IE 415 Control Of Autonomous System

Project Report

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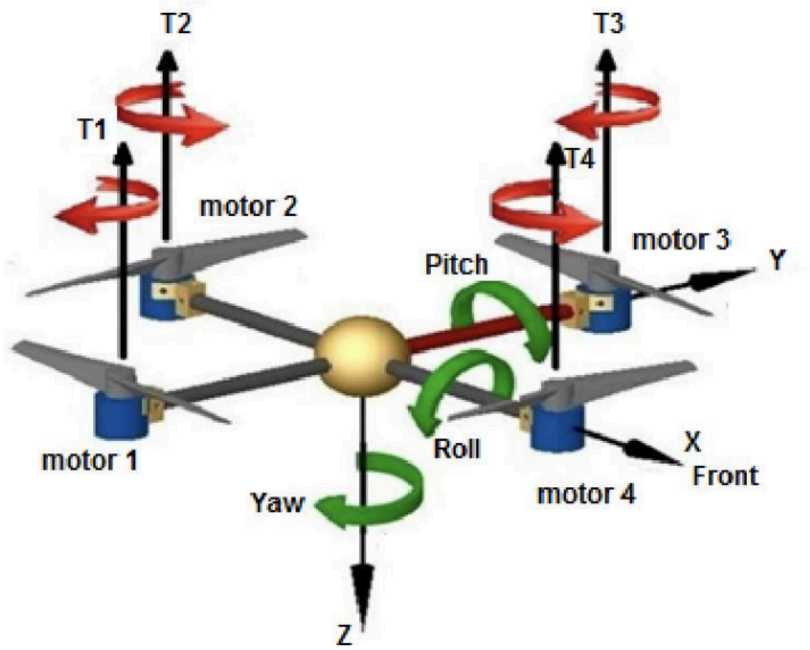
Modelling and Nonlinear Control of Quadcopter

Introduction

A quadcopter is an aerial vehicle equipped with four rotors, allowing it to move in various directions. Understanding its system dynamics necessitates familiarity with the concept of 6 degrees of freedom, which describes the position and orientation of the quadcopter in three-dimensional (3-D) space.

The concept of 6 degrees of freedom defines the position and orientation of a body using six coordinates, divided into two reference frames. The first is the fixed reference frame, known as the inertial or earth frame, represented by the x, y, and z axes aligned with the cardinal directions: North, East, and Down. The second is the moving reference frame, called the body frame, characterized by the angles ϕ, θ, and ψ, measured relative to the body's center of gravity.



The quadcopter is configured with four rotors arranged in a cross pattern, as shown in Figure above. Two opposite rotors rotate in the same direction, and the quadcopter's altitude and position are controlled by adjusting the angular speed of these rotors. The quadcopter can maintain a stable position without spinning when the torque generated by motors T1, T2, T3, and T4 is equal. Thrust, which controls the quadcopter's altitude for ascending or descending, is achieved by simultaneously increasing or decreasing the rotational speed of all four motors. Roll refers to the tilting of the quadcopter left or right, allowing side-to-side movement. Pitch refers to the tilting of the quadcopter forward or backward, enabling forward or backward movement. Yaw refers to the rotational movement of the quadcopter in a clockwise or counterclockwise direction, while remaining level with the ground, to change its orientation. These flight maneuvers are accomplished by varying the rotational speeds of the four motors.

<u>System Model Of QuadCopter</u>

The quadcopter is considered an underactuated nonlinear system, as it has four inputs but six outputs. Due to the system's complexity, the quadcopter is modeled based on the following assumptions for control purposes.

- The structure is rigid
- The structure is axis symmetrical
- The Centre of Gravity and the body fixed frame origin coincide
- The propellers are rigid
- Thrust and drag are proportional to the square of the propeller's speed

Euler Angle:

Euler angles, introduced by Leonhard Euler, are three angles used to describe the orientation of a rigid body within a coordinate system. They also help describe the relationship between two different reference frames and facilitate the conversion of a point's coordinates from one reference frame to another. The Euler angles are represented as ϕ, θ, and ψ, corresponding to roll, pitch, and yaw, respectively. These angles describe the rotations of a body about the axes of a coordinate system. Any rigid body orientation can be achieved by combining these three basic Euler angles. The corresponding rotation matrices are provided as:-

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\phi) & -s(\phi) \\ 0 & s(\phi) & c(\phi) \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} c(\theta) & 0 & s(\theta) \\ 0 & 1 & 0 \\ -s(\theta) & 0 & c(\theta) \end{bmatrix}$$

$$R_z(\psi) = \begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Where $c(\phi) = \cos(\phi)$, $s(\phi) = \sin(\phi)$, $c(\theta) = \cos(\theta)$, $s(\theta) = \sin(\theta)$, $c(\psi) = \cos(\psi)$, and $s(\psi) = \sin(\psi)$. The rotation matrix that represents the relationship between the inertial frame and the body frame is given as:

$$R = Rz\ (\psi) \times Ry\ (\theta) \times Rx\ (\phi)$$

$$R = \begin{bmatrix} c(\theta)c(\psi) & s(\phi)s(\theta)c(\psi) - c(\phi)s(\psi) & c(\phi)s(\theta)c(\psi) + s(\phi)s(\psi) \\ c(\theta)s(\psi) & s(\phi)s(\theta)s(\psi) + c(\phi)c(\psi) & c(\phi)s(\theta)s(\psi) - s(\phi)c(\psi) \\ -s(\theta) & s(\phi)c(\theta) & c(\phi)c(\theta) \end{bmatrix}$$

R is a rotational matrix and is orthogonal such that $R^{-1} = R^{T}$.

Reference Frame Transformation:

Let $[x, y, z, \phi, \theta, \psi]^T$ represent the vector of linear and angular positions in the inertial frame, and $[u, v, w, p, q, r]^T$ denote the vector of linear and angular velocities in the body frame. Although the derivative of angular positions typically gives angular velocities, the angular positions and velocities above are expressed in different reference frames. Therefore, a transformation matrix is required to convert between these frames.

Let $\xi = [x\ y\ z]^T$ *and* $\eta = [\phi\ \theta\ \psi]$
Let $vI = [\dot{x}\ \dot{y}\ \dot{z}]^T$ *and* $\omega I = [\dot{\phi}\ \dot{\theta}\ \dot{\psi}]$
Let $vB = [u\ v\ w]^T$ *and* $\omega B = [p\ q\ r]^T$

From this statement, $vI \neq vB$ *and* $\omega I \neq \omega B$, instead:

$$vI = R \cdot v\mathrm{B}$$
$$\omega I = W\eta\ {-1} \cdot \omega\mathrm{B}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & s(\phi)t(\theta) & c(\phi)t(\theta) \\ 0 & c(\phi) & -s(\phi) \\ 0 & \frac{s(\phi)}{c(\theta)} & \frac{c(\phi)}{c(\theta)} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Alternatively,

$$\omega B = W\eta \cdot \omega\mathrm{I}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -s(\theta) \\ 0 & c(\phi) & s(\phi)c(\theta) \\ 0 & -s(\phi) & c(\phi)c(\theta) \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

Rotational Motion:-

Assuming the quadcopter is a rigid body and applying Euler's equations for rigid bodies, the dynamics equation in the body frame can be expressed as:

$$I\dot{\omega}B + [\omega B \times (I\omega B)] + \Gamma = \tau_{\mathrm{B}}$$

Additionally, we assume the quadcopter has a symmetric structure with its four arms aligned along the body's x and y axes. As a result, the inertia matrix I is diagonal, with $I_{xx} = I_{yy}$.

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

The gyroscopic forces Γ result from the combined rotation of both the four rotors and the quadcopter body, such that:

$$\Gamma = \sum Jr\ (\omega B \wedge \hat{e}3\ )(-1)^{i+1}\omega_{ri}$$

Where i is from 1 to 4.

