Assignment 6

Dishank Jain - AI20BTECH11011

Download all python codes from

```
https://github.com/Dishank422/AI1103−Probability
    −and−random−variables/blob/main/
    Assignment_6/codes
```

and latex-tikz codes from

```
https://github.com/Dishank422/AI1103−Probability
    −and−random−variables/blob/main/
    Assignment_6/main.tex
```

Symbol	Representation
X_1	Arrival time of P_1
$X_1 + X_2$	Arrival time of P_2
$X_1 + X_2 + X_3$	Arrival time of P_3
$Y_1, ..., Y_{10}$	Departure times of the 10 people in park currently
$X_1 + Y_{11}$	Departure time of P_1
$X_1 + X_2 + Y_{12}$	Departure time of P_2

TABLE 3: Notations

1 Problem

(Gate 2021 ST Q. 50) Consider an amusement park where visitors are arriving according to a Poisson process with rate 1. Upon arrival, a visitor spends a random amount of time in the park and then departs. The time spent by the visitors is independent of one another, as well as of the arrival process and have common probability density function

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & otherwise \end{cases} \tag{1.0.1}$$

If at a given point, there are 10 visitors in the park, and p is the probability that there will be exactly two more arrivals before the next departure, then $\frac{1}{p}$ equals.....

2 Solution

According to the question, we want the following events to occur in order:

1) First visitor, P_1 arrives while no one leaves
2) Second visitor P_2 arrives while no one leaves
3) One or more person leaves before the third visitor P_3 arrives

Let the above events be E_1, E_2 and E_3 respectively. Thus the required probability

$$= \Pr(E_1E_2E_3) \tag{2.0.1}$$
$$= \Pr(E_1)\Pr(E_2|E_1)\Pr(E_3|E_1E_2) \tag{2.0.2}$$

$$\Pr(E_1) = \Pr(Y_1, ..., Y_{10} > X_1) \tag{2.0.3}$$
$$= \int_{-\infty}^{\infty} \Pr(Y_1, ..., Y_{10} > x|X_1 = x) \tag{2.0.4}$$
$$= \int_{-\infty}^{\infty} (1 - F_{Y_1}(x))^{10} f_{X_1}(x)dx \tag{2.0.5}$$

Note that (2.0.5) is obtained because $Y_1, ...Y_{12}$ are all identical random variables.

The visitors are arriving according to a Poisson process with rate 1. Thus for X_1,

$$\lambda = 1 * X_1 = X_1 \tag{2.0.6}$$
$$k = 1 \tag{2.0.7}$$
$$\implies f_{X_1}(x) = \begin{cases} \dfrac{x^1e^{-x}}{1!} = xe^{-x} & x > 0 \\ 0 & otherwise \end{cases} \tag{2.0.8}$$

As given in question

$$f_{Y_1}(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & otherwise \end{cases} \tag{2.0.9}$$
$$\implies F_{Y_1}(x) = \begin{cases} 1 - e^{-x} & x > 0 \\ 0 & otherwise \end{cases} \tag{2.0.10}$$

Thus from (2.0.5)

$$\Pr(E_1) = \int_0^{\infty} xe^{-11x}dx \tag{2.0.11}$$

Using Integration by parts

$$\Pr(E_1) = -\frac{xe^{-11x}}{11}\Big|_0^\infty + \frac{1}{11}\int_0^\infty e^{-11x}dx \quad (2.0.12)$$

$$= -\frac{e^{-11x}}{121}\Big|_0^\infty \quad (2.0.13)$$

$$= \frac{1}{121} \quad (2.0.14)$$

We can write $\Pr(E_2|E_1) =$

$$\Pr(Y_1, ..., Y_{10}, X_1 + Y_{11} > X_1 + X_2 | Y_1, ..., Y_{10} > X_1) \quad (2.0.15)$$

Using memoryless property of exponential distribution

$$\Pr(E_2|E_1) = \Pr(Y_1, ..., Y_{11} > X_2) \quad (2.0.16)$$

$$= \int_{-\infty}^{\infty} \Pr(Y_1, ...Y_{11} > x | X_2 = x) \quad (2.0.17)$$

$$= \int_{-\infty}^{\infty} (1 - F_{Y_1}(x))^{11} f_{X_2}(x)dx \quad (2.0.18)$$

Since X_2 is identical to X_1, we can directly substitute the values to obtain

$$\Pr(E_2|E_1) = \int_0^\infty xe^{-12x}dx \quad (2.0.19)$$

Using integration by parts

$$\Pr(E_2|E_1) = -\frac{xe^{-12x}}{12}\Big|_0^\infty + \frac{1}{12}\int_0^\infty e^{-12x}dx \quad (2.0.20)$$

$$= -\frac{e^{-12x}}{144}\Big|_0^\infty \quad (2.0.21)$$

$$= \frac{1}{144} \quad (2.0.22)$$

And finally, $\Pr(E_3|E_1E_2) =$

$$\Pr(min(Y_1, ..., Y_{10}, X_1 + Y_{11}, X_1 + X_2 + Y_{12}) < X_1 + X_2 + X_3 | Y_1, ..., Y_{10}, X_1 + Y_{11} > X_1 + X_2) \quad (2.0.23)$$

We can simplify and write $\Pr(E_3|E_1E_2) =$

$$1 - \Pr(Y_1, ..., Y_{10}, X_1 + Y_{11}, X_1 + X_2 + Y_{12} > X_1 + X_2 + X_3 | Y_1, ..., Y_{10}, X_1 + Y_{11} > X_1 + X_2) \quad (2.0.24)$$

Using memoryless property of exponential distribution

$$\Pr(E_3|E_1E_2) = 1 - \Pr(Y_1, ..., Y_{12} > X_3) \quad (2.0.25)$$

$$= 1 - \int_{-\infty}^{\infty} \Pr(Y_1, ...Y_{12} > x | X_3 = x) \quad (2.0.26)$$

$$= 1 - \int_{-\infty}^{\infty} (1 - F_{Y_1}(x))^{12} f_{X_3}(x)dx \quad (2.0.27)$$

Since X_3 is identical to X_1, we can directly substitute the values to obtain

$$\Pr(E_3|E_1E_2) = 1 - \int_0^\infty xe^{-13x}dx \quad (2.0.28)$$

Using integration by parts

$$\Pr(E_3|E_1E_2) = 1 - \left(-\frac{xe^{-13x}}{13}\Big|_0^\infty + \frac{1}{13}\int_0^\infty e^{-13x}dx\right) \quad (2.0.29)$$

$$= 1 - \left(-\frac{e^{-13x}}{169}\Big|_0^\infty\right) \quad (2.0.30)$$

$$= 1 - \frac{1}{169} \quad (2.0.31)$$

$$= \frac{168}{169} \quad (2.0.32)$$

Thus on substituting values in (2.0.2),

$$\Pr(E_1E_2E_3) = \frac{1}{121} \times \frac{1}{144} \times \frac{168}{169} \quad (2.0.33)$$