Uniform distribution

Let X be a continuous random variable assuming all values in the interval [a, b] where a and b are finite. If the pdf of X is given by

$$f(x) = \begin{cases} \frac{1}{(b-a)} & a \le x \le b \\ 0 & else\ where \end{cases}$$

then we say that X has uniform distribution defined over [a, b].

Note that, for any sub interval [c, d],

$$\mathrm{P}\,(\mathrm{c} < \mathrm{X} < \mathrm{d}) = \int_c^d f(x)dx = \int_c^d \frac{1}{(b-a)}dx = \frac{(d-c)}{(b-a)}$$

Cdf = $F(x) = \begin{cases} 0 & x \le a \\ \frac{x-a}{(b-a)} & a < x < b \\ 1 & x \ge b \end{cases}$

Mean E(X) = $\int_a^b x f(x) dx = \frac{1}{(b-a)} \left\{ \frac{x^2}{2} \right\} \begin{matrix} b \\ a \end{matrix} = \frac{(b-a)(a+b)}{2(b-a)} = \frac{(a+b)}{2}$

E(X^2) = $\int_a^b x^2 f(x) dx = \frac{1}{(b-a)} \left\{ \frac{x^3}{3} \right\} \begin{matrix} b \\ a \end{matrix} = \frac{(b^3 - a^3)}{3(b-a)} = \frac{(a^2 + b^2 + ab)}{3}$

Variance V(X) = E(X^2) – $[E(X)]^2$ = $\frac{(a^2 + b^2 + ab)}{3} - \{ \frac{(a+b)}{2} \}^2 = \frac{(b-a)^2}{12}$

Problems:

1. If X is uniformly distributed over (-2, 2) then find i) P(X<1) ii)P $(|X-1| \geq \frac{1}{2})$.

Solution: Given that $X \in U(-2,2)$.

Therefore, $f(x) = \begin{cases} \frac{1}{4} & -2 \leq x \leq 2 \\ 0 & else\ where \end{cases}$

i) P(X<1) = $\int_{-\infty}^{1} f(x)dx = \int_{-2}^{1} \frac{1}{4}\ dx = \frac{3}{4}$

ii) P $(|X-1| \geq \frac{1}{2})$ = 1 - P $(|X-1| < \frac{1}{2})$

= 1- P($-\frac{1}{2}$ < X – 1< $\frac{1}{2}$)

= 1 – P($\frac{1}{2}$ < X < $\frac{3}{2}$)

= 1 - $\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{4}\ dx = 1 - \frac{1}{4} = \frac{3}{4}$

2. If K is uniformly distributed over (0, 5) then what is the probability that the roots of the equation $4x^2 + 4xK + K + 2 = 0$ are real ?

Solution: Given that K$\in U(0,5)$.

Therefore, $f(k) = \begin{cases} \frac{1}{5} & 0 \le k \le 5 \\ 0 & else\ where \end{cases}$

P { Roots are real} = P{ $(4\text{K})^2 - 4.4(k+2) \ge 0$ }

= P{ $K^2 - K - 2 \ge 0$ } = P{ $(K+1)(K-2) \ge 0$ }

= P{ $(K+1) \ge 0, (K-2) \ge 0$ or $(K+1) \le 0, (K-2) \le 0$}

= P{ $K \ge -1, K \ge 2$ or $K \le -1, K \le 2$ }

= P{ $K \ge 2$ or $K \le -1$ }

= P{ $K \ge 2$ } + P{ $K \le -1$ }

$= \int_2^5 \frac{1}{5} dx + \int_{-\infty}^{-1} 0\, dx = 3/5$

Problems on Variance and Expectation:

1. A student takes a multiple choice test consisting of 2 problems. The first one has 3 possible answers and the second one has 5. The student chosen at random, one answer as the right answer for each of the 2 problems. Let X denote the number of right answers of student. Find V(X).

Solution: Let X : the number of right answers

X:	0	1	2
P(X=x):	8/15	6/15	1/15

{P(X =0) = 2/3 . 4/5 and P(X =1) = 1/3 . 4/5 + 2/3.1/5 = 6/15 and P(X =2) = 1/3 . 1/5 =1/15}

E(X) = 0 + 1.(6/15) + 2. (1/15) = 8/15, $E(X^2)$ = 0 + 6/15 + 4.(1/15) =10/15 = 2/3 and

V(X) = $E(X^2) - \{E(X)\}^2$ = 2/3 – $\{8/15\}^2$ = 86/225=0.3822

2. Three balls are randomly selected from an urn containing 3 white, 3 red, 5 black balls. The person who selects the ball wins \$ 1.00 for each white ball selected and lose \$1.00 for each red ball selected. Let X be the total winnings from the experiment. Find the Probability distribution of X and V(X).