In matrix form,

$$\Gamma = J_r\omega_B\omega_r \text{ where } \omega r = -\omega 1 + \omega 2 - \omega 3 + \omega 4$$

The external torque,

$$\tau B = [\tau\phi\ \tau\theta\ \tau\psi]\ T$$

The roll torque component $\tau\phi$, and the pitch torque component $\tau\theta$, are derived from standard mechanics, where motors $i = 1$ and $i = 3$ are arbitrarily assigned to the roll axis, while motors $i = 2$ and $i = 4$ are arbitrarily assigned to the pitch axis.

$$\tau\phi = \Sigma r \times T = l(-kt\omega_2^2 + kt\omega_4^2\ ) = lkt\ (-\omega_2^2 + \omega_4^2\ )$$

$$\tau\theta = \Sigma r \times T = l(-kt\omega_1^2 + kt\omega_3^2\ \ ) = lkt\ (-\omega_1^2 + \omega_3^2)$$

For the yaw-axis, the rotor axis is aligned with the z-direction in the body frame. The torque generated around the rotor axis is expressed as:

$$\tau\psi = (-1)^{i+1}kb\omega_i^2 + Im\dot{\omega}_i$$

where $(-1)^{i+1}$ is positive for the i-th propeller when it spins clockwise and negative when it spins counter-clockwise. The term $Im\dot{\omega}_i$ can be neglected since, in a steady state, $\dot{\omega}_i$ approx 0. Consequently, the total torque around the z-axis is the sum of the torques produced by each propeller:

$$\tau\psi = kb\,(-\omega_1^2 + \omega_2^2 - \omega_3^2 + \omega_4^2)$$

Therefore, the torque matrix can be written as:

$$\tau_B = \begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} lk_t(-\omega_2^2 + \omega_4^2) \\ lk_t(-\omega_1^2 + \omega_3^2) \\ k_b(-\omega_1^2 + \omega_2^2 - \omega_3^2 + \omega_4^2) \end{bmatrix}$$

where k_t is the thrust coefficient, k_b is the drag coefficient and l is the distance between rotor and the center of mass of the quadcopter.

Translation Motion:

Using the Newtonian equations to model the linear dynamics, the external forces acting on the quadcopter are expressed as:

$$\dot{m}v_1 = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + RT_B + F_D$$

The drag force, F_D , is the force exerted on the system due to fluid friction (air resistance). A simplified equation is used where the friction is modeled as being proportional to the linear velocity in all directions:

$F_D = -k_d\ v_l$

The angular velocity of the i-th rotor generates a force F_i along the rotor axis (z-direction). The combined forces result in a thrust T along the body's z-axis, such that:

$$T = \sum_{i=1}^{4} F_i = k_t \sum_{i=1}^{4} \omega_i^2$$

Since it acts in the z-axis:

$$T_B = \begin{bmatrix} 0 \\ 0 \\ T \end{bmatrix} = k_t \begin{bmatrix} 0 \\ 0 \\ \Sigma_{i=1}^{4} \omega_i^2 \end{bmatrix}$$

State Space Model:

$$\dot{\omega}_B = I^{-1}(-\omega_B \times (I\omega_B) - \Gamma + \tau_B)$$

$$\dot{\omega}_B = I^{-1}\left(\begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} I_{xx}p \\ I_{yy}q \\ I_{zz}r \end{bmatrix} - J_r \begin{bmatrix} p \\ q \\ r \end{bmatrix} \omega_r + \tau_B \right)$$

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \frac{(I_{yy}-I_{zz})qr}{I_{xx}} \\ \frac{(I_{zz}-I_{xx})pr}{I_{yy}} \\ \frac{(I_{xx}-I_{yy})pq}{I_{zz}} \end{bmatrix} - J_r \begin{bmatrix} \frac{q}{I_{xx}} \\ \frac{-p}{I_{yy}} \\ 0 \end{bmatrix} \omega_r + \begin{bmatrix} \frac{\tau_\phi}{I_{xx}} \\ \frac{\tau_\theta}{I_{yy}} \\ \frac{\tau_\psi}{I_{zz}} \end{bmatrix}$$

For the linear dynamics:-

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = -g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{T}{m} \begin{bmatrix} c(\psi)s(\theta)c(\phi) + s(\psi)s(\phi) \\ s(\psi)s(\theta)c(\phi) + c(\psi)s(\phi) \\ c(\theta)c(\phi) \end{bmatrix} - \frac{1}{m} \begin{bmatrix} k_{dx} & 0 & 0 \\ 0 & k_{dy} & 0 \\ 0 & 0 & k_{dz} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

QuadCopter System Control:-

The Control implemented is PID Controller:-

PID Controller:-

The general form of a PID controller is given as:

$$e(t) = r(t) - y(t)$$

$$u(t) = K_p e(t) + K_I \int e(\tau)d\tau + K_D d/dt(e(t))$$

Here, $u(t)$ represents the control input, $r(t)$ is the desired state, and $y(t)$ is the current or actual state. K_p , K_I , and K_D are the gain parameters for the proportional, integral, and derivative terms of the PID controller.

For the quadcopter control, the proportional and derivative terms are utilized. The generated torque is proportional to the angular velocities, so we set the torques to be proportional to the output of the controller as follows:

$$\tau = I \times u(t)$$

$$\begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} I_{xx}\left(k_{P\phi}(\phi_{des} - \phi) + k_{D\phi}(\dot{\phi}_{des} - \dot{\phi})\right) \\ I_{yy}\left(k_{P\theta}(\theta_{des} - \theta) + k_{D\theta}(\dot{\theta}_{des} - \dot{\theta})\right) \\ I_{zz}\left(k_{P\psi}(\psi_{des} - \psi) + k_{D\psi}(\dot{\psi}_{des} - \dot{\psi})\right) \end{bmatrix}$$

The input to the quadcopter system is the angular velocity of the rotors.

Recalling,

$$\tau_B = \begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} lk_t(-\omega_2^2 + \omega_4^2) \\ lk_t(-\omega_1^2 + \omega_3^2) \\ k_b(-\omega_1^2 + \omega_2^2 - \omega_3^2 + \omega_4^2) \end{bmatrix}$$

The torque is related to the square of the angular velocity of the rotors. There are three equations but four unknowns. To simplify, the total thrust T , which affects the acceleration in the z-direction, is set equal to mg . This constraint ensures that the quadcopter remains airborne. By converting this thrust equation to the appropriate reference frame and using a PD controller to minimize the error along the z-axis, we can achieve the desired dynamics:

$$T = (g + k_{PZ}(zdes - z) + k_{Dz}(zdes^{\cdot} - \dot{z}))*(m\,/(c(\phi)c(\theta)))$$

Solving for the angular velocities of the rotor ω_1^2 , ω_2^2 , ω_3^2 , *and* ω_4^2

$$\begin{bmatrix} w_1^2 \\ w_2^2 \\ w_3^2 \\ w_4^2 \end{bmatrix} = \begin{bmatrix} k_t & k_t & k_t & k_t \\ 0 & -lk_t & 0 & lk_t \\ -lk_t & 0 & lk_t & 0 \\ -k_b & k_b & -k_b & k_b \end{bmatrix}^{-1} \begin{bmatrix} T \\ \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix}$$

