:

$$(a+b)^n = C(n, 0)a^n + \cdots + C(n, k)a^{n-k}b^k + \cdots + C(n, n)b^n.$$

Of the numerous properties of binomial coefficients, we concentrate on two:

$$C(n, k) = C(n-1, k-1) + C(n-1, k) \quad \text{for } n > k > 0 \qquad \textbf{(8.3)}$$

and

$$C(n, 0) = C(n, n) = 1. \qquad \textbf{(8.4)}$$

Value of $C(n,k)$ can be computed by filling a table:

	0	1	2	. . . .	k-1	k
0	1					
1	1	1				
2	1	2	1			
.						
k	1					1
.						
n-1					$C(n-1,k-1)$	$C(n-1,k)$
n						$C(n,k)$

Computing $C(n,k)$: pseudocode and analysis

ALGORITHM *Binomial*(n, k)

```
//Computes C(n, k) by the dynamic programming algorithm
//Input: A pair of nonnegative integers n ≥ k ≥ 0
//Output: The value of C(n, k)
for i ← 0 to n do
    for j ← 0 to min(i, k) do
        if j = 0 or j = i
            C[i, j] ← 1
        else C[i, j] ← C[i − 1, j − 1] + C[i − 1, j]
return C[n, k]
```

Time efficiency: Θ(*nk*)

What is the time efficiency of this algorithm? Clearly, the algorithm's basic operation is addition, so let $A(n, k)$ be the total number of additions made by this algorithm in computing $C(n, k)$. Note that computing each entry by formula (8.3) requires just one addition. Also note that because the first $k + 1$ rows of the table form a triangle while the remaining $n - k$ rows form a rectangle, we have to split the sum expressing $A(n, k)$ into two parts:

$$A(n, k) = \sum_{i=1}^{k} \sum_{j=1}^{i-1} 1 + \sum_{i=k+1}^{n} \sum_{j=1}^{k} 1 = \sum_{i=1}^{k} (i - 1) + \sum_{i=k+1}^{n} k$$

$$= \frac{(k-1)k}{2} + k(n - k) \in \Theta(nk).$$

	0	1	2	3
0	1			
1	1	1		
2	1	2	1	
3	1	3	3	1
4	1	4	6	4
5	1	5	10	10
6	1	6	15	**20**

Warshall's Algorithm

Recall that the adjacency matrix $A = \{a_{ij}\}$ of a directed graph is the boolean matrix that has 1 in its ith row and jth column if and only if there is a directed edge from the ith vertex to the jth vertex. We may also be interested in a matrix containing the information about the existence of directed paths of arbitrary lengths between vertices of a given graph.

DEFINITION The ***transitive closure*** of a directed graph with n vertices can be defined as the n-by-n boolean matrix $T = \{t_{ij}\}$, in which the element in the ith row $(1 \le i \le n)$ and the jth column $(1 \le j \le n)$ is 1 if there exists a nontrivial directed path (i.e., a directed path of a positive length) from the ith vertex to the jth vertex; otherwise, t_{ij} is 0.

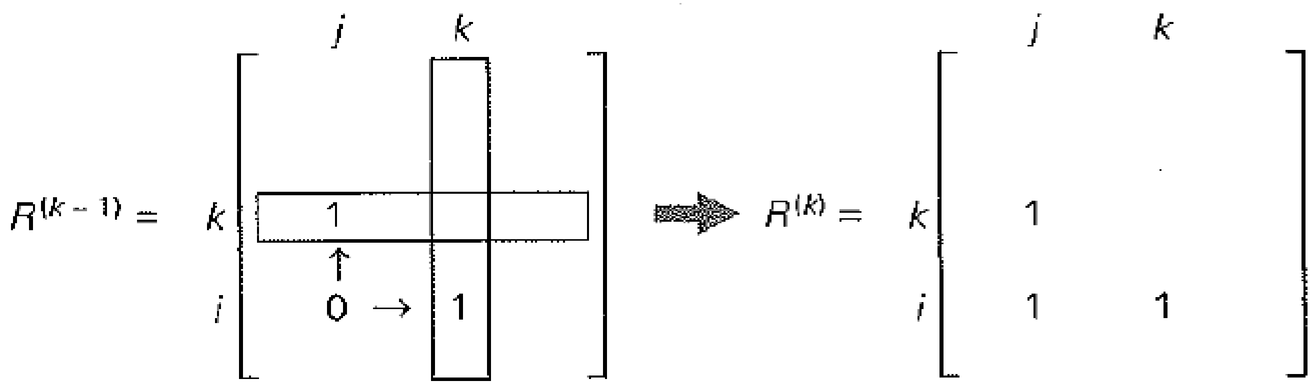
Warshall's algorithm after S. Warshall [War62]. Warshall's algorithm constructs the transitive closure of a given digraph with n vertices through a series of n-by-n boolean matrices:

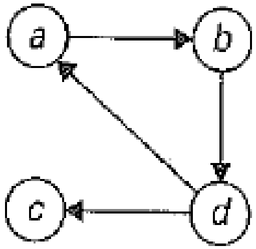
$$R^{(0)}, \ldots, R^{(k-1)}, R^{(k)}, \ldots, R^{(n)}. \tag{8.5}$$

$$r_{ij}^{(k)} = r_{ij}^{(k-1)} \text{ or } \left(r_{ik}^{(k-1)} \text{ and } r_{kj}^{(k-1)} \right). \tag{8.7}$$

Formula (8.7) is at the heart of Warshall's algorithm. This formula implies the following rule for generating elements of matrix $R^{(k)}$ from elements of matrix $R^{(k-1)}$, which is particularly convenient for applying Warshall's algorithm by hand:

- If an element r_{ij} is 1 in $R^{(k-1)}$, it remains 1 in $R^{(k)}$.
- If an element r_{ij} is 0 in $R^{(k-1)}$, it has to be changed to 1 in $R^{(k)}$ if and only if the element in its row i and column k and the element in its column j and row k are both 1's in $R^{(k-1)}$. (This rule is illustrated in Figure 8.3.)





$$R^{(0)} = \begin{array}{cc} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Ones reflect the existence of paths with no intermediate vertices ($R^{(0)}$ is just the adjacency matrix); boxed row and column are used for getting $R^{(1)}$.

$$R^{(1)} = \begin{array}{cc} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & \mathbf{1} & 1 & 0 \end{bmatrix} \end{array}$$

Ones reflect the existence of paths with intermediate vertices numbered not higher than 1, i.e., just vertex *a* (note a new path from *d* to *b*); boxed row and column are used for getting $R^{(2)}$.

$$R^{(2)} = \begin{array}{cc} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & \mathbf{1} \end{bmatrix} \end{array}$$

Ones reflect the existence of paths with intermediate vertices numbered not higher than 2, i.e., *a* and *b* (note two new paths); boxed row and column are used for getting $R^{(3)}$.

$$R^{(3)} = \begin{array}{cc} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Ones reflect the existence of paths with intermediate vertices numbered not higher than 3, i.e., *a*, *b*, and *c* (no new paths); boxed row and column are used for getting $R^{(4)}$.

$$R^{(4)} = \begin{array}{cc} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} \mathbf{1} & 1 & \mathbf{1} & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Ones reflect the existence of paths with intermediate vertices numbered not higher than 4, i.e., *a*, *b*, *c*, and *d* (note five new paths).

ALGORITHM *Warshall*($A[1..n, 1..n]$)

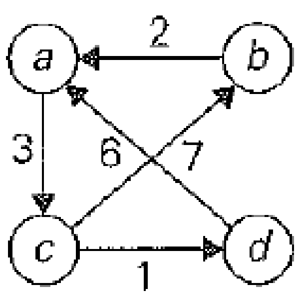
//Implements Warshall's algorithm for computing the transitive closure
//Input: The adjacency matrix A of a digraph with n vertices
//Output: The transitive closure of the digraph

