Assignment 9

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Download all python codes from

```
https://github.com/Dishank422/AI1103−Probability
    −and−random−variables/blob/main/
    Assignment_9/codes
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and latex-tikz codes from

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https://github.com/Dishank422/AI1103−Probability
    −and−random−variables/blob/main/
    Assignment_9/main.tex
```

1 Problem

(CSIR UGC NET, June 2016, Q.107)Suppose X and Y are independent and identically distributed random variables and let $Z = X + Y$. Then the distribution of Z is in the same family as that of X and Y if X is

1) Normal
2) Exponential
3) Uniform
4) Binomial

2 Solution

1) Let X and Y be independent and identically distributed normal random variables. Then the characteristic function of X and Y is given by

$$\Phi_X(\omega) = e^{j\eta\omega - \sigma^2\omega^2/2} \tag{2.0.1}$$

The characteristic function of Z is given by

$$\Phi_Z(\omega) = \Phi_X^2(\omega) \tag{2.0.2}$$
$$= e^{2j\eta\omega - \sigma^2\omega^2} \tag{2.0.3}$$

Thus Z is a normal random variable with parameters 2η and $2\sigma^2$. Thus option (1) is correct.

2) Let X and Y be independent and identically distributed exponential random variables. Then the pdf of X and Y is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & otherwise \end{cases} \tag{2.0.4}$$

Using convolution,

$$f_z(z) = \begin{cases} \int_0^z f_X(x) f_Y(z-x)dx & z > 0 \\ 0 & otherwise \end{cases} \tag{2.0.5}$$

Thus if $z > 0$,

$$f_Z(z) = \int_0^z \lambda^2 e^{-\lambda x} e^{-\lambda(z-x)} dx \tag{2.0.6}$$
$$= z\lambda^2 e^{-\lambda z} \tag{2.0.7}$$

Thus Z is not an exponential random variable. Therefore option (2) is wrong.

3) Let X and Y be independent and identically distributed uniform random variables such that X, Y ~ U(a,b). Then

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & otherwise \end{cases} \tag{2.0.8}$$

Then, using convolution, for $2a < z < a + b$,

$$f_Z(z) = \int_{2a}^z f_X(x) f_Y(z-x)dx \tag{2.0.9}$$
$$= \frac{z-2a}{(b-a)^2} \tag{2.0.10}$$

For $a + b < z < 2b$,

$$X, Y > z - (a+b) \tag{2.0.11}$$
$$Z = X + Y \tag{2.0.12}$$
$$\implies X, Y < (b-a) \tag{2.0.13}$$
$$\implies f_Z(z) = \int_{z-(a+b)}^{b-a} f_X(x) f_Y(z-x)dx \tag{2.0.14}$$
$$= \frac{2b-z}{(b-a)^2} \tag{2.0.15}$$

For $z < 2a$ or $z > 2b$,

$$f_Z(z) = 0 \tag{2.0.16}$$

Thus Z is not a uniform random variable. Thus option (3) is wrong.

4) Let X and Y be independent and identically

distributed binomial random variables. Then the pmf of X and Y is given by

$$p_X(k) = \binom{n}{k} p^k q^{n-k} \tag{2.0.17}$$

Taking Z-transform

$$p_X(z) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} z^{-k} \tag{2.0.18}$$

$$= \left(\frac{p}{z} + q\right)^n \tag{2.0.19}$$

Thus

$$p_Z(z) = p_X^2(z) \tag{2.0.20}$$

$$= \left(\frac{p}{z} + q\right)^{2n} \tag{2.0.21}$$

$$= \sum_{k=0}^{2n} \binom{2n}{k} p^k q^{2n-k} z^{-k} \tag{2.0.22}$$

Thus taking inverse Z-transform,

$$p_Z(k) = \binom{2n}{k} p^k q^{2n-k} \tag{2.0.23}$$

Thus Z is a binomial random variable. Thus option (4) is correct.