$$w_1^2 = \frac{T}{4k_t} - \frac{\tau_\theta}{2lk_t} - \frac{\tau_\psi}{4k_b}$$

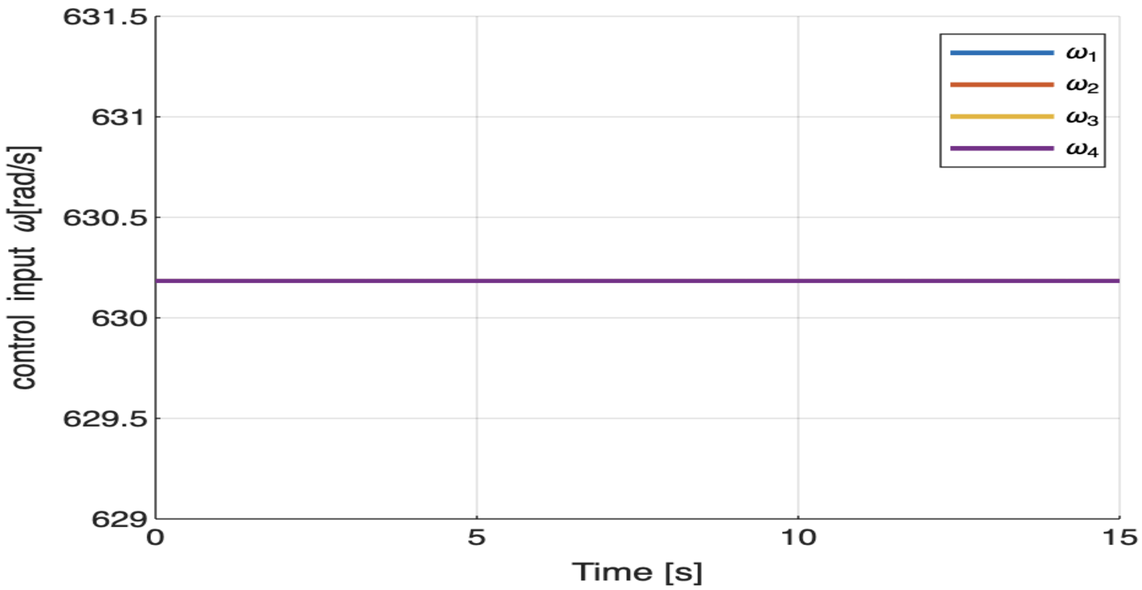
$$w_2^2 = \frac{T}{4k_t} - \frac{\tau_\phi}{2lk_t} + \frac{\tau_\psi}{4k_b}$$

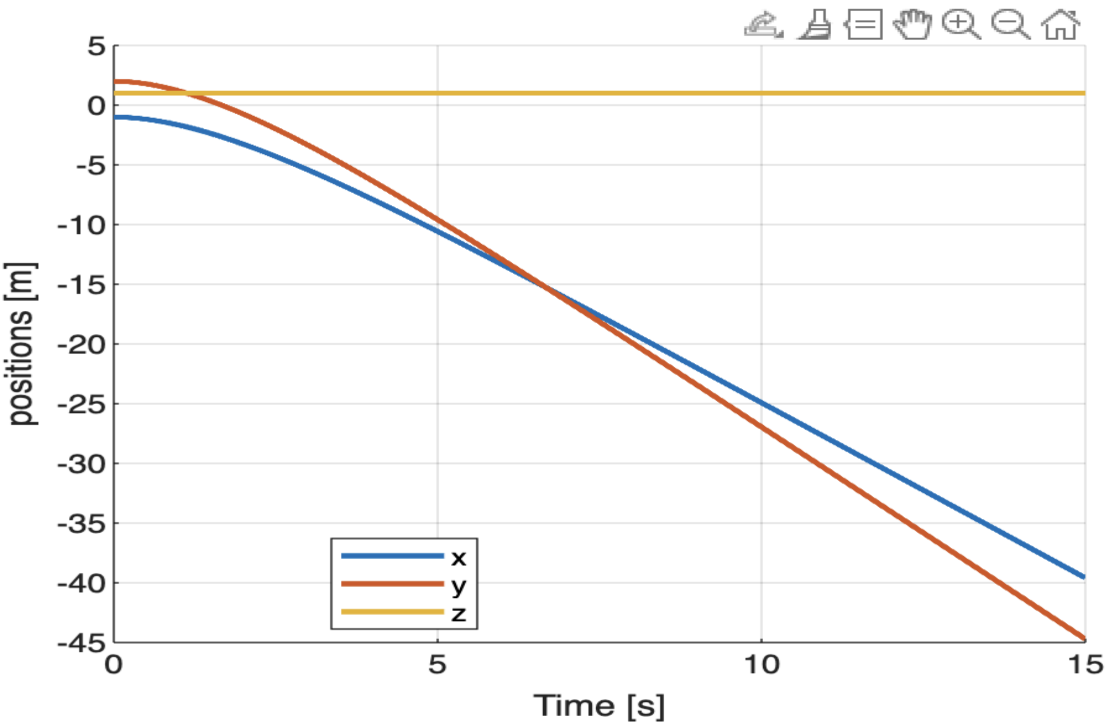
$$w_3^2 = \frac{T}{4k_t} + \frac{\tau_\theta}{2lk_t} - \frac{\tau_\psi}{4k_b}$$

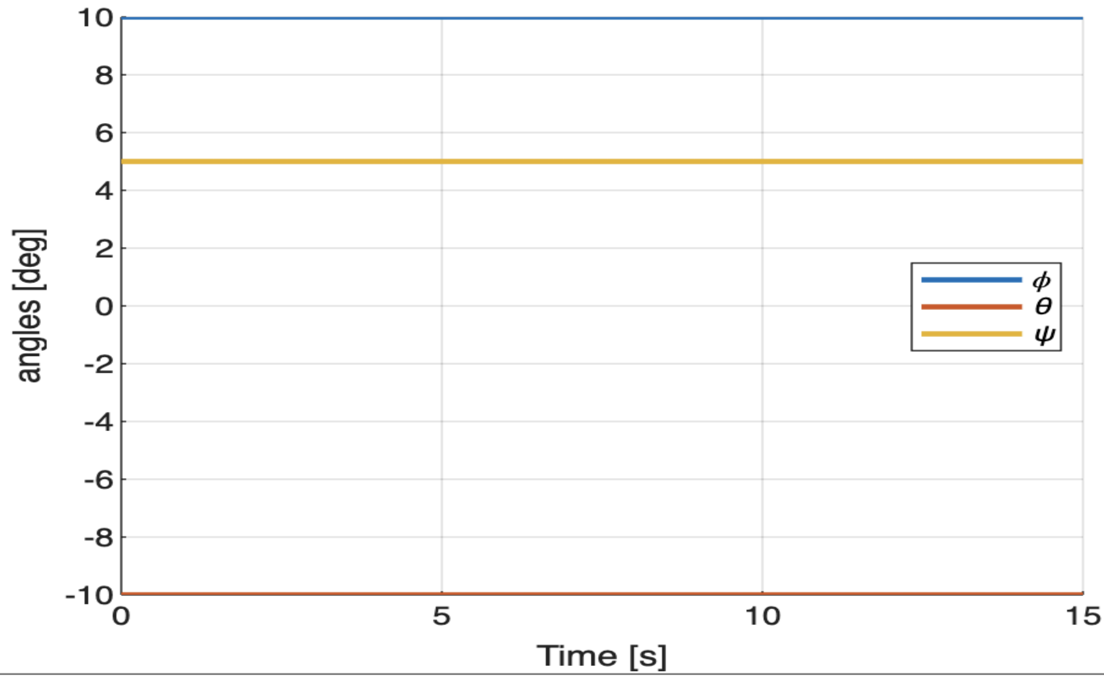
$$w_4^2 = \frac{T}{4k_t} + \frac{\tau_\phi}{2lk_t} + \frac{\tau_\psi}{4k_b}$$