$R^{(0)} \leftarrow A$
for $k \leftarrow 1$ **to** n **do**
    for $i \leftarrow 1$ **to** n **do**
        for $j \leftarrow 1$ **to** n **do**
            $R^{(k)}[i, j] \leftarrow R^{(k-1)}[i, j]$ **or** $(R^{(k-1)}[i, k]$ **and** $R^{(k-1)}[k, j])$
return $R^{(n)}$

$$R^{(4)} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Floyd's Algorithm for the All-Pairs Shortest-Paths Problem

Given a weighted connected graph (undirected or directed), the ***all-pairs shortest-paths problem*** asks to find the distances (the lengths of the shortest paths) from each vertex to all other vertices. It is convenient to record the lengths of shortest paths in an n-by-n matrix D called the ***distance matrix***: the element d_{ij} in the ith row and the jth column of this matrix indicates the length of the shortest path from the ith vertex to the jth vertex $(1 \le i, j \le n)$. For an example, see Figure 8.5.



(a)

$$W = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \infty & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \infty & 0 \end{bmatrix} \end{array}$$

(b)

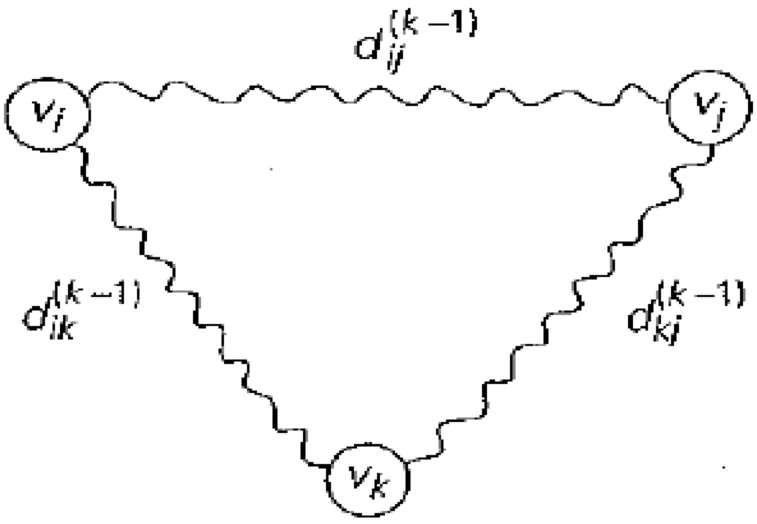
$$D = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 10 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ 7 & 7 & 0 & 1 \\ 6 & 16 & 9 & 0 \end{bmatrix} \end{array}$$

(c)

FIGURE 8.5 (a) Digraph. (b) Its weight matrix. (c) Its distance matrix.

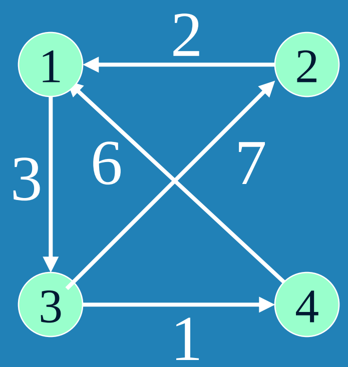
Floyd's algorithm computes the distance matrix of a weighted graph with n vertices through a series of n-by-n matrices:

$$D^{(0)}, \ldots, D^{(k-1)}, D^{(k)}, \ldots, D^{(n)}. \qquad \textbf{(8.8)}$$



$$d_{ij}^{(k)} = \min\{d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\} \quad \text{for } k \geq 1, \qquad d_{ij}^{(0)} = w_{ij}.$$

Floyd's Algorithm (example)



