Assignment 8

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Download latex-tikz codes from

https://github.com/Dishank422/AI1103−Probability−and−random−variables/blob/main/Assignment_8/main.tex

1 Problem

(CSIR-UGC-NET-DEC 2015, Q. 108) Suppose that (X, Y) has a joint probability distribution with the marginal distribution of X being N(0,1) and $E(Y|X = x) = x^3$ for all $x \in R$. Then, which of the following statements are true?

1) Corr$(X, Y) = 0$
2) Corr$(X, Y) > 0$
3) Corr$(X, Y) < 0$
4) X and Y are independent

2 Solution

The following result shall be useful later. For $n \in N$

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = \begin{cases} 0 & n \text{ is odd} \\ (n-1) \times ... \times 3 \times 1 & n \text{ is even} \end{cases} \tag{2.0.1}$$

The proof for the above can be found at the end of the solution.

$$Corr(X, Y) = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y} \tag{2.0.2}$$

We know $X \sim N(0, 1)$. Thus,

$$f_X(x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} \tag{2.0.3}$$

$$E(X) = 0 \tag{2.0.4}$$

$$\sigma_X^2 = 1 \tag{2.0.5}$$

$$\sigma_Y^2 = E(Y^2) - E(Y)^2 \tag{2.0.6}$$

$$E(Y) = \int_{-\infty}^{\infty} E(Y|X = x) f_X(x) dx \tag{2.0.7}$$

$$= \int_{-\infty}^{\infty} \frac{x^3 e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \tag{2.0.8}$$

$$= 0 \tag{2.0.9}$$

$$E(Y^2) = \int_{-\infty}^{\infty} E(Y^2|X = x) f_X(x) dx \tag{2.0.10}$$

$$= \int_{-\infty}^{\infty} \frac{x^6 e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \tag{2.0.11}$$

$$= 15 \tag{2.0.12}$$

Substituting in (2.0.6)

$$\sigma_Y^2 = 15 \tag{2.0.13}$$

$$\sigma_{XY}^2 = E(XY) - E(X)E(Y) \tag{2.0.14}$$

$$E(XY) = \int_{-\infty}^{\infty} E(XY|X = x) f_X(x) dx \tag{2.0.15}$$

$$= \int_{-\infty}^{\infty} E(xY|X = x) f_X(x) dx \tag{2.0.16}$$

$$= \int_{-\infty}^{\infty} xE(Y|X = x) f_X(x) dx \tag{2.0.17}$$

$$= \int_{-\infty}^{\infty} \frac{x^4 e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \tag{2.0.18}$$

$$= 3 \tag{2.0.19}$$

Substituting in (2.0.14)

$$\sigma_{XY}^2 = 3 \tag{2.0.20}$$

Substituting in (2.0.2)

$$Corr(X, Y) = \frac{3}{\sqrt{15}} > 0 \tag{2.0.21}$$

Since $Corr(X, Y) \neq 0$, X and Y are dependent. Thus option 2 is the only correct option.

Proof for the integral:

If n is odd, $\frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$ is an odd function, thus

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = 0 \tag{2.0.22}$$

If n is even, let $n = 2k$. We differentiate the following identity k times w.r.t. α.

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\left(\frac{\pi}{\alpha}\right)} \tag{2.0.23}$$

On differentiating k times, we get

$$\int_{-\infty}^{\infty} x^{2k} e^{-\alpha x^2} = \frac{1 \times 3 \times ... \times (2k-1)}{2^k} \sqrt{\left(\frac{\pi}{\alpha^{2k+1}}\right)} \tag{2.0.24}$$

On substituting $\alpha = \frac{1}{2}$, we get

$$\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} = 1 \times 3 \times ... \times (n-1) \sqrt{2\pi} \tag{2.0.25}$$

Thus

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = (n-1) \times ... \times 3 \times 1 \tag{2.0.26}$$