Simulations and Results:-

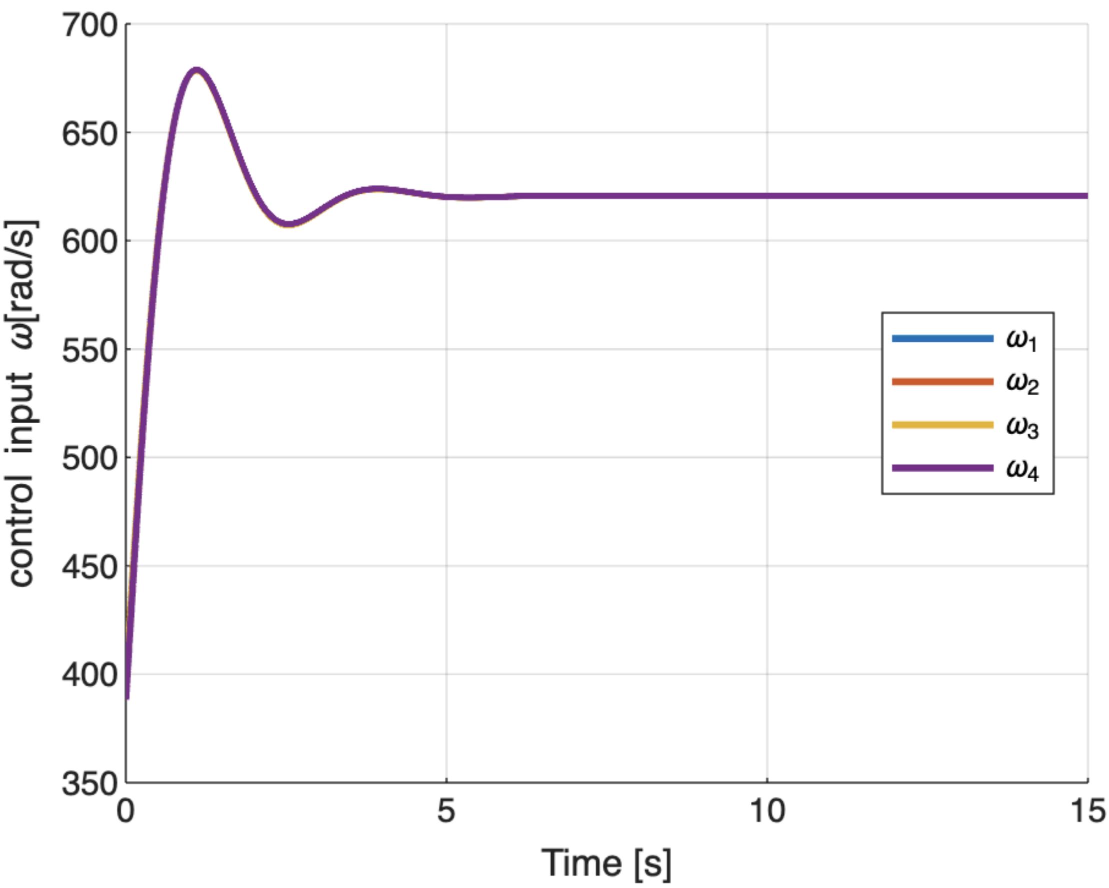
Simulation without PD Control:-

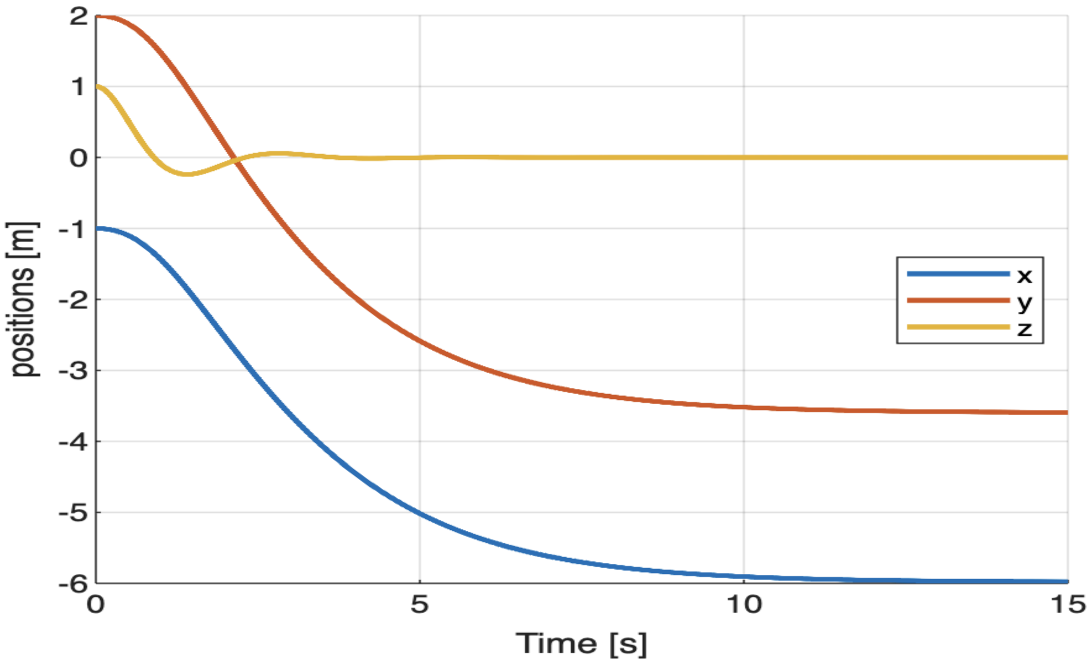




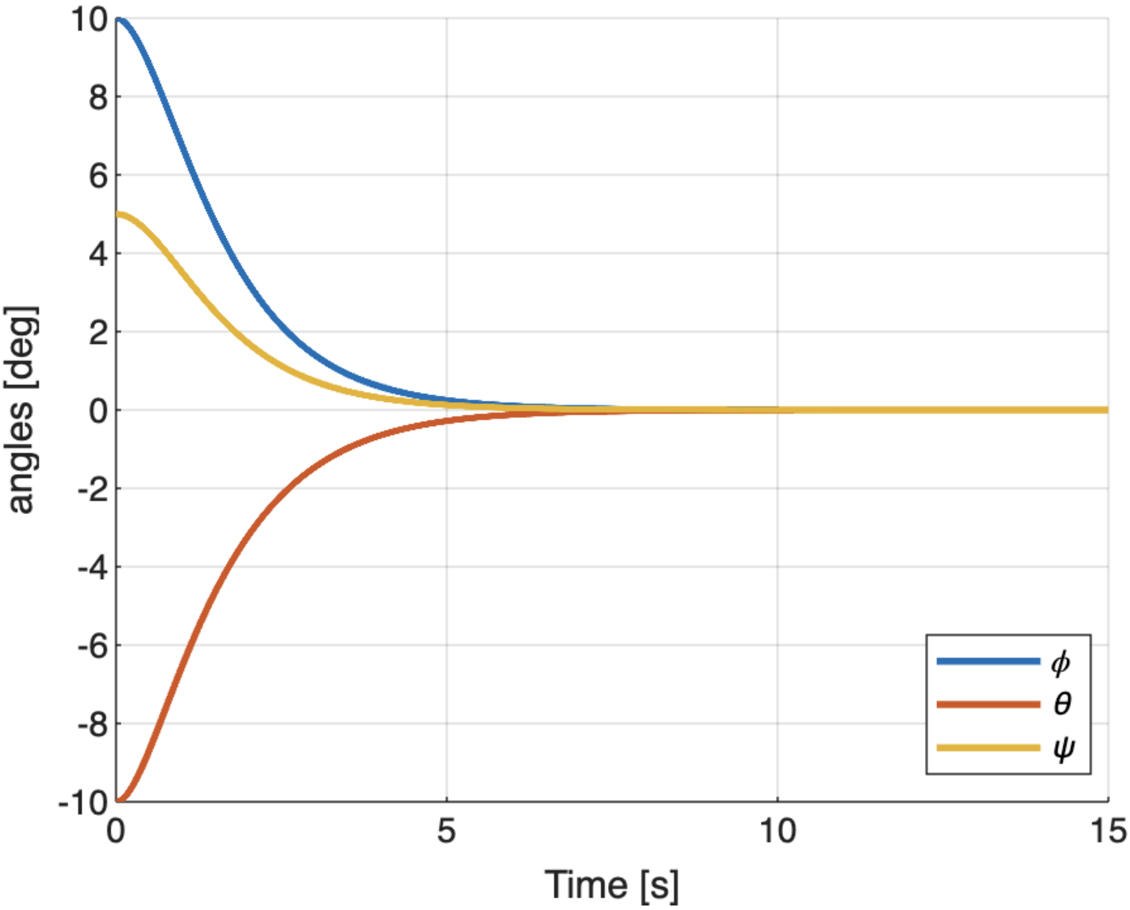