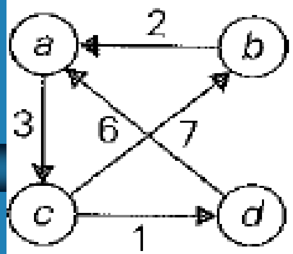
$$D^{(0)} = \begin{matrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \infty & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \infty & 0 \end{matrix}$$

$$D^{(1)} = \begin{matrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \mathbf{5} & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \mathbf{9} & 0 \end{matrix}$$

$$D^{(2)} = \begin{matrix} 0 & \infty & 3 & \infty \\ 2 & 0 & 5 & \infty \\ \mathbf{9} & 7 & 0 & 1 \\ 6 & \infty & 9 & 0 \end{matrix}$$

$$D^{(3)} = \begin{matrix} 0 & \mathbf{10} & 3 & \mathbf{4} \\ 2 & 0 & 5 & \mathbf{6} \\ 9 & 7 & 0 & 1 \\ 6 & \mathbf{16} & 9 & 0 \end{matrix}$$

$$D^{(4)} = \begin{matrix} 0 & 10 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ \mathbf{7} & 7 & 0 & 1 \\ 6 & 16 & 9 & 0 \end{matrix}$$



$$D^{(0)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc} 0 & \infty & 3 & \infty \\ 2 & 0 & \infty & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \infty & 0 \end{array}\right] \end{array}$$

Lengths of the shortest paths with no intermediate vertices ($D^{(0)}$ is simply the weight matrix).

$$D^{(1)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc} 0 & \infty & 3 & \infty \\ 2 & 0 & \mathbf{5} & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \mathbf{9} & 0 \end{array}\right] \end{array}$$

Lengths of the shortest paths with intermediate vertices numbered not higher than 1, i.e. just *a* (note two new shortest paths from *b* to *c* and from *d* to *c*).

$$D^{(2)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc} 0 & \infty & 3 & \infty \\ 2 & 0 & 5 & \infty \\ \mathbf{9} & 7 & 0 & 1 \\ 6 & \infty & 9 & 0 \end{array}\right] \end{array}$$

Lengths of the shortest paths with intermediate vertices numbered not higher than 2, i.e. a and *b* (note a new shortest path from *c* to *a*).

$$D^{(3)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc} 0 & \mathbf{10} & 3 & \mathbf{4} \\ 2 & 0 & 5 & \mathbf{6} \\ 9 & 7 & 0 & 1 \\ 6 & \mathbf{16} & 9 & 0 \end{array}\right] \end{array}$$

Lengths of the shortest paths with intermediate vertices numbered not higher than 3, i.e. a, *b*, and *c* (note four new shortest paths from *a* to *b*, from a to *d*, from *b* to *d*, and from *d* to *b*).

$$D^{(4)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc} 0 & 10 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ \mathbf{7} & 7 & 0 & 1 \\ 6 & 16 & 9 & 0 \end{array}\right] \end{array}$$

Lengths of the shortest paths with intermediate vertices numbered not higher than 4, i.e. a, *b*, *c*, and *d* (note a new shortest path from *c* to *a*).

Floyd's Algorithm (pseudocode and analysis)

```
ALGORITHM  Floyd(W[1..n, 1..n])
    //Implements Floyd's algorithm for the all-pairs shortest-paths problem
    //Input: The weight matrix W of a graph with no negative-length cycle
    //Output: The distance matrix of the shortest paths' lengths
    D ← W //is not necessary if W can be overwritten
    for k ← 1 to n do
        for i ← 1 to n do
            for j ← 1 to n do
                D[i, j] ← min{D[i, j], D[i, k] + D[k, j]}
    return D
```

Problem

- Solve the all-pairs shortest-path problem for the digraph with the weight matrix.

$$\begin{bmatrix} 0 & 2 & \infty & 1 & 8 \\ 6 & 0 & & & \\ \infty & & & & \\ \infty & & & & \\ 3 & & & & \end{bmatrix}$$

$$D^{(5)} = \begin{bmatrix} 0 & 2 & 3 & 1 & 4 \\ 6 & 0 & 3 & 2 & 5 \\ 10 & 12 & 0 & 4 & 7 \\ 6 & 8 & 2 & 0 & 3 \\ 3 & 5 & 6 & 4 & 0 \end{bmatrix}$$

