Assignment 8

Dishank Jain - AI20BTECH11011

Download all python codes from

```
https://github.com/Dishank422/AI1103−Probability
    −and−random−variables/blob/main/
    Assignment_8/codes
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and latex-tikz codes from

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https://github.com/Dishank422/AI1103−Probability
    −and−random−variables/blob/main/
    Assignment_8/main.tex
```

1 Problem

(CSIR-UGC-NET-DEC 2015, Q. 108) Suppose that (X, Y) has a joint probability distribution with the marginal distribution of X being N(0,1) and $E(Y|X = x) = x^3$ for all $x \in R$. Then, which of the following statements are true?

1) Corr$(X, Y) = 0$
2) Corr$(X, Y) > 0$
3) Corr$(X, Y) < 0$
4) X and Y are independent

2 Solution

$$Corr(X, Y) = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y} \tag{2.0.1}$$

We know $X \sim N(0, 1)$. Thus,

$$f_X(x) = \phi(x) \tag{2.0.2}$$
$$E(X) = 0 \tag{2.0.3}$$
$$\sigma_X^2 = 1 \tag{2.0.4}$$

$$\sigma_Y^2 = E(Y^2) - E(Y)^2 \tag{2.0.5}$$

$$E(Y) = \int_{-\infty}^{\infty} E(Y|X = x) f_X(x) dx \tag{2.0.6}$$
$$= \int_{-\infty}^{\infty} x^3 \phi(x) dx \tag{2.0.7}$$
$$= 0 \tag{2.0.8}$$

$$E(Y^2) = \int_{-\infty}^{\infty} E(Y^2|X = x) f_X(x) dx \tag{2.0.9}$$
$$= \int_{-\infty}^{\infty} x^6 \phi(x) dx \tag{2.0.10}$$
$$= 15 \tag{2.0.11}$$

Substituting in (2.0.5)

$$\sigma_Y^2 = 15 \tag{2.0.12}$$

$$\sigma_{XY}^2 = E(XY) - E(X)E(Y) \tag{2.0.13}$$

$$E(XY) = \int_{-\infty}^{\infty} E(XY|X = x) f_X(x) dx \tag{2.0.14}$$
$$= \int_{-\infty}^{\infty} x E(Y|X = x) f_X(x) dx \tag{2.0.15}$$
$$= \int_{-\infty}^{\infty} x^4 \phi(x) dx \tag{2.0.16}$$
$$= 3 \tag{2.0.17}$$

Substituting in (2.0.13)

$$\sigma_{XY}^2 = 3 \tag{2.0.18}$$

Substituting in (2.0.1)

$$Corr(X, Y) = \frac{3}{\sqrt{15}} > 0 \tag{2.0.19}$$

Since $Corr(X, Y) \neq 0$, X and Y are dependent. Thus option 2 is the only correct option.
The integrals can be solved using the following result for $n \in N$

$$\int_{-\infty}^{\infty} x^n \phi(x) dx = \begin{cases} 0 & n \text{ is odd} \\ (n-1) \times ... \times 3 \times 1 & n \text{ is even} \end{cases} \tag{2.0.20}$$

Proof:
If n is odd, $x^n \phi(x)$ is odd function, thus

$$\int_{-\infty}^{\infty} x^n \phi(x) dx = 0 \tag{2.0.21}$$

If n is even,

$$\int_{-\infty}^{\infty} x^n \phi(x)dx = \int_{-\infty}^{\infty} (x^{n-1})(x\phi(x))dx \qquad (2.0.22)$$

$$= \left(x^{n-1} \int x\phi(x)dx\right)\Bigg|_{-\infty}^{\infty} - (n-1) \int_{-\infty}^{\infty} x^{n-2} \left(\int x\phi(x)dx\right) dx \qquad (2.0.23)$$

$$= \left(x^{n-1}(-\phi(x))\right)\Big|_{-\infty}^{\infty} - (n-1) \int_{-\infty}^{\infty} x^{n-2}(-\phi(x))dx \qquad (2.0.24)$$

$$= (n-1) \int_{-\infty}^{\infty} x^{n-2}\phi(x)dx \qquad (2.0.25)$$

$$= (n-1)(n-3) \int_{-\infty}^{\infty} x^{n-4}\phi(x)dx \qquad (2.0.26)$$

$$= (n-1) \times ... \times 3 \times 1 \int_{-\infty}^{\infty} x^{0}\phi(x)dx \qquad (2.0.27)$$

$$= (n-1) \times ... \times 3 \times 1 \qquad (2.0.28)$$