Simulation with PD Control:-



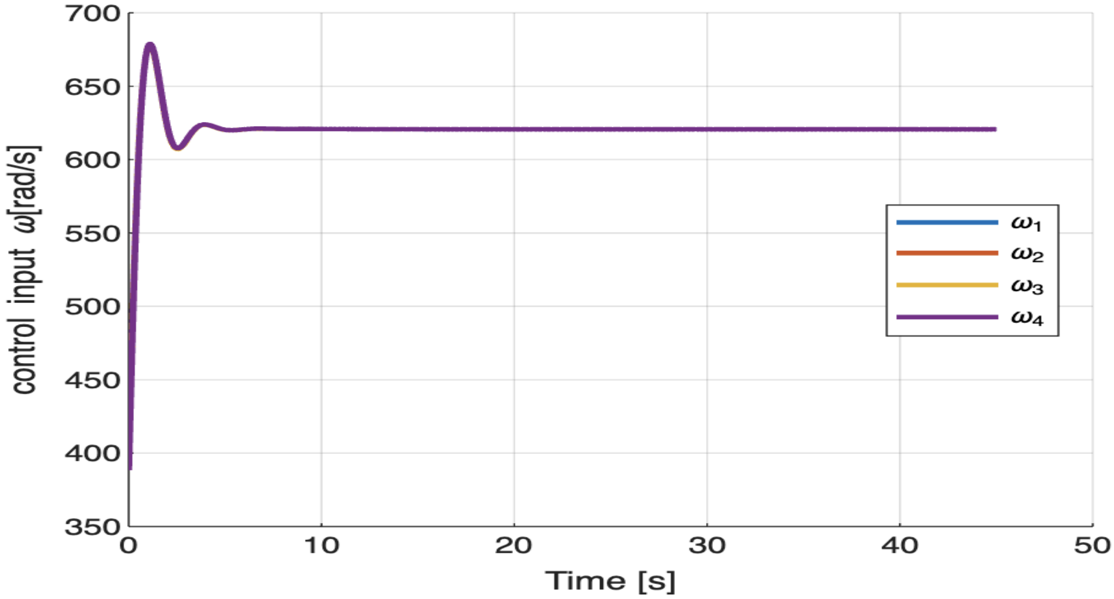


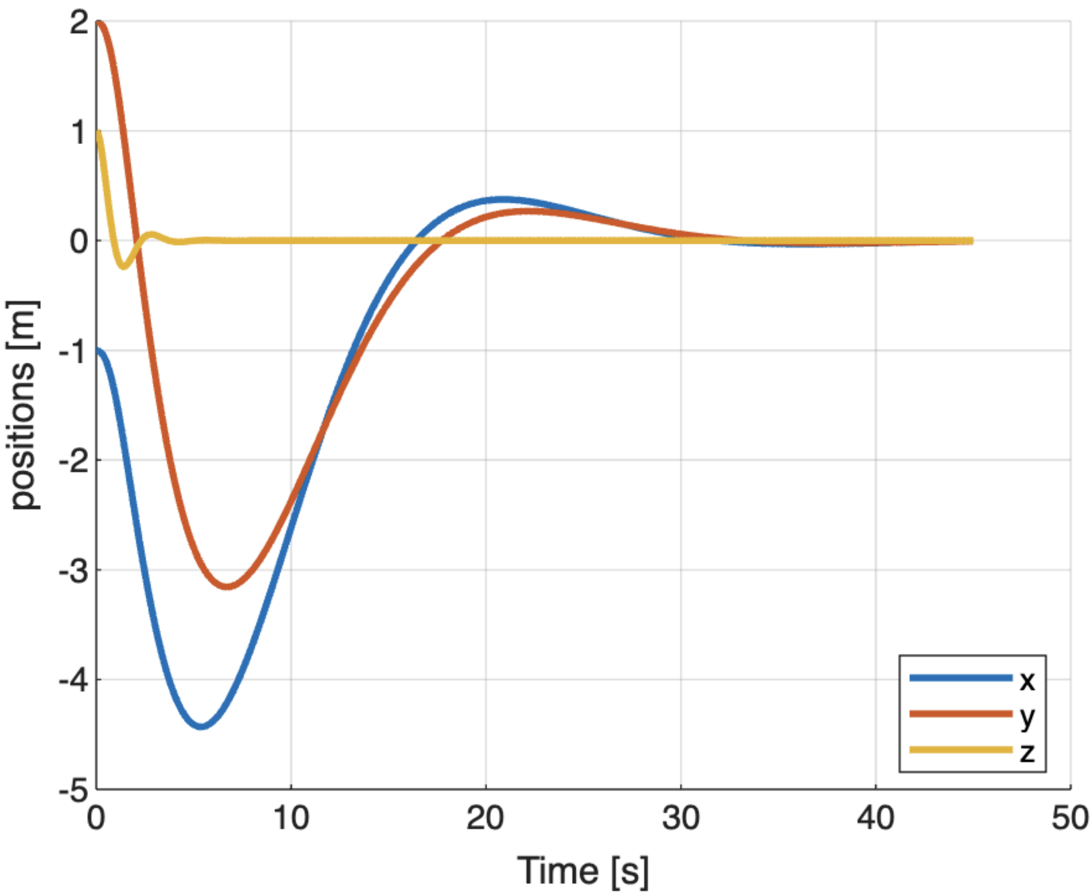
positions [m]
Time [s]
x
y
z
2
1
0
-1
-2
-3
-4
-5
-6
0
5
10
15