Knapsack Problem by DP

Given n items of

integer weights: $w_1 \quad w_2 \quad \ldots \quad w_n$

values: $v_1 \quad v_2 \quad \ldots \quad v_n$

a knapsack of integer capacity W

find most valuable subset of the items that fit into the knapsack

To design a dynamic programming algorithm, we need to derive a recurrence relation that expresses a solution to an instance of the knapsack problem in terms of solutions to its smaller subinstances. Let us consider an instance defined by the first i items, $1 \le i \le n$, with weights $w_1, \ldots, w_i$, values $v_1, \ldots, v_i$, and knapsack capacity j, $1 \le j \le W$. Let $V[i, j]$ be the value of an optimal solution to this instance, i.e., the value of the most valuable subset of the first i items that fit into the knapsack of capacity j. We can divide all the subsets of the first i items that fit the knapsack of capacity j into two categories: those that do not include the ith item and those that do. Note the following:

1. Among the subsets that do not include the ith item, the value of an optimal subset is, by definition, $V[i-1, j]$.
2. Among the subsets that do include the ith item (hence, $j - w_i \ge 0$), an optimal subset is made up of this item and an optimal subset of the first $i-1$ items that fit into the knapsack of capacity $j - w_i$. The value of such an optimal subset is $v_i + V[i-1, j-w_i]$.

		0	$j-w_i$	j	W
	0	0	0	0	0
	$i-1$	0	$V[i-1, j-w_i]$	$V[i-1, j]$	
w_i, v_i	i	0		$V[i, j]$	
	n	0			goal

Consider instance defined by first i items and capacity j ($j \leq W$).

Let $V[i,j]$ be optimal value of such an instance. Then

$$V[i,j] = \begin{cases} \max \{V[i-1,j],\ v_i + V[i-1,j-w_i]\} & \text{if } j-w_i \geq 0 \\ V[i-1,j] & \text{if } j-w_i < 0 \end{cases}$$

Initial conditions: $V[0,j] = 0$ and $V[i,0] = 0$

Knapsack Problem by DP (example)

Example: Knapsack of capacity $W = 5$

item	weight	value
1	2	\$12
2	1	\$10
3	3	\$20
4	2	\$15

$$V[i,j] = \begin{cases} \max\{V[i-1,j],\ v_i + V[i-1,j-w_i]\} & \text{if } j - w_i \ge 0 \\ V[i-1,j] & \text{if } j - w_i < 0 \end{cases}$$

capacity *j*

		0	1	2	3	4	5
	0	0	0	0	0	0	0
$w_1 = 2, v_1 = 12$	1	0	0	12	12	12	12
$w_2 = 1, v_2 = 10$	2	0	10	12	22	22	22
$w_3 = 3, v_3 = 20$	3	0	10	12	22	30	32
$w_4 = 2, v_4 = 15$	4	0	10	15	25	30	37 ?

Backtracing finds the actual optimal subset, i.e. solution.

Memory functions

ALGORITHM *MFKnapsack*(i, j)

//Implements the memory function method for the knapsack problem
//Input: A nonnegative integer i indicating the number of the first
// items being considered and a nonnegative integer j indicating
// the knapsack's capacity
//Output: The value of an optimal feasible subset of the first i items
//Note: Uses as global variables input arrays $Weights[1..n]$, $Values[1..n]$,
//and table $V[0..n, 0..W]$ whose entries are initialized with -1's except for
//row 0 and column 0 initialized with 0's

if $V[i, j] < 0$
  if $j < Weights[i]$
    $value \leftarrow MFKnapsack(i - 1, j)$
  else
    $value \leftarrow \max(MFKnapsack(i - 1, j),$
      $Values[i] + MFKnapsack(i - 1, j - Weights[i]))$
  $V[i, j] \leftarrow value$
return $V[i, j]$

Example

Example: Knapsack of capacity $W = 5$

item	weight	value
1	2	\$12
2	1	\$10
3	3	\$20
4	2	\$15

i/j	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	12	12	12	12
2	0	-	12	22	-	22
3	0	-	-	22	-	32
4	0	-	-	-	-	37

item	weight	value
1	3	\$25
2	2	\$20
3	1	\$15
4	4	\$40
5	5	\$50

capacity $W = 6$.