The following figures show the experimental distributions for Z in each case. The simulation length was kept one million.

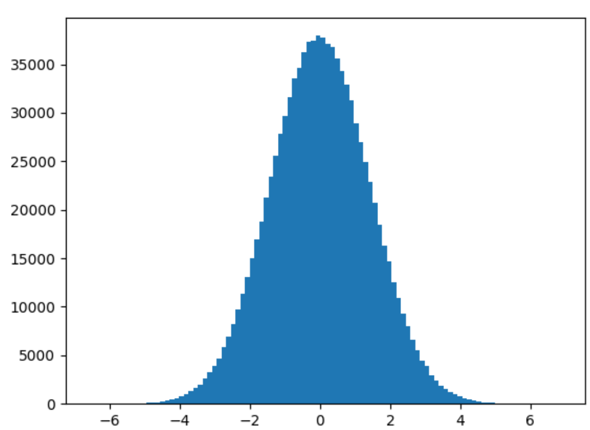

Fig. 4: Z when X is standard normal

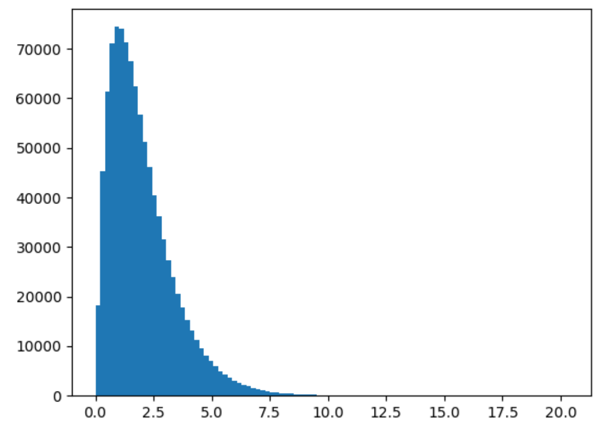

Fig. 4: Z when X is exponential with $\lambda = 1$

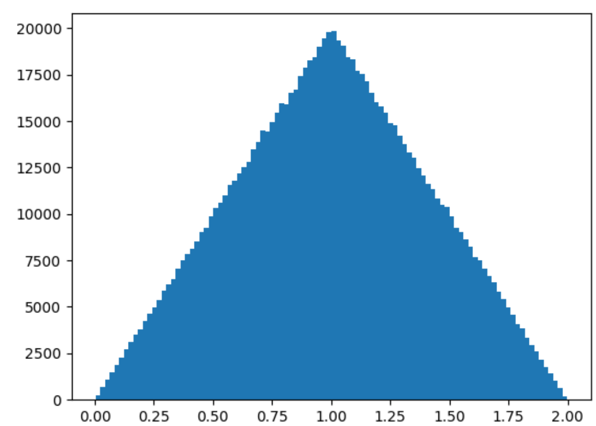

Fig. 4: Z when X $\sim$ U(0,1)

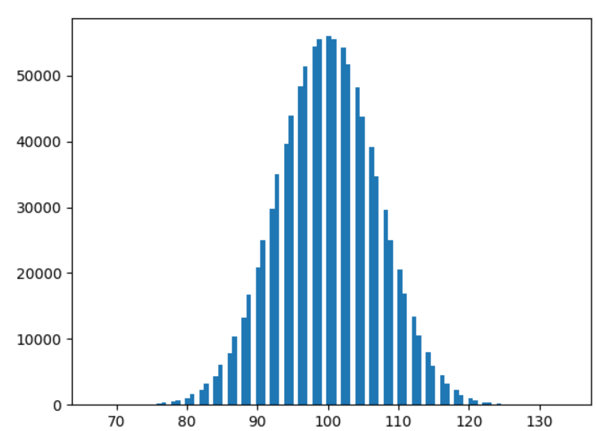

Fig. 4: Z when X $\sim$ B(100,0.5)