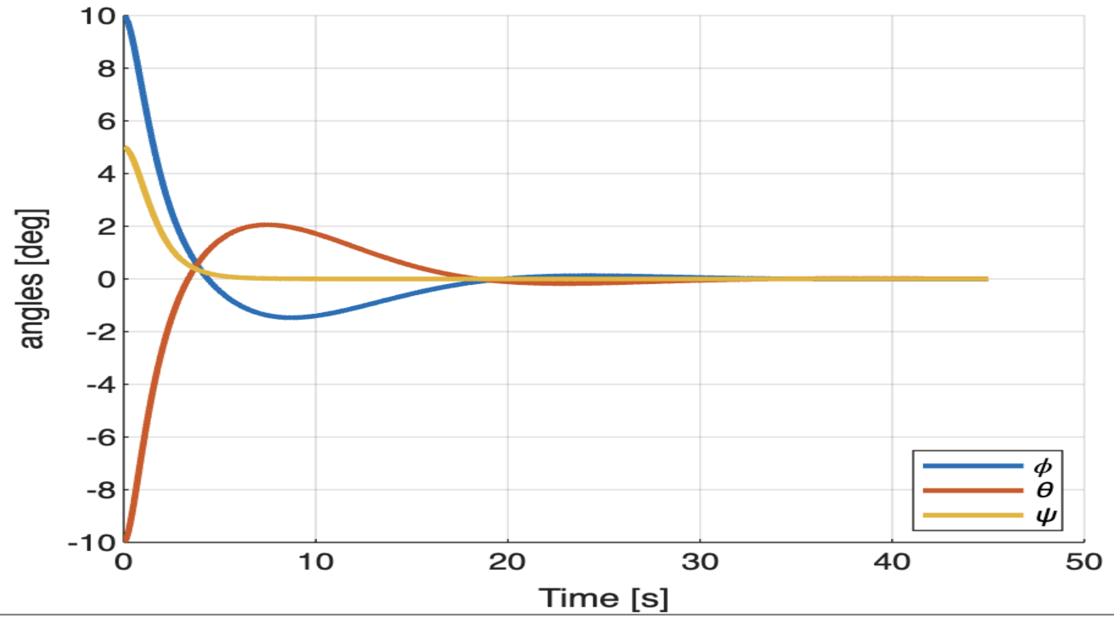


angles [deg]
Time [s]
ϕ
θ
ψ
10
8
6
4
2
0
-2
-4
-6
-8
-10
0
5
10
15

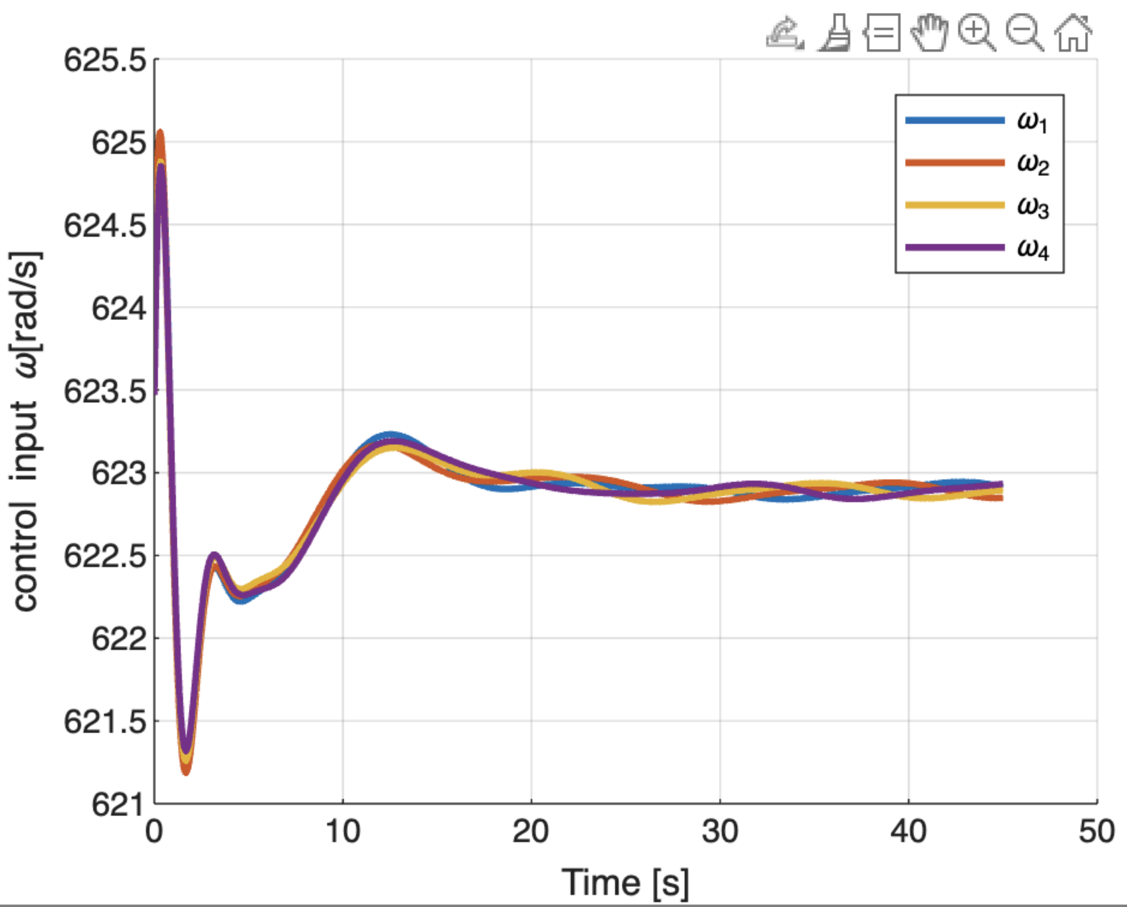
Simulation of stabilization with Zero Mapping of Position States:

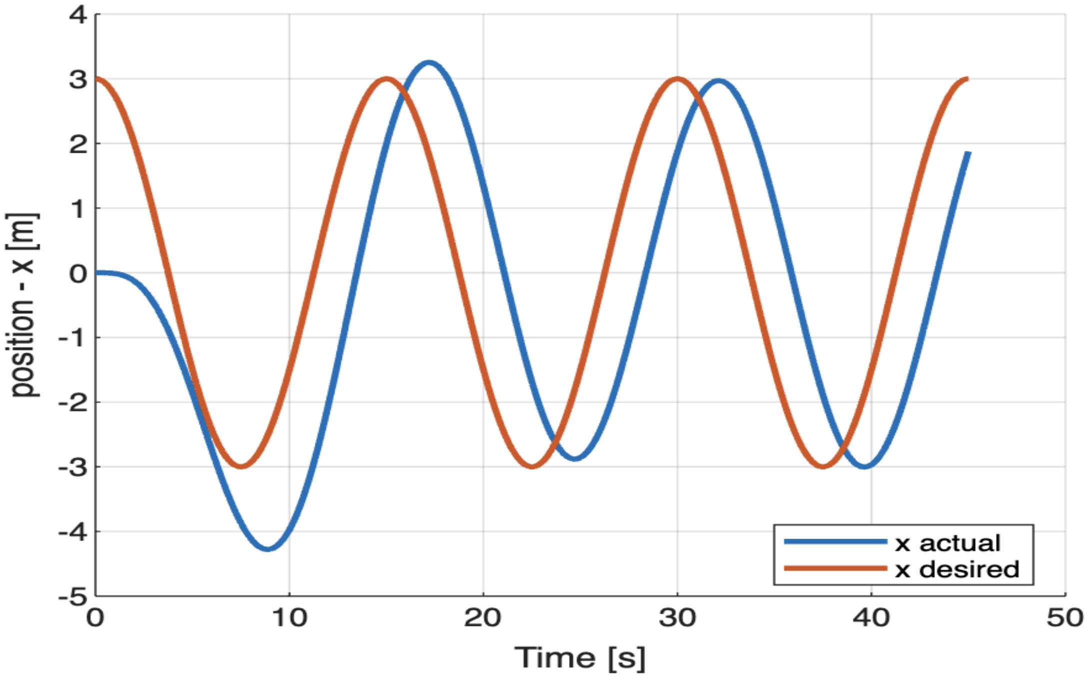


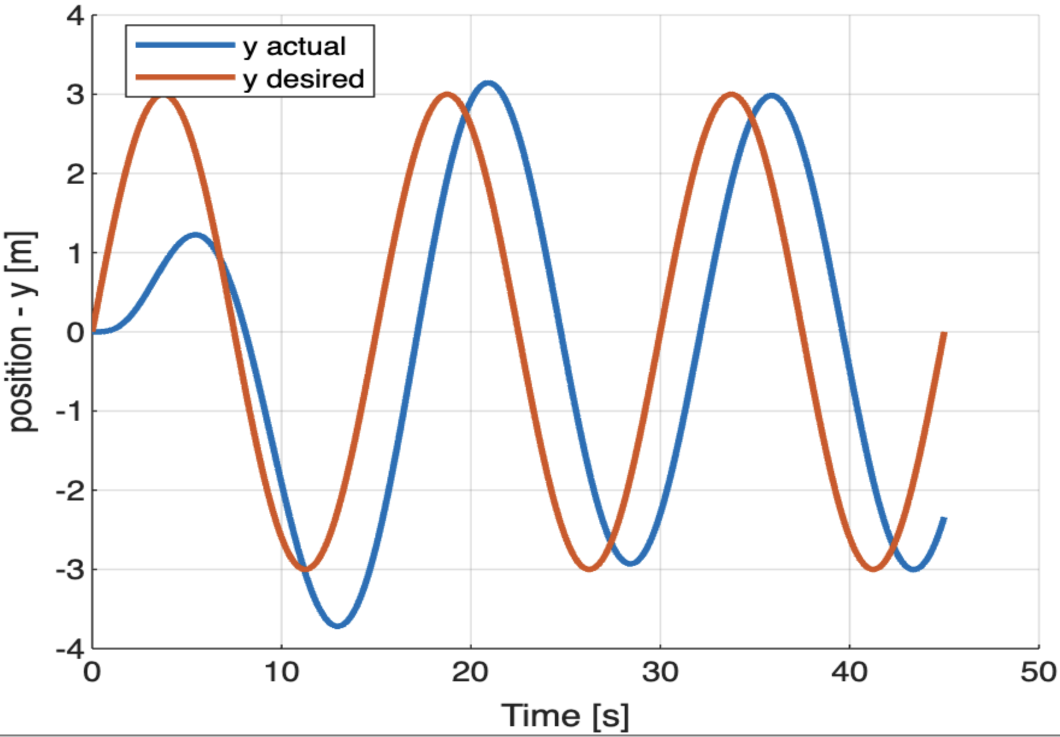


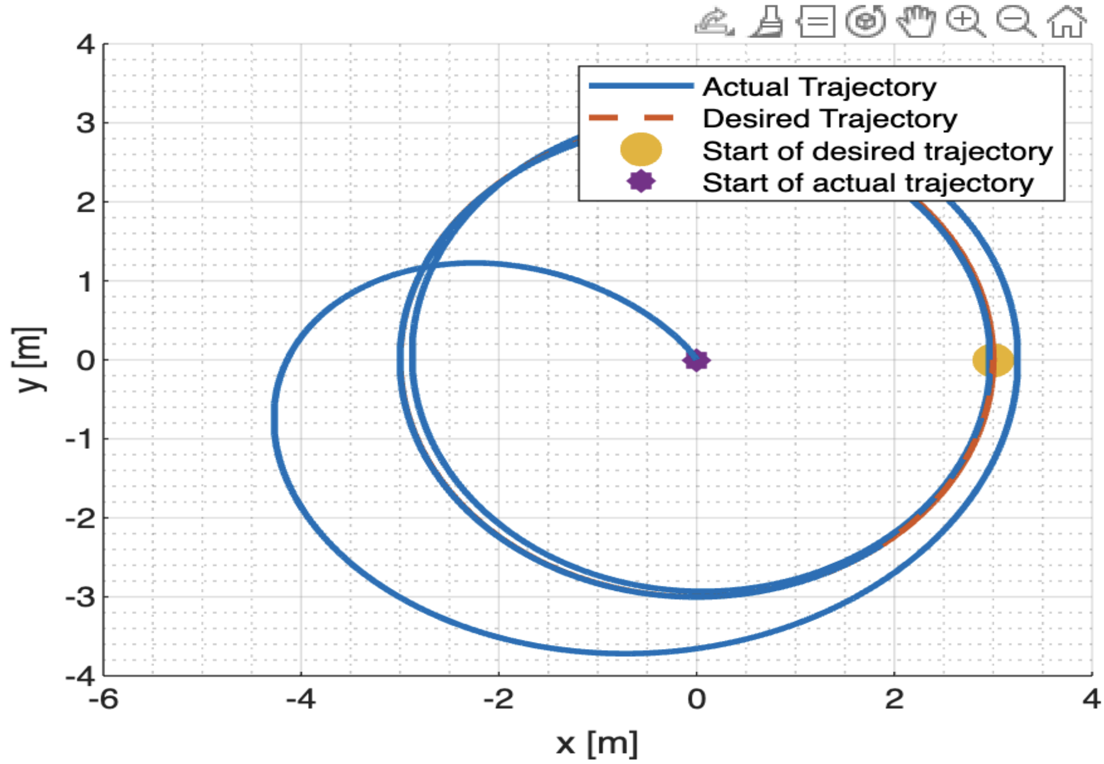


Trajectory Tracking with PD Control (Circular Trajectory):