Solution: Let X : total winnings

X:	-3	-2	-1	0	1	2	3
P(X = x):	1/165	15/165	39/165	55/165	39/165	15/165	1/165

{P(X =0) = P{Selection of 3B or 1W, 1B, 1R balls} = $\frac{5C_3}{11C_3}$ + $\frac{3C_1 \times 3C_1 \times 5C_1}{11C_3}$ = 55/165

P(X =1) = P(X = -1) = P{Selection of 2B, 1W or 2W, 1R balls} = $\frac{3C_1 \times 3C_2}{11C_3}$ + $\frac{3C_1 \times 5C_2}{11C_3}$ = 39/165

P(X =2) = P(X = - 2) = P{Selection of 1B, 2W balls} = $\frac{5C_1 \times 3C_2}{11C_3}$ = 15/165

and P(X =3) =P(X = -3) = P{Selection of 3W balls} = $\frac{3C_3}{11C_3}$ = 1/165}

E(X) = 0 , E(X^2) = 2{9/165 + 15X4/165 + 39/165 }=216/165 and V(X)= 216/165

3. A coin is tossed till head appears then find the probability distribution on number of tosses. Let X denote the number of tosses. Find E(X).

Solution:

X denote the number of tosses

P(H)= p

P(T)= q

X	1	2	3	...
P(X)	p	qp	q^2p	...

Therefore, P(X=k)= pq^{k-1}

Hence, E(X) = $\sum_1^\infty k\ pq^{k-1}$ = p $\sum_1^\infty k\ q^{k-1}$= $\frac{p}{(1-q)^2} = \frac{1}{p}$

4. Suppose that an electronic device has a life length X (in units' of 1000 hours) which is considered as a continuous random variable with the following pdf: f(x) = $e^{-x}, x > 0$, Suppose that the cost of manufacturing one such item is 2 rupees. The manufacturer sells the item for 5 rupees but guarantees a total refund if $x \leq 0.9$. What is the manufacturer's expected profit per item?

Solution:

Y- expected profit

Profit, p= $\begin{cases} 3 & x > 0.9 \\ -2 & x \leq 0.9 \end{cases}$

E(Y)= 3 (probability of x>0.9) +(-2) (probability of $x \leq 0.9$)

E(Y)= 3 ($\int_{0.9}^{\infty} e^{-x}\, dx$) +(-2) ($\int_{-\infty}^{0.9} e^{-x}\, dx$)

E(Y)= 3 ($\int_{0.9}^{\infty} e^{-x}\, dx$) +(-2) (1-$\int_{0.9}^{\infty} e^{-x}\, dx$)

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E(Y)= 3 (0.4066) +(-2) (1-0.466)

E(Y)=0.033

5. Let X be a random variable with probability function

$$P(X = k) = p\,(1-p)^{k-1}\ , k = \quad 1,2,3 \ldots n. \text{ Find V(X).}$$

Solution: Given $P(X = k) = p\,(1-p)^{k-1}\ , k = 1,2,3 \ldots n$

E(X) = $\sum_1^n x\,p(x) \ = \sum_1^n k\,p(1-p)^{k-1}$

= p{ 1 + 2 (1-p) + 3 $(1-p)^2$+….}

= p . $\frac{1}{(1-(1-p))^2}$ = 1/p

P(H)is not equal to P(T). P(H) =p, P(T) = 1-p

1+ x + $(x)^2$ +……..=1/1-x

1+2x+3$(x)^2$+ ….= $1/(1-x)^2$

X	1	2	3	…	N
P(X=k)	p	P(1-p)	(1-p)(1-p)p	…	(1-p) (1-p)….. (1-p) (1-p)p

$\sum_1^n p(x)$ =p [1+ (1-p)+$(1-p)^2$+….] =p/1-(1-p)=1

E(X^2)= $\sum_1^n x^2\, p(x)$ $= \sum_1^n k^2\, p(1-p)^{k-1}$ = p{ 1 + 4 (1-p) + 9 $(1-p)^2$+….} =pS

Where S = { 1 + 4 (1-p) + 9 $(1-p)^2$+….} ……………..(1)

Multi ply (1) by 1-p we get,

S (1-p) = { 1-p + 4 $(1-p)^2$+ 9 $(1-p)^3$+….} …………(2)

Eq(1)- Eq (2) simplifies to Sp = {1 +3(1-p) + 5 $(1-p)^2$ +………..} …………………..(3)

Multi ply (3) by 1-p we get,

Sp (1-p) = {1-p +3$(1-p)^2$ + 5 $(1-p)^3$ +………..} …………………..(4)

Eq(3)- Eq (4) simplifies to

Sp^2 = {1 +2(1-p) + 2 $(1-p)^2$ +………..} …………………..(3)

Therefore, Sp^2 = {1 +2(1-p) { 1+ (1-p) +$(1-p)^2$ +………..} }

S = $\frac{1}{p^2}$ [1+ 2(1-p) $\frac{1}{1-(1-p)}$] =$\frac{2}{p^3} - \frac{1}{p^2}$

E(X^2)= p($\frac{2}{p^3} - \frac{1}{p^2}$) $= \frac{2}{p^2} - \frac{1}{p}$

V(X) = $\frac{1-p}{p^2}$