

Assignment 8

Dishank Jain - AI20BTECH11011

Download latex-tikz codes from

https://github.com/Dishank422/AI1103-Probability-and-random-variables/blob/main/Assignment_8/main.tex

1 PROBLEM

(CSIR-UGC-NET-DEC 2015, Q. 108) Suppose that (X, Y) has a joint probability distribution with the marginal distribution of X being $N(0,1)$ and $E(Y|X = x) = x^3$ for all $x \in R$. Then, which of the following statements are true?

- 1) $\text{Corr}(X, Y) = 0$
- 2) $\text{Corr}(X, Y) > 0$
- 3) $\text{Corr}(X, Y) < 0$
- 4) X and Y are independent

2 SOLUTION

$$\text{Corr}(X, Y) = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y} \quad (2.0.1)$$

We know $X \sim N(0, 1)$. Thus,

$$f_X(x) = \phi(x) \quad (2.0.2)$$

$$E(X) = 0 \quad (2.0.3)$$

$$\sigma_X^2 = 1 \quad (2.0.4)$$

$$\sigma_Y^2 = E(Y^2) - E(Y)^2 \quad (2.0.5)$$

$$E(Y) = \int_{-\infty}^{\infty} E(Y|X = x)f_X(x)dx \quad (2.0.6)$$

$$= \int_{-\infty}^{\infty} x^3 \phi(x)dx \quad (2.0.7)$$

$$= 0 \quad (2.0.8)$$

$$E(Y^2) = \int_{-\infty}^{\infty} E(Y^2|X = x)f_X(x)dx \quad (2.0.9)$$

$$= \int_{-\infty}^{\infty} x^6 \phi(x)dx \quad (2.0.10)$$

$$= 15 \quad (2.0.11)$$

Substituting in (2.0.5)

$$\sigma_Y^2 = 15 \quad (2.0.12)$$

$$\sigma_{XY}^2 = E(XY) - E(X)E(Y) \quad (2.0.13)$$

$$E(XY) = \int_{-\infty}^{\infty} E(XY|X = x)f_X(x)dx \quad (2.0.14)$$

$$= \int_{-\infty}^{\infty} xE(Y|X = x)f_X(x)dx \quad (2.0.15)$$

$$= \int_{-\infty}^{\infty} x^4 \phi(x)dx \quad (2.0.16)$$

$$= 3 \quad (2.0.17)$$

Substituting in (2.0.13)

$$\sigma_{XY}^2 = 3 \quad (2.0.18)$$

Substituting in (2.0.1)

$$\text{Corr}(X, Y) = \frac{3}{\sqrt{15}} > 0 \quad (2.0.19)$$

Since $\text{Corr}(X, Y) \neq 0$, X and Y are dependent. Thus option 2 is the only correct option.

The integrals can be solved using the following result for $n \in N$

$$\int_{-\infty}^{\infty} x^n \phi(x)dx = \begin{cases} 0 & n \text{ is odd} \\ (n-1) \times \dots \times 3 \times 1 & n \text{ is even} \end{cases} \quad (2.0.20)$$

Proof:

If n is odd, $x^n \phi(x)$ is odd function, thus

$$\int_{-\infty}^{\infty} x^n \phi(x)dx = 0 \quad (2.0.21)$$

If n is even,

$$\int_{-\infty}^{\infty} x^n \phi(x)dx = \int_{-\infty}^{\infty} (x^{n-1})(x\phi(x))dx \quad (2.0.22)$$

$$= \left(x^{n-1} \int x \phi(x) dx \right) \Big|_{-\infty}^{\infty} - (n-1) \int_{-\infty}^{\infty} x^{n-2} \left(\int x \phi(x) dx \right) dx \quad (2.0.23)$$

$$= \left(x^{n-1} (-\phi(x)) \right) \Big|_{-\infty}^{\infty} - (n-1) \int_{-\infty}^{\infty} x^{n-2} (-\phi(x)) dx \quad (2.0.24)$$

$$= (n-1) \int_{-\infty}^{\infty} x^{n-2} \phi(x) dx \quad (2.0.25)$$

$$= (n-1)(n-3) \int_{-\infty}^{\infty} x^{n-4} \phi(x) dx \quad (2.0.26)$$

$$= (n-1) \times \dots \times 3 \times 1 \int_{-\infty}^{\infty} x^0 \phi(x) dx \quad (2.0.27)$$

$$= (n-1) \times \dots \times 3 \times 1 \quad (2.0.28)$$