

Assignment 9

Dishank Jain - AI20BTECH11011

Download latex-tikz codes from

https://github.com/Dishank422/AI1103-Probability-and-random-variables/blob/main/Assignment_9/main.tex

1 PROBLEM

(CSIR UGC NET, June 2016, Q.107) Suppose X and Y are independent and identically distributed random variables and let $Z = X + Y$. Then the distribution of Z is in the same family as that of X and Y if X is

- | | |
|------------|----------------|
| 1) Normal | 2) Exponential |
| 3) Uniform | 4) Binomial |

2 SOLUTION

- 1) Let X and Y be independent and identically distributed normal random variables. Then the characteristic function of X and Y is given by

$$\Phi_X(\omega) = e^{j\eta\omega - \sigma^2\omega^2/2} \quad (2.0.1)$$

The characteristic function of Z is given by

$$\Phi_Z(\omega) = \Phi_X^2(\omega) \quad (2.0.2)$$

$$= e^{2j\eta\omega - \sigma^2\omega^2} \quad (2.0.3)$$

Thus Z is a normal random variable with parameters 2η and $2\sigma^2$. Thus option (1) is correct.

- 2) Let X and Y be independent and identically distributed exponential random variables. Then the pdf of X and Y is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.0.4)$$

Using convolution,

$$f_Z(z) = \begin{cases} \int_0^z f_X(x)f_Y(z-x)dx & z > 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.0.5)$$

Thus if $z > 0$,

$$f_Z(z) = \int_0^z \lambda^2 e^{-\lambda x} e^{-\lambda(z-x)} dx \quad (2.0.6)$$

$$= z\lambda^2 e^{-\lambda z} \quad (2.0.7)$$

Thus Z is not an exponential random variable. Therefore option (2) is wrong.

- 3) Let X and Y be independent and identically distributed uniform random variables such that $X, Y \sim U(a, b)$. Then

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases} \quad (2.0.8)$$

Then, using convolution, for $2a < z < a + b$,

$$f_Z(z) = \int_{2a}^z f_X(x)f_Y(z-x)dx \quad (2.0.9)$$

$$= \frac{z-2a}{(b-a)^2} \quad (2.0.10)$$

For $a + b < z < 2b$,

$$X, Y > z - (a + b) \quad (2.0.11)$$

$$Z = X + Y \quad (2.0.12)$$

$$\Rightarrow X, Y < (b - a) \quad (2.0.13)$$

$$\Rightarrow f_Z(z) = \int_{z-(a+b)}^{b-a} f_X(x)f_Y(z-x)dx \quad (2.0.14)$$

$$= \frac{2b-z}{(b-a)^2} \quad (2.0.15)$$

For $z < 2a$ or $z > 2b$,

$$f_Z(z) = 0 \quad (2.0.16)$$

Thus Z is not a uniform random variable. Thus option (3) is wrong.

- 4) Let X and Y be independent and identically distributed binomial random variables. Then the pmf of X and Y is given by

$$p_X(k) = \binom{n}{k} p^k q^{n-k} \quad (2.0.17)$$

Taking Z-transform

$$p_X(z) = \sum_{k=0}^n \binom{n}{k} p^k q^{n-k} z^{-k} \quad (2.0.18)$$

$$= \left(\frac{p}{z} + q \right)^n \quad (2.0.19)$$

Thus

$$p_Z(z) = p_X^2(z) \quad (2.0.20)$$

$$= \left(\frac{p}{z} + q \right)^{2n} \quad (2.0.21)$$

$$= \sum_{k=0}^{2n} \binom{2n}{k} p^k q^{2n-k} z^{-k} \quad (2.0.22)$$

Thus taking inverse Z-transform,

$$p_Z(k) = \binom{2n}{k} p^k q^{2n-k} \quad (2.0.23)$$

Thus Z is a binomial random variable. Thus option (4) is correct.