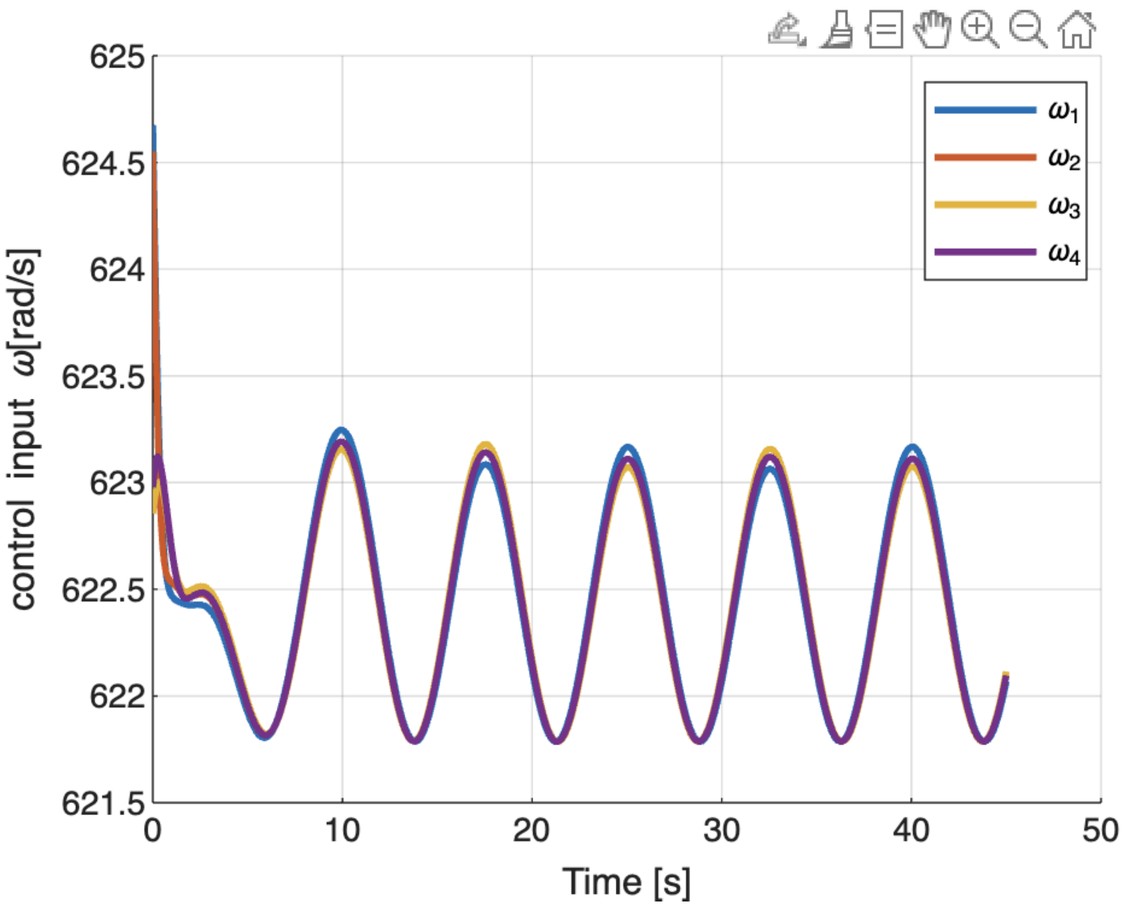


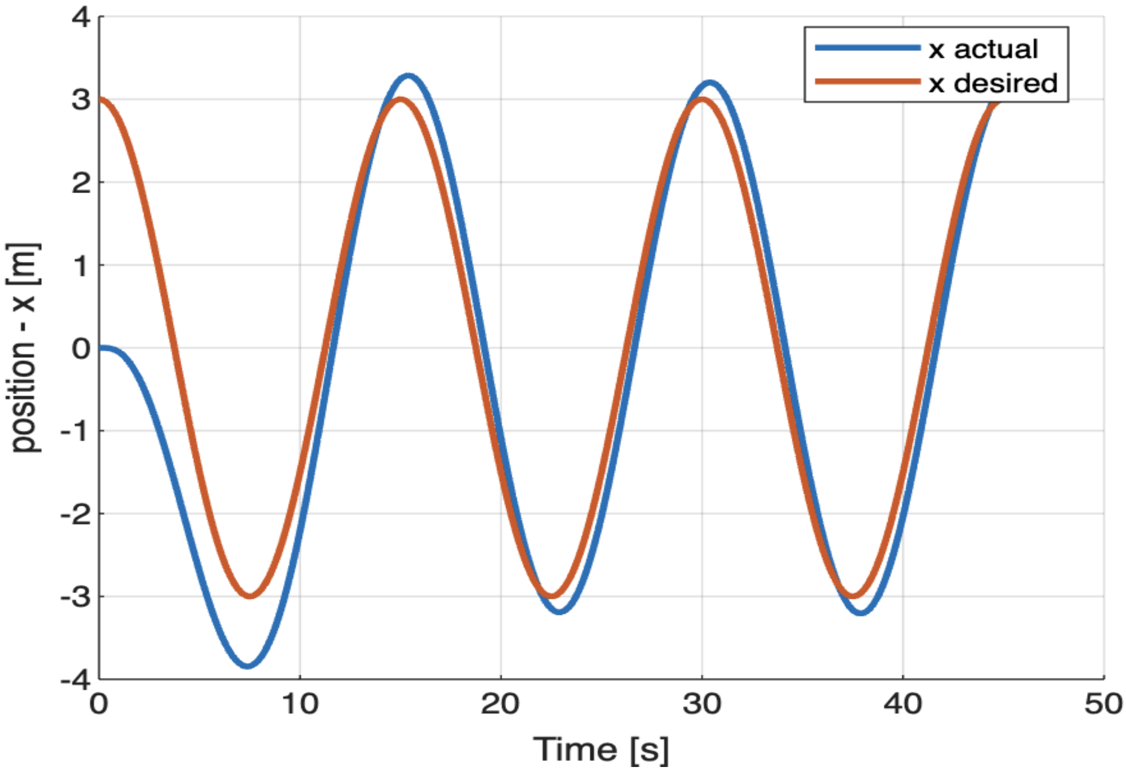

position - x [m]
Time [s]
x actual
x desired
4
3
2
1
0
-1
-2
-3
-4
-5
0
10
20
30
40
50


position - y [m]
Time [s]
y actual
y desired
4
3
2
1
0
-1
-2
-3
-4
0
10
20
30
40
50

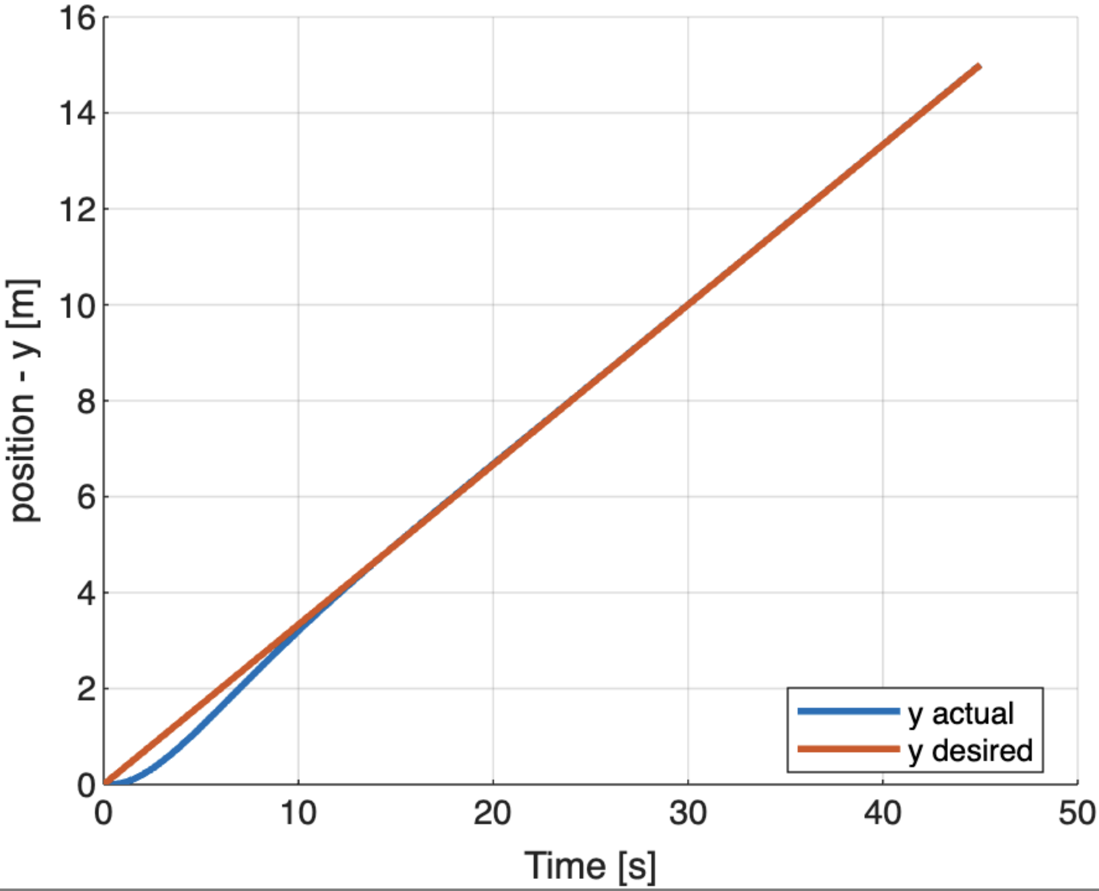


Trajectory Tracking with PD Control (Spiral Trajectory):

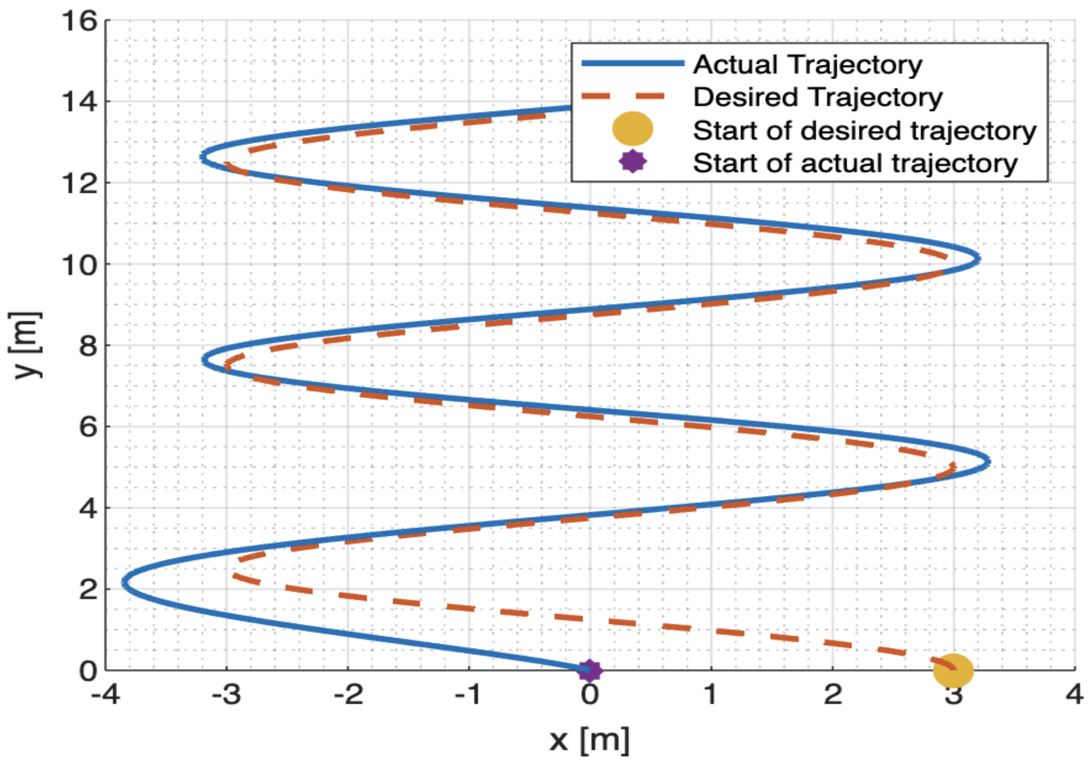




x actual
x desired
position - x [m]
Time [s]
4
3
2
1
0
-1
-2
-3
-4
0
10
20
30
40
50



position - y [m]
Time [s]
y actual
y desired



y [m]
x [m]
Actual Trajectory
Desired Trajectory
Start of desired trajectory
Start of actual trajectory

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